

Binomial Coefficients

CS/APMA 202

Rosen section 4.4

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Binomial Coefficients

- It allows us to do a quick expansion of $(x+y)^n$
- Why it's really important:
- It provides a good context to present proofs
 - Especially combinatorial proofs

Review: corollary 1 from section 4.3

- Let n and r be non-negative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$

- Or,

$$\binom{n}{r} = \binom{n}{n-r}$$

- Proof (from last slide set):

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{r!(n-r)!}$$

Review: combinatorial proof

- A *combinatorial proof* is a proof that uses counting arguments to prove a theorem, rather than some other method such as algebraic techniques
- Essentially, show that both sides of the proof manage to count the same objects
 - Usually in the form of an English explanation with supporting formulae

Polynomial expansion

- Consider $(x+y)^3$:
- Rephrase it as:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)(x+y)(x+y) = x^3 + [x^2y + x^2y + x^2y] + [xy^2 + xy^2 + xy^2] + y^3$$

- When choosing x twice and y once, there are $C(3,2) = C(3,1) = 3$ ways to choose where the x comes from
- When choosing x once and y twice, there are $C(3,2) = C(3,1) = 3$ ways to choose where the y comes from

Polynomial expansion

Consider $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

To obtain the x^5 term

- Each time you multiple by $(x+y)$, you select the x
- Thus, of the 5 choices, you choose x 5 times
 - $C(5,5) = 1$
- Alternatively, you choose y 0 times
 - $C(5,0) = 1$

To obtain the x^4y term

- Four of the times you multiply by $(x+y)$, you select the x
 - The other time you select the y
- Thus, of the 5 choices, you choose x 4 times
 - $C(5,4) = 5$
- Alternatively, you choose y 1 time
 - $C(5,1) = 5$

To obtain the x^3y^2 term

- $C(5,3) = C(5,2) = 10$

Etc...

Polynomial expansion

- For $(x+y)^5$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4y + \binom{5}{3}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + \binom{5}{0}y^5$$

Polynomial expansion: The binomial theorem

- For $(x+y)^n$

$$\begin{aligned}(x+y)^n &= \binom{n}{n} x^n y^0 + \binom{n}{n-1} x^{n-1} y^1 + \cdots + \binom{n}{1} x^1 y^{n-1} + \binom{n}{0} x^0 y^n \\&= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n \\&= \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j\end{aligned}$$

- The book calls this Theorem 1

Examples

What is the coefficient of $x^{12}y^{13}$ in $(x+y)^{25}$?

$$\binom{25}{13} = \binom{25}{12} = \frac{25!}{13!12!} = 5,200,300$$

What is the coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$?

- Rephrase it as $(2x+(-3y))^{25}$

$$(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

- The coefficient occurs when $j=13$:

$$\binom{25}{13} 2^{12} (-3)^{13} = \frac{25!}{13!12!} 2^{12} (-3)^{13} = -33,959,763,545,702,400$$

Rosen, section 4.4, question 4

- Find the coefficient of x^5y^8 in $(x+y)^{13}$

Answer: $\binom{13}{5} = \binom{13}{8} = 1287$

Pascal's triangle

(%)

