

1 Problem

Compute the derivatives of the least squares loss $L[\phi]$ with respect to ϕ_0 and ϕ_1 for the Gabor model equation 6.8.

2 Answer

$$f[x, \phi] = \sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x)^2}{32.0}\right) \quad (1)$$

$$L[\phi] = \sum_{i=1}^I (f[x_i, \phi] - y_i)^2 \quad (2)$$

$$\text{Let } u_i = f[x_i, \phi] - y_i$$

$$\text{Then } L[\phi] = \sum_{i=1}^I (u_i)^2$$

$$\frac{\partial L[\phi]}{\partial \phi_0} = \sum_{i=1}^I \frac{\partial L[\phi]}{\partial u_i} \cdot \frac{\partial u_i}{\partial \phi_0}$$

$$\text{Let us break it down } \frac{\partial L[\phi]}{\partial u_i} = 2 \cdot u_i = 2 \cdot \left(\sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) - y_i \right)$$

$$\frac{\partial u_i}{\partial \phi_0} = \frac{\partial \sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right)}{\partial \phi_0}$$

$$\text{Let } p = \sin(\phi_0 + 0.06 \cdot \phi_1 x_i)$$

$$\text{Let } q = \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right)$$

$$\text{So that } \frac{\partial u_i}{\partial \phi_0} = p \frac{\partial q}{\partial \phi_0} + q \frac{\partial p}{\partial \phi_0}$$

$$\text{Let } r = \phi_0 + 0.06 \cdot \phi_1 x_i$$

$$\text{So that } \frac{\partial p}{\partial \phi_0} = \frac{\partial \sin(r)}{\partial r} \cdot \frac{\partial r}{\partial \phi_0} = \cos(r) \cdot (1) = \cos(\phi_0 + 0.06 \cdot \phi_1 x_i)$$

$$\text{Let } s = \left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right)$$

$$\text{So that } \frac{\partial q}{\partial \phi_0} = \frac{\partial \exp(s)}{\partial s} \cdot \frac{\partial s}{\partial \phi_0} = \exp(s) \cdot \frac{\partial s}{\partial \phi_0}$$

$$\text{Let } t = \phi_0 + 0.06 \cdot \phi_1 x_i$$

$$\text{So that } \frac{\partial s}{\partial \phi_0} = \frac{\partial \frac{-1}{32} t^2}{\partial t} \cdot \frac{\partial t}{\partial \phi_0} = \frac{-1}{16} \cdot (\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot (1) = -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16}$$

$$\text{Substituting } \frac{\partial q}{\partial \phi_0} = \exp(s) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} = -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right)$$

$$\text{Similarly } \frac{\partial p}{\partial \phi_0} = \cos(\phi_0 + 0.06 \cdot \phi_1 x_i)$$

$$\begin{aligned} \text{So that } \frac{\partial u_i}{\partial \phi_0} &= \sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) \\ &\quad + \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) \cdot \cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \end{aligned}$$

$$\begin{aligned}
\text{Finally } \frac{\partial L[\phi]}{\partial \phi_0} &= \sum_{i=1}^I 2 \cdot \left(\sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) - y_i \right) \\
&\quad \cdot \left[\sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) \right. \\
&\quad \left. + \exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) \cdot \cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \right]
\end{aligned}$$