1 Problem

Compute the derivatives of the least squares loss $L[\phi]$ with respect to ϕ_0 and ϕ_1 for the Gabor model equation 6.8.

2 Answer

$$f[x, \phi] = \sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x)^2}{32.0}\right) \tag{1}$$

$$L[\phi] = \sum_{i=1}^{I} (f[x_i, \phi] - y_i)^2 \tag{2}$$

$$\text{Let } u_i = f[x_i, \phi] - y_i$$

$$\text{Then } L[\phi] = \sum_{i=1}^{I} (u_i)^2$$

$$\frac{\partial L[\phi]}{\partial \phi_0} = \sum_{i=1}^{I} \frac{\partial L[\phi]}{\partial u_i} \cdot \frac{\partial u_i}{\partial \phi_0}$$

$$\text{Let us break it down } \frac{\partial L[\phi]}{\partial u_i} = 2 \cdot u_i = 2 \cdot \left(\sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) - y_i \right)$$

$$\frac{\partial u_i}{\partial \phi_0} = \frac{\partial \sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right)}{\partial \phi_0}$$

$$\text{Let } p = \sin(\phi_0 + 0.06 \cdot \phi_1 x_i)$$

$$\text{Let } q = \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right)$$

$$\text{So that } \frac{\partial u_i}{\partial \phi_0} = p \frac{\partial q}{\partial \phi_0} + q \frac{\partial p}{\partial \phi_0}$$

$$\text{Let } r = \phi_0 + 0.06 \cdot \phi_1 x_i$$

$$\text{So that } \frac{\partial p}{\partial \phi_0} = \frac{\partial \sin(r)}{\partial r} \cdot \frac{\partial r}{\partial \phi_0} = \cos(r) \cdot (1) = \cos(\phi_0 + 0.06 \cdot \phi_1 x_i)$$

$$\text{Let } s = \left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right)$$

$$\text{So that} \frac{\partial q}{\partial \phi_0} = \frac{\partial \exp(s)}{\partial s} \cdot \frac{\partial s}{\partial \phi_0} = \exp(s) \cdot \frac{\partial s}{\partial \phi_0}$$

$$\text{Let } t = \phi_0 + 0.06 \cdot \phi_1 x_i$$

$$\text{So that } \frac{\partial s}{\partial \phi_0} = \frac{\partial \frac{-1}{32} t^2}{\partial t} \cdot \frac{\partial t}{\partial \phi_0} = \frac{-1}{16} \cdot (\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot (1) = -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16}$$

$$\text{Substituting } \frac{\partial q}{\partial \phi_0} = \exp(s) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} = -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right)$$

$$\text{Similarly } \frac{\partial p}{\partial \phi_0} = \cos(\phi_0 + 0.06 \cdot \phi_1 x_i)$$

$$\begin{aligned} \text{So that } \frac{\partial u_i}{\partial \phi_0} &= \sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\ &+ \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \cdot \cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \end{aligned}$$

$$\text{Finally } \frac{\partial L[\phi]}{\partial \phi_0} = \sum_{i=1}^{I} 2 \cdot \left(\sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) - y_i \right)$$
$$\cdot \left[\sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot - \frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) \right.$$
$$\left. + \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) \cdot \cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \right]$$

3 Using SymPy

```
>>> L
    (-y + exp(-0.03125*(phi0 + 0.06*phi1*x)**2)*sin(phi0 + 0.06*phi1*x))**2
>>> print(latex(diff(L,phi0)))
    \left(- y + e^{- 0.03125 \left(\phi_{0} + 0.06 \phi_{1} x\right)^{2}}
    \sin{\left(\phi_{0} + 0.06 \phi_{1} x \right)}\right) \left(2
    \left(- 0.0625 \phi_{0} - 0.00375 \phi_{1} x\right) e^{- 0.03125
    \left(\phi_{0} + 0.06 \phi_{1} x\right)^{2}} \sin{\left(\phi_{0} + 0.06
    \phi_{1} x \right)} + 2 e^{- 0.03125 \left(\phi_{0} + 0.06 \phi_{1}
    x\right)^{2}} \cos{\left(\phi_{0} + 0.06 \phi_{1} x \right)}\right)
>>> print(latex(diff(L,phi1)))
    \left(- y + e^{- 0.03125 \left(\phi_{0} + 0.06 \phi_{1} x\right)^{2}}
    \sin{\left(\phi_{0} + 0.06 \phi_{1} x \right)}\right) \left(- 0.0075 x
    \left(\phi_{0} + 0.06 \phi_{1} x\right) e^{- 0.03125 \left(\phi_{0} +
    0.06 \phi_{1} x\right)^{2}} \sin{\left(\phi_{0} + 0.06 \phi_{1} x
    \right)} + 0.12 x e^{- 0.03125 \left(\phi_{0} + 0.06 \phi_{1} x\right)^{2}}
    \cos{\left(\phi_{0} + 0.06 \phi_{1} x \right)}\right)
    >>> 1/16
    0.0625
    >>> 1/32
    0.03125
    >>> 3/800
    0.00375
    >>> 3/400
    0.0075
    >>> 3/25
    0.12
```

$$\frac{\partial L[\phi]}{\partial \phi_0} = \left(-y_i + e^{-\frac{1}{32}(\phi_0 + 0.06\phi_1 x_i)^2} \sin\left(\phi_0 + 0.06\phi_1 x_i\right)\right) \cdot$$
$$\left(2\left(-\frac{1}{16}\phi_0 - \frac{3}{800}\phi_1 x_i\right) e^{-\frac{1}{32}(\phi_0 + 0.06\phi_1 x_i)^2} \sin\left(\phi_0 + 0.06\phi_1 x_i\right)\right.$$
$$\left. + 2e^{-\frac{1}{32}(\phi_0 + 0.06\phi_1 x_i)^2} \cos\left(\phi_0 + 0.06\phi_1 x_i\right)\right)$$

$$
\begin{aligned}
\frac{\partial L[\phi]}{\partial \phi_1} = &\left(-y_i + e^{-\frac{1}{32}(\phi_0+0.06\phi_1 x_i)^2}\sin\left(\phi_0+0.06\phi_1 x_i\right)\right)\cdot \\
&\left(-\frac{3}{400}x_i\left(\phi_0+0.06\phi_1 x_i\right)e^{-\frac{1}{32}(\phi_0+0.06\phi_1 x_i)^2}\sin\left(\phi_0+0.06\phi_1 x_i\right)\right. \\
&\qquad\left. + 0.12x_i e^{-\frac{1}{32}(\phi_0+0.06\phi_1 x_i)^2}\cos\left(\phi_0+0.06\phi_1 x_i\right)\right)
\end{aligned}
$$