Representation Theory of Finite Groups

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Abstract. These lecture notes are for the Michaelmas term of the Representation theory module. They are a reworked version of notes I received from Sam Edwards (who in turn based them off those of Jack Shotton which were based on those of Jens Funke for a previous version of the course), and follow the book by Fulton and Harris quite closely.

Contents

0. Prerequisites

We give a brief overview of some key concepts from group theory and linear algebra that will be used throughout the module. This is not an exhaustive list; there may well be other facts we will need!

0.1. Group theory.

0.1.1. *Definition of a group.*

Definition 0.1. *A group is a triple $(G, \cdot, e)$, where G is a set, "$\cdot$" is a binary operation on G (i.e. a function from $G \times G$ to G), and $e \in G$ is a distinguished element such that $e \cdot a = a \cdot e = a$ for all $a \in G$.*

Furthermore, we require that

(i) $\cdot$ is associative, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$; this lets us write $a \cdot b \cdot c$ (normally we just write abc).

(ii) For every $a \in G$, there exists some $b \in G$ s.t. $ab = e$. (It follows that $ba = e$, and that such an element b is unique).

We often just write G for a group, as it is normally clear from the context which binary operation is being used for the definition. We will generally either use additive or multiplicative notation for the binary operation, i.e. $g + h$ or gh.

0.1.2. *Examples of groups.*

Example 0.2. $(\mathbb{Z}, +, 0)$, $(\mathbb{Q}, +, 0)$, $(\mathbb{R}, +, 0)$, $(\mathbb{C}, +, 0)$, $(\mathbb{Q}^\times, \cdot, 1)$, $(\mathbb{R}^\times, \cdot, 1)$, $(\mathbb{C}^\times, \cdot, 1)$, $(\mathbb{Z}/n\mathbb{Z}, +, 0)$, $((\mathbb{Z}/n\mathbb{Z})^\times, \cdot, 1)$

These are all Abelian! Some non-Abelian examples are as follows:

Example 0.3. *Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Here $i^2 = j^2 = k^2 = -1$ and $ij = k$.*

Example 0.4. *Let X be a set, and denote by S_X the set of invertible (i.e. bijective) functions from X to X. Then $(S_X, \circ, \mathrm{id})$ is a group (here "$\circ$" denotes composition of functions and id is the identity map $\mathrm{id}(x) = x$ for all $x \in X$).*

The group S_X is called the *permutation* or *symmetric* group on X. In the special case $X = \{1, 2, \ldots, n\}$, we simply write S_n.

A number of groups that will be important for us arise from linear algebra:

Example 0.5. *Let V be a vector space over a field k. We denote by $\mathrm{GL}(V)$ the group of invertible linear maps from V to V. In the case $V = k^n$, we write $\mathrm{GL}(k^n) = \mathrm{GL}_n(k)$.*

Observe that $GL_n(k)$ consists of invertible $n \times n$ matrices with entries from k. If V has dimension n, we may identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(k)$ by choosing a basis of V and using coordinate transformations.

0.1.3. *Presentations of groups.* A convenient way of defining a group is in terms of *generators* and *relations*; we write

$$G = \langle g_1, g_2, g_3, \ldots g_n \,|\, r_1 = r_2 = \ldots = r_m = e \rangle;$$

here $g_1, \ldots, g_n$ are called the generators and the $r_1, \ldots, r_m$ are finite words in the generators, called relations. The elements of the group are all finite words in the generators modulo the equivalence relation that two words are equivalent if one can be obtained from the other by substituting in or out the relations; we say that a word is <u>reduced</u> if it cannot be made any shorter by using the relations.

Example 0.6.

$$G = \langle a \mid a^n = e \rangle$$

This is just the cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$*;* a^m *denotes the word* $aaa\dots a$ *(*m *times). Then using the relation* $a^n = e$*, we see that* $a^{n+1} = a^n a = ea = a$*. Thus,* G *consists of the elements* $e, a, a^2, \dots, a^{n-1}$*.*

Example 0.7.

$$S = \langle s, t \mid s^2 = t^2 = e, sts = tst \rangle.$$

Since both elements have order 2, we see that any word in s *and* t *can be reduced to just alternating words with each element having order one, for example* $s^4t^5s^3t^2s^3 = t$*. From this, we obtain that the only possible reduced words are of the form* $stst\dots$ *or* $tstst\dots$*. However, for any of these words of length greater than 3, we can use the relation* $sts = tst$ *to shorten the word:*

$$stst\dots = (tst)t\dots = tst^2\dots = ts\dots$$

(and similarly for words of the form $tsts\dots$*). Thus, any reduced word has length at most 3. The elements of* S *are therefore*

$$S = \{e, s, t, st, ts, sts\}.$$

Example 0.8. *Let* $D_n = \langle s, r \mid s^2 = r^n = e,\ sr = r^{n-1}s \rangle$*. This is called the dihedral group of order* $2n$*, and can be viewed as the group of symmetries of the regular* n*-gon.*

Example 0.9. *Let* $Dic_n = \langle a, b \mid a^{2n} = e,\ b^2 = a^n,\ b^{-1}ab = a^{-1} \rangle$*. This is called the dicyclic group of order* $4n$*.*

0.1.4. *Homomorphisms and isomorphisms.*

Definition 0.10. *Let* $(G, \cdot)$ *and* $(H, *)$ *be two groups. A function* $\varphi : G \to H$ *is called a (group) homomorphism if*

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) * \varphi(g_2)$$

for all $g_1, g_2 \in G$*.*

Note that it follows from the definition that $\varphi(e_G) = e_H$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Example 0.11. *For any groups* G *and* H*, the trivial map* $\mathrm{triv} : G \to H$*,* $\mathrm{triv}(g) = e_H$ *for all* $g \in G$ *is a homomorphism.*

Example 0.12. *The determinant map* $\det : \mathrm{GL}_n(k) \to k^\times$ *is a homomorphism.*

An invertible (bijective) homomorphism is called an *isomorphism*. If there exists an isomorphism from a group G to a group H, then G and H are said to be *isomorphic*, and we write $G \cong H$. Isomorphic groups are essentially identical from a group-theoretic point of view.

Example 0.13. *Given* $n \in \mathbb{N}$*, let* $G = \{e^{2\pi ji/n} : j = 0, 1, \dots, n-1\} \subseteq \mathbb{C}^\times$ *and* $H = \mathbb{Z}/n\mathbb{Z}$*. The map* $\varphi : H \to G$ *given by* $\varphi(j + n\mathbb{Z}) = e^{2\pi ij/n}$ *is an isomorphism.*

0.1.5. *Subgroups and cosets.*

DEFINITION 0.14. *Given a group G, a subset $H \subseteq G$ is called a* subgroup *of G if H is a group with the same binary operation as the group G.*

In order to check whether a subset $H \subseteq G$ is a subgroup, one needs to check that $e_G \in H$, and that H is closed under multiplication and when taking inverses.

EXAMPLE 0.15. *Let $\varphi : G \to H$ be a group homomorphism. Then $\ker \varphi = \{g \in G \mid \varphi(g) = e_H\}$ is a subgroup of G and $\operatorname{im} \varphi = \{h \in H \mid \exists g \in G \ s.t.\ \varphi(g) = h\}$.*

A nice example of this is the following: $\mathrm{SL}_n(k) = \ker \det$ is the subgroup of $\mathrm{GL}_n(k)$ consisting of all matrices with determinant one.

Given a group G with a subgroup H, one can form the coset space G/H. The elements of G/H are called cosets; these are *subsets* of G of the form

$$gH = \{s \in G \mid \exists h \in H \ s.t.\ s = gh\}.$$

If a subgroup $N < G$ has the property that $gNg^{-1} = N$ for all $g \in G$, then N is said to be *normal*. The coset space G/N has a natural group structure, with multiplication given by $g_1N \cdot g_2N = g_1g_2N$. You can check this is a well-defined group operation due to N being normal.

THEOREM 0.16 (First Isomorphism Theorem). *Given a homomorphism $\varphi : G \to H$, $\ker \varphi$ is a normal subgroup of G and*

$$G/\ker \varphi \cong \mathrm{im}\varphi,$$

with the map $g \ker \varphi \mapsto \varphi(g)$ being an isomorphism between the two groups.

0.1.6. *Group actions and conjugacy classes.*

DEFINITION 0.17. *A* group action *of a group G on a set X is a function $f : G \times X \to X$ with the following properties:*

(i) $f(e, x) = x$ for all $x \in X$.
(ii) $f(gh, x) = f\big(g, f(h, x)\big)$ for all $g, h \in G$ and $x \in X$.

We normally don't write out the function "f"; instead, we simply denote $f(g, x)$ by $g \cdot x$. In this notation, property (2) reads

$$gh \cdot x = g \cdot (h \cdot x).$$

EXAMPLE 0.18. *The group S_n acts on the set $\{1, 2, \ldots, n\}$: $\sigma \cdot j = \sigma(j)$ for all $\sigma \in S_n$ and $j \in \{1, \ldots, n\}$. More generally, S_X acts on the set X by evaluation.*

EXAMPLE 0.19. *The group $D_n = \langle s, r \mid s^2 = r^n = e,\ sr = r^{n-1}s\rangle$ acts on the regular n-gon inscribed in the plane centred at the origin with a vertex at $(0, 1)$. The element r acts by rotating the polygon counterclockwise $\frac{2\pi}{n}$ radians, and s mirrors the polygon in the y-axis.*

EXAMPLE 0.20. *The group G acts on itself by conjugation: $g \cdot h = ghg^{-1}$ for all $g, h \in G$ (note that here the "$\cdot$" does not mean the group multiplication, but instead the conjugation action).*

Given a group action of a group G on a set X, we let $\mathcal{O}_G(x)$ denote the *orbit* of a point $x \in X$ under G, that is $\mathcal{O}_G(x) = \{g \cdot x \,|\, g \in G\}$. The *stabiliser* of a point x is defined as $\mathrm{Stab}_G(x) = \{g \in G \,|\, g \cdot x = x\}$.

In the special case of a group acting on itself by conjugation, the orbits are called *conjugacy classes*, and are instead denoted $\mathcal{C}_g$, i.e. $\mathcal{C}_g = \{hgh^{-1} \,|\, h \in G\}$.

Example 0.21. *Every element of S_n may be written as a product of disjoint cycles. For example, let $\sigma \in S_5$ be $\sigma = (12)(345)$. Then the conjugacy class of σ consists of all elements with the same cycle type. For the given element σ, we have that $\mathcal{C}_\sigma$ consists of all permutations of five elements that may be written as a disjoint 2-cycle and 3-cycle.*

0.2. Linear algebra.

0.2.1. *Vector spaces and linear maps.* Recall that a vector space V over a field k is a set combined with two operations:

(i) vector addition: given two vectors $\mathbf{v}, \mathbf{w} \in V$, we can "add" them together to obtain a new vector $\mathbf{v} + \mathbf{w} \in V$.
(ii) scalar multiplication: given a vector $\mathbf{v} \in V$ and an element $\lambda \in k$, we can "multiply" $\mathbf{v}$ by λ to obtain a new vector $\lambda\mathbf{v} \in V$.

These two operations must be compatible with each other and satisfy some natural properties, as listed in the axioms for vector spaces.

We will almost always just consider vector spaces over $\mathbb{C}$, however other examples that one might consider are $k = \mathbb{Q}, \mathbb{R}, \mathbb{F}_q$ (here $q = p^r$ where p is prime and $r \in \mathbb{N}$, and $\mathbb{F}_q$ denotes the field with q elements).

Given two vector spaces V, W (over the same field k), a function $T : V \to W$ is said to be *linear* if

$$T(\alpha\mathbf{v} + \beta\mathbf{u}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{u})$$

for all $\mathbf{v}, \mathbf{u} \in V$ and $\alpha, \beta \in k$. Two vector spaces are said to *isomorphic* if there exists an invertible linear map from one to the other.

The set of all linear maps from V to W is denoted $\mathrm{Hom}(V, W)$, and we write $\mathrm{Hom}(V)$ instead of $\mathrm{Hom}(V, V)$. These spaces carry a natural vector space structure inherited from V and W; given $S, T \in \mathrm{Hom}(V, W)$ and $\lambda \in k$, we define $S + T \in \mathrm{Hom}(V, W)$ and $\lambda T \in \mathrm{Hom}(V, W)$ by

$$(S + T)(\mathbf{v}) := S(\mathbf{v}) + T(\mathbf{v}), \qquad (\lambda T)(\mathbf{v}) = \lambda T(\mathbf{v})$$

for all $\mathbf{v} \in V$.

0.2.2. *Subspaces and sums and quotients of vector spaces.*

Definition 0.22. *A subset U of a vector space V is said to be a* <u>subspace</u> *of V if it is a vector space with the same addition and scalar multiplication operations as on V.*

Note that to show that a nonempty subset is a subspace, one simply needs to check that it is closed under vector addition and scalar multiplication.

Example 0.23. *Given $T \in \mathrm{Hom}(V, W)$, recall that*

$$\ker(T) = \{\mathbf{v} \in V \,|\, T(\mathbf{v}) = 0\}, \quad \mathrm{im}(T) = \{\mathbf{w} \in W \,|\, \exists \mathbf{v} \in V \text{ s.t. } \mathbf{w} = T(\mathbf{w})\}.$$

Then $\ker(T)$ is a subspace of V and $\mathrm{im}(T)$ is a subspace of W.

A vector space V is an Abelian group with respect to the vector addition operation. A subspace U of V is therefore a normal subgroup with respect to this operation, allowing us to consider the quotient group $(V/W, +, \mathbf{0})$. We can then define a scalar multiplication on this quotient by the formula

$$\lambda(\mathbf{v} + W) := \lambda\mathbf{v} + W$$

for all $\lambda \in k$ and $\mathbf{v} + W \in V/W$. This definition gives the quotient group a vector space structure, and we call this the quotient space of V with respect to W.

Given $T \in \mathrm{Hom}(V, W)$, define a new map $\widetilde{T} : V/\ker(T) \to W$ by

$$\widetilde{T}\big(\mathbf{v} + \ker(T)\big) := T(\mathbf{v})$$

for all $\mathbf{v} + \ker(T) \in V/\ker(T)$. Then $\widetilde{T} : T/\ker(T) \to \mathrm{im}(T)$ is an isomorphism of vector spaces.

The external direct sum of two vector spaces V, U is the set

$$V \oplus U = \{(\mathbf{v}, \mathbf{u}) \,|\, \mathbf{v} \in V, \, \mathbf{u} \in U\},$$

with the addition and scalar multiplication operations being defined component-wise.

If V and U are subspaces of a vector space W such that $V \cap U = \{\mathbf{0}\}$ and every element of W may be written as the sum of an element of V and an element of U, then we say that W is the internal direct sum of V and U, and the map $(\mathbf{v}, \mathbf{u}) \mapsto \mathbf{v} + \mathbf{u}$ is an isomorphism from $V \oplus U$ to W.

One can define sums $V_1 \oplus V_2 \oplus V_3 \oplus \ldots$ of multiple vector spaces in a similar way.

0.2.3. *Eigenvalues and eigenvectors.*

Definition 0.24. *A number λ is said to be an* <u>eigenvalue</u> *of $T \in \mathrm{Hom}(V)$ if there exists a non-zero vector $\mathbf{v} \in V$ such that*

$$T(\mathbf{v}) = \lambda\mathbf{v}.$$

The vector $\mathbf{v}$ is said to be a λ-eigenvector of T.

A basis $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ of V is said to be an *eigenbasis* of $T \in \mathrm{Hom}(V)$ if every element of the basis is an eigenvector of T. If T has an eigenbasis, then T is said to be *diagonalisable.* We will make use of the following result at a few key moments in the module:

Proposition 0.25. *Let A, B be two commuting elements of $\mathrm{Hom}(V)$, where V is a finite dimensional vector space. If A and B are both diagonalisable, then there is a* joint *eigenbasis of V, i.e. a basis of V such that every element of the basis is an eigenvector of both A and B.*

0.3. Exercises.

Problem 1. Let $(G, \cdot, e)$ be a group. Given $g \in G$, define $*_g : G \times G \to G$ by

$$x *_g y := xgy.$$

Show that $(G, *_g, g^{-1})$ is a group, where $g \cdot g^{-1} = e$.

Problem 2. Find all subgroups of D_4.

Problem 3. Compute the conjugacy classes of Q_8.

Problem 4. Compute the conjugacy classes of D_4.

Problem 5. Find the conjugacy classes of D_n.

Problem 6. Let a group G act on a set X. Show that the stabiliser of a point x, $\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$, is a subgroup of G. Note that it is often called the *stabiliser subgroup.*

Problem 7. Show that $D_3 \cong S_3$. *Hint: Label the vertices of the triangle by* $1, 2, 3$ *and consider the action of* D_3 *on them (cf. Example 0.19).*

Problem 8. Identify, with proof, the group given in Example 0.7.

Problem 9. Show that any finite group G is isomorphic to a subgroup of $S_{|G|}$.

Problem 10. For a prime p, compute the orders of $\mathrm{SL}_2(\mathbb{F}_p)$ and $\mathrm{GL}_2(\mathbb{F}_p)$.

Problem 11. Prove Proposition 0.25.

CHAPTER 1

Representations

1. Lecture 1

1.1. Definition of a representation. Let k be a field (we will *almost always* take $k = \mathbb{C}$), and let G be a group.

DEFINITION 1.1. *A* representation *of G over k is a pair (π, V), where*

(i) V is a vector space over k, and
(ii) $\pi : G \to \mathrm{GL}(V)$ is a group homomorphism.

We also say that π is a representation of G on V, or simply that π is a representation of G; this makes sense since the vector space V is part of the definition of π.

The *dimension* of a representation (π, V) is the dimension of V. We will often denote this by $\dim \pi$.

There is another way to think of this. Suppose that (ρ, V) is a representation of G. Then we may define an action of G on V by letting

$$g \cdot \mathbf{v} := \rho(g)\mathbf{v}$$

for all $g \in G$ and $\mathbf{v} \in V$. This is an action because ρ is a homomorphism, and it is linear, meaning that for every $g \in G$ the map taking $\mathbf{v} \mapsto \rho(g)\mathbf{v}$ is a linear map on V. Conversely, given a linear action of G on V, we can define a representation (π, V) of G by $\pi(g)\mathbf{v} := g \cdot \mathbf{v}$. In other words:

A representation of G is a linear action on a vector space.

Suppose that (π, V) is a representation of a group G and $\dim \pi = n$. By choosing a basis for V, we may identify V with k^n, and under this identification any (invertible) linear map $V \to V$ is just the same thing as an (invertible) $n \times n$ matrix.

Thus, once you choose a basis, a representation is just the same as a homomorphism $G \to \mathrm{GL}_n(k)$. In particular:

A 1-dimensional representation of G is a homomorphism $G \to k^\times$.

1.2. Examples.

EXAMPLE 1.2. *If V is any vector space, then we can always take $\rho : G \to \mathrm{GL}(V)$ to be the homomorphism sending every element to the identity. We call this the* trivial representation *of G on V.*

EXAMPLE 1.3. *Let $G = S_n$. Recall that there is a homomorphism*

$$\epsilon : S_n \to \{\pm 1\}$$

taking a permutation to its sign. Since $\{\pm 1\} \subseteq \mathbb{C}^\times$, this gives a 1-dimensional representation of S_n, called the sign representation*.*

Example 1.4. *Suppose that* $G = (\mathbb{Z}, +)$. *Then, if* ρ *is a representation of* G*, it is completely determined by* V *and the invertible linear map* $\rho(1) : V \to V$ *(which may be any element of* $\mathrm{GL}(V)$*). This is because we then have*

$$\rho(n) = \rho(1 + \ldots + 1) = \rho(1)^n.$$

Thus, a representation of $\mathbb{Z}$ *is just a vector space* V *together with an invertible linear map from* V *to itself.*

We can push this a bit further. Suppose that G *is a cyclic group of order* n *with generator* a*, hence*

$$G = \langle a \,|\, a^n = e \rangle .$$

A representation (π, V) *of* G *is once again determined by* V *and* $\rho(a)$*, which can be any linear map* $T : V \to V$ *such that* $T^n = \mathrm{Id}$.

Many interesting examples arise from geometry.

Example 1.5. *Let* $G = D_n$ *be the dihedral group of order* $2n$*, the group of symmetries (rotations and reflections) of a regular* n*-gon. Since each rotation/reflection is an invertible linear map from* $\mathbb{R}^2 \to \mathbb{R}^2$*, we get a representation* ρ *of* G *on* $\mathbb{R}^2$*. Letting* r *be counter-clockwise rotation by* $2\pi/n$ *radians and* s *be reflection in the horizontal axis, recall that* D_n *has the presentation*

$$\left\langle r, s \,|\, r^n = s^2 = e, sr = r^{-1}s \right\rangle .$$

As an explicit homomorphism $\rho : D_n \to \mathrm{GL}_2(\mathbb{R})$*, we have*

$$\rho(r) = \begin{pmatrix} \cos(\theta_n) & -\sin(\theta_n) \\ \sin(\theta_n) & \cos(\theta_n) \end{pmatrix}, \qquad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\theta_n = 2\pi/n$*. Since* $\mathrm{GL}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{C})$*, we may also view this as a representation of* D_n *on* $\mathbb{C}^2$*; we call* $(\rho, \mathbb{C}^2)$ *the* defining representation *of* D_n*.*

Example 1.6. *Let* $G = S_4$*. You might remember that this is isomorphic to the group of symmetries (rotations/reflections) of the regular tetrahedron in* $\mathbb{R}^3$*. We therefore get a representation*

$$\rho : S_4 \to \mathrm{GL}_3(\mathbb{R}).$$

It would be a slightly unpleasant exercise to work the matrices out explicitly.

Note that S_4 *is also isomorphic to the group of rotations of the cube, giving another (different!) 3-dimensional representation.*

Another source of representations comes from actions of groups on (usually finite) sets.

Example 1.7. *Define a representation* (π, k^n) *of* S_n *via*

$$\pi(\sigma)\left(x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n\right) = x_1\mathbf{e}_{\sigma(1)} + \ldots + x_n\mathbf{e}_{\sigma(n)},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *is the standard basis. This is called the* permutation representation *of* S_n *on* k^n*.*

Important: If we write elements of k^n as $(x_1, \ldots, x_n)^{\mathrm{t}}$ (as we commonly do), then it is **not** the case that

$$\pi(\sigma)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix}.$$

This actually would define a *right* action, not a left action. The **correct** formula is

$$\pi(\sigma)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix}.$$

This may be generalised whenever we have a group action on a set. Firstly, we associate to any set X an k-vector space:

DEFINITION 1.8. *Let X be a set. The* <u>free vector space</u> *$k(X)$ over k generated by X is the vector space consisting of all* formal sums

$$\sum_{x\in X} z_x x,$$

where $z_x \in k$ and $z_x = 0$ for all but finitely many $x \in X$. The vector space operations are as follows: vector addition is given by

$$\left(\sum_{x\in X} z_x x\right) + \left(\sum_{x\in X} y_x x\right) := \left(\sum_{x\in X} (z_x + y_x) x\right),$$

and scalar multiplication

$$\lambda\left(\sum_{x\in X} z_x x\right) := \left(\sum_{x\in X} \lambda z_x x\right)$$

for all $\lambda \in k$.

Observe that a basis for $k(X)$ is $\{x\}_{x\in X}$ (here $x = 1x$, i.e. the formal sum $\sum_{y\in X} z_y y$, with $z_y = 1$ if $y = x$ and $z_y = 0$ otherwise)

DEFINITION 1.9. *Given a group G acting on a set X, we define the* <u>permutation representation</u> *$\big(\pi, k(X)\big)$ for this action by*

$$\pi(g)\left(\sum_{x\in X} z_x x\right) := \left(\sum_{x\in X} z_x (g\cdot x)\right) = \left(\sum_{x\in X} z_{g^{-1}\cdot x} x\right).$$

1.3. Exercises.

REMARK 1.10. *We elaborate on two ways of checking whether a map $\pi : G \to \mathrm{GL}(V)$ is actually a representation. This point was discussed a bit during the lecture, but since it will be used frequently throughout the course, I would like to emphasise this a bit further:*

(i) Assume that a rule or formula is given for $\pi(g)$ for every *$g \in G$. In order to verify that (π, V) really is a representation, one needs to either check or show that $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in G$.*

(ii) Often, we will define π by where it sends the generators $g_1, \ldots, g_k$ of a group G. Since every element of G can be written as a word in $g_1, \ldots, g_k$, if π is required to be a homomorphism, then there is no choice *in the values of $\pi(g)$ for the other group elements g: $\pi(g)$ must be the product of the $\pi(g_i)$s corresponding to the decomposition of g into a product of g_is. In order for this procedure to give a well-defined homomorphism π on all of G, it is* both necessary and sufficient *that the $\pi(g_i)$s satisfy the same relations as the g_is. (This is a general fact about any group homomorphism, not just ones into $\mathrm{GL}(V)$).*

Compare the phrasing of problems 12 and 13 below!

Problem 12. Verify that $(\pi, \mathbb{C}^3)$ is a representation of S_3, where $\pi(\sigma) \in \mathrm{GL}_3(\mathbb{C})$ is defined via

$$\pi(\sigma)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_{\sigma^{-1}(1)} \\ z_{\sigma^{-1}(2)} \\ z_{\sigma^{-1}(3)} \end{pmatrix} \qquad \forall \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{C}^3.$$

Problem 13. Writing $D_3 = \langle r, s \,|\, r^3 = s^2 = rsrs = e\rangle$, let $\rho(r), \rho(s) \in \mathrm{GL}_2(\mathbb{C})$ be given by

$$\rho(r) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}, \qquad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that this defines a representation $(\rho, \mathbb{C}^2)$ of D_3.

Problem 14.

(a) Find the matrices of all the elements of S_3 for the permutation representation $(\pi, \mathbb{C}^3)$, with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

(b) Find another basis such that the matrices all take the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & ? & ? \\ 0 & ? & ? \end{pmatrix},$$

and determine the unknown entries for your basis.

Problem 15. Let (π, V) and (ρ, W) be two representations of a group G. Show that $(\pi \oplus \rho, V \oplus W)$ is a representation of G, where

$$\pi \oplus \rho(g)(v, w) := (\pi(g)v, \rho(g)w) \qquad \forall g \in G,\ v \in V,\ w \in W.$$

Remark: this is called the (direct) sum of two representations.

Problem 16. Let (π, V) be a representation of a group G. Given a subgroup $H \leq G$, show that $(\pi|_H, V)$ is a representation of H.

