

Chapter 1

1D Theory of Thermoelasticity

1.1 The governing equations of the linear theory of thermoelasticity

The governing equation of the general theory of thermoelasticity reduce to a much simple form in the case some simplifying assumptions are considered to be valid. This is particularly true if we assume the linearity conditions for the equilibrium equations, the strain-displacement (compatibility equations and the constitutive equations). According to what already established for the general theory, we assume as state variables of the problem, and as unknowns of the mathematical model of the thermoelastic behavior:

- the displacement components $u_i = u_i(x_i, t)$ of the material point having coordinates x_i in the reference configuration (lagrangian approach);
- the Green-Lagrange strain components $\varepsilon_{ij} = \varepsilon_{ij}(x_i, t)$;
- the Cauchy stress components $\sigma_{ij} = \sigma_{ij}(x_i, t)$.

The forcing actions on the 3D continuum body are represented by:

- the external volume forces per unit volume $X_i(x_i, t)$ acting on the volume V occupied by the body;
- the external surface forces $f_i(x_i, t)$ acting on the surface S_f of V where the external forces are applied;
- the applied displacements $\bar{u}_i(x_i, t)$, acting on S_u , the portion of S where the displacement are prescribed with $S_f + S_u = S$;

- the variation of temperature $\Delta T(x_i, t)$, assumed to be given for any point P of the body, in the hypothesis that the mechanical problem has no influence on the heat conduction problem.

Also initial conditions are prescribed for displacement

$$u_i(x_i, 0) = u_i^o(x_i) \quad (1.1)$$

and velocities

$$\dot{u}_i(x_i, 0) = \dot{u}_i^o(x_i) \quad (1.2)$$

We now assume that the displacement u_i are small as compared to the dimension of the body. In this circumstance, for the establishment of the equilibrium conditions of the body, the actual configuration of the body at time t , C^t can be considered coincident with the reference configuration C^o , that is assumed to be known. The equilibrium equation can be written with reference to the initial volume V , and also the Cauchy stress, defined in general on the actual configuration, is now referred to a known configuration. The equilibrium equations read

$$\sigma_{ij,j} + \bar{X}_i = 0 \quad i = 1, 2, 3 \quad \text{on} \quad V \quad (1.3)$$

with

$$\bar{X}_i = X_i - \rho \ddot{u}_i \quad (1.4)$$

being ρ the density (mass per unit volume) of the body. In general we might assume a non homogeneous body with $\rho = \rho(x_i)$. If also a small displacement gradient assumption with respect to unity is formulated, that is a small rotations and small strain hypothesis, we can disregard the second (non linear) terms of the Green Lagrange strain tensor that reduces to the engineering strain term, which can be expressed now, in terms of displacement as:

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad i = 1, 2, 3 \quad \text{on} \quad V \quad (1.5)$$

for a total of 6 equations, considering the symmetry condition of $\varepsilon_{ij} = \varepsilon_{ji}$. The boundary conditions for the stress can be expressed as

$$\sigma_{ij} n_j = f_i \quad i = 1, 2, 3 \quad \text{on} \quad S_f \quad (1.6)$$

with $n_j = \cos \widehat{nx_j}$ being \underline{n} the normal to the boundary surface, and the one on displacement can be written as

$$u_i = \bar{u}_i \quad i = 1, 2, 3 \quad \text{on} \quad S_u \quad (1.7)$$

The constitutive relations assume the form, for an isotropic body,

$$\sigma_{ij} = E_{ijhk} (\varepsilon_{hk} - \alpha \delta_{hk} \Delta T) \quad i, j = 1, 2, 3 \quad (1.8)$$

where α is the thermal expansion coefficient, δ_{hk} is the Kronecker δ ($\delta_{hk} = 0$ for $h \neq k$, $\delta_{hk} = 1$ for $h = k$), E_{ijhk} is the elastic tensor. In conclusion, for a total of 15 unknown components, 15 equations represent the physics of the problem. One possible technique of solution is to keep the displacement u_i as the unknown and express the equilibrium equation in terms of the 3 displacement components after expressing the stress as formulated in the constitutive relations and the strain as expressed in the strain-displacement equations.

With this substitutions the equilibrium equations, expressed in displacement components, read By assuming \ddot{E} and \ddot{E} , that is an homogeneous thermoelastic body, the equations will now read

$$\{E_{ijhk} \left[\frac{1}{2} (u_{h,k} + u_{k,h}) - \alpha \delta_{hk} \Delta T \right]\}_{,j} + X_i - \rho \ddot{u}_i = 0 \quad i = 1, 2, 3 \quad (1.9)$$

By assuming $E_{ijhk} = \text{const}$ and $\alpha = \text{const}$, that is an homogeneous thermoelastic body, the equations will now read

$$\frac{1}{2} E_{ijhk} (u_{h,k} + u_{k,h}) - E_{ijhk} \alpha \delta_{hk} \Delta T_{,j} + X_i - \rho \ddot{u}_i = 0 \quad i = 1, 2, 3 \quad (1.10)$$

The stress boundary conditions expressed in terms of displacement read

$$E_{ijhk} \left[\frac{1}{2} (u_{h,k} + u_{k,h}) - \alpha \delta_{hk} \Delta T \right] n_j = f_i \quad i = 1, 2, 3 \quad \text{on} \quad S_f \quad (1.11)$$

While on S_u $u_i = \bar{u}_i$ and the initial conditions

$$u_i(x_i, 0) = u_i^o(x_i) \quad \text{and} \quad \dot{u}_i(x_i, 0) = \dot{u}_i^o(x_i) \quad (1.12)$$

The governing equations represents a system of three partial differential equations of second order in time t and of second order in space variables x_i . The temperature variation term appears in the field equation derived at the first order while appears underived in the boundary conditions. Since the variation of the temperature is a given function of x and t , one can also see this term as a special component of the body forces X_i and the surface forces f_i In fact if we assume novel forcing terms

$$\tilde{X}_i = X_i - E_{ijhk} \alpha \delta_{hk} \Delta T_{,j} \quad (1.13)$$

and

$$\tilde{f}_i = f_i - E_{ijhk} \alpha \delta_{hk} \Delta T n_j \quad (1.14)$$

the governing equations of the linear thermoelastic problem will reduce to the linear elastic 3D problem, provided that \tilde{X}_i and \tilde{f}_i are assumed as forcing terms.

