Chapter 4

Predicate Logic and Proof

The material on predicates discussed in this chapter can be found in Section 1.3 of Rosen.

So far we have only considered simple propositions involving specific objects. Propositional logic is not sufficient to express all sentences or arguments. For example, "Peter likes football" is a proposition which may be true or false, and we can write it as likes(peter,football). (This is a relation as defined in CS1860.) But how do we write sentences such as

- Everyone likes football
- Some people like the Simpsons
- Someone has stolen John's car
- Every student is doing some course

And how do we work out whether such sentences are true or false?

4.1 Predicates

To deal with real applications we need to have propositions which make statements about collections of objects. For example, 'all integers which are divisible by 4 are even' and 'there exists an integer n such that $n^2 = 36$'.

- variables to stand for individual objects
- quantifiers to express how many objects are involved
- domain of discourse to define the range of objects

A statement about a property of an object is called a *predicate*. We use the notation $P(x)$ to denote a statement which concerns properties of the object x.

The set of values over which the quantifier ranges is called the *domain of discourse*.

For "Everyone likes football" and for "Some people like cats" the domain of discourse is the set of all people. The first sentence means all members of the set like football, the second is saying at least one of the domain like cats, but not necessarily everyone.

In the following examples the domain of discourse is the set of integers. For example, we may have

$$P(n) = ((n \text{ divisible by } 4) \Rightarrow (n \text{ even})),$$

where n is an integer.

Then we write the statement 'all integers which are divisible by 4 are even' as 'for all integers n, $P(n)$'.

Another example: $\quad P(n) = (n^2 = 36).$

Then we write the statement 'there is an integer such that $n^2 = 36$' as 'there exists an integer n such that $P(n)$'.

4.2 Universal and existential quantifiers

There are two cases: $P(x)$ may be true for all choices of x from a particular set, or $P(x)$ may be true for some choice of x from the set. (The case when $P(x)$ is never true is not interesting.) There are two symbols, called *predicate quantifiers*, which are used to denote these two cases.

Universal quantifier

The symbol $\forall$, called *for all*, is used to indicate that a statement applies to all choices of object.

$\forall x\ P(x)$ – for all x, $P(x)$ is true.

If the domain of discourse is the set of all people, "Everyone likes football" can be written $\forall x\ likes(x, football)$ or as $\forall x\ (person(x) \Rightarrow likes(x, football))$.

Examples

Domain of discourse: set of all people, "Everyone likes the Big Bang Theory" could be written $\forall x\ likes(x, TheBigBangTheory)$.

Domain of discourse: set of all people, "Everyone on the escalator must wear shoes" could be written $\forall x\ (on\_escalator(x) \Rightarrow wears(x, shoes))$.

Domain of discourse: $\mathbb{Z}$, "All integers which are divisible by 4 are even" could be written $\forall n\ (divisible(n, 4) \Rightarrow even(n))$.

Domain of discourse: set of all cats, "All cats are friendly and have four legs" could be written $\forall x\ (friendly(x)\ \wedge\ has\_legs(x, 4))$.

Strictly speaking we should include the set that n ranges over, so when

$$P(n) = (divisible(n, 4) \Rightarrow even(n))$$

we would write $\forall(\text{integers } n)P(n)$,
but this becomes complicated so it is more usual to state the domain of discourse,

$$\forall n P(n), \quad \text{where } n \text{ is an integer.}$$

Exercises 4.1
For the following suggest a suitable domain of discourse
and then express each as a formula using the universal quantifier:
(i) All horses eat hay
(ii) Dogs have teeth
(iii) The square of an odd number is odd

Existential quantifier

The symbol $\exists$, called *there exists*, is used to indicate that a statement is true for at least one choice of object in the domain of discourse.

$\exists x\ P(x)$ – there exists an x such that $P(x)$ is true.

Examples

Domain of discourse: set of all people, “Some people like cats” could be written $\exists x$ $likes(x, cats)$.

Domain of discourse: set of all people, “Someone has stolen the car” could be written $\exists x\ stolen(x, the\_car)$.

Domain of discourse: $\mathbb{Z}$, “Some integer is equal to 4^2” could be written $\exists n\ (n = 4^2)$.

Truth values in predicate logic

Like all propositions, a predicate may be true or false.

$\exists x P(x)$ is **true** if there is at least one x in the domain of discourse with $P(x)$ is true.
$\exists x P(x)$ is **false** if $P(x)$ is false for all possible values of x.
$\forall x P(x)$ is **true** if $P(x)$ is true for all possible values of x.
$\forall x P(x)$ is **false** if there is at least one value of x for which $P(x)$ is false.

