Continuous Portfolio Rebalancing as a System of ODEs

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Motivation: Why Rebalancing Matters

- Portfolio allocations drift over time due to differing asset returns.
- Long–term strategies (pension funds, target–date funds) require maintaining a stable risk profile.
- Continuous rebalancing models systematic transfers between risky and risk–free assets.
- Goal: maintain a target allocation while reducing volatility exposure.

Definition of Variables

- $x(t)$: dollars invested in the risky asset.
- $y(t)$: dollars in the risk–free asset
- x^*: target (desired) dollar amount in the risky asset.
- $k > 0$: rebalancing rate (strength of adjustment toward x^*).
- r_s: instantaneous rate of return on the risky asset.
- r_f: risk–free interest rate (lending/borrowing rate in the model).

Excess risky allocation: $x(t) - x^*$

Deriving the ODEs: Why These Growth Equations Make Sense

- The risky asset grows at its instantaneous return rate r_s.
  - If you invest $x(t)$ dollars in a risky asset earning rate r_s, the change in value is proportional to how much you currently hold.
  - Therefore:
    $$\frac{dx}{dt} = r_s\, x.$$
- The risk–free asset grows at the risk–free rate r_f.
  - A risk–free position behaves like a continuously compounding bank account: the balance earns interest at rate r_f.
  - If $y(t)$ is positive, it grows at r_f; if negative, you are borrowing and pay interest at the same rate.
  - Therefore:
    $$\frac{dy}{dt} = r_f\, y.$$

These natural growth equations form the foundation before adding any rebalancing flows.

Deriving the ODEs: Adding Rebalancing

Capital moves from one asset to the other at rate $k(x - x^*)$:

- If $x > x^*$: shift dollars out of risky asset.
- If $x < x^*$: shift dollars into risky asset.

$$\frac{dx}{dt} = r_s x - k(x - x^*)$$
$$\frac{dy}{dt} = r_f y + k(x - x^*)$$

This ensures flows sum to zero.

Matrix Representation of the System

We write the system as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r_s - k & 0 \\ k & r_f \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} kx^* \\ -kx^* \end{pmatrix}$$

This form helps compute equilibrium and stability.

Coefficient matrix:

$$A = \begin{pmatrix} r_s - k & 0 \\ k & r_f \end{pmatrix}.$$

Eigenvalues:

$$\lambda_1 = r_s - k < 0, \qquad \lambda_2 = r_f > 0.$$

Interpretation:

- The direction associated with $\lambda_1 < 0$ (the x-direction) is **stable**: rebalancing pulls $x(t)$ back toward the equilibrium allocation.
- The direction associated with $\lambda_2 > 0$ (the y-direction) is **unstable**: the risk–free asset grows exponentially.
- With one stable and one unstable direction, the equilibrium is a **saddle point**.

Finding the Equilibrium Point (1/2)

Set $x' = 0$, $y' = 0$:

$$0 = r_s x_e - k(x_e - x^*),$$
$$0 = r_f y_e + k(x_e - x^*).$$

Step 1: Solve for x_e.

$$0 = r_s x_e - k(x_e - x^*) = r_s x_e - kx_e + kx^*$$

$$(r_s - k)x_e + kx^* = 0$$

$$x_e = \frac{-kx^*}{r_s - k} = \frac{k}{k - r_s} \cdot x^*$$

Thus,

$$\boxed{x_e = \frac{k}{k - r_s} x^*}$$

Finding the Equilibrium Point (2/2)

Step 2: Solve for y_e.
Start with:

$$0 = r_f y_e + k(x_e - x^*) \quad \Rightarrow \quad r_f y_e = -k(x_e - x^*).$$

Compute the difference:

$$x_e - x^* = \frac{k}{k - r_s}x^* - x^* = x^* \left(\frac{k}{k - r_s} - 1 \right) = x^* \left(\frac{r_s}{k - r_s} \right).$$

Substitute:

$$y_e = -\frac{k}{r_f} \left(\frac{r_s}{k - r_s} \right) x^*.$$

Thus the equilibrium is:

$$\boxed{(x_e, y_e) = \left(\frac{k}{k - r_s}x^*, \; -\frac{k r_s}{r_f(k - r_s)}x^* \right)}$$

Phase Plane for Continuous Rebalancing

Parameters:

