Perturbative Mellin correlators in N=4 Supersymmetric Yang-Mills Theory

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ABSTRACT

Mellin space provides the most natural representation of CFT correlators by making manifest the conformal symmetry and obedience to OPE of the theory. It has also proven useful in the context of the AdS-CFT correspondence providing a natural way to interpret holographic CFT correlators as scattering amplitudes in the flat space limit of AdS. We explore the Mellin representation of correlators in the weak coupling limit of $\mathcal{N} = 4$ SYM. We analyse the 4-point functions of weight two half-BPS operators and calculate the first-order correction to the anomalous dimensions and OPE co-efficients for twist two single-trace multiplets. The rather cumbersome approach to calclulate these in position space is reduced to the use of orthogonality relations for Mellin blocks (continuous Hahn polynomials), again demonstrating the simplicity of the Mellin space formalism for CFTs.

CONTENTS

1. INTRODUCTION

Quantum field theory is our best understood framework to study elementary particles. Every QFT has to respect the symmetries of special relativity, i.e. it has to be invariant under space-time translations and Lorentz transformations. Some QFTs however enjoy additional symmetries characterized by the fact that their coupling constants are independent of any length or energy scale. Such theories are called conformal field theories (CFTs). Conformal invariance is a very special feature. In usual QFTs, the couplings change as the scale of the theory is changed; this is called Renormalization Group (RG) flow and the rate of change of couplings with scale is measured by the β-function. CFTs are therefore characterized by a vanishing β-function and are "fixed-points" of the RG flow. One can imagine then that in the space of couplings, any UV complete QFT can be considered a point on the path under RG flow between a CFT in the IR and a CFT in the UV. This basically means that if we zoom out sufficiently enough, the microscopic details of the theory become irrelevant or if we go to extremely high-energies, the macroscopic details become irrelevant. Thus many QFTs that look very different microscopically may actually flow to the same IR CFT when zoomed out or vice versa. This is the phenomenon of *universality* and the QFTs belonging to the same universality class in say the IR are said to be IR-equivalent. Examples of IR equivalences are abundant in high-energy as well as condensed-matter physics. E.g, the QFT describing the 3D Ising model, the ϕ^4 theory of scalar bosons and the critical point of water, all flow to the same CFT in the IR. Universality can thus be a very useful tool that allows us to calculate physically interesting data like critical exponents of liquids by studying the 3D Ising model without worrying about microscopic details.

Usually, the correlation functions in a QFT can only be calculated perturbatively by expanding the path integral in powers of the interaction coupling using Feynman diagrams.

This approach requires one to completely specify the Lagrangian of the theory, something that is not always possible. Enhanced symmetries in CFTs however allows us to calculate or highly constrain the form of the full (non-perturbative) correlation functions just by requiring proper transformation under symmetries and internal consistency in the theory. This approach is called the *conformal bootstrap* and has been highly successful in studying CFTs non-perturbatively. As we shall see, the bootstrap approach effectively allows us to understand the dynamics of any CFT just by the knowledge of scaling dimensions of local operators and 3-point coefficients - called the CFT data. Studying CFTs using the bootstrap approach is therefore an important step towards the ultimate goal of a non-perturbative description of QFTs removing the need for knowledge of the Lagrangian.

Symmetries if properly used can be a powerful tool not only for calculational simplicity, but also to bring forth the otherwise obscure physical content of the theory. A rather pedestrian example of this is the transition from position space to momentum space description of QFTs. The Feynman rules are much simpler in momentum space and the analytic structure of correlators is much more explicit and easily studied in momentum space. The poles of the 2-point functions lie at the masses of the particles, and the higher-point correlators have a factorization property at the poles. As we shall see, a very similar role in the case of CFTs is played by the Mellin space representation of correlators. Analogous to momentum space correlators, the poles in the Mellin representation of CFT correlators lie at the scaling dimensions of local operators and the higher-point correlators factorize at the poles with the residues given in terms of lower point Mellin coorelators.

In this thesis, we study the Mellin representation of four-point correlators in $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory. Apart from being a highly symmetric theory leading to an underlying integrable structure, $\mathcal{N} = 4$ SYM is also on the CFT side of one of the most studied example of the AdS-CFT correpondence, dual to Type IIB string theory on AdS. It has been shown that for large N CFTs with a weak AdS dual, Mellin space correlators can be interpreted as scattering amplitudes in the flat space limit of AdS. [5]. We want to analyze the implications of the representation in the opposite limit, the highly-curved or tensionless limit of string theory on AdS. In this limit, the CFT dual to Type IIB string theory is weakly coupled $N = 4$ SYM. Working perturbatively in the small t'Hooft coupling, we study the 4-point correlators of weight two half-BPS operators in Mellin space. Using the Mellin block decomposition of these correlators, we calculate the first-order anomalous dimension and OPE co-efficient correction for single-trace twist 2 operators. The Mellin space method turns out be structurally much simpler than the cumbersome approach in position space [12] essentially relying on the orthogonality properties of the Mellin blocks.

The first chapter of the thesis describes the necessary background for CFTs. This is followed by a description of the Mellin space formalism for CFTs and implications for the AdS-CFT correspondence. In the next chapter, we briefly discuss the basics of supersymmetry and then $N = 4$ SYM. Finally, we develop a reformulation of perturbative $N = 4$ SYM in Mellin space and end with some conclusions and possibilities for future work

2. CONFORMAL FIELD THEORY

In this chapter, we'll discuss the basics of Conformal field theory. Much of the content is based on [1–4]

2.1 Conformal Transformations

Under a general infinitesimal transformation of coordinates in $\mathbb{R}^{\mathrm{d}}$, $x_\mu \to x_\mu^{'} = x_\mu - \epsilon_\mu(x)$, the metric transforms as

$$g_{\mu\nu}^{'} = \frac{\partial x_\mu}{\partial x_\alpha^{'}} \frac{\partial x_\nu}{\partial x_\beta^{'}} g_{\alpha\beta} \quad \Longrightarrow \quad \delta_\epsilon g^{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \tag{2.1}$$

Infinitesimal isometries therefore satisfy the Killing equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 0 \tag{2.2}$$

In flat space, the spacetime transformations that solve this equation are translations and Lorentz transformations :

$$x_\mu^{'} = x_\mu + a_\mu \quad \text{(Translation)} \tag{2.3}$$

$$x_\mu^{'} = M_{\mu\nu} x^\nu \quad \text{(Lorentz transformation)} \tag{2.4}$$

Conformal field theories have a larger symmetry group called the Conformal Group which consists of transformations that change the metric only upto a scaling factor.

$$g_{\mu\nu}^{'} = \Lambda(x) g_{\mu\nu} \tag{2.5}$$

In this case thus, (2.2) is relaxed to

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} \tag{2.6}$$

This is the Conformal Killing equation. Contracting by $g_{\mu\nu}$ on both sides gives

$$\partial_\mu \epsilon^\mu = \frac{d}{2} f(x) \tag{2.7}$$

The equation has in addition to (2.3), (2.4), the following solutions:

$$x^{'}_\mu = \lambda x_\mu \quad \text{(Dilatation)} \tag{2.8}$$

$$x^{'}_\mu = \frac{x_\mu - b_\mu x^2}{1 - 2b.x + b^2 x^2} \quad \text{(Special Conformal Transformation (SCT))} \tag{2.9}$$

The SCTs look complicated. Its useful to thin of them as an (inversion + translation + inversion) transformation where inversion is simply $x^\mu \to x^\mu / x^2$. Thus if a theory is translation + inversion symmetric, it is also symmetric under SCTs. Despite their complicated form, SCTs are used instead of Inversions because inversion is not a continuous symmetry and hence, we can't define a generator for inversions.
The Euclidean generators that give the action of these transformations on the fields are of the form

$$P^\mu = \partial^\mu \qquad M^{\mu\nu} = (x^\mu \partial^\nu - x^\nu \partial \mu) \tag{2.10}$$

$$D = x^\mu \partial_\mu \qquad K^\mu = 2 x_\nu \partial^\nu x^\mu - x^2 \partial^\mu \tag{2.11}$$

The Lorentzian generators can be obtained by multiplying the above generators with a factor of $-i$. The algebra of the Euclidean generators is given as

$$[D, P^\mu] = P^\mu, \quad [D, K^\mu] = -K^\mu, \quad [K^\mu, P^\nu] = 2\delta^{\mu\nu} D - 2M^{\mu\nu} \tag{2.12}$$

$$[M^{\mu\nu}, P^\alpha] = \delta^{\mu\alpha} P^\nu - \delta^{\nu\alpha} P^\mu, \qquad [K^{\mu\nu}, P^\alpha] = \delta^{\mu\alpha} K^\nu - \delta^{\nu\alpha} K^\mu \tag{2.13}$$

$$[M^{\alpha\beta}, M^{\mu\nu}] = \delta^{\alpha\mu} M^{\beta\nu} + \delta^{\beta\nu} M^{\alpha\mu} - \delta^{\beta\mu} M^{\alpha\nu} - \delta^{\alpha\nu} M^{\beta\mu} \tag{2.14}$$

The Euclidean conformal group in d dimensions is isomorphic to the Lorentz group in $(d+1, 1)$ dimensions and is written as $SO(d+1, 1)$. This can be seen explicitly by defining the generators

$$L_{\mu\nu} = M_{\mu\nu}, \quad L_{1,0} = D, \quad L_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \quad L_{1,\mu} = \frac{1}{2}(P_\mu - K_\mu) \tag{2.15}$$

One can check that these satisfy the commutation relations of the Lorentz group. This isomorphism will become important when we study the embedding space formalism for CFTs in section 2.8.

2.2 Representations of the Conformal Group

With knowledge of the conformal algebra, we can classify all local operators in a CFT under representations of the Conformal Group. The commutation relations (2.12) allow us to define creation and annihilation operators. Consider an operator that satisfies:

$$[D, \mathcal{O}(0)] = \Delta\mathcal{O}(0) \tag{2.16}$$

Then (2.12) implies

$$[D, P^\mu\mathcal{O}(0)] = (\Delta+1)P^\mu\mathcal{O}(0) \qquad [D, K^\mu\mathcal{O}(0)] = (\Delta-1)K^\mu\mathcal{O}(0) \tag{2.17}$$

The eigenvalue Δ is called the scaling dimension of the operator. One can show that in a unitary theory the scaling dimensions of local operators have a positive lower bound. This means we can define families of operators such that each family contains an operator with the minimum scaling dimension that satisfies:

$$[K^\mu, \mathcal{O}(0)] = 0 \tag{2.18}$$

This is called the primary operator. And other operators in the family with higher scaling dimensions can be generated from the primary by acting with P^μ. These are called descendents.

$$\mathcal{O}^{'}(0) = P^{\mu_n}...P^{\mu_2}P^{\mu_1}\mathcal{O}(0) \tag{2.19}$$

Each family labelled uniquely by the scaling dimension of the primary operator forms an irreducible representation of the conformal group. It is helpful to think of the primary like the ground state of a particle and descendants like the excited states. As we'll see later, this analogy becomes precise in the context of the AdS-CFT correspondence.

Using (2.5) along with the conformal algebra, we find that under $x \to x' = \lambda x$,

$$\mathcal{O}(x) \to \mathcal{O}'(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x) \tag{2.20}$$

2.3 Conformal Correlation Functions

We'll now study the consequences of conformal symmetry for the correlation functions in a CFT. For convenience, we'll stick to correlators of scalar primaries. We know that under a translation $x \to x' = x + a$, a scalar field transforms trivially $\mathcal{O}(x) \to \mathcal{O}'(x+a) = \mathcal{O}(x)$. And thus the correlator satisfies

$$\langle \mathcal{O}_1(x_1 + a)...\mathcal{O}_n(x_n + a)\rangle = \langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle \tag{2.21}$$

This means that the correlator should depend of the relative positions only. Similarly under a rotation $x^\mu \to x'^\mu = \Lambda^\mu_\mu x^\nu$, $\mathcal{O}(x) \to \mathcal{O}'(\Lambda x) = \mathcal{O}(x)$

$$\langle \mathcal{O}_1(\Lambda x_1)...\mathcal{O}_n(\Lambda x_n)\rangle = \langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle \tag{2.22}$$

which combined with (2.19) implies that the correlator should depend only on the distances between pairs of operator insertions.

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = f(|x_i - x_j|) \tag{2.23}$$

Under a conformal transformation, the correlators transform as

$$\langle \mathcal{O}_1(x_1')...\mathcal{O}_n(x_n')\rangle = \left|\frac{dx'}{dx}\right|_{x_1} ... \left|\frac{dx'}{dx}\right|_{x_n} \langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle \tag{2.24}$$

Lets see what the above transformation rules imply for the conformal correlators of scalar primaries. First, consider the 2-point function. We know,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = f(|x_1 - x_2|) = f(|x_{12}|) \tag{2.25}$$

Under Dilations, (2.23) implies

$$\langle \mathcal{O}_1(sx_1)\mathcal{O}_2(sx_2)\rangle = s^{-(\Delta_1+\Delta_2)}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \tag{2.26}$$

This fixes the form of the correlator upto a constant C,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C}{|x_{12}|^{\Delta_1+\Delta_2}} \tag{2.27}$$

Further transformation rule under inversion is

$$\left\langle \mathcal{O}_1\Big(\frac{x_1}{x_1^2}\Big)\mathcal{O}_2\Big(\frac{x_2}{x_2^2}\Big)\right\rangle = (x_1^2)^{\Delta_1}(x_2^2)^{\Delta_2}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \tag{2.28}$$

This fixes the form to

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{\Delta_i,\Delta j}}{|x_1 - x_2|^{2\Delta_i}} \tag{2.29}$$

where we have normalized the fields to absorb the constant factor. Similarly, demanding invariance under dilatation and inversion, the form of the 3-point scalar primary correlator can be fixed upto a constant.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.30}$$

Here we can't get rid of the constant by renomalizing as we've already renormalized the fields to absorb the constant that shows up in the two-point function.

The four-point correlator has more structure. This is because with 4 points one can construct two "cross-ratios" that are invariant under conformal transformations. These are:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{2.31}$$

This conformal symmetry can fix 4-point correlators only upto some function of the conformal cross-ratios. For example, the 4-point coorelator scalar primary $\mathcal{O}(x)$ can be written as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{G}(u,v)}{(x_{12}^2 x_{34}^2)^{\Delta}} \tag{2.32}$$

2.4 Radial Quantization and the State-Operator Correspondence

Usually in a QFT, we foliate spacetime by picking a time-direction and choosing orthogonal equal time spatial slices. The spatial slices run from $t = -\infty$ to $t = \infty$. We define local fields that take operator-values at each point in spacetime. We define a Hilbert space of states on each spatial slice. It is important to point out that while local fields take values at each point, a state is defined on the entire spatial slice. Two different spatial slices are related by a symmetry, i.e. time-translation. The Hamiltonian generates time-translations that evolves states on one spatial slice to another. A correlation function is interpreted as the vacuum expectation value of the "time-ordered product" of local operators, where the time-ordering is with respect to our choice of foliation and the vacuum states live on the $-\infty$ and ∞ spatial slices.

$$\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \langle 0|\hat{\mathcal{T}}\{\mathcal{O}(t_1,\mathbf{x}_1)...\hat{\mathcal{O}}(t_n,\mathbf{x}_n)\}|0\rangle \tag{2.33}$$

The choice of foliation defines a quantization procedure. However, this is not the only choice for the quantization procedure. E.g. in a rotationally (SO(d)) invariant theory on $\mathbb{R}^{\mathrm{d}}$, any direction can be chosen as the time-direction and spatial slices can be taken orthogonal to it. Similarly, in a Lorentz invariant theory, any time-like vector can be chosen as the time-direction. Due to the symmetries of the theory, all these quantization procedures are physically equivalent. This means that no matter what quantization procedure we choose, the correlation function (2.46) should be the same. This is equivalent to the demand that within a particular quantization scheme, the correlation function be invariant under the space-time symmetries of the theory.

