1 Problem

Compute the derivatives of the least squares loss $L[\phi]$ with respect to ϕ_0 and ϕ_1 for the Gabor model equation 6.8.

2 Answer

$$f[x,\phi] = sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x)^2}{32.0}\right) \tag{1}$$

$$L[\phi] = \sum_{i=1}^{I} (f[x_i, \phi] - y_i)^2 \tag{2}$$

$$
\begin{aligned}
\text{Let } u_i &= f[x_i, \phi] - y_i \\
\text{Then } L[\phi] &= \sum_{i=1}^{I} (u_i)^2 \\
\frac{\partial L[\phi]}{\partial \phi_0} &= \sum_{i=1}^{I} \frac{\partial L[\phi]}{\partial u_i} \cdot \ \frac{\partial u_i}{\partial \phi_0} \\
\text{Let us break it down } \frac{\partial L[\phi]}{\partial u_i} &= 2 \cdot u_i = 2 \cdot \left(sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) - y_i \right) \\
\frac{\partial u_i}{\partial \phi_0} &= \frac{\partial sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right)}{\partial \phi_0} \\
\text{Let } p &= sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \\
\text{Let } q &= exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\
\text{So that } \frac{\partial u_i}{\partial \phi_0} &= p \frac{\partial q}{\partial \phi_0} + q \frac{\partial p}{\partial \phi_0} \\
\text{Let } r &= \phi_0 + 0.06 \cdot \phi_1 x_i \\
\text{So that } \frac{\partial p}{\partial \phi_0} &= \frac{\partial sin(r)}{\partial r} \cdot \frac{\partial r}{\partial \phi_0} = cos(r) \cdot (1) = cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \\
\text{Let } s &= \left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\
\text{So that} \frac{\partial q}{\partial \phi_0} &= \frac{\partial exp(s)}{\partial s} \cdot \frac{\partial s}{\partial \phi_0} = exp(s) \cdot \frac{\partial s}{\partial \phi_0} \\
\text{Let } t &= \phi_0 + 0.06 \cdot \phi_1 x_i \\
\text{So that } \frac{\partial s}{\partial \phi_0} &= \frac{\partial \frac{-1}{32} t^2}{\partial t} \cdot \frac{\partial t}{\partial \phi_0} = \frac{-1}{16} \cdot (\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot (1) = -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \\
\text{Substituting } \frac{\partial q}{\partial \phi_0} &= exp(s) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} = -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\
\text{Similarly } \frac{\partial p}{\partial \phi_0} &= cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \\
\text{So that } \frac{\partial u_i}{\partial \phi_0} &= sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \\
&\quad + exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0} \right) \cdot cos(\phi_0 + 0.06 \cdot \phi_1 x_i)
\end{aligned}
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Finally $\dfrac{\partial L[\phi]}{\partial \phi_0} = \sum_{i=1}^{I} 2 \cdot \left(sin(\phi_0 + 0.06 \cdot \phi_1 x) \cdot exp\left(\dfrac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) - y_i \right)$

$$\cdot \Bigg[sin(\phi_0 + 0.06 \cdot \phi_1 x_i) \cdot -\frac{\phi_0 + 0.06 \cdot \phi_1 x_i}{16} \cdot exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right)$$

$$+ exp\left(\frac{-(\phi_0 + 0.06 \cdot \phi_1 x_i)^2}{32.0}\right) \cdot cos(\phi_0 + 0.06 \cdot \phi_1 x_i) \Bigg]$$