Binomial Coefficients

CS/APMA 202

Rosen section 4.4

Aaron Bloomfield

Binomial Coefficients

- It allows us to do a quick expansion of $(x+y)^n$
- Why it's really important:
- It provides a good context to present proofs
  - Especially combinatorial proofs

Review: corollary 1 from section 4.3

- Let *n* and *r* be non-negative integers with $r \le n$. Then $C(n,r) = C(n,n-r)$
- Or, $\boxed{\binom{n}{r} = \binom{n}{n-r}}$
- Proof (from last slide set):

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{r!(n-r)!}$$

Review: combinatorial proof

- A *combinatorial proof* is a proof that uses counting arguments to prove a theorem, rather than some other method such as algebraic techniques
- Essentially, show that both sides of the proof manage to count the same objects
  - Usually in the form of an English explanation with supporting formulae

Polynomial expansion

- Consider $(x+y)^3$:
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
- Rephrase it as:

$$(x+y)(x+y)(x+y) = x^3 + \left[x^2y + x^2y + x^2y\right] + \left[xy^2 + xy^2 + xy^2\right] + y^3$$

- When choosing *x* twice and *y* once, there are C(3,2) = C(3,1) = 3 ways to choose where the *x* comes from
- When choosing *x* once and *y* twice, there are C(3,2) = C(3,1) = 3 ways to choose where the *y* comes from

Polynomial expansion

- Consider $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
- To obtain the x^5 term
  - Each time you multiple by $(x+y)$, you select the x
  - Thus, of the 5 choices, you choose x 5 times
    - C(5,5) = 1
  - Alternatively, you choose y 0 times
    - C(5,0) = 1
- To obtain the x^4y term
  - Four of the times you multiply by $(x+y)$, you select the x
    - The other time you select the y
  - Thus, of the 5 choices, you choose x 4 times
    - C(5,4) = 5
  - Alternatively, you choose y 1 time
    - C(5,1) = 5
- To obtain the x^3y^2 term
  - C(5,3) = C(5,2) = 10
- Etc…

Polynomial expansion

- For $(x+y)^5$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4y + \binom{5}{3}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + \binom{5}{0}y^5$$

Polynomial expansion: The binomial theorem

- For $(x+y)^n$

$$(x+y)^n = \binom{n}{n} x^n y^0 + \binom{n}{n-1} x^{n-1} y^1 + \cdots + \binom{n}{1} x^1 y^{n-1} + \binom{n}{0} x^0 y^n$$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

- The book calls this Theorem 1

Examples

- What is the coefficient of $x^{12}y^{13}$ in $(x+y)^{25}$?

$$\binom{25}{13} = \binom{25}{12} = \frac{25!}{13!12!} = 5,200,300$$

- What is the coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$?
  - Rephrase it as $(2x+(-3y))^{25}$

$$\left(2x+(-3y)\right)^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

  - The coefficient occurs when j=13:

$$\binom{25}{13} 2^{12}(-3)^{13} = \frac{25!}{13!12!} 2^{12}(-3)^{13} = -33,959,763,545,702,400$$

Rosen, section 4.4, question 4

- Find the coefficient of x^5y^8 in $(x+y)^{13}$
- Answer: $\binom{13}{5} = \binom{13}{8} = 1287$

Pascal's triangle

$n =$		
0	$\binom{0}{0}$	1
1	$\binom{1}{0}\ \binom{1}{1}$	1 1
2	$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$	1 2 1
3	$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$	1 3 3 1
4	$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$	1 4 6 4 1
5	$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$	1 5 10 10 5 1
6	$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$	1 6 15 20 15 6 1
7	$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$	1 7 21 35 35 21 7 1
8	$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$	1 8 28 56 70 56 28 8 1
	…	…

By Pascal's identity:

$$\binom{6}{4}+\binom{6}{5}=\binom{7}{5}$$

Pascal's Identity

- By Pascal's identity: $\binom{7}{5} = \binom{6}{4} + \binom{6}{5}$ or 21=15+6
- Let n and k be positive integers with $n \geq k$.
- Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$
  - or $C(n+1,k) = C(n,k-1) + C(n,k)$
- The book calls this Theorem 2
- We will prove this via two ways:
  - Combinatorial proof
  - Using the formula for $\binom{n}{k}$

Combinatorial proof of Pascal's identity

- Prove $C(n+1,k) = C(n,k-1) + C(n,k)$
- Consider a set T of $n+1$ elements
  - We want to choose a subset of k elements
  - We will count the number of subsets of k elements via 2 methods
- Method 1: There are $C(n+1,k)$ ways to choose such a subset
- Method 2: Let a be an element of set T
- Two cases
  - a is in such a subset
    - There are $C(n,k-1)$ ways to choose such a subset
  - a is not in such a subset
    - There are $C(n,k)$ ways to choose such a subset
- Thus, there are $C(n,k-1) + C(n,k)$ ways to choose a subset of k elements
- Therefore, $C(n+1,k) = C(n,k-1) + C(n,k)$