Radial Quantization

Due to scale-invariance, the most natural choice of foliation in a CFT is by spheres around the origin, with the radial-direction acting like the time-direction. One can the evolve states from one sphere to another by scaling using the dilatation operator. The correlators are taken to be radially ordered.

$$\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \langle \mathcal{R}\{\hat{\mathcal{O}}(t_1, \mathbf{x}_1)...\hat{\mathcal{O}}(t_n, \mathbf{x}_n)\}\rangle \tag{2.34}$$

State-Operator Correspondence

In a CFT, there is a one-one correspondence between the local operators and states. This is a remarkable feature. As mentioned before, in the usual quantization procedure, states and local operators are very distinct objects. While local operators are defined at each point in spacetime, states are defined over an entire spatial slice. More precisely, defining a state on a slice is actually defining a particular field configuration on the slice. One can for example define a field eigenstate $|\phi_e\rangle$ living on a slice at some time t (or radius r) and interpret it as saying that the field $\phi(x)$ has the particular configuration $\phi_e(x)$ on that slice, i.e.

$$\hat{\phi}(\mathbf{x})|\phi_e\rangle = \phi_e(\mathbf{x})|\phi_e\rangle \tag{2.35}$$

or in radial quantization,

$$\hat{\phi}(\hat{\mathbf{n}})|\phi_e\rangle = \phi_e(\hat{\mathbf{n}})|\phi_e\rangle \tag{2.36}$$

where $\hat{\mathbf{n}}$ is a unit vector specifying a point on the sphere. Any other state on the slice can then be written in terms of field eigenstates.

$$|\psi\rangle = \int D\phi_e \langle \phi_e|\psi\rangle |\phi_e\rangle \tag{2.37}$$

What allows states to be mapped to operators in a CFT is the fact that given any point x where a local operator is inserted and a sphere around it on which a state lives, one can shrink that sphere to the point x s.t both the state and the local operator live at the same point and this process will not change the physics (correlators) due to local scale invariance. Lets put this idea to use and explicitly see the mapping.

Operator $\implies$ State

We begin by defining the vacuum on the sphere of radius r.

$$|0\rangle = \int D\phi_e \langle \phi_e|0\rangle |\phi_e\rangle \tag{2.38}$$

The co-efficient is simply given by the path integral over the interior of the sphere with the "boundary condition" $|\phi_e\rangle$ at radius r.

$$\langle \phi_e|0\rangle = \int_{\phi(\hat{\mathbf{n}},r)=\phi_e(\hat{\mathbf{n}})} D\phi e^{-S[\phi]} \tag{2.39}$$

The state $|\Delta\rangle$ is defined by inserting an operator of scaling dimension Δ at the origin.

$$\langle \phi_e|\Delta\rangle = \int_{\phi(\hat{\mathbf{n}},r)=\phi_e(\hat{\mathbf{n}})} D\phi \mathcal{O}(0) e^{-S[\phi]} \tag{2.40}$$

Equation (2.47) explicitly tells us how to get a state from an operator. (2.46) can be seen as a special case where the operator inserted is that of scaling dimension 0 i.e. identity. Again the idea is that a state on a sphere of radius r around origin can be shrunk to the origin where the local operator is inserted.

State $\implies$ Operator

Defining an operator simply means defining its correlator with all other operators. Lets look at an n-point correlator of operators $\mathcal{O}_i$ with scaling dimension Δ_i

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = \frac{1}{Z}\int D\phi \mathcal{O}(x_1)...\mathcal{O}_n(x_n) e^{-S[\phi]} \tag{2.41}$$

We can rewrite rhs of (2.54) without talking about operator insertions. Instead we consider balls $\mathcal{B}_i$ centred around the points x_i and specify states $|\mathcal{O}_i\rangle$ on their spherical boundaries $\partial\mathcal{B}_i$. Then the correlator (2.54) is computed by path integrating over all paths but with field configurations fixed on the spheres $\partial\mathcal{B}_i$ by the states defined. The states $|\mathcal{O}_i\rangle$ can be expanded in field eigenstates

$$|\mathcal{O}_i\rangle = \int D\phi_{ei}\langle\phi_{ei}|\mathcal{O}_i\rangle|\phi_{ei}\rangle \tag{2.42}$$

where ϕ_{ei} is the field configuration on $\partial\mathcal{B}_i$. Therefore the full path integral is performed on the region outside the balls $\mathcal{B}_i$ and is written as

$$\langle\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = \frac{1}{Z}\int\prod_i D\phi_{ei}\langle\phi_{ei}|\mathcal{O}_i\rangle\int_{\phi|_{\partial\mathcal{B}_i}=\phi_{ei}} D\phi(x)e^{-S[\phi]} \quad (x \notin \mathcal{B}_i) \tag{2.43}$$

Now the balls $\mathcal{B}_i$ can be infinitesimally small and so the insertion points x_i can be brought arbitrarily close together. (2.50) then explicitly tells us how to compute correlators of local operators from states.

2.5 Operator Product Expansion

Another remarkable feature of CFTs is the existence of a convergent Operator Product expansion. Using the state-operator correspondence, we can construct a state at radius r by operator insertions at points 0 and x.

$$|\psi\rangle = \mathcal{O}(x)\mathcal{O}(0)|0\rangle \qquad (|x| < r) \tag{2.44}$$

But this state can also be expanded in a basis of Dilatation eigenstates, i.e. primaries and descendants.

$$|\psi\rangle = \sum_k C_{ij}^k(x, \partial_x)\mathcal{O}_k(0)|0\rangle$$

where k runs over all primaries in the thoery and $C_{ij}^k(x, \partial_x)$ packages derivatives that lead to desceandants in the expansion. We therefore have the operator equation

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k C_{ij}^k(x, \partial_x)\mathcal{O}_k(0) \tag{2.45}$$

This is called the Operator Product Expansion (OPE). The sum is convergent as long as $|x| < r$ where r is the radius of the biggest sphere that contains only 0 and x and no other operator insertions. Also, the expansion is general as any two points x_1 and x_2 can be brought to 0 and $x_2 - x_1$ by translation. This gives

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ij}^k(x_{12}, \partial_2)\mathcal{O}_k(x_2) \tag{2.46}$$

The form of C_{ij}^k is also constrained by conformal symmetry. Acting by the Dilatation operator on both sides, we see that it takes the form

$$C_{ij}^k(x,\partial_x) \propto |x|^{\Delta_k-\Delta_i-\Delta_j}(1+a_1x^\mu\partial_\mu+a_2x^2\partial^2+a_3x^\mu x^\nu\partial_\mu\partial_\nu+...) \tag{2.47}$$

Again acting by K^μ will fix the form upto a constant, but this calculation is cumbersome. An easier way is to multiply the (2.45) by another primary $\mathcal{O}_l(x_3)$. This gives the 3-point function on LHS and on the RHS, using$\langle\mathcal{O}_k(x_2)\mathcal{O}_l(x_3)\rangle = \delta_{k,l}|x_{23}|^{-2\Delta_l}$, we get

$$\frac{C_{ijk}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} = C_{ij}^k(x_{12},\partial_2)|x_{23}|^{-2\Delta_k} \tag{2.48}$$

This tells us that C_{ijk} is proportional to the 3-point constant f_{ijk} times a differential operator, which can be obtained by acting using the form (2.46) in (2.47) and finding the co-efficients order by order in small $|x_{12}|/|x_{23}|$ expansion.

2.6 Conformal Block Decomposition

Conformal Block Decomposition

We'll use the OPE to compute 4-point correlator of scalar primaries ϕ. Given the 4-points x_i are configured properly (ensuring convergent OPEs), we can write the 4-point correlator as a product of two OPEs.

$$\langle\overbrace{\phi(x_1)\phi(x_2)}\,\overbrace{\phi(x_3)\phi(x_4)}\rangle = \sum_{k,k',l,l'} f_{\phi\phi k}f_{\phi\phi k'}C_l(x_{12},x_2)C_{l'}(x_{34},x_4)\langle\mathcal{O}_k(x_2)\mathcal{O}_k'(x_4)\rangle \tag{2.49}$$

$$= \sum_{k,l} f_{\phi\phi k}^2 C_l(x_{12},x_2)C_{l'}(x_{34},x_4)\frac{I^{l,l'}(x_{24})}{x_{24}^{2\Delta_k}} \tag{2.50}$$

$$= \frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\sum_{\Delta_k,l_k} f_{\phi\phi k}^2 G_{\Delta_k,l_k}(x_i) \tag{2.51}$$

where we define

$$G_{\Delta_k,l_k}(x_i) = x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}C_l(x_{12},x_2)C_{l'}(x_{34},x_4)\frac{I^{l,l'}(x_{24})}{x_{24}^{2\Delta_k}} \tag{2.52}$$

$G_{\Delta,l}(x_i)$ are called Conformal Blocks and they represent the contribution of an entire family of primary and its descendants to the 4-point correlator. We already know from (2.45) the form of the 4-point correlator fixed by conformal invariance. We therefore have

$$\mathcal{G}(u,v) = \sum_{\Delta_k,l_k} f_{\phi\phi k}^2 G_{\Delta_k,l_k}(x_i) = \sum_{\Delta_k,l_k} f_{\phi\phi k}^2 G_{\Delta_k,l_k}(u,v) \tag{2.53}$$

This is called the Conformal Block Decomposition of the 4-point correlator.

Computing Conformal Blocks

We saw before (eqn. 2.15) that the d dimensional conformal group generators can be written as generators of the $(d+1,1)$ dimensional Lorentz group. The conformal Casimir is thus written as $C = -\frac{1}{2}L^{\mu\nu}L_{\mu\nu}$, $\mu = -1, 0, .., d+1$. The eigenvalues of the Casimir label the irreducible representations of the conformal group. Therefore, a primary and it's family of descendants are all eigenvectors of the Casimir with the same eigenvalue given as

$$C|\mathcal{O}\rangle = \lambda_{\Delta,l}|\mathcal{O}\rangle = (\Delta(\Delta-d) + l(l+d-2))|\mathcal{O}\rangle \tag{2.54}$$

Using this to find the action of the Casimir on the 4-point correlator, we see that the conformal blocks satisfy the eigenvalue equation

$$\mathcal{D}G_{\Delta,l}(u,v) = \lambda_{\Delta,l}G_{\Delta,l}(u,v) \tag{2.55}$$

where D is a differential operator given in terms of $z, \bar{z}$ variables as

$$\begin{aligned}\mathcal{D} = 2(z^2(1-z)\partial_z^2 - z^2\partial_z) + 2(\bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}) \\ +2(d-2)\frac{z\bar{z}}{1-z)(1-\bar{z})}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}})\end{aligned} \tag{2.56}$$

The differential equation can be solved analytically in even dimensions. The explicit forms on 2d and 4d are given as

$$\begin{aligned}G^{(2d)}_{\Delta,l}(u,v) = k_{\Delta+l}(z)k_{\Delta l}(\bar{z}) - k_{\Delta-l}(z)k_{\Delta+l}(\bar{z}) \\ G^{(4d)}_{\Delta,l}(u,v) = \frac{z\bar{z}}{z-\bar{z}}k_{\Delta+l}(z)k_{\Delta l-2}(\bar{z}) - k_{\Delta-l-2}(z)k_{\Delta+l}(\bar{z})\end{aligned} \tag{2.57}$$

where $k_n(x) = x^{n/2}{}_2F_1(\frac{n}{2}, \frac{n}{2}, n, x)$.

2.7 Crossing Symmetry and the Conformal Boostrap

We've seen that using the OPE between two local operators, we can reduce n-point correlation functions into $n-1$-point correlation functions. For example, in (2.46), we used the OPE to write a 3-point correlator of primaries in terms of 2-point functions of all the primaries and descendants in the theory. Similarly, the conformal block decomposition came from doing a double OPE expansion to write a 4-point correlator in terms of 2-point functions. One can do this recursively for an n-point function to write it as a sum of one-point functions which are completely fixed by conformal symmetry.

$$\langle \mathcal{O}(x) \rangle = \begin{cases} 1 & \mathcal{O} = \text{Identity operator} \\ 0 & \text{otherwise} \end{cases} \tag{2.58}$$

Thus the only information necessary to fully determine all the correlators in a CFT are the scaling dimensions Δ , spins l (two-point functions) and co-efficients of three-point functions f_{ijk}. This is called the *CFT data*. Using restrictions imposed by conformal invariance, OPE and other consistency conditions, one can highly constraint the CFT data. This is the idea of the Conformal bootstrap program. One of the well-known constraints that the CFT data must follow is *crossing symmetry*. In the last section, we expanded the four-point correlator into conformal blocks by doing a double OPE between operators $(\phi(x_1), \phi(x_2))$ and $(\phi(x_3), \phi(x_4))$. This is called the s-channel OPE. But in general we can do an OPE between between different pairs of operators in different orders and the result should be the same. For example, we could've done a t-channel, i.e. between $(\phi(x_1), \phi(x_4))$ and $(\phi(x_2), \phi(x_3))$. Then

$$\langle \overbracket{\phi(x_1)\phi(x_2)}\,\overbracket{\phi(x_3)\phi(x_4)} \rangle = \langle \overbracket{\phi(x_1)\,\overbracket{\phi(x_2)\phi(x_3)}\phi(x_4)} \rangle \tag{2.59}$$

The above equation means that the OPE is associative and the equality of OPEs in various channels is called crossing symmetry. Sometimes in analogy with flat space scattering amplitudes, we express equation (2.59) diagrammatically. Lets see what constraints the

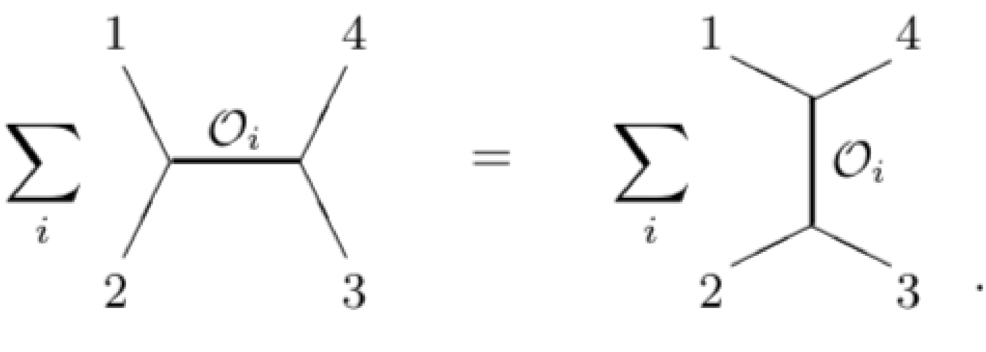

Figure 2.1: Equivalence of the s-channel and t-channel OPEs

crossing symmetry of the 4-point correlator of identical scalar primaries puts on the CFT data. One only needs to consider two other permutations apart from the s-channel. All the rest can be generated from these two permutations. First, we just consider a $(3 \leftrightarrow 4)$ exchange. This doesn't change the pairs of operators between which the OPE is done, but changes the ordering. From (2.32), we can easily see that this implies:

$$\mathcal{G}(u,v) = \mathcal{G}(\frac{u}{v}, \frac{1}{v}) \tag{2.60}$$

Doing a $(2 \leftrightarrow 4)$ exchange gives the t-channel OPE. This implies

$$\mathcal{G}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \mathcal{G}(v,u) \tag{2.61}$$

One can show that the constraint (2.60) is actually true block by block, i.e. all the blocks individually satisfy

$$G_{\Delta,l}(u,v) = G_{\Delta,l}(\frac{u}{v}, \frac{1}{v}) \tag{2.62}$$

This means that a $(3 \leftrightarrow 4)$ exchange doesn't place any constraints on the CFT data. (2.61) however is not satisfied by individual blocks and thus puts non-trivial constraints on the CFT data.