Once solution is obtained for the displacement u_i , the strain ε_{ij} can be obtained from the strain-displacement relations and the stress from the constitutive equations. In general it is extremely difficult to obtain a closed form solution for the governing equations in terms of displacement. This is due to the intrinsic difficulty of finding solutions of partial differential equation for general shape of the volume occupied by the body.

Alternative approaches to the solution can also be followed, as the one in which the stress σ_{ij} or the strain ε_{ij} are assumed as unknown of the problem. In this case the compatibility equations in terms of strain, that represents the integrability conditions of the strain-displacement equations for obtaining continuous u_i from assigned ε_{ij} , need to be imposed too. As well known closed form solution can be obtained for just few cases of linear elastic problem. The same is true for the linear thermoelastic problem. For practical purposes of using their model of the thermoelastic physics for quantitative evaluation of the mechanical response of a body under thermal actions, procedures for approximate solutions need to be set up. One family of numerical methods consider directly the governing equations of the problem and, in order to transform the differential nature of the equation and reduce the complexity of a volume of general shape substitute the differentials with the finite increment of the unknown functions in a finite increment of the variables x_i . The governing equations are written many times in correspondence to every point identified by the grid of increments Δx_i .

A family of mechanical techniques, called finite differences, follows this kind of approach. Intuitively, the consequences to the closed form solution is reached progressively as the finite increments and $\tilde{\epsilon}$ on a smaller and smaller grid of x_i variables. An alternative approach for finding approximate solution is based on the use of another form of the governing equations, that is possible to demonstrate to be equivalent to them, the virtual work equation.

1.2 Thermoelastic problems in 1D solids

Let us now consider the thermo-elastic problem in a 1D solid, like the one depicted in figure, a bar subjected to a load per unit length $p(x, t)$ and to a temperature variation $T(x, t)$ with respect the ambient temperature. A fixed end is assumed at the left and $x = 0$, with no lack of generality, since only rigid body motions are suppressed due to it.

The body has section $A(x)$, density per unit length $\rho(x)$ and is thermally isolated in any directions normal to x and is supposed to have a linear thermoelastic behavior with an elastic modulus $E(x)$ and an the thermoelastic response of their body expansion coefficient $\alpha(x)$ can be studied by assuming as state variables the displacement function $u(x, t)$, assumed to

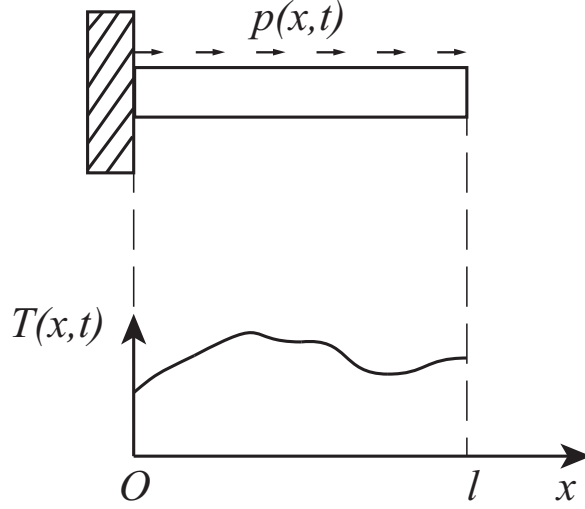


Figure 1.1: The thermoelastic problem in a 1D solid.

be continuous along x , the axial strain $\varepsilon(x, t)$ and the normal force $N(x, t)$.

The governing equations of the problem can be set up as follows:

- strain/displacement relation

$$\varepsilon(x, t) = \frac{\partial u(x, t)}{\partial x} \quad (1.15)$$

- equilibrium equation

$$\frac{\partial N(x, t)}{\partial x} = -\bar{p}(x, t) \quad (1.16)$$

obtained by setting the translation equation along x

$$\frac{\partial N(x, t)}{\partial x} dx + \bar{p}(x, t) dx = 0 \quad (1.17)$$

where $\bar{p}(x, t) = p(x, t) - \rho \frac{\partial^2 u}{\partial t^2}$ including the inertial term constitutive equation (linear thermoelasticity)

$$N(x, t) = E(x) \{ \varepsilon(x) - \alpha(x) T(x, t) \} \quad (1.18)$$

where $\varepsilon^T(x, t) = \alpha(x) T(x, t)$ can be defined as thermal strain and the difference $\varepsilon(x, t) - \varepsilon^T(x, t) = \varepsilon^M(x, t)$ as mechanical strain. Considering the boundary conditions we can either apply the imposed displacement, the imposed force or a combination of the two:

$$\begin{aligned} u(0, t) &= \bar{u}_o(t) \\ u(l, t) &= \bar{u}_l(t) \end{aligned} \quad (1.19)$$

$$N(0, t) = \bar{N}_o(t)$$

$$N(l, t) = \bar{N}_l(t)$$

In the case an elastic spring of stiffness K or a pointwise translational mass M is applied to one of the ends, say the left one at $x = l$, we will have