Problem 17. Given a group action $G \circlearrowright X$, let $V = \{f : X \to \mathbb{C}\}$.

a) Show that (π, V) is a representation of G, where for every $g \in G$ and $f \in V$, $\pi(g)f \in V$ is defined via the formula

$$\big(\pi(g)f\big)(x) = f(g^{-1} \cdot x) \qquad \forall x \in X.$$

b) Show that $(\widetilde{\pi}, V)$ is *not* a representation of G, where for every $g \in G$ and $f \in V$, $\widetilde{\pi}(g)f \in V$ is defined via the formula

$$\big(\widetilde{\pi}(g)f\big)(x) = f(g \cdot x) \qquad \forall x \in X.$$

Problem 18. Prove the following:

PROPOSITION 1.11. *Let G be a finite group and $(\pi, \mathbb{C})$ a representation of G. Show that for every $g \in G$, there exists $n_g \in \{0, 1, 2, \dots |G| - 1\}$ such that*

$$\pi(g) = e^{2\pi i n_g/|G|}.$$

Problem 19. Find all 1-dimensional representations of D_4.

2. Lecture 2

2.1. Subrepresentations and irreducible representations.

DEFINITION 2.1. *A* subrepresentation *of a representation* (ρ, V) *of* G *is a pair* $(\rho|_W, W)$ *consisting of a subspace* $W \subseteq V$ *such that* $\rho(g)\mathbf{w} \in W$ *for all* $\mathbf{w} \in W$ *together with the group homomorphism*

$$\rho|_W : G \longrightarrow \mathrm{GL}(W)\ ;\ \rho|_W(g)\mathbf{w} := \rho(g)\mathbf{w},$$

for all $w \in W$*.*

DEFINITION 2.2. *A representation* (ρ, V) *of* G *is* irreducible *if it is non-zero and has no subrepresentations except for* $\{0\}$ *and* V *itself.*

EXAMPLE 2.3. *Since 1-dimensional vector spaces have no non-trivial subspaces, every 1-dimensional representation is irreducible.*

EXAMPLE 2.4. *Consider the permutation representation* $(\pi, \mathbb{C}^3)$ *of* S_3*. This is the representation which acts on via* $\pi(\sigma)\mathbf{e}_i = \mathbf{e}_{\sigma(i)}$*, where* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ *are the standard basis vectors. It is not irreducible!*

Let $\mathbf{v}_1 = (1,1,1)^{\mathrm{t}}$*. Then* $\pi(\sigma)\mathbf{v}_1 = \mathbf{v}_1$ *for every* $\sigma \in S_3$*, so* $(\pi, \mathbb{C}\mathbf{v}_1)$ *is a subrepresentation of* $(\pi, \mathbb{C}^3)$*. Being 1-dimensional, it is in fact an* irreducible subrepresentation*.*

Denote by W_0 *the subspace of* $\mathbb{C}^3$ *consisting of all vectors* $(z_1, z_2, z_3)^{\mathrm{t}}$ *such that* $z_1+z_2+z_3 = 0$*. Then since* S_3 *permutes the coordinates of a vector, it doesn't change their sum; so* $\pi(\sigma)W_0 \subseteq W_0$*, and* W_0 *is a subrepresentation (of dimension 2).*

We claim that W_0 *is irreducible. Indeed, suppose that* $U \subseteq W_0$ *is a non-zero subrepresentation; we have to show that* $U = W_0$*. Let* $(x, y, z)^{\mathrm{t}} \in U$ *be non-zero. As* $x = y = z$ *can't happen, we can apply an element of* S_3 *to permute the coordinates so that* $x \neq y$*. Then applying* $\pi(12)$*, we have* $(y, x, z)^{\mathrm{t}} \in U$*. Taking the difference,* $(x-y, y-x, 0)^{\mathrm{t}} \in U$*; scaling,* $(1, -1, 0)^{\mathrm{t}} \in U$*. Applying* $\pi(23)$*, we have* $(1, 0, -1)^{\mathrm{t}} \in U$*. But these vectors span* W_0 *(e.g. because they are linearly independent and* $\dim W_0 = 2$*), so* $U = W_0$ *as required.*

EXAMPLE 2.5. *If* (ρ, V) *is a finite-dimensional representation of* $G = \mathbb{Z}$*, with* $T = \rho(1)$*, then* T *has an eigenvector* $\mathbf{v}$ *which spans a 1-dimensional subrepresentation of* V*. Thus* V *is not irreducible unless* $\dim V = 1$*. The irreducible subrepresentations of* V *are exactly the lines spanned by* T*-eigenvectors.*

2.2. Homomorphisms and isomorphisms.

DEFINITION 2.6. *Suppose that* (ρ, V) *and* (σ, W) *are representations of* G*. Then a* G-homomorphism *(or homomorphism of representations of* G*, or map of representations of* G*, or, if we are being lazy, just a homomorphism)* $V \to W$ *is a linear map* $\phi : V \to W$ *such that*

$$\phi(\rho(g)\mathbf{v}) = \sigma(g)\phi(\mathbf{v})$$

for all $\mathbf{v} \in V$*,* $g \in G$*.*

In other words, ϕ "commutes" with the action of G. We write $\mathrm{Hom}_G(V, W)$ for the (vector space) of G-homomorphisms from V to W.

There is another word that is sometimes used for G-homomorphism: *intertwiner*, or G-intertwiner.

DEFINITION 2.7. *A* G-isomorphism *(or just an isomorphism) is a bijective* G*-homomorphism.*

If (π, V) and (ρ, W) are two representations of a group G, and there exists a G-isomorphism $V \to W$ then we write $(\pi, V) \cong (\rho, W)$, and say that they are *isomorphic*. The collection of all representations isomorphic to a representation (π, V) is called the *isomorphism class* of (π, V).

LEMMA 2.8. *Suppose that V and W are representations of G. If $T \in \mathrm{Hom}_G(V, W)$ is an isomorphism, then $T^{-1} \in \mathrm{Hom}_G(W, V)$.*

PROOF. Exercise. $\square$

Isomorphic representations are "different pictures of the same object"; if (π, V) and (ρ, W) are isomorphic, with $T : V \to W$ begin a G-isomorphism, then we have

$$\rho(g) = T\pi(g)T^{-1}$$

for all $g \in G$, i.e. π and T tells us everything about ρ.

LEMMA 2.9. *Let (π, V) and (ρ, W) be two representations of a group G. If $\phi \in \mathrm{Hom}_G(V, W)$, then $(\pi, \ker(\phi))$ and $(\rho, \mathrm{im}(\phi))$ are subrepresentations of V and W, respectively.*

PROOF. We know from linear algebra that $\ker(\phi)$ and $\mathrm{im}(\phi)$ are subspaces, so we just have to show that they are preserved by the G-actions.

For the kernel: suppose that $\mathbf{v} \in \ker(\phi)$. Then $\phi(\mathbf{v}) = \mathbf{0}$, and so for any $g \in G$, we have

$$\phi(\pi(g)\mathbf{v}) = \rho(g)\phi(\mathbf{v}) = \rho(g)\mathbf{0} = \mathbf{0},$$

so $\pi(g)\mathbf{v} \in \ker(\phi)$. Thus $\ker(\phi)$ is G-invariant, as required.

For the image: suppose that $\mathbf{w} \in \mathrm{im}(\phi)$. Then $\mathbf{w} = \phi(\mathbf{v})$ for some $\mathbf{v} \in V$. Then for any $g \in G$,

$$\rho(g)\mathbf{w} = \rho(g)\phi(\mathbf{v}) = \phi(\pi(g)\mathbf{v}) \in \mathrm{im}(\phi),$$

so the image of ϕ is also G-invariant. $\square$

EXAMPLE 2.10. *Let $G = D_3 \cong S_3$ where the isomorphism takes $r \mapsto (123)$ and $s \mapsto (23)$. Let $(\pi, \mathbb{C}^3)$ be the permutation representation of G, so in the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ we have*

$$\pi(r) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \pi(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $(\rho, \mathbb{C}^2)$ denote the defining representation of G; thus

$$\rho(r) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \qquad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $T : \mathbb{C}^3 \to \mathbb{C}^2$ be the linear map defined through

$$T(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix},$$

(i.e. T maps the basis vectors $\mathbf{e}_i$ to the vertices of the equilateral triangle around the origin with a vertex at $(1, 0)^{\mathrm{t}}$). We want to show that T is a G-homomorphism.

To do this we first consider the matrix of T is

$$\begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}.$$

We need to show that $T\pi(g) = \rho T(g)$ for all $g \in D_3$. As D_3 is generated by just r and s we only need to check two cases. We need to check that

$$\begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix},$$

which is *true, and a similar equation coming from s.*

The kernel of the homomorphism T is the subspace $\mathbb{C}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ and in fact T defines an isomorphism from the subrepresentation

$$W_0 = \{(a, b, c)^{\,\mathrm{t}} \in \mathbb{C}^3 : a + b + c = 0\} \subseteq \mathbb{C}^3$$

to $\mathbb{C}^2$.

2.3. Exercises.

Problem 20. Let (π, V) be a representation of a group G. Show that if (π, W_1), (π, W_2) are two irreducible subrepresentations of (π, V), then either $W_1 \cap W_2 = 0$ or $W_1 = W_2$.

Problem 21. Prove Lemma 2.8.

Problem 22. Let (π, V) be an n-dimensional representation of a group G. Show that there exists a representation $(\widetilde{\pi}, \mathbb{C}^n)$ that is isomorphic to (π, V).

Problem 23. Generalise Example 2.4 to the permutation representation $(\pi, \mathbb{C}^n)$ of S_n.

Problem 24. Let

$$C_0(X) = \{f : X \to \mathbb{C} \mid f(x) = 0 \text{ for all but finitely many } x \in X\}.$$

The *quasi-regular* representation of G on $C_0(X)$ is denoted $(\lambda, C_0(X))$ and is defined via the formula

$$\big(\lambda(g)f\big)(x) := f(g^{-1} \cdot x)$$

for all $g \in G$, $x \in X$ and $f \in C_0(X)$. This is called the *functional* point of view. Let G be a group acting on a set X. Show that $(\pi, \mathbb{C}(X))$, the representation arising from this group action, is isomorphic to $(\lambda, C_0(X))$.

3. Lecture 3

One of the main goals of representation theory is to *classify* all the irreducible representations of a group G (or more correctly, the isomorphism classes of irreducible representations). This entails producing a list of representations of G and showing that

(i) any irreducible representation of G is isomorphic to some entry of the list, and
(ii) there are no redundancies in the list, i.e all entries are non-isomorphic.

3.1. Example: dihedral groups. We list the elements of the dihedral group D_n as

$$\{r^k, sr^k \mid k = 0, \dots, n-1\}.$$

We aim to show that Table 1 gives the complete list of irreducible representations of D_n, for n odd. We leave the case of n even as an exercise (there are two more 1-dimensional representations in this case).

TABLE 1. Representations of D_n.

	Dimension	$\rho(r)$	$\rho(s)$
$(\mathrm{Id}, \mathbb{C})$	1	1	1
$(\epsilon, \mathbb{C})$	1	1	-1
$\underset{1 \le k < n/2}{\rho_k}$	2	$\begin{pmatrix} e^{\frac{2\pi ik}{n}} & 0 \\ 0 & e^{\frac{-2\pi ik}{n}} \end{pmatrix}$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

We start by noting the following:

PROPOSITION 3.1. *Let (π, V) be an irreducible representation of a finite group G. Then $\dim \pi \le |G|$; in particular, V is finite-dimensional.*

PROOF. Let $\mathbf{v} \in V$ be a non-zero vector. Consider the subspace $U = \mathrm{span}\{\pi(g)\mathbf{v} \mid g \in G\} \subseteq V$. Since U is spanned by the vectors $\pi(g)\mathbf{v}$ as g runs over all elements of G, we have $1 \le \dim U \le |G|$. We claim that (π, U) is a subrepresentation of (π, V). Given $\mathbf{u} \in U$, we may write it as a linear combination of the spanning elements, i.e.

$$\mathbf{u} = \sum_{h \in G} z_h \pi(h)\mathbf{v},$$

where each $z_h \in \mathbb{C}$. Given any $g \in G$, we then have

$$\pi(g)\mathbf{u} = \pi(g)\left(\sum_{h \in G} z_h \pi(h)\mathbf{v}\right) = \sum_{h \in G} z_h \pi(g)\pi(h)\mathbf{v} = \sum_{h \in G} z_h \pi(gh)\mathbf{v} = \sum_{h \in G} z_{g^{-1}h} \pi(h)\mathbf{v}.$$

The vector $\sum_{h \in G} z_{g^{-1}h}\pi(h)\mathbf{v}$ is a linear combination of vectors in U, and so is also in U. The subspace U is therefore G-invariant, and hence a subrepresentation. Since (π, V) is irreducible and $U \neq 0$, $V = U$. $\square$

THEOREM 3.2. *Let n be odd. Then Table 1 is a complete list of non-isomorphic irreducible representations of D_n.*

PROOF. Firstly, we observe that the matrices in the table satisfy the group relations for D_n, and so do in fact define representations of D_n.

Now let (π, V) to be an irreducible (complex) representation of D_n. Since V is finite-dimensional, by Proposition 3.1, $\pi(r)$ has an eigenvector $\mathbf{v} \in V$ with eigenvalue λ. Note that $\mathbf{v} = \pi(e)\mathbf{v} = \pi(r^n)\mathbf{v} = \pi(r)^n\mathbf{v} = \lambda^n\mathbf{v}$; λ is therefore an n-th root of unity.

Consider the vector $\mathbf{w} = \pi(s)\mathbf{v}$. The key calculation is:

$$\pi(r)\mathbf{w} = \pi(r)\pi(s)\mathbf{v} = \pi(rs)\mathbf{v} = \pi(sr^{-1})\mathbf{v} = \pi(s)\pi(r)^{-1}\mathbf{v} = \pi(s)(\lambda^{-1}\mathbf{v}) = \lambda^{-1}\mathbf{w}.$$

We also have $\pi(s)\mathbf{w} = \pi(s)^2\mathbf{v} = \mathbf{v}$ and so $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$ is a subrepresentation of V. As V is irreducible, we see that $V = \operatorname{span}\{\mathbf{v}, \mathbf{w}\}$.

Case 1: Suppose that $\lambda \neq \lambda^{-1}$. Then $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of $\pi(r)$ with distinct eigenvalues, and so are linearly independent. Thus $\dim V = 2$. In the basis $\mathbf{v}, \mathbf{w}$, the representation is

$$\pi(r) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

$$\pi(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $\lambda = e^{2\pi i k/n}$ for some $1 \leq k < n/2$, then we get the representation ρ_k. Otherwise, $\lambda = e^{-2\pi i k/n}$ for some $1 \leq k < n/2$ and we instead take the basis $\mathbf{w}, \mathbf{v}$ to get ρ_k again.

Case 2: Suppose that $\lambda = \lambda^{-1}$. Then $\lambda = 1$ as n is odd. Since

$$\pi(r)(\mathbf{v} + \mathbf{w}) = \pi(s)(\mathbf{v} + \mathbf{w}) = \mathbf{v} + \mathbf{w},$$

we see that $\mathbf{v} + \mathbf{w}$ spans a subrepresentation of V. If $\mathbf{v} + \mathbf{w} \neq 0$, then $V = \mathbb{C}(\mathbf{v} + \mathbf{w})$ is the trivial representation. Otherwise, $\pi(s)\mathbf{v} = \mathbf{w} = -\mathbf{v}$, and we get the representation ϵ.

It remains to show that the representations in the table are non-isomorphic; this is left as Exercise 25 below.

□

3.2. Exercises.

Problem 25.

(a) Show that if (π, V) and (ρ, W) are two isomorphic finite-dimensional representations of a group G, then $\pi(g)$ and $\rho(g)$ have the same eigenvalues for all $g \in G$.
(b) Complete the proof of Theorem 3.2.

Problem 26. Find all the irreducible representations of D_n for n even.

Problem 27. Show that the irreducible representations of S_3 consist of *(a)* The 1-dimensional trivial representation *(b)* the *sign* representation $(\operatorname{sgn}, \mathbb{C})$, where $\operatorname{sgn}(\sigma) \in \{\pm 1\}$ is the *sign* of the permutation σ, and *(c)* the representation (π, W_0), where π is the usual permutation representation of S_3 on $\mathbb{C}^3$, and $W_0 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0\}$. *Hint: use the fact that we have a complete classification of the irreducible representations of D_n.*

Problem 28. Classify the irreducible representations of Q_8. *Hint: adapt the strategy we used for D_n.*

4. Lecture 4

4.1. Schur's Lemma. We move on to more theoretical considerations. Recall that if G is a group and (π, V) is a representation of G, then $\mathrm{Hom}_G(V) = \mathrm{Hom}_G(V, V)$ is the vector space of linear maps from V to V that commute with every $\pi(g)$ (for all $g \in G$).

Lemma 4.1 (Schur's Lemma, 1907). *If (π, V) is an* irreducible *finite-dimensional complex representation of a group G, then*

$$\mathrm{Hom}_G(V) = \mathbb{C}\,\mathrm{Id}.$$

Proof. Let $T(\neq 0) \in \mathrm{Hom}_G(V)$. Since V is a finite-dimensional vector space over $\mathbb{C}$, T has an eigenvalue $\lambda \in \mathbb{C}$. Consider the morphism $T - \lambda\,\mathrm{Id}$. Then for any $g \in G$,

$$\pi(g)(T - \lambda\,\mathrm{Id}) = \pi(g)T - \lambda\pi(g) = T\pi(g) - \lambda\pi(g) = (T - \lambda\,\mathrm{Id})\pi(g);$$

$T - \lambda\,\mathrm{Id}$ is therefore in $\mathrm{Hom}_G(V)$. By Lemma 2.9, $\big(\pi, \ker(T - \lambda\,\mathrm{Id})\big)$ is a subrepresentation of (π, V). The subspace $\ker(T - \lambda\,\mathrm{Id}) \subseteq V$ is non-zero, since any λ-eigenvector of T is in $\ker(T - \lambda\,\mathrm{Id})$, and so (π, V) being irreducible then gives $V = \ker(T - \lambda\,\mathrm{Id})$, hence $T = \lambda\,\mathrm{Id} \in \mathbb{C}\,\mathrm{Id}$. □

Corollary 4.2. *Let (π, V) and (ρ, W) be two* irreducible *finite-dimensional complex representations of a group G. Then*

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } (\pi, V) \cong (\rho, W) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We start by assuming that the representations are not isomorphic. Given $T \in \mathrm{Hom}_G(V, W)$, by Lemma 2.9, $(\pi, \ker(T))$ is a subrepresentation of (π, V) and $(\rho, \mathrm{im}(T))$ is a subrepresentation of (ρ, W). By assumption, (π, V) and (ρ, W) are irreducible, hence

$$\ker(T) = 0 \text{ or } V \quad \text{and} \quad \mathrm{im}(T) = 0 \text{ or } W.$$

If $\mathrm{im}(T) = 0$, then $T = 0$, and we are done. If $\mathrm{im}(T) = W$, then $\ker(T) \neq V$, hence $\ker(T) = 0$. The map T is therefore both injective ($\ker(T) = 0$) and surjective ($\mathrm{im}(T) = W$), i.e. invertible. This contradicts the assumption that the representations are non-isomorphic; hence we must have $\mathrm{im}(T) = 0$.

Assume now that the representations are isomorphic. Then by assumption there exists an invertible element $T \in \mathrm{Hom}_G(V, W)$. Let S be any other element of $\mathrm{Hom}_G(V, W)$. We then consider the map $T^{-1}S : V \to V$. By Problem 21, $T^{-1} \in \mathrm{Hom}_G(W, V)$, hence for any $g \in G$, we have

$$T^{-1}S\pi(g) = T^{-1}\rho(g)S = \pi(g)T^{-1}S,$$

hence $T^{-1}S \in \mathrm{Hom}_G(V)$. By Lemma 4.1, $\mathrm{Hom}_G(V) = \mathbb{C}\,\mathrm{Id}$, hence $T^{-1}S = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$. It follows that $S \in \mathbb{C}T$, showing that $\dim \mathrm{Hom}_G(V, W) = 1$. □

4.2. Abelian groups. Schur's lemma has a particularly striking application to Abelian groups:

Theorem 4.3. *Let G be an* Abelian *group. Then every finite-dimensional irreducible complex representation of G is 1-dimensional.*

Proof. Let (ρ, V) be an irreducible representation of G. For $h \in G$, set $T_h = \rho(h) \in \mathrm{GL}(V)$. Then $T_h \in \mathrm{Hom}_G(V)$, since it commutes with $\rho(g)$ for all $g \in G$. Indeed,

$$T_h\rho(g) = \rho(h)\rho(g) = \rho(hg) = \rho(gh) = \rho(g)\rho(h) = \rho(g)T_h.$$

Hence by Schur's Lemma, $T_h = \rho(h)$ acts by a scalar $\chi(h)$ on V:

$$\rho(h)\mathbf{v} = \chi(h)\mathbf{v}$$

for all $\mathbf{v} \in V$ and a non-zero scalar $\chi(h)$. But now, any non-zero $\mathbf{v} \in V$ spans a G-invariant subspace. Since V is irreducible, this implies that V is 1-dimensional, and $\chi = \rho$ is a homomorphism $G \to \mathbb{C}^\times$. □

Remark 4.4. *It is possible to give an alternative proof of this using the fact from linear algebra that any commuting set of linear maps from a finite-dimensional vector space to itself has a simultaneous eigenvector.*

A homomorphism $\chi : G \to \mathbb{C}^\times$ is often called a *character* (though this will later cause an unfortunate clash of notation). If G is Abelian, then the group

$$\widehat{G} = \{\text{homomorphisms } \chi : G \to \mathbb{C}^\times\}$$

is called the character group, or *dual group*, of G. It is a group under the operation $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$.

Example 4.5. *Let $G = C_n$ be a cyclic group of order n. Then $\widehat{G} \cong C_n$. Indeed, pick a generator g of G and let $\omega = e^{2\pi i/n}$ be a primitive n-th root of unity. Then a character χ of G is determined uniquely by $\chi(g)$, which must be an n-th root of unity ω^a. If we let $\chi_a \in \hat{G}$ be the homomorphism such that $\chi_a(g) = \omega^a$, then the map*

$$a \mapsto \chi_a$$

determines a group isomorphism $\mathbb{Z}/n\mathbb{Z} \to \widehat{G}$. This is a homomorphism because

$$\chi_{a+b}(g) = \omega^{a+b} = \omega^a\omega^b = \chi_a(g)\chi_b(g).$$

In fact, if G is any finite Abelian group, then $\widehat{G} \cong G$. You can prove this using the cyclic case and the fundamental theorem of finite Abelian groups.

For arbitrary groups G the same method of proof gives:

Proposition 4.6. *Let (ρ, V) be an irreducible finite-dimensional representation of G and let*

$$Z = Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$$

be the center of G. Then Z acts on V as a character: there is a character $\chi : Z \to \mathbb{C}^\times$ such that

$$\rho(z)\mathbf{v} = \chi(z)\mathbf{v}.$$

for all $z \in Z$ and $\mathbf{v} \in V$.

We call χ the central character *of ρ.*

Proof. See Problem 30 below. □

Finally, we can use our classification of the irreducible representations of Abelian groups to get a bound on the dimension of the irreducible representations of any finite group.

Proposition 4.7. *Let G be a finite group, A be an Abelian subgroup of G, and (π, V) be an irreducible representation of G. Then*

$$\dim V \leq \frac{|G|}{|A|} = [G : A].$$

Proof. Restricting the representation to A, we find an irreducible A-subrepresentation W of V (cf. Problem 29). By Theorem 4.3, W is 1-dimensional, spanned by a vector $\mathbf{v}$. There is then a character χ of A such that

$$\pi(h)\mathbf{v} = \chi(h)\mathbf{v}$$

for all $h \in A$. Now, $\{\pi(g)\mathbf{v} \,|\, g \in G\}$ spans a non-zero subrepresentation of V, hence is equal to V by irreducibility. Write $g_1A, g_2A, \ldots, g_rA$ for the left cosets of A in G, where $r = [G : A]$. Then for any $h \in A$, we have

$$\pi(g_ih)\mathbf{v} = \pi(g_i)\pi(h)\mathbf{v} = \pi(g_i)\chi(h)\mathbf{v} = \chi(h)(\pi(g_i)\mathbf{v})$$

But this implies that $V = \operatorname{span}\{\pi(g)v \,|\, g \in G\}$ is already spanned by

$$\{\pi(g_i)v \,|\, 1 \leq i \leq r\},$$

so has dimension at most $r = [G : A]$. $\square$

Example 4.8. *The group D_n has an abelian subgroup C_n of index 2, and so every irreducible representation of D_n has dimension at most* 2.

4.3. Exercises.

Problem 29. Let (π, V) be a finite-dimensional representation of a group G. Show that there is an irreducible subrepresentation (π, W) of (π, V).

Problem 30. Let (π, V) be an irreducible finite-dimensional representation of a group G. Denoting the centre of G by $Z(G)$ (i.e. $Z(G) = \{g \in G \,|\, gh = hg \quad \forall h \in G\}$), show that there exists a homomorphism $\psi : Z(G) \to \mathbb{C}^\times$ such that

$$\pi(g) = \psi(g)\,\mathrm{Id}$$

for all $g \in Z(G)$. *Hint: adapt the proof of Theorem 4.3.*

5. Lecture 5

5.1. Direct sums and reducibility. We start by defining sums of representations:

Definition 5.1. *Let (π, V) and (ρ, W) be representations of a group G. The* <u>direct sum</u> *of the two representations is $(\pi \oplus \rho, V \oplus W)$, where*

$$(\pi \oplus \rho)(g)(\mathbf{v}, \mathbf{w}) := (\pi(g)\mathbf{v}, \rho(g)\mathbf{w})$$

for all $g \in G$, $\mathbf{v} \in V$, $\mathbf{w} \in W$.

We also write $(\pi, V) \oplus (\rho, W)$ for the direct sum of the representations.

Example 5.2. *Let $(\chi_1, \mathbb{C})$ and $(\chi_2, \mathbb{C})$ be representations of $C_n = \langle a \,|\, a^n = e \rangle$ given by $\chi_1(a^j) = e^{2\pi ij/n}$ and $\chi_2(a^j) = e^{4\pi ij/n}$. Then $(\chi_1 \oplus \chi_2, \mathbb{C}^2)$ is the representation of C_n on $\mathbb{C}^2$ given by $\chi_1 \oplus \chi_2(a^j) = (e^{2\pi ij/n}, e^{4\pi ij/n})$.*

Conversely, we also have the following definition.

Definition 5.3. *If (π, V) is a representation with non-trivial subrepresentations (π, U) and (π, W) such that $V = U \oplus W$, then we say that (π, V) is* <u>decomposable</u> *or* reducible.

Note that if (π, V) is decomposable , then

$$(\pi, V) \cong (\pi, U) \oplus (\pi, W),$$

with a G-isomorphism from $U \oplus W$ to V given by

$$(\mathbf{u}, \mathbf{w}) \mapsto \mathbf{u} + \mathbf{w}$$

for all $\mathbf{u} \in U, \mathbf{w} \in W$ (see Problem 31 below). We say that the representation (π, V) has been *decomposed* into a direct sum of subrepresentations.