2.8 AdS-CFT in the Embedding Space Formalism

2.8.1 CFT in Embedding space

We've seen that the Conformal group in d dimensions is isomorphic to the Lorentz group in $(d+1,1)$ dimensions $SO(d+1,1)$. This means that there should be a way to embed CFT_d in $\mathbb{R}^{d+1,1}$ such that Lorentz transformations on the embedding space $\mathbb{R}^{d+1,1}$ act as conformal transformations on the CFT_d. This is very useful. Conformal tranformations like SCTs act non-linearly on the fields, and therefore make calculations cumbersome. In the embedding space formalism, this is omply replaced by the linear action of the Lorentz group. Lets see how this works.

The metric on $\mathbb{R}^{d+1,1}$ is simply given by

$$ds^2_{ES} = -(dX^0)^2 + \delta_{\mu,\nu} dX^\mu dX^\nu + (dX^{d+1})^2 \qquad \mu = 1, 2, ..d. \tag{2.63}$$

To reach a d-dimensional space, we need to get rid of two co-ordinates. We first restrict

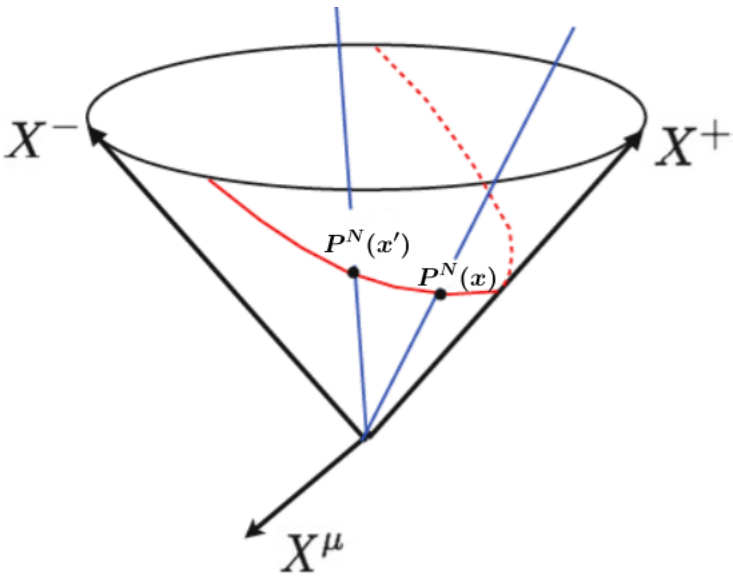

Figure 2.2: Poincare slice in embedding space

to the space of null vectors in $\mathbb{R}^{d+1,1}$, $X^2 = 0$, which is invariant under Lorentz transformations. Now, to get rid of another co-ordinate we need to restrict to a sub-space of the null cone which is flat. This can be achieved by slicing the null cone by the plane $X^0 + X^{d+1} = 1$. The resultant sub-space is called the *Poincare slice*. To see that it is indeed flat, lets parametrize the slice as follows (we use P to denote points on the Poincare slice)

$$P^0(x) = \frac{1+x^2}{2}, \quad P^\mu(x) = x^\mu, \quad P^{d+1}(x) = \frac{1-x^2}{2} \qquad \mu = 1, 2, ..d. \tag{2.64}$$

It is easy to see that the metric on the slice is simply that of $\mathbb{R}^d$.

$$ds_P^2 = \delta_{\mu,\nu} dP^\mu dP^\nu = \delta_{\mu,\nu} dx^\mu dx^\nu \tag{2.65}$$

Also, using points on the Poincare slice, we can parametrize the full null cone

$$X^N = \lambda(X) P^N(x) \qquad N = 0, 1, ..., d+1. \tag{2.66}$$

The metric on the null cone is given as

$$\begin{aligned} ds_{NC}^2 = d(\lambda(X)P(x))^2 = \lambda(X)^2 dP^2(x) + d\lambda(X)^2 P(X)^2 \\ +2\lambda(x) d\lambda(X) P(X) dP(X) = \lambda(X)^2 dP^2(x) \end{aligned} \tag{2.67}$$

where we have used $P^2(x) = 0$.

Now define an action of the Lorentz group on the Poincare slice as follows.

$$\begin{aligned} P^N(x) &\xrightarrow{\text{Lorentz off}} X^N(x) = \Lambda^N_M P^N(x) = \Omega^{-1}(X) P^N(x') \\ &\xrightarrow{\text{Scale back}} P^N(x') = \Omega(X) X^N(x) \end{aligned} \tag{2.68}$$

We first do a Lorentz transformation that takes $P(x)$ to some point $X(x)$ off the slice and on to the (rest of the) null cone. Then using (2.66), we appropriately scale $X(x)$ to bring it back to the slice. Now lets check how the action affects the metric on the slice.

$$\begin{aligned} dP(x)^2 \xrightarrow{\text{Lorentz}} dX(x)^2 = dP(x)^2 &\xrightarrow{\text{Scale}} dP(x')^2 = (d\Omega(X)X(x))^2 \\ &= \Omega(X)^2 dX(x)^2 = \Omega(X)^2 dP(x)^2 \end{aligned} \tag{2.69}$$

This confirms that the action defined in (2.68) indeed implements a conformal transformation on the slice as required. It also justifies why we chose to restrict to the null cone, and not to any other Lorentz invariant subspace of $\mathbb{R}^{d+1,1}$. The scaling back transformation that induces the conformal transformation on the slice is only possible because we can parametrize the null cone as in (2.66). We also need to make sure fields on the slice transform appropriately under conformal transformations. We'll focus on scalar fields. Consider a field defined on the full null cone $\phi(X)$. We simply demand that it changes homogeneously under scaling, i.e.

$$\phi(\lambda X) = \lambda^{-\Delta}\phi(X) \tag{2.70}$$

It is easy to see that this gives the appropriate transformation for scalar fields under conformal transformations when ϕ is restricted to the slice. The homogeneity property lets us extend the fields from the slice to the null cone. We can then work with scalar fields and their correlators on the full null cone and finally restrict to the physical slice using the homogeneity property. Extending vector and spinor fields to the null cone requires conditions in addition to (2.70). We'll not discuss them here since they're not required in any future discussions.

2.8.2 AdS in Embedding Space

Euclidean AdS_d can be embedded as a hyperboloid in $\mathbb{R}^{d+1,1}$ given by the equation

$$-(X^0)^2 + \delta_{\mu,\nu} X^\mu X^\nu + (X^{d+1})^2 = -R^2 \tag{2.71}$$

where R is the AdS radius. Clearly, the isometry group of AdS_d is the $(d+1,1)$ dimensional Lorentz group. We can also write (2.71) in Poincare co-ordinates defined by

$$X^0 = R\frac{1+x^2+z}{2z}, \quad x^\mu = R\frac{X^\mu}{z}, \quad X^{d+1} = R\frac{1-x^2-z}{2z} \quad (z>0) \tag{2.72}$$

The metric in the Poincare co-ordinates is

$$ds^2 = \frac{1}{z^2}(dz^2 + \delta_{\mu,\nu}dx^\mu dx^\nu) \tag{2.73}$$

In Poincare co-ordinates, AdS is seen to be conformal to the half-cylinder $\mathbb{R}_+ \times \mathbb{R}^{\mathrm{d}}$. The boundary of the cylinder then is conformal to $\mathbb{R}^{\mathrm{d}}$ and is reached in the $z \to 0$ limit. We can also directly see that as $z \to 0$, (2.72) approaches a scaling factor times the co-ordinates of the Poincare slice on which the CFT resides.

2.8.3 AdS-CFT in Embedding Space

The previous discussion suggests that working in embedding space one can compute correlators of operator insertions in AdS and take the limit where the insertions go to the boundary of AdS and this should give correlators of operators in a conformal theory. But it is not clear which operators in AdS (or the bulk) will map to which operators in the boundary conformal theory. Again the answer is provided by the fact the symmetry group of the boundary theory and the bulk theory are the same.
To see this, it is convenient to work in the global AdS co-ordinates. These are given by

$$X^0 = R\cosh\tau\cosh\rho \quad X^\mu = R\Omega_\mu \sinh\rho \quad X^{d+1} = -R\sinh\tau\cosh\rho \tag{2.74}$$

where Ω_μ parametrizes a unit sphere S^{d-1}, $\Omega.\Omega = 1$, τ is the Euclidean (imaginary) time and ρ is the radial co-ordinate. The Dilatation operator in these co-ordinates looks like

$$D = -iJ_{0,d+1} = -\frac{\partial}{\partial\tau} \tag{2.75}$$

To understand the global structure better, it is useful to change $\tanh\rho = \sin r$ so that $r \in [0, \pi/2]$. The metric then becomes

$$ds^2_{AdS} = \frac{R^2}{\cos^2 r}(d\tau^2 + dr^2 + \sin^2 r d\Omega^2_{d-1}) \tag{2.76}$$

In these co-ordinates, AdS is conformally equivalent to a solid cylinder whose boundary at $r = \pi/2$ is conformal to $\mathbb{R} \times S^d$. The imaginary time direction forms the axis of the cylinder, and time evolution is generated by the Dilatation operator.

In the discussion of CFTs in embedding space, it was useful to think of the d-dimensional conformal group in terms of the $(d+1,1)$ dimensional Lorentz group. Now, we'll do the opposite, i.e. think of the action of the conformal group generators on scalar fields in AdS.

Consider the Klein-Gordon equation for a free massive scalar in AdS.

$$\nabla^2_{AdS}\phi = m^2\phi \tag{2.77}$$

The Laplacian in AdS can be written in terms of the quadratic Casimir of the Lorentz group. To see this, write the Casimir in the embedding space

$$\frac{1}{2}J^{AB}J_{BA}\phi = [(-X^2\partial X^2 + X.\partial X(d + X.\partial X)]\phi \tag{2.78}$$

If we foliate the embedding space with AdS surfaces of different radii, it is natural that we have

$$\partial^2 X = -\frac{1}{R^{d+1}}\frac{\partial}{\partial R}R^{d+1}\frac{\partial}{\partial R}\phi + \nabla^2_{AdS}\phi \tag{2.79}$$

Using (2.78) and $X.\partial X = R.\partial R$ in (2.77), we get

$$\frac{1}{2}J^{AB}J_{BA}\phi = R^2\nabla^2_{AdS}\phi \tag{2.80}$$

Thus we can identify the Casimir eigenvalue of the scalar field ϕ with its mass m

$$\Delta(\Delta - d) = m^2 \tag{2.81}$$

Equation (2.75) tells us to identify Δ as the energies of states. We can then look for the primary state defined by $K^\mu\phi(\tau = 0)|0\rangle = 0$ which is the ground state of the field. Excited states correspond to the descendants $P^\mu\phi$ and all have the same Casimir eigenvalue as the primary and therefore the same mass. This answers the question we raised in the beginning. It is natural to map the one-particle spectrum of a scalar field of mass m in the bulk to the scaling dimensions of a primary and it's descendants in the boundary conformal theory such that

$$m^2 = \Delta(\Delta - d) \qquad E = \Delta + n \tag{2.82}$$

AdS-CFT Correlators in Embedding Space

Lets now explicitly check the expectation that boundary limit of AdS correlators should give us conformal correlators. Consider a free scalar field in Euclidean AdS with action

$$S = \int_{AdS} dX \frac{1}{2}((\nabla\phi)^2 + m^2) \tag{2.83}$$

The propagator $\Pi(X_1, X_2)$ satisfies

$$[\nabla_X^2 - m^2]\Pi(X_1, X_2) = -\delta(X_1, X_2) \tag{2.84}$$

and is called the bulk-bulk propagator. Solving the equation gives

$$\Pi(X_1, X_2) = \frac{C_\Delta}{\xi^\Delta} {}_2F_1(\Delta, \Delta - \frac{d-1}{2}, 2\Delta - d + 1, -\frac{4}{\xi}) \tag{2.85}$$

where $\xi = (X_1 - X_2)^2/R^2$ gives the distance between two points in embedding space, $\Delta(\Delta - d) = m^2$ and

$$C_\Delta = \frac{\Gamma(\Delta)}{2^{d/2}\Gamma(\Delta - \frac{d}{2} + 1)} \tag{2.86}$$

From now on, we'll set $R = 1$ and measure all lengths in units of AdS radius. Note that for points on AdS, $\xi = -2 - 2X_1.X_2$ and for points on the Poincare slice $\xi = -2X_1.X_2$. Now, lets consider the limit where the points X_1, X_2 go to the boundary. As $z \to 0$, the AdS metric goes as

$$\lim_{z\to 0} ds^2_{AdS} \approx \frac{1}{z^2} dP(x)^2 \tag{2.87}$$

Setting $\lambda = 1/z$, we take the boundary limit of a bulk operator as follows

$$\mathcal{O}(P(x)) = \frac{1}{\sqrt{C_\Delta}} \lim_{\lambda\to\infty} \lambda^\Delta O(X = \lambda P(x) + ...) \tag{2.88}$$

where we only keep the terms that grow as $\lambda \to \infty$.The additional terms in the argument ensure $X^2 = -1$. The *boundary-boundary* propagator is given by

$$\begin{aligned} \langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle &= \frac{1}{C_\Delta} \lim_{\lambda\to\infty} \lambda^{2\Delta} \langle \mathcal{O}(X_1)\mathcal{O}(X_2)\rangle \\ &= \frac{1}{C_\Delta} \lim_{\lambda\to\infty} \lambda^{2\Delta} \frac{C_\Delta}{(-2\lambda^2 P_1.P_2)^\Delta} = \frac{1}{(-2P_1.P_2)^\Delta} \end{aligned} \tag{2.89}$$

which is exactly a two-point correlator of two identical scalar primaries of dimension Δ as $-2P_1.P_2 = x_{12}^2$.

Similarly, the *bulk-boundary* propagator is given by

$$\langle \mathcal{O}(P)\mathcal{O}(X)\rangle = \frac{\sqrt{C_\Delta}}{(-2P.X)^\Delta} \tag{2.90}$$

We can also compute higher-point CFT correlators via an AdS calculation. E.g., if the action has a $g\phi^3$ interaction. We can compute the 3-point CFT correlator by evaluating a 3-point contact Witten diagram as follows

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\rangle = -gC_\Delta^{-\frac{3}{2}} \int_{AdS} DX \Pi(X,P_1)\Pi(X,P_2)\Pi(X,P_3) + O(g^3) \quad (2.91)$$

3. MELLIN SPACE FORMALISM FOR ADS-CFT

Often the physical content of a theory can be expressed through different but equivalent mathematical representations. The most convenient and illuminating representation is often the one that makes the symmetries and dynamics of the theory manifest. In usual QFTs, this is the reason we switch over to momentum space from position space. Apart from the fact that the Feynman diagrams in momentum space are much simpler and easier to calculate, the Fourier transform also has some nice analytic properties. For example, simple poles in p correspond to single-particle states in the theory while as branch cuts correspond to a multi-particle state. CFTs however do not have any inherent scale. The CFT correlators behave as $(p^2)^{\Delta-d/2}$ and therefore have a branch cut at $p = 0$. As we shall see, the most natural representation for CFT correlators is achieved by going to Mellin space. The Mellin representation for CFT correlators provides structurally the same advantages as the momentum space representation does for usual QFT correlators. Moreover, in the context of large N CFTs with weakly coupled bulk duals, there is a natural way to interpret the Mellin correlators as scattering amplitudes in the flat space limit of AdS. For a review and some useful applications see, [4–6].