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

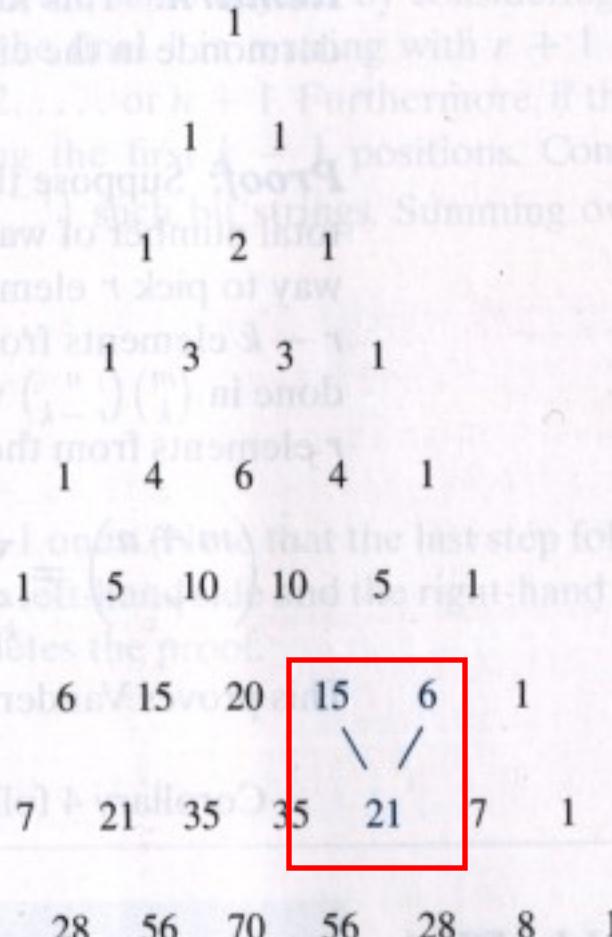
$$\binom{6}{0} \binom{6}{1} \binom{6}{2} \binom{6}{3} \binom{6}{4} \binom{6}{5} \binom{6}{6}$$

$$\left(\begin{array}{c} 7 \\ 0 \end{array}\right) \left(\begin{array}{c} 7 \\ 1 \end{array}\right) \left(\begin{array}{c} 7 \\ 2 \end{array}\right) \left(\begin{array}{c} 7 \\ 3 \end{array}\right) \left(\begin{array}{c} 7 \\ 4 \end{array}\right) \left(\begin{array}{c} 7 \\ 5 \end{array}\right) \left(\begin{array}{c} 7 \\ 6 \end{array}\right) \left(\begin{array}{c} 7 \\ 7 \end{array}\right)$$

$$\left(\begin{array}{c} 8 \\ 0 \end{array}\right) \left(\begin{array}{c} 8 \\ 1 \end{array}\right) \left(\begin{array}{c} 8 \\ 2 \end{array}\right) \left(\begin{array}{c} 8 \\ 3 \end{array}\right) \left(\begin{array}{c} 8 \\ 4 \end{array}\right) \left(\begin{array}{c} 8 \\ 5 \end{array}\right) \left(\begin{array}{c} 8 \\ 6 \end{array}\right) \left(\begin{array}{c} 8 \\ 7 \end{array}\right) \left(\begin{array}{c} 8 \\ 8 \end{array}\right)$$

n =

By Pasca



Pascal's Identity

- By Pascal's identity: $\binom{7}{5} = \binom{6}{4} + \binom{6}{5}$ or $21 = 15 + 6$
- Let n and k be positive integers with $n \geq k$.
- Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$
 - or $C(n+1, k) = C(n, k-1) + C(n, k)$
- The book calls this Theorem 2
- We will prove this via two ways:
 - Combinatorial proof
 - Using the formula for $\binom{n}{k}$

Combinatorial proof of Pascal's identity

- ➊ Prove $C(n+1, k) = C(n, k-1) + C(n, k)$
- ➋ Consider a set T of $n+1$ elements
 - We want to choose a subset of k elements
 - We will count the number of subsets of k elements via 2 methods
- ➌ Method 1: There are $C(n+1, k)$ ways to choose such a subset
- ➍ Method 2: Let a be an element of set T
 - ➎ Two cases
 - a is in such a subset
 - ➏ There are $C(n, k-1)$ ways to choose such a subset
 - a is not in such a subset
 - ➏ There are $C(n, k)$ ways to choose such a subset
 - ➏ Thus, there are $C(n, k-1) + C(n, k)$ ways to choose a subset of k elements
 - ➐ Therefore, $C(n+1, k) = C(n, k-1) + C(n, k)$

