LOCAL FUNCTIONS ON BLOCKS AND AUTOMORPHISMS OF PARTIAL GROUPS

by

JAMIE MASON

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School of Mathematics
College of Engineering and Physical Sciences
The University of Birmingham
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Abstract

In the first chapter of this thesis we define a block-by-block version of Isaacs and Navarro's chain local condition and then prove that the Alperin–McKay conjecture is equivalent to a certain function on groups having this property. We then go on to prove several other block-by-block versions of results from Isaacs and Navarro's paper. The results in this chapter have also been published in [35].

The second part concerns automorphisms of partial groups, specifically which groups can arise as automorphisms of different types of partial group. We show that for any finite group one can construct a finite partial group that has this finite group as an automorphism group. We also prove an analogous result for groups and objective partial groups as well as a partial result for finite groups and finite objective partial groups. Lastly we show that there are no automorphism groups of localities that do not arise as automorphism groups of groups, a rephrasing of the same result for fusion systems.

This thesis is split into two entirely self-contained chapters.

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CONTENTS

CHAPTER 1

LOCAL FUNCTIONS ON BLOCKS

1.1 Introduction

We define an integer-valued function, f, on pairs (G, B), where G is a group and B is a p-block of G, to be *block chain local* if

$$\sum_{C \in \mathcal{R}} (-1)^{|C|} \sum_{b | B_C} f(G_C, b) = 0,$$

where the first sum runs over representatives of G-orbits of chains of p-subgroups in G, G_C is the stabiliser group of C, B_C is a sum of blocks of G_C—which we will define later—and b is a summand of B_C. This definition is motivated as a block-by-block version of the definition of local functions given by Isaacs and Navarro [31]. In their paper they show that the McKay conjecture is equivalent to a function on groups having this local property and here we extend this result to show that the Alperin–McKay conjecture is equivalent to a function on pairs being block chain local.

As Isaacs and Navarro note, their work is not the first occurrence of a local-global conjecture being restated as a condition on chains of subgroups. This is partially as a result of the local-global conjectures' resistance to being proved in full generality, with the exceptions being Brauer's height zero conjecture [34] and the McKay conjecture, a proof of which has been announced by Späth. Effort has therefore been directed towards finding equivalent statements to the conjectures. Perhaps the most well-known example of this is Knörr and Robinson's restatement of Alperin's weight conjecture [33, Theorem 3.8], which in our language is equivalent to $\mathrm{l}(G, B)$, the number of irreducible Brauer characters in B, being block chain local. They also show that $\mathrm{k}(G, B) - \mathrm{l}(G, B)$ is, in our language, block chain local [33, Corollary 4.3] and thus $\mathrm{k}(G, B)$, the number of irreducible ordinary characters in B, being block chain local is dependent on Alperin's weight conjecture holding. We extend some of the ideas in their work to show that the Alperin–McKay conjecture can also be restated in a similar way.

In order to do this we first begin by introducing the concepts within modular representation theory that are fundamental to understanding the rest of this chapter, namely blocks and the Brauer correspondence. We then introduce chains, an involution on the set of chains and normalising triples. These are defined by Isaacs and Navarro [31] but we define them here so that this chapter can be read independently of their paper. We can then go on to define what we mean by block chain local and introduce the Alperin–McKay function. We then prove that this function is block chain local if and only if the conjecture holds. This proof method has much the same structure as [31, Section 4] but note that most results will be different as we are considering blocks.

The final section of this chapter is dedicated to proving several other block-by-block versions of results from Isaacs and Navarro, namely sufficient conditions for functions to be block chain local and that the function k_1, counting characters of defect one in a block, is block chain local.

Throughout this chapter G will be a finite group with order divisible by a prime p and k will be an algebraically closed field of characteristic p unless stated otherwise.

1.2 Block theory

Many of the definitions and results in this section can be given for an arbitrary algebra A over k, but one has to be slightly more rigorous in their constructions than we are. For brevity we have just considered group algebras of finite groups.

1.2.1 Brauer characters

Before we discuss blocks we must first define Brauer characters. Recall that when $k = \mathbb{C}$ we can associate to each irreducible module a class function, called a character, and these characters form a basis of the space of class functions. Details of this standard set up can be found in any introduction to representation theory course. From now on we shall call these characters ordinary characters as we are going to define their counterparts when k is of characteristic p, Brauer characters. As with many objects in modular representation theory we cannot construct these Brauer characters in the same way as if $k = \mathbb{C}$.

Definition 1.2.1. Let G_{reg} denote the set of elements of G with order not divisible by p and let $\rho : G \to \mathrm{GL}_n(k)$ be a representation of G. The *Brauer character* corresponding to ρ is the map

$$\psi_\rho : G_{reg} \longrightarrow \mathbb{C},$$

given by mapping the eigenvalues of $\rho(g)$ into $\mathbb{C}$, and then summing, for all g in G_{reg}. This is done by fixing a bijection from $|G_{reg}|$th roots of unity in k to $|G_{reg}|$th roots of unity in $\mathbb{C}$.

From this definition we see how these relate to ordinary characters and how they avoid the problem that characteristic p gives, namely that if we were to take just the trace of $\rho(g)$ we would lose information. Much like ordinary characters, Brauer characters are called irreducible when the corresponding module is irreducible. We can also write any Brauer character as a sum of irreducible Brauer characters, again as with ordinary characters. What is less clear is how to find these irreducible Brauer characters, but fortunately we do not need to consider that problem. More detail on work in this area can be found in [16, Section 2.2].

Given an irreducible ordinary character χ of G, if we restrict χ to G_{reg} we still have a class function, so we obtain a Brauer character. We denote this by χ^0, and because of our above discussion we have

$$\chi^0 = \sum_{\psi} d_{\chi,\psi}\psi,$$

where the sum runs over all irreducible Brauer characters of G and each $d_{\chi,\psi}$ is in $\mathbb{C}$. We call these $d_{\chi,\psi}$ *decomposition numbers.*

This is as far as our discussion of Brauer characters needs to go. Crucially we have a way to connect the irreducible ordinary characters of G with the irreducible Brauer characters of G, using these decomposition numbers. This will further allow us to link irreducible ordinary characters with other aspects of the local representation theory of G. This skeletal introduction to Brauer characters follows that of [16, Section 2.2], where a much more detailed explanation of the theory can be found.

1.2.2 Blocks

Many results of local representation theory have two versions: block-by-block or block-free. As we will be working with the former it makes sense to start by defining what a block is.

Definition 1.2.2. The group algebra kG has a decomposition into a direct sum of indecomposable two-sided ideals

$$kG = B_1 \oplus \cdots \oplus B_n.$$

Each B_i, for $1 \le i \le n$, is called a *block* of kG.

These are the fundamental components of local representation theory and allow us to partition the representation theory of a group algebra kG. Before we discuss that in more detail we quote the following result.

Theorem 1.2.3 (See [2, Theorem 13.1])**.** *The decomposition of kG into its blocks is unique up to the ordering of the decomposition.*

The proof of this is relatively simple and can be found in all introductory books on the subject. We have omitted it for this reason and point the reader to [2, Theorem 13.1] if they need the proof. We can now begin to discuss what we mean by the blocks partitioning the representation theory of a group algebra.

Definition 1.2.4. We say a module, M, of kG lies in the block B_i of kG if $B_iM = M$ and $B_jM = 0$ for all $j \neq i$. We say a Brauer character χ lies in a block B_i if its corresponding module does.

We see that this condition implies that submodules and quotients of modules lying in a block also lie in the same block. Furthermore direct sums of modules in a given block also lie in that block. The following result gives us the significance of this definition.

Proposition 1.2.5 (see [2, Theorem 13.2])**.** *If M is a finite-dimensional kG-module then M has a unique direct sum decomposition*

$$M = M_1 \oplus \cdots \oplus M_n,$$

where each M_i lies in B_i.

Again we have omitted the proof but one is given in [2, Theorem 13.1]. This result now shows how the blocks of kG partition its representation theory. In fact one can go further and show that the simple modules of kG are partitioned among the blocks and thus so are the irreducible Brauer characters. Further details of this can be found in [2, Section 13].

We can extend this idea of belonging in a block to irreducible ordinary character as well. Let χ be an irreducible ordinary characters of G. We say χ belongs to the same block as some Brauer character ψ if $d_{\chi,\psi}$ is non-zero. This is in fact well-defined as the other irreducible Brauer characters of G with non-zero decomposition numbers with respect to χ also belong to the same block as ψ. A proof of this can be found in [20, p.147]. This is why we needed to define Brauer characters, to provide this link between irreducible ordinary characters and block through the decomposition numbers.

1.2.3 Primitive central idempotents

What we have described in the previous section is not the only way to characterise the blocks of a group algebra. In order to discuss another way we must first recall the definition of an idempotent, a non-zero element e in kG such that $e^2 = e$.

Definition 1.2.6. We say an idempotent e in kG is *central* if it commutes with all other elements in kG, in other words $ea = ae$ for all a in kG. It is *primitive* if it cannot be written as the sum of two idempotents e_1 and e_2 where $e_1e_2 = 0$. A *primitive central* idempotent is primitive with respect to only central idempotents.

One should easily see from the definition that any central idempotent can be written as the sum of primitive central idempotents.

Now we will explore the significance of these primitive central idempotents in relation to the block decomposition of kG. From the block decomposition we see that the identity in kG splits into a sum

$$1 = e_1 + \cdots + e_n,$$

where each e_i is in B_i. As each B_i is a two-sided ideal we have that e_ie_j is in the intersection of B_i and B_j, so is zero, for all $i \neq j$. Further as 1 is an idempotent we have

$$\sum_{i=1}^{n} e_i^2 = \left(\sum_{i=1}^{n} e_i\right)^2 = 1^2 = 1 = \sum_{i=1}^{n} e_i,$$

so each e_i is an idempotent. As 1 is central in kG we see that

$$\sum_{i=1}^{n} e_i a = 1 \cdot a = a \cdot 1 = \sum_{i=1}^{n} a e_i,$$

for all a in kG. However as e_i is in the two-sided ideal B_i we have that e_ia and ae_i are both in B_i and are the only elements in each sum with this property. Therefore $e_ia = ae_i$

and e_i is central. This also gives us that $e_ikG = B_i$ as

$$kG = 1 \cdot kG = \sum_{i=1}^{n} e_ikG,$$

but $e_ikG \subseteq B_i$ so we must have equality. Lastly to see that e_i is primitive among central idempotents we suppose that it is not, in other words $e_i = e + e'$ where e and e' are non-zero central idempotents and $ee' = 0$. However this implies

$$kGe \oplus kGe' = B_i = ekG \oplus e'kG.$$

As e and e' are central we have $kGe = ekG$ and $kGe' = e'kG$. Thus they are both two-sided ideals which is a contradiction as B_i is indecomposable among two-sided ideals.

So what we have shown is that the decomposition of 1 among the blocks of kG gives us a set of primitive central idempotents with each one corresponding to one block of kG. One can go further and show that in fact these are all the primitive central idempotents of kG as the following result details.

Theorem 1.2.7 (See [15, Proposition 2.3])**.** *There is a bijection between the set of blocks of kG and the set of primitive central idempotents of kG given by $e \mapsto ekG$.*

A proof of this result, in slightly more generality, is given in [15, Proposition 2.3]. We will call the primitive central idempotent corresponding to a block B the *block idempotent* of B. While this correspondence often means authors can use block to refer to both the 2-sided ideal and the corresponding idempotent, we will try to avoid doing this.

1.3 The Brauer correspondence

We can now move on to discuss one of the most fundamental results of local representation theory, the Brauer correspondence. This result allows us to determine a global property of the representation theory of kG using only local information. It may be of no surprise then that the Brauer correspondence forms the basis of the many local-global conjectures within the field. We will introduce two of these here, namely the McKay and Alperin–McKay conjectures.

1.3.1 The Brauer morphism and Brauer pairs

In order to discuss the Brauer correspondence we first need to define several fundamental objects that are also attributed to Brauer. The first of these is the Brauer morphism.

Definition 1.3.1. For Q a p-subgroup of G, the *Brauer morphism* is the k-linear map given by

$$\mathrm{Br}_Q : kG \longrightarrow kC_G(Q) \; ; \; \sum_{g \in G} \alpha_g g \longmapsto \sum_{g \in C_G(Q)} \alpha_g g.$$

One can easily see that the Brauer morphism is a surjective map. As well as being very useful in defining Brauer pairs and the defect group, the Brauer morphism will continue to be useful throughout this chapter. This is because, in vague terms, it gives us a way to link blocks of a subgroup of G to blocks of G by looking at their idempotents. We will start to explore that with Brauer pairs; however, first we need to discuss several properties of the Brauer morphism.

This requires us to define what it means for an element of kG to be Q-stable. We say x in kG is *Q-stable* if for all q in Q we have $x^q = x$, in other words x is fixed under the conjugation action of Q. If we restrict the Brauer morphism to just these elements we obtain the following result.

Proposition 1.3.2 (See [15, Propostion 2.12])**.** *For Q a p-subgroup of G, the Brauer morphism* Br_Q *is multiplicative when restricted to Q-stable elements of kG.*

A proof of this is given in [15, Propostion 2.12]. We can now use this to prove the property of the Brauer morphism we require.

Corollary 1.3.3 (See [15, Corollary 2.13])**.** *If Q is a p-subgroup of G and e_B a block idempotent of kG then $Br_Q(e_B)$ is a sum of block idempotents of* $k\mathrm{C}_G(Q)$*.*

Proof. As e_B is a block idempotent then it is central in kG and thus commutes with all of Q. Hence e_B is Q-stable. Since Br_Q is multiplicative on Q-stable elements

$$\mathrm{Br}_Q(e_B) = \mathrm{Br}_Q\big(e_B^2\big) = \mathrm{Br}_Q(e_B)^2\,,$$

and thus $\mathrm{Br}_Q(e_B)$ is also an idempotent of $k\mathrm{C}_G(Q)$. Note that $k\mathrm{C}_G(Q)$ is Q-stable so

$$\mathrm{Br}_Q(e_B) = \mathrm{Br}_Q(e_B^x) = \mathrm{Br}_Q(x)^{-1}\,\mathrm{Br}_Q(e_B^x)\,\mathrm{Br}_Q(x) = \mathrm{Br}_Q(e_B)^x$$

for all x in $k\mathrm{C}_G(Q)$, meaning $\mathrm{Br}_Q(e_B)$ is in $\mathrm{Z}(k\mathrm{C}_G(Q))$. Thus as $\mathrm{Br}_Q(e_B)$ is a central idempotent it is a sum of block idempotents by definition. $\square$

Crucially this corollary means that, given a block of kG, the Brauer morphism gives us a set of blocks of $k\mathrm{C}_G(Q)$ related to that block through the image of its idempotent. This connection will be fundamental in many of the arguments we make later in the chapter, including the definition of a B-Brauer pair which follows in this section. However, first we must define a Brauer pair.

Definition 1.3.4. Let G be a group with order divisible by p. A *Brauer pair* is a pair (Q, e) where Q is a p-subgroup of G and e is a block idempotent of $kC_G(Q)$.

Although useful in some settings in their own right, we only require the definition of Brauer pairs in order to define B-Brauer pairs.

Definition 1.3.5. If B is a block of kG with block idempotent e_B then we define a *B-Brauer pair* to be a Brauer pair (Q, e) with the additional condition that $\mathrm{Br}_Q(e_B)\,e = e$.

This is where we see the usefulness of the Brauer morphism. What the condition $\mathrm{Br}_Q(e_B)\,e = e$ tells us is that e is one of the block idempotents in the sum of block idempotents that is $\mathrm{Br}_Q(e_B)$. So for (Q, e) to be a B-Brauer pair we require the block corresponding to e to be related to B through the Brauer morphism.

1.3.2 The defect of a block

The main purpose of B-Brauer pairs for us in this chapter is to use them to define the defect and the defect group of a block. Note that this is not the only way to define these; however, we hope that this way is easier to follow for someone unfamiliar with block theory. An alternate explanation, using vertex modules, is given in [3].

We say that a B-Brauer pair (D, e) is *maximal* if $|D|$ is maximal among all subgroups of G that appear in B-Brauer pairs. We can now define the defect group of a block.

Definition 1.3.6. A subgroup D of G is a *defect group* of a block B of kG if there exists a maximal B-Brauer pair (D, e), by inclusion, for some e. We say that a block B has *defect* d where $|D| = p^d$ and D is a defect group of B.

What this means is that to each block of kG we associate a set of p-subgroups of G which will prove crucial in the following sections. By the definition it is clear that the defect groups of a block B all have the same order, namely p^d, where d is the defect of B. In fact one can show that the defect groups of a block are even more intrinsically linked than that.

Theorem 1.3.7 (See [15, Theorem 2.27])**.** *All the defect groups of a block B of kG are G-conjugate.*

We will not prove this here as the proof involves using the relative trace map which we have avoided defining for brevity. A proof of this is given in [15, Theorem 2.27]. This is why, in this chapter, we will often say a defect group of B is the defect group of B, as we will generally be working up to G-conjugacy.

The fact that all defect groups of a block are conjugate in G may remind the reader of another set of p-groups in G that are conjugate, namely the Sylow p-subgroups. In fact one can show these groups are the defect groups of a block of kG, namely the principal block of kG or the block containing the trivial module. To show this let us first state the following result.

Theorem 1.3.8. *The defect of a block B is the smallest non-negative integer d such that p^{a-d} divides $\chi(1)$ for all χ an irreducible character in B, where a is the power of p in the p-part of G.*

This is how the defect of a block is defined in many texts. We have avoided this definition as it leads to a definition of the defect group which is more opaque than our own. A proof that these definitions are equivalent is given in [39, Theorem 4.6].

However, this equivalent definition makes our prior observation more clear. As the principal block contains the trivial module, it also contains the trivial character, in other words the character where $\chi(x) = 1$ for all x in kG. So for p^{a-d} to divide $\chi(1) = 1$ we require $a = d$, in other words the order of the defect group of the principal block is the same as the order of a maximal p subgroup of G. Thus the defect groups are precisely the Sylow p-subgroups.

Now that we have defined the defect of a block we can use this to define a property of the irreducible ordinary characters of G which will be useful in stating conjectures in the following sections.

Definition 1.3.9. Let χ be an irreducible ordinary character of G lying in a block B. The *height* of χ is given as the exact power of p dividing

$$\frac{\chi(1)}{p^{a-d}},$$

where a is the power of the p-part of G and d is the defect of B.

1.3.3 Brauer's first main theorem

We have now given sufficient definitions for us to encounter one of the more significant results of local representation theory, Brauer's first main theorem. Giving rise to the Brauer correspondence this result will be the first indication of the local control that is exerted on the representation theory of G.

Theorem 1.3.10 (Brauer's First Main Theorem)**.** *Let G be a finite group with D a p-subgroup. There is a bijection between the blocks of kG with defect group D and blocks of $\mathrm{N}_G(D)$ with defect group D given by the map Br_D, the Brauer morphism at D, on the block idempotents.*

Originally proved by Brauer [7, 8, 9] this result, along with Brauer's two other main theorems, formed the backbone of early work in local representation theory. We have not introduced the necessary apparatus to prove it here as much of it is not useful in achieving our final goal. A proof is given in [15, Theorem 2.30] which, because this chapter uses this book as a source, should be the easiest for a reader to follow.

1.3.4 The McKay and Alperin–McKay conjectures

The crucial significance of the Brauer correspondence will become abundantly clear with its use throughout this chapter. However in short it is most important as it gives us a vehicle in which to relate properties of subgroups of G with properties of G itself. This is why it forms the basis of many of the so-called local-global conjectures, which in turn is where much of the current focus for research in this field is found. We will be exploring one of these conjectures in this chapter, namely the Alperin–McKay conjecture. However, first we will introduce its predecessor, the McKay conjecture.

Theorem 1.3.11 (McKay conjecture)**.** *Let S be a Sylow p-subgroup of G. The number of irreducible ordinary characters of G with degree prime to p is the same as the number of irreducible ordinary characters of $\mathrm{N}_G(S)$ with degree prime to p.*

The original conjecture was formulated by McKay [36, 37] but only referred to Sylow 2-subgroups. This more general case was first explicitly given by Alperin [1] but was initially suggested by Isaacs [30]. Much progress has been made towards proving this since its inception and now a proof of the final cases has been announced by Späth. We have stated this conjecture in the format that it is most commonly given, however if we rephrase it, the fact that the Alperin–McKay conjecture relates to it becomes more clear.

Recall that, as all these characters have degree prime to p, they all lie in blocks where the defect is maximal, by Theorem 1.3.8. We can also see that they will be of height zero in those blocks and in fact will be all the irreducible characters of height zero in these blocks. Also note that because these blocks have maximal defect they all have the same defect groups, the Sylow p-subgroups of G. We can therefore restate the conjecture as the following: the number of height zero irreducible ordinary characters in the blocks of G with maximal defect is the same as the number of height zero irreducible ordinary characters in the blocks of $\mathrm{N}_G(S)$ with maximal defect, where S is a Sylow p-subgroup of G.

The McKay conjecture links the two sets of blocks through the Brauer correspondence. However, it crucially considers a set of blocks of G and their set of corresponding blocks. This is where the Alperin–McKay conjecture is stronger than the McKay conjecture, it considers blocks of G individually.

Conjecture 1.3.12 (Alperin–McKay conjecture)**.** *Let D be a p-subgroup of G. The number of irreducible height zero ordinary characters of a block B of kG is the same as the number of irreducible height zero ordinary characters of the block B' of $\mathrm{N}_G(D)$ with defect group D, where B' is the Brauer correspondent of B.*

First given by Alperin [2], the Alperin–McKay conjecture not only splits up the McKay conjecture block-by-block but also postulates the same result for blocks that are not of maximal defect. It is easy to see from this that the Alperin–McKay conjecture implies the McKay conjecture but the Alperin–McKay conjecture could still fail to hold on blocks of maximal defect even though the McKay conjecture is true.

At the time of writing only the Alperin–McKay conjecture remains open although much progress has been made towards proving it is true in certain cases. Several reduction arguments have also been made. For further details of these, as well as reduction arguments for the McKay conjecture, see [16, Section 4.2]. We will however state one reduction of the Alperin–McKay conjecture which we will rely on later in this text.

Theorem 1.3.13 ([32, Proposition 5])**.** *If G is a minimal counterexample to the Alperin–McKay conjecture, then $O_p(G) = 1$.*

This result is due to Kessar and Linckelmann [32, Proposition 5]. We have omitted the proof as again it requires apparatus we do not have the space to define.