3.1 Mellin Representation

The Mellin transform of a function $f(x)$ is defined as

$$M(f(x))(s) = \int_0^\infty dx x^{s-1} f(x) \tag{3.1}$$

and the inverse Mellin transform which we'll use more frequently as

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds x^{-s} M(s) \tag{3.2}$$

where the contour lies parallel to the imaginary axis. Two common Mellin integrals that we'll use are

$$x^{-\Delta} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds x^{-s}\frac{1}{s-\Delta} \qquad e^{-x} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds x^{-s}\Gamma(s) \tag{3.3}$$

For CFT correlators, we define the Mellin representation as follows

$$\langle O(x_1)...O(x_n)\rangle = \int_{-i\infty}^{i\infty} [d\gamma_{ij}]\mathcal{M}_n(\gamma_{ij})\prod_{i<j}^{n} x_{ij}^{-2\gamma_{ij}}\Gamma(\gamma_{ij}) \tag{3.4}$$

where γ_{ij} are the Mellin variables. The measure $[d\gamma_{ij}]$ denotes that the integration is to be done only over the Mellin variables that are independent, given the following constraints

$$\sum_j \gamma_{ij} = 0, \qquad \gamma_{ij} = \gamma_{ji}, \qquad \gamma_{ii} = -\Delta_i \tag{3.5}$$

The constraints come by demanding that the correlators transform properly under conformal transformations. The above constraints give $n(n-3)/2$ independent variables. The $\Gamma(\gamma_{ij})$ functions in the measure are a convention but a very useful one. We'll discuss their role in more detail later. Often the whole integrand along with the Γ-functions is called the "full" Mellin amplitude while just $\mathcal{M}_n(\gamma_{ij})$ is called the "reduced" Mellin amplitude. The contour runs parallel to the imaginary axis and its position on the real axis is such that it passes to the right/left of the semi-infinite sequences of poles of the integrand that run to the left/right. To understand this, lets focus on the case of 4-point function of identical scalar primaries. Solving the constraints (3.5) leaves us with two independent Mellin variables - pick γ_{12}, γ_{14}. The Mellin representation is given as

$$\begin{aligned}\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \frac{1}{(x_{12}^2)^{2\Delta}(x_{34}^2)^{\Delta}}\int \frac{ds_{12}ds_{14}}{(2\pi i)^2} u^{\Delta-\gamma_{12}}v^{-\gamma_{14}} \\ [\Gamma^2(\gamma_{12})\Gamma^2(\gamma_{14})\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})]\mathcal{M}_4(\gamma_{12},\gamma_{14})\end{aligned} \tag{3.6}$$

From the expression, we see that the Gamma functions give two semi-infinite sequences of double poles. Later we'll see that the Mellin transform also has same type of semi-infinite sequences of poles, and the contour must pass through an analytic strip between the two sequences as shown in Fig.(3.1) for the γ_{12} plane.

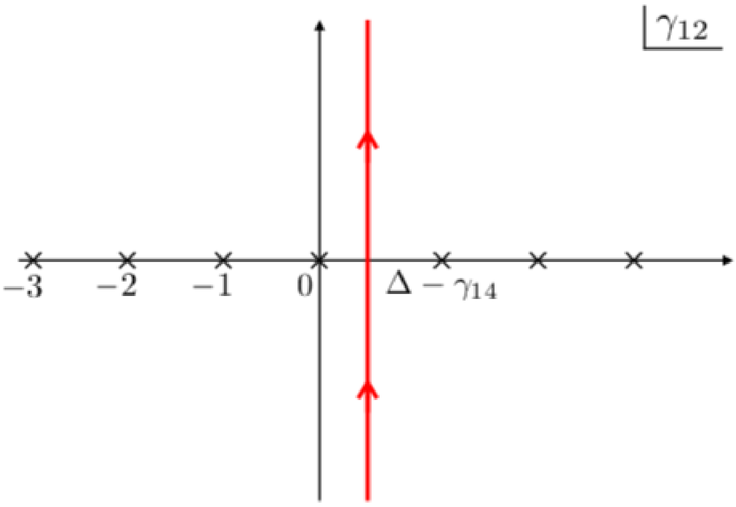

Figure 3.1: Contour prescription for Mellin variable γ_{12}

3.2 Properties of the Mellin Representation

Lets now study some important universal properties of the Mellin representation.

3.2.1 Crossing Symmetry

From (3.4), it is easy to see that the the Mellin representation is invariant under $x_i \leftrightarrow x_j$ if we make the same change in the Mellin variables, i.e. do a $i \leftrightarrow j$ transformation in all the Mellin variables. For example, in the case of four-point correlator of identical scalar primaries (Eqn. (3.6)), equivalence of s-channel and t-channel results, i.e. invariance under $x_2 \leftrightarrow x_4$ transformation can be expressed in Mellin space as

$$\mathcal{M}_4(\gamma_{12}, \gamma_{13}, \gamma_{14}) = \mathcal{M}_4(\gamma_{14}, \gamma_{13}, \gamma_{12}) \tag{3.7}$$

Thus we see that crossing symmetry in Mellin space takes a much simpler form than in position space.

3.2.2 Location of poles

It is useful to identify the Mellin variables with fictitious momenta via $\gamma_{ij} = p_i.p_j$. With this the constraints (3.5) become

$$p_i^2 = -\Delta_i, \qquad \sum_{j=1}^{n} p_j = 0 \tag{3.8}$$

If we identify $\Delta \approx m^2$, quite interestingly, we can interpret the constraints as an on-shell condition and momentum conservation. This provides a way to think off Mellin correlators in terms of flat space scattering amplitudes. Is this relation deeper? We'll see soon that this is indeed the case. If we take the above identification seriously, we expect that just like for scattering amplitudes, if we consider a tree-level k going to $n-k$ particle scattering amplitude as shown in Fig. 3.2, the mass m of the intermediate exchanged particle shows up as a simple pole in the momentum space representation at

$$(\sum_{i=1}^{k} p_i)^2 = -m^2 \tag{3.9}$$

By analogy, we'd expect that the poles in the Mellin diagram where a dimension $\Delta + 2n$,

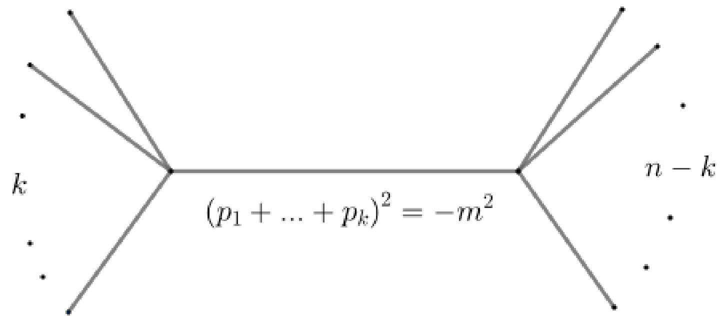

Figure 3.2: k to $n-k$ tree level scattering

spin l operator connects k-external points to $n-k$ external points should be given by

$$\begin{aligned}(\sum_{i=1}^{k} p_i)^2 = -m^2 &\implies \sum_{i=1}^{k} p_i^2 + 2\sum_{i<j=1}^{k} p_i.p_j = -m^2 \\ &\longrightarrow \sum_{i=1}^{k} \Delta_i - 2\sum_{i<j=1}^{k} \gamma_{ij} = \Delta - l + 2n\end{aligned} \tag{3.10}$$

Lets confirm that this expectation holds true by explicitly finding the poles. Consider an n-point CFT correlator of scalar primaries,

$$\mathcal{A}_n(x_i) = \langle \prod_{i=1}^{k} \mathcal{O}_i(x_i) \prod_{i=k+1}^{n} \mathcal{O}_i(x_i) \rangle \tag{3.11}$$

We recursively use the OPE to write the product of first k operators as follows

$$\prod_{i=1}^{k} \mathcal{O}_i(x_i) = \sum_{p} \sum_{m=0}^{\infty} C^{\mu_1,..\mu_m}_{p,\nu_1,..,\nu_l}(x_1,..,0)\partial_{\mu_1}..\partial_{\mu_m}\mathcal{O}_p^{\nu_1,..,\nu_l}(0) \tag{3.12}$$

where we've used translation to set $x_k = 0$. Acting by the Dilatation on both sides gives

$$\prod_{i=1}^{k} \mathcal{O}_i(e^{-\lambda}x_i) = \sum_p \sum_{m=0}^{\infty} e^{\lambda \sum_{i=1}^{k} \Delta i} e^{-\lambda(\Delta_p+2m)} C_{p,\nu_1,..,\nu_l}^{\mu_1,..\mu_m}(x_1,..,0)\partial_{\mu_1}..\partial_{\mu_m}\mathcal{O}_p^{\nu_1,..,\nu_l}(0) \tag{3.13}$$

Here to ensure that the OPE makes sense, we must choose λ large enough so that the points $x_1,..,x_{k-1}$ are closer to $x_k = 0$ than any of the other points. Plugging this back into (3.10), we get

$$\langle \prod_{i=1}^{k} \mathcal{O}_i(e^{-\lambda}x_i) \prod_{i=k+1}^{n} \mathcal{O}_i(x_i)\rangle = \sum_p \sum_{m=0}^{\infty} e^{\lambda \sum_{i=1}^{k} \Delta i} e^{-\lambda(\Delta_p+2m)} F_{p,m}(x_1,..,x_n) \tag{3.14}$$

where

$$F_{p,m}(x_1,..,x_n) = C_{p,\nu_1,..,\nu_l}^{\mu_1,..\mu_m}(x_1,..,0)\langle \partial_{\mu_1}..\partial_{\mu_m}\mathcal{O}_p^{\nu_1,..,\nu_l}(0)\prod_{i=k+1}^{n}\mathcal{O}_i(x_i)\rangle \tag{3.15}$$

This gives the scaling we expect from the OPE. Now lets perform the same scaling but in the Mellin representation of the coorelator in equation (3.4). We get

$$\begin{aligned}\langle \prod_{i=1}^{k} \mathcal{O}_i(e^{-\lambda}x_i) \prod_{i=k+1}^{n} \mathcal{O}_i(x_i)\rangle = \int [d\gamma_{ij}], M_n(\gamma_{ij}) e^{2\lambda \sum_{i<j}^{k}\gamma_{ij}} \prod_{i<j}^{n}\Gamma(\gamma_{ij})\prod_{i<j}^{k}(x_{ij}^2)^{-\gamma_{ij}} \\ \prod_{i\leq k<j}^{n}\left(x_j^2 - e^{-\lambda}2x_i\cdot x_j + e^{-2\lambda}x_i^2\right)^{-\gamma_{ij}} \prod_{k<i<j}^{n}(x_{ij}^2)^{-\gamma_{ij}}\end{aligned} \tag{3.16}$$

We want to compare the large λ behaviour of the OPE result with that from the Mellin representation. For this we expand,

$$\prod_{i\leq k<j}^{n}\left(x_j^2 - e^{-\lambda}2x_i\cdot x_j + e^{-2\lambda}x_i^2\right)^{-\gamma_{ij}} = \sum_{q=0}^{\infty} e^{-q\lambda}Q_q(x_1,\ldots,x_n) \tag{3.17}$$

where Q_q is a polynomial of degree q in x_i. The $q = l$ term in the expansion gives the contribution of a spin l operator in the OPE. Matching the powers of $e^{-\lambda}$ from both results, we reproduce as expected

$$\sum_{i=1}^{k}\Delta_i - 2\sum_{i<j=1}^{k}\gamma_{ij} = \Delta - l + 2m \tag{3.18}$$

3.2.3 Meromorphicity

Now lets come to the importance of the Gamma functions in the measure. The Gamma functions are useful when we're dealing with a large N CFT where N is the number of degrees of freedom. For example, as we shall study later, $\mathcal{N} = 4$ SYM with a large N gauge group $SU(N)$ is one such theory with $O(N^2)$ degrees of freedom. We'll restrict to the case of correlators of scalar primaries, and for convenience, lets focus on the four-point correlator.

Considering the s-channel OPE, equation (3.18) tells us that the Mellin correlator has poles at

$$\Delta_1 + \Delta_2 - 2\gamma_{12} = \Delta - l + 2m \tag{3.19}$$

The pole structure is better expressed in terms of new variables

$$s = \frac{\Delta_1 + \Delta_2}{2} - \gamma_{12} \qquad t = -\gamma_{14} \tag{3.20}$$

which keeping in mind the analogy with fictitious momenta, play the role of the Mandelstam invariants that appear in flat space scattering amplitudes.

On the other hand, the poles of the Γ-functions are fixed. For example, $\Gamma(\gamma_{12})$ has poles at $2s = \Delta_1 + \Delta_2 + 2m$. In a general CFT, there are no operators with exactly these dimensions. But in a large N CFT, we have composite operators of the form $\mathcal{O}_1(\partial^2)^n\mathcal{O}_2$. These have dimension $\Delta = \Delta_1 + \Delta_2 + 2n + O(1/N)$, where $n = 0$ gives the primary. Such operators are usually called double trace operators in analogy with the operators of Yang-Mills theories. One can show that in a large N CFT, the only multitrace contribution to the $\mathcal{O}_1$, $\mathcal{O}_2$ OPE comes from the conformal family of the $\mathcal{O}_1(\partial^2)^n\mathcal{O}_2$ operators. The $\Gamma(\gamma_{12})$ function having poles exactly at the scaling dimensions of this family fully captures its contribution to OPE. In general, the Γ-functions account for all multi-trace OPE contributions at the planar level and thus the planar Mellin correlator is a meromorphic function with only simple poles associated to the single-trace operators.

In the next section, we'll explicitly see the importance of the Γ-functions when we compute the Mellin representation of an n-point contact Witten diagram. In this case, the pole structure is completely captured by the Γ-functions and the Mellin correlator is just a constant independent of the Mellin variables!

3.2.4 Factorization at the Poles

We've seen that for large N CFTs, the poles of the Mellin correlator exhibit simple poles at the dimensions of the single trace primaries that contribute to the OPE, while the contribution of multi-trace operators are accounted for by the Gamma functions that are already present in the definition of the Mellin representation. Another interesting feature of the OPE that manifests in the Mellin correlators comes from looking at the residues $_m$ at the simple poles. Roughly speaking, the OPE lets us combine a k-point correlator with a $(n-k)$-point correlator by inserting a complete basis of primaries and descendants in between and summing over. Since the n-point Mellin correlator has poles at the locations of these primaries and descendants, it is natural that the residues at the poles are related to the lower point correlators which were combined to form it.

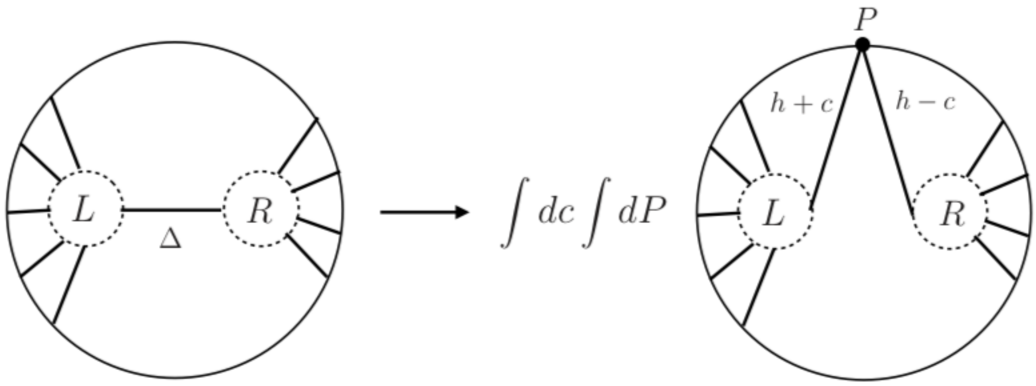

Figure 3.3: Pictorial representation of Factorization

This intuition can be made precise by considering a Witten diagram with a scalar bulk-bulk propagator that divides it into a left and a right piece (Fig. 3.3). The trick is to break the bulk-bulk propagator into two bulk-boundary propagators with a common boundary point that is integrated over. This is realized using the following formula

$$G_{BB}(X,Y) = \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \frac{2c^2}{c^2-(\Delta-h)^2} \int dP \frac{\mathcal{C}_{h+c}}{(-2P\cdot X)^{h+c}} \frac{\mathcal{C}_{h-c}}{(-2P\cdot Y)^{h-c}} \tag{3.21}$$

Here $h = d/2$ and the two bulk-boundary propagators have weights $\Delta_L = h + c$, $\Delta_R = h - c$. These correspond to operators whose dimensions are related by $\Delta_L + \Delta_R = d$ and therefore have the same Casimir eigenvalue. These are sometimes called *shadows* of eachother. Although it has not been shown, the additional integral over c is claimed to be necessary for correctly taking into account the contributions of multi-trace operators.