Rosen, section 4.4, question 19: algebraic proof of Pascal's identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\frac{(n+1)!}{k!(n+1-k)!} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!}$$

$$\frac{(n+1)[n!] \quad \quad \quad n! \quad \quad \quad n!}{k[(k-1)!(n+1-k)(n-k)!] = [(k-1)!(n-k+1)(n-k)!] + [k(k-1)!(n-k)!]}$$

$$\frac{(n+1)}{k(n+1-k)} = \frac{1}{(n-k+1)} + \frac{1}{k}$$

$$\frac{(n+1)}{k(n+1-k)} = \frac{k}{k(n-k+1)} + \frac{(n-k+1)}{k(n-k+1)}$$

$$n+1 = k + n - k + 1$$

$$n+1 = n+1$$

Substitutions:

$$(n+1-k)! = (n+1-k)*(n-k)!$$

$$(n+1)! = (n+1)n!$$

$$(n-k+1) = (n-k+1)(n-k)!$$



Http://www.google.com/googlegulp/



Google Gulp

April Fools Day Jokes

<http://www.google.com/googlegulp/>
(or do a Google search for 'gulp')



Pascal's triangle

view it was the same values by considering the individual summing with $\beta = 1$ one

$-1, r+2, \dots$, or -1 . Furthermore, if the j -th entries among the first $r+1$ positions. Consequently

entity: 1 2 1

$$= \begin{pmatrix} 7 \\ 5 \end{pmatrix} \quad 1 \quad 3 \quad 3 \quad 1$$

1 4 6 4 1

Since the last step follows

2' above 1 6 15 20 15 6 1

$$\begin{array}{ccccccccc} & & & & & \diagdown & \diagup & & \\ & & & & & 21 & 7 & 1 & \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & \end{array}$$

1 8 28 56 70 56 28 8 1

1 8 28 50 70 50 28 8 1

$$\text{sum} = 1 = 2^n$$

A log-log plot showing the relationship between the number of nodes (N) and the number of iterations required for convergence. The x-axis is labeled N and ranges from 2 to 256. The y-axis is labeled "Iterations" and ranges from 16 to 2. The data points show a clear linear trend on the log-log scale, indicating a power-law relationship.