Rosen, section 4.4, question 19: algebraic proof of Pascal’s identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\frac{(n+1)!}{k!(n+1-k)!} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!}$$

$$\frac{(n+1)\boxed{n!}}{k\boxed{(k-1)!}(n+1-k)\boxed{(n-k)!}} = \frac{\boxed{n!}}{\boxed{(k-1)!}(n-k+1)\boxed{(n-k)!}} + \frac{\boxed{n!}}{k\boxed{(k-1)!}\boxed{(n-k)!}}$$

$$\frac{(n+1)}{k(n+1-k)} = \frac{1}{(n-k+1)} + \frac{1}{k}$$

$$\frac{(n+1)}{k(n+1-k)} = \frac{k}{k(n-k+1)} + \frac{(n-k+1)}{k(n-k+1)}$$

$$n+1 = k+n-k+1$$

$$n+1 = n+1$$

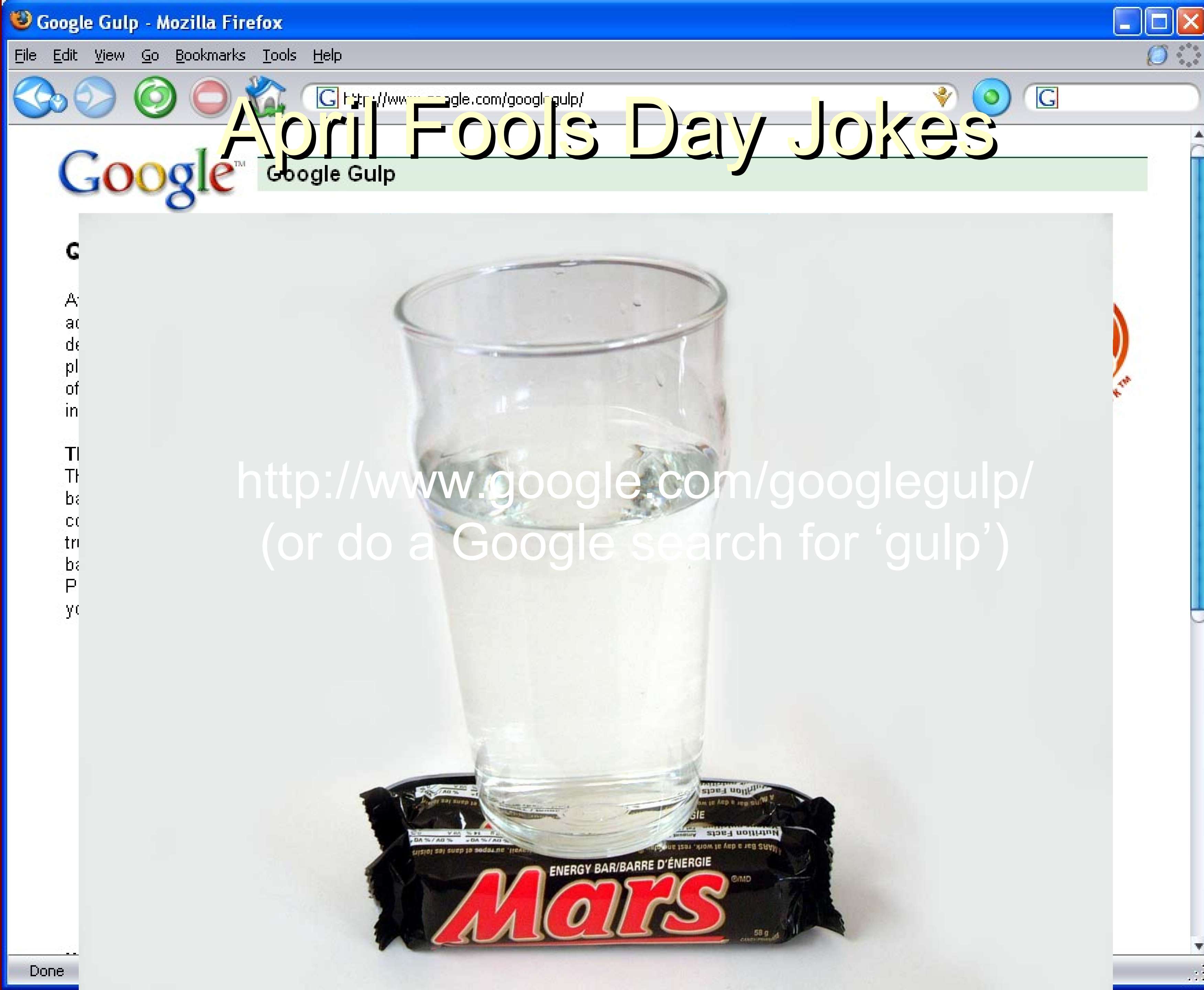
Substitutions:

$$(n+1-k)! = (n+1-k)*(n-k)!$$

$$(n+1)! = (n+1)n!$$

$$(n-k+1) = (n-k+1)(n-k)!$$

April Fools Day Jokes



http://www.google.com/googlegulp/
(or do a Google search for ‘gulp’)

Pascal's triangle

$n =$		sum =	$= 2^n$
0	1	1	
1	1 1	2	
2	1 2 1	4	
3	1 3 3 1	8	
4	1 4 6 4 1	16	
5	1 5 10 10 5 1	32	
6	1 6 15 20 15 6 1	64	
7	1 7 21 35 35 21 7 1	128	
8	1 8 28 56 70 56 28 8 1	256	
	…		

Proof practice: corollary 1

- Let *n* be a non-negative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- Algebraic proof

$$2^n = (1+1)^n \longrightarrow (x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

Proof practice: corollary 1

- Let n be a non-negative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- Combinatorial proof
  - A set with n elements has 2^n subsets
    - By definition of power set
  - Each subset has either 0 or 1 or 2 or … or n elements
    - There are $\binom{n}{0}$ subsets with 0 elements, $\binom{n}{1}$ subsets with 1 element, … and $\binom{n}{n}$ subsets with n elements
    - Thus, the total number of subsets is $\sum_{k=0}^{n} \binom{n}{k}$
  - Thus, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Pascal’s triangle

$n =$	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1

…

Proof practice: corollary 2

- Let n be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$
- Algebraic proof

$$
\begin{aligned}
0 &= 0^n \\
&= \left((-1)+1\right)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k
\end{aligned}
$$

- This implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Proof practice: corollary 3

- Let n be a non-negative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

- Algebraic proof

$$3^n = (1+2)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Vandermonde's identity

- Let *m*, *n*, and *r* be non-negative integers with *r* not exceeding either *m* or *n*. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- The book calls this Theorem 3

Combinatorial proof of Vandermonde's identity

- Consider two sets, one with *m* items and one with *n* items
  - Then there are $\binom{m+n}{r}$ ways to choose *r* items from the union of those two sets
- Next, we'll find that value via a different means
  - Pick *k* elements from the set with *n* elements
  - Pick the remaining *r-k* elements from the set with *m* elements
  - Via the product rule, there are $\binom{m}{r-k}\binom{n}{k}$ ways to do that for **EACH** value of *k*
  - Lastly, consider this for all values of *k*: $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$
- Thus, $\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$

Review of Rosen, section 4.3, question 11 (a)

- How many bit strings of length 10 contain exactly four 1's?
  - Find the positions of the four 1's
  - The order of those positions does not matter
    - Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
  - Thus, the answer is C(10,4) = 210

- Generalization of this result:
  - There are C(*n*,*r*) possibilities of bit strings of length *n* containing *r* ones

Yet another combinatorial proof

- Let n and r be non-negative integers with $r \le n$. Then $$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$
  - The book calls this Theorem 4
- We will do the combinatorial proof by showing that both sides show the ways to count bit strings of length n+1 with r+1 ones
- From previous slide: $\binom{n+1}{r+1}$ achieves this

Yet another combinatorial proof

- Next, show the right side counts the same objects
- The final one must occur at position r+1 or r+2 or … or n+1
- Assume that it occurs at the k^{th} bit, where $r+1 \le k \le n+1$
  - Thus, there must be r ones in the first k-1 positions
  - Thus, there are $\binom{k-1}{r}$ such strings of length k-1
- As k can be any value from r+1 to n+1, the total number of possibilities is

$$\sum_{k=r+1}^{n+1}\binom{k-1}{r} = \sum_{k=r}^{n}\binom{k}{r} = \sum_{j=r}^{n}\binom{j}{r}$$