In the Mellin representation, the above integral becomes easy to perform and the residues at the poles factorize in terms of $k+1$ and $n-k+1$-point Mellin correlators as expected. The exact formula for the residues are [5]

$$Q_m = -4\pi^h \frac{\Gamma(\Delta-h+1)m!}{(\Delta-h+1)_m} L_m(\delta_{ij}) \, R_m(\delta_{ij}) \tag{3.22}$$

where

$$L_m = \sum_{n_{ij}=m} M_{k+1}(\gamma_{ij}+n_{ij}) \prod_{i<j}^{k} \frac{(\gamma_{ij})_{n_{ij}}}{n_{ij}!}, \quad R_m = \sum_{n_{ij}=m} M_{n-k+1}(\gamma_{ij}+n_{ij}) \prod_{k<i<j}^{n} \frac{(\gamma_{ij})_{n_{ij}}}{n_{ij}!} \tag{3.23}$$

For the residues at poles corresponding to primaries, the residue is simply a product of the lower point correlators.

$$Q_0 = -4\pi^h \Gamma(\Delta - h + 1) M^L_{k+1} M^R_{n-k+1} \tag{3.24}$$

This factorization property of residues has been used to develop a recursion relation for higher point amplitudes in terms of lower ones.

3.3 Contact Witten Diagram in Mellin Space

Consider an interaction of the form $g\phi_1\phi_2...\phi_n$ of n scalar fields in AdS given by the n-point Witten diagram as shown in Fig. 3.4. It contributes to the n-point correlation function of the dual boundary operators $\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)...\mathcal{O}_n(P_n)\rangle$. To demonstrate the simplicity of the Mellin representation for AdS-CFT correlators and also because we'd need it in the last chapter, we'll compute the Mellin representation of the diagram. For simplicity, we'll do this calculation in the embedding space formalism.

The expression in position space is given by

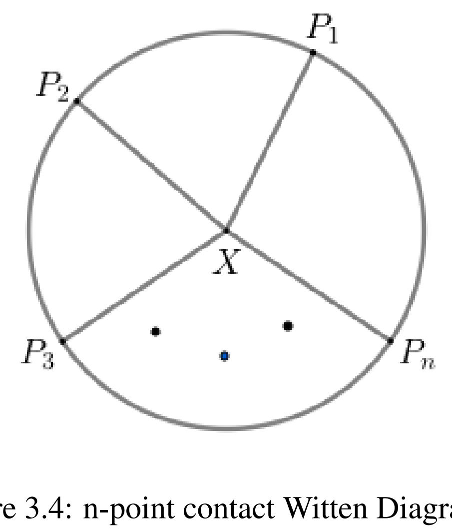

Figure 3.4: n-point contact Witten Diagram

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)...\mathcal{O}_n(P_n)\rangle = \int_{AdS} dX \prod_{i=1}^{n} \Pi(X, P_1) \tag{3.25}$$

where $\Pi(X,P_i) = \dfrac{\sqrt{C_{\Delta_i}}}{(-2X.P_i)}$ is the Bulk-boundary propagator. From the formula (3.3), we can write

$$\frac{1}{(-2X.P)^{\Delta}} = \frac{1}{\Gamma(\Delta)}\int_0^{\infty} ds s^{\Delta-1}e^{2sX.P} \tag{3.26}$$

Using this in tha above equation, we get

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)...\mathcal{O}_n(P_n)\rangle = \int_{AdS} dX \prod_{i=1}^{n}\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\int_0^{\infty} ds_i s_i^{\Delta_i-1}e^{2\sum_i s_i P_i.X} \tag{3.27}$$

Lets call the the vector $\sum_i s_i P_i = Q$ for convenience, and take Q to be along X_{d+1}. This gives

$$\prod_{i=1}^{n}\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\int_{AdS} dX \int_0^{\infty} ds_i s_i^{\Delta_i-1}e^{2QX_{d+1}} \tag{3.28}$$

We now shift to Poincare co-ordinates on AdS

$$\prod_{i=1}^{n}\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\int_{AdS} \frac{dzd^dx}{z^{d+1}} \int_0^{\infty} ds_i s_i^{\Delta_i-1}e^{\frac{Q(1-z^2-x^2)}{z}} \tag{3.29}$$

Performing the Gaussian integrals over x and then rescaling $z \to zQ$, we get

$$\pi^{d/2}\prod_{i=1}^{n}\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\int \frac{dz}{z}z^{-d/2}e^{-z} \int_0^{\infty} ds_i s_i^{\Delta_i-1}e^{\frac{Q^2}{z}} \tag{3.30}$$

We'd like to perform the integral over z, but the $1/z$ factor in the exponential is hard to handle. To get rid of it, we perform the change of variables

$$s_i = \frac{\sqrt{zt_1...t_n}}{t_i} \tag{3.31}$$

With this, the $1/z$ in the exponential goes away and integral becomes

$$\pi^{d/2}\prod_{i=1}^{n}\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\int \frac{dz}{z}z^{\frac{\sum_i^n \Delta_i - d}{2}}e^{-z} \int_0^{\infty} \frac{dt_i}{t_i}\frac{(t_1...t_n)^{\sum_i \Delta_i/2}}{t_1^{\Delta_1}...t_n^{\Delta_n}}e^{\sum_{i,j}\frac{t_1...t_n}{t_it_j}P_i.P_j} \tag{3.32}$$

We can now perform the z integral which gives a Γ-function, and do a sort of inverse transformation $s_i = \frac{\sqrt{t_1...t_n}}{t_i}$ to get back to the previous form except the $1/z$ in the exponential.

$$\pi^{d/2}\prod_{i=1}^{n}\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\Gamma\left(\frac{\sum_i \Delta_i - d}{2}\right)\int_0^{\infty} ds_i s_i^{\Delta_i-1}e^{2\sum_{i<j}s_is_jP_i.P_j} \tag{3.33}$$

where we also used that $e^{\sum_{i,j}s_is_jP_i.P_j} = e^{2\sum_{i<j}s_is_jP_i.P_j}$ because the sum is symmetric in i,j and $P^2 = 0$ since P is on the null cone in embedding space. Till now, we've

done everything in position space. We'll now introduce Mellin variables using the Mellin integral (3.3)

$$\pi^{d/2}\prod_{i=1}^{n}\left(\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\right)\Gamma\left(\frac{\sum_i \Delta_i - d}{2}\right)\prod_{i<j}\int d\gamma_{ij}\Gamma(\gamma_{ij})(-2P_i.P_j)^{-\gamma_{ij}} \\ \prod_{k=1}^{n}\int_0^\infty ds_k s_k^{\Delta_k-1}(s_i s_j)^{-\gamma_{ij}} \tag{3.34}$$

Now we only have the integrals over s_k to do. These integrals can be easily done using (3.3) and give rise to k simple poles or k δ-functions each imposing the constraint $\sum_{j\neq i}\gamma_{ij} = \Delta_i$ or $\sum_j \gamma_{ij} = 0 \;\; \gamma_{ii} = -\Delta_i$ on the Mellin variables as required in the definition of Mellin representation. We therefore finally get

$$\pi^{d/2}\prod_{i=1}^{n}\left(\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\right)\Gamma\left(\frac{\sum_i \Delta_i - d}{2}\right)\prod_{i<j}\int [d\gamma_{ij}]\Gamma(\gamma_{ij})(-2P_i.P_j)^{-\gamma_{ij}} \tag{3.35}$$

Comparing with the definition of the Mellin representation and recalling that $-2P_i.P_j = x_{ij}^2$, we find that the Mellin correlator is simply a constant.

$$\mathcal{M}(s,t) = \pi^{d/2}\prod_{i=1}^{n}\left(\frac{\sqrt{C_{\Delta_i}}}{\Gamma(\Delta_i)}\right)\Gamma\left(\frac{\sum_i \Delta_i - d}{2}\right) \tag{3.36}$$

This is in agreement with the expectation that a contact Witten diagram only receives contribution from multi-trace operator exchanges which is why we don't see any poles (correspoding to single trace primary exchanges) in the Mellin correlator. For the case when $n = 4$, the contact interaction in position space is denoted by $D_{\Delta_1,\Delta_2\Delta_3\Delta_4}$, so called D-functions.

3.4 Conformal Block Decomposition in Mellin Space

Now we'll focus on the case of four-point correlators of scalar primaries which we'll need in the final chapter. As mentioned before, in position space the correlator is

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\Delta_{12}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\Delta_{34}}\mathcal{G}(u,v) \tag{3.37}$$

Considering again the s-channel OPE, from the definition of the Mellin representation, after solving the constraints in terms of the variables s and t defined in (3.20), we find that $\mathcal{G}(u,v)$ is given as

$$\mathcal{G}(u,v) = \frac{1}{(2\pi i)^2}\int_{\gamma-i\infty}^{\gamma+i\infty} u^s v^t \Gamma(\frac{\Delta_1+\Delta_2}{2}-s)\Gamma(\frac{\Delta_3+\Delta_4}{2}-s)\Gamma[-t]\Gamma(-\Delta_{12}-\Delta_{34}-t)$$
$$\Gamma(s+t+\Delta_{12})\Gamma(s+t+\Delta_{34})\mathcal{M}(s,t) \quad (3.38)$$

The poles in $\mathcal{M}(s,t)$ should appear at

$$s = \frac{\Delta - l}{2} + m \quad (3.39)$$

Just as we did the conformal block decomposition in position space, we can decompose $\mathcal{M}(s,t)$ into Mellin blocks $B^{(s)}_{\Delta,l}(s,t)$ which are Mellin transforms of the conformal blocks $G^{(s)}_{\Delta,l}(u,v)$.

$$\mathcal{M}(s,t) = \sum_{\Delta,l} c_{\Delta,l} B^{(s)}_{\Delta,l}(s,t) \quad (3.40)$$

The Mellin blocks were computed in [] and are given as

$$B^{(s)}_{\Delta,l}(s,t) = e^{i\pi(s-\frac{\Delta-l}{2})}\frac{\Gamma(\frac{\Delta-l}{2}-s)}{\Gamma(\frac{\Delta_3+\Delta_4}{2}-s)\Gamma(\frac{\Delta_3+\Delta_4}{2}-s)\Gamma(s+1-h+\frac{\Delta-l}{2})}P^{(s)}_{\Delta-h,l}(s,t) \quad (3.41)$$

where $h = d/2$.

As expected, the $\Gamma(\frac{\Delta-l}{2}-s)$ in the numerator gives poles in s at the same locations as expected from (3.27). The crucial part of the Mellin blocks are the Mack Polynomials $P^{(s)}_{\Delta-h,l}(s,t)$. [15]. The explicit form is given in the Appendix.

3.5 Mellin Representation in the Flat Space Limit of AdS

In the previous section we had seen that setting $\gamma_{ij} = p_i.p_j$ where the p_i are some fictitious momenta lets us interpret many of the properties of Mellin correlators as being analogous to flat space scattering amplitudes. We'll now see that it is indeed true that the Mellin correlator is related to the flat space scattering amplitude. Consider an scattering amplitude $\mathcal{T}_n$ of n massless scalar particles in AdS. In general, $\mathcal{T}_n$ depends on the length scale l_s of the theory and and on the relativistic invariants $k_i.k_j$, where k_i are the momenta of external particles. In AdS, the flat space limit of a scattering amplitude is when the characteristic length scale is much smaller than the AdS radius, i.e. $R/l_s \to \infty$ so that the curvature effects are negligible. In other words we have a very high energy scattering

$ER \to \infty$ in AdS. Now, we've seen that the energies of particles in the bulk map to the scaling dimensions of the dual operators. And since the Mellin variables γ_{ij} are related to scaling dimensions, it is natural to define the flat space limit of the Mellin correlator as $\gamma_{ij} \to \infty$.'

More precisely, the claim is [7]

$$(l_s)^{n\frac{1-d}{2}+d+1}\mathcal{T}_n(l_s, k_i) = \mathcal{N}^{-1} \lim_{\theta\to\infty} (\theta)^{n\frac{d-1}{2}-d-1} \int_{\mathcal{C}} \frac{d\alpha}{2\pi i} \alpha^{\frac{d-\sum_i \Delta_i}{2}} e^{\alpha} \mathcal{M}_n(\theta, \gamma_{ij} = \frac{\theta^2}{2\alpha} l_s^2 k_i.k_j) \tag{3.42}$$

where $\mathcal{N}$ is a normalization constant given as

$$\mathcal{N} = \frac{\pi^{d/2}}{2} \prod_{i=1}^{n} \frac{\sqrt{\mathcal{C}_{\Delta_i}}}{\Gamma(\Delta_i)} \tag{3.43}$$

The contour $\mathcal{C}$ runs parallel to the imaginary axis, goes through the right of the branch point at $\alpha = 0$ and to the left of all poles of $\mathcal{M}_n$. The powers of l_s are included to make both sides of the equation dimensionless. The external particles must be massless in flat space although in AdS they can have any scaling dimension Δ_i of order 1. The formula is not applicable to massive flat space scattering amplitudes.

4. SUPERSYMMETRY

The Coleman-Madula *no-go* theorem states that if one allows for only bosonic (non-spinor) generators, then the most general symmetry of the S-matrix can be Poincare $\otimes$ Internal symmetries. These symmetries do not affect the spin of a particle. However, if we extend the spacetime symmetries by including fermionic (spinor) generators, the spin of the particles can be changed by the action of these generators, allowing for a symmetry between the bosons and fermions in a theory. This boson-fermion symmetry is called Supersymmetry (SUSY) and the extended Poincare group is called the Super-Poincare Group. For a review of supersymmetry and $\mathcal{N} = 4$ SYM, see [9–11]

4.1 Supersymmetry algebra

SUSY generators are given as a pair of two-component left and right Weyl spinors. The number of pairs is denoted by $\mathcal{N}$.

$$Q^I_\alpha \quad \alpha = 1, 2 \quad Left \tag{4.1}$$

$$Q^I_{\dot{\alpha}} \quad \dot{\alpha} = 1, 2 \quad Right \tag{4.2}$$

and $I = 1, 2, .., \mathcal{N}$. The two-component Weyl spinor notation is related to the 4-component Dirac spinor notation as

$$Q^I = \begin{pmatrix} Q^I_\alpha \\ Q^I_{\dot{\alpha}} \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{4.3}$$

and the following matrices are used to raise and lower spinor indices

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.4}$$

With notations in place, the SUSY algebra is given as

$$[Q^I_\alpha, M^{\mu\nu}] = i(\sigma^{\mu\nu})^\beta_\alpha Q^I_\beta \qquad [\bar{Q}^I_{\dot{\alpha}}, M^{\mu\nu}] = i(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}} \bar{Q}^I_{\dot{\beta}} \qquad [Q^I_\alpha, P^\mu] = [\bar{Q}^I_{\dot{\alpha}}, P^\mu] = 0 \tag{4.5}$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P^\mu \delta^{IJ} \qquad \{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^\star \tag{4.6}$$

A few comments are in order:

1. *Boson-Fermion symmetry*: The commutation with spin generators $J^3 = M^{12}$ is found to be

$$[J^3, Q^I_1] = \frac{1}{2} Q^I_1 \qquad [J^3, Q^I_2] = -\frac{1}{2} Q^I_2 \tag{4.7}$$

and

$$[J^3, \bar{Q}^I_1] = -\frac{1}{2} \bar{Q}^I_1 \qquad [J^3, \bar{Q}^I_2] = \frac{1}{2} \bar{Q}^I_2 \tag{4.8}$$

Therefore the SUSY charges act as creation and annihilation operators for spin, changing the spin by $\pm\frac{1}{2}$. This leads to a boson-fermion duality in any supersummetric theory. Each boson has a dual superpartner fermion and vice versa.