Example

If we have $P(n) = (n^2 = 36)$ and domain of discourse $\mathbb{Z}$ then
$\forall n\ P(n)$ is false, it is not true when $n = 1$
$\exists n\ P(n)$ is true, it is true for $n = 6$

4.2.1 Proof by existence and counter example

To prove $\forall x P(x)$ is true you have to prove $P(x)$ is true for all x in the domain of discourse. Techniques such as proof by induction and contradiction can be used for this.

To prove that $\forall x P(x)$ is false you just have to find one value of x in the domain of discourse for which $P(x)$ is false. This is called a proof by *counter example*.

To prove that $\exists x P(x)$ is true you just have to find one value of x in the domain of discourse for which $P(x)$ is true. This is called a proof by *existence*.

To prove $\exists x P(x)$ is false you have to prove $P(x)$ is false for all x in the domain of discourse. Again, techniques such as induction and contradiction can be used.

Examples

In the following the domain of discourse is $\mathbb{Z}$.
(i) $\exists n(n^2 = 16)$ is true, proof by existence $n = 4$, $4^2 = 16$ is true.
(ii) $\forall n(n^2 = 4)$ is false, proof by counterexample $n = 1$, $1^2 \neq 4$.
(iii) $\exists n(n^2 = 3)$ is false (but it would be true if the domain of discourse were $\mathbb{R}$). This can be proved by contradiction using properties of division by primes.

Exercises 4.2
Give the truth values of the following, if the domain of discourse is $\mathbb{Z}$.
1. $\forall n((n \text{ is divisible by } 4) \Rightarrow (n \text{ is even}))$
2. $\exists n(n^2 = 4)$
3. $\forall n(n^2 > 0)$
4. $\exists n(integer(n) \wedge (n^2 = 5))$
5. $\exists y \forall x(x + y = 0)$
6. $\forall x \exists y(x + y = 0)$

4.3 Multiple parameters

What about statements with more than one variable?

"Any dog on the escalator must be carried by somebody"

$\forall x \exists y((on\_escalator(x) \wedge dog(x)) \Rightarrow (person(y) \wedge carried\_by(x, y)))$

Predicates can involve more than one object. For example, the property $y = x^2$. In this case we write $P(x, y)$. We can then have a quantifier for each object.

$\forall x \exists y\ P(x, y) \qquad \forall x \exists y(y = x^2) \qquad \forall x \exists y(\text{integer}(x) \Rightarrow (\text{integer}(y) \wedge (y = x^2)))$

are all versions of the proposition:

for all x there exists some y such that, $y = x^2$ (all squares of integers are integers)

$\exists y \forall x\ P(x, y) \qquad \exists y \forall x(y = x^2) \qquad \exists y \forall x(\text{integer}(x) \Rightarrow (\text{integer}(y) \wedge (y = x^2)))$

are all versions of the proposition:

there exists some y such that for all x, $y = x^2$ (some integer is the square of every integer)

If x, y are integers then the first proposition is true, the second is false.

Exercise 4.3
Decide whether each of the propositions are true or false, when x, y are integers
$\forall y \exists x\ (y = x^2)$ and $\exists y \exists x\ (y = x^2)$

We can also form the negation of predicates, if $\neg \forall x\ P(x)$ is true exactly when $\forall x\ P(x)$ is false. In fact we have

$$\neg \forall x\ P(x) = \exists x\ \neg P(x)$$
$$\neg \exists x\ P(x) = \forall x\ \neg P(x)$$

4.4 n-ary Predicates

Predicates were used in CS1860 to define sets and relations. Predicates in CS1860 usually had one variable and relations had two.

For example: $odd(x)$ (meaning that x is an odd number) is a predicate; $less(x, y)$ (meaning that $x < y$) is a relation.

Both words are often used with more parameters, and then the difference between them disappears. For example, $parents(x, y, z)$ (meaning that x is z's father and y is z's mother) can be referred to as a ternary predicate or a ternary relation.

Generally, an *n-ary predicate* (also referred to as an n-ary relation), where n is a positive integer, is a property of n-tuples $(x_1, \ldots, x_n)$.

We use the terms "unary" for a predicate with 1 variable, "binary" for a predicate with 2 variables, and "ternary" for a predicate with 3 variables.