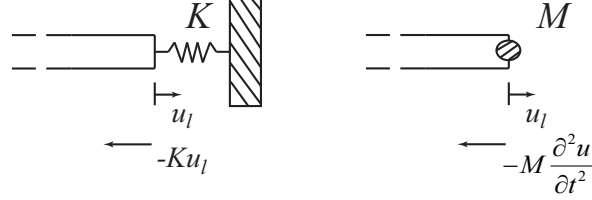


Figure 1.2: Different boundary conditions at one end of the structure.

$$N(l, t) = -Ku_l \quad \text{or} \quad N(l, t) = -M \frac{\partial^2 u}{\partial t^2} \quad (1.20)$$

as to the initial condition

$$u(x, 0) = u_i(x) \quad (1.21)$$

The resulting governing equation can be obtained by substituting in the equilibrium equation the remaining two, thus obtaining, expressed in the primitive state variable $u(x, t)$, the following expression

$$\frac{\partial}{\partial x} \left\{ E(x) A(x) \left[\frac{\partial u(x, t)}{\partial x} - \alpha(x) T(x, t) \right] \right\} = -p(x, t) + \rho \frac{\partial^2 u(x, t)}{\partial t^2} \quad (1.22)$$

with, as initial condition on $x = 0$

$$u(0, t) = u_0(t) \quad \text{or} \quad E(0) \left\{ \frac{\partial u(x, t)}{\partial x} \Big|_0 - \alpha(0) T(0, t) \right\} A(0) = N_0 \quad (1.23)$$

and as boundary condition on $x = l$

$$u(l, t) = u_l(t) \quad \text{or} \quad E(l) \left\{ \frac{\partial u(x, t)}{\partial x} \Big|_l - \alpha(l) T(l, t) \right\} A(l) = N_l \quad (1.24)$$

or other combination of the primitive and its first derivative as in the case corresponding to the spring or the point mass. The governing equations described alone (field equation and limit condition) have a linear nature.

In the static case the equation reads as follows

$$\frac{\partial}{\partial x} \left\{ E(x) A(x) \left[\frac{\partial u(x)}{\partial x} - \alpha(x) T(x) \right] \right\} = -p(x) \quad (1.25)$$

if the body has an homogeneous nature both for the elasticity and for the thermal expansion phenomena we can further simplify the expression:

$$E \frac{\partial}{\partial x} \left(A(x) \frac{\partial u(x)}{\partial x} - A(x) \alpha(x) T(x) \right) = -p(x) \quad (1.26)$$

with the boundary conditions

$$u(0) = u_0 \quad \text{or} \quad EA(0)\{\partial u \partial x|_0 - \alpha T(0)\} = N_0 \quad (1.27)$$

$$u(l) = u_l \quad \text{or} \quad EA(l)\{\partial u \partial x|_l - \alpha T(l)\} = N_l \quad (1.28)$$

Note that $T(x)$ appears under a first order derivation in the field equation.

The governing equation is in this case an ordinary second order differential equation that, if $A(x) = \text{const}$, is also characterized by constant coefficients. Let us now try to find the solution in simple cases for which $p(x)$ is assumed to be zero. If also $A(x) = \text{const}$, the field equation reads

$$EAu''(x) - EA\alpha T'(x) = 0 \quad (1.29)$$

with

$$\begin{aligned} A(x) &= \text{const} & u(0) &= 0 \\ T(x) = \bar{T} & & N(l) &= 0 \end{aligned} \quad (1.30)$$

In this case the equations read, with $(\prime) = \frac{\partial}{\partial x}()$ and $(\prime\prime) = \frac{\partial^2}{\partial x^2}()$

$$EAu''(x) = 0 \quad (1.31)$$

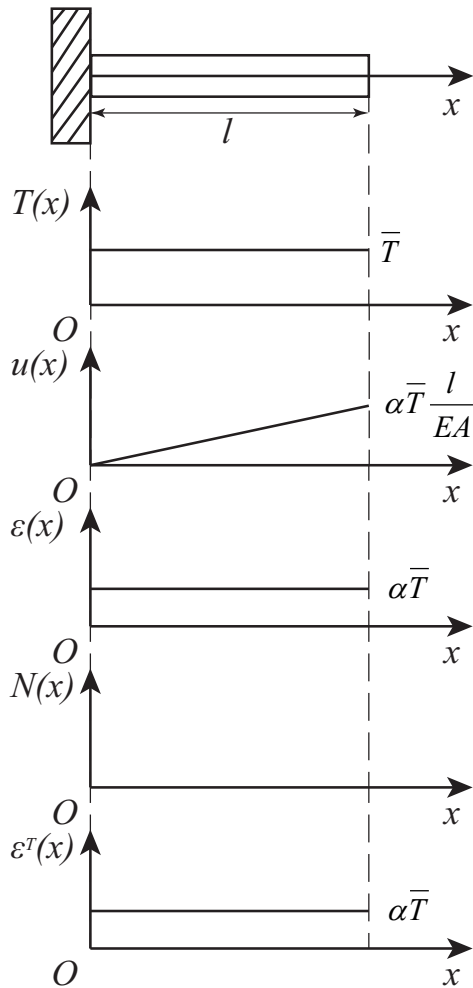
$$u(0) = 0 \quad (1.32)$$

$$N(l) = EA(u'_l - \alpha \bar{T}) = 0 \quad (1.33)$$

The solution has the form $u(x) = A + Bx$ with A and B obtained by imposing the boundary conditions.