2. *Supergravity*: The relation $\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P^\mu \delta^{IJ}$ has important consequences. It says that we can generate infinitesimal translations using SUSY charges. If we have a theory with local SUSY (i.e. the SUSY transformation parameter depends on x^μ, then using this commutation rule means that the theory is invariant under infinitesimal translations whose parameter depends on x^μ. In other words, we have a theory invariant under general co-ordiante transformations, a theory of gravity! Theories of local SUSY therefore incorporate gravity automatically and are called *Supergravity* (SUGRA) theories.

3. *Central Charges*: The Z^{IJ} must satisy $Z^{IJ} = -Z^{JI}$ for the anti-commutator to make sense. This means that they vanish for $\mathcal{N} = 1$. These generators are actually Lorentz scalars that commute within themselves and with all other generators. This is why they are called *central charges*.[1] They'll become important when we discuss BPS operators.

[1]The term "central" refers to the centre of a group which is a sub-group whose elements commute with all other elements of the group.

4. *R-symmetry*: The SUSY algebra is invariant under a global phase rotation of supercharges denoted as $U(1)_R$ as well as a $SU(\mathcal{N})_R$ rotation of supercharges among themselves (left among left and right among right). As indicated by the sub-scripts, these two internal symmetries together are called the R-symmetry group.

4.2 Representations of the Super-Poincare Group

The Poincare algebra has two Casimirs whose eigenvalues ”label” its irreducible representations. These are P^2 and W^2 where

$$W^{\mu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^{\nu}M^{\rho\sigma} \tag{4.9}$$

These representations are what we call particles with the ”labels” being their properties. We can evaluate the eigenvalues in rest frame since P^2 and W^2 are Lorentz scalars. For a massive particle in the rest frame, $P^2 = m^2$ and $W^2 = -m^2\vec{J}^2 = -j(j+1)m^2$. So every massive particle is an irreducible representation of the Poincare group labelled as $|m, j, p^{\mu}, j_3\rangle$. Similarly, for a massless particle in rest frame, $P^2 = 0$ and $W^2 = 0$. But $P^{\mu} = (E, 0, 0, E)$ and $W^{\mu} = \pm M^{12}P^{\mu} = \pm jm$. So, j, the spin is fixed and $\pm j$ gives the helicity of the particle. Massless particles are therefore labelled as $|m = 0, j, E, \pm j\rangle$. $\pm j$ label the same particle because they are related by CPT.

When we extended to SuperPoincare, it is clear that W^2 is no longer a Casimir, since SUSY generators change spin. Irreducible SUSY representations are therefore classified by their mass. Now since, the Poincare algebra is a sub-algebra of the SuperPoincare, in general, any irreducible representation of the SuperPoincare can be decomposed into irreducibles of Poincare. For this reason, the SuperPoincare irreducibles are called *superparticles* or *supermultiplets*. In other words, one can think of a superparticle as a composite of many (Poincare) particles with the same mass but different spins, related to eachother by the action of SUSY charges.

In general, greater the number of SUSY charges, more the number of higher spin (or helicity) particles in a supermultiplet. Although algebraically there is no upper limit to the value of $\mathcal{N}$, but physically to have interacting local field theories in $d = 4$, we have the conditions

- $\mathcal{N} \leq 4$ for theories without gravity (spin ≤ 1 allowed)

- $\mathcal{N} \leq 8$ for theories with gravity (SUGRA) (spin ≤ 2 allowed).

We only need to discuss massless $\mathcal{N} = 4$ representations since we need these to study $\mathcal{N} = 4$ SYM later. But we'll also discuss first massless $\mathcal{N} = 1$ to get a better idea of how to contruct supermultiplets. The general procedure is to define minimum helicity λ_0 state called the *Clifford vacuum*. The Clifford vacuum is annihilated by helicity-lowering SUSY charges and can be acted upon by helicty-increasing charges to generate the supermultiplet until we reach the maximum helicity. Lets see this explicitly.

4.2.1 Massless Representations

For massless particles $P^\mu = (E, 0, 0, E)$ and therefore,

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma_\mu)_{\alpha\dot{\beta}} P^\mu = 2(-\sigma^0 + \sigma^4)_{\alpha\dot{\beta}} E = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}_{\alpha\dot{\beta}} \tag{4.10}$$

This means that

$$\{Q_2, \bar{Q}_2\} = 0 \qquad \Longrightarrow \qquad Q_2 = \bar{Q}_2 = 0 \tag{4.11}$$

Only Q_1 and $\bar{Q}_1$ supercharges survive therefore in each pair of generators. We can use these to define creation and annihilation operators

$$a = \frac{Q_1}{2\sqrt{E}} \qquad a^\dagger = \frac{\bar{Q}_1}{2\sqrt{E}} \tag{4.12}$$

such that

$$\{a, a^\dagger\} = 1, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \text{ and } \quad \{J^3, a\} = -\frac{1}{2}a, \quad \{J^3, a^\dagger\} = \frac{1}{2}a^\dagger \tag{4.13}$$

Therefore $a^\dagger$ raises helicity by $\frac{1}{2}$ and a lowers it by $\frac{1}{2}$. Another important consequence of (3.11) is that for massless representations, the central charges Z^{IJ} must vanish (3.6).

- $\mathcal{N} = 1$ **Supermultiplets**

We have only 1 creation operator. Therefore, we can only increase helicity by $\frac{1}{2}$. We will not directly consider $\lambda_0 = -1, -\frac{1}{2}$. Instead we'll get them by demanding CPT invariance.

Matter Multiplet (or Chiral Multiplet)

$$\lambda_0 = 0 \qquad \Longrightarrow \qquad \left(0, +\frac{1}{2}\right) \underbrace{\oplus}_{CPT} \left(0, -\frac{1}{2}\right) \tag{4.14}$$

Gauge Multiplet (or Vector Multiplet)

$$\lambda_0 = \frac{1}{2} \quad \Longrightarrow \quad \Big(+\frac{1}{2}, +1\Big) \underbrace{\oplus}_{CPT} \Big(-\frac{1}{2}, -1\Big) \tag{4.15}$$

Only these two multiplets can exist in a $\mathcal{N} = 1$ supersymmetric theory without gravity. For completeness, we will also write down multiplets upto helicity 2.

Gravitino Multiplet

$$\lambda_0 = 1 \quad \Longrightarrow \quad \Big(+1, +\frac{3}{2}\Big) \underbrace{\oplus}_{CPT} \Big(-1, -\frac{3}{2}\Big) \tag{4.16}$$

Graviton Multiplet

$$\lambda_0 = \frac{3}{2} \quad \Longrightarrow \quad \Big(+\frac{3}{2}, +2\Big) \underbrace{\oplus}_{CPT} \Big(-\frac{3}{2}, -2\Big) \tag{4.17}$$

The particle with helicty $\frac{3}{2}$ is called the gravitino and is the supersymmetric partner of graviton, helicity 2.

- $\mathcal{N} = 4$ **Supermultiplet**

In this case, we have 4 supercharges and can therefore increase helicity by 2. But that means in a non-gravity theory, we can have only 1 supermultiplet with all particle helicities ≤ 1, the one with $\lambda_0 = -1$.

Gauge Multiplet (or Vector Multiplet)

$$\lambda_0 = -1 \quad \Longrightarrow \quad \Big(-1, (-\frac{1}{2}) \times 4, (0) \times 6, (+\frac{1}{2}) \times 4, +1\Big) \tag{4.18}$$

The Gauge multiplet is self-conjugate under CPT.

4.3 $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

$\mathcal{N} = 4$ SYM consists of particles that belong to the massless vector representation of $\mathcal{N} = 4$ SUSY . As we have seen, this makes it the maximally supersymmetric non-gravitational theory possible in 4 dimensions.

$$\begin{aligned}\mathcal{L} = tr\Bigg[&-\frac{1}{2g_{YM}^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i \\ &+ \sum_{a,b,i} g_{YM} C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g_{YM} \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g_{YM}^2}{2} \sum_{ij} [X^i, X^j]^2 \Bigg]\end{aligned} \tag{4.19}$$

The field content is that of the $\mathcal{N}=4$ gauge multiplet having 1 gauge field, 4 Weyl spinors and 6 scalar bosons. Since all the fields in the Lagrangian belong to the supermultiplet, they all transform in the adjoint representation of the gauge group. Further, the theory has an $SU(4)$ R-symmetry. Classically, the theory is scale invariant as can be seen since the Lagrangian only contains dimension four terms. What makes $\mathcal{N}=4$ SYM remarkable is that the renormalization group β-function of the theory vanishes identically, i.e. there is no scale dependence introduced during the renormalization process. The theory is exactly conformally invariant at the quantum level.

4.3.1 Superconformal Symmetry

Conformal symmetry together with Supersymmetry forms a larger group called the Superconformal group. The algebra has in addition to conformal generators and the usual "Poincare" supercharges, additonal supersymmetry generators called *Conformal supercharges* that arise because the SCTs do not commute with Poincare supercharges. The $U(1)_R$ and $SU(4)_R$ R-symmetry is also extended, so the Superconformal algebra is invariant under the rotation of conformal supercharges among themselves. The Superconformal group is denoted as $SU(2,2|4)$ and algebra in addition to (3.5),(3.6) is as follows

$$\{S_\alpha, S_\beta\} = \{\bar{S}_{\dot{\alpha}}, \bar{S}_{\dot{\beta}}\} = 0, \qquad \{Q_\alpha, \bar{S}_{\dot{\beta}}\} = 0 \tag{4.20}$$

$$\{S_\alpha, \bar{S}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} K_\mu, \qquad \{Q^I_\alpha, S^J_\beta\} = \epsilon_{\alpha\beta}(\delta^{IJ} D + T^{IJ} + \frac{1}{2}\delta^{IJ}(\sigma^{\mu\nu})_{\alpha\beta} M_{\mu\nu} \tag{4.21}$$

$$[D, Q_\alpha] = \frac{1}{2}Q_\alpha \quad [D, \bar{Q}_{\dot{\alpha}}] = \frac{1}{2}\bar{Q}_{\dot{\alpha}} \quad [D, S_\alpha] = -\frac{1}{2}S_\alpha \quad [D, \bar{S}_{\dot{\alpha}}] = -\frac{1}{2}\bar{S}_{\dot{\alpha}} \tag{4.22}$$

Additional symmetries means that we need to consider irreducible representations of the Superconformal group. And again these can be decomposed into Poincare irreducibles and hence, a "superconformal particle" can be thought of as a composite of Poincare particles in the massless $\mathcal{N}=4$ gauge multiplet. Just like for the conformal group, we build the representations by defining *superconformal primaries*. (4.4) tells us that the SUSY charges can change the scaling dimension of dilatation eigenstates by $\pm\frac{1}{2}$. Therefore in a superconformal theory, there are operators that satisfy

$$[D, O(0)] = \Delta O(0) \quad and \quad [S_\alpha, O(0)] = [\bar{S}_{\dot{\alpha}}, O(0)] = 0 \tag{4.23}$$

These are superconformal primaries as opposed to conformal primaries which are annihilated by K^μ. (4.3) tells us that all conformal primaries and superconformal primaries but not vice versa. We also define superconformal descendents by acting on the superconformal primary by conformal supercharges Q that increase the scaling dimension by $\frac{1}{2}$.

$$O^{'}(0) = [Q_\alpha, O(0)] \quad and \quad [D, O^{'}(0)] = (\Delta + \frac{1}{2})O^{'}(0) \tag{4.24}$$

4.3.2 Half-BPS operators in $\mathcal{N} = 4$ SYM

In this work, we'll be interested in a special class of superconformal primaries called half-BPS operators. The "half" is because these commute with half of the 16 supercharges in the $\mathcal{N} = 4$ superconformal algebra due to which their supermultiplets are shortened (hence the name BPS). The simplest half-BPS operators are the single trace primaries given by

$$\mathcal{O}^p_{i_1...i_p} = tr(X_{\{}i_1...X_{i_p\}}) \qquad p \geq 2 \tag{4.25}$$

where each of the R-symmetry indices $i_1, ..i_p$ goes from 1 to 6, the notation $\{i_1...i_p\}$ means to take the R-traceless and R-symmetric part of the tensor, the trace tr is over the gauge algebra indices (not the R-symmetry algebra) which we have suppressed for simplicity. These have dimension $\Delta = p$ and transform in the $[0, p, 0]$ irrep of the $SU(4)$ R-symmetry group (where $[r_1, r_2, r_3]$ are the Dynkin labels). To keep track of the R-symmetry indices, it is often convenient to contract them with a $SO(6)$ null vector $T_i, i = 1, .., 6, T^2 = 0$

$$\mathcal{O}^p = T^{i_1}T^{i_2}...T^{i_p}\mathcal{O}^p_{i_1...i_p} \tag{4.26}$$

For example, the simplest single trace half-BPS operator is given as

$$\mathcal{O}^2 = T^{i_1}T^{i_2}tr[\frac{X_{i_1}X_{i_2} + X_{i_2}X_{i_1}}{2} - \frac{\delta_{i_1,i_2}}{6}X_iX_i] = T^{i_1}T^{i_2}tr[X_{i_1}X_{i_2}] = tr[(T.X)^2] \tag{4.27}$$

where we've used i_1, i_2 symmetry and $\delta_{i_1,i_2}T^{i_1}T^{i_2} = T^2 = 0$. So if say $T^i = (1, i, 0, 0, 0, 0)$, then the operator is $tr(Z^2)$ where $Z = X_1 + iX_2$.

The half-BPS multiplets being short are protected from receiving quantum corrections by superconformal symmetry. Thus their scaling dimensions are fixed at the free theory value. In terms of correlation functions, this means that the two-point functions of half-BPS operators do not receive any quantum corrections. An even stronger result is that their three-point functions are also protected. Non-trivial dynamics starts appearing at the level of the four-point functions where one sees the rich spectrum of the OPE of these operators, which contains protected as well as unprotected (long multiplet) states.

Half-BPS operators also play an important role in the AdS-CFT correspondence. The compactification of type IIB supergravity on S^5 results in an infinite tower of Kaluza-Klein modes. According to the AdS/CFT conjecture, IIB supergravity is dual to the $\mathcal{N} = 4$ SYM in the string coupling regime. In this regime, the half-BPS multiplet corresponding to $\mathcal{O}^p$ ia dual to the KK modes transforming also transforming in the $[0, p, 0]$ irrep of the $SU(4)$ R-symmetry group. Also, the type IIB string spectrum on the $AdS_5 \times S^5$ is presently unknown. This is dual to the weak coupling limit of $\mathcal{N} = 4$ SYM, and therefore half-BPS correlators in this regime contain information about the dual string scattering amplitudes.

In this thesis, we'll focus on the correlators of $p = 2$ half-BPS operators. Apart from being the simplest, these are the most widely studied correlators because the $\mathcal{O}^2$ supermultiplet contains the conserved R symmetry current, the stress-energy tensor and the Lagrangian of $\mathcal{N} = 4$ SYM. It is dual to the graviton supermultiplet of the $AdS_5 \times S^5$ supergravity which comprises of massless KK modes.

