1 Problem

Compute the derivatives of the least squares loss $L[\phi]$ with respect to ϕ_0 and ϕ_1 for the Gabor model equation 6.8.

2 Answer

$$f[x, \phi] = \sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x)^2}{32.0}\right) \tag{1}$$

$$L[\phi] = \sum_{i=1}^{I} (f[x_i, \phi] - y_i)^2 \tag{2}$$

$$
\begin{aligned}
\text{Let } u_i &= f[x_i, \phi] - y_i \\
\text{Then } L[\phi] &= \sum_{i=1}^{I} (u_i)^2 \\
\frac{\partial L[\phi]}{\partial \phi_0} &= \sum_{i=1}^{I} \frac{\partial L[\phi]}{\partial u_i} \cdot \ \frac{\partial u_i}{\partial \phi_0} \\
\text{Let us break it down } \frac{\partial L[\phi]}{\partial u_i} &= 2 \cdot u_i = 2 \cdot \left(\sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) - y_i \right) \\
\frac{\partial u_i}{\partial \phi_0} &= \frac{\partial \sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right)}{\partial \phi_0} \\
\text{Let } p &= \sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \\
\text{Let } q &= \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\
\text{So that } \frac{\partial u_i}{\partial \phi_0} &= p \frac{\partial q}{\partial \phi_0} + q \frac{\partial p}{\partial \phi_0} \\
\text{Let } r &= \phi_0 + 0.06 \cdot \phi_1 x_i \\
\text{So that } \frac{\partial p}{\partial \phi_0} &= \frac{\partial \sin(r)}{\partial r} \cdot \frac{\partial r}{\partial \phi_0} = \cos(r) \cdot (1) = \cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \\
\text{Let } s &= \left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\
\text{So that} \frac{\partial q}{\partial \phi_0} &= \frac{\partial \exp(s)}{\partial s} \cdot \frac{\partial s}{\partial \phi_0} = \exp(s) \cdot \frac{\partial s}{\partial \phi_0} \\
\text{Let } t &= \phi_0 + 0.06 \cdot \phi_1 x_i \\
\text{So that } \frac{\partial s}{\partial \phi_0} &= \frac{\partial \frac{-1}{32} t^2}{\partial t} \cdot \frac{\partial t}{\partial \phi_0} = \frac{-1}{16} \cdot (\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot (1) = -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \\
\text{Substituting } \frac{\partial q}{\partial \phi_0} &= \exp(s) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} = -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\
\text{Similarly } \frac{\partial p}{\partial \phi_0} &= \cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \\
\text{So that } \frac{\partial u_i}{\partial \phi_0} &= \sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\
&\quad + \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \cdot \cos(\phi_0 + 0.06 \cdot \phi_1 x_i)
\end{aligned}
$$

$$\text{Finally } \frac{\partial L[\phi]}{\partial \phi_0} = \sum_{i=1}^{I} 2 \cdot \left(\sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) - y_i \right)$$
$$\cdot \left[\sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot - \frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) + \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \cdot \cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \right]$$

3 Using SymPy

```
 ~ python
Python 3.11.9 | packaged by conda-forge | (main, Apr 19 2024, 18:34:54) [Clang 16.0.6 ] on darwin
Type "help", "copyright", "credits" or "license" for more information.
>>> from sympy import *
>>> x = Symbol('x')
>>> phi0 = Symbol('phi0')
>>> phi1 = Symbol('phi1')

>>> print(latex(diff(sin(phi0 + 0.06 * phi1 * x) * exp((-(phi0 + 0.06 * phi1*x)**2)/32.0), phi0)))
\left(- 0.0625 \phi_{0} - 0.00375 \phi_{1} x\right) e^{- 0.03125
\left(\phi_{0} + 0.06 \phi_{1} x\right)^{2}} \sin{\left(\phi_{0}
+ 0.06 \phi_{1} x \right)} + e^{- 0.03125 \left(\phi_{0} + 0.06
\phi_{1} x\right)^{2}} \cos{\left(\phi_{0} + 0.06 \phi_{1} x
\right)}
>>> 1/16
0.0625
>>> 1/32
0.03125
>>>
>>> print(latex(diff(sin(phi0 + 0.06 * phi1 * x) * exp((-(phi0 + 0.06 * phi1*x)**2)/32.0), phi1)))
- 0.00375 x \left(\phi_{0} + 0.06 \phi_{1} x\right)
e^{- 0.03125 \left(\phi_{0} + 0.06 \phi_{1} x\right)^{2}}
\sin{\left(\phi_{0} + 0.06 \phi_{1} x \right)} + 0.06 x
e^{- 0.03125 \left(\phi_{0} + 0.06 \phi_{1} x\right)^{2}}
\cos{\left(\phi_{0} + 0.06 \phi_{1} x \right)}
>>> 3/800
0.00375
```

$$\frac{\partial L[\phi]}{\partial \phi_0} = \left(-\frac{1}{16}\phi_0 - \frac{1}{32}\phi_1 x \right) e^{-\frac{1}{32}(\phi_0 + 0.06\phi_1 x)^2} \sin\left(\phi_0 + 0.06\phi_1 x\right) + e^{-\frac{1}{32}(\phi_0 + 0.06\phi_1 x)^2} \cos\left(\phi_0 + 0.06\phi_1 x\right)$$

$$
\begin{aligned}
\frac{\partial L[\phi]}{\partial \phi_1} = & -\frac{3}{800}x\left(\phi_0+0.06\phi_1 x\right)e^{-\frac{1}{32}\left(\phi_0+0.06\phi_1 x\right)^2}\sin\left(\phi_0+0.06\phi_1 x\right) \\
& +0.06xe^{-\frac{1}{32}\left(\phi_0+0.06\phi_1 x\right)^2}\cos\left(\phi_0+0.06\phi_1 x\right)
\end{aligned}
$$