1.3 Examples

Example n.1



$$u(0) = 0$$

$$N(l) = 0$$

$$T(x) = \bar{T}$$

$$p(x) = 0$$

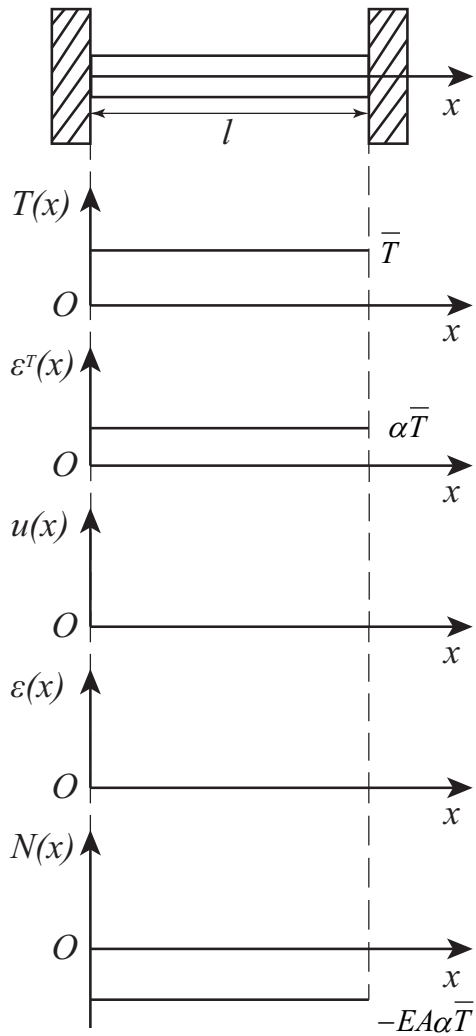
$$u(0) = A = 0$$

$$u'(l) - \alpha \bar{T} = B - \alpha \bar{T} \quad B = \alpha \bar{T}$$

$$u(x) = \alpha \bar{T} x$$

$$\varepsilon(x) = \alpha \bar{T}$$

$$N(x) = EA (\alpha \bar{T} - \alpha \bar{T}) = 0$$

Example n.2

$$u(0) = 0$$

$$u(l) = 0$$

$$T(x) = \bar{T}$$

$$p(x) = 0$$

$$u(x) = A + Bx \quad u'(x) = B$$

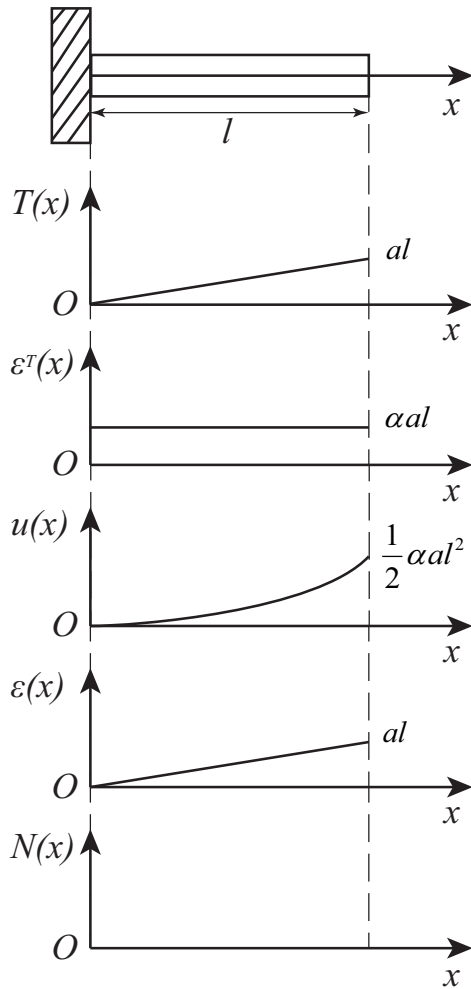
$$u(0) = A = 0$$

$$u(l) = Bl = 0$$

$$u(x) = 0$$

$$\varepsilon(x) = 0$$

$$N(x) = EA(-\alpha\bar{T}) = -EA\alpha\bar{T}$$

Example n.3

$$u(0) = 0$$

$$N(l) = 0$$

$$T(x) = ax$$

$$p(x) = 0$$

$$EAu''(x) = EA\alpha T'(x) = EA\alpha a$$

$$u(x) = A + Bx + cX^2$$

$$u'(x) = B + 2Cx$$

$$u''(x) = 2Cx = \alpha a \quad C = \frac{\alpha a}{2}$$

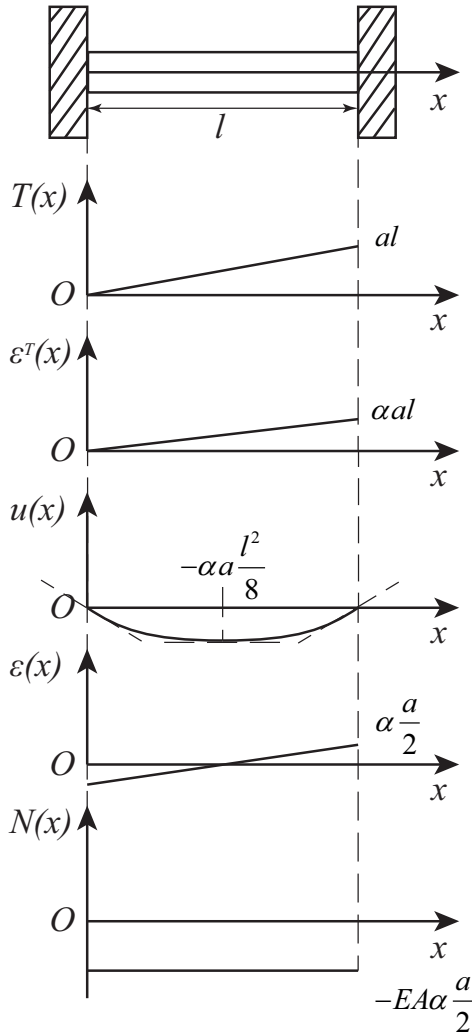
$$u(0) = A = 0$$

$$u'(x) = B = 0$$

$$u(x) = \frac{1}{2}\alpha ax^2$$

$$\varepsilon(x) = \alpha ax$$

$$N(x) = EA(\alpha ax - \alpha ax) = 0$$

Example n.4

$$u(0) = 0$$

$$u(l) = 0$$

$$T(x) = ax$$

$$p(x) = 0$$

$$EAu''(x) = EA\alpha T'(x) = EA\alpha a$$

$$u(x) = A + Bx + cX^2$$

$$u'(x) = B + 2Cx$$