4.3.3 Half-BPS correlators in Position space

Two-Point Function

We need to find

$$\langle \mathcal{O}^p(x_1)\mathcal{O}^p(x_2)\rangle = \langle tr(T_1.X)^p(x_1)tr(T_2.X)^p(x_2)\rangle \tag{4.28}$$

The two-point function of the scalar fields X_i is fixed by conformal symmetry

$$\langle X_i(x_1)X_j(x_2)\rangle = \frac{\delta_{ij}}{x_{12}^2} \tag{4.29}$$

Thus

$$\langle (T_1.X)^p(x_1)(T_2.X)^p(x_2)\rangle = \left(\frac{T_1.T_2}{x_{12}^2}\right)^p = \left(\frac{t_{12}^2}{x_{12}^2}\right)^p \tag{4.30}$$

where $t_{ij}^2 = T_i.T_j$. In the above equations, the gauge symmetry indices have been suppressed. Keeping the indices, we can expand the fields $X_{ab} = X_A T_{ab}^A$ where $a = 1, .., N$ and $A = 1, 2, ..., N^2 - 1$ labels the $SU(N)$ generators. Then using $\sum_A T_{ab}^A T_{cd}^A \propto$

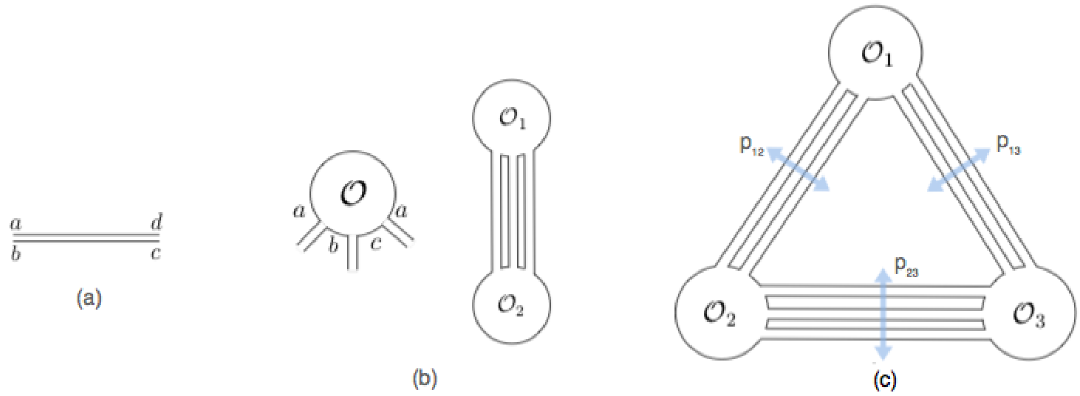

Figure 4.1: (a) Propagator of X_i fields. (b) Half-BPS operator and a planar diagram contributing to the two-point correlator. (c) A planar diagram contributing to the three-point correlator.

$\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd} \approx \delta_{ad}\delta_{bc}$, we get upto $1/N$ suppressed terms

$$\langle X_{ab}(x_1)X_{cd}(x_2)\rangle = \langle X^A T_{A,ab}(x_1)X^B T_{B,cd}(x_2)\rangle \propto \frac{\delta_{ad}\delta_{bc}}{x_{12}^2} \tag{4.31}$$

As usual, the diagrams correspoding to these propagators are written using the double line notation. The two-point function of half-BPS operators can then be computed by drawing all the possible planar diagrams connecting two operators. These diagrams involve contractions between the constituent X_i operators (Fig. 4.1). For dimension p half-BPS operators, there are p different ways of planar Wick contractions which are related to each other by the cyclic permutation of one of the operators. The full two-point function is then

$$\langle tr(T_1.X)^p(x_1)tr(T_2.X)^p(x_2)\rangle \propto pN^p\left(\frac{t_{12}^2}{x_{12}^2}\right)^p \tag{4.32}$$

We normalize the operator as $\mathcal{O}^p \to \mathcal{O}^p/\sqrt{pN^p}$ so that

$$\langle \mathcal{O}^p(x_1)\mathcal{O}^p(x_2)\rangle = \left(\frac{t_{12}^2}{x_{12}^2}\right)^p \tag{4.33}$$

Three-point function

In the case of three-point functions, there are $p_1p_2P_3$ inequivalent ways of planar Wick contractions which are related to each other by the cyclic permutations of the individual operators (Fig.4.1). The three-point function (in the above normalization) is therefore given by

$$\langle \mathcal{O}^{p_1}(x_1)\mathcal{O}^{p_2}(x_2)\mathcal{O}^{p_3}(x_3)\rangle = \frac{\sqrt{p_1p_2p_3}}{N}c_{123}\left(\frac{t_{12}^2}{x_{12}^2}\right)^{p12}\left(\frac{t_{23}^2}{x_{23}^2}\right)^{p23}\left(\frac{t_{31}^2}{x_{31}^2}\right)^{p31} \tag{4.34}$$

where $p_{ij} = (p_i + p_j - p_k)/2$ denotes the number of the Wick contractions between the operators $\mathcal{O}_i$ and $\mathcal{O}_j$. We can see that the three point function is $1/N$ suppressed relative to the two-point function.

Four-point function

As mentioned before, the two and three-point functions of half-BPS operators are protect and hence don't capture any dynamics. The four-point functions receive unprotected contributions as well and hence are the more interesting objects to study. We'll now restrict to the four point correlators of equal weight p half-BPS operators, and later restrict to $p = 2$.

Just as with 4 spatial points, we can construct conformal invariant cross-ratios, with 4 T-vectors also, we can construct R-symmetry invariant cross-ratios. These are given by

$$\sigma = \frac{(t_{12}^2)(t_{34}^2)}{(t_{13}^2)(t_{24}^2)} \qquad \tau = \frac{(t_{14}^2)(t_{23}^2)}{(t_{13}^2)(t_{24}^2)} \tag{4.35}$$

R-symmetry and conformal symmetry then fix the form of the four point function upto an arbitrary function of u, v, σ, τ.

$$\langle \mathcal{O}^p(x_1)\mathcal{O}^p(x_2)\mathcal{O}^p(x_3)\mathcal{O}^p(x_4)\rangle = \Big(\frac{t_{12}^2 t_{34}^2}{x_{12}^2 x_{34}^2}\Big)^p \mathcal{G}(u, v, \sigma, \tau) \tag{4.36}$$

We can write [12]

$$\mathcal{G}(u, v, \sigma, \tau) = \sum_{0 \geq m \geq n \leq p} a_{nm}(u, v) Y_{nm}(\sigma, \tau) \tag{4.37}$$

where Y_{nm} are two variable harmonic polynomials which are given explicitly in terms of single variable Legendre polynomials by

$$Y_{nm}(\sigma, \tau) = \frac{P_{n+1}(y)P_m(\bar{y}) - P_m(y)P_{n+1}(\bar{y})}{y - \bar{y}}\,, \quad \sigma = (1+y)(1+\bar{y})\,,\ \tau = (1-y)(1-\bar{y})\,. \tag{4.38}$$

And the function $a_{nm}(u, v)$ is expanded in conformal blocks as

$$a_{nm}(u, v) = \sum_{\Delta, l} a_{nm,l}^{\Delta} u^{\frac{\Delta - l}{2}} G_{\Delta}^{(l)}(u, v) \tag{4.39}$$

Defining variable $t' = (\Delta - l)/2$, we can rewrite

$$a_{nm}(u, v) = \sum_{t', l} a_{nm,l}^{2t'+l} u^{t'} G_{2t'+l}^{(l)}(u, v) \tag{4.40}$$

The four-point function is additionally constrained by the bigger superconformal group. Requiring satisfaction of the superconformal Ward identity gives the following decomposition [17]

$$\mathcal{G}(u,v,\sigma,\tau) = k + \mathcal{G}_{\hat{f}}(u,v,\sigma,\tau) + R(u,v,\sigma,\tau)\mathcal{H}(u,v,\sigma,\tau) \tag{4.41}$$

where

$$\begin{aligned} R(u,v,\sigma,\tau) = \frac{(z-\alpha)(\bar{z}-\alpha)(z-\bar{\alpha})(\bar{z}-\bar{\alpha})}{(\alpha\bar{\alpha})^2} &= \frac{1}{\sigma^2}(\sigma\tau(\sigma^2-\sigma-\sigma\tau)v \\ + (-\tau-\sigma\tau+\tau^2)u + (1-\tau-\sigma)uv + \sigma v^2 + \tau u^2) \end{aligned} \tag{4.42}$$

The constant k and the function $\mathcal{G}_{\hat{f}}$ are completely determined by the free part of $\mathcal{G}$ whereas dynamical effects, which lead to anomalous dimensions, are contained in the function $\mathcal{H}$.

Just as we expanded $a_{nm}(u,v)$, we can expand $\mathcal{H}$ as

$$\mathcal{H}(u,v;\sigma,\tau) = \sum_{0\geq m\geq n\leq p-2}\sum_{t',l} A^{2t'+l}_{nm,l} u^{t'} G^{(l)}_{2t'+4+l}(u,v) Y_{nm}(\sigma,\tau) \tag{4.43}$$

$\mathcal{G}_{\hat{f}}$ can be explicitly given in terms of $\hat{f}$ which has an expansion of the form

$$\hat{f}(x,y) = -2\sum_{\substack{0\leq n\leq p-1 \\ \ell}} b_{n,\ell}\, g^{(0)}_{0,\ell+2}(x) P_n(y) \qquad \begin{cases} l \text{ odd if } n \text{ even} \\ l \text{ even if } n \text{ odd} \end{cases} \tag{4.44}$$

where

$$g^{(0)}_{t,\ell}(x) = \big(-x\big)^{\ell}\, {}_2F_1\big(t+\ell, t+\ell; 2t+2\ell; x\big) \tag{4.45}$$

is the zeroth term in the u expansion of the conformal block

$$G^{(l)}_{2t'+l}(u,v) = \sum_m u^m g^{(m)}_{t',l}(1-v) \tag{4.46}$$

5. PERTURBATIVE MELLIN CORRELATORS IN N=4 SYM

In this chapter, we'll study the four-point correlator of half-BPS operators with each external dimension two. We'll compute the Mellin representation of these operators upto first order in t'Hooft coupling λ, and expand it in (first order) Mellin Blocks. Then using the orthogonality properties of the Mellin blocks, we'll extract the first-order corrections to the scaling dimension and OPE coefficients corresponding to twist two exchange primaries.

5.1 Perturbative results in Position Space

The superconformal decomposition (4.41) simplifies for $p = 2$. The external sum in (4.43) goes away and since $Y_{00}(\sigma,\tau) = 1$, we simply get

$$\mathcal{H}(u,v,\sigma,\tau) = \sum_{t',l} A^{2t'+l}_{00,l} u^{t'} G^{(l)}_{2t'+4+l}(u,v) = \sum_{t',l} A_{t',l} u^{t'} G^{(l)}_{2t'+4+l}(u,v) \quad (p=2) \tag{5.1}$$

where for simplicity we define $A_{t',l} = A^{2t'+l}_{00,l}$.

Now lets expand the correlator upto first order in t'Hooft coupling $\lambda = g^2N/4\pi^2$. We identify three terms

$$\mathcal{G}(u,v,\sigma,\tau) = \mathcal{G}^{(0)}_D(u,v,\sigma,\tau) + \frac{1}{N^2}(\mathcal{G}^{(0)}_C(u,v,\sigma,\tau) + \lambda\mathcal{G}^{(1)}_C(u,v,\sigma,\tau)) \tag{5.2}$$

which are the free disconnected, free connected and first-order connected terms respectively. The expressions for each are given as [17]

$$\mathcal{G}^{(0)}_D(u,v,\sigma,\tau) = 1 + \left(\frac{u}{\sigma}\right)^2 + \left(\frac{\tau u}{\sigma v}\right)^2 \tag{5.3}$$

$$\mathcal{G}_C^{(0)}(u,v,\sigma,\tau) = \frac{u}{\sigma} + \Big(\frac{\tau u}{\sigma v}\Big)^2 + \frac{\tau u^2}{\sigma^2 v} \tag{5.4}$$

$$\mathcal{G}_C^{(1)}(u,v,\sigma,\tau) = R(u,v,\sigma,\tau)\frac{u}{v}\Phi^{(1)}(u,v) \tag{5.5}$$

where

$$-\frac{i\pi^2}{x_{13}^2 x_{24}^2}\Phi^{(1)}(u,v) = \int \frac{dx_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \tag{5.6}$$

where is the so called conformal box integral. The integral was performed for the first time in [14] and is given as

$$\Phi^{(1)}(u,v) = \frac{1}{\lambda}\Big\{2Li_2(-\rho u) + 2Li_2(-\rho v) + ln(\rho u)ln(\rho v) + ln\frac{u}{v}ln\frac{1+\rho u}{1+\rho v} + \frac{\pi^2}{3}\Big\} \tag{5.7}$$

where $\lambda(u,v) = \sqrt{(1-u-v)^2 - 4uv}$ and $\rho(u,v) = 2(1-u-v+\lambda)^{-1}$ and $Li_2(x)$ is the Dilogarithm function.

As mentioned before, the dynamic information that leads to anomalous dimensions and co-efficient corrections is in $\mathcal{H}(u,v,\sigma,\tau)$. So we need to do the superconformal decomposition of each part separately. This was done in [12].

For $\mathcal{G}_D^{(0)}$,

$$k_D^{(0)} = 3, \quad \hat{f}_D^{(0)}(z,y) = \frac{1}{2}y\big(z^2 + z'^2\big) + \frac{1}{2}\big(z^2 - z'^2 + 2(z-z')\big), \quad \mathcal{H}_D^{(0)} = 1 + \frac{1}{v^2} \tag{5.8}$$

where $z' = z/(z-1)$.

For $\mathcal{G}_C^{(0)}$

$$k_C^{(0)} = 12, \quad \hat{f}_C^{(0)}(z,y) = -2y(x+x') + 6(x-x'), \quad \mathcal{H}_C^{(0)} = \frac{4}{v} \tag{5.9}$$

For $\mathcal{G}_C^{(1)}$,

$$\mathcal{H}_C^{(1)} = \frac{u}{v}\Phi^{(1)}(u,v) \tag{5.10}$$

Now we perturbatively expand the conformal block decomposition in (5.1). Denote

$$\Delta = \Delta^{(0)} + \lambda\Delta^{(1)} \qquad A_{t',l} = A_{t'^{(0)},l}(1+\lambda B_{t'^{(0)},l}) \tag{5.11}$$

Upon expanding upto first order in λ, we get

$$\mathcal{H}^{(0)}(u,v) = \sum_{t'^{(0)},l} A_{t'^{(0)},l}u^{t'^{(0)}}G^{(l)}_{2t'^{(0)}+4+l}(u,v) \tag{5.12}$$

$$\mathcal{H}_C^{(1)}(u,v) = \sum_{t'^{(0)},l} A_{t'^{(0)},l}u^{t'^{(0)}}\Big[\Big(\Delta^{(1)}\frac{\partial}{\partial\Delta}G^{(l)}_{2t'+4+l}(u,v)|_{t'=t'^{(0)}} + B_{t'^{(0)},l}G^{(l)}_{2t'^{(0)}+4+l}(u,v)\Big) + ln(u)\Big(\frac{\Delta^{(1)}}{2}G^{(l)}_{2t'^{(0)}+4+l}(u,v)\Big)\Big] \tag{5.13}$$

We're interested in finding the anomalous dimension and co-efficient correction for twist 2 primaries from (5.13). To extract the twist 2 contribution, we take the limit $u \to 0$ of (5.13) and then compare powers of $ln(u)$ to get

$$\sum_l A_{1,l}\Big(\Delta^{(1)}\frac{\partial}{\partial\Delta}g^{(0)}_{t'+2,l}(1-v)|_{t'=1} + B_{1,l}g^{(0)}_{3,l}(1-v)\Big) = -\frac{Li_2(1-v)}{v(1-v)} \tag{5.14}$$

$$\sum_l A_{1,l}\Big(\frac{\Delta^{(1)}}{2}g^{(0)}_{3,l}(1-v)\Big) = -\frac{1}{2}\frac{ln(v)}{v(1-v)} \tag{5.15}$$

The free theory coefficient has been computed [12] and is given by

$$A_{1,l} = \frac{4}{N^2}\frac{2^{l+1}(l+2)!^2}{(2l+4)!} \tag{5.16}$$

The two equations can then be used to extract the anomalous dimension $\Delta^{(1)}$ and the first order coefficient. Doing this in position space is in general an ardous task. We'll instead do this by writing these equations in Mellin space, and using the orthogonality properties of the Mellin representaton of $g^{(0)}_{t',l}$.