1.3.5 Block induction

We now wish to extend the idea of the Brauer correspondence to relating blocks of G to blocks of any subgroup of G. We do this in the following way. Let H be a subgroup of G and define the map

$$s_H^* : \mathrm{Z}(kG) \longrightarrow \mathrm{Z}(kH) \ ; \ \sum_{g \in G} \alpha_g g \longmapsto \sum_{g \in H} \alpha_g g,$$

in a similar fashion to the Brauer morphism. Let b be a block of H and let χ be an irreducible ordinary character of b. To χ we can associate a unique central character $\omega_\chi^* : \mathrm{Z}(kH) \to k$. The details of this are rather convoluted and have been omitted, a complete construction can be found in [38, Section 3.6]. We have kept our notation consistent with theirs. The idea is that these characters are the objects we want to keep track of when we restrict from elements of $\mathrm{Z}(kG)$ to elements of $\mathrm{Z}(kH)$. First we note that the central character ω_χ^*, much like χ, is uniquely associated to the block b, so it is also denoted ω_b^*. In fact for any choice of χ in b we get the same central character ω_χ^* so this notation is well defined. This is [38, Theorem 3.6.2(ii)], where a proof can also be found.

The composition $\omega_b^* \circ s_H^* : \mathrm{Z}(kG) \to k$, is a k-homomorphism. If it is a k-algebra homomorphism then there exist a block B of kG such that $\omega_b^* \circ s_H^* = \omega_B^*$. If this is the case we write $B = b^G$ and say b^G is *defined*. We call this process *block induction*. This type of induction is often called Brauer induction to distinguish it from the several other definitions of induction that are not equivalent. Further details of Brauer induction can be found in [38, Section 5.3] and details of these other definitions are give in Breuer and

Horváth [10]. We have chosen to not fully detail the method used to construct the central characters as we do not use them again in the text. One can follow the rest of this chapter by understanding block induction as a well defined way to relate blocks of a group and its subgroups.

We can now give some basic results relating to block induction.

Lemma 1.3.14 (See [38, Lemmas 5.3.3 and 5.3.4])**.** *Let H be a subgroup of G and b a block of kH with defect group D.*

(i) If b^G is defined then D is contained in a defect group of b^G.

(ii) If H is a subgroup of some p-subgroup K of G and the blocks b^K, $\left(b^K\right)^G$ and b^G are defined then $b^G = \left(b^K\right)^G$.

We will not prove this result here and instead a proof can be found in [38]. The most significant of these is (ii) as it shows that block induction is transitive. We will make use of this property later in the chapter.

1.4 Chains of subgroups

Here we introduce what we mean by chains, an involution on the set of chains and normalising triples. These are all defined by Isaacs and Navarro [31] but some of the results here differ from theirs.

1.4.1 Chain stabilisers

Definition 1.4.1. Given a group G we say a *chain* C in G is a totally ordered set of p-subgroups, $C = \left\{ Q_i \mid 0 \leq i \leq n \right\}$, with ordering $Q_0 < Q_1 < \cdots < Q_n$ and where Q_0 is the trivial group.

We define the *length* of C, denoted $|C|$, to be the number of non-trivial groups in C, in other words for C as above $|C| = n$. Our requirement that $Q_0 = \{1\}$ means all our chains will have non-negative length. Note that this is not universal and other authors may not require $\{1\}$ to be in the chain.

These chains are the same as those used by Isaacs and Navarro [31, p.2] and Knörr and Robinson [33, Definition 2.1]. Our choice to use chains of arbitrary p-subgroups is not universal and in fact a result by Knörr and Robinson [33, Proposition 3.3] shows that for many purposes normal chains, chains of radical subgroups and chains of elementary abelian subgroups can be used. Recall a p-subgroup P of G is said to be *radical* if $P = O_p(\mathrm{N}_G(P))$ and *elementary abelian* if it is abelian and all non-trivial elements have order p.

Definition 1.4.2. The *chain stabiliser*, G_C, of a chain C in G is the set of elements of G that stabilise all elements in C under the conjugation action. In other words G_C is the intersection of the normalisers of each Q_i in C.

The chain stabiliser subgroups are what Isaacs and Navarro use to define chain local and with that obtain many results. As we are working with blocks we also need to introduce a way to check when a block of kG_C inducts up to a block of kG.

Let B be a block of kG. We define B_C to be $\mathrm{Br}_{Q_n}(e_B)\, kG_C$, where e_B is the central idempotent corresponding to B and Br_{Q_n} is the Brauer morphism at Q_n. Note that $\mathrm{Br}_{Q_n}(e_B)$ is either zero or is a central idempotent of kG_C, as $\mathrm{C}_G(Q_n) \leq G_C \leq \mathrm{N}_G(Q_n)$, and thus B_C is either zero or a sum of blocks of G_C. This sum of blocks was defined by Knörr and Robinson [33], who proved the following result.

Lemma 1.4.3 ([33, Lemma 3.2])**.** *Let C be a chain in G. If b is a block of kG_C we have that b^G is defined and further $b^G = B$ if and only if b is a summand of B_C.*

We write $b \mid B_C$ if b is a summand of B_C. Thus we can now link blocks of kG_C to blocks of G.

1.4.2 An involution on the set of chains

Let $P \leq G$ be a non-trivial p-subgroup. Consider the set of all chains of p-subgroups of G where either P normalises every group in C, for each C in this set. We can define a permutation on the set of chains as follows.

Let $C = \{ Q_i \mid 0 \leq i \leq n \}$ be in this set and note that as $P \nsubseteq Q_0$ there is a maximal m such that $0 \leq m \leq |C|$ and $P \nsubseteq Q_m$. Thus $Q_m < Q_m P$ and $Q_m P \subseteq Q_i$ for all $m+1 \leq i \leq n$. Now we consider two cases: if $Q_m P$ is in C, we set $C^* = C \setminus \{Q_m P\}$ and otherwise, if $Q_m P$ is not in C we set $C^* = C \cup \{Q_m P\}$.

The chain C^* depends on the group P as well as on C, however the P in question will generally be clear from context so we do not include it in the notation for C^*. We can now state a few basic results about this permutation.

Lemma 1.4.4 ([31, Lemma 4.1])**.** *If C, P and G are as above, then the following properties hold:*

(i) Either $C \subsetneq C^*$ and $|C^*| = |C| + 1$ or $C^* \subsetneq C$ and $|C^*| = |C| - 1$;

(ii) $(C^*)^* = C$;

(iii) Every subgroup of G contained in G_C and normalising P is also contained in G_{C^*}.

All parts of this lemma should be easy to see from our construction.

1.4.3 Normalising triples

Definition 1.4.5. Let C be a chain of p-subgroups of G. Let P be a non-trivial p-subgroup of G_C and let X be a non-empty subset of $G_C \cap \mathrm{N}_G(P)$. We call (C, P, X) a *normalising triple.*

Again these are first defined by Isaacs and Navarro [31, p.16]. We have a well-defined action of G on the set of normalising triples given by conjugation of C, P and X by some g in G. Let $\mathcal{O}$ be an orbit of normalising triples under this action. All chains that form the left-hand component of triples in $\mathcal{O}$ have the same length so we can set $s(\mathcal{O}) = (-1)^{|C|}$, where C is the first component of a triple in $\mathcal{O}$. If $\mathcal{R}$ is a set of representatives of the orbits of chains of p-subgroups of G under conjugation then one can see that for each orbit $\mathcal{O}$ exactly one member of $\mathcal{R}$ appears as the first component in a member of $\mathcal{O}$.

Notice that as P is contained in G_C it can be used to define C^*. As X is contained in $\mathrm{N}_G(P)$ and G_C, it is also in G_{C^*}. Thus (C^*, P, X) is also a normalising triple uniquely determined by (C, P, X). This gives us a permutation on the set of normalising triples, namely $(C, P, X)^* := (C^*, P, X)$.

We can extend the idea of this permutation as follows. If $\mathcal{O}$ is a G-orbit of normalising triples, then set

$$\mathcal{O}^* := \left\{ (C^*, P, X) \;\middle|\; (C, P, X) \in \mathcal{O} \right\}.$$

The set $\mathcal{O}^*$ is an orbit of normalising triples as $((C, P, X)^g)^* = ((C, P, X)^*)^g$ for all g in G. Additionally $s(\mathcal{O}) = -s(\mathcal{O}^*)$ so $\mathcal{O} \neq \mathcal{O}^*$ but $|\mathcal{O}| = |\mathcal{O}^*|$. Finally note that $(\mathcal{O}^*)^* = \mathcal{O}$ and we say that $\mathcal{O}$ and $\mathcal{O}^*$ are paired.

We now quote a further result by Isaacs and Navarro [31, Lemma 4.4] that describes sets of normalising triples.

Lemma 1.4.6 ([31, Lemma 4.4])**.** *For G a finite group, let $\mathcal{T}$ be a set of normalising triples in G, invariant under the action of G. Let C be a chain in G and $\mathcal{Q}$ be the set of pairs (Q, Y) such that (C, Q, Y) is in $\mathcal{T}$. Let $\mathcal{U}$ be a G_C-orbit on $\mathcal{Q}$ and let*

$$\mathcal{S} = \left\{ (C, Q, Y) \;\middle|\; (Q, Y) \in \mathcal{U} \right\}.$$

There exists a unique G-orbit $\mathcal{O}$ on $\mathcal{T}$ such that $\mathcal{S}$ is contained in $\mathcal{O}$. Furthermore the map τ, where we set $\tau(\mathcal{U}) = \mathcal{O}$, is a bijection from the set of G_C-orbits on $\mathcal{Q}$ onto the set of G-orbits $\mathcal{O}$ on $\mathcal{T}$ such that C is the first component of some triple in $\mathcal{O}$.

Note that this makes sense because $\mathcal{Q}$ is G_C-invariant as well as G-invariant. The proof of this result is again due to Isaacs and Navarro [31, Lemma 4.4]. The following result gives us some properties of a normalising triple provided $X = \mathrm{N}_{G_C}(P)$.

Lemma 1.4.7. *Let G be a group and B a block of kG. If (C, P, X) is a normalising triple where $X = \mathrm{N}_{G_C}(P)$, then the following hold:*

(i) $X = \mathrm{N}_{G_{C^*}}(P)$;

(ii) If P is a defect group of the block b of kG_C with positive defect and $b^G = B$ then there is a unique block b^* of G_{C^*} associated to b with defect group P and $(b^*)^G = B$.

Proof. Part (i) is due to Isaacs and Navarro [31, Lemma 4.3(a)]. For part (ii) first we note that as $\mathrm{N}_{G_C}(P) = \mathrm{N}_{G_{C^*}}(P)$ we can apply the Brauer correspondence twice to get from blocks of G_C with defect group P to blocks of G_{C^*} with defect group P. Thus the block of kG_{C^*} corresponding to b is the block b^* where $\mathrm{Br}_P(e_{b^*}) = \mathrm{Br}_P(e_b)$ and we know this is unique by the Brauer correspondence. Also, as either $C^* \subsetneq C$ or $C \subsetneq C^*$ we have that either $(b^*)^{G_C} = b$ or $b^{G_{C^*}} = b^*$ again by the Brauer correspondence. Thus by transitivity of block induction, as stated in Lemma 1.3.14, we have $(b^*)^G = B$. □

Notice that the reason we need to exclude blocks of defect zero in (ii) is because they have trivial defect group and normalising triples with trivial second component would not make sense.

1.5 Block chain local functions

In order to prove an equivalence for the Alperin–McKay conjecture we first needed to define a block-by-block version of chain local, which we have called block chain local. This section follows the general structure of [31, Section 4], but the results themselves and their proofs are original, unless stated otherwise.

1.5.1 Functions on blocks

We will consider a collection of pairs (G, B), where G is a finite group of order divisible by p and B is a block of kG, which we will call a family, denoted $\mathcal{F}$. We require that if

(G, B) is in $\mathcal{F}$ then for all $H \leq G$ and b a block of H with $b^G = B$ defined we have (H, b) is in $\mathcal{F}$. We can define maps from $\mathcal{F}$ to an abelian group U, which in most cases in this text will be $\mathbb{Z}$. We require that if there exist pairs (G, B) and (H, B') in $\mathcal{F}$ such that there exists an isomorphism $\phi : G \to H$ with $\phi(B) = B'$ then each of these maps f must have the property $f(G, B) = f(H, B')$. We call functions with this property *isomorphism constant*. Note that these pairs in general are not Brauer pairs.

Isaacs and Navarro [31] considered a family consisting of just groups and functions on that family. From there they defined chain local and used it to show certain functions were locally determined. We wish to do something similar, but with functions on our family $\mathcal{F}$. In order to do this, we define block chain local, a block-by-block version of chain local. Lemma 1.4.3 allows us to determine which blocks of G_C induce up to G, which will be very useful.

Definition 1.5.1. An isomorphism-constant function f on $\mathcal{F}$ is *block chain local* if for all pairs (G, B) in $\mathcal{F}$ we have

$$\sum_{C \in \mathcal{R}} (-1)^{|C|} \sum_{b | B_C} f(G_C, b) = 0,$$

where $\mathcal{R}$ is a set of representatives of G-orbits of chains of p-subgroups of G and B_C is as defined above.

We will also sometimes say a function is block chain local on just a pair (G, B), in which case we are referring to the smallest family $\mathcal{F}$ containing (G, B). In other words $\mathcal{F}$ is (G, B) and all pairs (H, b) where $H \leq G$ and $b^G = B$. We will call the double sum above the *alternating chain sum*. It is easy to see that sums and integer multiples of block chain local functions with values in an abelian group are also block chain local. We now prove a further property of block chain local functions.

Lemma 1.5.2. *Let f and g be block chain local functions on some family $\mathcal{F}$. If we have that $f(H, b) = g(H, b)$ for all subgroups H of G with $O_p(H) > 1$ and blocks b of kH with $b^G = B$, then $f(G, B) = g(G, B)$.*

Proof. We may assume that $O_p(G) = 1$ as otherwise the result holds trivially. As f and g are block chain local we have

$$\sum_{C \in \mathcal{R}} (-1)^{|C|} \sum_{b|B_C} f(G_C, b) = 0 = \sum_{C \in \mathcal{R}} (-1)^{|C|} \sum_{b|B_C} g(G_C, b),$$

where $\mathcal{R}$ is a set of representatives of G-orbits of chains of p-subgroups. Notice that the trivial chain C_0 is always in $\mathcal{R}$ and that for the trivial chain we have $G_{C_0} = G$. When $G_C = G$ we have $B_C = B$ and thus B is trivially the only summand of B_{C_0}. Also note that, as $O_p(G) = 1$, the trivial chain is the only such chain where $G_C = G$. We can therefore rearrange the left-hand side of the above to be

$$f(G, B) = - \sum_{C \in \mathcal{R} \setminus \{C_0\}} (-1)^{|C|} \sum_{b|B_C} f(G_C, b),$$

as $|C_0| = 0$. We can also do the same with the right-hand side. Note that for C in $\mathcal{R} \setminus \{C_0\}$ we have $O_p(G_C) > 1$ so

$$\sum_{b|B_C} f(G_C, b) = \sum_{b|B_C} g(G_C, b).$$

Summing over all of $\mathcal{R} \setminus \{C_0\}$ we obtain

$$f(G, B) = - \sum_{C \in \mathcal{R} \setminus \{C_0\}} (-1)^{|C|} \sum_{b|B_C} f(G_C, b) = - \sum_{C \in \mathcal{R} \setminus \{C_0\}} (-1)^{|C|} \sum_{b|B_C} g(G_C, b) = g(G, B).$$

□

We can also, using our involution on chains, prove a strong result about all conjugacy-constant functions on pairs.

Proposition 1.5.3. *If G is a finite group such that $O_p(G) > 1$ and B is a block of kG, then every conjugacy-constant function on (G, B) and pairs (H, b), where H is a subgroup of G and $b^G = B$, is block chain local.*

Proof. If $P = O_p(G) > 1$, then for any chain C in G we have that P is normalised by every group of C, and thus C^* is defined with respect to P. We therefore obtain a bijection from the set of chains of odd length to the set of chains of even length given by sending C to C^*. Each of these sets is invariant under G-conjugation. Also note that we have $(C^g)^* = (C^*)^g$ for all g in G, and thus $G_{C^*} = G_C$. Further, we must also have $B_{C^*} = B_C$.

Let $\mathcal{R}$ be a set of representatives of G-orbits of even-length chains in G and notice that $\mathcal{R}^* = \left\{ C^* \;\middle|\; C \in \mathcal{R} \right\}$. Thus for f a conjugacy-constant function on (G, B) and pairs (H, b), where H is a subgroup of G and $b^G = B$, we can write the alternating chain sum as

$$\sum_{C \in \mathcal{R}} \left((-1)^{|C|} \sum_{b | B_C} f(G_C, b) + (-1)^{|C^*|} \sum_{b | B_{C^*}} f(G_{C^*}, b) \right).$$

Because $|C^*| = |C| \pm 1$, the pairs of terms cancel and the sum above goes to zero. We therefore have that f is block chain local on (G, B) and pairs (H, b), where H is a subgroup of G and $b^G = B$. □

This means from now on when proving a conjugacy-constant function is chain local we need only consider groups G in $\mathcal{F}$ where $O_p(G) = 1$.

1.5.2 The Brauer correspondence and block chain local functions

In this section we will explore a specific condition for when a function is block chain local, arising from the Brauer correspondence. To do this we define several functions on pairs and show that they are block chain local. The proof of this result will follow roughly the same structure as the proof by Isaacs and Navarro of Theorem D in [31, Section 4].

Let G be a finite group and B be a block of kG with positive defect. Let P be a non-trivial p-subgroup of G and X a subset of G. For all subgroups H of G and blocks b of kH with $b^G = B$, define $\omega_{(P,X)}(H, b)$ to be the number of H-orbits of pairs (Q, Y) that are G-conjugate to (P, X), where Q is a subgroup of H, $Y = \mathrm{N}_H(Q)$ and Q is a defect group of b.

Proposition 1.5.4. *Let G be a group and B a block of kG with positive defect. The function $\omega_{(P,X)}$ is block chain local on (G, B).*

Proof. Consider $H = G_C$, the stabiliser of a chain C of p-subgroups in G. By definition Y normalises Q and both Y and Q are subsets of G_C. Thus (C, Q, Y) is a normalising triple. Let $\mathcal{T}$ be the set of normalising triples (C, Q, Y) such that (Q, Y) is G conjugate to (P, X), $Y = \mathrm{N}_{G_C}(Q)$ and Q is a defect group of b, where b is a block of kG_C and $b^G = B$. For a triple (C, Q, Y) in $\mathcal{T}$ note that $(C, Q, Y)^*$ is also in $\mathcal{T}$ by Lemma 1.4.7(ii). By their definitions one can see that $\omega_{(P,X)}(G_C, b)$ is the number of G_C-orbits of pairs (Q, Y) such that (C, Q, Y) lies in $\mathcal{T}$ and Q is a defect group of b. Thus we see that

$$\sum_{b|B_C} \omega_{(P,X)}(G_C, b)$$

is the number of G_C-orbits of pairs (Q, Y) such that (C, Q, Y) lies in $\mathcal{T}$. We can therefore apply Lemma 1.4.6 to show that this is also the number of G-orbits $\mathcal{O}$ of elements of $\mathcal{T}$ such that C appears as the first component of a triple in $\mathcal{O}$. Recall that, as C is the first component of a triple in the G-orbit $\mathcal{O}$, by definition $(-1)^{|C|} = s(\mathcal{O})$ and we have

$$(-1)^{|C|} \sum_{b|B_C} \omega_{(P,X)}(G_C, b) = \sum_{\mathcal{O}} s(\mathcal{O}),$$

where the sum on the right is over all G-orbits $\mathcal{O}$ in $\mathcal{T}$ that contain a triple with first component C. If we now consider the set $\mathcal{R}$ of representatives of G-orbits of chains in G then for each $\mathcal{O}$ in $\mathcal{T}$ there is exactly one C in $\mathcal{R}$ and b in B_C such that C is the first component of a triple in $\mathcal{O}$. Thus if we sum over $\mathcal{R}$ we have

$$\sum_{C \in \mathcal{R}} (-1)^{|C|} \sum_{b|B_C} \omega_{(P,X)}(G_C, b) = \sum_{\mathcal{O}} s(\mathcal{O}),$$

where now the sum on the right is over all G-orbits $\mathcal{O}$ in $\mathcal{T}$. As we have established, every G-orbit $\mathcal{O}$ in $\mathcal{T}$ is paired with a unique G-orbit $\mathcal{O}^*$ which is also in $\mathcal{T}$. Thus, as

$s(\mathcal{O}) = -s(\mathcal{O}^*)$ we see that the above sum is zero and $\omega_{(P,X)}$ is block chain local on the arbitrary pair (G, B), where B has positive defect. $\square$

We can now define the next function. Let N and G be finite groups and B a block of kG with non-trivial defect group P. Define $\Omega_N(G, B) = 1$ if $\mathrm{N}_G(P) \cong N$ and $\Omega_N(G, B) = 0$ otherwise.

Corollary 1.5.5. *The function Ω_N is block chain local.*

Proof. If $O_p(N) = 1$ then Ω_N is identically zero and hence this case is trivial. Hence we may assume $O_p(N) > 1$. Let $\mathcal{S}$ be a set of pairs of subgroups (P, X) of G such that $P \trianglelefteq X$ and $X \cong N$. Let ω denote the restriction of Ω_N to pairs (H, b), where H is a subgroup of G and $b^G = B$. We proceed by showing all the restrictions ω are block chain local on the arbitrary pair (G, B), which in turn will show that Ω_N is block chain local.

Let H be a subgroup of G with order divisible by p and b be a block of kH such that $b^G = B$. By definition we have $\omega(H, b) = 1$ if $\mathrm{N}_H(Q) \cong N$ where Q is a defect group of b and $\omega(H, b) = 0$ otherwise. Since all defect groups of b are H-conjugate, $\omega(H, b)$ is the number of H-orbits on the set $\mathcal{P}$ of pairs (Q, Y) such that $Y = \mathrm{N}_H(Q)$, Q is a defect group of b and $Y \cong N$. Further, every member of each H-orbit on $\mathcal{P}$ is G-conjugate to some unique (P, X) in $\mathcal{S}$. The total number of H-orbits on $\mathcal{P}$ whose members are G-conjugate to some given member (P, X) of $\mathcal{S}$ is $\omega_{(P,X)}(H, b)$, where $\omega_{(P,X)}$ is as in Proposition 1.5.4. Thus

$$\omega(H, b) = \sum_{(P,X)\in\mathcal{S}} \omega_{(P,X)}(H, b),$$

and as $\omega_{(P,X)}$ is block chain local ω is as well. Thus as ω is block chain local and, as (G, B) in $\mathcal{F}$ was arbitrary, we have that Ω_N is block chain local for any finite group N. $\square$

Theorem 1.5.6. *Let f be an isomorphism-constant function on a family $\mathcal{F}$ with values in an abelian group. If, for all (G, B) in $\mathcal{F}$, we have $f(G, B) = f(N, b)$, where $N = \mathrm{N}_G(P)$, P is a defect group of B and b is the Brauer correspondent of B, then f is block chain local for all pairs (G, B) where B has positive defect.*

Proof. We make use of the function Ω_M, which we proved is block chain local for any finite group M in Corollary 1.5.5. Let G be a group in $\mathcal{F}$ and let B be a block of that group with positive defect. Let N be the normaliser of a defect group of B, so (N, B') is in $\mathcal{F}$ and $f(N, B')$ is defined, where B' is the Brauer correspondent of B. We have that $f(G, B) = f(N, B')$. However note that we can also write

$$f(N, B') = \sum_{[M]} \Omega_M(G, B) \sum_{b'} f(M, b'),$$

where the left sum runs over all isomorphism classes $[M]$ of groups in pairs in $\mathcal{F}$ and the right sum over all blocks b' of M such that $b'^G = B$. This is because $\Omega_M(G, B) = 0$ unless $M \cong N$ in which case $\Omega_M(G, B) = 1$ and the only block of kM where $b'^G = B$ is B'.