$$u''(x) = 2Cx = \alpha a \quad C = \frac{\alpha a}{2}$$

$$u(0) = A = 0$$

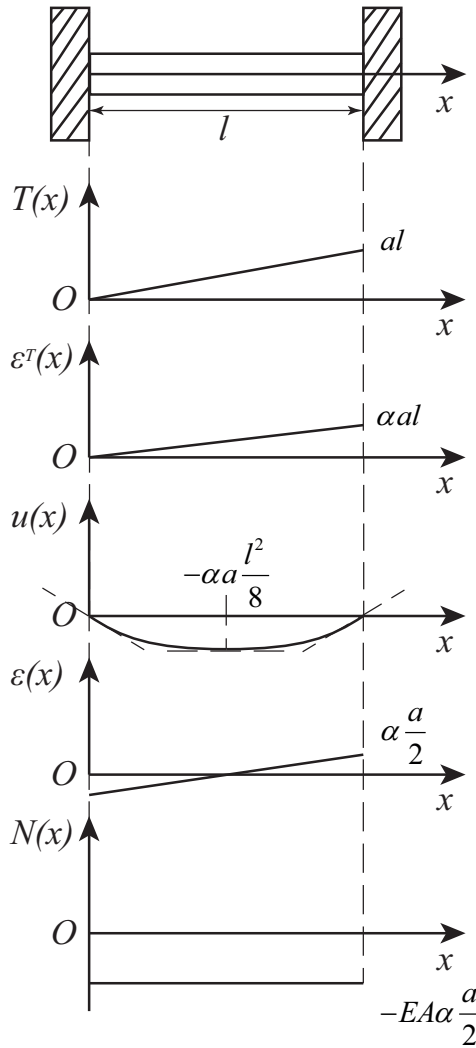
$$u(l) = Bl + \frac{\alpha a}{2}l^2 \quad B = -\frac{\alpha a}{2}l$$

$$u(x) = -\frac{\alpha a}{2}lx + \frac{\alpha a}{2}x^2$$

$$u'(x) = -\frac{\alpha a}{2}l + \alpha ax \quad u'\left(\frac{l}{2}\right) = 0$$

$$N(x) = EA\left(-\frac{\alpha a}{2}l + \alpha ax - +\alpha ax\right) = -EA\frac{\alpha a}{2}l$$

The expression for $T(x)$ can also be written as $T(x) = \frac{T_b}{l}x$

Example n.5

$$u(0) = 0$$

$$u(l) = 0$$

$$T(x) = T_a + \frac{T_b - T_a}{l}x$$

$$p(x) = 0$$

This case can be solved by superimposition of effects by superimposing the solution for

$$T_1(x) = T_a$$

and the one for

$$T_2(x) = \frac{T_b - T_a}{l}x$$

obtained for the example 2 and 4 respectively, as follows

$$u(x) = u_1(x) + u_2(x)$$

that is

$$u_1(x) = 0$$

$$\varepsilon_1(x) = 0$$

$$N_1(x) = EA(-\alpha \bar{T}) = -EA\alpha \bar{T}$$

$$u_2(x) = -\frac{\alpha}{2} \left(\frac{T_b - T_a}{l} \right) lx + \alpha \left(\frac{T_b - T_a}{2l} \right) x^2$$

$$\varepsilon_2(x) = -\alpha \left(\frac{T_b - T_a}{2} \right) + \alpha \left(\frac{T_b - T_a}{l} \right) x$$