An *atomic formula* in predicate logic has the form $predicate\_name(a_1, a_2, \ldots, a_n)$. The atomic formula may contain variables, so that if precise objects are supplied for the variables the formula becomes a proposition. For example, the atomic formula "$likes(x, football)$" becomes a proposition for a particular x, such as "$likes(pete, football)$".

Names such as "pete" used to refer to particular values of variables are *constants.*

Summary on predicates

We can use predicate symbols such as P, Q, R to represent predicates in formulas. These symbols are predicate names.

- 1 argument: $P(x)$ is a unary predicate or property e.g., even(n), friendly(x)
- 2 arguments: $P(x, y)$ is a binary predicate or relation e.g., likes(x, y), divisible(n, m)
- 3 arguments: $P(x, y, z)$ is a ternary predicate e.g., parents(x, y, z)
- n arguments: $P(a_1, a_2, \ldots, a_n)$ is an n-ary predicate, with n arguments

4.5 Well formed formulae (wff)

We can use the logical symbols of propositional calculus to build logical expressions from predicates.

By convention, variables are written using x, y, z and constants are written $a, b, c, \ldots$ We can also use predicate symbols such as P, Q, R. For example:

$\forall x(P(x) \Rightarrow \exists y Q(x, y))$
$\exists x(P(x) \wedge Q(x, c))$

The formal definition of a *well formed formula* (wff) is by structural induction (see Section 3.3 of Rosen) but for this course it is sufficient to think of a wff as any sentence constructed using predicates, variables, quantifiers and the logical connectives, $\wedge$, $\vee$, $\Rightarrow$ and $\neg$.

Examples

Suppose that the domain is all people, and that $L(x, y)$ means "x likes y", $F(x)$ means "x can speak French", and $J(x)$ means "x knows Java". Then
(i) "some people know Java and speak French" has formula $\exists x(J(x) \wedge F(x))$
(ii) "everyone who can speak French knows Java" has formula $\forall x(F(x) \Rightarrow J(x))$
(iii) "everyone who speaks French likes someone who knows Java" has formula $\forall x(F(x) \Rightarrow \exists y(L(x, y) \wedge J(y)))$

Exercises 4.4 Using the predicates in the previous example, write formulas for:
1. Some people don't like Fred.
2. There is a person who can speak French and knows Java.
3. Some people can speak French but don't know Java.
4. Everyone can speak French or knows Java.
5. No one can speak French or knows Java.
6. Everyone likes everyone else and themselves.
7. Some people like everyone except themselves.

Logical equivalences

Wff manipulation forms the basis of *predicate calculus*, the study of equality of logical expressions. We consider only first order predicate calculus in which the quantifiers can only range over variables, not functions.

We have the following logical equivalences for wwfs A and B.

Re-naming:

$\forall x A(x) \Leftrightarrow \forall y A(y)$
$\exists x A(x) \Leftrightarrow \exists y A(y)$

Negation:

$\neg \forall x A(x) \Leftrightarrow \exists x(\neg A(x))$
$\neg \exists x A(x) \Leftrightarrow \forall x(\neg A(x))$

Distributive laws:

$\forall x(A(x) \wedge B(x)) \Leftrightarrow \forall x A(x) \wedge \forall x B(x)$
$\exists x(A(x) \vee B(x)) \Leftrightarrow \exists x A(x) \vee \exists x B(x))$

Convention: $\forall$ and $\exists$ have higher precedence than $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$.

Example

Show that $\neg \forall x(P(x) \wedge Q(x)) \Leftrightarrow (\exists x \neg P(x) \vee \exists y \neg Q(y))$

$$\begin{aligned}
\neg \forall x(P(x) \wedge Q(x)) &\Leftrightarrow \exists x \neg(P(x) \wedge Q(x)) \\
&\Leftrightarrow \exists x(\neg P(x) \vee \neg Q(x)) \\
&\Leftrightarrow \exists x(\neg P(x)) \vee \exists x(\neg Q(x))) \\
&\Leftrightarrow \exists x(\neg P(x)) \vee \exists y(\neg Q(y)))
\end{aligned}$$

Exercises 4.5 Show that
$\forall x(P(x) \wedge \neg Q(x)) \Leftrightarrow (\forall x P(x) \wedge \neg \exists y Q(y))$
$\exists x(P(x) \Rightarrow Q(x)) \Leftrightarrow (\forall y P(y) \Rightarrow \exists x Q(x))$

4.6 Interpretations

A formula in propositional logic has a truth value associated with each possible truth value, T or F, of each of its propositional variables. For example, $P \Rightarrow Q$ is true for $P = T, Q = T$, but false for $P = T, Q = F$.