If we now consider a chain C in G with stabiliser G_C and b a block of kG_C we have

$$f(G_C, b) = \sum_{[M]} \Omega_M(G_C, b) \sum_{b'} f(M, b'),$$

where the left sum is as above and the right sum is over all b' of M where $b'^{G_C} = b$. Now, for $\mathcal{R}$ a set of representative of G-orbits of chains, we have

$$\begin{aligned}
\sum_{C \in \mathcal{R}} (-1)^{|C|} \sum_{b | B_C} f(G_C, b) &= \sum_{C \in \mathcal{R}} (-1)^{|C|} \sum_{b | B_C} \sum_{[M]} \Omega_M(G_C, b) \sum_{b'} f(M, b') \\
&= \sum_{[M]} \left(\sum_{C \in \mathcal{R}} (-1)^{|C|} \sum_{b | B_C} \Omega_M(G_C, b) \right) \sum_{b'} f(M, b') \\
&= 0,
\end{aligned}$$

as Ω_M is block chain local for all M in pairs in $\mathcal{F}$. Thus as G and B were arbitrary we have that f is block chain local on all pairs (G, B) where B has positive defect. $\square$

1.5.3 The Alperin–McKay function

Recall the Alperin–McKay conjecture states that for a group G the number of irreducible height zero characters in a block B of kG with defect group D is the same as the number of irreducible height zero characters of the Brauer correspondent B' of B in $\mathrm{N}_G(D)$. We can rephrase this in terms of a function on $\mathcal{F}$. Let f_{AM} be the function given by

$$f_{\mathrm{AM}}(G,B) := \ \#\text{ of irreducible height zero characters of } B,$$

for (G,B) a pair in $\mathcal{F}$. This function is obviously isomorphism-constant and maps to the integers. The Alperin–McKay conjecture is the statement $f_{\mathrm{AM}}(G,B) = f_{\mathrm{AM}}(N,B')$, where $N = \mathrm{N}_G(D)$ and D is the defect group of B and B' is its Brauer correspondent, for all (G,B) in the family $\mathcal{F}$ and $\mathcal{F}$ being arbitrary. We now have everything necessary to state our motivating result.

Theorem 1.5.7. *The Alperin–McKay conjecture holds if and only if the Alperin–McKay function is block chain local on all pairs (G,B) such that B has positive defect.*

Note we do not need to consider blocks of defect zero as the conjecture is vacuous for blocks with normal defect group, in particular defect zero.

Proof. The forward direction is a clear result of Theorem 1.5.6: recall that the Alperin–McKay conjecture can be stated as $f_{\mathrm{AM}}(G,B) = f_{\mathrm{AM}}(N,B')$ for all G a group and all B a block of kG where $N = \mathrm{N}_G(P)$, P is a defect group of B and B' is the Brauer correspondent of B.

Now let us consider the other direction, in other words suppose that the Alperin–McKay function is block chain local and show this implies the Alperin–McKay conjecture holds. We do this by contradiction. Let (G,B) be a minimal counterexample to the conjecture, so $f_{\mathrm{AM}}(G,B) \neq f_{\mathrm{AM}}(N,B')$. This implies that $N \neq G$ and thus $|G|$ is divisible by p. By a result of Kessar and Linckelmann [32, Proposition 5] we have that if (G,B) is a minimal counterexample to the Alperin–McKay conjecture then $O_p(G) = 1$, so we can assume this as well.

Define f^0_{AM} on the pair (X, b) in $\mathcal{F}$ to be the isomorphism-constant function given by setting $f^0_{\mathrm{AM}}(X, b) = f_{\mathrm{AM}}(M, b')$, where M is the normaliser of the defect group of b and b' is the Brauer correspondent of b. By definition $f^0_{\mathrm{AM}}(X, b) = f_{\mathrm{AM}}(M, b') = f^0_{\mathrm{AM}}(M, b')$, for all (X, b) in $\mathcal{F}$, so by Theorem 1.5.6 f^0_{AM} is chain local.

Note for $X \leq G$ with $O_p(X) > 1$ this implies X is a proper subgroup of G as $O_p(G) = 1$. As G is minimal we have $f_{\mathrm{AM}}(X, b) = f_{\mathrm{AM}}(M, b') = f^0_{\mathrm{AM}}(X, b)$, where b is a block of kX such that $b^G = B$ and M and b' are as before. By Lemma 1.5.2 we therefore have that $f_{\mathrm{AM}}(G, B) = f^0_{\mathrm{AM}}(G, B) = f_{\mathrm{AM}}(N, B')$, where N is the normaliser of a defect group of B and B' is a block of kN such that $B'^G = B$. This is a contradiction and the result is proved. □

The proof in itself relies heavily on the result of Kessar and Linckelmann [32, Proposition 5] as without this information about a minimal counterexample our method would not work.

This new way which we have written the Alperin–McKay conjecture as an alternating sum will hopefully help with setting this conjecture in a more general context. Although this might not directly lead to a proof it may help with a theoretical proof in the future.

1.5.4 Sufficient conditions for block chain local

Here we collect several more results that are also adapted from those given by Isaacs and Navarro. We start with a direct corollary of Theorem 1.5.6.

Corollary 1.5.8. *A constant function f is block chain local on all pairs (G, B) where B has positive defect.*

Proof. As f is constant for all pairs (G, B) in $\mathcal{F}$ we have $f(G, B) = f(N, b)$ where b is the Brauer correspondent of B and N is the normaliser of a defect group of B. Thus Theorem 1.5.6 applies and the result follows. □

The next two results tell us about functions that are entirely described by the values they take on p-subgroups and their block.

Theorem 1.5.9. *Let f be a conjugacy-constant function from some family $\mathcal{F}$ to an abelian group U such that*

$$f(G, B) = \sum_{Q} \sum_{b} h(\mathrm{N}_G(Q), b),$$

where the first sum runs over a set of representatives of G-orbits of p-subgroups, the second sum runs over all blocks b of $\mathrm{N}_G(Q)$ such that $b^G = B$ and the function h is an isomorphism-constant function on $\mathcal{F}$. Then f is block chain local.

This theorem is the block chain local version of Theorem F(b) in [31]. Our proof follows the same structure as theirs and thus in order to prove it we need to introduce one of the functions that they use.

Let N be a finite group. For any group G, let $g_N(G)$ be the number of conjugacy classes of non-trivial p-subgroups Q of G such that $\mathrm{N}_G(Q) \cong N$. Isaacs and Navarro prove that this function is chain local [31, Corollary 4.6], in other words we have

$$\sum_{C \in \mathcal{P}} g_N(G_C) = 0,$$

where G is any group with order divisible by p and $\mathcal{P}$ is a set of representatives of G-orbits of chains in G.

Proof of Theorem 1.5.9. Using the function g_N from above we see that for all pairs (G, B) in $\mathcal{F}$ we can write f as

$$f(G, B) = \sum_{[N]} g_N(G) \sum_{b'} h(N, b'),$$

where the first sum runs over isomorphism classes of normalisers of non-trivial p-subgroups of G and the second sum runs over all blocks b of $\mathrm{N}_G(Q)$ such that $b^G = B$. If we now consider the alternating chain sum of the arbitrary pair (G, B) then we obtain

$$\begin{aligned}\sum_{C\in\mathcal{P}}\sum_{b|B_C} f(G_C, b) &= \sum_{C\in\mathcal{P}}\sum_{b|B_C}\sum_{[N]} g_N(G_C)\sum_{b'} h(N, b') \\ &= \sum_{[N]}\sum_{C\in\mathcal{P}} g_N(G_C)\sum_{b|B_C}\sum_{b'} h(N, b'),\end{aligned}$$

where are sums are defined as before. As g_N is chain local [31, Corollary 4.6(b)] we have that $\sum_{C\in\mathcal{P}} g_N(G_C) = 0$ for all groups N. Thus for each $[N]$ in the first sum everything is zero and so

$$\sum_{C\in\mathcal{P}}\sum_{b|B_C} f(G_C, b) = 0,$$

for all pairs (G, B) in $\mathcal{F}$. □

This gives us a further sufficient condition to determine when a function is block chain local along with Theorem 1.5.6. There is a third such result we can adapt from Isaacs and Navarro [31, Theorem F(c)].

Theorem 1.5.10. *Let f be a conjugacy-constant function from some family $\mathcal{F}$ to an abelian group U such that*

$$f(G, B) = \sum_{Q}\sum_{b} h(\mathrm{N}_G(Q), b),$$

where the first sum runs over a set of representatives of G-orbits of radical p-subgroups, the second sum runs over all blocks b of $\mathrm{N}_G(Q)$ such that $b^G = B$ and the function h is an isomorphism-constant function on $\mathcal{F}$. Then f is block chain local.

The proof of this result has exactly the same structure as that of Theorem 1.5.9; however, instead of using the function g_N it uses r_N, also from Isaacs and Navarro. For N a finite group and G a finite group with order divisible by p we define $r_N(G)$ to be the number of conjugacy classes of non-trivial radical p-subgroups Q of G such that $\mathrm{N}_G(Q) \cong N$. This function is also chain local [31, Corollary 4.6(c)] and thus the proof follows in the same way.

1.5.5 Blocks with defect one

Here we offer a block-by-block version of a result by Isaacs and Navarro [31, Theorem E(d)]. Let $\mathcal{F}$ be a family of pairs of the form (G, B) and let $\mathrm{k}_1(G, B)$ be the number of irreducible ordinary characters in B with defect one.

Theorem 1.5.11. *The function* k_1 *is block chain local.*

Proof. First note that if B a block of defect zero it contains only one irreducible ordinary character and it has degree divisible by the order of a Sylow p-subgroup of G [7], so $\mathrm{k}_1(G, B) = 0$. From here we assume B has non-zero defect. By Lemma 1.3.14(i) all blocks b of subgroups of G where $b^G = B$ have defect less than or equal to B. Thus for all pairs (G_C, b) in the alternating chain sum for (G, B) we have that b has defect zero, so $\mathrm{k}_1(G_C, b) = 0$. The alternating chain sum is therefore zero and k_1 is block chain local on pairs where the block has defect zero. Additionally Brauer showed that if a block contains an irreducible ordinary character of defect one then all irreducible ordinary characters have defect one [6, Theorem 3]. Thus for all blocks B with defect greater than one we obtain $\mathrm{k}_1(G, B) = 0$ and k_1 is block chain local in this case, by Corollary 1.5.8.

We now only need to consider blocks B of kG with defect one as these are the only blocks where irreducible ordinary characters of defect one lie. Also only irreducible ordinary characters of defect one lie in these blocks. Let D be the defect group of B. Dade showed in [17, Theorem 1(i)] that the number of ordinary irreducible characters of a block with defect one are entirely determined by the defect group. Thus, as B and b have the same defect group, we have $\mathrm{k}_1(G, B) = \mathrm{k}_1(\mathrm{N}_G(D), b)$. Then, by noticing that B and b are Brauer correspondents, we can apply Theorem 1.5.6. Therefore k_1 is block chain local on pairs (G, B) where B has defect one and therefore is on all pairs. □

CHAPTER 2

SOME AUTOMORPHISMS OF PARTIAL GROUPS

2.1 Introduction

Partial groups were introduced by Chermak [11] with the goal of providing a more workable language in which to generalise of the p-structure of a finite group than fusion systems. In part this goal has already been realised in several papers and preprints by Henke and other authors [25, 26, 27, 28, 29] and work is continuing in these areas. These results will in turn hopefully help in the programme to understand all simple saturated fusion systems which itself is expected to form part of a new classification of finite simple groups.

The main idea of a partial group is to generalise the idea of a group by considering a set with an operation that is not defined on all words in letters in this set. Inverses still exist everywhere but because multiplication is not defined everywhere associativity does not hold in general. Partial groups themselves do not in general have much useful structure so Chermak introduces a hierarchy of definitions, each adding some further structure. Objective partial groups have a set of subgroups through which the multiplication in the partial group is defined and then localities restrict this set to p-subgroups and add a maximal p-subgroup to it, for p some prime. This gives an object with a fusion system structure on it. The final step in this hierarchy is a linking locality which is rather difficult to define but results in an object with a unique saturated fusion system associated to it.

This is why partial groups and localities are useful in work with fusion systems.

One way to investigate a category is to look at which groups can be automorphism groups of objects in this category. When every finite group is the automorphism group of some object in the category we say it is finitely universal. One of the most well-known examples of a finitely universal category is the category of simply connected finite graphs. This is due to a classical result of Frucht [24]. At the other end of the scale one of the most well-known categories that is not finitely universal is the category of finite groups. There are many finite groups that do not arise as the automorphism group of a group but one of the most well-known families is C_n, for $n > 1$ odd. With partial groups, and other objects in Chermak's hierarchy, being defined as weakening some of the axioms and structure of a group, it is natural to see if these categories are finitely universal or if they share the same exceptions as the category of groups does.

This question has already been answered by Díaz, Molinier and Viruel [19] for one category, that of partial groups. However, while they show this category is universal, the partial groups they construct are almost always infinite. As one of the interests of partial groups is their relationship to fusion systems, it is therefore beneficial to just consider finite partial groups. We begin by showing the category of finite partial groups is finite universal. We then consider the next step in the hierarchy: objective partial groups. Although we are not able to show finite objective partial groups are finite universal, we find a family of finite objective partial groups each containing one object isomorphic to C_n for each $n > 1$ odd which suggests this category may be finite universal. We do however show that the category of infinite objective partial groups is universal. Lastly we consider localities and show that one cannot construct a locality with automorphism group C_n for any $n > 1$ odd. Thus we find when objects in the hierarchy has too much structure for this family of groups to arise as automorphism groups. In addition to checking if categories are universal and finite universal we also give examples showing that the finite partial groups and finite objective partial groups we construct are not unique with respect to their automorphism groups.

This chapter will mainly discuss the objects in this hierarchy below a linking locality, namely partial groups, objective partial groups and localities. Here we assume a graduate-level knowledge of finite group theory as well as a very basic knowledge of graph theory and formal language theory. Any standard text in each of these fields would provide a suitable reference. Some understanding of fusion systems would also help the reader motivate much of the discussion in this chapter. A good introduction to the theory can be found in either Aschbacher, Kessar and Oliver [4] or Craven [15].

2.2 A short introduction to the theory of partial groups

Partial groups are relatively new objects and therefore there are no standard books setting out their theory. On two separate occasions Chermak [11, 13] and once with Henke [14] lays out the framework of this theory. We will give a much abridged version of this, only including results that are necessary for later discussion, but if the reader wishes for a more complete introduction they can look at any one of these papers. We will most closely follow [13].

2.2.1 Partial groups

We begin with the object in the hierarchy with the least structure, partial groups. First let us introduce some notation. Let $\mathcal{L}$ be a set and $\mathbf{W}(\mathcal{L})$ be the set of words in letters in $\mathcal{L}$. We will consider $\mathcal{L}$ as a subset of $\mathbf{W}(\mathcal{L})$ and denote the concatenation of two words v and w in $\mathbf{W}(\mathcal{L})$ as $v \circ w$. Let $\emptyset$ denote the empty word.

Definition 2.2.1. Let $\mathcal{L}$ be a non-empty set and $\mathbf{D} \subseteq \mathbf{W}(\mathcal{L})$ with two maps $\Pi : \mathbf{D} \to \mathcal{L}$ and an involutionary bijection $\cdot^{-1} : \mathcal{L} \to \mathcal{L}$ that extends to

$$\cdot^{-1} : \mathbf{W}(\mathcal{L}) \longrightarrow \mathbf{W}(\mathcal{L}) \ ; \ g_1 \dots g_k \longmapsto g_k^{-1} \dots g_1^{-1},$$

for all $g_1, \ldots, g_k$ in $\mathcal{L}$. We say $\mathcal{L}$ is a partial group with the above product and inverse if for all u, v, w in $\mathbf{W}(\mathcal{L})$ we have

(P1) $\mathcal{L} \subseteq \mathbf{D}$ and if $u \circ v$ in $\mathbf{D}$ then u and v are in $\mathbf{D}$;

(P2) Π restricts to the identity on $\mathcal{L}$;

(P3) If $u \circ v \circ w$ is in $\mathbf{D}$ then $u \circ \Pi(v) \circ w$ is in $\mathbf{D}$ and $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$;

(P4) If w is in $\mathbf{D}$ then $w^{-1} \circ w$ is in $\mathbf{D}$ and $\Pi(w^{-1} \circ w) = \mathbf{1}$ where $\mathbf{1} := \Pi(\emptyset)$.

We will call $\mathbf{D}$ the set of allowed words. We do not in general require $\mathcal{L}$ to be finite and in fact we will consider both finite and infinite partial groups in this text. However, much of the theory related to partial groups requires finiteness and certainly when one relates this to fusion systems it makes sense for $\mathcal{L}$ to be finite as much of the work in fusion systems considers just finite fusion systems.

This definition in itself is rather opaque and it is not necessarily clear what properties this product map possesses and what conditions we have on $\mathbf{D}$. However with not much work one can unpack several more expected properties. For example, by applying (P4) to any word w in $\mathbf{D}$ we have that $w^{-1} \circ w$ is in $\mathbf{D}$ and then applying (P1) we have that w^{-1} is in $\mathbf{D}$ as it is a subword, so $\mathbf{D}$ is closed upon taking inverses. The following lemma gives several more examples of where this is the case. We include the proof as it is a good opportunity for the reader to familiarise themselves with the notation.

Lemma 2.2.2 ([13, Lemma 1.4])**.** *Let $\mathcal{L}$ be a partial group. The following hold for all words u, v, w in $\mathbf{D}$;*

(i) Π *is* $\mathbf{D}$*-multiplicative, in other words if* $u \circ v$ *is in* $\mathbf{D}$ *then* $\Pi(u) \circ \Pi(v)$ *is in* $\mathbf{D}$ *and* $\Pi(u \circ v) = \Pi(\Pi(u) \circ \Pi(v))$.

(ii) Π *is* $\mathbf{D}$*-associative, in other words if* $u \circ v \circ w$ *is in* $\mathbf{D}$ *then*

$$\Pi(\Pi(u \circ v) \circ w) = \Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v \circ w)).$$

(iii) $u \circ v$ *in* $\mathbf{D}$ *if and only if* $u \circ \mathbf{1} \circ v$ *in* $\mathbf{D}$ *and if this is the case then* $\Pi(u \circ v) = \Pi(u \circ \mathbf{1} \circ v)$.

(iv) *If* $u \circ v$ *is in* $\mathbf{D}$ *then both* $u^{-1} \circ u \circ v$ *and* $u \circ v \circ v^{-1}$ *are in* $\mathbf{D}$ *and* $\Pi(u^{-1} \circ u \circ v) = \Pi(v)$ *and* $\Pi(u \circ v \circ v^{-1}) = \Pi(u)$.

(v) *If* $u \circ v$ *and* $u \circ w$ *are in* $\mathbf{D}$ *then* $\Pi(v) = \Pi(w)$ *if and only if* $\Pi(u \circ v) = \Pi(u \circ w)$. *The same holds for right cancellation.*

(vi) *We have* u^{-1} *in* $\mathbf{D}$ *and* $\Pi(u)^{-1} = \Pi(u^{-1})$.

Proof. (i) We have $u \circ v = \emptyset \circ u \circ v \circ \emptyset$ so applying axiom (P3) from Definition 2.2.1 twice we obtain

$$\Pi(u \circ v) = \Pi(\emptyset \circ u \circ v \circ \emptyset) = \Pi(\emptyset \circ \Pi(u) \circ v \circ \emptyset) = \Pi(\emptyset \circ \Pi(u) \circ \Pi(v) \circ \emptyset) = \Pi(\Pi(u) \circ \Pi(v)).$$

(ii) Similarly we can add the empty word to either end of $u \circ v \circ w$ and apply (P3) again to obtain

$$\Pi(u \circ v \circ w) = \Pi(\emptyset \circ u \circ v \circ w) = \Pi(\emptyset \circ \Pi(u \circ v) \circ w) = \Pi(\Pi(u \circ v) \circ w),$$
$$\Pi(u \circ v \circ w) = \Pi(u \circ v \circ w \circ \emptyset) = \Pi(u \circ \Pi(v \circ w) \circ \emptyset) = \Pi(u \circ \Pi(v \circ w)).$$

(iii) Using (P3) it is clear that if $u \circ v = u \circ \emptyset \circ v$ is in $\mathbf{D}$ then so is $u \circ \Pi(\emptyset) \circ v = u \circ \mathbf{1} \circ v$. Suppose $u \circ \mathbf{1} \circ v$ is in $\mathbf{D}$. If u is empty then we have $u \circ v = \emptyset \circ v = v$ which is in $\mathbf{D}$, by (P1), as it is a subword of $u \circ \mathbf{1} \circ v$. We therefore assume u is non-empty so $u = \widetilde{u} \circ x$, where x is an element of $\mathcal{L}$. Thus $\widetilde{u} \circ \Pi(x \circ 1) \circ v$ is in $\mathbf{D}$ by (P3) and we have

$$\Pi(x \circ 1) = \Pi(x \circ 1 \circ \emptyset) = \Pi(x \circ \Pi(\emptyset) \circ \emptyset) = \Pi(x \circ \emptyset \circ \emptyset) = \Pi(x) = x,$$

again by (P3). Lastly, it is easy to see by (P3) that we have

$$\Pi(u \circ \mathbf{1} \circ v) = \Pi(u \circ \Pi(\emptyset) \circ v) = \Pi(u \circ \emptyset \circ v) = \Pi(u \circ v).$$

(iv) If $u \circ v$ is in $\mathbf{D}$ then so is $v^{-1} \circ u^{-1} \circ u \circ v$, by (P4), and thus so is $u^{-1} \circ u \circ v$, by (P1). Using part (i) of this lemma we have

$$\Pi(u^{-1} \circ u \circ v) = \Pi(\Pi(u^{-1} \circ u) \circ \Pi(v)) = \Pi(\mathbf{1} \circ \Pi(v)) = \Pi(\emptyset \circ \Pi(v)) = \Pi(\emptyset \circ v) = \Pi(v).$$

We also have that $v^{-1} \circ u^{-1}$ is in $\mathbf{D}$ so, by (P4), we have $u \circ v \circ v^{-1} \circ u^{-1}$ in $\mathbf{D}$ and thus, by (P1), so is $u \circ v \circ v^{-1}$. In a similar fashion to above we apply part (i) of this lemma to obtain

$$\Pi(u \circ v \circ v^{-1}) = \Pi(\Pi(u) \circ \Pi(v \circ v^{-1})) = \Pi(\Pi(u) \circ 1) = \Pi(\Pi(u) \circ \emptyset) = \Pi(u \circ \emptyset) = \Pi(u).$$

(v) Suppose both $u \circ v$ and $u \circ w$ are in $\mathbf{D}$. If we have $\Pi(u \circ v) = \Pi(u \circ w)$ then applying part (iv) of this lemma along with part (ii) gives us

$$\Pi(v) = \Pi(u^{-1} \circ u \circ v) = \Pi(u^{-1} \circ \Pi(u \circ v)) = \Pi(u^{-1} \circ \Pi(u \circ w)) = \Pi(u^{-1} \circ u \circ w) = \Pi(w).$$

If we now let $\Pi(v) = \Pi(w)$ then we have

$$\Pi(u \circ v) = \Pi(\Pi(u) \circ \Pi(v)) = \Pi(\Pi(u) \circ \Pi(w)) = \Pi(u \circ w),$$

by applying part (i) of this lemma. A similar argument holds for right cancellation.