N	Iterations
2	2
4	4
8	8
16	16
32	32
64	64
128	128
256	256
512	512
1024	1024
2048	2048
4096	4096
8192	8192
16384	16384
32768	32768
65536	65536
131072	131072
262144	262144
524288	524288
1048576	1048576
2097152	2097152
4194304	4194304
8388608	8388608
16777216	16777216
33554432	33554432
67108864	67108864
134217728	134217728
268435456	268435456
536870912	536870912
1073741824	1073741824
2147483648	2147483648
4294967296	4294967296
8589934592	8589934592
17179869184	17179869184
34359738368	34359738368
68719476736	68719476736
137438953472	137438953472
274877906944	274877906944
549755813888	549755813888
1099511627776	1099511627776
2199023255552	2199023255552
4398046511104	4398046511104
8796093022208	8796093022208
17592186044416	17592186044416
35184372088832	35184372088832
70368744177664	70368744177664
140737488355328	140737488355328
281474976710656	281474976710656
562949953421312	562949953421312
1125899906842624	1125899906842624
2251799813685248	2251799813685248
4503599627370496	4503599627370496
9007199254740992	9007199254740992
18014398509481984	18014398509481984
36028797018963968	36028797018963968
72057594037927936	72057594037927936
14411518807585968	14411518807585968
28823037615171936	28823037615171936
57646075230343872	57646075230343872
115292150460687744	115292150460687744
230584300921375488	230584300921375488
461168601842750976	461168601842750976
922337203685501952	922337203685501952
184467407337000384	184467407337000384
368934814674000768	368934814674000768
737869629348001536	737869629348001536
147573925869600312	147573925869600312
295147851739200624	295147851739200624
590295703478401248	590295703478401248
118059140695680256	118059140695680256
236118281391360512	236118281391360512
472236562782721024	472236562782721024
944473125565442048	944473125565442048
1888946251130884096	1888946251130884096
3777892502261768192	3777892502261768192
7555785004523536384	7555785004523536384
1511157000904707368	1511157000904707368
3022314001809414736	3022314001809414736
6044628003618829472	6044628003618829472
1208925600723765896	1208925600723765896
2417851201447531792	2417851201447531792
4835702402895063584	4835702402895063584
9671404805790127168	9671404805790127168
19342809611580254336	19342809611580254336
38685619223160508672	38685619223160508672
77371238446321017344	77371238446321017344
154742476892642034688	154742476892642034688
309484953785284069376	309484953785284069376
618969907570568138752	618969907570568138752
1237939815141136275504	1237939815141136275504
2475879630282272551008	2475879630282272551008
4951759260564545102016	4951759260564545102016
9903518521129090204032	9903518521129090204032
19807037042258180408064	19807037042258180408064
39614074084516360816128	39614074084516360816128
79228148169032721632256	79228148169032721632256
158456283338065443264512	158456283338065443264512
316912566676130886529024	316912566676130886529024
633825133352261773058048	633825133352261773058048
1267650266704523546116096	1267650266704523546116096
2535300533409047092232192	2535300533409047092232192
5070601066818094184464384	5070601066818094184464384
10141202133636188368928768	10141202133636188368928768
20282404267272376737857536	20282404267272376737857536
40564808534544753475715072	40564808534544753475715072
81129617069089506951430144	81129617069089506951430144
162259234138179013902860288	162259234138179013902860288
324518468276358027805720576	324518468276358027805720576
649036936552716055611441152	649036936552716055611441152
1298073873105432111222882304	1298073873105432111222882304
2596147746210864222445764608	2596147746210864222445764608
5192295492421728444891529216	5192295492421728444891529216
1038459098484345688978308832	1038459098484345688978308832
2076918196968691377956617664	2076918196968691377956617664
4153836393937382755913235328	4153836393937382755913235328
8307672787874765511826470656	8307672787874765511826470656
16615345575749531023652941312	16615345575749531023652941312
33230691151498562047305882624	33230691151498562047305882624
66461382302997124094611765248	66461382302997124094611765248
13292276460598248198922353048	13292276460598248198922353048
26584552921196496397844706096	26584552921196496397844706096
53169105842392992795689412192	53169105842392992795689412192
106338211684785985591378824384	106338211684785985591378824384
212676423369571971182757648768	212676423369571971182757648768
425352846739143942365515297536	425352846739143942365515297536
850705693478287884731030595072	850705693478287884731030595072
1701411386956575769462061190144	1701411386956575769462061190144
3402822773913151538924122380288	3402822773913151538924122380288
6805645547826303077848244760576	6805645547826303077848244760576
1361129109565260615569648952152	1361129109565260615569648952152
2722258219130521231139297904304	2722258219130521231139297904304
5444516438261042462278595808608	5444516438261042462278595808608