- Thus,

$$\binom{n+1}{r+1} = \sum_{j=r}^{n}\binom{j}{r}$$

Rosen, section 4.4, question 24

- Show that if p is a prime and k is an integer such that $1 \leq k \leq p-1$, then p divides $\binom{p}{k}$
- We know that $\binom{p}{k} = \frac{p!}{k!(p-k)!}$
- p divides the numerator ($p!$) once only
  - Because p is prime, it does not have any factors less than p
- We need to show that it does **NOT** divide the denominator
  - Otherwise the p factor would cancel out
- Since $k < p$ (it was given that $k \leq p-1$), p cannot divide $k!$
- Since $k \geq 1$, we know that $p-k < p$, and thus p cannot divide $(p-k)!$
- Thus, p divides the numerator but not the denominator
- Thus, p divides $\binom{p}{k}$

Rosen, section 4.4, question 38

- Give a combinatorial proof that if *n* is positive integer then
$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}$$
- Provided hint: show that both sides count the ways to select a subset of a set of *n* elements together with two not necessarily distinct elements from the subset
- Following the other provided hint, we express the right side as follows:
$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n-1)2^{n-2} + n2^{n-1}$$

Rosen, section 4.4, question 38

- Show the left side properly counts the desired property

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = \cdots$$

Consider each of the possible subset sizes *k*

Choosing one of the *k* elements in the subset twice

Choosing a subset of *k* elements from a set of *n* elements

Rosen, section 4.4, question 38

- Two cases to show the right side: $n(n-1)2^{n-2}+n2^{n-1}$
  - Pick the same element from the subset
    - Pick that one element from the set of *n* elements: total of *n* possibilities
    - Pick the rest of the subset
      - As there are n-1 elements left, there are a total of 2^{n-1} possibilities to pick a given subset
    - We have to do both
      - Thus, by the product rule, the total possibilities is the product of the two
      - Thus, the total possibilities is $n*2^{n-1}$
  - Pick different elements from the subset
    - Pick the first element from the set of *n* elements: total of *n* possibilities
    - Pick the next element from the set of n-1 elements: total of n-1 possibilities
    - Pick the rest of the subset
      - As there are n-2 elements left, there are a total of 2^{n-2} possibilities to pick a given subset
    - We have to do all three
      - Thus, by the product rule, the total possibilities is the product of the three
      - Thus, the total possibilities is $n*(n-1)*2^{n-2}$
  - We do one or the other
    - Thus, via the sum rule, the total possibilities is the sum of the two
    - Or $n*2^{n-1}+n*(n-1)*2^{n-2}$

Quick survey

- I felt I understood the material in this slide set…

a) Very well

b) With some review, I'll be good

c) Not really

d) Not at all

Quick survey

- The pace of the lecture for this slide set was…

a) Fast
b) About right
c) A little slow
d) Too slow

Quick survey

- How interesting was the material in this slide set? Be honest!

a) Wow! That was SOOOOOO cool!
b) Somewhat interesting
c) Rather borting
d) Zzzzzzzzzzz

Becoming an IEEE author

Do You Want to Become an IEEE Author?

Suppose you want to publish something that is as simple as

$$1+1=2 \tag{1}$$

This is not a very impressive. If you want your article to be accepted by IEEE reviewers, you have to be more abstract. So, you could complicate the left hand side of the expression by using

$$1=\ln(e) \quad \text{and} \quad 1=\sin^2 x+\cos^2 x$$

The right hand side can be stated as

$$2=\sum_{n=0}^{\infty}\frac{1}{2^n}$$

Therefore, Eq. (1) can be expressed more “scientifically” as:

$$\ln(e)+(\sin^2 x+\cos^2 x)=\sum_{n=0}^{\infty}\frac{1}{2^n} \tag{2}$$

which is far more impressive. However, you should not stop here. The expression can be have to be more abstract. So, you could complicate the left hand side of the expression by using

$$1=\ln(e) \quad \text{and} \quad 1=\sin^2 x+\cos^2 x$$

The right hand side can be stated as

$$2=\sum_{n=0}^{\infty}\frac{1}{2^n}$$

Therefore, Eq. (1) can be expressed more "scientifically" as:

$$\ln(e)+(\sin^2 x+\cos^2 x)=\sum_{n=0}^{\infty}\frac{1}{2^n} \tag{2}$$

which is far more impressive. However, you should not stop here. The expression can be further complicated by using

$$1=\cosh(y)\sqrt{1-\tanh^2(y)} \qquad \text{and} \qquad e=\lim_{z\to 0}\left(1+\frac{1}{z}\right)^z$$

Eq. (2) may therefore be written as

$$\ln\left[\lim_{z\to 0}\left(1+\frac{1}{z}\right)^z\right]+(\sin^2 x+\cos^2 x)=\sum_{n=0}^{\infty}\frac{\cosh\left(y\sqrt{1-\tanh^2 y}\right)}{2^n} \tag{3}$$

Note: Other methods of a similar nature could also be used to enhance your prestige, once you grasp the underlying principles.