(vi) As already discussed u^{-1} is in $\mathbf{D}$ if u is. As $u^{-1} \circ u$ is in $\mathbf{D}$, by part (i) of this lemma so is $\Pi(u^{-1}) \circ \Pi(u)$. But as $\Pi(u)$ is in $\mathbf{D}$, we have $\Pi(u)^{-1} \circ \Pi(u)$ in $\mathbf{D}$ by (P4). Therefore, as $\Pi(\Pi(u^{-1}) \circ \Pi(u)) = \Pi(\Pi(u)^{-1} \circ \Pi(u))$, we apply right cancellation from part (v) of the lemma.

$\square$

From this point onward we will not be as rigorous in stating which axioms and which parts of this lemma we use but we do it here to demonstrate how more expected properties of a product arise from the axioms. We see from this lemma that inverses behave entirely as they would in a group. It also appears that we have associativity as we would with a group but one needs to be careful. If we have $u \circ v$ and $\Pi(u \circ v) \circ w$ in $\mathbf{D}$ one would like to assume we have $u \circ \Pi(v \circ w)$ and even $\Pi(\Pi(u \circ v) \circ w) = \Pi(u \circ \Pi(v \circ w))$ but this is not in general the case. In fact we do not even necessarily have $u \circ v \circ w$ in $\mathbf{D}$. Thus we have that associativity holds on triples where the multiplication is defined and does not hold in general.

We have already stated that in some sense partial groups are a weakening of the idea of a group. If this is the case it is natural to consider whether partial groups are ever also groups and if this is the case under what conditions. The following lemma gives us precisely when this is the case.

Lemma 2.2.3 ([13, Lemma 1.3]). *If $\mathcal{L}$ is a partial group then $\mathcal{L}$ is a group if and only if $\mathbf{D} = \boldsymbol{W}(\mathcal{L})$.*

The proof of this has been omitted for brevity but can be found in Chermak's paper. However from our discussion so far it should not be hard to convince oneself this is true; we have seen that associativity holds for all triples where the multiplication is defined, that there is an identity element and that inverses in partial groups behaves as they do for groups. Thus we have associativity on all possible triples if and only if $\mathbf{D}$ contains all possible triples. What is somewhat less obvious is how the binary operation of a group translates to the product map on words in a partial group. However this is just a case of carefully redefining each map, the details of which can be found in Chermak [13]. We now give, as an example, the smallest partial group that is not a group.

Example 2.2.4. Certainly $\mathcal{L}$ must contain 1 and another element not equal to 1, which we call a. If $a^{-1} = a$ then we have something isomorphic to C_2 so let a^{-1} and a be distinct. We now construct $\mathbf{D}$. Note that, by (P4), we need $a^{-1} \circ a$ to be in $\mathbf{D}$ as a needs to be in $\mathbf{D}$, by (P1). But then we see that, again by (P4), we need $a^{-1} \circ a \circ a^{-1} \circ a$ to be in $\mathbf{D}$ and thus we need $\mathbf{D}$ to contain all words $(a^{-1} \circ a)^n$, with n in $\mathbb{N}$. But by (P1) we need $\mathbf{D}$ to be closed under taking subwords, thus $\mathbf{D}$ must be precisely all words of alternating

a and a^{-1}. The sets $\mathcal{L}$ and $\mathbf{D}$ now satisfy (P1) and (P4) so we turn our attention to the remaining axioms. It is clear we must define $\Pi(a) = a$ and $\Pi(a^{-1}) = a^{-1}$ to satisfy axiom (P2). Lastly if we let u be an empty word $v = (a^{-1} \circ a)^n)$ for n in $\mathbb{N}$, and $w = a^{-1}$ then

$$\Pi(u \circ \Pi(v) \circ w) = \Pi(\emptyset \circ \Pi((a^{-1} \circ a)^n)) \circ a^{-1}) = \Pi(\emptyset \circ \emptyset \circ a^{-1}) = \Pi(a^{-1}) = a^{-1}.$$

Thus to satisfy axiom (P3) we need $\Pi((a^{-1} \circ a)^n \circ a^{-1}) = a^{-1}$ and repeating this argument again we see that $\Pi(a \circ (a^{-1} \circ a)^n) = a$ and we have satisfied all axioms. Therefore the set of allowed words is

$$\mathbf{D} = \left\{ \, a \circ (a^{-1} \circ a)^n, a^{-1} \circ (a \circ a^{-1})^n \;\middle|\; n \in \mathbb{N} \, \right\} \cup \{\emptyset\}.$$

Note that any word containing $a \circ a$ is not in $\mathbf{D}$ and therefore multiplication with these words is not defined. If $a \circ a$ is included in $\mathbf{D}$ then applying all the axioms gives the free group on one letter. Hence this partial group is sometimes called the free partial group on one generator.

Now we define some structures of partial groups that are analogous to common structures of groups. When one is introduced to groups the first definition that follows is that of a subgroup, so next we shall define a partial subgroup.

Definition 2.2.5 ([13, Definition 1.7])**.** A subset $\mathcal{H}$ of $\mathcal{L}$ is a *partial subgroup* if it is closed under the inverse map and $\Pi(w)$ in $\mathcal{H}$ for all w in $\mathbf{D} \cap \mathbf{W}(\mathcal{H})$.

Notice that this is more similar to the idea of full subcategory than subgroup. Given a subset $\mathcal{H}$ we need to include all possible multiplication that exists in $\mathcal{L}$ for $\mathcal{H}$ to be a partial subgroup, but it can still be a partial group without including all multiplication. We denote this in the same way as we do for groups, in other words $\mathcal{H} \leq \mathcal{L}$. Notice that $\mathcal{H}$ forms a group if $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}$ and in such case we call it a *subgroup* of $\mathcal{L}$. Furthermore we call $\mathcal{H}$ a *p-subgroup* of $\mathcal{L}$ if it is a p-group. The following lemma details some properties of partial subgroups that are again similar to those of subgroups.

Lemma 2.2.6 ([13, Lemma 1.8])**.** *Let $\mathcal{H}$ be a partial subgroup of a partial group $\mathcal{L}$.*

(i) If $\mathcal{K}$ is a subset of $\mathcal{H}$ then $\mathcal{K}$ is a partial subgroup of $\mathcal{H}$ if and only if it is a partial subgroup of $\mathcal{L}$.

(ii) If $\mathcal{K}$ is a subgroup of $\mathcal{L}$ then $\mathcal{H} \cap \mathcal{K}$ is a subgroup of $\mathcal{H}$ and of $\mathcal{K}$.

(iii) If $\left\{ \mathcal{H}_i \mid i \in I \right\}$ is a collection of partial subgroups of $\mathcal{L}$, then $\bigcap_{i \in I} \mathcal{H}_i$ is also a partial subgroup of $\mathcal{L}$.

The proof of each of these is simply a case of applying the definition of partial subgroup several times and can be found in Chermak [13, Lemma 1.8].

In order to define any further substructure of partial groups we need a notion of conjugation. For g in $\mathcal{L}$ set

$$\mathbf{D}(g) := \left\{ x \in \mathcal{L} \mid g^{-1} \circ x \circ g \in \mathbf{D} \right\},$$

in other words the set of all x in $\mathcal{L}$ where x^g is defined, where here we use x^g to denote $\Pi(g^{-1} \circ x \circ g)$. We now state several properties of $\mathbf{D}(g)$.

Lemma 2.2.7 ([13, Lemma 1.6])**.** *Let $\mathcal{L}$ be a partial group and g some element of the set $\mathcal{L}$.*

(i) We have $\mathbf{1}$ in $\mathbf{D}(g)$ and $\mathbf{1}^g = \mathbf{1}$.

(ii) For all x in $\mathbf{D}(g)$ we have x^{-1} is in $\mathbf{D}(g)$ and $(x^{-1})^g = (x^g)^{-1}$.

(iii) Conjugation by g gives a bijection from $\mathbf{D}(g)$ to $\mathbf{D}(g^{-1})$ and its inverse is conjugation by g^{-1}.

(iv) $\mathbf{D}(\mathbf{1}) = \mathcal{L}$ and $x^{\mathbf{1}} = x$ for all x in $\mathcal{L}$.

The proofs of these are relatively simple, and follows from applying the definition of a partial group and Lemma 2.2.2 so are omitted. For some subset $\mathcal{X}$ of $\mathbf{D}(g)$ we define $\mathcal{X}^g := \left\{ x^g \mid x \in \mathcal{X} \right\}$. We now quote two small results that tell us conjugation behaves as expected, where it is defined.

Lemma 2.2.8 ([13, Lemma 1.5])**.** *Let $\mathcal{L}$ be a partial group and let x and y be in $\mathcal{L}$.*

(i) Let $x \circ y$ and $y \circ x$ be in $\mathbf{D}$ with $\Pi(x \circ y) = \Pi(y \circ x)$. If x is in $\mathbf{D}(y)$ then we have $x^y = x$.

(ii) If x is in $\mathbf{D}(y)$ and $x^y = x$, then y is in $\mathbf{D}(x)$, $\Pi(x \circ y) = \Pi(y \circ x)$ and $y^x = y$.

Note that we have to be a little more careful here, we only have $x^y = x$ is equivalent to $\Pi(x \circ y) = \Pi(y \circ x)$ when all the necessary words are in $\mathbf{D}$. This highlights another general difficulty with partial groups; we can only multiply by an element on both sides of an equality, be it on the left or the right, if the resulting words on both sides are defined.

With this discussion of conjugation we can go on to define a notion of normality.

Definition 2.2.9 ([13, Definition 1.7])**.** Let $\mathcal{N}$ be a partial subgroup of $\mathcal{L}$. We say $\mathcal{N}$ is a *partial normal subgroup* of $\mathcal{L}$ if n^g is in $\mathcal{N}$ for all g in $\mathcal{L}$ and all n in $\mathcal{N} \cap \mathbf{D}(g)$.

Where conjugation is defined this definition is equivalent to that of a normal subgroup, so it is the natural generalisation of normality to the setting of partial groups. We denote it in the same way as we do for groups, $\mathcal{N} \trianglelefteq \mathcal{L}$. A partial subgroup $\mathcal{H}$ is *subnormal* if there exists a chain of normal subgroups from $\mathcal{H}$ to $\mathcal{L}$. We have the following immediate properties of partial normal subgroups.

Lemma 2.2.10 ([13, Lemma 1.8])**.** *If $\mathcal{N}$ is a partial normal subgroup of $\mathcal{L}$ and if $\mathcal{H}$ is a partial subgroup of $\mathcal{L}$ then $\mathcal{N} \cap \mathcal{H}$ is a partial normal subgroup of $\mathcal{H}$. If additionally $\mathcal{H}$ is a subgroup of $\mathcal{L}$ then $\mathcal{N} \cap \mathcal{H}$ is a normal subgroup of $\mathcal{H}$. Furthermore the intersection of partial normal subgroups is a partial normal subgroup.*

The last structures we shall define for general partial groups are analogues of normalisers and centralisers. For $\mathcal{X}$ a subset of $\mathcal{L}$ set

$$\mathrm{N}_{\mathcal{L}}(\mathcal{X}) := \left\{ g \in \mathcal{L} \;\middle|\; \mathcal{X} \subseteq \mathbf{D}(g) \text{ and } \mathcal{X}^g = \mathcal{X} \right\}$$

and

$$\mathrm{C}_{\mathcal{L}}(\mathcal{X}) := \left\{ g \in \mathcal{L} \;\middle|\; \mathcal{X} \subseteq \mathbf{D}(g) \text{ and } x^g = x \text{ for all } x \in \mathcal{X} \right\}.$$

Again these sets only make sense when we have conjugation defined and again they are the generalisations of centralisers and normalisers of subsets of groups. In general we cannot say anything meaningful about $\mathrm{N}_{\mathcal{L}}(\mathcal{X})$ without returning to the setting of group theory. We certainly have $\mathrm{N}_{\mathcal{L}}(\mathcal{X})$ non-empty as $\mathbf{1}$ is in $\mathrm{N}_{\mathcal{L}}(\mathcal{X})$, by Lemma 2.2.7(iv). We also have that $\mathrm{N}_{\mathcal{L}}(\mathcal{X})$ is closed under taking inverses. If g is in $\mathrm{N}_{\mathcal{L}}(\mathcal{X})$, then for all x in $\mathcal{X}$ we have $g^{-1} \circ x \circ g$ is in $\mathbf{D}$ and x^g is in $\mathcal{X}$. Thus by applying Lemma 2.2.2(iv) twice we have $g \circ g^{-1} \circ x \circ g \circ g^{-1}$ is in $\mathbf{D}$ and $(x^g)^{g^{-1}} = x$. As any x' in $\mathcal{X}$ is the image of $g^{-1} \circ x \circ g$ under the product map, for some x in $\mathcal{X}$, we have g^{-1} is in $\mathrm{N}_{\mathcal{L}}(\mathcal{X})$. Where $\mathrm{N}_{\mathcal{L}}(\mathcal{X})$ fails to be a partial subgroup is closure under the product map for every word in $\mathbf{D} \cap \mathbf{W}(\mathrm{N}_{\mathcal{L}}(\mathcal{X}))$; it is not true in general for w in $\mathbf{D} \cap \mathbf{W}(\mathrm{N}_{\mathcal{L}}(\mathcal{X}))$ that $w^{-1} \circ x \circ w$ is in $\mathbf{D}$ for arbitrary x in $\mathcal{X}$. The normaliser, however, will be more useful once we have defined what we mean by an objective partial group.

Similarly, in general $\mathrm{C}_{\mathcal{L}}(\mathcal{X})$ does not have any meaningful structure. It should be easy to see that again this fails to be a partial subgroup in the same way $\mathrm{N}_{\mathcal{L}}(\mathcal{X})$ does. For w in $\mathbf{D} \cap \mathbf{W}(\mathrm{C}_{\mathcal{L}}(\mathcal{X}))$ we again do not necessarily have $w^{-1} \circ x \circ w$ in $\mathbf{D}$ so w may not be in $\mathrm{C}_{\mathcal{L}}(\mathcal{X})$ even if each letter in w is. This is the case regardless of whether $\mathcal{X}$ is a subgroup, partial subgroup or just a set. Again once we add more structure to our definitions the centraliser will become more useful. We shall also define $\mathrm{Z}(\mathcal{L}) = \mathrm{C}_{\mathcal{L}}(\mathcal{L})$ to be the *centre* of $\mathcal{L}$.

2.2.2 Objective partial groups

We now move on to define the next step in the hierarchy of Chermak, objective partial groups. As the name suggests these have objects associated to them and these are used to define the product map of the partial group.

Definition 2.2.11 ([13, Definition 2.1]). Let $\mathcal{L}$ be a partial group with set of allowed words $\mathbf{D}$ and let Δ be a non-empty set of subgroups of $\mathcal{L}$. We define $(\mathcal{L}, \Delta)$ to be an *objective partial group* if

(O1) The set $\mathbf{D} = \mathbf{D}_\Delta$, where $\mathbf{D}_\Delta$ is the set of all words $g_1 \circ \cdots \circ g_n$ in $\mathbf{W}(\mathcal{L})$ for which there exists a chain of groups $X_0, \ldots, X_n$ each in Δ with $X_{i-1} \subseteq \mathbf{D}(g_i)$ and $X_{i-1}^{g_i} = X_i$ for all $1 \leq i \leq n$.

(O2) For X and Y in Δ and g in $\mathcal{L}$ with X^g a subgroup of Y then all subgroups of Y containing X^g are also in Δ.

Here we call Δ the *set of objects* of the objective partial group. Axiom (O1) allows us to conjugate between subgroups if there exists a chain of conjugate subgroups between them. Axiom (O2) just makes sure we include all non-trivial subgroups of subgroups where conjugation is defined. Note that Δ need not be unique and one can construct objective partial groups where there are multiple sets of objects that define the same $\mathbf{D}_\Delta$. With that in mind we will often just refer to an objective partial group as $\mathcal{L}$. We still do not require $\mathcal{L}$ to be finite for any of the results in the section but note that in most texts, including Chermak [13], it is assumed that objective partial groups are finite.

The additional structure afforded to us by defining multiplication through conjugation means we can go back and say something more meaningful about the centraliser and normaliser of specific subgroups of an objective partial group.

Lemma 2.2.12. *Let $(\mathcal{L}, \Delta)$ be an objective partial group. If X is in Δ then $\mathrm{N}_\mathcal{L}(X)$ and $\mathrm{C}_\mathcal{L}(X)$ are both subgroups of $\mathcal{L}$.*

Proof. Let X be in Δ. We have already discussed how both normalisers and centralisers are in general non-empty and closed under taking inverses. To show that $\mathrm{N}_\mathcal{L}(X)$ is closed under the product of $\mathcal{L}$ and all words in letters in $\mathrm{N}_\mathcal{L}(X)$ are in $\mathbf{D}$ note that for every word $w = g_1 \circ g_2 \circ \cdots \circ g_n$ in $\mathbf{W}(\mathrm{N}_\mathcal{L}(X))$, we have that $w^{-1} \circ x \circ w$ is in $\mathbf{D}_\Delta$ by definition of $\mathbf{D}_\Delta$. By applying axiom (P3) several times we have that $\Pi(w^{-1} \circ x \circ w)$ is

$$\Pi(g_n^{-1} \circ \cdots \circ \Pi(g_2^{-1} \circ \Pi(g_1^{-1} x \circ g_1) \circ g_2) \circ \cdots \circ g_n) = \Pi(g_n^{-1} \circ \cdots \circ \Pi(g_2^{-1} \circ x_1 \circ g_2) \circ \cdots \circ g_n) = x_n,$$

where each x_i is in X. Thus $\mathrm{N}_\mathcal{L}(X)$ is a subgroup. Furthermore, as $\mathbf{W}(\mathrm{C}_\mathcal{L}(X))$ is a subset of $\mathbf{W}(\mathrm{N}_\mathcal{L}(X))$, certainly $\mathrm{C}_\mathcal{L}(X)$ is also a subgroup. □

Lastly we will quote an important structural result for objective partial groups. For some S in Δ and $w = g_1 \circ \cdots \circ g_n$ in $\mathbf{W}(\mathcal{L})$ define S_w to be the set of all x in S such that $w^{-1} \circ x \circ w$ is in $\mathbf{D}$ and x^w is in S.

Proposition 2.2.13 ([13, Corollary 2.6])**.** *Let $(\mathcal{L}, \Delta)$ be an objective partial group and let S be in Δ. For some w in $\boldsymbol{W}(\mathcal{L})$ we have S_w is a subgroup of S and is in Δ if and only if w is in $\mathbf{D}$.*

2.2.3 Localities and fusion systems

We now move to the next step in the hierarchy: localities. In some sense this is where partial groups begin to have useful applications, as localities can be associated to fusion systems, but it is also where they become less interesting for our purposes as they have too much structure.

Definition 2.2.14 ([13, Definition 2.8])**.** Let $(\mathcal{L}, \Delta)$ be an objective partial group and p be a prime. We say $(\mathcal{L}, \Delta, S)$ is a *locality* if Δ is a set of p-subgroups of $\mathcal{L}$ that contains S, where S is the maximal p-subgroup of $\mathcal{L}$ under inclusion.

Again we will often only use $\mathcal{L}$ to refer to a locality with the rest of the structure being implicit. Note that, because of axiom (O2), Δ must be closed upon taking p-overgroups.

The structure added by this definition means that localities are much closer to groups. The following result gives us a sufficient condition for a locality being a group which may be familiar to the reader if they have worked with fusion systems as it is a restatement of Burnside's fusion theorem.

Lemma 2.2.15. *Let $(\mathcal{L}, \Delta, S)$ be a locality. If S is abelian then $\mathcal{L} = \mathrm{N}_{\mathcal{L}}(S)$. Furthermore $\mathcal{L}$ is a group.*

Proof. If $\mathcal{L} = S$ then this is trivial so let α be in $\mathcal{L}$ but not in S. For $\alpha^{-1} \circ \alpha$ to be defined there must exist an X and Y in Δ such that $X^\alpha = Y$. However, as $X^g = X$ for all g in S, we have that $\alpha^{-1} \circ g \circ \alpha$ is defined through Y, X, X, Y. Furthermore, for any natural number n, $(\alpha^{-1} \circ g \circ \alpha)^{p^n}$ is also defined by p^n copies of Y, X, X followed by a Y. By the definition of Π we have that

$$\Pi((\alpha^{-1} \circ g \circ \alpha)^{p^n}) = \Pi(\alpha^{-1} \circ \Pi(g \circ \Pi(\alpha \circ \alpha^{-1}) \circ \ldots \circ g \circ \Pi(\alpha \circ \alpha^{-1}) \circ g) \circ \alpha) = \Pi(\alpha^{-1} \circ \Pi(g^{p^n}) \circ \alpha),$$

which is $\mathbf{1}$, for some suitable n, so g^α is an element of p-power order and thus generates a p-subgroup of $\mathcal{L}$. But by definition S contains all p subgroups of $\mathcal{L}$ so g^α is in S. Hence α is in $\mathrm{N}_{\mathcal{L}}(S)$ and as α was arbitrary $\mathcal{L} = \mathrm{N}_{\mathcal{L}}(S)$ and, by Lemma 2.2.12, $\mathcal{L}$ is a group. $\square$

We have mentioned the connection that localities have with fusion systems several times so far. We will now explore this connection, but first let us define a fusion system.