10889032865322084924557911617168	10889032865322084924557911617168
21778065730644169849115823234336	21778065730644169849115823234336
43556131461288339698231646468672	43556131461288339698231646468672
87112262922576679396463292937344	87112262922576679396463292937344
174224525845153358792926585874688	174224525845153358792926585874688
348449051690306717585853171749376	348449051690306717585853171749376
696898103380613435171706343498752	696898103380613435171706343498752
139379620676122687034341268697504	139379620676122687034341268697504
278759241352245374068682533395008	278759241352245374068682533395008
557518482704490748137365066790016	557518482704490748137365066790016
111503696540895149274473013358032	111503696540895149274473013358032
223007393081790298548946026716064	223007393081790298548946026716064
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892029572327160194195784106864256	892029572327160194195784106864256
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3568118289308640776783136427457024	3568118289308640776783136427457024
7136236578617281553566272854914048	7136236578617281553566272854914048
1427247315723456310713255770982896	1427247315723456310713255770982896
2854494631446912621426511541965792	2854494631446912621426511541965792
5708989262893825242853023083931584	5708989262893825242853023083931584
1141797852578765048566046616786368	1141797852578765048566046616786368
2283595705157530097132093233572736	2283595705157530097132093233572736
4567191410315060194264186467145472	4567191410315060194264186467145472
9134382820630120388528372934290944	9134382820630120388528372934290944
1826876564126024077656674586858188	1826876564126024077656674586858188
3653753128252048155313349173716376	3653753128252048155313349173716376
7307506256504096310626698347432752	7307506256504096310626698347432752
1461501253208193262013396668866504	1461501253208193262013396668866504
2923002506416386524026793337733008	2923002506416386524026793337733008
5846005012832772648053586675466016	5846005012832772648053586675466016
1169201002566554529610717335092032	1169201002566554529610717335092032
2338402005133109059221434670184064	2338402005133109059221434670184064
4676804010266218118442869340368128	4676804010266218118442869340368128
9353608020532436236885738680736256	9353608020532436236885738680736256
18707216041064872473771477361472512	18707216041064872473771477361472512
37414432082129544947542954722945024	37414432082129544947542954722945024
7482886416425908989508590944589008	7482886416425908989508590944589008
14965772832851817979017181889178016	14965772832851817979017181889178016
29931545665703635958034363778356032	29931545665703635958034363778356032
59863091331407271916068727556712064	59863091331407271916068727556712064
11972618266281454383213745511344128	11972618266281454383213745511344128
23945236532562908766427491022688256	23945236532562908766427491022688256
47890473065125817532854982045376512	47890473065125817532854982045376512
95780946130251635065709964090753024	95780946130251635065709964090753024
191561892260503270131419928181506048	191561892260503270131419928181506048
383123784521006540262839856363012096	383123784521006540262839856363012096
76624756904201308052567971272602496	76624756904201308052567971272602496
153249513808402616105139542545204924	153249513808402616105139542545204924
306498527616805232210279085090409848	306498527616805232210279085090409848
612997055233610464420558170180819696	612997055233610464420558170180819696
122599411046722092884111634036163992	122599411046722092884111634036163992
245198822093444185768223268072327984	245198822093444185768223268072327984
49039764418688837153644653614465968	49039764418688837153644653614465968
98079528837377674307289307228911936	98079528837377674307289307228911936
19615905767475534861457861445783972	19615905767475534861457861445783972
3923181153495106972291572289157744	3923181153495106972291572289157744
7846362306985213944583144578315488	7846362306985213944583144578315488
15692726613970427889162889156630976	15692726613970427889162889156630976
31385453227940855778325778311261952	31385453227940855778325778311261952
62770906455881711556651556625243904	62770906455881711556651556625243904
12554181291176342311310303050487808	12554181291176342311310303050487808
25108362582352684622620606100975616	25108362582352684622620606100975616
50216725164705369245241212201951232	50216725164705369245241212201951232
10043345032941073849048242440382464	10043345032941073849048242440382464
20086690065882147698096484880764928	20086690065882147698096484880764928
40173380131764295396192969761529856	40173380131764295396192969761529856
80346760263528590792385939523059712	80346760263528590792385939523059712
160693520527057181584771879060119424	160693520527057181584771879060119424
321387041054114363169543758120238848	321387041054114363169543758120238848
642774082108228726	