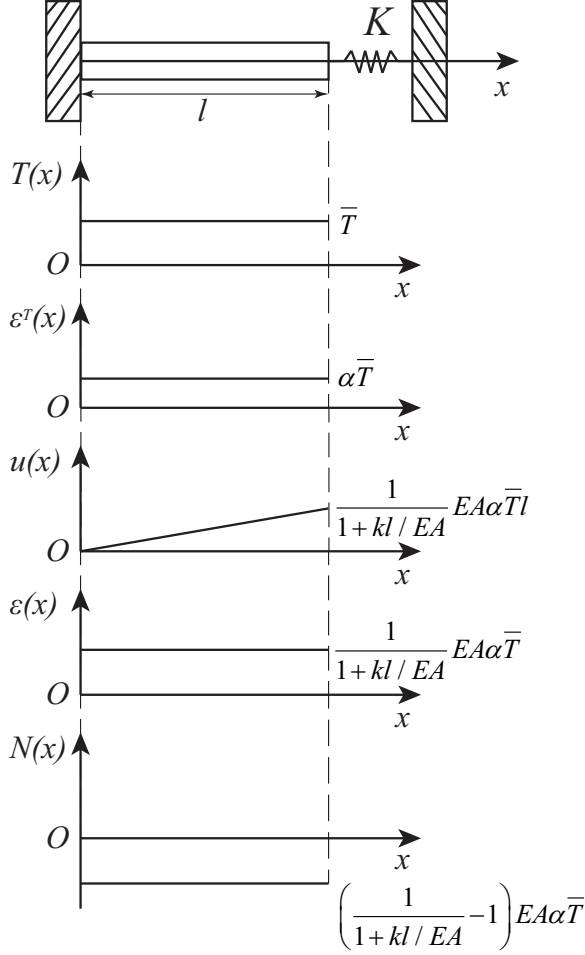
$$N_2(x) = -EA \frac{\alpha a}{2} l \left(\frac{T_b - T_a}{2l} \right)$$

In conclusion

$$u(x) = -\frac{\alpha}{2} \left(\frac{T_b - T_a}{l} \right) lx + \alpha \left(\frac{T_b - T_a}{2l} \right) x^2$$

$$\varepsilon(x) = -\alpha \left(\frac{T_b - T_a}{2} \right) + \alpha \left(\frac{T_b - T_a}{l} \right) x$$

$$N(x) = -EA \frac{\alpha a}{2} (T_a + T_b)$$

Example n.6

$$u(0) = 0$$

$$N(l) = -ku_l$$

$$T(x) = \bar{T}$$

$$p(x) = 0$$

$$u(x) = A + Bx$$

$$u(0) = A = 0$$

$$N(l) = EA(u'_l - \alpha \bar{T})$$

$$u'(x) = B$$

$$u(l) = Bl$$

from which we obtain

$$EA(B - \alpha \bar{T}) = -kBl$$

$$B = \frac{\alpha \bar{T} EA / l}{EA / l + k}$$

$$u(x) = \frac{EA / l}{EA / l + k} \alpha \bar{T} x$$

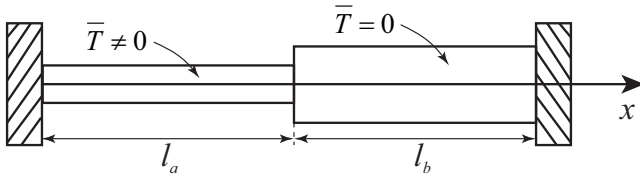
$$\varepsilon(x) = \frac{EA / l}{EA / l + k} \alpha \bar{T}$$

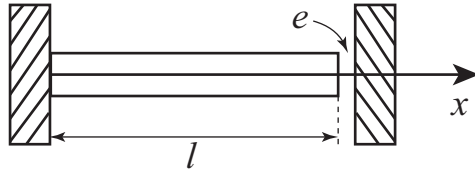
$$N(x) = EA \left(\frac{EA / l}{EA / l + k} \alpha \bar{T} - \alpha \bar{T} \right) EA$$

that for $k \rightarrow 0$ will be equal to $N(x) = 0$

while for $k \rightarrow \infty$ will be equal to $N(x) = -\alpha \bar{T} EA$

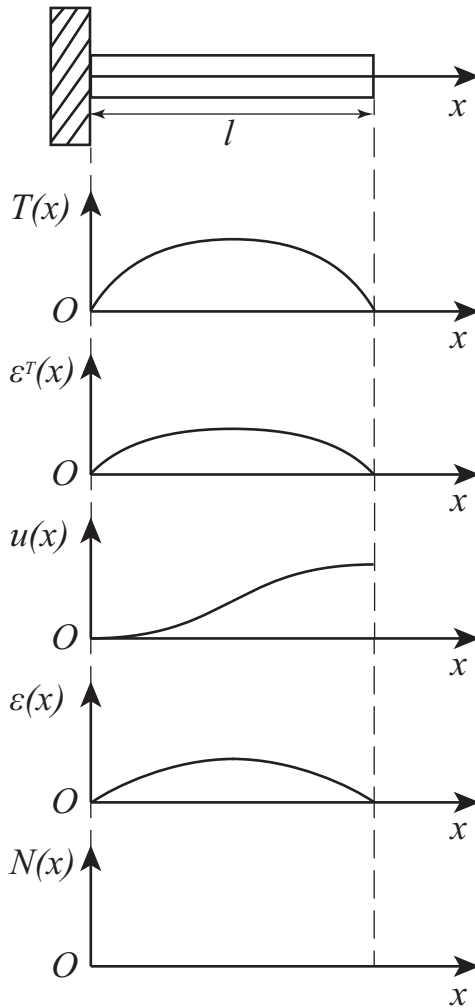
The stiffness k may represent the stiffness of another bar of different characteristics, measured at its end, as in the following picture. In this case the bar a has a variation in temperature \bar{T} whereas b remains undisturbed from the thermal viewpoint.



Example n.7

$$T(x) = \bar{T}$$

In the presence of a small gap e the boundary conditions change from $N(l) = 0$ to $u(l+e)$ after the displacement has reached the value e . It is a nonlinear case for which, in case e is small as compared to l one way just change the boundary condition and add the solution obtained for displacement and strain to the one with fixed right end.