Definition 2.2.16. Let S be a p-group. A *fusion system* $\mathcal{F}$ on S is a category with objects p-subgroups of S and, for all P and Q objects in $\mathcal{F}$, $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ contains injective group homomorphisms with composition given by composition of group homomorphisms such that:

(F1) For each g in S such that P^g is a subgroup of Q we have the map given by conjugation by g is in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$;

(F2) For any morphism ϕ in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ the isomorphism induced from P to its image is in $\mathrm{Hom}_{\mathcal{F}}(P, \phi(P))$ and its inverse is in $\mathrm{Hom}_{\mathcal{F}}(\phi(P), P)$.

If P and Q are groups in $\mathcal{F}$ such that there exists a $\phi : P \to Q$ in $\mathcal{F}$ that is an isomorphism then we say P and Q are *$\mathcal{F}$-conjugate*. This convention comes from the fact that fusion systems are considered generalisations of fusion within groups so much of the language comes from this context.

If we consider a locality $(\mathcal{L}, \Delta, S)$ then we denote by $\mathcal{F}_S(\mathcal{L})$ the fusion system on S which has morphisms given by all isomorphisms from S_g to $S_{g^{-1}}$, which is defined by conjugation by g in $\mathcal{L}$, and all inclusion maps of subgroups. By Proposition 2.2.13 we can describe all isomorphisms in $\mathcal{F}_S(\mathcal{L})$ as follows. For all w in $\mathbf{D}$ and Q a subgroup of

S_w we have an isomorphism from Q to Q^w, a subgroup of $S_{w^{-1}}$ given by applying the conjugation isomorphism corresponding to each letter of w successively.

2.2.4 Proper localities and saturated fusion systems

This general definition of fusion systems is not that useful to group theorists so we introduce more structure to give us objects that correspond to so called saturated fusion systems. We include a brief discussion of this for completeness, but note that these objects will not be discussed further in this text. For further information the reader is directed to Chermak and Henke [14]. First let us define a saturated fusion system.

Definition 2.2.17. Let $\mathcal{F}$ be a fusion system on a p-group S. We say $\mathcal{F}$ is *saturated* if every $\mathcal{F}$-conjugacy class of subgroups of S contains a group P such that:

(S1) Every isomorphism $\phi : Q \to P$ in $\mathcal{F}$ can be extended to the set of all g in $\mathrm{N}_S(Q)$ such that there exists h in $\mathrm{N}_S(P)$ with $\phi(Q)^h = \phi(Q^g)$;

(S2) $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

This definition gives fusion systems the additional structure needed for them to be of interest to group theorists and also topologists. Information on these applications, and more detail about this definition, can be found in either Aschbacher, Kessar and Oliver [4] or Craven [15].

We now go on to define a proper locality. For a group P in a fusion system $\mathcal{F}$ on a p-group S we define P to be *fully $\mathcal{F}$-normalised* if, for all Q conjugate in $\mathcal{F}$ to P, we have $|\mathrm{N}_S(P)| \leq |\mathrm{N}_S(Q)|$. Also recall $O_p(G)$ denotes the unique maximal normal p-subgroup of G. We say a subgroup P in a fusion system $\mathcal{F}$ is *$\mathcal{F}$-centric* if for all Q conjugate in $\mathcal{F}$ to P we have $\mathrm{C}_S(Q) \leq Q$ and we call P *$\mathcal{F}$-radical* if $O_p(\mathrm{N}_{\mathcal{F}}(P)) = P$. Let $\mathcal{F}_S(\mathcal{L})^{cr}$ denote the set of *centric radical subgroups* of the fusion system $\mathcal{F}_S(\mathcal{L})$, in other words the subgroups $P \leq S$ that are both $\mathcal{F}$-centric and $\mathcal{F}$-radical. We say a finite group G is of *characteristic* p if $\mathrm{C}_G(O_p(G)) \leq O_p(G)$.

Definition 2.2.18 ([12, Defintion 2.4])**.** A locality $(\mathcal{L}, \Delta, S)$ is *objective characteristic p* if $\mathrm{N}_{\mathcal{L}}(P)$ is of characteristic p for every P in Δ. A locality $(\mathcal{L}, \Delta, S)$ is a *proper locality* if it is of objective characteristic p and $\mathcal{F}_S(\mathcal{L})^{cr} \subseteq \Delta$.

In the literature these are sometimes called linking localities. In Chermak and Henke [14] and much of Henke's papers they are defined to be localities whose fusion system is saturated, the following result of Chermak shows that this definition is equivalent to ours.

Proposition 2.2.19 ([12, Corollary 6.5])**.** *The fusion system of a proper locality is saturated.*

These are the objects in the hierarchy of partial groups which have been found to have the most fruitful application, mainly in place of fusion systems as this novel way of phrasing a saturated fusion system appears to be more intuitive. In this area Henke has made much progress [26, 27, 28, 29] and has not only greatly expanded the theory of proper localities but has proved many novel results for saturated fusion systems.

2.2.5 Morphisms and categories of partial groups

Up until this point we have not yet given a way to map from one of the objects we have defined to another. We will now amend that by introducing morphisms between partial groups.

Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ both be partial groups with corresponding products $\Pi : \mathbf{D} \to \mathcal{L}$ and $\widetilde{\Pi} : \widetilde{\mathbf{D}} \to \widetilde{\mathcal{L}}$. If there exists a map $\phi : \mathcal{L} \to \widetilde{\mathcal{L}}$ on sets then we have an induced map $\phi^* : \mathbf{W}(\mathcal{L}) \to \mathbf{W}(\widetilde{\mathcal{L}})$ on words given by $\phi^*(f_1 \circ \cdots \circ f_n) = \phi(f_1) \circ \cdots \circ \phi(f_n)$.

Definition 2.2.20. A map $\phi : \mathcal{L} \to \widetilde{\mathcal{L}}$ on sets is a *homomorphism of partial groups* if $\phi^*(\mathbf{D}) \subseteq \widetilde{\mathbf{D}}$ and $\phi(\Pi(w)) = \widetilde{\Pi}(\phi^*(w))$, for every w in $\mathbf{D}$.

The *kernel* of a homomorphism $\phi : \mathcal{L} \to \widetilde{\mathcal{L}}$ is the set of g in $\mathcal{L}$ such that $\phi(g) = \widetilde{\mathbf{1}}$ in $\widetilde{\mathcal{L}}$. We now quote several lemmas due to Chermak that show homomorphisms of partial groups in many cases behave as homomorphisms of groups.

Lemma 2.2.21 ([13, Lemmas 1.13, 1.14 and 1.15]). *Let $\phi : \mathcal{L} \to \widetilde{\mathcal{L}}$ be a homomorphism of partial groups. The following hold.*

(i) The image of the identity of $\mathcal{L}$ is the identity of $\widetilde{\mathcal{L}}$ and ϕ commutes with the inverse map.

(ii) The kernel of ϕ is a partial normal subgroup of $\widetilde{\mathcal{L}}$.

(iii) If $\mathcal{H}$ is a partial subgroup of $\mathcal{L}$ then $\phi(\mathcal{H})$ is a partial subgroup of $\widetilde{\mathcal{L}}$.

The proofs of these are a straightforward application of the appropriate definitions and can be found in [13]. A homomorphism ϕ is an *isomorphism* if $\phi^*(\mathbf{D}) = \widetilde{\mathbf{D}}$ and ϕ is a bijection of sets. It is easy to see that this is an equivalent condition to the map on sets $\phi^{-1} : \widetilde{\mathcal{L}} \to \mathcal{L}$ being a homomorphism of partial groups and $\phi^{-1} \circ \phi$ and $\phi \circ \phi^{-1}$ being the identity on $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ respectively. We write $\mathrm{Aut}(\mathcal{L})$ for the set of isomorphisms from $\mathcal{L}$ to itself, the *automorphisms* of $\mathcal{L}$.

We have a category $\mathcal{P}art$ with objects partial groups and morphisms homomorphisms of partial groups, with composition of morphisms given by composing homomorphisms as expected. This category structure also means $\mathrm{Aut}(\mathcal{L})$ has a group structure. It should also be easy to see that if we restrict the objects to finite partial groups and allow all possible morphisms we obtain a full subcategory of $\mathcal{P}art$, which we shall denote $\mathcal{F}in\mathcal{P}art$. These categories have some 'nice' structure as shown by Salati.

Proposition 2.2.22 ([41, Theorem A]). *The category $\mathcal{P}art$ is complete and cocomplete, in other words all limits and colimits exist. Furthermore $\mathcal{F}in\mathcal{P}art$ is finitely complete and finitely cocomplete.*

The category $\mathcal{P}art$ was first given by Chermak and has also been explored in [41] and [19]. What has not been investigated, or even defined, is a category of objective partial groups. We will now attempt to do this.

One might be tempted to consider objective partial groups as pairs $(\mathcal{L}, \Delta)$, where Δ is part of their definition, so if Δ and $\widetilde{\Delta}$ are suitable sets of objects for $\mathcal{L}$, $(\mathcal{L}, \Delta)$ and $(\mathcal{L}, \widetilde{\Delta})$

are different objective partial groups. Now if we wish for the image of a morphism to be objective and in this category we must define a morphism $\phi : (\mathcal{L}, \Delta) \to (\mathcal{L}', \Delta')$ to be a homomorphism of partial groups where the set of objects of $\phi(\mathcal{L})$ is a subset of Δ'. This however has other issues; given a partial group $\mathcal{L}$ with two potential sets of objects Δ and Δ' where Δ is a subset of Δ' we clearly have a morphism $(\mathcal{L}, \Delta) \to (\mathcal{L}, \Delta')$ but not one in the other direction. Hence these two objects would not be isomorphic even though they are both the same sets with the same multiplication. This may end up being a useful definition for a category of partial groups but we wish for objects to be isomorphic if and only if they have the same underlying set and multiplication as this seems like the more natural option.

The objects should therefore be objective partial groups in any potential category but we have several choices for what morphisms we allow. Let $\mathfrak{O}\mathcal{P}art$ denote the full subcategory of $\mathcal{P}art$ with objects all objective partial groups, in other words we allow all homomorphisms of partial groups between objects. This however is not a very useful definition of morphisms between objective partial groups. In general the image of a homomorphism of partial groups from an objective partial group is not objective. We therefore seek to restrict what is a morphism in our subcategory.

Now consider the case where we only include morphisms of partial groups, $\mathcal{L} \to \mathcal{L}'$, that map a set of objects of $\mathcal{L}$ to a subset of a set of subgroups of $\mathcal{L}'$. Unfortunately this also does not guarantee the image is objective as the following example shows.

Example 2.2.23. Consider $G = \langle\, a \,\rangle$, the free group on one generator, which as an objective partial group can be considered to have set of objects $\{1\}$. Consider $\mathcal{L} = \{1, a, a^{-1}\}$, the free partial group on one generator, as defined in Example 2.2.4, which is clearly a subset of G. The map $\phi : G \to \mathcal{L}$ given by $\phi(a^n) = a$, $\phi(a^{-n}) = a^{-1}$ and $\phi(1) = 1$, for n in $\mathbb{N}$ is clearly a homomorphism of partial groups with image $\mathcal{L}$. It also maps $\{1\}$ to $\{1\}$ so the image of a set of objects is a set of subgroups of $\mathcal{L}$. However $\mathcal{L}$ is clearly not objective, it only contains one subgroup $\{1\}$ and if this were in the set of objects for $\mathcal{L}$ all words in a and a^{-1} inverse would be in $\mathbf{D}$ so we would have $\mathcal{L} = G$.

We therefore settle for the category $\mathcal{OP}art$ with objects objective partial groups and morphisms of the form $\phi : \mathcal{L} \to \mathcal{L}'$ where ϕ is a homomorphism of partial groups, $\phi(\mathcal{L})$ is an objective partial group and has a set of object that is a subset of a set of objects of $\mathcal{L}'$. This category has all the nice properties we require so is the best candidate for a category of objective partial groups. Note that it is certainly not a full subcategory of $\mathcal{P}art$.

Ultimately it is not clear what the 'correct' definition for the category of partial groups is and what is suitable for this text may not be suitable for other authors investigating other properties of objective partial groups. Fortunately we can take some solace in the following result, which shows that in fact the choice of category does not matter when it comes to considering automorphisms, so long as we do not consider objective partial groups as pointed objects. As we will only be considering automorphisms for the remainder of this text either category definition is suitable for us.

Lemma 2.2.24. *Let $\mathcal{L}$ and $\mathcal{L}'$ be objective partial groups. If $\phi : \mathcal{L} \to \mathcal{L}'$ is an isomorphism of partial groups then $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic in $\mathcal{OP}art$.*

Proof. We simply need to show that ϕ is also a morphism in $\mathcal{OP}art$. Clearly $\phi(\mathcal{L}) = \mathcal{L}'$ is objective so it is just a case of showing the image of a set of objects of $\mathcal{L}$ is a set of objects of $\mathcal{L}'$. If Δ is a set of objects of $\mathcal{L}$ then clearly $\phi(\Delta)$ is a set of subgroups of $\mathcal{L}'$. As ϕ extends to a bijection from $\mathcal{D}_{\mathcal{L}}$ to $\mathcal{D}_{\mathcal{L}'}$ we have that the image of any multiplication defined by conjugating an element of Δ is defined by conjugating an element of $\phi(\Delta)$. Thus ϕ is a morphism in $\mathcal{OP}art$ and as it was arbitrary so is its inverse and their composition is the identity. $\square$

For each of these categories we obtain the corresponding full subcategory by considering just finite objective partial groups as our objects, denoted by the prefix $\mathcal{F}in$.

For localities a sensible definition for morphisms between them would be one that agrees with the definition of morphisms of fusion systems.

Definition 2.2.25. Let $\mathcal{F}$ and $\mathcal{F}'$ be fusion systems on groups S and S'. A *morphism of* *fusion systems* is a functor $\Phi : \mathcal{F} \to \mathcal{F}'$ such that there exists a group homomorphisms $\theta : S \to S'$ with $\theta(Q) = \Phi(Q)$, for all Q a subgroup of S, and for $\psi : Q \to R$ a morphism in $\mathcal{F}$ we have $\Phi(\psi) \circ \theta = \theta \circ \psi$. We say θ *induces* the morphism $\mathcal{F}$ to $\mathcal{F}'$ and that it is the *underlying* group homomorphism of Φ.

Definition 2.2.26. Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$ be localities. A homomorphism of localities is a homomorphism of partial groups $\phi : \mathcal{L} \to \mathcal{L}'$ such that $\phi|_S : S \to S'$ is a group homomorphism.

Lemma 2.2.27. *Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$ be localities and let $\phi : \mathcal{L} \to \mathcal{L}'$ be a homomorphism of localities. The maps*

$$\Phi : \mathcal{F}_S(\mathcal{L}) \longrightarrow \mathcal{F}_{S'}(\mathcal{L}'),$$
$$Q \longmapsto \phi(Q),$$
$$(Q \mapsto Q^w) \longmapsto \left(\phi(Q) \mapsto \phi(Q)^{\phi(w)}\right),$$

where w is a word in $\mathbf{D}$, are a functor. Furthermore $\phi|_S$ and Φ is a morphism of fusion systems.

Proof. First note that, because ϕ is a homomorphism of partial groups, we have $\phi \circ \Pi_{\mathcal{L}} = \Pi_{\mathcal{L}'} \circ \phi$ so $\phi(Q)^{\phi(w)} = \phi(Q^w)$. As $\phi|_S$ is a group homomorphism both $\phi(Q)$ and $\phi(Q^w)$ are subgroups of S' so $\phi(Q) \mapsto \phi(Q)^{\phi(w)}$ is a morphism in $\mathcal{F}_{S'}(\mathcal{L}')$ and hence Φ is a functor.

By definition $\Phi(Q) = \phi|_S(Q)$ for all Q a subgroup of S. Let $\psi : Q \to Q^w \hookrightarrow R$, for w in $\mathbf{D}$, be a morphism in $\mathcal{F}_S(\mathcal{L})$. Thus it is clear that

$$\Phi(\psi) \circ \phi|_S(Q) = \Phi(\psi) \circ \phi(Q) = \phi(R) = \phi|_S(R) = \phi|_S \circ \psi(Q),$$

and $\phi|_S$ and Φ is a morphism of fusion systems. $\square$

Lastly this means we have a final pair of categories, $\mathcal{L}oc$ and $\mathcal{F}in\mathcal{L}oc$. The first of these has objects localities and morphisms homomorphisms of localities and the second has objects finite localities and morphism all homomorphisms of localities. Again $\mathcal{F}in\mathcal{L}oc$ is clearly a subcategory of $\mathcal{L}oc$ and they are respectively subcategories of $\mathcal{F}in\mathcal{OP}art$ and $\mathcal{OP}art$.

The richer structure of the category of localities means we can define an analogous idea of inner automorphism for localities. Recall an inner automorphism of a groups G is an isomorphism $\phi : G \to G$ such that $\phi(g) = g^h$ for some h in G.

Definition 2.2.28 ([13, Defintion 2.15])**.** Let $\mathcal{L}$ be a locality and let ψ be an automorphism of $\mathcal{L}$. We say ψ is *inner* if it is given by conjugation by some g in $\mathcal{L}$, in other words for all x in $\mathcal{L}$ we have $g^{-1} \circ x \circ g$ in $\mathbf{D}$ and $\psi(x) = x^g$ and if $x_1 \circ \ldots \circ x_n$ in $\mathbf{D}$ then $x_1^g \circ \ldots \circ x_n^g$ is also in $\mathbf{D}$ with $\Pi(x_1^g \circ \ldots \circ x_n^g) = \Pi((x_1 \circ \ldots \circ x_n)^g)$.

As with groups we use $\mathrm{Inn}(\mathcal{L})$ to denote the set of inner automorphisms of $\mathcal{L}$. We now quote a few expected results about $\mathrm{Inn}(\mathcal{L})$. Proofs of these can be found in Chermak [13].

Proposition 2.2.29 ([13, Proposition 2.17])**.** *Let $\mathcal{L}$ be a locality.*

(i) The set of all g in $\mathcal{L}$ such that $(\,\cdot\,)^g$ is in $\mathrm{Inn}(\mathcal{L})$ *is a subgroup of $\mathcal{L}$.*

(ii) We have that $\mathrm{Inn}(\mathcal{L}) \trianglelefteq \mathrm{Aut}(\mathcal{L})$ *and any automorphism of $\mathcal{L}$ can be factored into an inner automorphism and an automorphism that leaves S invariant.*

2.3 Some further preliminaries

This final section of preliminaries contains some more miscellaneous background information necessary for the sections that follow. The first of these is a discussion on universal categories that gives details of the property which we will be showing our categories possess and the second of these is a brief introduction to graph theory in order to avoid any ambiguity in our definitions.

2.3.1 Universal categories

We now introduce the concept of a universal category. Universal, when used to describe categories, has several different meanings depending on the context. The following definition details what we mean when we call a category universal.

Definition 2.3.1. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *universal* if, for any group G, there exists an object X in $\mathcal{C}$ such that $\mathrm{Aut}(X) = G$. It is *finitely universal* if we restrict G to any finite group.

A much more complete discussion of the property of universality can be found in [5]. We shall now give one of the more quoted examples of a universal category.

Example 2.3.2. The category of simple connected undirected graphs is universal. The proof in the finite case is down to Frucht [24] and the infinite both Sabidussi [40] and De Groot [18, Theorem 6].

This is arguably the most useful example of a universal category and has been used to show other categories themselves are universal, for example the category of field extensions. The proof, due to Fried and Kollár [22, 23], takes a graph with a given automorphism group and constructs a field from this. In general most proofs of universality follow this method.

We can now quote the result that motivated this work.

Theorem 2.3.3 ([19, Theorem 5.3])**.** *The category $\mathcal{P}art$ is universal.*

The proof of this follows the method mentioned above: from each graph they construct a partial group with the same automorphism group. Note however this result only holds for the category of all partial groups and in fact necessitates infinite partial groups be included even for finite automorphism groups.

In contrast the category of finite groups is one of the most common examples of a category that fails to be universal. The following result demonstrates just one family of objects that cause this category to fail to be universal.

Theorem 2.3.4 (see [21, Theorem 1(ii)]). *If G is a finite group then* $\mathrm{Aut}(G)$ *is not isomorphic to C_n for any $n > 1$ odd.*

There are many other classes of groups for which the category fails to be universal: however these groups are in some way the most basic for which is fails and crucially C_3 is the smallest. The class of groups C_n, for $n \geq 3$ odd, are therefore very useful when trying to see if some category containing the category of groups is universal. These are the easiest groups to check if they appear as automorphisms of objects. We will repeatedly consider this class throughout this chapter.

2.3.2 Some basic graph theory definitions

As a result of the previous subsection's discussion we will make considerable use of graphs in this text and because this is an algebra text we include some basic definitions from graph theory so as to avoid any ambiguity. We define a *graph* Γ to be a pair (V, E) consisting of a set of *vertices*, V, and a set of *edges*, E, where each edge has an associated pair of vertices. All graphs here will be undirected so the ordering of vertices associated to an edge is not important. We will also only be considering finite simple connected graphs, in other words graphs with finite edge and vertex sets, graphs with only one edge between a pair of vertices and no edges from a vertex to itself, and graphs with only one connected component.

We will characterise edge sets in two different ways throughout this text depending on the construction they will be a part of. Given some v and u in V we will either consider the edge joining them to be the word uv or a singleton e_{uv} indexed by u and v. Note that in both cases, as our graphs are undirected we have $uv = vu$ or $e_{uv} = e_{vu}$ in E. For the following two definitions we will use the first characterisation but note that everything discussed can be adapted to the second characterisation.

Definition 2.3.5. We define a *walk* in Γ to be an ordered list $\{v_1, \ldots, v_n\}$ of V such that $v_i v_{i+1}$ is in E, for each $1 \leq i \leq n-1$.

Definition 2.3.6. We define a *path* in Γ to be a walk with the additional condition that $v_i \neq v_j$ for each $i \neq j$.

We will often abuse notation slightly and identify walks and paths with the corresponding ordered set of edges rather than vertices. However, it is easy to see these are in bijection.

2.4 Finite partial groups

Díaz, Molinier and Viruel showed in [19] that the category $\mathcal{P}art$ is universal and thus the natural next step would be to show that $\mathcal{F}in\mathcal{P}art$ is finite universal. Unfortunately their construction necessitates the use of infinite partial groups, even for graphs with finite automorphism groups. In this section we therefore introduce a new construction that gives us a finite partial group from each finite graph and therefore shows that $\mathcal{F}in\mathcal{P}art$ is finite universal. We then go on to show that there is in fact a non-isomorphic family of finite partial groups associated to each graph with the same automorphism group as each graph.

