Safe Labeling of Graphs with Minimum Span

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Abstract. Let $G = (V, E)$ be a graph with vertex set V and edge set E such that $|V| = n$ and $|E| = m$. $G' = (V', E')$ is a subgraph of G if $V = V'$, $E' \subseteq E$. I wish to label all vertices of G with a positive integer such that each vertex will receive a distinct integer and the difference of the labels of two adjacent vertices in G will be at least k. We will call such a labeling of G a k-safe labeling of G. The range from the smallest to the largest integers assigned to the vertices of G in a k-safe labeling is called the span of the k-safe labeling. k-safe labeling problem finds a k-safe labeling of a graph with the minimum span. In this paper I will give upper bounds on k -safe labelings of trees and bipartite graphs. k-safe labeling has many practical applications in radio frequency assignment $[4, 5]$.

Keywords: Safe labeling, Minimum, Small Span, Bipartite Graphs.

1 Introduction

Graph labeling is an interesting research topic in graph theory which concerns the assignment of values to the edges and/or vertices of a graph such that the assignments meet some conditions in the graph[1]. Let G be a graph of n vertices and k be a positive integer . I wish to label the vertices of G with positive integers such that each vertex receives a distinct integer and the difference of the labels of two adjacent vertices is at least k. Call such a labeling of G a *k-safe labeling* of g. We call the range from the largest to the smallest integers assigned to the vertices of G in a k-safe labeling the *span* of the k-safe labeling. $I_l - I_s + 1$ is the span of the k-safe labeling of G, where I_s and I_l be the smallest and the largest integer.

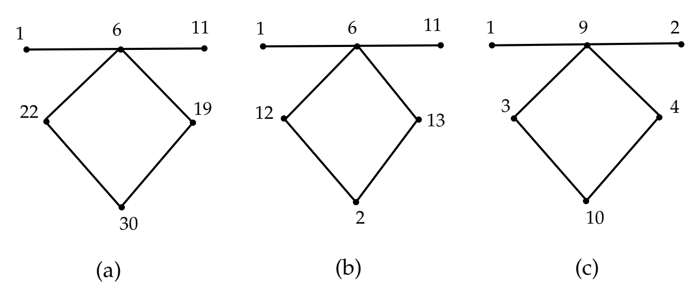

Fig. 1. For $k = 5$ (a) labeling of a graph with span 30, (b) labeling of a graph with span 13 and (c) labeling of a graph with minimum span 10.

The *k-safe labeling problem* asks to find a k-safe labeling of a graph with the minimum span . In figure 1 we show three k-safe labelings of the same graph of 6 vertices. The spans of the k-safe labelings in figure 1(a), (b) and (c) are thirty, 13 and 10, respectively . In fact the k-safe labeling in Figure 1(c) is a k-safe labeling with the minimum span. K_n needs span at least $(n-1)k+1$ for a k-safe labeling. Again for a complete bipartite graph $K_{p,q}$ the span for a k-safe labeling is at least $p+q+k-1 = n+k-1$, since $p+q=n$. Thus one may expect to find k-safe labelings of sparse graphs using narrow spans.

2 Preliminaries

Let $G=(V,E)$ be a simple graph . A *path* is a sequence $v_1, v_2, \cdots, v_q$ of distinct vertices (except possibly v_1 and v_q) such that $v_1, \cdots v_q \in V$ and $(v_{i-1}, v_i) \in E$ for all $2 \leq i \leq q$. The *length* of the path is the number of edges on the path. A path $v_1, v_2, \cdots, v_q$ is *closed* if $v_1 = v_q$. A closed path containing at least one edge is called a *cycle*. An *odd cycle* is a cycle with odd length, that is, with an odd number of edges . An *even cycle* is a cycle with even length, that is, with an even number of edges. A graph G is *connected* if there is a path between every pair of vertices in G. If G is a connected graph containing no cycle then G is called a tree T.

Let $G=(V,E)$ be a graph. A subset V' of V is called *independent* if the vertices in V' are pair-wise non-adjacent. If the vertex set V of G can be partitioned into two nonempty subsets P and Q such that both P and Q are independent sets then G is called a *bipartite graph*. The partition $V = P \cup Q$ is called a *bipartition* of the bipartite graph G. Note that $P \cup Q = V$ and $P \cap Q = \emptyset$. The following Lemmas on bipartite graphs are known [6].

Lemma 1. *G is bipartite if and only if G has no odd cycle.*

Lemma 2. *A graph G is bipartite if and only if the vertices of g can be colored by two colors such that every pair of adjacent vertices receive different colors.*

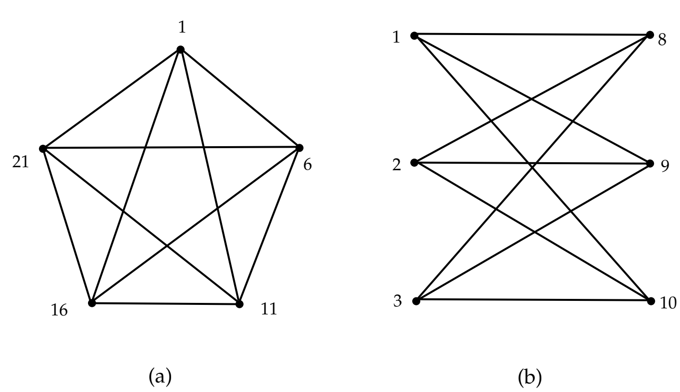

Fig. 2. For $k=5$ (a) A k-safe labeling of K_5 with span 21, (b) a k-safe labeling of $K_{3,3}$ with span 10.

The characterization of bipartite graphs in lemma 2, leads to a linear-time algorithm to determine whether a graph is bipartite or not, and also compute a bipartition of a bipartite graph.

3 k-safe Labelings of Sparse Graphs

Theorem 1. *Let $G = (V, E)$ be a bipartite graph of n vertices with the bipartition $V = P \cup Q$. Then G admits a k-safe labeling with a span $n + k - 1$.*

It is interesting to observe that the trivial span $n + k - 1$ in theorem 1 is optimal for bipartite graphs, since a complete bipartite graph does not admit a k-safe labeling with a span smaller than $n + k - 1$.

We have the following Theorem on k-safe labelings of trees.

Theorem 2. *Let T be a tree of n vertices. Let m be the number of edges in T and $\Delta(T)$ is the maximum degree of T. Every tree of n vertices admits a k-safe labeling with span $n + k - 1$.*

Proof. Since a tree has no cycle, a tree is a bipartite graph by lemma 1 in section 2. Hence every tree of n vertices has a k-safe labeling with span $n + k - 1$ by theorem 1. $\square$

4 Conclusions

We study safe labeling here which has many applications. So this is a great paper.

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