Example 5.4. *The permutation representation $(\pi, \mathbb{C}^n)$ decomposes as $(\pi, W_1) \oplus (\pi, W_0)$, where $W_0 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 + \ldots + x_n = 0\}$ and W_1 is the linear span of $(1, \ldots, 1)$.*

The following example shows that there are representations that are neither decomposable or irreducible:

Example 5.5. *Let $(\rho, \mathbb{C}^2)$ be the representation of $(\mathbb{Z}, +)$ given by*

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

This representation is not irreducible, as $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a non-trivial subrepresentation.

The representation is not decomposable , as then there would be two non-zero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbb{C}^2 = \mathbb{C}\mathbf{v}_1 \oplus \mathbb{C}\mathbf{v}_2$, and $(\rho, \mathbb{C}\mathbf{v}_1)$, $(\rho, \mathbb{C}\mathbf{v}_2)$ are subrepresentations. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ would then be two linearly independent eigenvectors of all the matrices $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, which is not possible - these matrices are not diagonalisable.

However, for finite groups, we have the following:

Theorem 5.6 (Maschke's Theorem, 1899). *Let (π, W) be a non-trivial subrepresentation of a finite-dimensional complex representation of a finite group G. Then there exists a subrepresentation (π, W') of (π, V) such that*

$$(\pi, V) = (\pi, W) \oplus (\pi, W').$$

Before proving this, we introduce another important notion:

5.2. Unitary representations.

Definition 5.7. *Let (π, V) be a representation of a group G on an inner product space V, with inner product $\langle \cdot, \cdot \rangle$. The representation is said to be* unitary *if*

$$\langle \pi(g)\mathbf{v}, \pi(g)\mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \tag{5.8}$$

for all $\mathbf{v}, \mathbf{u} \in V$ and $g \in G$.

We say that the inner product $\langle \cdot, \cdot \rangle$ is *G-invariant* if (5.8) holds for all $\mathbf{v}, \mathbf{u} \in V$ and $g \in G$. Other ways of writing (5.8) include

$$\langle \pi(g)\mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{v}, \pi(g)^* \mathbf{u} \rangle$$

and $\pi(g)^* = \pi(g^{-1}) = \pi(g)^{-1}$.

Proposition 5.9. *Let (π, W) be a subrepresentation of a unitary representation (π, V). Then $(\pi, W^\perp)$ is also a subrepresentation, and hence*

$$(\pi, V) = (\pi, W) \oplus (\pi, W^\perp).$$

Consequently, all unitary representations with a nontrivial subrepresentation are decomposable.

Proof. We simply need to verify that $W^\perp$ is G-invariant. Recall that

$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

Given $g \in G$, $\mathbf{v} \in W^\perp$ and any $\mathbf{w} \in W$, we have $\langle \pi(g)\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \pi(g^{-1})\mathbf{w} \rangle$. However, (π, W) is a subrepresentation, so $\pi(g^{-1})\mathbf{w} \in W$. Since $\mathbf{v} \in W^\perp$, we thus have $\langle \mathbf{v}, \pi(g^{-1})\mathbf{w} \rangle = 0$, hence $\pi(g)\mathbf{v} \in W^\perp$. $\square$

The following shows that every finite-dimensional representation of a finite group may be "made" unitary:

Proposition 5.10. *Let (π, V) be a finite-dimensional complex representation of a finite group G. Then there exists a G-invariant inner product on V.*

Proof. Since V is finite-dimensional, it has an inner product $\langle \cdot, \cdot \rangle$ (see Problem 33 below). We now define $\langle \cdot, \cdot \rangle_G : V \times V \to \mathbb{C}$ by the formula

$$\langle \mathbf{v}, \mathbf{u} \rangle_G = \frac{1}{|G|} \sum_{h \in G} \langle \pi(h)\mathbf{v}, \pi(h)\mathbf{u} \rangle,$$

and claim that this is a G-invariant inner product. We leave it as an exercise for the reader to verify that this is an inner product (see Problem 34). The G-invariance is shown as follows:

$$\begin{aligned} \langle \pi(g)\mathbf{v}, \pi(g)\mathbf{u} \rangle_G &= \frac{1}{|G|} \sum_{h \in G} \langle \pi(h)\pi(g)\mathbf{v}, \pi(h)\pi(g)\mathbf{u} \rangle \\ &= \frac{1}{|G|} \sum_{h \in G} \langle \pi(hg)\mathbf{v}, \pi(hg)\mathbf{u} \rangle \\ &= \frac{1}{|G|} \sum_{\widetilde{h} \in G} \langle \pi(\widetilde{h})\mathbf{v}, \pi(\widetilde{h})\mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle_G. \end{aligned}$$

$\square$

We note that as a consequence of the *spectral theorem* for unitary matrices, we have

PROPOSITION 5.11. *Let (π, V) be a finite-dimensional unitary representation of a group G. Then for any $g \in G$, $\pi(g)$ is diagonalisable, i.e. has an eigenbasis.*

PROOF. See Exercise 35 below $\square$

5.3. Decompositions into irreducibles. Note that Theorem 5.6 now follows directly from Propositions 5.9 and 5.10. In fact, we can give a stronger statement:

THEOREM 5.12. *Let (π, V) be a representation of G. Then there exist irreducible subrepresentations (π, W_i) such that*

$$(\pi, V) = (\pi, W_1) \oplus (\pi, W_2) \oplus \ldots \oplus (\pi, W_n).$$

Moreover, the number of times each isomorphism class of an irreducible representation shows up in the above decomposition is independent of the exact choice of decomposition.

The existence of such a decomposition follows as a corollary from repeated application of Maschke's Theorem (Theorem 5.6). To complete the proof of the second part of the theorem we need the following two lemmas:

LEMMA 5.13. *Let (π_1, U), (π_2, V), and (π_3, W) be representations of a group G. Then*
(i) $\mathrm{Hom}_G(U \oplus V, W) \cong \mathrm{Hom}_G(U, W) \oplus \mathrm{Hom}_G(V, W)$
(ii) $\mathrm{Hom}_G(W, U \oplus V) \cong \mathrm{Hom}_G(W, U) \oplus \mathrm{Hom}_G(W, V)$

PROOF. Exercise: Problem 36 below. $\square$

LEMMA 5.14. *Let $(\pi, V) = \bigoplus_{i=1}^n (\pi, W_i)$ be a decomposition of (π, V) into irreducible subrepresentations. For any irreducible representation (ρ, U), the number of (π, W_i) that are isomorphic to (ρ, U) is $\dim \mathrm{Hom}_G(V, U)$.*

PROOF. By Lemma 5.13, $\dim \mathrm{Hom}_G(V, U) = \sum_i \dim \mathrm{Hom}_G(W_i, U)$. By Corollary 4.2 to Schur's lemma, $\dim \mathrm{Hom}_G(W_i, U) = 1$ if $(\pi, W_i) \cong (\rho, U)$, and zero otherwise. $\square$

PROOF OF THEOREM 5.12. The existence of such a decomposition follows from repeated application of Theorem 5.6. The second part of the statement follows from Lemma 5.14. $\square$

5.4. Exercises.

Problem 31. Let (π, U) and (π, W) be subrepresentations of a representation (π, V) such that $V = U \oplus W$. Verify that the map

$$(\mathbf{u}, \mathbf{w}) \mapsto \mathbf{u} + \mathbf{w}$$

is a G-isomorphism from $(\pi, U) \oplus (\pi, W)$ to (π, V).

Problem 32. Verify that

$$\langle \pi(g)\mathbf{v}, \pi(g)\mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \qquad \forall \mathbf{v}, \mathbf{u} \in V,\ g \in G$$

is equivalent to

$$\langle \pi(g)\mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{v}, \pi(g^{-1})\mathbf{u} \rangle \qquad \forall \mathbf{v}, \mathbf{u} \in V,\ g \in G.$$

Problem 33. Let V be a finite-dimensional complex vector space with a basis $\{\mathbf{b}_i\}_{i=1,\dots,n}$, where $n = \dim V$. Show that

$$\left\langle \sum_{i=1}^{n} x_i \mathbf{b}_i, \sum_{j=1}^{n} y_j \mathbf{b}_j \right\rangle := \sum_{i=1}^{n} x_i \overline{y_i},$$

for x_i, $y_j \in \mathbb{C}$, defines an inner product on V.

Problem 34. Fill in the remaining details of the proof of Proposition 5.10, i.e. show that $\langle \cdot, \cdot \rangle_G$ is linear in the first argument, conjugate-symmetric, and positive-definite.

Problem 35. Let (π, V) be a finite-dimensional unitary representation of a group G, with G-invariant inner product $\langle \cdot, \cdot \rangle$. Show that the matrix of any $\pi(g)$ with respect to an orthonormal basis for $\langle \cdot, \cdot \rangle$ is unitary (recall that a matrix is unitary if $M^*(= \overline{M}^{\,\mathrm{t}}) = M^{-1}$).

Problem 36. Prove Lemma 5.13.

6. Lecture 6

6.1. The group algebra and the regular representation. Let G be a finite group. Recall (cf. Definition 1.8) that we defined the free vector space $\mathbb{C}(G)$ to be the vector space of formal sums

$$\sum_{g\in G} z_g g,$$

where all $z_g \in \mathbb{C}$, with vector addition and scalar multiplication given by

$$\left(\sum_{g\in G} z_g g\right) + \left(\sum_{g\in G} w_g g\right) := \left(\sum_{g\in G} (w_g + z_g) g\right), \quad \lambda \left(\sum_{g\in G} z_g g\right) := \sum_{g\in G} (\lambda z_g)\, g.$$

The group multiplication lets us define a multiplication operation on $\mathbb{C}(G)$:

$$\left(\sum_{g\in G} z_g g\right)\left(\sum_{h\in G} w_h h\right) := \sum_{g,h\in G} z_g w_h gh = \sum_{g\in G}\left(\sum_{h\in G} z_{gh^{-1}} w_h\right) g.$$

Definition 6.1. *Let G be a finite group. The free vector space $\mathbb{C}(G)$ equipped with the multiplication rule defined above is called the* group algebra *$\mathbb{C}G$ (or $\mathbb{C}[G]$) of G.*

Example 6.2. *Let $\mathbf{x} = e - (12)$ and $\mathbf{y} = 2(23) + (123)$ be elements of $\mathbb{C}S_3$. Then*

$$\begin{aligned} \mathbf{x}\cdot\mathbf{y} &= \big(e - (12)\big)\big(2(23) + (123)\big) \\ &= 2(23) + (123) - 2(12)(23) - (12)(123) \\ &= 2(23) + (123) - 2(123) - (23) \\ &= (23) - (123). \end{aligned}$$

Remark 6.3. *A vector space V over a field k with a multiplication rule "$\cdot$": $V\times V \to V$ that satisfies*

(i) $\mathbf{v}\cdot(\mathbf{u}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{w}$
(ii) $(\mathbf{u}+\mathbf{w})\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{v}$
(iii) $(\alpha\mathbf{v})\cdot(\beta\mathbf{u}) = (\alpha\beta)\mathbf{v}\cdot\mathbf{u}$

for all $\mathbf{v},\mathbf{u},\mathbf{w}\in V$ and $\alpha,\beta\in k$ is called an algebra. *Note that one does not require $\cdot$ to be associative however we will only consider algebras with an associative product. Examples of algebras include $\mathbb{R}^3$ with the cross product, the space of all polynomials $k[X]$ with coefficients in k with multiplication being pointwise multiplication of polynomials, and $\mathrm{Hom}(V)$ for any vector space V with multiplication being composition of linear maps.*

The formal sums $g = 1g \in \mathbb{C}G$ form a basis of $\mathbb{C}G$, hence $\dim \mathbb{C}G = |G|$.

Any representation (ρ, V) of G gives rise to an *algebra* homomorphism from $\mathbb{C}G$ to $\mathrm{Hom}(V)$, which we also denote as ρ:

$$\rho\left(\sum_{g\in G} z_g g\right) := \sum_{g\in G} z_g \rho(g).$$

Remark 6.4. *For many this is the definition of representation of a finite group that they use, in other words an algebra homomorphism from the group algebra to a vector space. This is because this definition can be extended to general algebra, not just group algebras (see Problem 41).*

Recall also that we have the permutation representation of G on $\mathbb{C}(X)$ (Definition 1.9); in this case we normally denote this by $(\lambda, \mathbb{C}G)$, and call it the *regular representation*, i.e.

$$\lambda(g)\left(\sum_{h\in G} z_h h\right) := \sum_{h\in G} z_h gh = \sum_{h\in G} z_{g^{-1}h} h.$$

THEOREM 6.5. *Let (π, V) be any representation of G. Then there is an isomorphism of vector spaces*

$$\operatorname{Hom}_G(\mathbb{C}G, V) \cong V.$$

Equivalently, $\dim \operatorname{Hom}_G(\mathbb{C}G, V) = \dim V$*.*

The proof of this is short, but can be difficult to wrap your head around. The idea is to provide a recipe to turn a G-homomorphism $\mathbb{C}G \to V$ into an element of V, and a recipe to turn an element of V into a G-homomorphism $\mathbb{C}G \to V$, and check that these recipes are inverse to each other.

PROOF. If $T : \mathbb{C}G \to V$ is a G-homomorphism, define

$$\Phi(T) := T(e) \in V.$$

Conversely, given $\mathbf{v} \in V$, let $\Psi(\mathbf{v}) \in \operatorname{Hom}_G(\mathbb{C}G, V)$ be the linear map defined by

$$\Psi(\mathbf{v})\left(\sum_{g\in G} z_g g\right) = \sum_{g\in G} z_g \pi(g)\mathbf{v};$$

you can check that $\Psi(\mathbf{v})$ is indeed a G-homomorphism (see Exercise 38).

We claim that the maps Φ and Ψ are linear maps between V and $\operatorname{Hom}_G(\mathbb{C}G, V)$ that are 2-sided inverses of each other, so these vector spaces are isomorphic. It is clear that they are linear maps. We must check that

$$\Phi(\Psi(\mathbf{v})) = \mathbf{v}$$

and

$$\Psi(\Phi(T)) = T$$

for all $\mathbf{v} \in V$, $T \in \operatorname{Hom}_G(\mathrm{C}\, G, V)$.

(i) Let $\mathbf{v} \in V$. Then

$$\Phi(\Psi(\mathbf{v})) = \Psi(\mathbf{v})(e) = \pi(e)\mathbf{v} = \mathbf{v},$$

as required.

(ii) Let $T \in \operatorname{Hom}_G(\mathbb{C}G, V)$ and define $S = \Psi(\Phi(T)) \in \operatorname{Hom}_G(\mathbb{C}G, V)$. We wish to show that $S = T$. To do this, we evaluate S at the basis vectors $g \in \mathbb{C}G$. Since S and T are linear, if they agree on a basis, then they are the same:

$$S(g) = \Psi\big(\Phi(T)\big)(g) = \pi(g)\Phi(T) = \pi(g)T(e) = T(\lambda(g)e) = T(g),$$

since T is a G-homomorphism. □

This has a beautiful consequence: the sum of the squares of the dimensions of the irreducible representations is equal to the order of the group. We write $\operatorname{Irr}(G)$ for the set of isomorphism classes of irreducible representations of G.

THEOREM 6.6.

(i) *Every irreducible representation* (ρ, W_ρ) *of* G *is a constituent of the regular representation with multiplicity* $\dim \rho$*. In other words,*

$$(\lambda, \mathbb{C}G) \cong \bigoplus_{\rho \in \mathrm{Irr}(G)} (\rho, W_\rho)^{\dim \rho}$$

(here $(\rho, W_\rho)^{\dim \rho} = (\rho, W_\rho) \oplus (\rho, W_\rho) \oplus \ldots \oplus (\rho, W_\rho)$*;* $\dim \rho$ *times).*

(ii) *(Sum of squares formula.) We have*

$$\sum_{\rho \in \mathrm{Irr}(G)} \dim(\rho)^2 = |G|,$$

where the sum runs over the isomorphism classes of irreducible representations of G*. In particular,* $\mathrm{Irr}(G)$ *is finite.*

PROOF.

(i) By Maschke's theorem (or more precisely Theorem 5.12), we can decompose $\mathbb{C}G$ as a direct sum of irreducibles. By Lemma 5.14, an isomorphism class of an irreducible representations ρ appears in the decomposition $\dim \mathrm{Hom}_G(\mathbb{C}G, \rho)$ times. By Theorem 6.5, $\dim \mathrm{Hom}_G(\mathbb{C}G, \rho) = \dim \rho$.

(ii) This follows from equating dimensions on both sides of the first part and noting that $\dim \mathbb{C}G = |G|$. ☐

6.2. Exercises.

Problem 37. Verify that the sum of squares formula holds for dihedral groups.

Problem 38. Fill in the following details from the proof of Theorem 6.5:

(a) Verify that $\Phi : \mathrm{Hom}_G(\mathbb{C}G, V) \to V$ is linear.

(b) Verify that $\Psi : V \to \mathrm{Hom}(\mathbb{C}G, V)$ is linear.

(c) Show that for every $\mathbf{v} \in V$, $\Psi(\mathbf{v})$ is a G-homomorphism, that is

$$\Psi(\mathbf{v})\lambda(g) = \pi(g)\Psi(\mathbf{v})$$

(hence $\Psi(\mathbf{v}) \in \mathrm{Hom}_G(\mathbb{C}G, V)$).

Problem 39. Decompose $(\lambda, \mathbb{C}(D_3))$ into irreducible subrepresentations.

Problem 40. Show the following:

THEOREM 6.7. *If every irreducible representation of a finite group* G *is 1-dimensional, then* G *is Abelian.*

Hint: Consider the representation $(\lambda, \mathbb{C}G)$*. Show that* $\ker \lambda = \{e\}$*, and thus* G *is isomorphic to* $\lambda(G) < \mathrm{GL}(\mathbb{C}G)$*. Use Theorem 6.6 to show that there is a basis of* $\mathbb{C}G$ *for which the matrices of all* $\lambda(g)$ *are diagonal, hence* $\lambda(G)$ *is Abelian.*

Remark: Combining this with Proposition 3.1 and Theorem 4.3 gives the following nice result:

THEOREM 6.8. *A finite group* G *is Abelian if and only if all its irreducible representations are 1-dimensional.*

Problem 41. Let A be an algebra over $\mathbb{C}$. An A-*module*, or *algebra representation* of A is a pair (π, V), where V is a $\mathbb{C}$-vector space and $\pi \in \mathrm{Hom}(A, \mathrm{Hom}(V))$ is such that

$$\pi(\mathbf{ab}) = \pi(\mathbf{a})\pi(\mathbf{b}) \qquad \forall\, \mathbf{a}, \mathbf{b} \in A.$$

In an analogous manner to group representations, a subspace $W \subseteq V$ is said to define a *submodule* (π, W) of (π, V) if $\pi(\mathbf{a})\mathbf{w} \in W$ for all $\mathbf{a} \in A$, $\mathbf{w} \in W$, and (π, V) is said to be *irreducible* if it has no non-trivial submodules.

(a) Show Schur's lemma for irreducible algebra-modules: if (π, V) is a finite dimensional irreducible algebra module and $T \in \mathrm{Hom}(V)$ is such that

$$\pi(\mathbf{a})T = T\pi(\mathbf{a})$$

for all $\mathbf{a} \in A$, then $T \in \mathbb{C}\,\mathrm{Id}$.

(b) Show that (π, V) is an irreducible representation of a group G if and only if (π, V) is an irreducible $\mathbb{C}G$-module.

Problem 42. Let A be an algebra, and define $Z(A) = \{\mathbf{a} \in A \,|\, \mathbf{ab} = \mathbf{ba}\ \forall \mathbf{b} \in A\}$. This set is called the *centre* of A.

(a) Show that $Z(A)$ is a subalgebra of A.

(b) Compute $Z(\mathbb{C}S_3)$.

(c) Show that if (π, V) is an irreducible A-module, then there exists a homomorphism of commutative algebras $\psi : Z(A) \to \mathbb{C}$ such that

$$\pi(\mathbf{z}) = \psi(\mathbf{z})\,\mathrm{Id}$$

for all $\mathbf{z} \in Z(A)$.

CHAPTER 2

Character Theory

7. Lecture 7

7.1. Characters. Throughout this section, G is a finite group and V is a finite-dimensional complex vector space.

DEFINITION 7.1. *Let (π, V) be a finite-dimensional complex representation of G. The* <u>character</u> *of (π, V) is the function $\chi_\pi : G \to \mathbb{C}$ defined by*

$$\chi_\pi(g) := \operatorname{tr}\big(\pi(g)\big).$$

REMARK 7.2. *Here we define* $\operatorname{tr} A$ *($A : V \to V$) as the sum of the eigenvalues of the matrix A (with multiplicities). It is a theorem from linear algebra that this is equal to the sum of the diagonal elements of the matrix of A with respect to* any *choice of basis of V*

EXAMPLE 7.3. *Let (Id, V) be the trivial representation of G on V. Then $\chi_{\mathrm{Id}}(g) = \dim V$ for all $g \in G$.*

EXAMPLE 7.4. *Let $(\rho, \mathbb{C}^2)$ be the defining representation of D_n. Then*

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(r) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix},$$

hence $\chi_\rho(e) = 2$, $\chi_\rho(s) = 0$, $\chi_\rho(r) = 2\cos(2\pi/n)$.

Important: $\chi_\pi(gh) \neq \chi_\pi(g)\chi_\pi(h)$! This is easily seen from the previous example, as $s^2 = e$.

The following lemma is quite important:

LEMMA 7.5. *Isomorphic representations have the same character.*

PROOF. By Problem 25, if π and ρ are isomorphic, then $\pi(g)$ and $\rho(g)$ have the same eigenvalues, hence also the same traces. □

It seems that we throw away a lot of information when we go from studying a representation to studying its character; the remarkable thing is that, in fact, the character completely determines the representation. Moreover, there is a lot of structure to the characters and it is often possible to find all of the characters of a group, even when it is not clear how to construct the representations!

EXAMPLE 7.6. *If $(\psi, \mathbb{C})$ is a 1-dimensional representation, then ψ is its own character (the trace of a scalar is just itself).*

EXAMPLE 7.7. *Let $(\pi, \mathbb{C}^n)$ be the permutation representation of S_n. Then*

$$\chi_\pi(\sigma) = \#\big\{j \in \{1, 2, \ldots, n\} \mid \sigma(j) = j\big\}.$$

Proof. Given $\sigma \in S_n$, consider the matrix of $\pi(\sigma)$ with respect to the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The j-th column of this matrix is $\pi(\sigma)\mathbf{e}_j = \mathbf{e}_{\sigma(j)}$. Thus, this column will contribute $+1$ to the trace if it has a 1 in the j-th row (i.e. if $\sigma(j) = j$), and zero otherwise. □

By a similar argument:

Example 7.8. *Let $(\lambda, \mathbb{C}G)$ be the regular representation of G. Then*

$$\chi_\lambda(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $h_1, h_2, \ldots, h_{|G|}$ be an enumeration of the elements of G; this forms a basis of $\mathbb{C}G$. Then for any j, we have

$$\lambda(g)h_j = gh_j = h_k$$

for some $k \in \{1, \ldots, |G|\}$. The j-th column of the matrix of $\lambda(g)$ is the coordinate vector of h_k with respect to the choice of basis, i.e. it has a 1 in row k, and zeroes elsewhere. This gives a non-zero contribution to the trace (in in that case $+1$) if and only if h_k is on the diagonal, i.e. $h_k = h_j$. But $h_k = gh_j$, so this happens if and only if $g = e$. □

We collect several useful properties of characters in the following proposition:

Proposition 7.9. *Let (π, V) and (ρ, W) be finite-dimensional representations of a finite group G. Then*

(i) $\chi_\pi(e) = \dim V$.
(ii) $\chi_\pi(gh) = \chi_\pi(hg)$ *and* $\chi_\pi(hgh^{-1}) = \chi_\pi(g)$ *for all* $g, h \in G$.
(iii) $\chi_{\pi\oplus\rho} = \chi_\pi + \chi_\rho$
(iv) $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$ *for all* $g \in G$.

Proof.

(i) $\chi_\pi(e) = \mathrm{tr}(\mathrm{Id}) = \dim V$.
(ii) This follows from the linear algebra identity $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
(iii) Let $\{\mathbf{v}_i\}_{i=1,\ldots,\dim V}$ be an an eigenbasis of V for $\pi(g)$ with eigenvalues λ_i, and $\{\mathbf{w}_j\}_{j=1,\ldots,\dim W}$ an eigenbasis of W for $\rho(g)$ with eigenvalues μ_j. Then $\{(\mathbf{v}_i, \mathbf{0})\}_i \cup \{(\mathbf{0}, \mathbf{w}_j)\}_j$ is a basis of $V \oplus W$. This is in fact an eigenbasis for $(\pi \oplus \rho)(g)$, since

$$(\pi \oplus \rho)(g)(\mathbf{v}_i, \mathbf{0}) = (\pi(g)\mathbf{v}_i, \rho(g)\mathbf{0}) = (\lambda_i \mathbf{v}_i, \mathbf{0}) = \lambda_i(\mathbf{v}_i, \mathbf{0}),$$

and similarly $(\pi \oplus \rho)(g)(\mathbf{0}, \mathbf{w}_j) = \mu_j(\mathbf{0}, \mathbf{w}_j)$. All the eigenvalues of $(\pi \oplus \rho)(g)$ are thus given by $\{\lambda_i\} \cup \{\mu_j\}$, hence

$$\chi_{\pi\oplus\rho}(g) = \sum_i \lambda_i + \sum_j \mu_j = \chi_\pi(g) + \chi_\rho(g).$$

(iv) This follows from the fact that for any eigenvector $\mathbf{v}_i$ of $\pi(g)$ with eigenvalue λ_i, we have

$$\mathbf{v}_i = \pi(e)\mathbf{v}_i = \pi(g^{-1})\pi(g)\mathbf{v}_i = \lambda_i \pi(g^{-1})\mathbf{v}_i,$$

so $\pi(g^{-1})\mathbf{v}_i = \lambda_i^{-1}\mathbf{v}_i$. Since $\pi(g)$ has finite order, $\lambda_i^{|G|} = 1$, giving $\lambda_i^{-1} = \overline{\lambda_i}$. This then yields

$$\chi_\pi(g^{-1}) = \sum_i \lambda_i^{-1} = \sum_i \overline{\lambda_i} = \overline{\sum_i \lambda_i} = \overline{\chi_\pi(g)}.$$

□

7.2. Exercises.

Problem 43. Let (π, W_0) denote the usual $(n-1)$-dimensional irreducible subrepresentation of the permutation representation of S_n on $\mathbb{C}^n$. Compute the character $\chi_{(\pi,W_0)}$ of this subrepresentation.

Hint: combine Example 7.7 with Proposition 7.9 (iii).

Problem 44. Let $g \in G$ be such that g and g^{-1} are both in the same conjugacy class. Show that $\chi_\pi(g)$ is real-valued for any representation π of G.

Problem 45. Let (π, V) be a representation of a finite group G. Show that

(a) $|\chi_\pi(g)| \leq \dim(\pi) \quad \forall g \in G$.

(b) $\ker \pi = \chi_\pi^{-1}\big(\{\dim(\pi)\}\big)$.