2.4.1 The category of finite partial groups is finite universal

Let Γ be a simply connected finite undirected graph. Notice that Γ can be entirely and uniquely described by just considering the vertex set V and the edge set E. Given two vertices v_1 and v_2 in V connected by an edge we identify the words v_1v_2 and v_2v_1 in E. We will now construct a finite partial group using these two sets. Let $\mathcal{L}_\Gamma = V \cup E$ and define $\mathbf{D}$ to be the set

$$\left\{ \, x^{\alpha_0}y^{2\beta_1}x^{2\alpha_1}\ldots y^{2\beta_n}x^{2\alpha_n}y^{\beta_0},\, x^{\alpha_0}y^{2\beta_1}x^{2\alpha_1}\ldots y^{2\beta_n}x^{\beta_0} \;\middle|\; xy \in E,\, \alpha_i, \beta_i \in \mathbb{N},\, n \in \mathbb{N} \cup \{0\} \, \right\}.$$

Let $\Pi : \mathbf{D} \to \mathcal{L}_\Gamma$ be given by

$$\Pi(x^{\alpha_0} y^{2\beta_1} x^{2\alpha_1} \dots y^{2\beta_n} x^{2\alpha_n} y^{\beta_0}) = \begin{cases} x & \text{if } \alpha_0 \text{ odd, } \beta_0 \text{ even,} \\ y & \text{if } \alpha_0 \text{ even, } \beta_0 \text{ odd,} \\ xy & \text{if } \alpha_0 \text{ odd, } \beta_0 \text{ odd,} \\ 1 & \text{if } \alpha_0 \text{ even, } \beta_0 \text{ even,} \end{cases}$$

and

$$\Pi(x^{\alpha_0} y^{2\beta_1} x^{2\alpha_1} \dots y^{2\beta_n} x^{\beta_0}) = \begin{cases} x & \text{if } \alpha_0 + \beta_0 \text{ odd,} \\ 1 & \text{if } \alpha_0 + \beta_0 \text{ even.} \end{cases}$$

This is a neat and easy characterisation of Π to write down but is not so useful for proofs. We therefore give the following result showing it is equivalent to applying the product to pairs. First note that we can write any word w in $\mathbf{D}$ in the form $w = x_1 \circ \cdots \circ x_l$ where each x_i is a vertex in V and l is in $\mathbb{N}$.

Lemma 2.4.1. *If $w = x_1 \circ \cdots \circ x_l$ is an arbitrary word in $\mathbf{D}$, where each x_i is in V and l is in $\mathbb{N}$, then $\Pi(w) = \Pi(\cdots \Pi(\Pi(x_1) \circ x_2) \cdots \circ x_l)$.*

Proof. We shall do this by induction. First when $l = 1$ the assumption holds vacuously. Let $l = 2$, then we have two cases to consider: xx; and xy where xy is an arbitrary edge in E. Note that as E is symmetric we can assume every word starts with x. This again holds vacuously as $\Pi(\Pi(x) \circ x) = \Pi(x \circ x)$ and $\Pi(\Pi(x) \circ y) = \Pi(x \circ y)$.

Now suppose the assumption holds for length $l-1$ and let w' be a word of this length. We will consider all possible cases. First suppose $\Pi(w') = x$: then all the y terms in w' have an even power and either w' ends in a y and the first term has an odd power or the sum of the first and last powers of x is odd. Note that in both cases $w' \circ x$ is in $\mathbf{D}$ and consider $\Pi(\Pi(w') \circ x) = \Pi(x \circ x) = 1$. The word $w' \circ x$ now starts and ends in a power of x and the powers of these sum to an even number so $\Pi(w' \circ x) = 1$. Now suppose $w' \circ y$ is in $\mathbf{D}$ and we have $\Pi(\Pi(w') \circ y) = \Pi(x \circ y) = xy$. For $w' \circ y$ to be in $\mathbf{D}$ we require w' to end in a y or an even power of x, thus the first term of w' must be an odd power. Therefore $w' \circ y$ starts with an odd power of x and ends with an odd power of y so $\Pi(w' \circ x) = xy$.

If $\Pi(w') = y$ then w' starts with an even power of x and ends with an odd power of y. Thus $w' \circ x$ is not in $\mathbf{D}$ and we only need to consider $w' \circ y$. We have that $\Pi(\Pi(w') \circ y) = \Pi(y \circ y) = 1$. The word $w' \circ y$ starts with an even power of x and now an ends in an even power of y so $\Pi(w' \circ y) = 1$.

Next, if $\Pi(w') = xy$ then w' starts with an odd power of x and ends with an odd power of y. Again we have that $w' \circ x$ is not in $\mathbf{D}$ so we only consider $w' \circ y$ in $\mathbf{D}$. We have that $\Pi(\Pi(w') \circ y) = \Pi(xy \circ y) = x$. The word $w' \circ y$ starts with an odd power of x and ends with an even power of y so $\Pi(w' \circ y) = x$.

Lastly suppose $\Pi(w') = 1$. Therefore w' either starts with an even power of x and ends with an even power of y or it ends with a power of x and the sum of the first and last power is even. Consider $\Pi(\Pi(w') \circ x) = \Pi(1 \circ x) = x$. The word $w' \circ x$ now ends with a power of x and the sum of the first and last power is odd so $\Pi(w' \circ x) = x$. Now suppose $w' \circ y$ is in $\mathbf{D}$ and consider $\Pi(\Pi(w') \circ y) = \Pi(1 \circ y) = y$. The word $w' \circ y$ is in $\mathbf{D}$ if w' starts with an even power of x and ends with an even power of y or x. Thus $w' \circ y$ ends with an odd power of y and $\Pi(w' \circ y) = y$. Thus if we suppose our hypothesis holds for all words of length $l - 1$ then it holds for all words of length l and we are done. $\square$

We need to show one additional property of Π before we proceed, that it is associative on $\mathcal{L}_\Gamma$.

Lemma 2.4.2. *For any u, v and w in $\mathcal{L}_\Gamma$ such that $u \circ v \circ w$ is in $\mathbf{D}$ we have $\Pi(u \circ v) \circ w$ and $u \circ \Pi(v \circ w)$ are in $\mathbf{D}$ and $\Pi(\Pi(u \circ v) \circ w) = \Pi(u \circ \Pi(v \circ w))$, in other words the product Π is associative on $\mathcal{L}_\Gamma$*

Proof. The proof is just a matter of considering several cases. Let xy be in E. Then the set of triples in $\mathbf{D}$ we need to consider are

$$\{x \circ x \circ x,\ x \circ x \circ y,\ x \circ y \circ y,\ x \circ x \circ xy,\ x \circ y \circ yx,\ x \circ xy \circ y,$$

$$x \circ xy \circ yx,\ xy \circ y \circ x,\ xy \circ y \circ y,\ xy \circ yx \circ x,\ xy \circ yx \circ xy\big\}.$$

Note that because $\mathcal{L}_\Gamma$ and $\mathbf{D}$ are symmetric we only need to consider all possible words starting in x. Applying the product to all cases we obtain

$$\Pi(\Pi(x \circ x) \circ x) = \Pi(1 \circ x) = x = \Pi(x \circ 1) = \Pi(x \circ \Pi(x \circ x)),$$

$$\Pi(\Pi(x \circ x) \circ y) = \Pi(1 \circ y) = y = \Pi(x \circ xy) = \Pi(x \circ \Pi(x \circ y)),$$

$$\Pi(\Pi(x \circ y) \circ y) = \Pi(xy \circ y) = x = \Pi(x1) = \Pi(x \circ \Pi(y \circ y)),$$

$$\Pi(\Pi(x \circ x) \circ xy) = \Pi(1 \circ xy) = xy = \Pi(x \circ y) = \Pi(x \circ \Pi(x \circ xy)),$$

$$\Pi(\Pi(x \circ y) \circ yx) = \Pi(xy \circ yx) = 1 = \Pi(x \circ x) = \Pi(x \circ \Pi(y \circ yx)),$$

$$\Pi(\Pi(x \circ xy) \circ y) = \Pi(y \circ y) = 1 = \Pi(x \circ x) = \Pi(x \circ \Pi(xy \circ y)),$$

$$\Pi(\Pi(x \circ xy) \circ yx) = \Pi(y \circ yx) = x = \Pi(x \circ 1) = \Pi(x \circ \Pi(xy \circ yx)),$$

$$\Pi(\Pi(xy \circ y) \circ x) = \Pi(x \circ x) = 1 = \Pi(xy \circ yx) = \Pi(xy \circ \Pi(y \circ x)),$$

$$\Pi(\Pi(xy \circ y) \circ y) = \Pi(x \circ y) = xy = \Pi(xy \circ 1) = \Pi(xy \circ \Pi(y \circ y)),$$

$$\Pi(\Pi(xy \circ yx) \circ x) = \Pi(1 \circ x) = x = \Pi(xy \circ y) = \Pi(xy \circ \Pi(yx \circ x)),$$

$$\Pi(\Pi(xy \circ yx) \circ xy) == \Pi(1 \circ xy)xy = \Pi(xy \circ 1) = \Pi(xy \circ \Pi(yx \circ xy)).$$

Note that it is clear that $\Pi(u \circ v) \circ w$ and $u \circ \Pi(v \circ w)$ are in $\mathbf{D}$ in each of the above cases. Thus, as xy in E was arbitrary, we have that Π is associative on $\mathcal{L}_\Gamma$. □

Lastly we need to define the inverse map on $\mathcal{L}_\Gamma$. For x in V we set $x^{-1} = x$ and for xy in E we set $(xy)^{-1} = yx$ which we note is also in E by definition. We then extend this to $W(\mathcal{L}_\Gamma)$ in the standard way.

Proposition 2.4.3. *Let Γ be a simply connected finite undirected graph. The set $\mathcal{L}_\Gamma$ is a finite partial group with product map Π and inversion $\cdot^{-1}$.*

Proof. Consider first for arbitrary xy in E, the word $x^{\alpha_0}y^{2\beta_1}x^{2\alpha_1}\cdots y^{2\beta_n}x^{2\alpha_n}y^{\beta_0}$ in $\mathbf{D}$. If we set $n = 0$, $\beta_0 = 0$ and $\alpha_0 = 1$ then we see that x is in $\mathbf{D}$ and as Γ is connected then for all x in V there exists a y in V such that xy is in E. Thus V is a subset of $\mathbf{D}$. If we now set $n = 0$, $\beta_0 = 1$ and $\alpha_0 = 1$ then we have xy is in $\mathbf{D}$ and as this is true for any edge in E we have that E, and further $\mathcal{L}_\Gamma$, are subsets of $\mathbf{D}$. It should also be clear from the definition of $\mathbf{D}$ that if a word is in $\mathbf{D}$ any of its subwords are also in $\mathbf{D}$ as one can always remove letters one at a time from the start or end of a word and still have a word in $\mathbf{D}$.

For any x in V we have that $\Pi(x) = x$ and for any xy in E we have that $\Pi(xy) = xy$ by definition of Π. Thus Π is the identity on $\mathcal{L}_\Gamma$.

Let xy be in E and let $u \circ v \circ w$ be in $\mathbf{D}$ in letters x and y. We will consider all possible cases for $\Pi(v)$. Note without loss of generality we can assume v starts with x. First if $\Pi(v) = y$ then v starts with an even power of x and ends with an odd power of y. Thus w either starts with an odd power of y, is just a power of y or is empty. So $\Pi(v) \circ w = y \circ w$ either starts with an even power of y or is just a power of y. As v starts with an even power of x, u must either end with an even power of x or y, be just a power of x or y or be empty. In all of these cases it is easy to see that $u \circ \Pi(v) \circ w = u \circ y \circ w$ is in $\mathbf{D}$. Now if $\Pi(v) = xy$ then v starts with an odd power of x and ends with an odd power of y. Thus w either starts with an odd power of y, is just a power of y or is empty. So $\Pi(v) \circ w = xy \circ w$ starts with an odd power of x followed by a word starting with an even power of y or is just a power of y. As v starts with an odd power of x, u must either end with an odd power of x be just a power of x or be empty. In each case one can again see that $u \circ \Pi(v) \circ w = u \circ xy \circ w$ is in $\mathbf{D}$. Next let $\Pi(v) = x$. If we assume v starts with an odd power of x then it ends with either an even power of x or y or is just an odd power of x. Thus u must either end with an odd power of x, be just a power of x or be empty. Thus $u \circ \Pi(v) = u \circ x$ either ends with an even power of x or y or is just a power of x. As v ends in an even power or is just a power of x, w either starts with an even power of x or y, is just a power of x or y or is empty. In each case we have $u \circ \Pi(v) \circ w = u \circ x \circ w$ is in $\mathbf{D}$. If we now assume v starts with an even power of x then it must end in an odd power of x. Thus w either starts with with an odd power of x, is just a power of x or is empty. Thus $\Pi(v) \circ w = x \circ w$ either starts with an even power of x or is just a power of x. Similar to above, as v starts with an even power, u must either end with an even power, be just a power of x or y or be empty. In each case we have $u \circ \Pi(v) \circ w = u \circ x \circ w$ is in $\mathbf{D}$. Lastly suppose $\Pi(v) = 1$. If we assume v starts with an even power of x then it ends with an even power of either x or y. Thus u must either end in an even power of x or y, be just a power of x or y or be empty and w must either start with an even power, be just a power of x or y or be empty. In all cases we have $u \circ \Pi(v) \circ w = u \circ 1 \circ w = u \circ w$ is in $\mathbf{D}$. Similarly if v starts with an odd power of x it must end with an odd power of x. Thus u must either end with an odd power of x, be just a power of x or be empty and w can either start with an odd power of x, be just a power of x or be empty. In each case again we have $u \circ \Pi(v) \circ w = u \circ 1 \circ w = u \circ w$ is in $\mathbf{D}$. Thus for $u \circ v \circ w$ in $\mathbf{D}$ arbitrary we have $u \circ \Pi(v) \circ w$ is in $\mathbf{D}$.

In order for us to show that $\mathcal{L}_\Gamma$ obeys the rest of the third axiom we use the fact that Π is associative on $\mathcal{L}_\Gamma$ where it is defined, as shown in Lemma 2.4.2. As already discussed, we can write $u \circ v \circ w = u_1 \circ \cdots \circ u_k \circ v_1 \circ \cdots \circ v_l \circ w_1 \circ \cdots \circ w_m$ where each u_i, v_i and w_i is a vertex in V and k, l and m are the length of u, v and w in letters in V respectively. Thus, from Lemma 2.4.1, we have $\Pi(u \circ v \circ w) = \Pi(\cdots \Pi(\Pi(u_1) \circ u_2) \cdots \circ w_m)$. As we have that Π is associative on $\mathcal{L}_\Gamma$ we can rewrite this as

$$\Pi(\cdots \Pi(\Pi(u_1) \circ u_2) \cdots \circ w_m) = \Pi(u_1 \circ \cdots \circ u_k \circ \Pi(v_1 \circ \cdots \circ v_l) \circ w_1 \circ \cdots \circ w_m) = \Pi(u \circ \Pi(v) \circ w).$$

Thus $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$ for all words $u \circ v \circ w$ in $\mathbf{D}$. Also note that as Π is associative on $\mathcal{L}_\Gamma$, by Lemma 2.4.2, each time we swap the order of a product the resulting concatenation of elements of $\mathcal{L}_\Gamma$ is in $\mathbf{D}$ if it was before swapping, so as $u \circ v \circ w$ is in $\mathbf{D}$ then so is $u \circ \Pi(v) \circ w$.

Now let xy in E be arbitrary and consider $w = x^{\alpha_0} y^{2\beta_1} x^{2\alpha_1} \cdots y^{2\beta_n} x^{2\alpha_n} y^{\beta_0}$ in $\mathbf{D}$. We also have yx in E by definition so $w^{-1} = y^{\beta_0} x^{2\alpha_n} y^{2\beta_n} \cdots x^{2\alpha_1} y^{2\beta_1} x^{\alpha_0}$ is also in $\mathbf{D}$. As $2\alpha_0$ is even we have that $w^{-1} \circ w$ is in $\mathbf{D}$ and, because $2\beta_0$ is even, $\Pi(w^{-1} \circ w) = 1$. For words of the form $w = x^{\alpha_0}y^{2\beta_1}x^{2\alpha_1}\cdots y^{2\beta_n}x^{\beta_0}$ in $\mathbf{D}$ it is clear that $w^{-1} = x^{\alpha_0}y^{2\beta_n}\cdots x^{2\alpha_1}y^{2\beta_1}x^{\alpha_0}$ is also in $\mathbf{D}$. Therefore, as both $2\alpha_0$ and $2\beta_0$ are even, we have $w^{-1}\circ w$ in $\mathbf{D}$ and $\Pi(w^{-1}\circ w) = 1$. Thus $\mathcal{L}_\Gamma$ is a partial group with product Π and inversion $\cdot^{-1}$. Lastly it is easy to see that $\mathcal{L}_\Gamma$ is finite as Γ is finite by definition so both V and E are finite sets. □

Theorem 2.4.4. *If* Γ *is a simply connected finite undirected graph then* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{L}_\Gamma)$.

Proof. If ϕ is in $\mathrm{Aut}(\Gamma)$ then ϕ is a bijection on V and E so induces a bijection on the set $\mathcal{L}_\Gamma$. As the image of any edge xy in E is also a unique edge $\psi(xy) = \psi(x)\psi(y)$ in E then the image of any word w in $\mathbf{D}$ in letters x and y is a unique word in $\mathbf{D}$ in letters $\psi(x)$ and $\psi(y)$. Thus ψ extends to a bijection on $\mathbf{D}$. Additionally in $\psi(w)$ all the powers of $\psi(x)$ and $\psi(y)$ will be the same as x and y respectively so $\Pi(\psi(w)) = \psi(\Pi(w))$. Thus ψ is an automorphism of $\mathcal{L}_\Gamma$. Now let ϕ be in $\mathrm{Aut}(\mathcal{L}_\Gamma)$. As the set V contains exactly all the elements of $\mathcal{L}_\Gamma$ that are self-inverse ϕ must be a bijection on V and thus also a bijection on E. Consider x and y in V with xy in E. We have that $\Pi(\phi(x)\circ\phi(y)) = \phi(\Pi(x\circ y)) = \phi(xy)$ so for every pair of vertices in V connected by an edge their images under ϕ are also connected by an edge and ϕ is in $\mathrm{Aut}(\Gamma)$. Thus $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{L}_\Gamma)$. □

Corollary 2.4.5. *The category* $\mathcal{F}in\mathcal{P}art$ *is finitely universal.*

Proof. By Frucht's theorem [24], we have that for any finite group G there exists a simply connected finite undirected graph, Γ such that $\mathrm{Aut}(\Gamma) \cong G$. Theorem 2.4.4 gives us that $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{L}_\Gamma)$ so $\mathrm{Aut}(\mathcal{L}_\Gamma) \cong G$ and as G was an arbitrary finite group then category of finite partial groups is universal. □

2.4.2 A family of non-isomorphic partial groups for any graph, each with the same automorphism group

We will now construct a finite partial group, this time using the vertices and all paths of length less than or equal to some n, in $\mathbb{N}$, of a simply connected finite undirected graph

Γ. For each n this partial group will have the same automorphism group as Γ. Let

$$P_n = \left\{ v_1 \ldots v_n \;\middle|\; v_i \in V \text{ for } 1 \leq i \leq n,\, v_i v_{i+1} \in E \text{ for } 1 \leq i \leq n-1, v_i \neq v_j \text{ for } i \neq j \right\},$$

in other words the set of all words of length n in vertices of Γ where adjacent letters are joined by an edge and no vertex appears more than once. We shall call this set the *paths of length* n. Notice that E is the same as P_2 and V the same as P_1. Let

$$\mathcal{L}_{\Gamma,n} = V \cup \bigcup_{i=2}^{n} P_i,$$

and define $\mathbf{D}_n$ to be the set of all words of the form

$$x_{i_0}^{\alpha_1} \ldots x_{i_1-1}^{2\alpha_{r-1}-1} x_{i_1}^{2\alpha_r} x_{i_1-1}^{2\alpha_{r+1}-1} \ldots x_{i_2+1}^{2\alpha_{s-1}-1} x_{i_2}^{2\alpha_s} x_{i_2+1}^{2\alpha_{s+1}-1} \ldots x_{i_l \pm 1}^{2\alpha_{t-1}-1} x_{i_l}^{\alpha_t},$$

where $1 \leq i_j \leq m$, for $1 \leq j \leq l$, each α_j is in $\mathbb{N}$ and $x_1 \ldots x_m$ is a path in P_m, for all $2 \leq m \leq n$. The set $\mathbf{D}_n$ can be thought of as walks back and forth along paths in P_m, for each $2 \leq m \leq n$. If we think of each walk taking an amount of time measured in intervals we will call 'ticks' and on the ith tick we record the ith vertex in the walk. The walk is allowed to 'wait' at the first and last vertices for any number of ticks, at any vertex where it changes direction for an even number of ticks and at any other vertex for an odd number of ticks. From this characterisation it is easy to see that for any word in $\mathbf{D}_n$ all of its subwords are also contained in $\mathbf{D}_n$. By definition we have that $\mathcal{L}_{\Gamma,n}$ contains $\mathcal{L}_{\Gamma,k}$ and $\mathbf{D}_n$ contains $\mathbf{D}_k$, for all $k < n$.

Note that if we set $n = 2$ then we have that $\mathcal{L}_{\Gamma,2} = V \cup E = \mathcal{L}_\Gamma$ from Section 2.4.1. Furthermore words in $\mathbf{D}_2$ can only be constructed from paths of length 2. For xy in E we have that any word starting with any power of x then followed by alternating even powers of y and x and ending in any power of x or y is in $\mathbf{D}_2$ so it is easy to see that we also have $\mathbf{D}_2 = \mathbf{D}$, where $\mathbf{D}$ is the set from Section 2.4.1.