Example n.8

$$u(0) = 0$$

$$N(l) = 0$$

$$T(x) = \sin\left(\frac{\pi x}{l}\right)$$

$$p(x) = 0$$

The solution is of the kind

$$u(x) = A + Bx - \frac{\alpha}{\pi/l} \cos\left(\frac{\pi x}{l}\right)$$

$$u'(x) = B + \alpha \sin\left(\frac{\pi x}{l}\right)$$

$$N(x) = EA(u'(x) - \alpha T(x)) = EAB$$

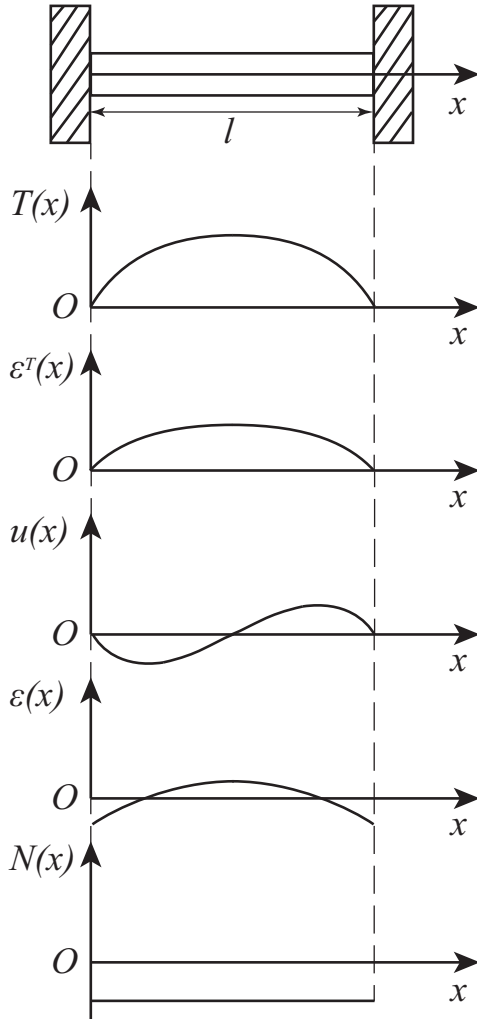
$$u(0) = A - \frac{\alpha}{\pi/l} = 0 \quad A = \frac{\alpha}{\pi/l}$$

$$N(l) = EAB = 0 \quad B = 0$$

$$u(x) = \frac{\alpha}{\pi/l} \left(1 - \cos\left(\frac{\pi x}{l}\right)\right)$$

$$\varepsilon(x) = \alpha \sin\left(\frac{\pi x}{l}\right)$$

$$N(x) = EA \left(\alpha \sin\left(\frac{\pi x}{l}\right) - \alpha \sin\left(\frac{\pi x}{l}\right) \right) = 0$$

Example n.9

$$u(0) = 0$$

$$u(l) = 0$$

$$T(x) = \sin\left(\frac{\pi x}{l}\right)$$

$$p(x) = 0$$

$$EAu'' = EA\alpha T' = EA\alpha \frac{\pi}{l} \cos\left(\frac{\pi x}{l}\right)$$

$$u_{part}(x) = a \cos\left(\frac{\pi x}{l}\right)$$

$$u_{part}''(x) = -a \frac{\pi^2}{l^2} \cos\left(\frac{\pi x}{l}\right)$$

$$-EAa \frac{\pi^2}{l^2} \cos\left(\frac{\pi x}{l}\right) = EA\alpha \frac{\pi}{l} \cos\left(\frac{\pi x}{l}\right)$$

$$a = -\frac{\alpha}{\pi/l}$$

$$u_{part}(x) = -\frac{\alpha}{\pi/l} \cos\left(\frac{\pi x}{l}\right)$$

$$u(x) = A + Bx - \frac{\alpha}{\pi/l} \cos\left(\frac{\pi x}{l}\right)$$

$$u'(x) = B + \alpha \sin\left(\frac{\pi x}{l}\right)$$

$$u(0) = A - \frac{\alpha}{\pi/l} = 0 \quad A = \frac{\alpha}{\pi/l}$$

$$u(l) = \frac{\alpha}{\pi/l} + Bl + \frac{\alpha}{\pi/l} = 0 \quad B = -2 \frac{\alpha}{\pi/l} \frac{1}{l}$$

$$u(x) = \frac{\alpha}{\pi/l} \left(1 - 2 \frac{x}{l} - \cos\left(\frac{\pi x}{l}\right)\right)$$

$$\varepsilon(x) = \frac{\alpha}{\pi/l} \left(-\frac{2}{l} + \frac{\pi}{l} \sin\left(\frac{\pi x}{l}\right)\right)$$

$$N(x) = -2EA \frac{\alpha}{\pi}$$

1.4 Thermoelastic effect in a 2D bending beam

Let us consider a beam of unitary width in the x, y plane subjected to a variation of the temperature $\Delta T(y) = \frac{T_1 + T_2}{2} + \frac{T_1 - T_2}{h} z$ uniform in the x direction, with $T(h/2) = T_2$, $T(-h/2) = T_1$.