Problem 46. Let the finite group G act on the finite set X, and consider the representation $\big(\pi, \mathbb{C}(X)\big)$ as in Definition 1.9. Show that

$$\chi_\pi(g) = \#\{x \in X \mid g \cdot x = x\}.$$

8. Lecture 8

8.1. Irreducible characters and the character table.

Definition 8.1. *A character is called* irreducible *if the representation corresponding to it is irreducible.*

Recall from Proposition 7.9 *(ii)* that the character of a representation is constant on conjugacy classes. Observe also that by Proposition 7.9 *(iii)* and Theorem 5.12, every character may be written as an finite sum of irreducible characters. Thus, all the characters are completely determined by the values of the irreducible characters on each conjugacy class of G.

We often collect this data in a *character table.* This has columns labelled by the conjugacy classes of G, and rows labelled by the irreducible representations. The entries are the values of the characters of the irreducible representations on elements of the conjugacy class.

Example 8.2. *Here is the character table of S_3. We label each column by a representative element of the conjugacy class. It is also common, as here, to write the number of elements in the conjugacy class in the second row.*

class:	e	(12)	(123)
size:	1	3	2
triv	1	1	1
sgn	1	-1	1
π	2	0	-1

Example 8.3. *Let $G = C_n$ and $\omega = e^{2\pi i/n}$. Then we can write down the character table of C_n; we will do this for $n = 5$ for concreteness. Since G is Abelian, all conjugacy classes are singletons so we will omit the second row.*

Class	e	g	g^2	g^3	g^4
triv	1	1	1	1	1
χ	1	ω	ω^2	ω^3	ω^4
χ^2	1	ω^2	ω^4	ω	ω^3
χ^3	1	ω^3	ω	ω^4	ω^2
χ^4	1	ω^4	ω^3	ω^2	ω

Remark 8.4. *It is standard practice to put the identity conjugacy class and the trivial representation in the the first column and row respectively.*

From these two examples, we conjecture:

Theorem 8.5. *The character table is square, i.e. the number of (isomorphism classes) of irreducible representations of a finite group is equal to the number of conjugacy classes.*

This will follow from a stronger theorem, which is the main result of this chapter.

8.2. Class functions.

Definition 8.6. *A* class function *is a function $G \to \mathbb{C}$ that is constant on conjugacy classes. The set of all ($\mathbb{C}$-valued) class functions on a group G is denoted $CF(G)$.*

Observe that $CF(G)$ is a vector space. It has dimension equal to the number of conjugacy classes of G (since the indicator functions of the conjugacy classes form a basis of $CF(G)$), and a natural inner product

$$\langle f_1, f_2\rangle_G := \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)} = \frac{1}{|G|}\sum_{\text{conj. classes } \mathcal{C}\subseteq G} |\mathcal{C}| f_1(\mathcal{C})\overline{f_2(\mathcal{C})}$$

for all $f_1, f_2 \in CF(G)$.

THEOREM 8.7. *The set $\{\chi_\rho\}_{\rho\in\mathrm{Irr}(G)}$ is an orthonormal basis of $CF(G)$ with respect to the inner product $\langle\cdot,\cdot\rangle_G$.*

We will prove this over the course of the next two lectures. First we give a corollary to it.

COROLLARY 8.8. *Let $\mathcal{C}$ be a conjugacy class of G. We denote the column of the character table corresponding to $\mathcal{C}$ by $[\mathcal{C}]$, when considered as an element of $\mathbb{C}^{|\mathrm{Irr}(G)|}$. We have*

$$[\mathcal{C}]\cdot[\mathcal{D}] = \begin{cases} 0 & \text{if } \mathcal{C}\neq\mathcal{D}, \\ \frac{|G|}{|\mathcal{C}|} & \text{if } \mathcal{C}=\mathcal{D}. \end{cases}$$

PROOF. Let $\mathbb{1}_\mathcal{C}$ denote the indicator function of $\mathcal{C}$, i.e. $\mathbb{1}_\mathcal{C}(g) = 1$ if $g\in\mathcal{C}$, and zero otherwise. Since conjugacy classes are disjoint, we have

$$\langle \mathbb{1}_\mathcal{C}, \mathbb{1}_\mathcal{D}\rangle_G = \frac{1}{|G|}\sum_{g\in G}\mathbb{1}_\mathcal{C}(g)\overline{\mathbb{1}_\mathcal{D}(g)} = \begin{cases} 0 & \text{if } \mathcal{C}\neq\mathcal{D} \\ \frac{|\mathcal{C}|}{|G|} & \text{if } \mathcal{C}=\mathcal{D}. \end{cases}$$

By Theorem 8.7, the irreducible characters form an orthonormal basis of $CF(G)$, hence

$$\mathbb{1}_\mathcal{C}(g) = \sum_{\rho\in\mathrm{Irr}(G)}\langle\mathbb{1}_\mathcal{C},\chi_\rho\rangle_G\,\chi_\rho(g), \quad \mathbb{1}_\mathcal{D}(g) = \sum_{\sigma\in\mathrm{Irr}(G)}\langle\mathbb{1}_\mathcal{D},\chi_\sigma\rangle_G\,\chi_\sigma(g).$$

Substituting in these two expressions into the previous equality gives

$$\frac{1}{|G|}\sum_{g\in G}\sum_{\rho,\sigma\in\mathrm{Irr}(G)}\langle\mathbb{1}_\mathcal{C},\chi_\rho\rangle_G\,\chi_\rho(g)\overline{\langle\mathbb{1}_\mathcal{D},\chi_\sigma\rangle_G\,\chi_\sigma(g)} = \begin{cases} 0 & \text{if } \mathcal{C}\neq\mathcal{D} \\ \frac{|\mathcal{C}|}{|G|} & \text{if } \mathcal{C}=\mathcal{D}. \end{cases}$$

By swapping the order of summation,

$$\begin{aligned}&\frac{1}{|G|}\sum_{g\in G}\sum_{\rho,\sigma\in\mathrm{Irr}(G)}\langle\mathbb{1}_\mathcal{C},\chi_\rho\rangle_G\overline{\langle\mathbb{1}_\mathcal{D},\chi_\sigma\rangle_G\,\chi_\sigma(g)} \\ &\quad= \sum_{\rho,\sigma\in\mathrm{Irr}(G)}\langle\mathbb{1}_\mathcal{C},\chi_\rho\rangle_G\overline{\langle\mathbb{1}_\mathcal{D},\chi_\sigma\rangle_G}\frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\overline{\chi_\sigma(g)}.\end{aligned}$$

Again using Theorem 8.7, we have $\sum_{g\in G}\chi_\rho(g)\overline{\chi_\sigma(g)} = 0$ unless $\sigma=\rho$, in which case it equals one. This gives

$$\sum_{\rho\in\mathrm{Irr}(G)}\langle\mathbb{1}_\mathcal{C},\chi_\rho\rangle_G\overline{\langle\mathbb{1}_\mathcal{D},\chi_\rho\rangle_G} = \begin{cases} 0 & \text{if } \mathcal{C}\neq\mathcal{D} \\ \frac{|\mathcal{C}|}{|G|} & \text{if } \mathcal{C}=\mathcal{D}, \end{cases}$$

and the claim then follows from the identity $\langle \mathbb{1}_{\mathcal{C}}, \chi_\rho \rangle_G = \frac{|\mathcal{C}|}{|G|}\chi_\rho(\mathcal{C})$ and noting that

$$[\mathcal{C}] \cdot [\mathcal{D}] = \sum_{\rho \in \mathrm{Irr}(G)} \chi_\rho(\mathcal{C})\overline{\chi_\rho(\mathcal{D})}.$$

. □

8.3. Exercises.

Problem 47. Verify that Theorem 8.7 holds for the character tables of S_3, C_5, and D_4.

Problem 48. Compute the character tables of the following groups (You shouldn't need to use Theorem 8.7 for this):

(a) C_9
(b) $C_3 \times C_3$
(c) D_5
(d) Q_8
(e) D_n

(*You will need to find the conjugacy classes and irreducible representations for each of these groups. See the exercises from previous lectures!*)

Problem 49. Write the character table of a group G as a matrix T. Use Theorem 8.7 to show that

$$|\det(T)| = \sqrt{\prod_{\text{con. classes } \mathbb{C} \subseteq G} \frac{|G|}{|\mathcal{C}|}}.$$

9. Lecture 9

9.1. Schur Orthogonality. The proof of Theorem 8.7 is split into two parts. First, in this section we show that the irreducible characters form an orthonormal set. Later on we will show that this set spans $CF(G)$.

Given two representations π, ρ, we can create a new representation:

LEMMA 9.1. *Let (π, V) and (ρ, W) be two representations of a group G. The pair $\big(c_\pi^\rho, \mathrm{Hom}(V,W)\big)$ is a representation of G given by*

$$c_\pi^\rho(g)(T) := \rho(g)T\pi(g^{-1}),$$

for $T \in \mathrm{Hom}(V,W)$.

PROOF. This is left as an exercise (see Problem 50). □

LEMMA 9.2.

(i) $\mathrm{Hom}_G(V,W) = \{T \in \mathrm{Hom}(V,W) \mid c_\pi^\rho(g)T = T, \forall g \in G\}$.

(ii) $\chi_{c_\pi^\rho} = \chi_\rho \overline{\chi_\pi}$.

PROOF. Part *(i)* follows directly from the definition of $\mathrm{Hom}_G(V,W)$:

$$\begin{aligned} T \in \mathrm{Hom}_G(V,W) &\Leftrightarrow \rho(g)T = T\pi(g)\ \forall g \in G \\ &\Leftrightarrow \rho(g)T\pi(g^{-1}) = T\ \forall g \in G \\ &\Leftrightarrow c_\pi^\rho(g)T = T\ \forall g \in G, \end{aligned}$$

as claimed.

For part *(ii)*, given $g \in G$, let $\{\mathbf{v}_i\}$ be an eigenbasis of V for $\pi(g)$ (with $\pi(g)\mathbf{v}_i = \lambda_i \mathbf{v}_i$) and $\{\mathbf{w}_j\}$ an eigenbasis of W for $\rho(g)$ (with $\rho(g)\mathbf{w}_j = \mu_j \mathbf{w}_j$). Then $\{T_{i,j}\}$ forms a basis of $\mathrm{Hom}(V,W)$, where $T_{i,j}$ is the linear map defined by

$$T_{i,j}(\mathbf{v}_k) = \begin{cases} \mathbf{0} & \text{if } k \neq i, \\ \mathbf{w}_j & \text{if } k = i. \end{cases}$$

The set $\{T_{i,j}\}$ forms an eigenbasis of $\mathrm{Hom}(V,W)$ for $c_\pi^\rho(g)$:

$$\begin{aligned} \big(c_\pi^\rho(g)T_{i,j}\big)(\mathbf{v}_k) = \rho(g)T_{i,j}\big(\pi(g^{-1})\mathbf{v}_k\big) = \rho(g)T_{i,j}(\overline{\lambda_k}\mathbf{v}_k) &= \overline{\lambda_k}\rho(g)T_{i,j}(\mathbf{v}_k) \\ &= \begin{cases} \overline{\lambda_k}\rho(g)\mathbf{0} & \text{if } k \neq i, \\ \overline{\lambda_i}\rho(g)\mathbf{w}_j & \text{if } k = i \end{cases} \\ &= \begin{cases} \mathbf{0} & \text{if } k \neq i, \\ \overline{\lambda_i}\mu_j\mathbf{w}_j & \text{if } k = i \end{cases} = \overline{\lambda_i}\mu_j T_{i,j}(\mathbf{v}_k). \end{aligned}$$

Thus,

$$\chi_{c_\pi^\rho}(g) = \sum_{i,j} \overline{\lambda_i}\mu_j = \left(\sum_i \overline{\lambda_i}\right)\left(\sum_j \mu_j\right) = \chi_\rho(g)\overline{\chi_\pi(g)}.$$

□

We make another useful definition:

DEFINITION 9.3. *Let* (π, V) *be a representation of* G. *Define*

$$V^G = \{\mathbf{v} \in G \mid \pi(g)\mathbf{v} = \mathbf{v},\, \forall g \in G\},$$

to be the set of fixed points of (π, V).

Note that (π, V^G) is a subrepresentation of G, and in fact $(\pi, V^G) = (\mathrm{Id}, V^G)$.

LEMMA 9.4. *Let* (π, V) *be a representation of* G. *Then*

$$\dim(V^G) = \frac{1}{|G|}\sum_{g\in G}\chi_\pi(g).$$

PROOF. Let $\langle\cdot,\cdot\rangle$ be a G-invariant inner product on V. We write

$$(\pi, V) = (\pi, V^G) \oplus \big(\pi, (V^G)^\perp\big),$$

and let $\mathbf{v}_1, \ldots, \mathbf{v}_{\dim(V^G)}$ be a basis of V^G, and $\mathbf{w}_1, \ldots, \mathbf{w}_{\dim\left((V^G)^\perp\right)}$ a basis of $(V^G)^\perp$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_{\dim(V^G)}, \mathbf{w}_1, \ldots, \mathbf{w}_{\dim\left((V^G)^\perp\right)}$ is a basis of V, which we denote by $\mathcal{B}$.

Defining $S = \frac{1}{|G|}\sum_{h\in G}\pi(h) \in \mathrm{Hom}(V,V)$, observe that for any $\mathbf{v} \in V$, we have

$$\pi(g)S\mathbf{v} = \pi(g)\frac{1}{|G|}\sum_{h\in G}\pi(h)\mathbf{v} = \frac{1}{|G|}\sum_{h\in G}\pi(gh)\mathbf{v} = \frac{1}{|G|}\sum_{h\in G}\pi(h)\mathbf{v} = S\mathbf{v},$$

hence $S\mathbf{v} \in V^G$. Moreover, if $\mathbf{v} \in V^G$, then

$$S\mathbf{v} = \frac{1}{|G|}\sum_{h\in G}\pi(h)\mathbf{v} = \frac{1}{|G|}\sum_{h\in G}\mathbf{v} = \frac{1}{|G|}|G|\mathbf{v} = \mathbf{v},$$

hence $S|_{V^G} = \mathrm{Id}$. On the other hand, note that since $(\pi, (V^G)^\perp)$ is a subrepresentation, we have $\pi(g)\mathbf{v} \in (V^G)^\perp$ for all $\mathbf{v} \in (V^G)^\perp$ and $g \in G$, hence $S\mathbf{v} \in (V^G)^\perp$ for all $\mathbf{v} \in (V^G)^\perp$. Since $V^G \cap (V^G)^\perp = \{0\}$, $S|_{(V^G)^\perp} = 0$, so writing the matrix of S with respect to the basis $\mathcal{B}$ gives

$$[S]_{\mathcal{B}} = \mathrm{diag}\,\big(1, 1, \ldots, 1, 0, 0\ldots, 0\big),$$

with $\dim(V^G)$ ones, and $\dim\big((V^G)^\perp\big)$ zeroes. In conclusion,

$$\dim(V^G) = \mathrm{tr}\,\big([S]_{\mathcal{B}}\big) = \mathrm{tr}\left(\frac{1}{|G|}\sum_{h\in G}\pi(h)\right) = \frac{1}{|G|}\sum_{h\in G}\mathrm{tr}(\pi(h)) = \frac{1}{|G|}\sum_{h\in G}\chi_\pi(h),$$

as claimed. □

REMARK 9.5. *The main part of the proof of Lemma 9.4 shows that* $\frac{1}{|G|}\sum_{g\in G}\pi(g)$ *is an* orthogonal projection *onto* V^G*; this is an important result that we will generalise further next time.*

LEMMA 9.6. *Let* (π, V) *and* (ρ, W) *be two representations of a group* G. *Then*

$$\langle\chi_\rho, \chi_\pi\rangle_G = \dim \mathrm{Hom}_G(V, W).$$

PROOF. We will apply Lemma 9.4 to the representation $\big(c_\pi^\rho, \mathrm{Hom}(V,W)\big)$. By Lemma 9.2(i) and Lemma 9.4,

$$\dim\big(\mathrm{Hom}_G(V,W)\big) = \dim\big(\mathrm{Hom}(V,W)^G\big) = \frac{1}{|G|}\sum_{g\in G}\chi_{c_\pi^\rho}(g).$$

Now applying Lemma 9.2(ii), we have $\chi_{c^\rho_\pi}(g) = \chi_\rho(g)\overline{\chi_\pi(g)}$, and so

$$\dim\big(\operatorname{Hom}_G(V,W)\big) = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\overline{\chi_\pi(g)} = \langle \chi_\rho, \chi_\pi\rangle_G.$$

□

9.2. Consequences of the inner product formula. We collect a few important consequences of Lemma 9.6:

Proof of orthogonality in Theorem 8.7. By Schur's Lemma (Corollary 4.2) and Lemma 9.6: if π and ρ are irreducible representations of G, then

$$\langle \chi_\pi, \chi_\rho\rangle_G = \dim\big(\operatorname{Hom}_G(V,W)\big) = \begin{cases} 1 & \text{if } (\pi,V)\cong(\rho,W), \\ 0 & \text{otherwise.}\end{cases}$$

□

Theorem 9.7.

(i) Two representations are isomorphic if and only if they have the same character.
(ii) A representation (π, V) is irreducible if and only if

$$\|\chi_\pi\|_G^2 = \langle \chi_\pi, \chi_\pi\rangle_G = 1.$$

Proof. The fact that isomorphic representations have the same character has been proved already (Lemma 7.5), so for *(i)*, we need to show the other direction. Letting (π, V) be any representation of G, we can decompose π into irreducibles by Theorem 5.12:

$$(\pi, V) \cong \bigoplus_{\rho\in\operatorname{Irr}(G)} (\rho, W_\rho)^{n_\rho}, \tag{9.8}$$

where the n_ρ are non-negative integers. Writing $\chi_\pi = \sum_\rho n_\rho\chi_\rho$, for any $\sigma\in\operatorname{Irr}(G)$, we then have

$$\langle \chi_\pi, \chi_\sigma\rangle_G = \sum_{\rho\in\operatorname{Irr}(G)} n_\rho\langle\chi_\rho,\chi_\sigma\rangle_G = n_\sigma.$$

The representation π is thus completely determined by its character.

For part *(ii)*, by (9.8) we have that $\|\chi_\pi\|_G^2 = \sum_\rho n_\rho^2$, which (being the sum of squares of non-negative integers) is equal to 1 if and only if a single n_ρ equals one, and the rest are zero.

□

9.3. Exercises.

Problem 50. Verify that $\big(c^\rho_\pi, \operatorname{Hom}(V,W)\big)$ is a representation of G.

The next few problems give an alternative proof of Theorem 8.7 (actually a slightly stronger result is proved).

Problem 51. Let (π, V) be a finite-dimensional representation of a group G. Suppose that V is equipped with a G-invariant inner product $\langle\cdot,\cdot\rangle$. Show that

$$\chi_\pi(g) = \sum_{i=1}^{\dim V}\langle \pi(g)\mathbf{v}_i, \mathbf{v}_i\rangle,$$

where $\{\mathbf{v}_i\}_{i=1,\dots,\dim(V)}$ is any orthonormal basis for V with respect to the invariant inner product.

Remark: A function $\Phi : G \to \mathbb{C}$ of the form $\Phi(g) = \langle \pi(g)\mathbf{v}_1, \mathbf{v}_2\rangle$, where $\mathbf{v}_1, \mathbf{v}_2 \in V$ for some unitary representation (π, V) of G is called a matrix coefficient.

Problem 52. Let (π, V) and (ρ, W) be two representations of a group G. For $T \in \mathrm{Hom}(V, W)$, define T^G by

$$T^G = \frac{1}{|G|}\sum_{g\in G} \pi_2(g) T \pi_1(g^{-1}).$$

Show that $T^G \in \mathrm{Hom}_G(V, W)$.

Problem 53. Show that if (π, V) and (ρ, W) are two non-isomorphic irreducible unitary representations of G, then their matrix coefficients are orthogonal with respect to the standard L^2-inner product on $\mathbb{C}$-valued functions on G, i.e.

$$\frac{1}{|G|}\sum_{g\in G} \left\langle \pi(g)\mathbf{v}_1, \mathbf{v}_2\right\rangle_V \overline{\left\langle \rho(g)\mathbf{w}_1, \mathbf{w}_2\right\rangle_W} = 0 \qquad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V,\ \mathbf{w}_1, \mathbf{w}_2 \in W.$$

Hint: Define the map $T \in \mathrm{Hom}(V, W)$ by

$$T(v) = \langle \mathbf{v}, \mathbf{v}_1\rangle_V \mathbf{w}_1 \qquad \forall \mathbf{v} \in V.$$

Then use Problem 52 and Schur's lemma to study $\langle \mathbf{w}_2, T^G\mathbf{v}_2\rangle_W$.

Problem 54. Let (π, V) be an irreducible unitary representation of G. Show that

$$\frac{1}{|G|}\sum_{g\in G} \left\langle \pi(g)\mathbf{v}_1, \mathbf{v}_2\right\rangle_V \overline{\left\langle \pi(g)\mathbf{u}_1, \mathbf{u}_2\right\rangle_V} = \frac{\langle \mathbf{v}_1, \mathbf{u}_1\rangle_V \overline{\langle \mathbf{v}_2, \mathbf{u}_2\rangle_V}}{\dim V} \qquad \forall\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1, \mathbf{u}_2 \in V.$$

Hint: Define the map $T \in \mathrm{Hom}(V)$ by

$$T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}_1\rangle_V \mathbf{u}_1 \qquad \forall\, \mathbf{v} \in V.$$

Then compute $\langle \mathbf{u}_2, T^G(\mathbf{v}_2)\rangle_V$ using Problem 52 and Schur's lemma (compare with the proof of Lemma 9.4; show that $\operatorname{tr} T^G = \operatorname{tr} T$).

Problem 55. Use Problems 51 to 54 to give an alternative proof of the fact that the characters of irreducible representations form an orthonormal set in $CF(G)$.

The following problems give a generalisation of Fourier analysis to non-commutative finite groups:

Problem 56. For each $(\sigma, W_\sigma) \in \mathrm{Irr}(G)$, let $\{\mathbf{w}_{\sigma,i}\}_{i=1,\dots,\dim\sigma}$ be an orthonormal basis of W_σ with respect to an invariant inner product $\langle\cdot,\cdot\rangle_{W_\sigma}$. Show that $\{\Phi_{\sigma,i,j}\}_{\sigma\in\mathrm{Irr}(G), 1\le i,j\le\dim\sigma}$ is an orthonormal basis of $L^2(G) = \{f : G \to \mathbb{C}\}$ with respect to the inner product $\langle f_1, f_2\rangle_G = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$, where

$$\Phi_{\sigma,i,j}(g) := \sqrt{\dim\sigma}\langle \sigma(g)\mathbf{w}_{\sigma,i}, \mathbf{w}_{\sigma,j}\rangle_{W_\sigma}.$$

Problem 57. *(Challenging!)* For each $(\sigma, W_\sigma) \in \mathrm{Irr}(G)$ and $f \in L^2(G) = \{f : G \to \mathbb{C}\}$, define $\widehat{f}(\sigma) \in \mathrm{Hom}(W_\sigma)$ by

$$\widehat{f}(\sigma) = \frac{1}{|G|} \sum_{g \in G} f(g)\sigma(g^{-1}).$$

(a) Show that $f \mapsto \widehat{f}(\sigma)$ is an intertwining operator between $(\rho, L^2(G))$, and $\big(\widetilde{\sigma}, \mathrm{Hom}(W_\sigma)\big)$, where $(\rho(g)f)(h) = f(hg)$, $\forall g, h \in G$ $f \in L^2(G)$, and $\widetilde{\sigma(g)}(T) = \sigma(g)T$ for all $T \in \mathrm{Hom}(W_\sigma)$, $g \in G$.

(b) Show the Plancherel identity

$$\langle f_1, f_2 \rangle_G = \sum_{\sigma \in \mathrm{Irr}(G)} \dim \sigma \ \mathrm{tr}\left(\widehat{f_1}(\sigma)\widehat{f_2}(\sigma)^*\right),$$

and use it to deduce the Fourier inversion formula

$$f(g) = \sum_{\sigma \in \mathrm{Irr}(G)} \dim \sigma \ \mathrm{tr}\left(\widehat{f}(\sigma)\sigma(g)\right).$$

10. Lecture 10

In this lecture we will finally define the last of the machinery needed to finish the proof of Theorem 8.7, and then we will finish said proof.

10.1. Universal projections. Recall that for any representation (π, V), the sum of matrices $\frac{1}{|G|}\sum_{g\in G}\pi(g)$ acts on V as the orthogonal projection onto V^G, the subspace of vectors in V that are G-invariant, i.e. the subrepresentation of (π, V) that is isomorphic to $\dim(V^G)$ copies of the irreducible trivial representation. We now generalise this. To start with, recall that for any element of the group algebra $\sum_{g\in G} z_g g \in \mathbb{C}G$, we define

$$\pi\left(\sum_{g\in G} z_g g\right) := \sum_{g\in G} z_g \pi(g).$$

The space of $\mathbb{C}$-valued functions on G is isomorphic to $\mathbb{C}G$; one possible choice of isomorphism is

$$f \mapsto \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}g.$$

Composition then lets us define $\pi(f)$:

$$\pi(f) := \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\pi(g).$$

Lemma 10.1. *If $f \in CF(G)$, then $\pi(f) \in \mathrm{Hom}_G(V)$.*

Proof. This is Problem 58 below. □

Proposition 10.2. *Let $f \in CF(G)$ and (ρ, W) be an irreducible representation of G. Then*

$$\rho(f) = \frac{1}{\dim(W)}\langle\chi_\rho, f\rangle_G \,\mathrm{Id}.$$

Proof. By Schur's Lemma 4.1, $\mathrm{Hom}_G(W) = \mathbb{C}\,\mathrm{Id}$. For any $f \in CF(G)$, Lemma 10.1 then gives $\rho(f) = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$. We compute this λ by taking the trace of $\rho(f)$:

$$\lambda\dim(W) = \mathrm{tr}\left(\rho(f)\right) = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\,\mathrm{tr}\left(\rho(g)\right) = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\chi_\rho(g) = \langle\chi_\rho, f\rangle.$$

□

Proof of Theorem 8.7, continued: We want to prove that $\{\chi_\rho\}_{\rho\in\mathrm{Irr}(G)}$ spans the space $CF(G)$. To do this, we assume that $f \in CF(G)$ is orthogonal to all χ_ρ; and need to show that then $f = 0$.