A mapping from each of the variables to T or F is called an *interpretation*, and gives a meaning to the formula.

Similarly formulae in predicate logic have a truth value when we have given values to all the variables. However, the process of giving an interpretation to a predicate logic formula is more complicated than in propositional logic.

4.6.1 Interpretations in predicate logic

Consider an interpretation of the formula $\forall x \forall y(P(x, y))$ where the domain of discourse for both x and y is $\{sue, ann, john, bill\}$, and where P is the relation *likes* and we have

$$likes = \{(sue, john), (john, sue), (sue, ann), (bill, ann), (sue, bill), (ann, sue)\}$$

In logic we often use the terminology

$$likes(sue, john), \quad likes(john, sue), \quad likes(sue, ann),$$

$$likes(bill, ann), \quad likes(sue, bill), \quad likes(ann, sue)$$

to specify the binary relation $likes$ and the convention is that $likes(x, y) = T$ in the above cases and $likes(x, y) = F$ for all other combinations of x and y.

The above formula is *false* under this interpretation because we can find a counter-example, e.g., $x = ann, y = bill$

Example

What is the truth value of $\exists x \forall y(((x \neq y) \Rightarrow P(x, y)) \wedge \neg P(x, x))$ under the above interpretation, i.e. when $P = likes$ and $Domain = \{sue, ann, john, bill\}$.

If we take x to be *sue* then we can check that $likes(sue, y)$ is true when y is $john$, ann or $bill$ so

$$(sue \neq y) \Rightarrow P(sue, y)$$

is true for every possible choice of y. Also, (sue, sue) is not in the relation $likes$, so $\neg P(sue, sue)$ is true. Thus the predicate is true as we can take $x = sue$.

Exercises 4.6 Consider the interpretation:
Domain = {margaret, george, tony, harriet}
Predicates:
P = leader, and is true for {george, tony, margaret}
Q = likes, and is true for {(george, tony), (tony, george), (margaret, tony)}

Find the truth values under this interpretation of:

1. $\exists x \exists y(Q(x, y) \wedge Q(y, x))$
2. $\forall x(P(x) \vee \exists y Q(x, y))$
3. $\exists x \forall y(\neg Q(x, y) \wedge \neg Q(y, x))$
4. $\forall x(P(x) \Rightarrow \exists y Q(x, y))$

4.6.2 Semantic entailment

Definitions An interpretation which makes a formula A true is a *model* for A. A formula which has at least one model is said to be *consistent* or *satisfiable*. A formula which has no models is said to be *inconsistent* or *unsatisfiable*. A formula which is true for all interpretations is said to be *valid*. A formula which is neither inconsistent nor valid is said to be *contingent*.

A formula A *semantically entails* formula B if and only if every model for A is also a model for B. That is, any interpretation which makes A true also makes B true.

We write:

$$A \models B$$

We can also say that A logically implies B, or B is a logical consequence of A.

Examples

The above interpretation, $P = likes$ and $Domain = \{sue, ann, john, bill\}$, is a model for the formula $\exists x \forall y(((x \neq y) \Rightarrow P(x, y)) \wedge \neg P(x, x))$.

The formula $\forall x P(x) \wedge \exists y \neg P(y)$ has no models.

The formula $\forall x P(x) \vee \exists y \neg P(y)$ is true for all interpretations.

4.7 Inference and proof

A proof is a logical argument which ends with the conclusion that some proposition is true. We state the assumptions, or hypotheses, and these form the axioms on which the proof is based. We then use rules to deduce consequences of these axioms, until we get the required proposition.

In the previous section we discussed the idea of the logical consequence of a formula in logic: $A \models B$ means any interpretation which makes formula A true also makes formula B true. We can also say that A *logically implies* B, or B is a *logical consequence* of A.

We can apply the same idea to a set of formulae: $S \models A$ means that any interpretation which makes all the formulae in set S true also makes A true. That is: A is *semantically entailed* by the set S, and A is a *logical consequence* of S.

4.7.1 Demonstrating logical consequences

In propositional logic, to demonstrate that $S \models U$, we can construct the truth tables.

Write the truth table with a column for each of the propositions in S and a column for U and check that that in *every* row in which all of the propositions in S are true, U is also true.

Example

Use truth tables to show that $\{P, Q, Q \Rightarrow (R \vee U), \neg R\} \models U$.