5.2 Anomalous dimension in Mellin space

Orthogonality of Mellin blocks

The terms $g^{(m)}_{t',l}$ that appear in (5.14) and (5.15) are known in Mellin space [13]

$$g^{(m)}_{t',l} = \frac{1}{m!(2t'+l-h+1)_m}\Big(\frac{\Gamma(2t'+2l)(2t'+l-1)_l}{(-4)^l\Gamma^4(t'+l)}\Big)\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}v^t\Gamma^2(t+t')\Gamma^2(-t)Q^{t'}_{l,m}(t) \tag{5.17}$$

where $Q^{t'}_{l,m}(t)$ are related to the Mack polynomials as [16]

$$Q^{t'}_{l,m}(t) = \frac{4^l}{(\Delta-1)_l(2h-\Delta-1)_l}P^{(s)}_{\Delta-h,l}(s=t'+m,t) \tag{5.18}$$

In particular, $Q^{t'}_{l,0}(t)$ turn out to be continuous Hahn polynomials written as

$$Q^{t'}_{l,0}(t) = \frac{2^l(t')_l(t')_l}{(2t'+l-1)_l}{}_3F_2[-l,2t'+l-1,t'+t;t',t';1] \tag{5.19}$$

and satisfy the orthogonality relation in spin

$$\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\Gamma^2(t+t')\Gamma^2(-t)Q^{t'}_{l,0}(t)Q^{t'}_{l',0}(t) = \kappa_l(t')\delta_{l,l'} \tag{5.20}$$

where

$$\kappa_l(t') = \frac{(-4)^l l!}{(2t'+l-1)_l^2}\frac{\Gamma^4(t'+l)}{(2t'+2l-1)\Gamma(2t'+l-1)} \tag{5.21}$$

We'll use this orthogonality property to extract the anomalous dimensions from (5.15).

First-order anomalous dimension

Using the Mellin representation for $g^{(0)}_{t',l}(1-v)$, (5.15) can be written in Mellin space as

$$\begin{aligned}\sum_l A_{1,l}\frac{\Delta^{(1)}}{2}\Big(\frac{\Gamma(6+2l)(6+l-1)_l}{(-4)^l\Gamma^4(3+l)}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{dt}{2\pi i}v^t\Gamma^2(t+3)\Gamma^2(-t)Q^3_{l,0}(t)\Big)\\ =\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{dt}{2\pi i}v^t\frac{1}{2}\Gamma^2(-t)\Gamma^2(1+t)\end{aligned} \tag{5.22}$$

where $-2<\gamma<-1$.

This gives

$$A_{1,l}\Delta^{(1)} = \frac{(-4)^l\Gamma^4(3+l)}{\Gamma(6+2l)(6+l-1)_l}[\kappa_l(3)]^{-1}\Big[\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\Gamma^2(-t)\Gamma^2(1+t)Q^3_{l,0}(t)\Big] \tag{5.23}$$

$$= \frac{2^l\Gamma^2(l+3)\Gamma(l+5)}{(2!)^2 l!\Gamma(2l+5)}\Big[\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\Gamma^2(-t)\Gamma^2(1+t)\,{}_3F_2[-l,l+5,3+t;3,3;1]\Big] \tag{5.24}$$

Using the integral representation,

$${}_3F_2[a_1,a_2,a_3;b_1,b_2;z] = \frac{\Gamma(b_2)}{\Gamma[a_3}\Gamma[b_2-a_3]\int_0^1 dt t^{a_3-1}(1-t)^{-a_3+b_2-1}\,{}_2F_1[a_1,a_2;b_1;tz] \tag{5.25}$$

the integral in (5.24) can be solved for $\Delta^{(1)}$ to obtain

$$\Delta^{(1)} = \frac{\Gamma(l+5)!}{2l!}\sum_{s=0}^l\frac{(-l)_s(l+5)_s}{\Gamma^2(s+3)}h(s+1) = \begin{cases}2h(l+2) & l=0,2,4,..\\ 0 & l=1,3,5,..\end{cases} \tag{5.26}$$

The result can also be expressed via a hypergeometric function as

$$\Delta^{(1)} = \frac{\Gamma(l+4)}{\Gamma(l+2)}\,{}_4F_3[1,1,-l-1,l+4;2,2;1] \tag{5.27}$$

5.3 OPE co-efficient correction in Mellin space

Now lets turn to the equation for co-efficient correction

$$\sum_l A_{1,l}\Big(\Delta^{(1)}\frac{\partial}{\partial\Delta}g^{(0)}_{t'+2,l}(1-v)|_{t'=1}+B_{1,l}g^{(0)}_{3,l}(1-v)\Big) = -\frac{Li_2(1-v)}{v(1-v)} \tag{5.28}$$

The first order correction $B_{1,l}$ to the free co-efficient can be broken into two parts: $B_{1,l} = b_l^{(1)} + b_l^{(2)}$ where $b_l^{(1)}$ captures the contribution to $B_{1,l}$ from the Li_2 term and $b_l^{(2)}$ captures the contribution from the derivative term.

To compute $b_l^{(1)}$, like before we write both sides of (5.28) in Mellin space

$$\sum_l b_l^{(1)}\Big(\frac{\Gamma(6+2l)(6+l-1)_l}{(-4)^l\Gamma^4(3+l)}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{dt}{2\pi i}v^t\Gamma^2(t+3)\Gamma^2(-t)Q^3_{l,0}(t)\Big) \\ =\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{dt}{2\pi i}v^t\frac{1}{2}\Gamma^2(-t)\Gamma^2(1+t)h(1+t) \tag{5.29}$$

where $-2<\gamma<-1$.

This gives

$$b_l^{(1)}=\frac{(-4)^l\Gamma^4(3+l)}{\Gamma(6+2l)(6+l-1)_l}[\kappa_l(3)]^{-1}\Big[\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\Gamma^2(-t)\Gamma^2(1+t)h(1+t)Q^3_{l,0}(t)\Big] \tag{5.30}$$

$$=\frac{2^l\Gamma^2(l+3)\Gamma(l+5)}{(2!)^2 l!\Gamma(2l+5)}\Big[\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\Gamma^2(-t)\Gamma^2(1+t)h(1+t)_3F_2[-l,l+5,3+t;3,3;1]\Big] \tag{5.31}$$

Using the integral representation for ${}_3F_2$ and the following integral representation for $h(z)$,

$$h(z)=\int_0^1 dt\frac{1-t^z}{1-t} \tag{5.32}$$

(5.31) can be simplified to obtain

$$b_l^{(1)}=\frac{\Gamma(l+5)}{2^2 l!}\sum_{s=0}^{l}\frac{(-l)_s(l+5)_s}{\Gamma^2(s+3)}(-h^{(2)}(s+1)-h^2(s+1)) \tag{5.33}$$

$$=(-1)^{l+1}h(l+2)^2+\sum_0^{l+2}(-1)^r\frac{1}{r^2} \tag{5.34}$$

Next, to compute $b_l^{(2)}$, we have to consider the derivative $\frac{\partial}{\partial\Delta}g^{(0)}_{t',l}(1-v)$. This gives us the equation

$$\sum_l A_{1,l}\Delta_l^{(1)}\frac{\partial}{\partial\Delta}g_{t'+2,l}(1-v)|_{t'=1}=-\sum_l A_{1,l}b_l^{(2)}g_{3,l}(1-v) \tag{5.35}$$

In Mellin space, the equation becomes

$$\sum_l A_{1,l}\frac{\Delta_l^{(1)}}{2}\frac{\partial}{\partial t'}\Big[\beta_l(t')\Gamma^2(-t)\Gamma^2(t+t'+2)Q_l^{t'+2}(t)\Big]_{t'=1} \\ =-\sum_l b_l^{(2)}A_{1,l}\Big[\beta_l(t')\Gamma^2(-t)\Gamma^2(t+t'+2)Q_l^{t'+2}(t)\Big]_{t'=1} \tag{5.36}$$

Using orthogonality, we get

$$\sum_l A_{1,l}\frac{\Delta_l^{(1)}}{2}\frac{\partial}{\partial t'}\Big[\beta_l(t'+2)\Gamma^2(-t)\Gamma^2(t+t'+2)Q_l^{t'+2}(t)\Big]_{t'=1}Q_{l'}^{t'+2}(t)|_{t'=1}=-b_{l'}^{(2)}A_{1,l'}l'! \tag{5.37}$$

Now we'll use the chain rule to rewrite this equation as

$$\begin{aligned}&\sum_l A_{1,l}\frac{\Delta_l^{(1)}}{2}\frac{\partial}{\partial t'}\Big[\beta_l(t'+2)\Gamma^2(-t)\Gamma^2(t+t'+2)Q_l^{t'+2}(t)Q_{l'}^{t'+2}(t)\Big]_{t'=1}\\&+\sum_l A_{1,l}\frac{\Delta_l^{(1)}}{2}\Big[\beta_l(t'+2)\Gamma^2(-t)\Gamma^2(t+t'+2)Q_l^{t'+2}(t)\Big]_{t'=1}\frac{\partial}{\partial t'}Q_{l'}^{t'+2}(t)|_{t'=1}=-b_{l'}^{(2)}A_{1,l'}l'!\end{aligned} \tag{5.38}$$

Now using orthogonality in the first term on rhs, the first term vanishes and for the second term, we use the result (5.22) to get

$$b_l^{(2)}=\frac{1}{A_{1,l}l!}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{dt}{2\pi i}\frac{1}{2}\Gamma^2(-t)\Gamma^2(1+t)\frac{\partial}{\partial t'}Q_l^{t'+2}(t)|_{t'=1} \tag{5.39}$$

where $-2<\gamma<-1$.

$$b_l^{(2)}=\frac{1}{A_{1,l}l!}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{dt}{2\pi i}\frac{1}{2}\Gamma^2(-t)\Gamma^2(1+t)\frac{\partial}{\partial t'}Q_l^{t'+2}(t)|_{t'=1} \tag{5.40}$$

Solving this equation gives for the rhs,

$$\begin{aligned}&\frac{\Gamma(l+5)!}{4l!}\sum_{s=0}^{l}\frac{(-l)_s(l+5)_s}{\Gamma^2(s+3)}\Big[h(s+1)[2h(l+2)-2h(l+4)]\\&+[-h^{(2)}(s+1)^2-h^2(s+1)]+2h(s+1)[h(l+s+4)-\frac{1}{s+2}]\Big]\end{aligned} \tag{5.41}$$

which using the following identity

$$\begin{aligned}&\frac{\Gamma(l+5)!}{2l!}\sum_{s=0}^{l}\frac{(-l)_s(l+5)_s}{\Gamma^2(s+3)}h(s+1)(h(l+s+4)-\frac{1}{s+2})\\&=\begin{cases}2h(l+2)^2-h^{(2)}(l+2)-2\sum_{r=1}^{l+2}\frac{(-1)^r}{r^2} & l=0,2,4,..\\ -2h(l+2)^2-2\sum_{r=1}^{l+2}\frac{(-1)^r}{r^2} & l=1,3,5,..\end{cases}\end{aligned} \tag{5.42}$$

gives

$$b_l^{(2)}=\begin{cases}3h(l+2)^2-2h(l+2)h(2l+4)-h^{(2)}(l+2)-\sum_{r=1}^{l+2}\frac{(-1)^r}{r^2} & l=0,2,4,..\\ -h(l+2)^2-\sum_{r=1}^{l+2}\frac{(-1)^r}{r^2} & l=1,3,5,..\end{cases} \tag{5.43}$$

Therefore the coefficient correction $B^1_{00,l} = b_l^{(1)} + b_l^{(2)}$ becomes

$$B_{1,l} = \begin{cases} 2h(l+2)^2 - 2h(l+2)h(2l+4) - h^{(2)}(l+2) & l = 0, 2, 4, .. \\ 0 & l = 1, 3, 5, .. \end{cases} \tag{5.44}$$

Our results for the anomalous dimensions and coefficient correction match exactly the results of [12] in position space.

6. CONCLUSIONS AND FUTURE WORK

The Mellin space method to extract the anomalous dimensions and co-efficient corrections is much simpler in principle compared to the rather cumbersome appraoch in poisiton space [12]. The calculation is reduced to the use of orthogonality properties of Mellin blocks. This reemphasizes the simplicity of the Mellin representation for CFT correlators and a need to explore this approach more both for its use to study the space of CFTs and also in the context of the AdS-CFT correspondence. We're working presently to extend this calculation to higher-loops and compute higher-order anomalous dimensions and co-efficient corrections. Another direction is to follow the bootstrap approach of [15]- [16]. Rather than expanding in conformal blocks using the OPE and then bootstrapping the correlators by demanding crossing symmetry, this approach involves an opposite strategy of expanding the correlator in a manifestly crossing symmetric basis of Witten diagrams and then constraining the CFT data by requiring consistency with the OPE.

A. THE MACK POLYNOMIAL

The Mack polynomials $P_{\nu,\ell}^{(s)}(s,t)$ are explicitly known, albeit in terms of a multiple sum

$$P_{\nu,\ell}^{(s)}(s,t) = \frac{1}{\prod_i \Gamma(l_i)} \widetilde{\sum} \gamma_{l_1,a_s}\gamma_{\bar{l}_1,b_s}(l_2-s)_k(\bar{l}_2-s)_k(s+t+a_s)_\beta(s+t+b_s)_\alpha \\ \times(-t)_{m-\alpha}(-a_s-b_s-t)_{\ell-2k-m-\beta} \tag{A.1}$$

where

$$\widetilde{\sum} \equiv \frac{\ell!}{2^\ell(h-1)_\ell} \sum_{k=0}^{[\frac{\ell}{2}]} \sum_{m=0}^{\ell-2k} \sum_{\alpha=0}^{m} \sum_{\beta=0}^{\ell-2k-m} \frac{(-1)^{\ell-k-\alpha-\beta}\Gamma(\ell-k+h-1)}{\Gamma(h-1)k!} \\ \times(\ell-2k)!\binom{\ell-2k}{m}\binom{m}{\alpha}\binom{\ell-2k-m}{\beta}. \tag{A.2}$$

And we've defined

$$\gamma_{x,y} = \Gamma(x+y)\Gamma(x-y), \quad a_s = \frac{\Delta_2-\Delta_1}{2}, \quad b_s = \frac{\Delta_3-\Delta_4}{2} \tag{A.3}$$

We also employ the notation

$$l_1 = \frac{h+\nu+\ell}{2}, \quad \bar{l}_1 = \frac{h-\nu+\ell}{2}, \quad l_2 = \frac{h+\nu-\ell}{2}, \quad \text{and} \quad \bar{l}_2 = \frac{h-\nu-\ell}{2} \tag{A.4}$$

where the l_i are given by,

$$l_1 = l_2 - a_s + \ell - k - m + \alpha - \beta, \quad l_2 = l_2 + a_s + k + m - \alpha + \beta. \tag{A.5}$$

$$l_3 = \bar{l}_2 + b_s + k + m, \quad l_4 = \bar{l}_2 - b_s + \ell - k - m \tag{A.6}$$

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