Now we define a map π on certain pairs of elements in $\mathcal{L}_{\Gamma,n}$, provided both elements are subpaths of an arbitrary path $p = v_1 \dots v_m$ in P_m for any $m \leq n$, or the path traced in reverse. Let $v_i \dots v_{i+j}$ be an arbitrary subpath of p, possibly of length 1, so $1 \leq i, i+j \leq m$. First we define π on any path and the empty word as

$$\pi(v_i \dots v_{i+j}) = \pi(v_i \dots v_{i+j} \circ \emptyset) = v_i \dots v_{i+j} = \pi(\emptyset \circ v_i \dots v_{i+j}).$$

Next we define the product applied to this path and any subpath of p of the form $v_{i+j+1} \dots v_{i+j+k}$, for $i+j \leq m-1$ and $1 \leq k \leq m-i-j$, or any subpath of p traced in reverse of the form $v_{i+j} \dots v_{i+j-k}$, for $0 \leq k \leq i+j-1$. In the first case we set

$$\pi(v_i \dots v_{i+j} \circ v_{i+j+1} \dots v_{i+j+k}) = v_i \dots v_{i+j+k},$$

and in the second case we set

$$\pi(v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j-k}) = \begin{cases} v_i \dots v_{i+j-k-1} & \text{if } k < j, \\ 1 & \text{if } k = j, \\ v_{i-1} \dots v_{i+j-k} & \text{if } k > j. \end{cases}$$

Note that the product of any two subpaths of p or p traced in reverse is also a subpath of p or p traced in reverse. From this definition we can show that π is associative on $\mathcal{L}_{\Gamma,n}$.

Lemma 2.4.6. *The product π is associative on $\mathcal{L}_{\Gamma,n}$, in other words for any u, v and w in $\mathcal{L}_{\Gamma,n}$ such that $u \circ v \circ w$ is in $\mathbf{D}_n$ we have $\pi(u \circ \pi(v \circ w)) = \pi(\pi(u \circ v) \circ w)$.*

Proof. As with the proof of Lemma 2.4.2 we proceed by considering all possible cases. Let $p = v_1 \dots v_m$ be an arbitrary path in P_m for any $m \leq n$ and let $v_i \dots v_{i+j}$, with $1 \leq i, i+j \leq m$, be an arbitrary subpath of p. As $\mathcal{L}_{\Gamma,n}$ is symmetric, we can assume this is the first word of any allowed triple of subpaths of p and p traced in reverse. Thus the only triples we can have are

$$v_i \dots v_{i+j} \circ v_{i+j+1} \dots v_{i+j+k} \circ v_{i+j+k+1} \dots v_{i+j+k+l}, \tag{2.4.1}$$

for $1 \le i+j \le m-2$, $1 \le k \le m-i-j-1$ and $1 \le l \le m-i-j-k$,

$$v_i \dots v_{i+j} \circ v_{i+j+1} \dots v_{i+j+k} \circ v_{i+j+k} \dots v_{i+j+k-l}, \tag{2.4.2}$$

for $1 \le i+j \le m-1$, $1 \le k \le m-i-j-2$ and $0 \le l \le i+j+k-1$,

$$v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j-k} \circ v_{i+j-k} \dots v_{i+j-k+l}, \tag{2.4.3}$$

for $1 \le i+j \le m$, $0 \le k \le i+j-1$ and $1 \le l \le m-i-j+k$, and

$$v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j-k} \circ v_{i+j-k-1} \dots v_{i+j-k-l}, \tag{2.4.4}$$

for $1 \le i+j \le m$, $0 \le k \le i+j-2$ and $0 \le l \le i+j-k-1$. The first of these cases, (2.4.1), is trivial as the product each time is simply the concatenation of the two paths and it does not matter which order these concatenation are made, so we obtain $v_i \dots v_{i+j+k+l}$ either way.

Similarly if we consider the (2.4.2) we have that the product of the first two words is just their concatenation, so if we apply the product to the first two and then the third we obtain

$$\pi(v_i \dots v_{i+j+k} \circ v_{i+j+k} \dots v_{i+j+k-l}) = \begin{cases} v_i \dots v_{i+j+k-l-1} & \text{if } l < j+k, \\ 1 & \text{if } l = j+k, \\ v_{i-1} \dots v_{i+j+k-l} & \text{if } l > j+k. \end{cases}$$

If we apply the product to the second pair in this triple we obtain

$$\pi(v_{i+j+1} \dots v_{i+j+k} \circ v_{i+j+k} \dots v_{i+j+k-l}) = \begin{cases} v_{i+j+1} \dots v_{i+j+k-l-1} & \text{if } l < k-1, \\ 1 & \text{if } l = k-1, \\ v_{i+j} \dots v_{i+j+k-l} & \text{if } l > k-1. \end{cases}$$

So for $l < k-1$ then clearly $l < j+k$ and we obtain

$$\pi(v_i \dots v_{i+j+k} \circ v_{i+j+k} \dots v_{i+j+k-l}) = v_i \dots v_{i+j+k-l-1} = \pi(v_i \dots v_{i+j} \circ v_{i+j+1} \dots v_{i+j+k-l}).$$

Now if $l = k-1$, then again we have $l < j+k$ and

$$\pi(v_i \dots v_{i+j+k} \circ v_{i+j+k} \dots v_{i+j+k-l}) = v_i \dots v_{i+j} = \pi(v_i \dots v_{i+j} \circ 1).$$

If we let $l > k-1$, then $\pi(v_{i+j+1} \dots v_{i+j+k} \circ v_{i+j+k} \dots v_{i+j+k-l}) = v_{i+j} \dots v_{i+j+k-l}$ and we have

$$\pi(v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j+k-l}) = \begin{cases} v_i \dots v_{i+j-k-l-1} & \text{if } l < j+k, \\ 1 & \text{if } l = j+k, \\ v_{i-1} \dots v_{i+j-k-l} & \text{if } l > j+k, \end{cases}$$

so clearly $\pi(v_i \dots v_{i+j+k} \circ v_{i+j+k} \dots v_{i+j+k-l}) = \pi(v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j+k-l})$ for all choices of l and thus associativity holds for (2.4.2).

In (2.4.3) the product of the first two paths is

$$\pi(v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j-k}) = \begin{cases} v_i \dots v_{i+j-k-1} & \text{if } k < j, \\ 1 & \text{if } k = j, \\ v_{i-1} \dots v_{i+j-k} & \text{if } k > j. \end{cases}$$

If $k < j$ or $k = j$ then taking the product of this with the third path is just concatenation and we obtain $v_i \dots v_{i+j-k+l}$ or $v_i \dots v_{i+l}$ respectively. If $k > j$ then taking the product of this with the third path is

$$\pi(v_{i-1} \dots v_{i+j-k} \circ v_{i+j-k} \dots v_{i+j-k+l}) = \begin{cases} v_{i-1} \dots v_{i+j-k+l+1} & \text{if } l < k-j-1, \\ 1 & \text{if } l = k-j-1, \\ v_i \dots v_{i+j-k+l} & \text{if } l > k-j-1. \end{cases}$$

Similarly if we compute the product of the second and third path in (2.4.3) we obtain

$$\pi(v_{i+j}\dots v_{i+j-k}\circ v_{i+j-k}\dots v_{i+j-k+l}) = \begin{cases} v_{i+j}\dots v_{i+j-k+l+1} & \text{if } l<k, \\ 1 & \text{if } l=k, \\ v_{i+j+1}\dots v_{i+j-k+l} & \text{if } l>k. \end{cases}$$

If $l=k$ or $l>k$ then taking the product with the first path gives $v_i\dots v_{i+j}$ or $v_i\dots v_{i+j-k+l}$ respectively. In the case where $l<k$ we have

$$\pi(v_i\dots v_{i+j}\circ v_{i+j}\dots v_{i+j-k+l+1}) = \begin{cases} v_{i-1}\dots v_{i+j-k+l+1} & \text{if } l<k-j-1, \\ 1 & \text{if } l=k-j-1, \\ v_i\dots v_{i+j-k+l} & \text{if } l>k-j-1. \end{cases}$$

Now we just compare for different values of k and l. If $k<j$ and $k\neq l$ then we have

$$\pi(v_i\dots v_{i+j-k-1}\circ v_{i+j-k}\dots v_{i+j-k+l}) = v_i\dots v_{i+j-k+l} = \pi(v_i\dots v_{i+j}\circ v_{i+j+1}\dots v_{i+j-k+l}),$$

and if $k=l$ then we have

$$\pi(v_i\dots v_{i+j-k-1}\circ v_{i+j-k}\dots v_{i+j-k+l}) = v_i\dots v_{i+j} = \pi(v_i\dots v_{i+j}\circ 1).$$

For $k=j$ and $k\neq l$ then we have

$$\pi(1\circ v_{i+j-k}\dots v_{i+j-k+l}) = v_i\dots v_{i+l} = \pi(v_i\dots v_{i+j}\circ v_{i+j+1}\dots v_{i+j-k+l}),$$

and if $k=l$ then we have

$$\pi(1\circ v_{i+j-k}\dots v_{i+j-k+l}) = v_i\dots v_{i+j} = \pi(v_i\dots v_{i+j}\circ 1).$$

We now consider the cases when $k > j$. If $l = k$ then we have

$$\pi(v_{i-1} \dots v_{i+j-k} \circ v_{i+j-k} \dots v_{i+j-k+l}) = v_i \dots v_{i+j} = \pi(v_i \dots v_{i+j} \circ 1),$$

and if $l > k$ we have

$$\pi(v_{i-1} \dots v_{i+j-k} \circ v_{i+j-k} \dots v_{i+j-k+l}) = v_i \dots v_{i+j-k+l} = \pi(v_i \dots v_{i+j} \circ v_{i+j+1} \dots v_{i+j-k+l}).$$

In the case where both $k > j$ and $l < k$ we have clearly have that $l > k - j - 1$ hence we have

$$\pi(v_i \dots v_{i+j-k-1} \circ v_{i+j-k} \dots v_{i+j-k+l}) = v_i \dots v_{i+j-k+l} = \pi(v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j-k+l+1}),$$

again. Thus associativity holds for (2.4.3).

Lastly we consider triples of the form (2.4.4). Notice that the product of the second and third path is just concatenation so if we take that product first then the product with the first path we obtain

$$\pi(v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j-k-l}) = \begin{cases} v_i \dots v_{i+j-k-l-1} & \text{if } l+k < j, \\ 1 & \text{if } l+k = j, \\ v_{i-1} \dots v_{i+j-k-l} & \text{if } l+k > j. \end{cases}$$

If we now consider the product of the first two paths in (2.4.4) we again obtain

$$\pi(v_i \dots v_{i+j} \circ v_{i+j} \dots v_{i+j-k}) = \begin{cases} v_i \dots v_{i+j-k-1} & \text{if } k < j, \\ 1 & \text{if } k = j, \\ v_{i-1} \dots v_{i+j-k} & \text{if } k > j. \end{cases}$$

When $k = j$ and $k > j$ the product of this with the third path is just concatenation so we have $v_{i-1}\dots v_{i-l}$ and $v_{i-1}\dots v_{i+j-k-l}$ respectively. If $k=j$ then certainly $l+k>j$ so we have

$$\pi(1\circ v_{i+j-k-1}\dots v_{i+j-k-l}) = v_{i-1}\dots v_{i-l} = \pi(v_i\dots v_{i+j}\circ v_{i+j}\dots v_{i+j-k}),$$

and similarly for $k>j$ we have

$$\begin{aligned}\pi(v_{i-1}\dots v_{i+j-k}\circ v_{i+j-k-1}\dots v_{i+j-k-l}) &= v_{i-1}\dots v_{i+j-k-l}\\ &= \pi(v_{i-1}\dots v_{i+j-k}\circ v_{i+j-k-1}\dots v_{i+j-k}).\end{aligned}$$

When $k<j$ the product of the first two paths with the third is

$$\pi(v_i\dots v_{i+j-k-1}\circ v_{i+j-k-1}\dots v_{i+j-k-l}) = \begin{cases} v_i\dots v_{i+j-k-l-1} & \text{if } l<j-k,\\ 1 & \text{if } l=j-k,\\ v_{i-1}\dots v_{i+j-k-l} & \text{if } l>j-k.\end{cases}$$

Thus we see that $\pi(v_i\dots v_{i+j-k-1}\circ v_{i+j-k-1}\dots v_{i+j-k-l}) = \pi(v_i\dots v_{i+j}\circ v_{i+j}\dots v_{i+j-k-l})$ for all choices of j, k and l in this case. Hence associativity holds for (2.4.4) and as we have now considered all cases we have that π is associative on $\mathcal{L}_{\Gamma,n}$. $\square$

We can restrict the definition of π to just be on pairs of elements in V as a subset of $\mathcal{L}_{\Gamma,n}$. Let xy be in $P_2=E$ and then we have $\pi(xx)=1$ and $\pi(xy)=xy$. We will also consider it on pairs where the first in the pair is a subpath of an arbitrary path $v_1\dots v_m$ in P_m and the second is in V and their concatenation is in $\mathbf{D}_n$. We do this as follows

$$\begin{aligned}\pi(v_i\dots v_{i+j-1}\circ v_{i+j}) &= v_1\dots v_{i+j-1}v_{i+j},\\ \pi(v_i\dots v_{i+j-1}v_{i+j}\circ v_{i+j}) &= v_1\dots v_{i+j-1},\end{aligned}$$

for $1\le i\le m-1$, $1\le j\le m-i$ and all $m>2$. Thus for w in $\mathbf{D}_n$ note we can write

$w = x_1 \circ \cdots \circ x_m$, where each x_i is in V. We then define

$$\Pi(w) := \pi(\cdots \pi(\pi(x_1) \circ x_2) \cdots \circ x_m),$$

for w an arbitrary word in $\mathbf{D}_n$. First we need to show that this second characterisation is well-defined on $\mathbf{D}_n$.

Lemma 2.4.7. *If $w = x_1 \circ x_2 \circ \cdots \circ x_l$ is a word in $\mathbf{D}_n$, where each letter is a vertex of Γ then $\Pi(x_1 \dots x_{k-1} \circ x_k) \circ x_{k+1} \circ \cdots \circ x_l$, for all $1 \leq k \leq n$, is in $\mathbf{D}_n$ and $\Pi : \mathbf{D}_n \to \mathcal{L}_{\Gamma,n}$ is a well-defined map.*

Proof. We shall prove this by induction on k. Let $v_1 \dots v_m$ be the path in P_m corresponding to w, for $m \leq n$. The case where $k = 1$, is trivial so consider $k = 2$. Note that both $\mathbf{D}_n$ and P_m are symmetric, so we may assume that there are only two words of length 2 that we need to consider in $\mathbf{D}_n$: $v_i \circ v_{i+1}$ and $v_i \circ v_i$, where $v_i v_{i+1}$ is any length 2 subpath of $v_1 \dots v_m$. If we first consider $v_i \circ v_{i+1}$ then we see that $\Pi(v_i \circ v_{i+1}) = v_i v_{i+1}$ so if these are the first two letters of w and w is in $\mathbf{D}_n$ it is clear that applying Π to the first two letters results in a word in $\mathbf{D}_n$. Similarly, note that $\Pi(v_i \circ v_i) = 1$ so if w starts with these letters then the word resulting in applying Π to the first two letters of w is in $\mathbf{D}_n$, as it is a subword of a word already in $\mathbf{D}_n$ and we have discussed that it is clear $\mathbf{D}_n$ is closed upon taking subwords. Lastly we have that $\Pi(v_i \circ v_i)$ and $\Pi(v_i \circ v_{i+1})$ are both in $\mathcal{L}_{\Gamma,n}$ by definition of Π.

Now suppose that $w' = \Pi(x_1 \circ \cdots \circ x_{k-1}) \circ x_k \circ \cdots \circ x_l$ is in $\mathbf{D}_n$ and that $\Pi(x_1 \circ \cdots \circ x_{k-1})$ is in $\mathcal{L}_{\Gamma,n}$, for some $k < l$. Let $\Pi(x_1 \circ \cdots \circ x_{k-1}) = v_i v_{i+1} \dots v_j$, a subpath of $v_1 \dots v_m$ and without loss of generality we assume $i < j$. If we have $v_i v_{i+1} \dots v_j \circ x_k \circ \cdots \circ x_l$ in $\mathbf{D}_n$ then there are three potential choices for x_k, either v_{j-1}, v_j or v_{j+1}. First, if $x_k = v_{j-1}$, then we have

$$\Pi(x_1 \circ \cdots \circ x_{k-1}) \circ x_k \circ \cdots \circ x_l = v_i \dots v_{j-1} v_j \circ v_{j-1} \circ x_{k+1} \circ \cdots \circ x_l$$

which is not a word in $\mathbf{D}_n$ as the subword $v_{j-1}v_jv_{j-1}$ is not allowed to occur.

Now, if $x_k = v_j$ then clearly

$$\Pi(\Pi(x_1 \circ \cdots \circ x_{k-1}) \circ x_k) = \Pi(v_i \dots v_j \circ v_j) = v_i \dots v_{j-1}$$

which is in $\mathcal{L}_{\Gamma,n}$. By our assumption we have that $w' = v_i \dots v_j \circ v_j \circ x_{k+1} \circ \cdots \circ x_l$ is in $\mathbf{D}_n$ so again we have three options for x_{k+1}. If $x_{k+1} = v_{j-1}$ and is followed by an odd number of v_{j-1}s then the next different vertex in w' must be v_j and we have that

$$\Pi(v_i \dots v_j \circ v_j) \circ x_{k+1} \circ \cdots \circ x_l = v_i \dots v_{j-2} \circ \underbrace{v_{j-1} \circ \cdots \circ v_{j-1}}_{\text{odd number}} \circ v_j \circ \cdots \circ x_l,$$

where the word is unaltered after v_j. Thus this word is still in $\mathbf{D}_n$. If $x_{k+1} = v_{j-1}$ is followed by an even, possibly zero, number of v_{j-1}s then the next different vertex in w' must be v_{j-2} and we have that

$$\Pi(v_i \dots v_j \circ v_j) \circ x_{k+1} \circ \cdots \circ x_l = v_i \dots v_{j-2} \circ \underbrace{v_{j-1} \circ \cdots \circ v_{j-1}}_{\text{even number}} \circ v_{j-2} \circ \cdots \circ x_l,$$

where the word is unaltered after v_j. Thus this word is again still in $\mathbf{D}_n$. We proceed in a similar way for the case where $x_{k+1} = v_j$. If it is followed by an odd number of v_js then the next letter has to be v_{j-1} for w' to be in $\mathbf{D}_n$. Thus

$$\Pi(v_i \dots v_j \circ v_j) \circ x_{k+1} \circ \cdots \circ x_l = v_i \dots v_{j-1} \circ \underbrace{v_j \circ \cdots \circ v_j}_{\text{even number}} \circ v_{j-1} \circ \cdots \circ x_l,$$

where the word is unaltered after the second v_{j-1}, so it is still in $\mathbf{D}_n$. If it were instead followed by an even number of v_js, possibly zero, then the next letter would have to be v_{j+1} for w' to be in $\mathbf{D}_n$. Thus

$$\Pi(v_i \dots v_j \circ v_j) \circ x_{k+1} \circ \cdots \circ x_l = v_i \dots v_{j-1} \circ \underbrace{v_j \circ \cdots \circ v_j}_{\text{odd number}} \circ v_{j+1} \circ \cdots \circ x_l,$$

where the word is unaltered after v_{j+1}, so it is still in $\mathbf{D}_n$. The last of the three cases to check is when $x_{k+1} = v_{j+1}$. Here we quickly reach a contradiction as $w' = v_i \dots v_{j-1}v_j \circ v_j \circ v_{j+1} \circ x_{k+2} \circ \cdots \circ x_l$ and $v_{j-1}v_j^2 v_{j+1}$ is not an allowed word in $\mathbf{D}_n$.

Finally we move to the case where $x_k = v_{j+1}$. We have $\Pi(\Pi(x_1 \circ \cdots \circ x_{k-1}) \circ x_k) = \Pi(v_i \dots v_j \circ v_{j+1}) = v_i \dots v_{j+1}$ which is in $\mathcal{L}_{\Gamma,n}$. By our assumption we have that $w' = v_i \dots v_j \circ v_{j+1} \circ x_{k+1} \circ \cdots \circ x_l$ is in $\mathbf{D}_n$ so now we consider the three choices for x_{k+1}. If $x_{k+1} = v_j$ then we have $w' = v_i \dots v_j \circ v_{j+1} \circ v_j \circ x_{k+2} \circ \cdots \circ x_l$. This is not in $\mathbf{D}_n$ regardless of our choice of x_{k+2}, so x_{k+1} cannot be v_j. If we have $x_{k+1} = v_{j+1}$ then notice that we recover the three cases we have when $x_k = v_j$, as j is arbitrary. Thus we only need to consider $x_{k+1} = v_{j+2}$. If x_{k+1} is followed by an odd power of v_{j+2} then for w' to be in $\mathbf{D}_n$ we have that the next letter must be v_{j+1}. Thus

$$\Pi(v_i \dots v_j \circ v_{j+1}) \circ x_{k+1} \circ \cdots \circ x_l = v_i \dots v_{j+1} \circ \underbrace{v_{j+2} \circ \cdots \circ v_{j+2}}_{\text{even number}} \circ v_{j+1} \circ \cdots \circ x_l,$$

where the word remains the same after the second v_{j+1}, is a word in $\mathbf{D}_n$. Lastly, if $x_{k+1} = v_{j+2}$ is followed by an even number of v_{i+2}s, then we must have the next letter of w' be v_{i+3} for w' to be in $\mathbf{D}_n$. Therefore

$$\Pi(v_i \dots v_j \circ v_{j+1}) \circ x_{k+1} \circ \cdots \circ x_l = v_i \dots v_{j+1} \circ \underbrace{v_{j+2} \circ \cdots \circ v_{j+2}}_{\text{odd number}} \circ v_{j+3} \circ \cdots \circ x_l,$$

where the word remains the same after the second v_{j+1}, is a word in $\mathbf{D}_n$. We have therefore checked all cases and provided $w' = \Pi(x_1 \circ \cdots \circ x_{k-1}) \circ x_k \circ \cdots \circ x_l$ is in $\mathbf{D}_n$ and that $\Pi(x_1 \circ \cdots \circ x_{k-1})$ is in $\mathcal{L}_{\Gamma,n}$ then $\Pi(x_1 \circ \cdots \circ x_k) \circ x_{k+1} \circ \cdots \circ x_l$ is in $\mathbf{D}_n$ and $\Pi(x_1 \circ \cdots \circ x_k)$ is in $\mathcal{L}_{\Gamma,n}$. Thus by induction the hypothesis holds for all k and thus all words in $\mathbf{D}_n$. □

We can now show the two products we have defined agree on pairs of elements in $\mathcal{L}_{\Gamma,n}$ where both definitions make sense.

Lemma 2.4.8. *If u and v are elements of $\mathcal{L}_{\Gamma,n}$ such that $u \circ v$ is in $\mathbf{D}_n$ and we write*

$u \circ v = x_1 \circ \cdots \circ x_m$, *where each* x_i *is in* V, *then* $\Pi(u \circ v) = \pi(u \circ v)$.