P	Q	R	U	R∨U	Q⇒(R∨ U)	¬R	
T	T	T	T	T	T	F	F
T	T	T	F	T	T	F	F
T	T	F	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	F	F
T	F	T	F	T	T	F	F
T	F	F	T	T	T	T	F
T	F	F	F	F	T	T	F

The last column is the truth value of the conjunction of propositions in the set S. (Note, we only need the rows for which $P = T$ so we have left the other rows out to fit the table onto the page.) The only row in which the LHS propositions are all true is row 3, and for this row $U = T$, as required.

Example

Use truth tables to show that $\{Q, P \Rightarrow Q, Q \Rightarrow R\} \models (R \wedge Q)$

P	Q	R	P⇒ Q	Q⇒R		R∧ Q
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	T	F	F
F	F	F	T	T	F	F

The 6th column is the truth value of the conjunction of propositions in the left hand side set. The only rows in which the LHS propositions are all true are row 1 and row 5, and for these rows $(R \wedge Q) = T$, as required.

4.7.2 Proof using logical inference

Using truth tables for checking large semantic entailments is cumbersome, and for predicate logic it is not possible if the domain of discourse is infinite.

It is often possible to prove semantic entailments by applying inference rules. You can think of truth tables as showing something is true by looking at every element, while applying inference rules is using mathematical reasoning to give a proof.

If S is a set of formulae and A is a single formula, then A *can be proved from* S if A can be obtained from S by application of sound inference rules.

We write $S \vdash A$

We can also say that A *can be derived* from S, or that S *syntactically entails* A.

We now describe the inference rules.

4.7.3 Modus ponens

If we know that a property P implies a property Q, and we know that P is true, then we can deduce that Q is true.

For example,

- ◇ If it is true that when I am at the seaside then I am happy,
- ◇ if it is true that I am at the seaside,

then you can deduce that I am happy.

Modus ponens is the inference rule: from (if P then Q) and P, deduce Q.

That is, from hypotheses $P \Rightarrow Q$ and P, we can infer Q.

We write either $\dfrac{P, P \Rightarrow Q}{Q}$ or $\{P, P \Rightarrow Q\} \vdash Q$

In addition to single propositions P and Q, we can have formulae A and B

$$\frac{A, A \Rightarrow B}{B}$$

Example

Consider the propositions

P = Interest rates increase $\quad Q$ = Mortgage rates increase $\quad R$ = House prices fall

and the hypotheses $\{P \Rightarrow Q, Q \Rightarrow R, P\}$

We can use modus ponens to show that $\{P \Rightarrow Q, Q \Rightarrow R, P\} \vdash R$
that is, if interest rate increase means mortgage rate increase, and mortgage rate increase means house prices fall, then if interest rates increase house prices will fall.

$$\frac{\dfrac{P, P \Rightarrow Q}{Q} \qquad Q \Rightarrow R}{R}$$

We can also write the proof using the $\vdash$ notation:

$$\{P, P \Rightarrow Q\} \vdash Q, \qquad \{Q, Q \Rightarrow R\} \vdash R$$

Example

We can use modus ponens and the rule $\{A \wedge B\} \vdash B$ to show
$\{R \Rightarrow ((\forall x P(x)) \wedge Q), R\} \vdash \forall x P(x)$

$$\frac{\dfrac{R, R \Rightarrow ((\forall x P(x)) \wedge Q)}{(\forall x P(x)) \wedge Q}}{\forall x P(x)}$$

4.7.4 Soundness

We only want to use an inference rule if things proved using it are correct in the sense that if A is proved from S then $S \models A$. If any formula A which can be derived from a set S of other formulae using one or more applications of a rule of inference is also a logical consequence of S, then the rule of inference is said to be *sound*.

In other words, anything that we derive from S using that rule of inference will be true for all interpretations which make all the formulae in S true (all models of S).

Definition A *sound inference rule* R is one for which: if $S \vdash A$ (using R) then $S \models A$.

Look at the truth table for $P \Rightarrow Q$

P	Q	P⇒ Q
T	T	T
T	F	F
F	T	T
F	F	T

We can see that if $P = T$ and $(P \Rightarrow Q) = T$ then only row 1 of the table is relevant and in this row $Q = T$.

We have shown above that $\{P \Rightarrow Q, Q \Rightarrow R, P\} \vdash R$ using modus ponens. It can be checked using truth tables that $\{P \Rightarrow Q, Q \Rightarrow R, P\} \models R$.