Given any representation (π, V), we may decompose (π, V) into irreducible subrepresentations:

$$(\pi, V) = (\pi, W_1) \oplus \ldots \oplus (\pi, W_r).$$

Now, $\pi(f)$ preserves each irreducible component, hence $\pi(f) = \sum_i \pi(f)|_{W_i}$. By Proposition 10.2, $\pi(f)|_{W_i} = \frac{\langle\chi_{\rho_i}, f\rangle_G}{\dim(W_i)}\,\mathrm{Id}|_{W_i}$, where $\pi|_{W_i} \cong \rho_i$. By assumption, $\langle\chi_{\rho_i}, f\rangle_G = 0$ for all i, hence $\pi(f) = 0$

Thus: $\pi(f) = 0$ for *any* representation (π, V) of G. In particular, considering the regular representation $(\lambda, \mathbb{C}G)$, we have

$$\mathbf{0} = \lambda(f)(e) = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\lambda(g)e = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}g.$$

The set $\{g\}_{g\in G}$ forms a basis of $\mathbb{C}G$, so the linear combination is only zero if and only if $\overline{f(g)} = 0$ for all $g \in G$, i.e. $f = 0$. □

We collect two more useful facts regarding maps $\pi(f)$:

Corollary 10.3. *If (π, V) and (ρ, W) are two irreducible representations, then*

$$\pi(\chi_\rho) = \begin{cases} \frac{1}{\dim(V)}\,\mathrm{Id} & \text{if } (\pi, V) \cong (\rho, W),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. This follows directly from Proposition 10.2 and Theorem 8.7. □

Corollary 10.4. *Let (π, V) be a representation of G with decomposition into irreducible subrepresentations*

$$(\pi, V) = (\pi, W_1) \oplus (\pi, W_2) \oplus \ldots \oplus (\pi, W_r).$$

For each $\rho \in \mathrm{Irr}(G)$, define

$$(\pi, V_\rho) := \bigoplus_{\substack{i\\ \pi|_{W_i}\cong\rho}} (\pi, W_i).$$

Then $\dim(\rho)\,\pi(\chi_\rho)$ is a projection onto V_ρ.

Proof. This follows directly from the previous corollary. □

The subrepresentation (π, V_ρ) is called the ρ-isotopic component of π, and $\dim(\rho)\pi(\chi_\rho)$ is called the *ρ-isotopic projector*. Observe that since $\dim(\rho)\,\pi(\chi_\rho)$ is independent of the choice of decomposition of (π, V) into irreducibles, (π, V_ρ) must also be independent of this choice. This implies that the only choices in the decomposition are given by the different decompositions of each (π, V_ρ) into copies of ρ.

Example 10.5. *We consider the permutation representation $(\pi, \mathbb{C}^3)$ of S^3, and compute the matrices of the isotopic projectors for all the irreducible representations of S_3:*

$$\pi(\chi_{\mathrm{triv}}) = \frac{1}{6}\big(\pi(e) + \pi(12) + \pi(13) + \pi(23) + \pi(123) + \pi(132)\big) = \frac{1}{6}\begin{pmatrix} 2 & 2 & 2\\ 2 & 2 & 2\\ 2 & 2 & 2\end{pmatrix}.$$

$$\pi(\chi_{\mathrm{sgn}}) = \frac{1}{6}\big(\pi(e) - \pi(12) - \pi(13) - \pi(23) + \pi(123) + \pi(132)\big) = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

$$2\,\pi(\chi_{(\pi,W_0)}) = \frac{1}{3}\big(2\pi(e) - \pi(123) - \pi(132)\big) = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}.$$

10.2. Example: the character table of S_4.

EXAMPLE 10.6. *We determine the character table of S_4. First, we have the 1-dimensional trivial and sign representations:*

	e	(12)	$(12)(34)$	(123)	(1234)
	1	6	3	8	6
triv	1	1	1	1	1
sgn	1	-1	1	1	-1

Next, we have the 3-dimensional irreducible permutation representation (π, W_0) of S_4 on $\{1,2,3,4\}$, whose character χ_π satisfies

$$\chi_\pi(g) = \#\{\textit{fixed points of } g\} - 1.$$

Notice that we also have an operation on representations known as twisting*: if (ρ, V) is any irreducible representation of G and ψ is a 1-dimensional character of G, then we can define a new representation, $\psi\rho$, of G on V by the formula $(\psi\rho)(g) = \psi(g)\rho(g)$. It has character $\psi\chi_\rho$. In this case, we can look at the representation $(\mathrm{sgn}\pi, W_0)$ and see that it will also be irreducible:*

	e	(12)	$(12)(34)$	(123)	(1234)
	1	6	3	8	6
π	3	1	-1	0	-1
$\mathrm{sgn}\pi$	3	-1	-1	0	1

Since $\|\chi_\pi\|^2_{S_4} = \|\mathrm{sgn}\chi_\pi\|^2_{S_4} = 1$, these are both irreducible. The characters of the two representations are different, hence they are non-isomorphic.

There is one more row of the table to find. This can be done using orthonormality with the first column, this is Corollary 8.8. If we recall that the dimension must be a positive integer we can test candidate cases and determine that it must be 2. Thus we obtain that the full character table is

	e	(12)	$(12)(34)$	(123)	(1234)
	1	6	3	8	6
triv	1	1	1	1	1
sgn	1	-1	1	1	-1
π	3	1	-1	0	-1
$\mathrm{sgn}\pi$	3	-1	-1	0	1
ρ	2	0	2	-1	0

Notice that we constructed the character of the final representation without constructing the representation itself!

10.3. Exercises.

Problem 58. Prove Lemma 10.1.

Problem 59. Let (π, V) and (ρ, U) be two representations of a group G. Show that $(\pi, V) \cong (\rho, U)$ if and only if

$$\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}_G(U, W)$$

for all representations (σ, W) of G.

Problem 60. Let (π, V) be a representation of a group G and H a subgroup of G. Show that

$$\chi_{\pi|_H}(h) = \chi_\pi(h) \qquad \forall\, h \in H.$$

Problem 61. Find the character table of A_4. Decompose the restriction of each irreducible representation of S_4 to A_4 into irreducibles.

Remark: When we say "decompose a representation into irreducibles", we simply mean find out how many times each irreducible representation occurs in the decomposition. This is in contrast to if we were to ask for a "decomposition into irreducible subrepresentations", which involves finding irreducible subrepresentations of the given representation and showing that their direct sum is the whole representation - this is a much harder task.

Problem 62. Decompose $\mathbb{C}[Q_8]$ into irreducible isotopic components.

Problem 63. Let X be a finite set on which a group G acts. Consider the permutation representation $(\pi, \mathbb{C}(X))$ of G on the free vector space of X.

(a) Show that $\dim(\mathbb{C}(X)^G)$ (recall that $\mathbb{C}(X)^G = \{\mathbf{v} \in \mathbb{C}(X) \,|\, \pi(g)\mathbf{v} = \mathbf{v}\, \forall g \in G\}$) is equal to the number of G-orbits in X.

(b) Compute $\langle \chi_\pi, \chi_{\mathrm{Id}} \rangle_G$, and use it to show *Burnside's lemma:* the number of G-orbits in X is equal to

$$\frac{1}{|G|} \sum_{g \in G} \#\{x \in X \,|\, g \cdot x = x\},$$

i.e. the average number of fixed points of the group elements.

CHAPTER 3

Further constructions

11. Lecture 11

We now turn to various ways of constructing new representations out of old ones. The methods will split into two general classes:

(i) Linear algebraic
(ii) Group theoretic

We start with linear algebra:

11.1. The dual representation. Given a vector space V, recall that V^* (the *dual space* of V) is the vector space consisting of all linear functionals from V to $\mathbb{C}$, i.e. $V^* = \mathrm{Hom}(V, \mathbb{C})$. We define a representation (π^*, V^*) of G by letting (for every $g \in G$ and $\lambda \in V^*$) $\pi^*(g)\lambda$ be the functional defined via the formula

$$[\pi^*(g)\lambda](\mathbf{v}) := \lambda\big(\pi(g^{-1})\mathbf{v}\big) \qquad \forall \mathbf{v} \in V.$$

Note that this may be seen as a special case of some representations we have seen before:

(i) (π^*, V^*) is a subrepresentation of the permutation representation on the space of $\mathbb{C}$-valued functions on V that arises from the linear π-action of G on V.
(ii) Recall the representation $(c_\pi^\rho, \mathrm{Hom}(V, W))$ (cf. Lemma 9.1). The dual representation is the special case $(\rho, W) = (\mathrm{triv}, \mathbb{C})$.

Being a special case of c_π^ρ, Lemma 9.2 *(b)* gives $\chi_{\pi^*} = \overline{\chi_\pi}$. Problem 64 below gives another proof of this fact.

11.2. Quotient representations. Let (π, V) be a representation of a group G and $(\pi|_W, W)$ a subrepresentation of (π, V). Recall that the quotient space V/W is the vector space V modulo the equivalence relation $\mathbf{v} \sim \mathbf{w} \Leftrightarrow \mathbf{v} - \mathbf{w} \in W$. We normally write elements of V/W in the form $\mathbf{v} + W$. We define a representation $(\hat{\pi}, V/W)$ of G on V/W via the formula $\hat{\pi}(g)(\mathbf{v}+W) := \pi(g)\mathbf{v}+W$. Note that it is critical that (π, W) is a subrepresentation for this to make sense (i.e. $\pi(g)W = W$).

The following proposition will let us avoid using quotient representations too often:

Proposition 11.1. *If* $(\pi, V) = (\pi|_W, W) \oplus (\pi|_{W'}, W')$, *then*

$$(\hat{\pi}, V/W) \cong (\pi|_{W'}, W').$$

As a consequence of the above, we have

Proposition 11.2. - *Let* (π, V) *be a finite dimensional representation of a finite group* G *and* $(\pi|_W, W)$ *a subrepresentation with corresponding quotient representation* $(\hat{\pi}, V/W)$. *Then*

$$\chi_{\hat{\pi}} = \chi_\pi - \chi_{\pi|_W}.$$

The proofs of both these results are left as exercises below.

We won't need to use quotient representations too often, but they are required to make what we do next rigorous.

11.3. Tensor Products. We now define the *tensor product* of two representations; this is a way of 'multiplying' two or more representations. Throughout this section, let (π, V) and (ρ, W) be two representations of a group G.

Firstly, we consider the permutation representation of G on the free vector space $\mathbb{C}(V \times W)$; we write this as $\pi \otimes \rho$, i.e. the elements of $\mathbb{C}(V \times W)$ are finite formal sums

$$\sum_{(\mathbf{v},\mathbf{w}) \in V \times W} z_{(\mathbf{v},\mathbf{w})}(\mathbf{v}, \mathbf{w}),$$

for $z_{(\mathbf{v},\mathbf{w})} \in \mathbb{C}$, where all but finitely many $z_{(\mathbf{v},\mathbf{w})}$ are non-zero. Note that since $V \times W$ is an infinite set, $\mathbb{C}(V \times W)$ is an infinite-dimensional vector space, with basis given by all possible pairs $(\mathbf{v}, \mathbf{w}) \in V \times W$. The group G acts via the formula

$$[\pi \otimes \rho](g) \left(\sum_{(\mathbf{v},\mathbf{w}) \in V \times W} z_{(\mathbf{v},\mathbf{w})}(\mathbf{v}, \mathbf{w}) \right) = \sum_{(\mathbf{v},\mathbf{w}) \in V \times W} z_{(\mathbf{v},\mathbf{w})} \big(\pi(g)\mathbf{v}, \rho(g)\mathbf{w}\big).$$

This may seem like a sensibly defined representation however note that in $\mathbb{C}(V \times W)$, the vectors $(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w})$ and $(\mathbf{v}_1, \mathbf{w}) + (\mathbf{v}_2, \mathbf{w})$ are *different*. In fact we don not have any relations that allow us to split up brackets or pull out scalar factors. We therefore define the vector space

$$U = \operatorname{span}\Big\{ (\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \lambda\mathbf{w}_1 + \mu\mathbf{w}_2) - \alpha\lambda(\mathbf{v}_1, \mathbf{w}_1) - \alpha\mu(\mathbf{v}_1, \mathbf{w}_2)$$
$$-\beta\lambda(\mathbf{v}_2, \mathbf{w}_1) - \beta\mu(\mathbf{u}_2, \mathbf{w}_2) \,|\, \mathbf{v}_1, \mathbf{v}_2 \in V,\ \mathbf{w}_1, \mathbf{w}_2 \in W,\ \alpha, \beta, \lambda, \mu \in \mathbb{C} \Big\}.$$

We claim that $(\pi \otimes \rho, U)$ is a subrepresentation of $\big(\pi \otimes \rho, \mathbb{C}(V \times W)\big)$, and define the *tensor product representation* as the quotient representation

$$\big(\pi \otimes \rho, V \otimes W\big) := \big(\pi \otimes \rho, \mathbb{C}(V \times W)/U\big).$$

Taking the quotient space by U allows us to identify vectors that differ by elements of U. If we consider our example notice that they differ by $(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) - (\mathbf{v}_1, \mathbf{w}) - (\mathbf{v}_2, \mathbf{w})$ which is one of the spanning elements of U, hence

$$(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) + U = (\mathbf{v}_1, \mathbf{w}) + (\mathbf{v}_2, \mathbf{w}) + U$$

as elements of $V \otimes W$. The same holds for the other linear relations in both components of $\mathbb{C}(V \times W)$. In other words, modding out the subspace U is a way of forcing desired linear relations on the space $\mathbb{C}(V \times W)$.

If $\mathbf{v} \in V$ and $\mathbf{w} \in W$, then we write $\mathbf{v} \otimes \mathbf{w}$ for the element $(\mathbf{v}, \mathbf{w}) + U$ in $V \otimes W$.

Important: Not every element of $V \times W$ may be written in the form $\mathbf{v} \otimes \mathbf{w}$ (see Problem 69 below). However, the vectors $\mathbf{v} \otimes \mathbf{w}$ do span $V \otimes W$, and the following proposition gives a convenient way of finding a basis from this spanning set:

Proposition 11.3. *Let V and W be finite-dimensional vector spaces, with $\mathbf{v}_1, \ldots, \mathbf{v}_n$ being a basis of V and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ being a basis of W. Then*

$$\{\mathbf{v}_i \otimes \mathbf{w}_j \,|\, 1 \leq i \leq n,\ 1 \leq j \leq m\}$$

is a basis of $V \otimes W$. In particular, $\dim(V \otimes W) = \dim(V)\dim(W)$.

Example 11.4. *The above basically says $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$.*

Remark 11.5. *An element in $V \otimes W$ of the form $\mathbf{v} \otimes \mathbf{w}$ ($\mathbf{v} \in V$, $\mathbf{w} \in W$) is called a* pure tensor.

The previous proposition can be used to find the characters of tensor products of representations:

Proposition 11.6. $\chi_{\pi\otimes\rho} = \chi_\pi \chi_\rho$

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an eigenbasis of V for $\pi(g)$ ($\pi(g)\mathbf{v}_i = \lambda_i \mathbf{v}_i$) and $\mathbf{w}_1, \ldots, \mathbf{v}_m$ be an eigenbasis of W for $\rho(g)$ $\rho(g)\mathbf{w}_j = \mu_j \mathbf{w}_j$). Then from the Proposition 11.3, $\{\mathbf{v}_i \otimes \mathbf{w}_j\}$ is a basis of $V \otimes W$, and is in fact an eigenbasis for $[\pi \otimes \rho](g)$, since

$$[\pi \otimes \rho](g)\mathbf{v}_i \otimes \mathbf{w}_j = \big(\pi(g)\mathbf{v}_i\big) \otimes \big(\rho(g)\mathbf{w}_j\big) = (\lambda_i \mathbf{v}_i) \otimes (\mu_j \mathbf{w}_j) = \lambda_i \mu_j (\mathbf{v}_i \otimes \mathbf{w}_j).$$

This gives $\chi_{\pi\otimes\rho}(g) = \sum_{ij} \lambda_i \mu_j = (\sum_i \lambda_i)\left(\sum_j \mu_j\right) = \chi_\pi(g)\chi_\mu(g)$. □

Example 11.7. *From Lemma 9.2 (b), we have that the character of $(c_\pi^\rho, \mathrm{Hom}(V, W))$ is $\chi_\rho \overline{\chi_\pi}$. From Proposition 11.6 and Problem 64, we then get*

$$\big(c_\pi^\rho, \mathrm{Hom}(V, W)\big) \cong (\rho, W) \otimes (\pi^*, V^*).$$

11.4. Exercises.

Problem 64. Let (π, V) be a finite-dimensional representation of a finite group G with dual representation (π^*, V^*).

(a) Let $\mathcal{B} = \{\mathbf{v}_i\}_{i=1,\ldots,\dim V}$ be a basis of V and $\mathcal{A} = \{\lambda_i\}_{i=1,\ldots,\dim V^*}$ the corresponding dual basis of V^*, i.e.

$$\lambda_i(\mathbf{v}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Show that $[\pi^*(g)]_\mathcal{A} = [\pi(g^{-1})]_\mathcal{B}^T$ and $\chi_{\pi^*} = \overline{\chi_\pi}$.

(b) Show that (π^*, V^*) is irreducible if and only if (π, V) is.

Problem 65. Let (π, V) be a representation of a group G and (π, W) a subrepresentation of (π, V). Verify that defining

$$\pi(g)(\mathbf{v} + W) := \pi(g)\mathbf{v} + W \qquad \forall\ \mathbf{v} + W \in V/W,\ g \in G$$

gives a G-representation $(\pi, V/W)$.

Problem 66. Prove Propositions 11.1 and 11.2.

Problem 67. Consider the representation of S_3 on (ρ, V), where $V = \mathbb{C}^2$, given by

$$\rho(1\ 2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(1\ 2\ 3) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$$

with $\omega = e^{2\pi i/3}$.

(a) Write down the matrices of $(\rho \otimes \rho)(1\ 2)$ and $(\rho \otimes \rho)(1\ 2\ 3)$ with respect to the basis

$$\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2, \mathbf{e}_2 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2$$

of $V \otimes V$ (where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard basis of V).
(b) Write the character of $(\rho, V) \otimes (\rho, V)$ as a sum of irreducible characters.
(c) For each of the irreducible characters occurring in the expression of $\chi_{\rho\otimes\rho}$ used in the previous part, find a subrepresentation of $V \otimes V$ with that character.
(d) Find an isomorphism from $(\mathrm{sgn}, \mathbb{C}) \otimes (\rho, V)$ to (ρ, V).
(e) Find an isomorphism from (ρ^*, V^*) to (ρ, V).
(f) Let $(\rho^{\otimes n}, V^{\otimes n}) = (\rho, V) \otimes \ldots \otimes (\rho, V)$ with n factors. Decompose $(\rho^{\otimes n}, V^{\otimes n})$ into irreducible representations.

Problem 68. Let (π, V), (ρ, U), and (σ, W) be three finite-dimensional representations of a finite group G. Show that
(a) $\big((\pi, V) \otimes (\rho, U)\big) \otimes (\sigma, W) \cong (\pi, V) \otimes \big((\rho, U) \otimes (\sigma, W)\big)$
(b) $\big((\pi, V) \oplus (\rho, U)\big) \otimes (\sigma, W) \cong \big((\rho, U) \otimes (\sigma, W)\big) \oplus \big((\pi, V) \otimes (\sigma, W)\big)$

Remark 11.8. *Note that (a) lets us write $(\pi, V) \otimes (\rho, U) \otimes (\sigma, W)$ unambiguously (and similarly for larger products of representations).*

Problem 69. Let $\mathbf{e}, \mathbf{f}$ form a basis of $\mathbb{C}^2$. Show that $\mathbf{e} \otimes \mathbf{e} + \mathbf{f} \otimes \mathbf{f} \in \mathbb{C}^2 \otimes \mathbb{C}^2$ cannot be written in the form $\mathbf{v} \otimes \mathbf{w} \in \mathbb{C}^2 \otimes \mathbb{C}^2$.

Problem 70. For two finite-dimensional representations (π, V) and (ρ, W) of a finite group G, construct an isomorphism to show that

$$\big(c_\pi^\rho, \mathrm{Hom}(V, W)\big) \cong (\pi^*, V^*) \otimes (\rho, W).$$

Problem 71. Given a finite group G, let $\big(\pi, \mathbb{C}(G)\big)$ denote the permutation representation of G on $\mathbb{C}(G)$ associated to the conjugation action of G on itself, i.e.

$$\pi(g)\left(\sum_{h\in G} z_h h\right) = \sum_{h\in G} z_h g h g^{-1} = \sum_{h\in G} z_{g^{-1}hg} h \qquad \forall g \in G,\ \sum_{h\in G} z_h h \in \mathbb{C}(G).$$

(a) Show that $\chi_\pi(g) = \frac{|G|}{|\mathcal{C}_g|}$, where $\mathcal{C}_g$ is the conjugacy class g belongs to. *Hint: use Problem 46.*
(b) Then show that

$$\big(\pi, \mathbb{C}(G)\big) \cong \bigoplus_{\rho\in\mathrm{Irr}(G)} (\rho, W_\rho) \otimes (\rho^*, W_\rho^*).$$

12. Lecture 12

12.1. Symmetric and alternating products. We defined the tensor product $V \otimes V$ as a quotient $\mathbb{C}(V \times V)/U$, where the subspace U is chosen to give the desired linear relations in $V \otimes V$. We will now do something similar to $V \otimes V$, to obtain vector spaces that have even more symmetries.

DEFINITION 12.1. *Let $U_1 \subseteq V \otimes V$ be the subspace*

$$U_1 = \operatorname{span}\{\mathbf{v}_1 \otimes \mathbf{v}_2 - \mathbf{v}_2 \otimes \mathbf{v}_1 \,|\, \mathbf{v}_1, \mathbf{v}_2 \in V\},$$

and $\operatorname{Sym}^2(V) := (V \otimes V)/U_1$. The vector space $\operatorname{Sym}^2(V)$ is called the (second) symmetric product of V.

If $\mathbf{v}_1 \otimes \mathbf{v}_2 \in V \otimes V$, then

$$\mathbf{v}_1 \otimes \mathbf{v}_2 + U_1 = \mathbf{v}_2 \otimes \mathbf{v}_1 + \big(\mathbf{v}_1 \otimes \mathbf{v}_2 - \mathbf{v}_2 \otimes \mathbf{v}_1\big) + U_1 = \mathbf{v}_2 \otimes \mathbf{v}_1 + U_1. \tag{12.2}$$

Thus, modding out the subspace U_1 enforces the relation "$\mathbf{v}_1 \otimes \mathbf{v}_2 = \mathbf{v}_2 \otimes \mathbf{v}_1$" on $V \otimes V$. We normally write $\mathbf{v}_1\mathbf{v}_2$ for the element $\mathbf{v}_1 \otimes \mathbf{v}_2 + U_1$ of $\operatorname{Sym}^2(V)$. We have $\mathbf{v}_1\mathbf{v}_2 = \mathbf{v}_2\mathbf{v}_1$, as well as linear relations in both arguments. As a consequence we have that, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis of V then

$$\{\mathbf{v}_i\mathbf{v}_j \,|\, 1 \le i \le n,\ 1 \le j \le i\}$$

is a basis of $\operatorname{Sym}^2(V)$. Thus $\dim\big(\operatorname{Sym}^2(V)\big) = \frac{n(n+1)}{2}$.

In a similar fashion, we will also define a subspace U_2 of $V \otimes V$ that enforces the relation "$\mathbf{v}_1 \otimes \mathbf{v}_2 = -\mathbf{v}_2 \otimes \mathbf{v}_1$" on $V \otimes V$:

DEFINITION 12.3. *Let $U_2 \subseteq V \otimes V$ be the subspace*

$$U_2 := \operatorname{span}\{\mathbf{v}_1 \otimes \mathbf{v}_2 + \mathbf{v}_2 \otimes \mathbf{v}_1 \,|\, \mathbf{v}_1, \mathbf{v}_2 \in V\},$$

and $\bigwedge^2 V := (V \otimes V)/U_2$. The vector space $\bigwedge^2(V)$ is called the (second) alternating product of V.

By the same method as the computation 12.2 above, we have $\mathbf{v}_1 \otimes \mathbf{v}_2 + U_2 = -\mathbf{v}_2 \otimes \mathbf{v}_1 + U_2$. We normally write $\mathbf{v}_1 \wedge \mathbf{v}_2$ for $\mathbf{v}_1 \otimes \mathbf{v}_2 + U_2$; and $\mathbf{v}_1 \wedge \mathbf{v}_2$ is linear both arguments, and $\mathbf{v}_1 \wedge \mathbf{v}_2 = -\mathbf{v}_2 \wedge \mathbf{v}_1$. Thus, if we let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V, we have

$$\{\mathbf{v}_i \wedge \mathbf{v}_j \,|\, 1 \le i \le n,\ 1 \le j < i\}$$

is a basis of $\bigwedge^2(V)$. Thus $\dim\big(\bigwedge^2(V)\big) = \frac{n(n-1)}{2}$.

We now look at representations on these spaces:

PROPOSITION 12.4. *Let (π, V) be a representation of a group G. Then $(\pi \otimes \pi, U_1)$ and $(\pi \otimes \pi, U_2)$ are subrepresentations of $(\pi \otimes \pi, V \otimes V)$, and*

$$(\pi \otimes \pi, V \otimes V) \cong \big(\operatorname{Sym}^2\pi, \operatorname{Sym}^2(V)\big) \oplus \big(\pi \wedge \pi, \textstyle\bigwedge^2(V)\big).$$

PROOF. See Problem 72 below. $\square$

PROPOSITION 12.5. *Let (π, V) be a finite-dimensional representation of a finite group G. Then for all $g \in G$,*

$$\chi_{\operatorname{Sym}^2\pi}(g) = \frac{1}{2}\big(\chi_\pi(g)^2 + \chi_\pi(g^2)\big), \qquad \chi_{\pi\wedge\pi}(g) = \frac{1}{2}\big(\chi_\pi(g)^2 - \chi_\pi(g^2)\big).$$

Proof. We only need to prove the first of these identities, the other follows from Proposition 12.4 and Proposition 7.9 *(iii)*.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an eigenbasis of V for $\pi(g)$, i.e. $\pi(g)\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Then

$$\{\mathbf{v}_i\mathbf{v}_j \mid 1 \leq i \leq n,\ 1 \leq j \leq i\}$$

is a basis of $\mathrm{Sym}^2(V)$. This is an eigenbasis for $\mathrm{Sym}^2\pi(g)$ as

$$\mathrm{Sym}^2\pi(g)\mathbf{v}_i\mathbf{v}_j = (\lambda_i\mathbf{v}_i)(\lambda_j\mathbf{v}_j) = (\lambda_i\lambda_j)\mathbf{v}_i\mathbf{v}_j.$$

The definition of $\chi_{\mathrm{Sym}^2\pi}(g)$ gives

$$\chi_{\mathrm{Sym}^2\pi}(g) = \sum_{i=1}^{n}\sum_{j=1}^{i}\lambda_i\lambda_j.$$

On the other hand,

$$\begin{aligned}\chi_\pi(g)^2 = \left(\sum_{k=1}^{n}\lambda_k\right)^2 &= \sum_{i=1}^{n}\lambda_i^2 + 2\sum_{i=1}^{n}\sum_{1\leq j<i}\lambda_i\lambda_j \\ &= \chi_\pi(g^2) + 2\left(\sum_{i=1}^{n}\sum_{1\leq j\leq i}\lambda_i\lambda_j - \sum_{i=1}^{n}\lambda_i^2\right) \\ &= 2\chi_{\mathrm{Sym}^2\pi}(g) - \chi_\pi(g^2).\end{aligned}$$

□

We conclude this section by noting that we can generalise these constructions to higher symmetric and alternating powers $\mathrm{Sym}^k(V)$, $\bigwedge^k V$ for $k = 3, 4, \ldots$.