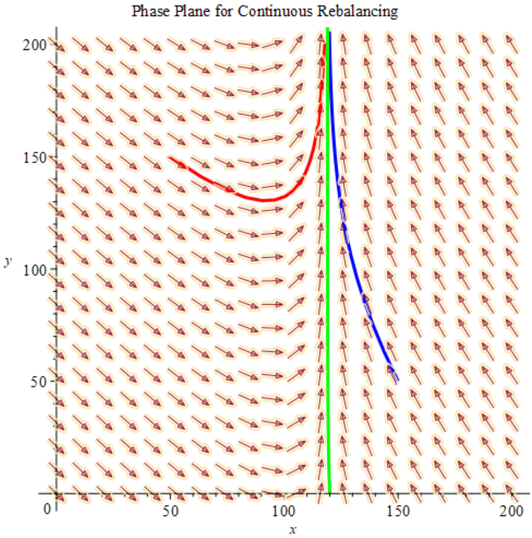
$$r_s = 0.08, \qquad r_f = 0.03, \qquad k = 0.50, \qquad x^* = 100$$

Equilibrium (computed in Maple):

$$(x_e, y_e) = (119.0476190,\ -317.4603175)$$

The equilibrium point shown above is the same point in the phase-plane diagram on the next slide.

Phase Plane for Continuous Rebalancing



Interpreting the Phase Plane

The phase plane illustrates how the portfolio behaves under continuous rebalancing when the investor constantly adjusts toward a target mix of risky and risk-free assets.

1. **Vertical green trajectory: perfect diversification.**
   - This curve represents the allocation where rebalancing is perfectly balanced; the portfolio simply grows over time without shifting money between assets.
2. **Blue trajectory: starting with too much risky asset.**
   - The system responds by selling risky assets to move back toward the desired mix; once the risky portion is corrected, the portfolio resumes normal growth.
3. **Red trajectory: starting with too little risky asset.**
   - The system responds by buying risky assets to restore the target mix; after adjusting, the portfolio again follows a stable growth pattern.

Why the Equilibrium Shows Saddle Behavior

The equilibrium portfolio behaves like a **saddle** in a qualitative, financial sense. This means the system is stable in one direction but unstable in another.

1. **Stable direction: allocation (horizontal movement).**
   - If the portfolio starts with too much or too little risky assets, the rebalancing rule automatically pushes it back toward the balanced-mix line.
   - This horizontal pull always corrects misallocation.
   - In this sense, the equilibrium is *stable*: allocation errors shrink over time.
2. **Unstable direction: total wealth (vertical movement).**
   - Once allocation is correct, the portfolio no longer rebalances.
   - From that point on, the portfolio simply grows or shrinks depending on the risk-free component.
   - Thus the system is *unstable* vertically: the portfolio drifts away from the equilibrium wealth level.

Verifying the Phase Plane

We found the equilibrium point:

$$x_e = \frac{k}{k - r_s} x^*, \qquad y_e = -\frac{k r_s}{r_f(k - r_s)} x^*.$$

Plug in the parameters:

$$r_s = 0.08, \quad r_f = 0.03, \quad k = 0.5, \quad x^* = 100.$$

Compute:

$$x_e = \frac{0.5}{0.5 - 0.08}(100) = \frac{0.5}{0.42}(100) \approx 119.0476,$$

$$y_e = -\frac{0.5(0.08)}{0.03(0.42)}(100) = -3.174603(100) \approx -317.4603.$$

Maple output:

$$(x_e, y_e) = (119.0476190,\ -317.4603175),$$

which matches our calculation exactly.

Implications for the Phase Plane

- The equilibrium point itself lies below the plotted region (its y_e value is negative). This is not a problem: Negative y means you are leveraged, you have borrowed at the risk-free rate to finance some of your risky-asset position.
- We do not need to plot negative values of y, because the only part of the equilibrium that matters for the dynamics,the vertical line

  $$x_e \approx 119.05$$

  does appear in the graph. Negative wealth is typically not considered in these models.

Financial Interpretation

- The model shows how continuous rebalancing automatically counters portfolio drift by shifting money between risky and risk-free assets.
- The rebalancing strength k determines how aggressively the system pulls the portfolio back toward the target risky allocation.
- The equilibrium allocation $x = x_e$ represents the mix where rebalancing is perfectly balanced: no money needs to be moved between assets.
- A negative value of y_e means the investor is leveraged: borrowing at the risk-free rate to maintain the desired risky exposure. This is common in continuous-time portfolio models.
- Phase-plane behavior:
  - If the portfolio starts too risky, rebalancing sells risky assets
  - If it starts too conservative, rebalancing buys risky assets
  - Once the mix is correct, total wealth drifts up or down depending on the risk-free component (vertical motion).

References

- Davis, Mark H. A., and Sébastien Lleo. "Risk-Sensitive and Robust Asset Allocation." *Annals of Finance*, 2008.
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- Perold, André F., and William F. Sharpe. "Dynamic Strategies for Asset Allocation." *Financial Analysts Journal*, 1995.