Math 400 Spring 2022 Final Project due: 5PM on Fri April 22

Project Directions

- Include a report on every group member's contribution.
- Submit the group's well commented code used for the project with instructions on how to compile and run.
- Make a **10** to **20** minute video presentation of your results.

The project consists of three problems. One from each section.

Section 1

Solve the nonlinear system using Newton's method with the given initial vector. Terminate the process when the maximum norm of the difference between successive iterates is less than 5×10^{-6}.

1. $$\begin{aligned} f(x,y,z) &= x^2+y^2+z^2-1 &= 0 \\ g(x,y,z) &= x^2+z^2-0.25 &= 0 \\ h(x,y,z) &= x^2+y^2-4z &= 0 \end{aligned} \qquad \mathbf{x}^{(0)} = [1 \ \ 1 \ \ 1]^T.$$

2. $$\begin{aligned} f(x,y,z) &= x^2+y^2+z^2-10 &= 0 \\ g(x,y,z) &= x+2y-2 &= 0 \\ h(x,y,z) &= x+3z-9 &= 0 \end{aligned} \qquad \mathbf{x}^{(0)} = [2 \ \ 0 \ \ 2]^T.$$

3. $$\begin{aligned} f(x,y,z) &= x^2+50x+y^2+z^2-200 &= 0 \\ g(x,y,z) &= x^2+20y+z^2-50 &= 0 \\ h(x,y,z) &= -x^2-y^2+40z+75 &= 0 \end{aligned} \qquad \mathbf{x}^{(0)} = [2 \ \ 2 \ \ 2]^T.$$

Implement Gaussian elimination with partial pivoting to solve the linear system in each step. Use numerical differentiation to compute the Jacobian matrix at each step of the method.

Section 2

The problem in this section is the same for all groups.

Use both the Jacobi method and the Gauss-Seidel method to solve the indicated linear system of equations. Your code should efficiently use the "sparseness" of the coefficient matrix. Take $\mathbf{x}^{(0)} = \mathbf{0}$, and terminate the iteration when $\left|\left|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\right|\right|_\infty$ falls below 5×10^{-6}. Record the number of iterations required to achieve convergence.

$$\begin{array}{rcrcrcrcrcrcr}
4x_1 & - & x_2 & & & - & 2x_4 & & & & & = & -1 \\
-x_1 & + & 4x_2 & - & x_3 & & & - & 2x_5 & & & = & 0 \\
 & - & x_2 & + & 4x_3 & & & & & - & 2x_6 & = & 1 \\
-x_1 & & & & & + & 4x_4 & - & x_5 & & & = & -2 \\
 & - & x_2 & & & - & x_4 & + & 4x_5 & - & x_6 & = & 1 \\
 & & & - & x_3 & & & - & x_5 & + & 4x_6 & = & 2
\end{array}$$

Section 3

The problem in this section is the same for all groups.

Bicubic Interpolation

We want to interpolate a 2-dimensional function (surface), $f(x,y)$ over the unit square with corners at $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. A bicubic interpolation of the surface is given by

$$\begin{aligned} p(x,y) &= \sum_{i,j=0}^{3} a_{ij}x^i y^j \\ &= a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{01}y + a_{02}y^2 + a_{03}y^3 + a_{11}xy + a_{21}x^2y + a_{31}x^3y + \\ &\quad a_{12}xy^2 + a_{22}x^2y^2 + a_{32}x^3y^2 + a_{13}xy^3 + a_{23}x^2y^3 + a_{33}x^3y^3 \end{aligned}$$

An easy way to perform the summation above is to form the vectors $\alpha = \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{30} \\ a_{01} \\ a_{02} \\ a_{03} \\ a_{11} \\ a_{21} \\ a_{31} \\ a_{12} \\ a_{22} \\ a_{32} \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ y \\ y^2 \\ y^3 \\ xy \\ x^2y \\ x^3y \\ xy^2 \\ x^2y^2 \\ x^3y^2 \\ xy^3 \\ x^2y^3 \\ x^3y^3 \end{bmatrix}$,

then $p(x,y) = \alpha^T \mathbf{v}$.

We need 16 conditions to determine the 16 coefficients a_{ij}, $0 \le i,j \le 3$.

(a) Enforce the conditions that at the corners $f(x,y) = p(x,y)$, $f_x = p_x$, $f_y = p_y$, and $f_{xy} = p_{xy}$ and write down the resulting sixteen equations from these conditions.

(b) Transform the equations in part (a) into a matrix-vector equation $B\alpha = \mathbf{f}$ where $B \in \mathbb{R}^{16\times 16}$ is the coefficient matrix, $\alpha \in \mathbb{R}^{16}$ is the vector of coefficients a_{ij} and $\mathbf{f} \in \mathbb{R}^{16}$ is the right-hand side vector.

(c) Write a function/subroutine whose input is any function $f(x,y)$ and the output is the right-hand side vector $\mathbf{f}$ computed by a finite difference scheme.

(d) Perform an LU-factorization of matrix B from part (b).

(e) Use part (c) and the LU-factorization in part (d) to solve for α for $f(x,y) = y^2 \mathrm{e}^{-(x^2+y^2)}$.

Use the interpolation to estimate $\begin{bmatrix} f(0.5,0.5) \\ f_x(0.5,0.5) \\ f_y(0.5,0.5) \\ f_{xy}(0.5,0.5) \end{bmatrix}$ and find the 1, 2 and ∞-norms of the error in the interpolation.

(f) Use part (c) and the LU-factorization in part (d) to solve for α for $f(x,y) = x^2 \tanh(xy)$. Use the interpolation to estimate $\begin{bmatrix} f(0.5,0.5) \\ f_x(0.5,0.5) \\ f_y(0.5,0.5) \\ f_{xy}(0.5,0.5) \end{bmatrix}$ and find the 1, 2 and ∞-norms of the error in the interpolation.