12.2. Example: the character table of S_5. We will now complete the character table of S_5. The irreducibles we already know are the trivial representation triv, the sign representation sgn, the irreducible permutation representation (π, W_0), and its twist $(\mathrm{sgn}\,\pi, W_0)$, as before. So we can put these entries into the table:

	e	(12)	$(12)(34)$	(123)	$(123)(45)$	(1234)	(12345)
	1	10	15	20	20	30	24
triv	1	1	1	1	1	1	1
sgn	1	-1	1	1	-1	-1	1
π	4	2	0	1	-1	0	-1
$\mathrm{sgn}\,\pi$	4	-2	0	1	1	0	-1

We then try $\pi \wedge \pi$, which has character as shown below. This is an irreducible character (as it has norm one).

	e	(12)	$(12)(34)$	(123)	$(123)(45)$	(1234)	(12345)
	1	10	15	20	20	30	24
$\pi \wedge \pi$	6	0	-2	0	0	0	1

A natural next step could be to try $\mathrm{sgn}(\pi \wedge \pi)$, but it turns out this is isomorphic to $\pi \wedge \pi$. We can also try $\mathrm{Sym}^2\,\pi$, which has character below; it isn't irreducible.

	e	(12)	$(12)(34)$	(123)	$(123)(45)$	(1234)	(12345)
	1	10	15	20	20	30	24
$\mathrm{Sym}^2\,\pi$	10	4	2	1	1	0	0

	e	(12)	$(12)(34)$	(123)	$(123)(45)$	(1234)	(12345)
	1	10	15	20	20	30	24
triv	1	1	1	1	1	1	1
sgn	1	-1	1	1	-1	-1	1
π	4	2	0	1	-1	0	-1
$\operatorname{sgn}\pi$	4	-2	0	1	1	0	-1
$\pi\wedge\pi$	6	0	-2	0	0	0	1
ρ	5	1	1	-1	1	-1	0
$\operatorname{sgn}\rho$	5	-1	1	-1	-1	1	0

TABLE 1. Character table of S_5

By taking inner products with the characters we've already found, we see that

$$\mathrm{Sym}^2\pi \cong \mathrm{triv} \oplus \pi \oplus \rho,$$

where ρ is another irreducible representation. Finally, we get one more from twisting ρ by sgn.

	e	(12)	$(12)(34)$	(123)	$(123)(45)$	(1234)	(12345)
	1	10	15	20	20	30	24
ρ	5	1	1	-1	1	-1	0
$\operatorname{sgn}\rho$	5	-1	1	-1	-1	1	0

This gives all of the irreducible characters, which we assemble into Table 1.

12.3. Exercises.

Problem 72. Let (π, V) be a finite-dimensional representation of a finite group G. Define $U_1, U_2 \subseteq V \otimes V$ by

$$U_1 = \mathrm{span}\{\mathbf{v}_1 \otimes \mathbf{v}_2 - \mathbf{v}_2 \otimes \mathbf{v}_1 \mid \mathbf{v}_1, \mathbf{v}_2 \in V\}, \quad U_2 = \mathrm{span}\{\mathbf{v}_1 \otimes \mathbf{v}_2 + \mathbf{v}_2 \otimes \mathbf{v}_1 \mid \mathbf{v}_1, \mathbf{v}_2 \in V\}$$

Show that $(\pi\otimes\pi, U_1)$ and $(\pi\otimes\pi, U_2)$ are subrepresentations of $(\pi\otimes\pi, V\otimes V)$ and thus show that

$$(\pi\otimes\pi, V\otimes V) \cong \big(\mathrm{Sym}^2\pi, \mathrm{Sym}^2(V)\big) \oplus \big(\pi\wedge\pi, \textstyle\bigwedge^2 V\big).$$

.

Problem 73. Let $(\pi, \mathbb{C}^2)$ be a representation of a group G. Suppose that the matrix of $\pi(g)$ with respect to the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ is given by

$$\pi(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(a) Compute the matrix of $\mathrm{Sym}^2\pi(g)$ with respect to the basis $\{\mathbf{b}_1^2, \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_2^2\}$.
(b) Compute the matrix of $\pi\wedge\pi(g)$ with respect to the basis $\{\mathbf{b}_1\wedge\mathbf{b}_2\}$.

Problem 74. Let (σ, V) be any irreducible 5-dimensional representation of S_5. Decompose $(\mathrm{Sym}^2\sigma, \mathrm{Sym}^2 V)$ and $(\sigma\wedge\sigma, \Lambda^2 V)$ into irreducible representations.

Problem 75. Let $(\pi, \mathbb{C}^4)$ be the permutation representation of S_4, and let (ρ, V) be the 2-dimensional irreducible representation of S_4.
(a) Show that $(\mathrm{Sym}^2\pi, \mathrm{Sym}^2\mathbb{C}^4)$ has a unique subrepresentation isomorphic to (ρ, V).

(b) Use the ρ-isotopic projector to find that subrepresentation.

Problem 76. A group G of order 168 has conjugacy classes C_1, C_2, C_3, C_4, C_{7A} and C_{7B}, where each conjugacy class is labelled by the order of any element in that class (so, for example, any element of C_{7A} or C_{7B} has order 7). The following shows one of the rows of the character table of G.

size:	1	21	56	42	24	24
class	C_1	C_2	C_3	C_4	C_{7A}	C_{7B}
(π_1, V)	3	-1	0	1	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$

(a) Show that if x is an element of C_{7A} or C_{7B}, then x is conjugate to x^2.
(b) Find the character table of G.

13. Lecture 13

13.1. Inflation. We now turn to group-theoretic methods to create new representations. Recall that if $N < G$ is a *normal* subgroup, the quotient $\widetilde{G} = G/N$ is a group with multiplication

$$g_1 N \cdot g_2 N = g_1 g_2 N \qquad \forall\, g_1 N,\ g_2 N \in \widetilde{G}.$$

We will show how one can create representations of G out of representations of the smaller group $\widetilde{G}$.

Definition 13.1. *Let (π, V) be a representation of $\widetilde{G} = G/N$. The* <u>inflation</u> *of π to G is denoted $(\widetilde{\pi}, V)$, and defined by*

$$\widetilde{\pi}(g) := \pi(gN)$$

for all $g \in G$.

This is also called the <u>*lift*</u> of the representation from $\widetilde{G}$ to G.

Proposition 13.2. *A representation (ρ, V) of G may be expressed as the inflation of a representation of $\widetilde{G} = G/N$ if and only if $N \subseteq \ker \rho$.*

Proof. Let $(\widetilde{\pi}, V)$ be a representation of G such that it is the inflation of a representation (π, V) of $\widetilde{G}$. Then for any $n \in N$, we have

$$\widetilde{\pi}(n) = \pi(nN) = \pi(eN) = \mathrm{Id},$$

so $N \subseteq \ker \widetilde{\pi}$.

On the other hand, if $N \subseteq \ker \rho$, then we define a representation (π, V) of $\widetilde{G}$ by

$$\pi(gN) = \rho(g),$$

for any $gN \in \widetilde{G}$. This is well-defined, since $\rho(gn) = \rho(g)\rho(n) = \rho(g)$ for all $g \in G$, $n \in N$. It is a homomorphism since

$$\pi(g_1 N \cdot g_2 N) = \pi(g_1 g_2 N) = \rho(g_1 g_2) = \rho(g_1)\pi(g_2) = \pi(g_1 N)\pi(g_2 N),$$

for all $g_1 N$, $g_2 N$ in $\widetilde{G}$. Finally, letting $\widetilde{\pi}$ be the lift of π to G, we have

$$\widetilde{\pi}(g) = \pi(gN) = \rho(g),$$

showing that ρ is indeed the lift of a representation. $\square$

Proposition 13.3. *Let $(\widetilde{\pi}, V)$ be a representation of G that is the inflation of a representation (π, V) of $\widetilde{G}$. Then $(\widetilde{\pi}, V)$ is irreducible if and only if (π, V) is.*

Proof. If W is an invariant subspace of V for π, then for any $g \in G$,

$$\widetilde{\pi}(g)W = \pi(gN)W = W,$$

and conversely, if W is an invariant subspace for for $\widetilde{\pi}$, then

$$\pi(gN)W = \widetilde{\pi}(g)W = W.$$

The representations thus have subrepresentations in common, and thus are both irreducible or decomposable together. $\square$

EXAMPLE 13.4. *$D_n/\langle r\rangle = e\langle r\rangle \sqcup s\langle r\rangle \cong C_2 = \mathbb{Z}/2\mathbb{Z}$. There are two irreducible representations of C_2, given by the choices $\pi(s\langle r\rangle) = \pm 1$. Inflating these to D_n gives the two 1-dimensional representations of D_n with $\widetilde{\pi}(r) = 1$ (i.e. the trivial representation and the representation ϵ: $\epsilon(r) = 1$, $\epsilon(s) = -1$).*

Recall that the kernel of any group homomorphism is a normal subgroup. By the previous proposition, we may thus realise any representation π as the inflation of a representation of $G/\ker\pi$. The representations that are not inflations are said to be *faithful*:

DEFINITION 13.5. *Let (π, V) be a representation of a group G. The representation π is said to be* faithful *if the kernel is trivial, i.e.*

$$\ker\pi = \{g \in G \mid \pi(g) = \mathrm{Id}\} = \{e\}.$$

REMARK 13.6. *Recall that by Problem 45(ii),* $\ker\pi = \{g \in G \mid \chi_\pi(g) = \dim(\pi)\}$.

Thus, every non-trivial irreducible representation of a simple group is faithful. The follow proposition provides a converse to this fact:

PROPOSITION 13.7. *Every normal subgroup N of a finite group G may be expressed as*

$$N = \bigcap_{i=1}^{n} \ker\widetilde{\rho}_i$$

for some collection of irreducible representations $(\widetilde{\rho}_i, V_i)$ of G.

PROOF. This is Problem 78 below. □

We conclude this section with the following result regarding characters of inflations:

PROPOSITION 13.8. *Let G be a finite group and N a normal subgroup. For any finite-dimensional representation (π, V) of $\widetilde{G} = G/N$ with inflation $(\widetilde{\pi}, V)$ to G, we have*

(i) $\chi_{\widetilde{\pi}}(g) = \chi_\pi(gN)$ for all $g \in G$.
(ii) $\|\chi_\pi\|^2_{\widetilde{G}} = \|\chi_{\widetilde{\pi}}\|^2_G$.

PROOF. Claim (i) is more or less immediate (left as an exercise below). For (ii), use (i) and compute:

$$\begin{aligned}
\|\chi_{\widetilde{\pi}}\|^2_G &= \frac{1}{|G|}\sum_{g\in G}|\chi_{\widetilde{\pi}}(g)|^2 = \frac{1}{|G|}\sum_{gN\in G/N}\sum_{n\in N}|\chi_{\widetilde{\pi}}(gn)|^2 \\
&= \frac{1}{|G|}\sum_{gN\in G/N}\sum_{n\in N}|\chi_\pi(gN)|^2 = \frac{1}{|G|}\sum_{gN\in G/N}|N||\chi_\pi(gN)|^2 \\
&= \frac{1}{|G|/|N|}\sum_{gN\in G/N}|\chi_\pi(gN)|^2 = \|\chi_\pi\|^2_{\widetilde{G}}.
\end{aligned}$$

□

REMARK 13.9. *Note that this proposition and Theorem 9.7 (ii) provide an alternative proof of Proposition 13.3.*

13.2. Exercises.

Problem 77. Show Proposition 13.8(i).

Problem 78. Prove Proposition 13.7 by proceeding as follows:

(a) Given a normal subgroup $N \subseteq G$, let $\{(\rho_i, V_i)\}$ denote the irreducible representations of $\widetilde{G} = G/N$, and let $(\widetilde{\rho}_i, V_i)$ be the lift of (ρ_i, V_i) to G. Define

$$K = \bigcap_{i=1}^{n} \ker \widetilde{\rho}_i.$$

Show that K is normal in G.

(b) Use Proposition 13.2 to show that $N \subseteq K$, and hence $|G/K| \leq |G/N|$.

(c) Use Theorem 6.6 to show that $|G/N| = \sum_i (\dim V_i)^2$.

(d) Use Propositions 13.2 and 13.3 to show that each $(\widetilde{\rho}_i, V_i)$ is the lift of an irreducible representation representation (σ_i, V_i) of G/K.

(e) Use Theorem 6.6 to show that $\sum_i (\dim V_i)^2 \leq |G/K|$.

(f) Conclude that $N = K$.

Problem 79. Using the character table of S_5, find the character table of A_5. Using the character table, show that A_5 is simple (that is, it has no non-trivial proper normal subgroups). *Hint: every element of A_5 is conjugate to its inverse.*

Problem 80. *(Challenging!)* Let (π, V) be a faithful finite-dimensional representation of a finite group G. Given an irreducible representation (ρ, W) of G, show that (ρ, W) is isomorphic to a subrepresentation of $(\pi^{\otimes n}, V^{\otimes n})$ for some $n \geq 1$.

14. Lecture 14

14.1. Induced representations. Let $H \leq G$ be a subgroup. Given a a representation (π, V) of H, we will "stitch together" this with the G-action on G/H to create a representation of G, which we call the *induced representation* of (π, V). This is often denoted $\mathrm{Ind}_H^G(\pi, V)$ or $(\pi, V)\uparrow_H^G$.

Our definition will depend on a choice of cosets for G/H: let $r_1, r_2, \ldots, r_n \in G$ be such that

$$G = r_1 H \sqcup r_2 H \sqcup \ldots \sqcup r_n H;$$

here $n = \frac{|G|}{|H|}$, and the "r"s stand for *representatives.* Note that each $g \in G$ may be written *uniquely* as $g = r_i h$ for some representative r_i and $h \in H$.

We use these to describe the G-action on G/H as follows. Define

$$j(\cdot,\cdot) : G \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}, \qquad h_{(\cdot,\cdot)} : G \times \{1, 2, \ldots, n\} \to H$$

by the formula

$$g r_i = r_{j(g,i)} h_{(g,i)}$$

for all $g \in G$ and $i \in \{1, \ldots, n\}$. These functions are well-defined by uniqueness of the decomposition of G into H-cosets.

LEMMA 14.1.

$$j(g_1 g_2, i) = j\big(g_1, j(g_2, i)\big)$$
$$h_{(g_1 g_2, i)} = h_{\left(g_1, j(g_2, i)\right)} h_{(g_2, i)}$$

for all $g_1, g_2 \in G$, $i \in \{1, \ldots, n\}$.

PROOF. From the definition of $j(\cdot,\cdot)$ and $h_{(\cdot,\cdot)}$, we have

$$g_1 g_2 r_i = r_{j(g_1 g_2, i)} h_{(g_1 g_2, i)}.$$

On the other hand,

$$\begin{aligned} g_1 g_2 r_i &= g_1(g_2 r_i) = g_1\big(r_{j(g_2,i)} h_{(g_2,i)}\big) = \big(g_1 r_{j(g_2,i)}\big) h_{(g_2,i)} \\ &= \Big(r_{j\left(g_1, j(g_2,i)\right)} h_{\left(g_1, j(g_2,i)\right)}\Big) h_{(g_2,i)} = r_{j\left(g_1, j(g_2,i)\right)} \Big(h_{\left(g_1, j(g_2,i)\right)} h_{(g_2,i)}\Big). \end{aligned}$$

By uniqueness of the decomposition of G into H-cosets, we must then have

$$r_{j(g_1 g_2, i)} = r_{j\left(g_1, j(g_2,i)\right)}, \quad h_{(g_1 g_2, i)} = h_{\left(g_1, j(g_2,i)\right)} h_{(g_2,i)}.$$

□

Given a representation (π, V) of H and choice of coset representatives $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ for G/H, we define a vector space

$$V_{\mathbf{r}} = r_1 V \oplus r_2 V \oplus \ldots \oplus r_n V,$$

with vector space operations

$$\alpha \left(\sum_{i=1}^n r_i \mathbf{v}_i \right) + \beta \left(\sum_{i=1}^n r_i \mathbf{u}_i \right) := \sum_{i=1}^n r_i (\alpha \mathbf{v}_i + \beta \mathbf{u}_i).$$

Proposition 14.2. *The* induced representation $\mathrm{Ind}_H^G(\pi, V) := \big(\mathrm{Ind}_H^G\pi, V_{\mathbf{r}}\big)$ *defined by extending the map*

$$\mathrm{Ind}_H^G\pi(g)\big(r_i\mathbf{v}\big) := r_{j(g,i)}\pi(h_{(g,i)})\mathbf{v}$$

linearly to all of $V_{\mathbf{r}}$ (for each $g \in G$) is *a representation of G on $V_{\mathbf{r}}$.*

Proof. We need to show that $\mathrm{Ind}_H^G\pi(g_1g_2) = \mathrm{Ind}_H^G\pi(g_1)\mathrm{Ind}_H^G\pi(g_2)$. Let $r_i\mathbf{v} \in V_{\mathbf{r}}$. Then

$$\begin{aligned}\mathrm{Ind}_H^G\pi(g_1)\mathrm{Ind}_H^G\pi(g_2)r_i\mathbf{v} &= \mathrm{Ind}_H^G\pi(g_1)r_{j(g_2,i)}\pi(h_{(g_2,i)})\mathbf{v}\\ &= r_{j\left(g_1,j(g_2,i)\right)}\pi\Big(h_{\left(g_1,j(g_2,i)\right)}\Big)\pi(h_{(g_2,i)})\mathbf{v}\\ &= r_{j(g_1g_2,i)}\pi(h_{(g_1g_2,i)})\mathbf{v}\\ &= \mathrm{Ind}_H^G\pi(g_1g_2)r_i\mathbf{v},\end{aligned}$$

where the previous lemma was used for the second-to-last equality. $\square$

Remark 14.3. *We will later see that making a different choice of coset representatives gives rise to an isomorphic representation.*

Example 14.4. *Let $H = \langle r\rangle < D_n$. Then $D_n = e\langle r\rangle \sqcup s\langle r\rangle$, and $H = C_n$. Given a representation $(\psi_j, \mathbb{C})$, $\psi_j(r^k) = e^{2\pi ijk/n}$, we consider the induced representation $\mathrm{Ind}_H^{D_n}\psi_j$ on $W = e\mathbb{C} \oplus s\mathbb{C}$: for $z, w \in \mathbb{C}$, we have*

$$\mathrm{Ind}_H^{D_n}\psi_j(s)(ez + sw) = sz + ew,$$

$$\mathrm{Ind}_H^{D_n}\psi_j(r)(ez + sw) = e\psi_j(r)z + s\psi_j(r^{-1})w = e^{2\pi ij/n}ez + e^{-2\pi ij/n}sw.$$

The matrices of these maps with respect to the basis $e1$, $s1$ of W are then

$$\mathrm{Ind}_H^{D_n}\psi_j(s) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \mathrm{Ind}_H^{D_n}\psi_j(r) = \begin{pmatrix}e^{2\pi ij/n} & 0\\ 0 & e^{-2\pi ij/n}\end{pmatrix},$$

hence

$$\mathrm{Ind}_H^{D_n}(\psi_j, \mathbb{C}) = (\rho_j, \mathbb{C}^2).$$

Note that $\dim \mathrm{Ind}_H^G\pi = \frac{|G|}{|H|}\dim\pi$.

Recall that since G acts on G/H, we have the quasi-regular representation $\big(\lambda, C(G/H)\big)$, where $C(G/H) = \{f : G/H \to \mathbb{C}\}$,

$$[\lambda(g)f](g_0H) = f(g^{-1}g_0H).$$

Proposition 14.5. $\big(\lambda, C(G/H)\big) \cong \mathrm{Ind}_H^G(\mathrm{triv}, \mathbb{C})$.

Proof. We first define $T : C(G/H) \to \mathbb{C}_{\mathbf{r}} = r_1\mathbb{C} \oplus \ldots \oplus r_n\mathbb{C}$ as follows:

$$T(f) := r_1f(r_1H) + r_2f(r_2H) + \ldots + r_nf(r_nH).$$

It is straightforward to verify that this is an invertible linear map, so it remains to show that it is a G-homomorphism. For $g \in G$ and $f \in C(G/H)$,

$$\mathrm{Ind}_H^G\mathrm{triv}(g)T(f) = \mathrm{Ind}_H^G\mathrm{triv}(g)\left(\sum_i r_if(r_iH)\right) = \sum_i r_{j(g,i)}f(r_iH).$$

Now reparameterising the sum with $j = j(g, i)$, hence $i = j(g^{-1}, j(g, i))$ (cf. Lemma 14.1), we obtain

$$\mathrm{Ind}_H^G\mathrm{triv}(g)T(f) = \sum_j r_jf(r_{(g^{-1},j)}H) = \sum_j r_jf(g^{-1}r_jH) = T(\lambda(g)f).$$

□

Exercise 81 generalises this type of isomorphism to any induced representation. We conclude with the following important observation about induced representations:

$$\mathrm{Ind}_H^G \pi(h) r_i \mathbf{v} = r_{j(h,i)} \pi(h_{(h,i)}) \neq r_i \pi(h)\mathbf{v}!$$

14.2. Exercises.

Problem 81. Let (π, V) be a representation of $H \leq G$. Define

$$C_\pi(G,V) = \{f : G \to V \mid f(gh) = \pi(h^{-1})f(g)\ \forall\ g \in G,\ h \in H\}.$$

(a) Compute $\dim C_\pi(G,V)$.
(b) Show that $\big(\lambda, C_\pi(G,V)\big)$ is a G-representation, where as usual $[\lambda(g)f](g') = f(g^{-1}g')$ for all $g, g' \in G$.
(c) Given a set of representatives $\mathbf{r} = r_1, \ldots, r_n$, let $T_{\mathbf{r}} : C_\pi(G,V) \to V_{\mathbf{r}}$ be the map

$$T_{\mathbf{r}}(f) = r_1 f(r_1) + r_2 f(r_2) + \ldots + r_n f(r_n).$$

Show that $T_{\mathbf{r}}$ is a G-isomorphism from $\big(\lambda, C_\pi(G,V)\big)$ to $\mathrm{Ind}_H^G(\pi, V)$.

Remark 14.6. *This shows that the induced representations of (π, V) for different choices of representatives for G/H give rise to isomorphic representations!*

15. Lecture 15

15.1. Frobenius Reciprocity. Let H be a subgroup of G and let (π, V) be a representation of G. Recall that we have already defined what we mean by the restriction of (π, V) to H, which we initially denoted as $(\pi|_H, V)$. It is also often denoted as $\mathrm{Res}^G_H(\pi, G)$ or $(\pi, V)\downarrow^G_H$. We will now see how restriction pairs with induction in Frobenius Reciprocity.

THEOREM 15.1 (Frobenius Reciprocity, 1898). *Let (π, V) be a representation of $H \leq G$. Then for any representation (ρ, W) of G, we have*

$$\mathrm{Hom}_G(V_{\mathbf{r}}, W) \cong \mathrm{Hom}_H(V, W),$$

where $(\mathrm{Ind}^G_H\pi, V_{\mathbf{r}})$ is the induced representation of π, and $(\rho|_H, W)$ is the restricted representation of ρ.

PROOF. We will create linear maps between these spaces, and show that they are inverses of each other. Firstly, define $\Phi : \mathrm{Hom}_H(V, W) \to \mathrm{Hom}(V_{\mathbf{r}}, W)$ by setting

$$\Phi(T)\left(\sum_i r_i\mathbf{v}_i\right) := \sum_i \rho(r_i)T(\mathbf{v}_i)$$

for all $T \in \mathrm{Hom}_H(V, W)$ and $\sum_i r_i\mathbf{v}_i \in V_{\mathbf{r}}$. On the other hand, WLOG we assume that $e \in r_1H$ (hence $r_1 \in H$), and define $\Psi : \mathrm{Hom}_G(V_{\mathbf{r}}, W) \to \mathrm{Hom}(V, W)$ by

$$\Psi(S)\mathbf{v} := \rho(r_1^{-1})S(r_1\mathbf{v})$$

for all $S \in \mathrm{Hom}_G(V_{\mathbf{r}}, W)$ and $\mathbf{v} \in V$.

Since they are defined using evaluation, both Φ and Ψ are linear, so it remains to verify that:

(i) $\Phi(T)$ is a G-homomorphism,
(ii) $\Psi(S)$ is an H-homomorphism,
(iii) & (iv) they are left/right inverse of each other (iii) .

(i) For any $g \in G$, we have

$$\begin{aligned}\Phi(T)\big(\mathrm{Ind}^G_H\pi(g)r_i\mathbf{v}\big) =&\Phi(T)\big(r_{j(g,i)}\pi(h_{(g,i)})\mathbf{v}\big) = \rho\big(r_{j(g,i)}\big)T\big(\pi(h_{(g,i)})\mathbf{v}\big)\\ &= \rho\big(r_{j(g,i)}\big)\rho(h_{(g,i)})T(\mathbf{v}) = \rho\big(r_{j(g,i)}h_{(g,i)}\big)T(\mathbf{v})\\ &= \rho(gr_i)T(\mathbf{v}) = \rho(g)\rho(r_i)T(\mathbf{v}) = \rho(g)\Phi(T)(r_i\mathbf{v}).\end{aligned}$$

(ii) Since $r_1 \in H$, for any $h \in H$, we have

$$\begin{aligned}\Psi(S)\pi(h)\mathbf{v} =&\rho(r_1^{-1})S(r_1\pi(h)\mathbf{v}) = \rho(r_1)^{-1}S\big(\mathrm{Ind}^G_H\pi(r_1hr_1^{-1})r_1\mathbf{v}\big)\\ &= \rho(r_1^{-1})\rho(r_1hr_1^{-1})S(r_1\mathbf{v}) = \rho(h)\rho(r_1^{-1})S(r_1\mathbf{v}) = \pi(h)\Psi(S)\mathbf{v}.\end{aligned}$$

(iii) For any $r_i\mathbf{v} \in r_iV$,

$$\Phi\big(\Psi(S)\big)r_i\mathbf{v} = \rho(r_i)\Psi(S)\mathbf{v} = \rho(r_i)\rho(r_1^{-1})S(r_1\mathbf{v}) = S(\mathrm{Ind}^G_H\pi(r_ir_1^{-1})r_1\mathbf{v}) = S(r_i\mathbf{v}),$$

hence $\Phi(\Psi(S)) = S$.

(iv) For any $\mathbf{v} \in V$,

$$\Psi(\Phi(T))\mathbf{v} = \rho(r_1^{-1})\Phi(T)(r_1\mathbf{v}) = \rho(r_1^{-1})(\rho(r_1)T(\mathbf{v})) = T\mathbf{v},$$

hence $\Psi(\Phi(T)) = T$.