Proof practice: corollary 1

- Let n be a non-negative integer. Then

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

- Algebraic proof

$$2^n = (1+1)^n$$

$$= \sum_{k=0}^n \binom{n}{k} 1^k 1^{n-k}$$

$$= \sum_{k=0}^n \binom{n}{k}$$

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j$$

Proof practice: corollary 1

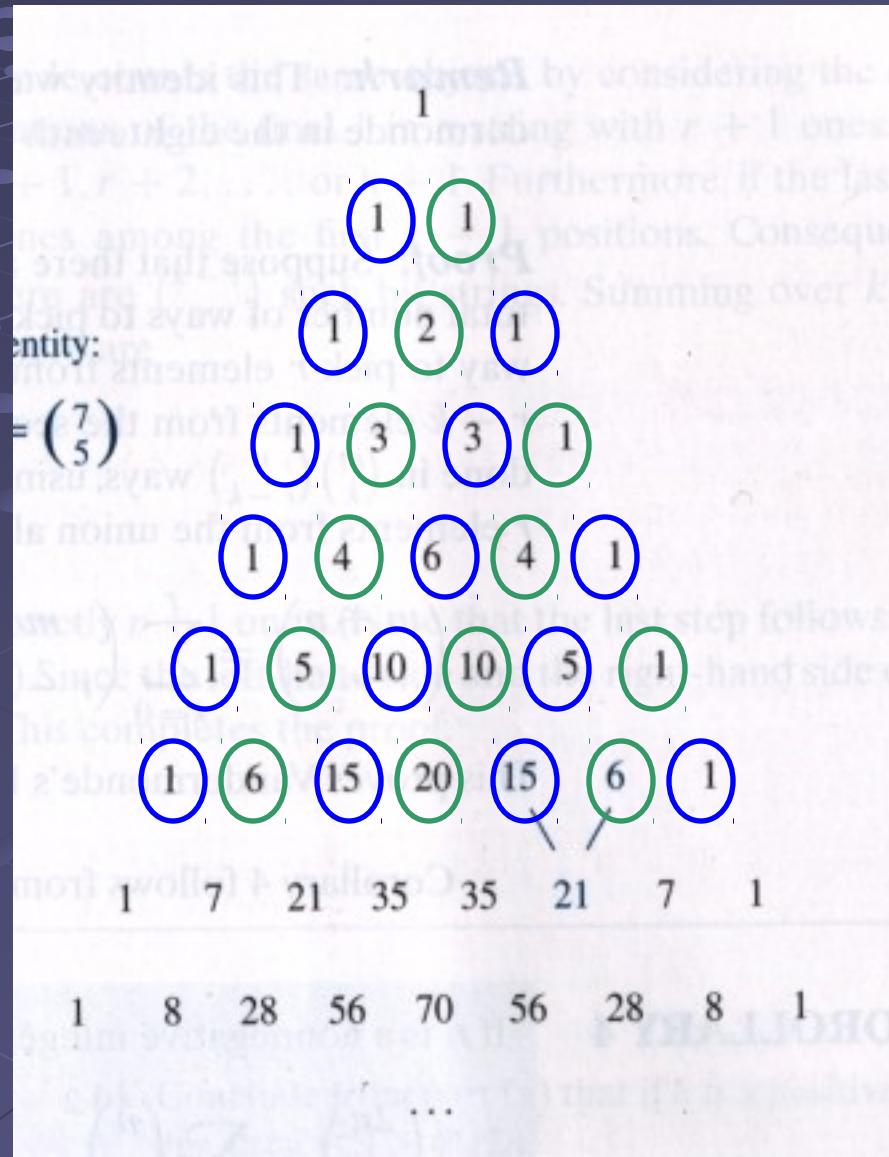
Let n be a non-negative integer. Then

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Combinatorial proof

- A set with n elements has 2^n subsets
 - By definition of power set
- Each subset has either 0 or 1 or 2 or ... or n elements
 - There are $\binom{n}{0}$ subsets with 0 elements, $\binom{n}{1}$ subsets with 1 element, ... and $\binom{n}{n}$ subsets with n elements
 - Thus, the total number of subsets is $\sum_{k=0}^n \binom{n}{k}$
- Thus, $\sum_{k=0}^n \binom{n}{k} = 2^n$