4.7.5 Some inference rules

There are lots of sound inference rules.

$$\frac{A}{A \vee B} \qquad \frac{\neg(A \wedge B)}{(\neg A) \vee (\neg B)}$$

$$\frac{A \wedge B}{A} \qquad \frac{\neg(A \vee B)}{(\neg A) \wedge (\neg B)}$$

$$\frac{A, B}{A \wedge B} \qquad \frac{A \vee (B \wedge C)}{A \vee B}$$

$$\frac{A, A \Rightarrow B}{B} \qquad \frac{A \wedge (B \vee C)}{(A \wedge B) \vee (A \wedge C)}$$

$$\frac{\neg B, A \Rightarrow B}{\neg A} \qquad \frac{A \Rightarrow B}{(\neg A) \vee B}$$

$$\frac{A \Rightarrow B, B \Rightarrow C}{A \Rightarrow C}$$

$$\frac{A \vee B, \neg A}{B}$$

The soundness of each of these inference rules can be shown using truth tables.

4.7.6 Direct proofs with inferences

We can derive a result using any combination of sound inference rules.

Example

If I have a cold, I sneeze. If I sneeze, either I have hay-fever or I should stay in bed. I have a cold and I don't have hay-fever. So I should stay in bed.

Model this with $P =$ "I have a cold", $Q =$ "I sneeze", $R =$ "I have hay fever", $U =$ "I should stay in bed"

Show that $\{P, P \Rightarrow Q, Q \Rightarrow (R \vee U), \neg R\} \vdash U$

$$\dfrac{\dfrac{\dfrac{P, P \Rightarrow Q}{Q} \qquad Q \Rightarrow (R \vee U)}{R \vee U} \qquad \neg R}{U}$$

Example

Using direct proof by inferences, show that $\{P \Rightarrow (Q \Rightarrow R), P \wedge Q\} \vdash R$

$$\dfrac{\dfrac{\dfrac{P \wedge Q}{P} \qquad P \Rightarrow (Q \Rightarrow R)}{Q \Rightarrow R} \qquad \dfrac{P \wedge Q}{Q}}{R}$$

4.7.7 Proof by contradiction

If we want to show that

$$S \models A$$

it is sufficient to show that

$$S \cup \{\neg A\} \models \perp$$

where $\perp$ is a symbol which in this case denotes the truth value FALSE.

If $S \cup \{\neg A\} \models \perp$ then this means that no row of the truth table for all the propositions in $S \cup \{\neg A\}$ has all Ts. This means that if a row has Ts in all the columns for S then $\neg A$ must be F, and hence A is T. So $S \models A$.

A similar argument shows that the converse is also true, if $S \models A$ then $S \cup \{\neg A\} \models \perp$. So if $S \cup \{\neg A\} \not\models \perp$ then $S \not\models A$.

So, to prove a result by contradiction, add the negation of what you want to prove to the hypotheses, and derive an expression which must be false.

We express this as the inference rule

$$\frac{A, \neg A}{\perp}$$

Example

We prove $\{P, P \Rightarrow Q, Q \Rightarrow R\} \models R \wedge Q$ by contradiction and inference rules. Assuming $\neg(R \wedge Q)$, we show $\{P, P \Rightarrow Q, Q \Rightarrow R, \neg(R \wedge Q)\} \vdash \perp$.

$$\dfrac{\dfrac{\dfrac{P, P \Rightarrow Q}{Q} \qquad \dfrac{\neg(R \wedge Q)}{\neg R \vee \neg Q}}{\neg R} \qquad \dfrac{\dfrac{P, P \Rightarrow Q}{Q} \qquad Q \Rightarrow R}{R}}{\perp}$$

4.8 Practice exercises

1. Let n be an integer and let $P(n)$ be the predicate '$n^2 - 2n + 1 = 0$'. What are the truth values of
   a. $\forall n P(n)$
   b. $\exists n P(n)$.

2. Write down, in predicate form, the negation of the following propositions, where x, y are real numbers.
   a. $\exists x \forall y (y \text{ exactly divides } x)$
   b. $\forall x \exists y (x - y = 17)$.

3. Write the following statements in predicate form.
   a. All dogs bark
   b. There does not exist an x such that $x = 2x$
   c. For each y, y^2 is greater than y
   d. No computer science student is not mathematically literate

4. Is the predicate in Question 2(b) above true? (Justify your answer.)

5. Use truth tables to show that $\{Q \wedge (P \vee R), Q \Rightarrow R\} \models R$.

6. Use inference rules to show that $\{Q \wedge (P \vee R), Q \Rightarrow R\} \vdash R$.