$\square$

This has a nice application to characters; we first introduce some convenient notation. Given a representation (π, V) and a class function $f \in CF(G)$, we write

$$\mathrm{Ind}_H^G \chi_\pi := \chi_{\mathrm{Ind}_H^G \pi}, \qquad \mathrm{Res}_H^G f := f|_H.$$

It is also common to see $\chi_\pi \uparrow_H^G$ and $f \downarrow_H^G$.

Corollary 15.2. *Let $H \leq G$. For all $f \in CF(G)$ and finite-dimensional representations (π, V) of H,*

$$\langle \mathrm{Ind}_H^G \chi_\pi, f \rangle_G = \langle \chi_\pi, \mathrm{Res}_H^G f \rangle_H.$$

Remark 15.3. *Let V, W be two inner product spaces. Given $A : V \to W$, we define $A^* : W \to V$ by the formula*

$$\langle A\mathbf{v}, \mathbf{w} \rangle_W = \langle \mathbf{v}, A^* \mathbf{w} \rangle_V \qquad \forall\ \mathbf{v} \in V,\ \mathbf{w} \in W.$$

The map A^* is called the adjoint *of A. Because of this, we say that Ind_H^G is the adjoint of Res_H^G.*

Proof. Since the irreducible characters of G form an orthonormal basis of $CF(G)$, we may write

$$f = \sum_{\rho \in \mathrm{Irr}(G)} \langle f, \chi_\rho \rangle_G \chi_\rho.$$

Then for a representation π of H, we have

$$\langle \mathrm{Ind}_H^G \chi_\pi, f \rangle_G = \sum_{\rho \in \mathrm{Irr}(G)} \overline{\langle f, \chi_\rho \rangle_G} \langle \mathrm{Ind}_H^G \chi_\pi, \chi_\rho \rangle_G.$$

By Lemma 9.6 and Theorem 15.1,

$$\langle \mathrm{Ind}_H^G \chi_\pi, \chi_\rho \rangle_G = \dim\big(\mathrm{Hom}_G(V_{\mathbf{r}}, W_\rho)\big) = \dim\big(\mathrm{Hom}_H(V, W_\rho)\big) = \langle \chi_\pi, \mathrm{Res}_H^G \chi_\rho \rangle_H.$$

Substituting this into the previous identity gives

$$\langle \mathrm{Ind}_H^G \chi_\pi, f \rangle_G = \left\langle \chi_\pi, \sum_{\rho \in \mathrm{Irr}(G)} \langle f, \chi_\rho \rangle_G \mathrm{Res}_H^G \chi_\rho \right\rangle_H = \langle \chi_\pi, \mathrm{Res}_H^G f \rangle_H.$$

□

Example 15.4. *Table 2 below shows the irreducible characters ψ_i of S_4 and χ_i of S_3. We regard S_3 as the subgroup of S_4 of elements that fix 4, and use Frobenius reciprocity to compute $\mathrm{Ind}_{S_3}^{S_4} \chi_2$.*

For each i, Frobenius reciprocity implies that

$$\langle \mathrm{Ind}_{S_3}^{S_4} \chi_2, \psi_i \rangle_{S_4} = \langle \chi_2, \psi_i|_{S_3} \rangle_{S_3}.$$

The right hand side is easily seen to be zero for $i = 0, 1$ and one for $i = 2, 3, 4$. We therefore have

$$\mathrm{Ind}_{S_3}^{S_4} \chi_2 = \psi_2 + \psi_3 + \psi_4.$$

Corollary 15.5. *Let (π, V) be a representation of $H \leq G$. The representation $(\mathrm{Ind}_H^G \pi, V_{\mathbf{r}})$ is independent of the choice of $\mathbf{r}$, up to isomorphism.*

Proof. By Corollary 15.2 the character of $(\mathrm{Ind}_H^G \pi, V_{\mathbf{r}})$ is entirely determined by the restrictions of irreducible characters of G and the character of π. Thus, by Theorem 9.7 any choice of $\mathbf{r}$ gives an isomorphic representation. □

	e	(12)	$(12)(34)$	(123)	(1234)
ψ_0	1	1	1	1	1
ψ_1	1	-1	1	1	-1
ψ_2	2	0	2	-1	0
ψ_3	3	1	-1	0	-1
ψ_4	3	-1	-1	0	1
χ_0	1	1	NA	1	NA
χ_1	1	-1	NA	1	NA
χ_2	2	0	NA	-1	NA

TABLE 2. Characters of S_3 and S_4

15.2. Exercises.

Problem 82. For each irreducible representation of S_4, decompose its induction to S_5 into irreducibles (where S_4 is regarded as the subgroup of elements of S_5 that fix $5 \in \{1, \dots, 5\}$).

Problem 83. Let (π, V) be an irreducible representation of G and H a subgroup of G. Show that (π, V) isomorphic to a subrepresentation of a representation induced from an irreducible representation of H.

Problem 84. Let χ_π be an irreducible character of $H \leq G$, and write

$$\mathrm{Ind}_H^G \chi_\pi = \sum_{\sigma \in \mathrm{Irr}(G)} d_\sigma \chi_\sigma.$$

Show that $\sum_\sigma d_\sigma^2 \leq [G : H]$.

Problem 85. Let H be a subgroup of G. Show

(a) If (π_1, V_1), (π_2, V_2) are representations of H, then

$$\mathrm{Ind}_H^G\big((\pi_1, V_1) \oplus (\pi_2, V_2)\big) \cong \mathrm{Ind}_H^G(\pi_1, V_1) \oplus \mathrm{Ind}_H^G(\pi_2, V_2).$$

(b) If $K \leq H \leq G$, and (ρ, U) is a representation of K, then

$$\mathrm{Ind}_K^G(\rho, U) \cong \mathrm{Ind}_H^G\big(\mathrm{Ind}_K^H(\rho, U)\big).$$

(c) If (π, V) is a representation of G, then

$$\mathrm{Ind}_H^G\big(\,\mathrm{Res}_H^G(\pi, V)\big) \cong (\pi, V) \otimes \mathrm{Ind}_H^G(\,\mathrm{Id}, \mathbb{C}).$$

16. Lecture 16

16.1. The character formula. Let $H \leq G$ be finite groups and (π, V) a representation of H. We now use Frobenius reciprocity to give a formula for $\operatorname{Ind}_H^G \chi_\pi$.

THEOREM 16.1. *Let $\mathcal{C}$ be a conjugacy class of G. Let H be a subgroup of G and write*

$$\mathcal{C} \cap H = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_n,$$

where $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are conjugacy classes of H. Then

$$(\operatorname{Ind}_H^G \chi_\pi)(\mathcal{C}) = \frac{|G|}{|H|} \sum_{i=1}^{n} \frac{|\mathcal{D}_i|}{|\mathcal{C}|} \chi_\pi(\mathcal{D}_i).$$

PROOF. Let $\mathbb{1}_{\mathcal{C}}$ be the indicator function of $\mathcal{C}$. Then for any class function f on G,

$$\langle f, \mathbb{1}_{\mathcal{C}} \rangle_G = \frac{|\mathcal{C}|}{|G|} f(\mathcal{C}).$$

Combining this with Frobenius reciprocity (Corollary 15.2), we have

$$\begin{aligned}
(\operatorname{Ind}_H^G \chi_\pi)(\mathcal{C}) &= \frac{|G|}{|\mathcal{C}|} \langle \operatorname{Ind}_H^G \chi_\pi, \mathbb{1}_{\mathcal{C}} \rangle_G \\
&= \frac{|G|}{|\mathcal{C}|} \langle \chi_\pi, \operatorname{Res}_H^G \mathbb{1}_{\mathcal{C}} \rangle_H \\
&= \frac{|G|}{|\mathcal{C}|} \sum_{i=1}^{n} \langle \chi_\pi, \mathbb{1}_{\mathcal{D}_i} \rangle_H \\
&= \frac{|G|}{|\mathcal{C}|} \sum_{i=1}^{n} \frac{|\mathcal{D}_i|}{|H|} \chi_\pi(\mathcal{D}_i) = \frac{|G|}{|H|} \sum_{i=1}^{n} \frac{|\mathcal{D}_i|}{|\mathcal{C}|} \chi_\pi(\mathcal{D}_i)
\end{aligned}$$

which is the claimed formula. ∎

REMARK 16.2. *We derived the character formula from Frobenius reciprocity. It is also possible to go the other way around: prove the character formula directly, then derive Frobenius reciprocity as a consequence.*

REMARK 16.3. *If $g \in G$, we write $C_G(g)$ for the* centraliser *of g:*

$$C_G(g) = \{x \in G \mid x^{-1} g x = g\}.$$

By the orbit-stabiliser theorem, if $\mathcal{C}_g$ is the conjugacy class of g, then

$$\frac{|G|}{|\mathcal{C}_g|} = |C_G(g)|$$

giving an interpretation for some of the factors in the above formula.

EXAMPLE 16.4. *We continue with the example of the dihedral group. Let $H = C_n \leq G = D_n$, and let $\psi : H \to \mathbb{C}^\times$ be a homomorphism with $\psi(r) = \omega$. Then:*

(i) $\operatorname{Ind}_H^G \psi(e) = [G : H] = 2$.
(ii) $\operatorname{Ind}_H^G \psi(s) = \operatorname{Ind}_H^G \psi(rs) = 0$, *as the conjugacy class of s or rs does not intersect H.*

(iii) If $0 < i < n/2$, then the conjugacy class $\{r^i, r^{-i}\}$ of G splits into two conjugacy classes, $\{r^i\}$ and $\{r^{-i}\}$, of H. We have

$$\mathrm{Ind}_H^G(\psi)(r^i) = 2\big(\frac{1}{2}\psi(r^i) + \frac{1}{2}\psi(r)^{-i}\big) = \omega^i + \omega^{-i}.$$

(iv) If $i = n/2$, then the conjugacy class $\{r^{n/2}\}$ of G remains as a single conjugacy class of H and

$$\mathrm{Ind}_H^G(\psi)(r^{n/2}) = 2\psi(r^{n/2}) = 2\omega^{n/2}.$$

Taking $\omega \neq \pm 1$ we again obtain all the irreducible 2-dimensional characters of D_n.

16.2. Example: $\mathbf{D_4}$ to $\mathbf{S_4}$. Let H be the subgroup of $G = S_4$ isomorphic to D_4, obtained by labeling the vertices of a square 1, ..., 4 and letting D_4 act on them. In other words, H is the image of the injective homomorphism $D_4 \to S_4$ sending

$$r \mapsto (1234), s \mapsto (12)(34).$$

Recall that D_4 has five irreducible representations with characters as shown:

	e	$r = (1234)$	$r^2 = (13)(24)$	$s = (12)(34)$	$rs = (13)$
	1	2	1	2	2
triv	1	1	1	1	1
ϵ	1	1	1	-1	-1
ϕ_+	1	-1	1	1	-1
ϕ_-	1	-1	1	-1	1
σ	2	0	-2	0	0

while recall that S_4 has character table

	e	(12)	(12)(34)	(123)	(1234)
	1	6	3	8	6
triv	1	1	1	1	1
sgn	1	-1	1	1	-1
π	3	1	-1	0	-1
$\mathrm{sgn}\,\pi$	3	-1	-1	0	1
ρ	2	0	2	-1	0

We want to first find out how the conjugacy classes of S_4 intersect with D_4. The result is as follows, writing $\mathcal{C}_g$ for the conjugacy class of g in S_4, and $\mathcal{D}_h$ for the conjugacy class of h in D_4.

$g \in S_4$	$\mathcal{C}_g \cap D_4$	sizes
e	$\mathcal{D}_e$	1
(12)	$\mathcal{D}_{rs}$	2
(12)(34)	$\mathcal{D}_{r^2} \cup \mathcal{D}_s$	1, 2
(123)	$\emptyset$	–
(1234)	$\mathcal{D}_r$	2

We use this and the character formula to determine $\mathrm{Ind}_H^G \phi_+$. The factor $|G|/|H|$ is constant, equal to 3. We obtain:

		$\mathrm{Ind}_H^G\phi_+$
e	1	3
(12)	6	$3\cdot 2/6\cdot(-1)=-1$
$(12)(34)$	3	$3(1/3+2/3)=3$
(123)	8	0
(1234)	6	$3(-2/6)=-1$

Decomposing this character, we see that

$$\mathrm{Ind}_H^G\phi_+ = \mathrm{sgn} + \chi_\rho.$$

But note that we did *not* need to know the character table of S_4 to find the induced character.

We check that this is consistent with Frobenius reciprocity: the restriction of sgn to D_4 is ϕ_+, so

$$\langle \mathrm{Res}_H^G \,\mathrm{sgn}, \phi_+\rangle_{D_4} = \langle \mathrm{sgn}, \mathrm{Ind}_H^G\phi_+\rangle_{S_4} = 1$$

as required. The restriction of χ_ρ to D_4 is triv $+ \phi_+$ so

$$\langle \mathrm{Res}_H^G \,\chi_\rho, \phi_+\rangle_{D_4} = \langle \chi_\rho, \mathrm{Ind}_H^G\phi_+\rangle_{S_4} = 1$$

(we could also check that the restrictions of the other irreducible characters of S_4 to D_4 do not contain ϕ_+).

16.3. Exercises.

Problem 86. Let χ_σ be the irreducible degree 3 character (i.e. $\chi_\sigma(e)=3$) of $H=A_5$ such that

$$\chi_\sigma((12345)) = \frac{1+\sqrt{5}}{2}.$$

Let $G=A_6$, with H as the subgroup of G fixing 6. Compute the character $\mathrm{Ind}_H^G\chi_\sigma$.

Problem 87. Let $H=\langle (12\dots p)\rangle \subseteq S_p$, for p a prime, and ψ a nontrivial character of H.

(a) Compute $\mathrm{Ind}_H^{S_p}\psi$.

(b) By considering $\langle \mathrm{Ind}_H^{S_p}\psi, \mathrm{Ind}_H^{S_p}\psi\rangle_{S_p}$, show that

$$(p-1)! \equiv -1 \pmod{p}.$$

(Recall that if ζ is a non-trivial p-th root of unity, we have $1+\zeta+\zeta^2+\ldots+\zeta^{p-1}=0$.)

Problem 88. Let

$$G = \left\langle x, y \mid y^7 = x^3 = e,\ xyx^{-1} = y^2 \right\rangle.$$

G has 21 elements, and may be expressed as $\{x^i y^j \mid 0\le i\le 2,\ 0\le j\le 6\}$.

(a) Show that $H=\langle y\rangle$ is a normal subgroup of G.

(b) By considering representations lifted from G/H and induced from H, find the character table of G.

CHAPTER 4

Irreducible representations of S_n

17. Lecture 17

17.1. Young diagrams, tableaux, and tabloids. By Theorem 8.7, we know that there is a bijection between the irreducible representations and conjugacy classes of any finite group G. However, for most groups there is no canonical way of giving an explicit construction of such a bijection without first finding all the irreducibles.

In the important case $G = S_n$, we can in fact construct every irreducible representation from the conjugacy classes. More exactly, given a conjugacy class $\mathcal{C}$, we will construct an irreducible representation $\pi_{\mathcal{C}}$ and show that these representations are all non-isomorphic for different choices of $\mathcal{C}$.

Recall that the conjugacy classes of S_n are given by all the permutations with the same cycle type.

DEFINITION 17.1. *Let $n \in \mathbb{N}$. A* partition *λ of n is an ordered tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where*

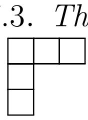
$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0 \quad \text{and} \quad \lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n.$$

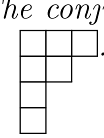
We write $\lambda \vdash n$ and $|\lambda| = \ell$ is called the length *of λ.*

DEFINITION 17.2 (Young Diagram, 1900). *Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n we can define a corresponding* Young diagram *as n boxes arranged into ℓ rows with λ_i in the ith row (from the top). A partition $\lambda \vdash n$ is also called the* shape *of the corresponding Young diagram.*

We will choose to have the row corresponding to λ_1 at the top and to draw our boxes left adjusted, so the left most column always has ℓ entries, but note that there are other conventions.

Notice that we can associate a cycle type in S_n to a unique partition, obtained by ordering the cycles by length. We can therefore also associate a Young diagram.

EXAMPLE 17.3. *The conjugacy class in S_5 of $(123) = (123)(4)(5)$ is illustrated by the Young diagram* .

EXAMPLE 17.4. *The conjugacy class in S_7 of $(12)(356) = (356)(12)(4)(7)$ is illustrated by the Young diagram* .

DEFINITION 17.5. *Given a partition $\lambda \vdash n$, a* Young tableau *is a Young diagram of shape λ with one of the numbers $1, 2, \ldots, n$ in each box such that no number occurs twice in the whole diagram. The set of all Young tableaux of shape λ is denoted $\mathbf{YT}^\lambda$*

Example 17.6. *The Young tableaux* $\begin{array}{|c|c|c|}\hline 1&3&2\\\hline 6&7&5\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}$ *and* $\begin{array}{|c|c|c|}\hline 4&7&1\\\hline 6&2&3\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}$ *are both elements of* $\mathbf{YT}^{(3,3,1)}$.

Observe that $|\mathbf{YT}^\lambda| = n!$. There is also a natural S_n-action on $\mathbf{YT}^\lambda$ inherited from the defining action on $1, \ldots, n$:

$$(354)(67) \cdot \begin{array}{|c|c|c|}\hline 4&7&1\\\hline 6&2&3\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 3&6&1\\\hline 7&2&5\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}.$$

In order to create an S_n-action that is different to the standard S_n action, we identify all Young tableaux with the same entries in each rows. This defines an equivalence relation on $\mathbf{YT}^\lambda$, and each equivalence class is called a *Young tabloid*. The set of Young tabloids of shape λ is denoted $\mathbf{YTD}^\lambda$. We illustrate a Young tabloid by removing the vertical lines in the corresponding Young tableau, as well as not caring about the ordering of the numbers in each row (though we normally write the numbers in each row in ascending order).

Example 17.7.

$$(23)(56) \cdot \begin{array}{ccc}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array} = \begin{array}{ccc}\hline 1&3&2\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array} = \begin{array}{ccc}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}.$$

As an exercise, you should verify that

$$|\mathbf{YTD}^\lambda| = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_\ell!} \tag{17.8}$$

17.2. Specht modules. Since we have an S_n-action on $\mathbf{YTD}^\lambda$, there is the corresponding permutation representation $\big(\pi, \mathbb{C}(\mathbf{YTD}^\lambda)\big)$ as in Definition 1.9. We write $\mathcal{M}^\lambda$ for the free vector space $\mathbb{C}(\mathbf{YTD}^\lambda)$, and call the vectors in this vector space *polytabloids*.

Example 17.9. *Let's look at the representation* $(\pi, \mathcal{M}^{(2,2)})$ *of* S_4. *From* (17.8), *we have* $\dim(\mathcal{M}^{(2,2)}) = 6$, *with a basis given by the elements of* $\mathbf{YTD}^{(2,2)}$*:*

$$\begin{array}{cc}\hline 1&2\\\hline 3&4\\\hline\end{array},\quad \begin{array}{cc}\hline 1&3\\\hline 2&4\\\hline\end{array},\quad \begin{array}{cc}\hline 1&4\\\hline 2&3\\\hline\end{array},\quad \begin{array}{cc}\hline 3&4\\\hline 1&2\\\hline\end{array},\quad \begin{array}{cc}\hline 2&4\\\hline 1&3\\\hline\end{array},\quad \begin{array}{cc}\hline 2&3\\\hline 1&4\\\hline\end{array}.$$

We compute the action of (12) on a polytabloid:

$$\pi(12)\left(\begin{array}{cc}\hline 1&2\\\hline 3&4\\\hline\end{array} - 2\begin{array}{cc}\hline 1&4\\\hline 2&3\\\hline\end{array} + 3\begin{array}{cc}\hline 2&3\\\hline 1&4\\\hline\end{array}\right) = \begin{array}{cc}\hline 1&2\\\hline 3&4\\\hline\end{array} - 2\begin{array}{cc}\hline 2&4\\\hline 1&3\\\hline\end{array} + 3\begin{array}{cc}\hline 1&3\\\hline 2&4\\\hline\end{array}.$$

The representation $(\pi, \mathcal{M}^\lambda)$ is not irreducible (for example, it has the trivial representation as a subrepresentation), it does have a special subrepresentation $(\pi, \mathcal{S}^\lambda)$ that is irreducible and the map $\lambda \mapsto (\pi, \mathcal{S}^\lambda)$ is a bijection between conjugacy classes of S_n and irreducible representations.

Before defining this representation, we need to introduce some more notation. Given a Young tableau $t \in \mathbf{YT}^\lambda$, we write $[t] \in \mathbf{YTD}^\lambda$ for the corresponding tabloid. We also let $C(t)$ be the set of elements that preserve the numbers in each of the columns of t (i.e. elements of $C(t)$ just permute the entries of each column individually).

Example 17.10. *We compute* $C\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array}\right)$*: elements have to leave each column of the tableau invariant, so* $C(t)$ *is generated by the elements* (12) *and* (34). *We therefore have that*

$$C\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array}\right) = \{e, (12), (34), (12)(34)\}.$$

DEFINITION 17.11. *For each $t \in \mathbf{YT}^\lambda$, let $\mathbf{e}_t \in \mathcal{M}^\lambda$ be the polytabloid*

$$\mathbf{e}_t := \sum_{\sigma \in C(t)} \operatorname{sgn}(\sigma)[\sigma \cdot t].$$

THEOREM 17.12. *Let $\mathcal{S}^\lambda = \operatorname{span}\{\mathbf{e}_t \,|\, t \in \mathbf{YT}^\lambda\}$. The subspace $\mathcal{S}^\lambda$ is an irreducible S_n-invariant subspace of $\mathcal{M}^\lambda$. If $\lambda \vdash n$ and $\mu \vdash n$ are two different partitions of n, then $(\pi, \mathcal{S}^\lambda) \ncong (\pi, \mathcal{S}^\mu)$.*

The representation $(\pi, \mathcal{S}^\lambda)$ is called the *Specht module* for λ. These are named after Wilhelm Specht who studied them first in 1935. We will prove this theorem next lecture.

REMARK 17.13. *Note that while the vectors $\mathbf{e}_t$ span $\mathcal{S}^\lambda$, they do not form a basis, as they are* not *linearly independent. A basis for $\mathcal{S}^\lambda$ is given by $\{\mathbf{e}_t \,|\, t$ is a* standard Young tableau$\}$ *(we will define these in a few lectures). Observe also that even if two tableaux t_1, t_2 give rise to the same tabloid (i.e. $[t_1] = [t_2]$; that is to say, they have the same entries in each row), in general, one has $\mathbf{e}_{t_1} \neq \mathbf{e}_{t_2}$, since $C(t_1)$ will be different from $C(t_2)$.*

17.3. Exercises.

Problem 89. Decompose $(\pi, \mathcal{M}^{(3,2)})$ into irreducible representations of S_5.

Problem 90. Show that $(\pi, \mathcal{S}^{(n)})$ is the trivial representation of S_n and that $(\pi, \mathcal{S}^{(1,\dots,1)})$ is the sign representation.

Problem 91. Show that $(\pi, \mathcal{M}^{(n-1,1)})$ is isomorphic to the standard permutation representation of S_n on $\mathbb{C}^n$ and that $(\pi, \mathcal{S}^{(n-1,1)})$ is isomorphic to the usual irreducible $n-1$-dimensional subrepresentation W_0 of $\mathbb{C}^n$.

Problem 92. Show that $(\pi, \mathcal{M}^\lambda)$ is a unitary representation of S_n with respect to the inner product

$$\left\langle \sum_{[t] \in \mathbf{YTD}^\lambda} z_{[t]}[t], \sum_{[s] \in \mathbf{YTD}^\lambda} w_{[s]}[s] \right\rangle_\lambda := \sum_{[t] \in \mathbf{YTD}^\lambda} z_{[t]} \overline{w_{[t]}}.$$

18. Lecture 18

18.1. Irreducibility of Specht modules. The goal is to prove the first part of Theorem 17.12. Recall that for any Young tableau t, $C(t) \subseteq S_n$ is the set of permutations that preserve the columns of t.

LEMMA 18.1. *For any $t \in \mathbf{YT}^\lambda$, $C(t)$ is a subgroup of S_n, and for any $\sigma \in S_n$,*

$$\sigma C(t)\sigma^{-1} = C(\sigma \cdot t).$$

PROOF. This is Problem 93. □

Recall that for $t \in \mathbf{YT}^\lambda$, we define $\mathbf{e}_t = \sum_{\sigma\in C(t)} \operatorname{sgn}(\sigma)[\sigma \cdot t]$, and $\mathcal{S}^\lambda = \operatorname{span}\{\mathbf{e}_t \,|\, t \in \mathbf{YT}^\lambda\} \subseteq \mathcal{M}^\lambda$.

PROPOSITION 18.2. *For any $\sigma \in S_n$ and $t \in \mathbf{YT}^\lambda$, $\pi(\sigma)\mathbf{e}_t = \mathbf{e}_{\sigma\cdot t}$. As a consequence, $(\pi, \mathcal{S}^\lambda)$ is a subrepresentation of $(\pi, \mathcal{M}^\lambda)$.*

PROOF. We have

$$\pi(\sigma)\mathbf{e}_t = \sum_{\tau\in C(t)} \operatorname{sgn}(\tau)[\sigma \cdot (\tau \cdot t)] = \sum_{\tau\in C(t)} \operatorname{sgn}(\tau)[\sigma\tau\sigma^{-1} \cdot (\sigma \cdot t)].$$

Using the previous lemma, we may rewrite this as follows:

$$\pi(\sigma)\mathbf{e}_t = \sum_{\tau\in C(\sigma\cdot t)} \operatorname{sgn}(\sigma^{-1}\tau\sigma)[\tau \cdot (\sigma \cdot t)] = \sum_{\tau\in C(\sigma\cdot t)} \operatorname{sgn}(\tau)[\tau \cdot (\sigma \cdot t)] = \mathbf{e}_{\sigma\cdot t}.$$

□

Given $t \in \mathbf{YT}^\lambda$, define $P_t : \mathcal{M}^\lambda \to \mathcal{M}^\lambda$ by

$$P_t = \sum_{\sigma\in C(t)} \operatorname{sgn}(\sigma) \cdot \pi(\sigma).$$

Observe that $P_t[t] = \mathbf{e}_t$ and if (π, W) is a subrepresentation of $(\pi, \mathcal{M}^\lambda)$, then $P_t\mathbf{w} \in W$ for all $t \in \mathbf{YT}^\lambda$ and $\mathbf{w} \in W$.