Pascal's triangle



Proof practice: corollary 2

- Let n be a positive integer. Then $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$
- Algebraic proof

$$\begin{aligned}0 &= 0^n \\&= ((-1) + 1)^n \\&= \sum_{k=0}^n \binom{n}{k} (-1)^k 1^{n-k} \\&= \sum_{k=0}^n \binom{n}{k} (-1)^k\end{aligned}$$

- This implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Proof practice: corollary 3

- Let n be a non-negative integer. Then

$$\sum_{k=0}^n 2^k \binom{n}{k} = 3^n$$

- Algebraic proof

$$\begin{aligned} 3^n &= (1+2)^n \\ &= \sum_{k=0}^n \binom{n}{k} 1^{n-k} 2^k \\ &= \sum_{k=0}^n \binom{n}{k} 2^k \end{aligned}$$

Vandermonde's identity

- Let m , n , and r be non-negative integers with r not exceeding either m or n . Then

$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{r-k} \binom{n}{k}$$

- The book calls this Theorem 3

Combinatorial proof of Vandermonde's identity

- Consider two sets, one with m items and one with n items

- Then there are $\binom{m+n}{r}$ ways to choose r items from the union of those two sets

- Next, we'll find that value via a different means

- Pick k elements from the set with n elements
- Pick the remaining $r-k$ elements from the set with m elements
- Via the product rule, there are $\binom{m}{r-k} \binom{n}{k}$ ways to do that for **EACH** value of k
- Lastly, consider this for all values of k :

$$\text{Thus, } \binom{m+n}{r} = \sum_{k=0}^r \binom{m}{r-k} \binom{n}{k}$$

$$\sum_{k=0}^r \binom{m}{r-k} \binom{n}{k}$$

Review of Rosen, section 4.3, question 11 (a)

- ➊ How many bit strings of length 10 contain exactly four 1's?
 - Find the positions of the four 1's
 - The order of those positions does not matter
 - ➊ Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
 - Thus, the answer is $C(10,4) = 210$
- ➋ Generalization of this result:
 - There are $C(n,r)$ possibilities of bit strings of length n containing r ones

Yet another combinatorial proof

- Let n and r be non-negative integers with $r \leq n$.
Then

$$\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}$$

- The book calls this Theorem 4
- We will do the combinatorial proof by showing that both sides show the ways to count bit strings of length $n+1$ with $r+1$ ones
- From previous slide: $\binom{n+1}{r+1}$ achieves this

Yet another combinatorial proof

- Next, show the right side counts the same objects
- The final one must occur at position $r+1$ or $r+2$ or ... or $n+1$
- Assume that it occurs at the k^{th} bit, where $r+1 \leq k \leq n+1$
 - Thus, there must be r ones in the first $k-1$ positions
 - Thus, there are $\binom{k-1}{r}$ such strings of length $k-1$
- As k can be any value from $r+1$ to $n+1$, the total number of possibilities is

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{k=r}^n \binom{k}{r} = \sum_{j=r}^n \binom{j}{r}$$

- Thus,

$$\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}$$

Rosen, section 4.4, question 24

- >Show that if p is a prime and k is an integer such that $1 \leq k \leq p-1$, then p divides $\binom{p}{k}$
- We know that $\binom{p}{k} = \frac{p!}{k!(p-k)!}$
- p divides the numerator ($p!$) once only
 - Because p is prime, it does not have any factors less than p
- We need to show that it does **NOT** divide the denominator
 - Otherwise the p factor would cancel out
- Since $k < p$ (it was given that $k \leq p-1$), p cannot divide $k!$
- Since $k \geq 1$, we know that $p-k < p$, and thus p cannot divide $(p-k)!$
- Thus, p divides the numerator but not the denominator
- Thus, p divides $\binom{p}{k}$