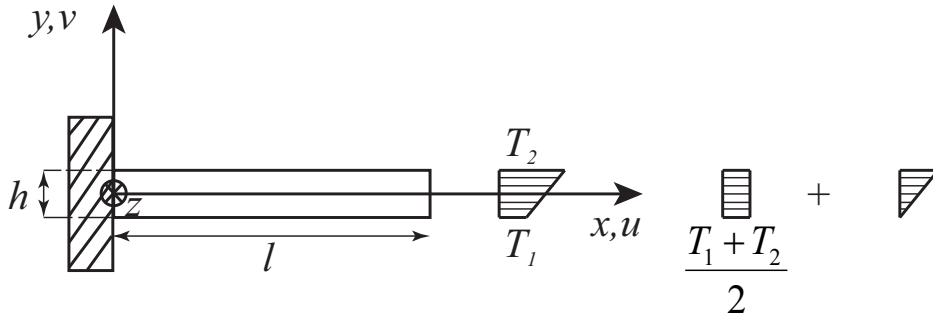


Figure 1.3: Thermoelastic effect in a 2D beam.

The corresponding thermal strain will be

$$\varepsilon^T = \alpha T(y) = \alpha \left[\frac{T_1 + T_2}{2} + \frac{\Delta T}{h} y \right] \quad (1.34)$$

with

$$\Delta T = T_2 - T_1 \quad (1.35)$$

The case can be studied by separately consider the uniform strain along y (that has been already considered in the analysis of the bar) and the linear symmetric variation along y . Let us now briefly recall some elements of the 2D beam theory.

DISPLACEMENT FIELD ASSUMPTIONS

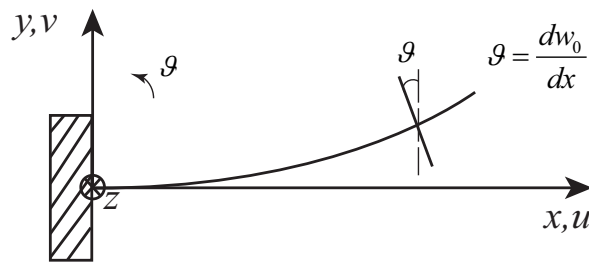


Figure 1.4: The rotation field in a 2D beam.

$$u(x, y) = u_0(x) - \vartheta(x)y$$

$$v(x, y) = v_0(x)$$

STRAIN DISPLACEMENT RELATIONS

$$\begin{aligned}
\varepsilon_x(x, y) &= \frac{du_0(x)}{dx} - \frac{d\vartheta(x)}{dx}y \Rightarrow \varepsilon_x(x, y) = \varepsilon_0(x) - \chi y \\
\varepsilon_y(x, y) &= 0 \\
\gamma_{xy}(x, y) &= 0
\end{aligned}$$

with

$$\begin{aligned}
\chi &= \frac{d^2 w_0}{dx^2} \quad \text{Curvature} \\
\varepsilon_0 &= \frac{du_0}{dx} \quad \text{Axial Strain}
\end{aligned}$$

EQUILIBRIUM EQUATIONS(bending equations only)

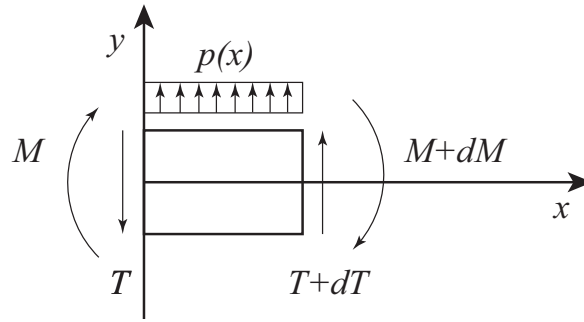


Figure 1.5: Forces and moments acting on an dx element of a 2D beam.

$$\begin{aligned}
M(x) &= -\frac{dT(x)}{dx} \\
T(x) &= -\frac{dp(x)}{dx}
\end{aligned}$$

from 2D continua:

$$M(x) = - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x(x) \cdot y \cdot dy \quad (1.36)$$

CONSTITUTIVE EQUATIONS

From 2D continua

$$\sigma_x = E(\varepsilon_x - \alpha T) \quad (1.37)$$

(notice that T expresses a variation of the temperature)

$$M(x) = - \int_{-\frac{h}{2}}^{\frac{h}{2}} E \varepsilon_x y dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} E \alpha T y dy \quad (1.38)$$

and by substituting the strain-displacement equations:

$$\begin{aligned} M(x) &= - \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(E \varepsilon_0(x) y - E \chi y^2 \right) dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} E \alpha \frac{\delta T}{h} y^2 dy = \\ &= E \chi \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy + E \alpha \frac{\delta T}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy = EI \chi + E \alpha \frac{\delta T}{h} \end{aligned} \quad (1.39)$$

In case $p(x) = \text{const} = 0$ and $T(0) = T(l) = 0$ also $T(x) = 0$ and $M(x) = 0$. In this case (thermal action only acting on the beam) we will have

$$EI \left(\chi + \alpha \frac{\Delta T}{h} \right) = 0 \Rightarrow \chi = -\alpha \frac{\Delta T}{h} \quad (1.40)$$

that in fact, corresponded to a curvature like in figure, where the top fibers of the beam elongate more due to the higher temperature.

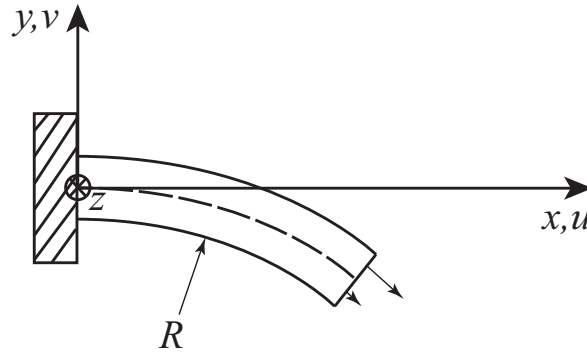


Figure 1.6: The deformed 2D beam.