LEMMA 18.3. *Let $\langle\cdot,\cdot\rangle$ be any S_n-invariant inner product on $\mathcal{M}^\lambda$. Then P_t is self-adjoint with respect to $\langle\cdot,\cdot\rangle$, i.e.*

$$\langle P_t\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, P_t\mathbf{v}\rangle \qquad \forall \mathbf{u}, \mathbf{v} \in \mathcal{M}^\lambda.$$

PROOF. Since $\{[s]\}_{s\in\mathbf{YTD}^\lambda}$ spans $\mathcal{M}^\lambda$, by linearity it suffices to show that $\langle P_t[s], [r]\rangle = \langle [s], P_t[r]\rangle$ for all $r, s, t \in \mathbf{YT}^\lambda$. By the definition of P_t, we then have

$$\begin{aligned}
\langle P_t[s], [r]\rangle &= \sum_{\sigma\in C(t)} \operatorname{sgn}(\sigma)\langle \pi(\sigma)[s], [r]\rangle = \sum_{\sigma\in C(t)} \operatorname{sgn}(\sigma)\langle [s], \pi(\sigma^{-1})[r]\rangle \\
&= \sum_{\sigma\in C(t)} \operatorname{sgn}(\sigma^{-1})\langle [s], \pi(\sigma^{-1})[r]\rangle = \sum_{\sigma^{-1}\in C(t)} \langle [s], \operatorname{sgn}(\sigma)\pi(\sigma)[r]\rangle \\
&= \sum_{\sigma\in C(t)} \langle [s], \operatorname{sgn}(\sigma)\pi(\sigma)[r]\rangle = \langle [s], P_t[r]\rangle;
\end{aligned}$$

the second to last equality holding since $C(t)$ is a subgroup of S_n. □

PROPOSITION 18.4. *For any $t \in \mathbf{YT}^\lambda$ and $\mathbf{v} \in \mathcal{M}^\lambda$, $P_t\mathbf{v} \in \mathbb{C}\mathbf{e}_t$.*

Proof. Again using the fact that the tabloids $[s]$ span $\mathcal{M}^\lambda$, we need only show that $P_t[s] \in \mathbb{C}\mathbf{e}_t$ for each $s, t \in \mathbf{YT}^\lambda$. Let i and j be in the same column of t, so (ij) is in $C(t)$. We consider two cases.

Firstly, suppose that s is such that $[s]$ has the numbers i, j in a single row. Let $H = \langle (ij) \rangle = \{e, (ij)\}$. We now let $r_1, \ldots, r_m \in C(t)$ be a set of representatives for $C(t)/H$, i.e.

$$C(t) = r_1 H \sqcup r_2 H \sqcup \ldots \sqcup r_m H.$$

Since $[s]$ has i and j in the same row, $\pi(ij)[s] = [(ij) \cdot s] = [s]$. We can now compute:

$$\begin{aligned} P_t[s] = \sum_{\sigma \in C(t)} \operatorname{sgn}(\sigma)\pi(\sigma)[s] &= \sum_{k=1}^{m} \operatorname{sgn}(r_k)\pi(r_k)\Big(\operatorname{sgn}(e)\pi(e) + \operatorname{sgn}(ij)\pi(ij)\Big)[s] \\ &= \sum_{k=1}^{m} \operatorname{sgn}(r_k)\pi(r_k)\big(1 + (-1)\big)[s] = 0. \end{aligned}$$

We now consider the remaining case, i.e. i and j are in different rows of $[s]$. Letting the first column of t be $a_{1,1}, a_{1,2}, \ldots, a_{1,M}$, we let $\sigma_1 \in C(t)$ be a permutation that moves each $a_{1,i}$ into the row that it is in in $[s]$, and fixes all other elements. Similarly, let σ_2 be a permutation that permutes the elements of the second column into the rows they occupy in $[s]$, σ_3 a permutation of the third column of t, and so on. Then $[\sigma_{\lambda_1} \ldots \sigma_2\sigma_1 \cdot t] = [s]$, and since each σ_i is just a permutation of column entries, $\widetilde{\sigma} = \sigma_{\lambda_1} \ldots \sigma_2\sigma_1 \in C(t)$. We now compute

$$\begin{aligned} P_t[s] = \sum_{\sigma \in C(t)} \operatorname{sgn}(\sigma)[\sigma \cdot s] &= \sum_{\sigma \in C(t)} \operatorname{sgn}(\sigma)[\sigma\widetilde{\sigma} \cdot t] \\ &= \sum_{\sigma \in C(t)} \operatorname{sgn}(\sigma\widetilde{\sigma}^{-1})[\sigma \cdot t] = \operatorname{sgn}(\widetilde{\sigma})\mathbf{e}_t. \end{aligned}$$

□

Proof of the first part of Theorem 17.12. Let (π, W) be a subrepresentation of $(\pi, \mathcal{S}^\lambda)$. Suppose that there exists $t \in \mathbf{YT}^\lambda$ and $\mathbf{w} \in W$ such that $P_t\mathbf{w} \neq 0$. Then $P_t\mathbf{w} = \lambda\mathbf{e}_t$, with $\lambda \neq 0$. Then $\mathbf{e}_t \in W$, hence

$$\mathcal{S}^\lambda = \operatorname{span}\{\mathbf{e}_s \mid s \in \mathbf{YT}^\lambda\} = \operatorname{span}\{\mathbf{e}_{\sigma\cdot t} \mid \sigma \in S_n\} = \operatorname{span}\{\pi(\sigma)\mathbf{e}_t \mid \sigma \in S_n\} \subseteq W,$$

i.e. $W = \mathcal{S}^\lambda$.

On the other hand, if $P_t\mathbf{w} = 0$ for all $t \in \mathbf{YT}^\lambda$ and $\mathbf{w} \in W$, then

$$0 = \langle P_t\mathbf{w}, [t] \rangle = \langle \mathbf{w}, P_t[t] \rangle = \langle \mathbf{w}, \mathbf{e}_t \rangle,$$

thus $W \subseteq (\mathcal{S}^\lambda)^\perp$. Since $(\mathcal{S}^\lambda)^\perp \cap \mathcal{S}^\lambda = 0$, we then have that $W = 0$. □

The proof above actually gives the slightly stronger statement:

Theorem 18.5. *Let (π, W) be a subrepresentation of $(\pi, \mathcal{M}^\lambda)$. Then either $\mathcal{S}^\lambda \subseteq W$ or $W \subseteq (\mathcal{S}^\lambda)^\perp$.*

18.2. Exercises.

Problem 93. Let $\lambda \vdash n$. Given a Young tableau $t \in \mathbf{YT}^\lambda$ and $\sigma \in S_n$, show that $C(t)$ is a subgroup of S_n and that

$$\sigma C(t)\sigma^{-1} = C(\sigma \cdot t).$$

Problem 94. Write out the details of the proof of Theorem 18.5.

19. Lecture 19

19.1. Morphisms between permutation modules. We aim to complete the proof of Theorem 17.12 by showing that $(\pi, \mathcal{S}^\lambda)$ and $(\pi, \mathcal{S}^\mu)$ are non-isomorphic representations of S_n if $\lambda \neq \mu$.

To do this, we introduce a *partial order* $\trianglelefteq$ on the set of all Young diagrams of size n.

Definition 19.1. *Let $\lambda, \mu \vdash n$. We write $\lambda \trianglelefteq \mu$ if*

$$\sum_{i=1}^{m} \lambda_i \leq \sum_{i=1}^{m} \mu_i,$$

for all $1 \leq m \leq \min\{|\lambda|, |\mu|\}$.

Proposition 19.2. *The relation $\trianglelefteq$ defines a partial order on the set of all Young diagrams of size n.*

Proof. This is Problem 95 (not examinable). $\square$

Example 19.3. *For the partitions of 6, we have*

$$(4,2) \trianglelefteq (5,1) \trianglelefteq (6),$$

and

$$(3,2,1) \trianglelefteq (4,1,1) \trianglelefteq (4,2),$$

as well as

$$(3,2,1) \trianglelefteq (3,3) \trianglelefteq (4,2).$$

Observe that we $(3,3) \not\trianglelefteq (4,1,1)$, since $4+1 < 3+3$, but also $(4,1,1) \not\trianglelefteq (3,3)$, since $3 < 4$.

Proposition 19.4. *Let $\lambda, \mu \vdash n$. Suppose that there exists $t \in \mathbf{YT}^\lambda$ such that the morphism*

$$P_t = \sum_{\sigma \in C(t)} \operatorname{sgn}(\sigma)\pi(\sigma)$$

on $\mathcal{M}^\mu$ is non-zero. Then $\mu \trianglelefteq \lambda$.

Proof. If $P_t \neq 0$, then there exists a Young tableau s of shape μ such that $P_t[s] \neq 0$. Let i and j be two numbers from the same row of s. If i and j were in the same column of t, then $(ij) \in C(t)$, and in a similar way to the proof of Proposition 18.4, writing $H = \langle (ij) \rangle = \{e, (ij)\}$ and letting $r_1, \dots, r_m \in C(t)$ be a set of representatives for $C(t)/H$ (i.e. $C(t) = r_1 H \sqcup r_2 H \sqcup \ldots \sqcup r_m H$), we would then have

$$P_t[s] = \sum_j \pi(r_j)\big(\pi(e) - \pi(ij)\big)[s] = 0.$$

Thus any two numbers from the same row of s are in different columns of t. There are μ_1 numbers in the first row of s, which must be placed into different columns of t; t has λ_1 columns, hence $\mu_1 \leq \lambda_1$. After placing out the numbers from the first row, we then must place out the μ_2 numbers into the columns of t. Having already placed out the numbers of the first row of s, there are then $(\lambda_1 - \mu_1) + \lambda_2$ columns available to put them in, hence $\mu_2 \leq (\lambda_1 - \mu_1) + \lambda_2$, i.e. $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$. Proceeding in this manner gives $\mu \trianglelefteq \lambda$. $\square$

Proposition 19.5. *If there exists an S_n-homomorphism from $(\pi, \mathcal{M}^\lambda)$ to $(\pi, \mathcal{M}^\mu)$ whose restriction to $\mathcal{S}^\lambda$ is non-zero, then $\mu \trianglelefteq \lambda$.*

Proof. Letting $T \in \mathrm{Hom}_{S_n}(\mathcal{M}^\lambda, \mathcal{M}^\mu)$ be such that $T|_{\mathcal{S}^\lambda} \neq 0$, i.e. we can find one of the spanning vectors $\mathbf{e}_t \in \mathcal{S}^\lambda$ such that $T(\mathbf{e}_t) \neq 0$. Since $\mathbf{e}_t = P_t[t]$, we then have

$$0 \neq T(\mathbf{e}_t) = T(P_t[t]) = P_t T([t]).$$

The morphism P_t evaluated at $T([t]) \in \mathcal{M}^\mu$ is non-zero, so by Proposition 19.4 $\mu \trianglelefteq \lambda$. $\square$

Finishing the proof of Theorem 17.12. Suppose that $(\pi, \mathcal{S}^\lambda)$ and $(\pi, \mathcal{S}^\mu)$ are isomorphic, with $T \in \mathrm{Hom}_G(\mathcal{S}^\lambda, \mathcal{S}^\mu)$. We decompose $\mathcal{M}^\lambda$ as

$$(\pi, \mathcal{M}^\lambda) = (\pi, \mathcal{S}^\lambda) \oplus (\pi, (\mathcal{S}^\lambda)^\perp),$$

and trivially extend T to an morphism $\widetilde{T}$ on all of $\mathcal{M}^\lambda$ by setting it to zero on $(\mathcal{S}^\lambda)^\perp$, i.e.

$$\widetilde{T}(\mathbf{v} + \mathbf{w}) := T(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathcal{S}^\lambda,\ \mathbf{w} \in (\mathcal{S}^\lambda)^\perp.$$

This is also a S_n-homomorphism since $(\pi, (\mathcal{S})^\perp)$ is a subrepresentation. Since T is non-zero, so is $\widetilde{T}$, allowing us to apply the Proposition 19.5 to get $\mu \trianglelefteq \lambda$. By symmetry (i.e. carrying out the same argument with T^{-1} in place of T), we also have $\lambda \trianglelefteq \mu$, thus $\lambda = \mu$. $\square$

19.2. Standard Young Tableaux.

Definition 19.6. *A Young tableau is said to be* <u>standard</u> *if the numbers in each row and column are increasing left to right and top to bottom.*

The set of all standard Young tableaux of shape λ is denoted $\mathbf{SYT}^\lambda$. We define

$$f_\lambda = |\mathbf{SYT}^\lambda|.$$

We have the following combinatorial result which is a generalisation of the Robinson–Schensted correspondence (1938,1961).

Theorem 19.7 (Robinson–Schensted–Knuth correspondence, 1970). *There is a bijection between S_n and $\bigcup_{\lambda \vdash n} \mathbf{SYT}^\lambda \times \mathbf{SYT}^\lambda$. In particular,*

$$n! = \sum_{\lambda \vdash n} (f_\lambda)^2.$$

We do not have the time to prove this here. A nice discussion for the interested reader can be found in Chapter 4 of Young Tableaux by William Fulton.

Theorem 19.8. *The set $\{\mathbf{e}_t \,|\, t \in \mathbf{SYT}^\lambda\}$ is a basis of $\mathcal{S}^\lambda$.*

Proof. We will show that for each $\lambda \vdash n$, the set $\{\mathbf{e}_t \,|\, t \in \mathbf{SYT}^\lambda\}$ is linearly independent. From this, we then get that

$$\dim(\mathcal{S}^\lambda) \geq |\{\mathbf{e}_t \,|\, t \in \mathbf{SYT}^\lambda\}| = f_\lambda.$$

By Theorem 6.6(ii), Theorem 17.12, and Theorem 19.7, we then have

$$n! = \sum_{\lambda \vdash n} \dim(\mathcal{S}^\lambda)^2 \geq \sum_{\lambda \vdash n} (f_\lambda)^2 = n!,$$

which gives $\dim(\mathcal{S}^\lambda) = f_\lambda$, i.e. the set $\{\mathbf{e}_t \,|\, t \in \mathbf{SYT}^\lambda\}$ spans $\mathcal{S}^\lambda$, and is therefore a basis.

So it remains to show that the $\mathbf{e}_t$, $t \in \mathbf{SYT}^\lambda$, are linearly independent. Before this, we need to introduce another order. $\square$

Definition 19.9. *Define the relation $\prec$ on $\mathbf{YTD}^\lambda$ as follows. We say $[s] \prec [t]$ if there exists an $1 \leq i \leq n$ such that for each $j < i$ we have j in the same row in both $[s]$ and $[t]$ and i appears in a row of $[s]$ below the row it appears in $[t]$.*

Example 19.10. *The six elements of $\mathbf{YTD}^{(2,2)}$ are*

$$\begin{array}{c}\hline 1\ 2\\\hline 3\ 4\\\hline\end{array},\quad \begin{array}{c}\hline 1\ 3\\\hline 2\ 4\\\hline\end{array},\quad \begin{array}{c}\hline 1\ 4\\\hline 2\ 3\\\hline\end{array},\quad \begin{array}{c}\hline 3\ 4\\\hline 1\ 2\\\hline\end{array},\quad \begin{array}{c}\hline 2\ 4\\\hline 1\ 3\\\hline\end{array},\quad \begin{array}{c}\hline 2\ 3\\\hline 1\ 4\\\hline\end{array}.$$

The order of these elements is

$$\begin{array}{c}\hline 3\ 4\\\hline 1\ 2\\\hline\end{array} \prec \begin{array}{c}\hline 2\ 4\\\hline 1\ 3\\\hline\end{array} \prec \begin{array}{c}\hline 2\ 3\\\hline 1\ 4\\\hline\end{array} \prec \begin{array}{c}\hline 1\ 4\\\hline 2\ 3\\\hline\end{array} \prec \begin{array}{c}\hline 1\ 3\\\hline 2\ 4\\\hline\end{array} \prec \begin{array}{c}\hline 1\ 2\\\hline 3\ 4\\\hline\end{array}.$$

Proposition 19.11. *The relation $\prec$ defines a total order on $\mathbf{YTD}^\lambda$*

Proof. This is Problem 96 (not examinable). ☐

Lemma 19.12. *For any $t \in \mathbf{SYT}^\lambda$ we have $[\sigma \cdot t] \prec [t]$ for all $\sigma \in C(t) \setminus \{e\}$.*

Proof. Given $\sigma \in C(t) \setminus \{e\}$, let i be the smallest number in $\{1, \dots, n\}$ such that i is in a different position in $\sigma \cdot t$ than in t. Since i is moved to a place in the same column, it must be moved down to a lower row, since otherwise it would displace a number above it in the same column, which is necessarily smaller than i, as t is standard. This would contradict the minimality in the choice of i. Thus, for every $j < i$, $[\sigma \cdot t]$ and $[t]$ have j in the same row, and $[\sigma \cdot t]$ has i in a lower row, hence $[\sigma \cdot t] \prec [t]$. ☐

Finishing the proof of Theorem 19.8. Suppose that $\{\mathbf{e}_t \,|\, t \in \mathbf{SYT}^\lambda\}$ is linearly dependent. Then there exist z_t (not all zero) such that

$$\sum_{t \in \mathbf{SYT}^\lambda} z_t \mathbf{e}_t = 0. \tag{19.13}$$

Suppose $t_0 \in \mathbf{SYT}^\lambda$ such that $[t_0]$ is maximal among all $[t]$ where $z_t \neq 0$. Since the tabloids form a basis of $\mathcal{M}^\lambda$, once all the $\mathbf{e}_t$ with non-zero z_t have been expanded into tabloids, the coefficient in front of $[t_0]$ must be zero. But $[t_0]$ cannot appear as a component in any of the other $\mathbf{e}_t$, since by assumption $[t] \prec [t_0]$, hence by Lemma 19.12, we have $[s] \prec [t] \prec [t_0]$ for any $[s]$ a component of $[\mathbf{e}_t]$. Thus $z_{t_0} = 0$, a contradiction. ☐

19.3. Exercises.

Problem 95. Verify that $\trianglelefteq$ is a partial order on the set of partitions of n (not examinable).

Problem 96. Verify that $\prec$ is a strict total order on $\mathbf{YTD}^\lambda$ (not examinable).

Problem 97. Show that if $s, t \in \mathbf{SYT}^\lambda$, then $[s] \neq [t]$ if and only if $s \neq t$.

Problem 98. Use Proposition 19.5 to compute the characters of $(\pi, \mathcal{S}^{(4,2)})$ and then $(\pi, \mathcal{S}^{(3,3)})$ *Challenging! Hint: Proposition 19.5 puts a restriction on the shapes of the $\mathcal{S}^\lambda$ that can appear as subrepresentations of $\mathcal{M}^{(4,2)}$ and $\mathcal{M}^{(3,3)}$.*

20. Lecture 20

20.1. The branching rule. We may view S_{n-1} as a subgroup of S_n by letting it consist of all permutations that fix n, i.e.

$$S_{n-1} = \{\sigma \in S_n \mid \sigma(n) = n\} = \mathrm{Stab}_{S_n}(n).$$

The goal of this section is to decompose $\mathrm{Res}^{S_n}_{S_{n-1}}(\pi, \mathcal{S}^\lambda)$ into irreducible representations of S_{n-1}, i.e. Specht modules.

We say that a box in the Young diagram of λ is *removable* if it has no boxes below or to the right of it.

THEOREM 20.1. *For any $\lambda \vdash n$,*

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\pi, \mathcal{S}^\lambda) = \bigoplus_{\lambda^- \vdash (n-1)} (\pi, \mathcal{S}^{\lambda^-}),$$

where the direct sum runs over all $\lambda^- \vdash (n-1)$ whose Young diagrams are obtained by removing a removable box from the Young diagram for λ.

EXAMPLE 20.2.

$$\mathrm{Res}^{S_5}_{S_4}\left(\pi, \mathcal{S}^{\ydiagram{3,2}}\right) = \left(\pi, \mathcal{S}^{\ydiagram{3,1}}\right) \oplus \left(\pi, \mathcal{S}^{\ydiagram{2,2}}\right).$$

PROOF. We label the removable boxes of λ by $\mathsf{c}_1, \dots \mathsf{c}_r$, with the numbering going from bottom to top. For each $i = 1, \dots, r$, we define a map $T_i' : \mathcal{M}^\lambda \to \mathcal{M}^{\lambda \setminus \mathsf{c}_i}$ by defining it on the basis given by tabloids (and then extending linearly):

$$T_i'([t]) := \begin{cases} 0 & \text{if } n \text{ and } \mathsf{c}_i \text{ are not in the same row} \\ [t] \setminus n & \text{otherwise,} \end{cases}$$

i.e. if not zero, then $T_i'([t])$ is the Young tabloid of shape $\lambda \setminus \mathsf{c}_i$, with the n removed from $[t]$. We claim that T_i' is an S_{n-1}-homomorphism from $(\mathrm{Res}^{S_n}_{S_{n-1}} \pi, \mathcal{M}^\lambda)$ to $(\pi, \mathcal{M}^{\lambda \setminus \mathsf{c}_i})$ (see Problem 101).

Observe that for elements of $\mathbf{SYT}^\lambda$, the number n must be in a removable box, since there are no greater numbers to place either to its right or below. Supposing that $t \in \mathbf{SYT}^\lambda$ is such that n is in c_j, for any element $\sigma \in C(t)$ the element n is then in in the same row of $[\sigma \cdot t]$ as c_j or above. This gives $T_i'([\sigma \cdot t]) = 0$ for all $j > i$, since c_j is in a higher row than c_i. In particular, we obtain $\mathbf{e}_t \in \ker T_i'$ for all $j > i$.

On the other hand, if $t \in \mathbf{SYT}^\lambda$ has the number n in the same row as c_i, then

$$T_i'(\mathbf{e}_t) = \sum_{\sigma \in C(t)} \mathrm{sgn}(\sigma) T_i'([\sigma \cdot t]) = \left(\sum_{\sigma \in C(t) \cap S_{n-1}} \mathrm{sgn}(\sigma) T_i'([\sigma \cdot t]) \right) + \left(\sum_{\sigma \in C(t) \setminus S_{n-1}} \mathrm{sgn}(\sigma) T_i'([\sigma \cdot t]) \right).$$

For $\sigma \in C(t) \setminus S_{n-1}$, $T_i'([\sigma \cdot t]) = 0$, since (as described above) σ must move n to a box in a higher row. For $\sigma \in S_{n-1}$, $T_i'[\sigma \cdot t] = [(\sigma \cdot t) \setminus n] = [\sigma \cdot (t \setminus n)]$, hence

$$T_i'(\mathbf{e}_t) = \sum_{\sigma \in C(t) \cap S_{n-1}} \mathrm{sgn}(\sigma)[\sigma \cdot (t \setminus n))] = \sum_{\sigma \in C(t \setminus n)} \mathrm{sgn}(\sigma)[\sigma \cdot (t \setminus n))] = \mathbf{e}_{t \setminus n}.$$

We note also that since every standard Young tableau of shape $\lambda \setminus \mathsf{c}_i$ may be obtained by deleting the n from a standard Young tableau of shape λ with n in c_i,

$$T_i'\big(\operatorname{span}\{\mathbf{e}_t \mid t \in \mathbf{SYT}^\lambda \text{ and } t \text{ has } n \text{ in box } \mathsf{c}_i\}\big) = \operatorname{span}\big\{\mathbf{e}_t \mid t \in \mathbf{SYT}^{\lambda\setminus\mathsf{c}_i}\big\} \tag{20.3}$$
$$= \mathcal{S}^{\lambda\setminus\mathsf{c}_i}.$$

We denote by T_1 the restriction of T_1' to $\mathcal{S}^\lambda$, and then for all other i,

$$T_{i+1} := T_{i+1}'|_{\ker(T_i)}.$$

This gives rise to a sequence of S_{n-1}-invariant subspaces of $\mathcal{S}^\lambda$:

$$\mathcal{S}^\lambda \supset \ker(T_1) \supset \ker(T_2) \supset \ldots \supset \ker(T_r).$$

By Lemma 2.9 and Proposition 11.1 (together with the isomorphism theorem for vector spaces),

$$(\pi, \mathcal{S}^\lambda) \cong \big(\pi, \operatorname{im}(T_1)\big) \oplus \big(\pi, \ker(T_1)\big),$$

and for $i \geq 1$,

$$\big(\pi, \ker(T_i)\big) \cong \big(\pi, \operatorname{im}(T_{i+1})\big) \oplus \big(\pi, \ker(T_{i+1})\big).$$

Noting that $\ker(T_r) = 0$, we then have

$$(\pi, \mathcal{S}^\lambda) \cong \big(\pi, \operatorname{im}(T_1)\big) \oplus \big(\pi, \operatorname{im}(T_2)\big) \oplus \ldots \oplus \big(\pi, \operatorname{im}(T_r)\big).$$

By (20.3), $\operatorname{im}(T_i) = \mathcal{S}^{\lambda\setminus\mathsf{c}_i}$, hence

$$(\pi, \mathcal{S}^\lambda) \cong \big(\pi, \mathcal{S}^{\lambda\setminus\mathsf{c}_1}\big) \oplus \big(\pi, \mathcal{S}^{\lambda\setminus\mathsf{c}_2}\big) \oplus \ldots \oplus \big(\pi, \mathcal{S}^{\lambda\setminus\mathsf{c}_r}\big),$$

as claimed. □

20.2. Induction. Using Theorem 20.1 and Frobenius reciprocity, we can now decompose the induction from S_{n-1} to S_n of Specht modules into irreducible representations of S_n:

THEOREM 20.4. *For any $\lambda \vdash n-1$,*

$$\operatorname{Ind}_{S_{n-1}}^{S_n}(\pi, \mathcal{S}^\lambda) = \bigoplus_{\lambda^+ \vdash n} (\pi, \mathcal{S}^{\lambda^+}),$$

where the direct sum runs over all $\lambda^+ \vdash n$ whose Young diagrams are obtained by adding a single box to the Young diagram for λ.

PROOF. Let χ_λ be the character of $(\pi, \mathcal{S}^\lambda)$. For any $\mu \vdash n$, by Corollary 15.2, we have

$$\langle \operatorname{Ind}_{S_{n-1}}^{S_n} \chi_\lambda, \chi_\mu \rangle_{S_n} = \langle \chi_\lambda, \operatorname{Res}_{S_{n-1}}^{S_n} \chi_\mu \rangle_{S_{n-1}}.$$

By Theorem 20.1, $\langle \chi_\lambda, \operatorname{Res}_{S_{n-1}}^{S_n} \chi_\mu \rangle_{S_{n-1}} = 1$ if λ is obtained by removing a box from μ (which is equivalent to μ being obtained by adding a box to λ), and zero otherwise. □

REMARK 20.5. *The relation between restriction and induction of Specht modules is displayed pictorially by* Young's lattice.

20.3. Exercises.

Problem 99. Decompose $\mathrm{Ind}_{S_4}^{S_6}\left(\pi, \mathcal{S}^{(3,1)}\right)$ into irreducible representations of S_6.

Problem 100. Let $T : V \to W$ be a G-homomorphism between two representations (π, V) and (ρ, W). Show that

$$(\pi, V) \cong \big(\pi, \ker(T)\big) \oplus \big(\rho, \mathrm{im}(T)\big).$$

Problem 101. Show that the maps T_i' and T_i defined during the proof of Theorem 20.1 are S_{n-1}-homomorphisms.