Rosen, section 4.4, question 38

- Give a combinatorial proof that if n is positive integer then

$$\sum_{k=0}^n k^2 \binom{n}{k} = n(n+1)2^{n-2}$$

- Provided hint: show that both sides count the ways to select a subset of a set of n elements together with two not necessarily distinct elements from the subset

- Following the other provided hint, we express the right side as follows:

$$\sum_{k=0}^n k^2 \binom{n}{k} = n(n-1)2^{n-2} + n2^{n-1}$$

Rosen, section 4.4, question 38

- Show the left side properly counts the desired property

Consider each of the possible subset sizes k

$$\sum_{k=0}^n k^2 \binom{n}{k} = \dots$$

Choosing one of the k elements in the subset twice

Choosing a subset of k elements from a set of n elements

Rosen, section 4.4, question 38

Two cases to show the right side: $n(n-1)2^{n-2} + n2^{n-1}$

- Pick the same element from the subset
 - Pick that one element from the set of n elements: total of n possibilities
 - Pick the rest of the subset
 - As there are $n-1$ elements left, there are a total of 2^{n-1} possibilities to pick a given subset
- We have to do both
 - Thus, by the product rule, the total possibilities is the product of the two
 - Thus, the total possibilities is $n * 2^{n-1}$
- Pick different elements from the subset
 - Pick the first element from the set of n elements: total of n possibilities
 - Pick the next element from the set of $n-1$ elements: total of $n-1$ possibilities
 - Pick the rest of the subset
 - As there are $n-2$ elements left, there are a total of 2^{n-2} possibilities to pick a given subset
- We have to do all three
 - Thus, by the product rule, the total possibilities is the product of the three
 - Thus, the total possibilities is $n * (n-1) * 2^{n-2}$
- We do one or the other
 - Thus, via the sum rule, the total possibilities is the sum of the two
 - Or $n * 2^{n-1} + n * (n-1) * 2^{n-2}$

Quick survey

- I felt I understood the material in this slide set...
 - a) Very well
 - b) With some review, I'll be good
 - c) Not really
 - d) Not at all

Quick survey

- The pace of the lecture for this slide set was...
 - a) Fast
 - b) About right
 - c) A little slow
 - d) Too slow

Quick survey

- How interesting was the material in this slide set? Be honest!
- a) Wow! That was SOOOOOO cool!
- b) Somewhat interesting
- c) Rather boring
- d) Zzzzzzzzzz

Becoming an IEEE author

Do You Want to Become an IEEE Author?

Suppose you want to publish something that is as simple as

$$1+1=2 \tag{1}$$

This is not a very impressive. If you want your article to be accepted by IEEE reviewers, you have to be more abstract. So, you could complicate the left hand side of the expression by using

$$1=\ln(e) \quad \text{and} \quad 1=\sin^2 x+\cos^2 x$$

The right hand side can be stated as

$$2 = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

Therefore, Eq. (1) can be expressed more “scientifically” as:

$$\ln(e) + (\sin^2 x + \cos^2 x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \tag{2}$$

which is far more impressive. However, you should not stop here. The expression can be

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Therefore, Eq. (1) can be expressed more "scientifically" as:

$$\ln(e) + (\sin^2 x + \cos^2 x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \quad (2)$$

which is far more impressive. However, you should not stop here. The expression can be further complicated by using

$$1 = \cosh(y)\sqrt{1 - \tanh^2(y)} \quad \text{and} \quad e = \lim_{z \rightarrow 0} \left(1 + \frac{1}{z}\right)^z$$

Eq. (2) may therefore be written as

$$\ln\left[\lim_{z \rightarrow 0} \left(1 + \frac{1}{z}\right)^z\right] + (\sin^2 x + \cos^2 x) = \sum_{n=0}^{\infty} \frac{\cosh(y\sqrt{1 - \tanh^2 y})}{2^n} \quad (3)$$

Note: Other methods of a similar nature could also be used to enhance your prestige, once you grasp the underlying principles.