Proof. If $u \circ v$ is in $\mathbf{D}_n$ then there exists some path $p = v_1 \ldots v_m$ in P_m, for $m \leq n$ such that u is a subpath of p and v is either a subpath of p or p traced backwards. If $u = v_i \ldots v_{i+j}$, for $1 \leq i, i+j \leq m$, then either $v = v_{i+j+1} \ldots v_{i+j+k}$, for $1 \leq i+j \leq m-1$, $1 \leq k \leq m-i-j$, or $v = v_{i+j} \ldots v_{i+j-k}$, for $1 \leq i+j \leq m$, $0 \leq k \leq i+j$. The first case is trivial as $\pi(u \circ v)$ is just their concatenation and $\Pi(u \circ v) = \pi(\cdots \pi(\pi(x_1) \circ x_2) \circ \cdots) \circ x_m)$ is the concatenation of each vertex in turn. When $v = v_{i+j} \ldots v_{i+j-k}$ we have

$$\pi(v_i \ldots v_{i+j} \circ v_{i+j} \ldots v_{i+j-k}) = \begin{cases} v_i \ldots v_{i+j-k-1} & \text{if } k < j, \\ 1 & \text{if } k = j, \\ v_{i-1} \ldots v_{i+j-k} & \text{if } k > j. \end{cases}$$

Our other definition of Π is the same as the first case for the first j vertices, each vertex is added to the end of the resulting path so we obtain

$$\Pi(u \circ v) = \pi(\cdots \pi(\pi(\cdots \pi(\pi(v_i) \circ v_{i+1}) \cdots \circ v_{i+j}) \circ v_{i+j}) \cdots \circ v_{i+j-k}),$$

is equal to $\pi(\cdots \pi(v_i \ldots v_{i+j} \circ v_{i+j}) \cdots \circ v_{i+j-k})$. Each subsequent application of the product shortens the path by one vertex. If k, the length of v, is less than j, the length of u then the resulting path is $v_i \ldots v_{i+j-k-1}$. If they are the same length then the result is the empty path and lastly if $k > j$ the resulting path is $v_{i-1} \ldots v_{i+j-k}$ which is in the other direction. Thus both definitions of the product agree on $\mathcal{L}_{\Gamma,n}$. $\square$

Our two characterisations of the product are therefore equivalent on $\mathcal{L}_{\Gamma,n}$ and we will use them interchangeably from now on.

Notice that, for $n = 2$, this characterisation of Π is the same as in Section 2.4.1. Hence, as the sets that Π maps between are the same in this case, we have that $\Pi : \mathbf{D}_2 \to \mathcal{L}_{\Gamma,2}$ is the same as Π from Section 2.4.1.

Now that we have a well-defined product map from $\mathbf{D}_n$ to $\mathcal{L}_{\Gamma,n}$, we need to define an inverse map. For any v in V we set $v = v^{-1}$ and for a general path $v_1 \dots v_m$ in P_m we set $(v_1 \dots v_m)^{-1} = v_m \dots v_1$, for any $m \leq n$. We then extend this to $\mathbf{D}_n$ in the natural way. We will now show this inverse map behaves in the expected way.

Lemma 2.4.9. *Let w be in $\mathbf{D}_n$, then $w^{-1} \circ w$ is also in $\mathbf{D}_n$ and $\Pi(w^{-1} \circ w) = 1$.*

Proof. If w is a word in $\mathbf{D}_n$, then we can characterise w as a walk back and forth along some path $p = v_1 \dots v_m$ in P_m for $m \leq n$, stopping at vertices where it does not change direction for an odd number of ticks and where it does for an even number. Say this walk starts at v_i and ends at v_j. By the definition of $\cdot^{-1}$ extended to $\mathbf{D}_n$ we have that w^{-1} can be characterised as the same walk in reverse stopping at each vertex for the same number of ticks. Thus $w^{-1} \circ w$ is the walk w^{-1} followed by the walk w and as it certainly waits at v_i, when it passes from w^{-1} to w and changes direction, for an even number of ticks it and waits at all other vertices for the same number of ticks as in both subpaths we have $w^{-1} \circ w$ is in $\mathbf{D}_n$.

If v is an arbitrary vertex that occurs in w then it must occur in $w^{-1} \circ w$ an even number of times as it occurs in w and w^{-1} the same number of times by definition. From Lemma 2.4.7 we have that Π maps elements of $\mathbf{D}_n$ to $\mathcal{L}_{\Gamma,n}$, which is the set of all paths in Γ of length less than or equal to n. Thus in each element of $\mathcal{L}_{\Gamma,n}$, v occurs at most once. By the first part of this proof $w^{-1} \circ w$ is in $\mathbf{D}_n$ so $\Pi(w^{-1} \circ w)$ is in $\mathcal{L}_{\Gamma,n}$ and v occurs at most once in $\Pi(w^{-1} \circ w)$. However, v occurring in $w^{-1} \circ w$ an even number of times means it must occur in $\Pi(w^{-1} \circ w)$ an even number of times as well. This is because $\Pi(v^{2\alpha-1}) = v$ and $\Pi(v^{2\alpha}) = 1$, for α in $\mathbb{N}$, by definition of Π. Hence v does not occur in $\Pi(w^{-1} \circ w)$ and as v was arbitrary we have $\Pi(w^{-1} \circ w) = 1$. $\square$

Before we show that $\mathcal{L}_{\Gamma,n}$ is a partial group we will show that the first part of the third axiom, Definition 2.2.1(P3), holds.

Lemma 2.4.10. *If $u \circ v \circ w$ is a word in $\mathbf{D}_n$ then $u \circ \Pi(v) \circ w$ is also in $\mathbf{D}_n$.*

Proof. Let $p = t_1 \dots t_m$ be the path in the P_m corresponding to $u \circ v \circ w$ in $\mathbf{D}_n$. First note this statement holds vacuously for v the empty word so we assume it is non-empty.

We can also assume both u and w are non-empty because, as discussed, $\mathbf{D}_n$ is closed upon taking subwords. Thus if we show $u \circ \Pi(v) \circ w$ is in $\mathbf{D}_n$ then certainly $u \circ \Pi(v)$ and $\Pi(v) \circ w$ are also in $\mathbf{D}_n$.

If $\Pi(v) = 1$ then v is of the form $y_1 \circ y_1^{-1} \circ \cdots \circ y_l \circ y_l^{-1}$, where each y_i is a word in $\mathbf{D}_n$ and l is in $\mathbb{N}$. Let $v = y_1 \circ y_1^{-1}$ and assume y_1 is just a power of one vertex. Without loss of generality set $y_1 = t_i^{\alpha}$, for some $1 \leq i \leq l$ and α in $\mathbb{N}$. Hence $v = t_i^{2\alpha}$ is an even power of a single letter and $u \circ v \circ w$ and $u \circ \Pi(v) \circ w = u \circ 1 \circ w$ both have either odd powers of t_i or even powers of t_i at the end of u. Hence the direction of the walk along p corresponding to $u \circ v \circ w$ is preserved in $u \circ \Pi(v) \circ w$ and $u \circ \Pi(v) \circ w$ is in $\mathbf{D}_n$.

Note that by our definition of Π, if we write $v = a \circ b$ then $\Pi(v) = \Pi(\Pi(a) \circ b)$. So in the case when v starts with an even power 2α of any vertex, say t_i, we can set $a = t_i^{2\alpha}$ and then

$$u \circ \Pi(v) \circ v = u \circ \Pi(\Pi(t_i^{2\alpha}) \circ b) \circ v = u \circ \Pi(1 \circ b) \circ v = u \circ \Pi(b) \circ v.$$

Furthermore, from above, we have that $u \circ \Pi(a) \circ b \circ w = u \circ b \circ w$ is in $\mathbf{D}_n$ so it suffices to assume that from this point onward v starts with a single t_i as we can always remove an even power of t_i from the start of v.

Now suppose that y_1 has at least two different vertices in it, and without loss of generality we may assume it starts with t_i followed by some power of t_{i+1}, as $\mathbf{D}_n$ is symmetric. Thus $v = t_i t_{i+1}^{\alpha} \ldots t_{i+1}^{\alpha} t_i$, for some α in $\mathbb{N}$. We need to consider all possible endings of u and starts of w such that $u \circ v \circ w$ is in $\mathbf{D}_n$. Note that at most we only need to know the last two vertices in the word u and the first two in w and the powers of the last and first respectively as we only need to know that the direction of the walk $u \circ v \circ w$ is preserved when we apply Π to v. We therefore consider all the possible ways u can end such that $u \circ v$ is in $\mathbf{D}_n$. These are $t_{i-1}^{\gamma_1} t_i^{\beta_1}$ and $t_i^{\gamma_1} t_{i-1}^{\beta_1}$, for β_1 even in both cases and $t_{i-2}^{\gamma_1} t_{i-1}^{\beta_1}$ and $t_{i+1}^{\gamma_1} t_i^{\beta_1}$, for β_1 odd in both cases. For the first two letters of w we have the same four options but with the order of the letters reversed; $t_i^{\beta_2} t_{i-1}^{\gamma_2}$ and $t_{i-1}^{\beta_2} t_i^{\gamma_2}$, for β_2 even in both cases and $t_{i-1}^{\beta_2}t_{i-2}^{\gamma_2}$ and $t_i^{\beta_2}t_{i+1}^{\gamma_2}$, for β_2 odd in both cases. Now it is just a case of checking all sixteen combinations to see if $u \circ \Pi(v) \circ w = u \circ w$ is in $\mathbf{D}_n$ in each. When β_1 and β_2 are both even then we have the cases $u \circ w = \ldots t_{i-1}^{\gamma_1}t_i^{\beta_1+\beta_2}t_{i-1}^{\gamma_2}\ldots$, $u \circ w = \ldots t_{i-1}^{\gamma_1}t_i^{\beta_1}t_{i-1}^{\beta_2}t_i^{\gamma_2}\ldots$, $u \circ w = \ldots t_i^{\gamma_1}t_{i-1}^{\beta_1}t_i^{\beta_2}t_{i-1}^{\gamma_2}\ldots$ and $u \circ w = \ldots t_i^{\gamma_1}t_{i-1}^{\beta_1+\beta_2}t_i^{\gamma_2}\ldots$ which are all words in $\mathbf{D}_n$. If β_1 is even and β_2 odd then we have the cases $u \circ w = \ldots t_{i-1}^{\gamma_1}t_i^{\beta_1}t_{i-1}^{\beta_2}t_{i-2}^{\gamma_2}\ldots$, $u \circ w = \ldots t_{i-1}^{\gamma_1}t_i^{\beta_1+\beta_2}t_{i+1}^{\gamma_2}\ldots$, $u \circ w = \ldots t_i^{\gamma_1}t_{i-1}^{\beta_1+\beta_2}t_{i-2}^{\gamma_2}\ldots$ and $u \circ w = \ldots t_i^{\gamma_1}t_{i-1}^{\beta_1}t_i^{\beta_2}t_{i+1}^{\gamma_2}\ldots$ which are all in $\mathbf{D}_n$. If β_1 is odd and β_2 even then we have the cases $u \circ w = \ldots t_{i-2}^{\gamma_1}t_{i-1}^{\beta_1}t_i^{\beta_2}t_{i-1}^{\gamma_2}\ldots$, $u \circ w = \ldots t_{i-2}^{\gamma_1}t_{i-1}^{\beta_1+\beta_2}t_i^{\gamma_2}\ldots$, $u \circ w = \ldots t_{i+1}^{\gamma_1}t_i^{\beta_1+\beta_2}t_{i-1}^{\gamma_2}\ldots$ and $u \circ w = \ldots t_{i+1}^{\gamma_1}t_i^{\beta_1}t_{i-1}^{\beta_2}t_i^{\gamma_2}\ldots$, again all words in $\mathbf{D}_n$. Lastly if both β_1 and β_2 are odd then we have the cases $u \circ w = \ldots t_{i-2}^{\gamma_1}t_{i-1}^{\beta_1+\beta_2}t_{i-2}^{\gamma_2}\ldots$, $u \circ w = \ldots t_{i-2}^{\gamma_1}t_{i-1}^{\beta_1}t_i^{\beta_2}t_{i+1}^{\gamma_2}\ldots$, $u \circ w = \ldots t_{i+1}^{\gamma_1}t_i^{\beta_1}t_{i-1}^{\beta_2}t_{i-2}^{\gamma_2}\ldots$ and $u \circ w = \ldots t_{i+1}^{\gamma_1}t_i^{\beta_1+\beta_2}t_{i+1}^{\gamma_2}\ldots$ which are all words in $\mathbf{D}_n$. Thus, if $v = t_it_{i+1}^{\alpha}\ldots t_{i+1}^{\alpha}t_i$, we have that $u \circ \Pi(v) \circ w$ is in $\mathbf{D}_n$.

We now have that the hypothesis holds for any v of the form $y_1 \circ y_1^{-1}$, where y_1 is any word in $\mathbf{D}_n$. Suppose $v = y_1 \circ y_1^{-1} \circ \cdots \circ y_l \circ y_l^{-1}$, for l in $\mathbb{N}$. From above we have

$$u \circ \Pi(y_1 \circ y_1^{-1}) \circ y_2 \circ y_2^{-1} \circ \cdots \circ y_l \circ y_l^{-1} \circ w = u \circ y_2 \circ y_2^{-1} \circ \cdots \circ y_l \circ y_l^{-1} \circ w$$

is in $\mathbf{D}_n$. Thus we can apply Π to $y_2 \circ y_2^{-1}$ and again remain in $\mathbf{D}_n$. Repeating this l times we have that $u \circ \Pi(v) \circ w = u \circ w$ is in $\mathbf{D}_n$ for any v such that $\Pi(v) = 1$.

Suppose now that $\Pi(v)$ is a single vertex, without loss of generality t_i. If v does not start with t_i then it starts with something of the form $y_1 \circ y_1^{-1} \circ \cdots \circ y_l \circ y_l^{-1}$ followed by a single t_i. Thus from the above discussion we can assume v starts t_i. Assuming $v = t_i$, it is clear that $u \circ \Pi(v) \circ w = u \circ v \circ w$ is in $\mathbf{D}_n$. Thus our hypothesis holds for all words of the form $y_1 \circ y_1^{-1} \circ \cdots \circ y_l \circ y_l^{-1} \circ t_i$.

If v is now arbitrary such that $\Pi(v)$ is not the identity, and without loss of generality it starts with t_i, then there exists an occurrence of t_i in v such that this t_i becomes the first letter of $\Pi(v)$. The subword of v, which we shall denote a, that precedes this must have the property $\Pi(a) = 1$. Hence a is of the form $y_1 \circ y_1^{-1} \circ \cdots \circ y_l \circ y_l^{-1}$. If we denote the remainder of v after the occurrence of this t_i as b then we have $v = a \circ t_i \circ b$ and $u \circ \Pi(a \circ t_i) \circ b \circ w = u \circ t_i \circ b \circ w$ is in $\mathbf{D}_n$ from above. If we write $b = b_1 \circ \cdots \circ b_k$, where each b_i is in V, then by definition of Π on $\mathbf{D}_n$ we have that $\Pi(t_i \circ b) = \Pi(\cdots \Pi(\Pi(t_i \circ b_1) \circ b_2) \cdots \circ b_k)$. Note that, by our assumption, each time we apply Π here the t_i remains at the start of the word and is never cancelled. Thus $\Pi(t_i \circ b) = t_i \Pi(b)$ and we can therefore write $u \circ \Pi(v) \circ w = u \circ t_i \circ \Pi(b) \circ w$. We have that $u \circ t_i \circ b \circ w$ is in $\mathbf{D}_n$, so if we relabel $u \circ t_i$ as u' we have $u' \circ b \circ w$ is in $\mathbf{D}_n$. As b is an arbitrary word then $\Pi(b) = 1$ and we are done or $\Pi(b)$ is non-trivial and we are in the same situation as the start of this paragraph. This process can therefore be repeated and as v is finite will terminate. Thus for any choice of u, v and w such that $u \circ v \circ w$ is in $\mathbf{D}_n$ we have $u \circ \Pi(v) \circ w$ in $\mathbf{D}_n$. □

Proposition 2.4.11. *Let Γ be a simply connected finite undirected graph and let n be in $\mathbb{N}$. The set $\mathcal{L}_{\Gamma,n}$ is a finite partial group with product map Π and inversion $\cdot^{-1}$.*

As already discussed $\mathcal{L}_{\Gamma,2}$ with product $\Pi : \mathbf{D}_2 \to \mathcal{L}_{\Gamma,2}$ is the same as that described in Section 2.4.1 so we have already shown this proposition holds for the case $n = 2$ in proving Proposition 2.4.3. The following proof, however, holds for all $n > 1$.

Proof of Proposition 2.4.11. Clearly $\mathcal{L}_{\Gamma,n}$ is contained in $\mathbf{D}_n$; just take the set of all walks in $\mathbf{D}_n$ that do not wait at any vertices or change direction. We have also already discussed how $\mathbf{D}_n$ is closed under taking subwords, as clearly removing a letter from either end of an arbitrary word in $\mathbf{D}_n$ results in a word in $\mathbf{D}_n$.

By the definition of Π on pairs in $\mathcal{L}_{\Gamma,n}$ we have that, for p in $\mathcal{L}_{\Gamma,n}$,

$$\Pi(p) = \Pi(p \circ 1) = \Pi(1 \circ p) = p.$$

Thus Π is the identity on $\mathcal{L}_{\Gamma,n}$.

If u, v and w are words in $\mathbf{D}_n$ such that $u \circ v \circ w$ is in $\mathbf{D}_n$ then from Lemma 2.4.10 we have that $u \circ \Pi(v) \circ w$ is in $\mathbf{D}_n$. We can write $u \circ v \circ w = u_1 \circ \cdots \circ u_k \circ v_1 \circ \cdots \circ v_l \circ w_1 \circ \cdots \circ w_m$ where each u_i, v_i and w_i is a vertex in V and k, l and m are the lengths of u, v and w in letters in V respectively. Hence, by the definition of Π on $\mathbf{D}_n$, we have

$\Pi(u \circ v \circ w) = \Pi(\cdots \Pi(\Pi(u_1) \circ u_2) \cdots \circ w_m)$. As we have that Π is associative on $\mathcal{L}_{\Gamma,n}$, by Lemma 2.4.6, we can rewrite this as

$$\Pi(\cdots \Pi(\Pi(u_1) \circ u_2) \cdots \circ w_m) = \Pi(u_1 \circ \cdots \circ u_k \circ \Pi(v_1 \circ \cdots \circ v_l) \circ w_1 \circ \cdots \circ w_m) = \Pi(u \circ \Pi(v) \circ w).$$

Thus $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$ for all words $u \circ v \circ w$ in $\mathbf{D}_n$.

We already have that $\mathcal{L}_{\Gamma,n}$ satisfies the final axiom by Lemma 2.4.9. Therefore, as it satisfies all axioms, $\mathcal{L}_{\Gamma,n}$ is a finite partial group with product map Π and inversion $\cdot^{-1}$. □

Notice we have not set a maximum value for n dependent on the longest path length in Γ. This is because for all n greater than this maximum length, M, we have that $\mathcal{L}_{\Gamma,n}$ and $\mathbf{D}_n$ are the same as $\mathcal{L}_{\Gamma,M}$ and $\mathbf{D}_M$ respectively. Thus the products are the same so we have that $\mathcal{L}_{\Gamma,n} \cong \mathcal{L}_{\Gamma,M}$ as partial groups for any $n \geq M$.

Theorem 2.4.12. *If Γ is a simply connected finite undirected graph then* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{L}_{\Gamma,n})$ *for all $n > 1$ in $\mathbb{N}$.*

Proof. If ψ is in $\mathrm{Aut}(\Gamma)$ then ψ is a bijection on V, and on each P_m for $m \leq n$, so induces a bijection on the set $\mathcal{L}_{\Gamma,n}$. As the image of any path $p = v_1 \dots v_m$ in P_m is also a unique path $\psi(p) = \psi(v_1) \dots \psi(v_m)$ in P_m, the image of any word w in $\mathbf{D}_n$ corresponding to p is a unique word in $\mathbf{D}$ corresponding to $\psi(p)$. Thus ψ extends to a bijection on $\mathbf{D}_n$. Additionally in $\psi(w)$ all the powers of each $\psi(v_i)$ will be the same as v_i, for $1 \leq i \leq m$ so $\Pi(\psi(w)) = \psi(\Pi(w))$. Hence ψ is an automorphism of $\mathcal{L}_{\Gamma,n}$.

Now let ϕ be in $\mathrm{Aut}(\mathcal{L}_{\Gamma,n})$. As the set V contains exactly all the elements of $\mathcal{L}_{\Gamma,n}$ that are self-inverse ϕ must be an a bijection on V and thus also a bijection on P_m, for each $m \leq n$. Consider x and y in V with xy in E. We have that $\Pi(\phi(x) \circ \phi(y)) = \phi(\Pi(x \circ y)) = \phi(xy)$, so for every pair of vertices in V connected by an edge their images under ϕ are also connected by an edge and ϕ is in $\mathrm{Aut}(\Gamma)$. Thus $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{L}_{\Gamma,n})$. □

2.5 Objective partial groups

Another way to extend the result of Díaz, Molinier and Viruel is to consider the categories of objective and finite objective partial groups and attempt to show versions of these results in each case. Recall, by Lemma 2.2.24, that in both $\mathfrak{O}\mathcal{P}art$ and $\mathcal{O}\mathcal{P}art$ isomorphisms, and thus automorphisms, are the same, It therefore does not matter which of the two we consider in the following sections.

We begin by investigating finite objective partial groups as these are the more useful objects that correspond to finite localities and thus fusion systems. Showing this category is universal has been elusive however we have succeeded in finding a family where each member has automorphism group some C_n for all $n \geq 3$. When n is odd there does not exist a finite group with automorphism group C_n so this shows that finite objective partial groups can have automorphism groups that groups cannot and is the first step towards showing the category is finite universal. We also construct a second family of examples of objective partial groups with cyclic automorphism groups, showing our first family was not unique in having this property. Lastly we consider all objective partial groups and in this case arrive at a result showing these categories are universal.

2.5.1 A family of finite objective partial groups with cyclic automorphism group

We will now construct an objective partial group with automorphism group C_n for $n \geq 3$ an integer. Note that the groups C_2 and C_3 are finite objective partial groups with automorphism group C_1 and C_2 respectively. We therefore still have an objective partial group with automorphism group C_n for all n but just not in the family we will describe.

Following from the ideas in the previous section one would hope to start with a graph with automorphism group C_n, for $n \geq 3$ odd, and then construct a finite objective partial group from it. If we use the same idea of setting each vertex to be a copy of C_2 and then keep track of conjugation by drawing an edge whenever one can conjugate between two groups by an element of $\mathcal{L}$ we unfortunately reach a problem. If any two vertices in this graph are connected by a path then there exists a word in $\mathbf{D}_\Delta$ that conjugates between them however this word must map to a letter in $\mathcal{L}$, by the definition of objective partial group, and thus an edge must connect these two vertices. Thus if we start with a connected graph we end up having to make it complete to satisfy the axioms of an objective partial group. One also may be drawn to considering directed graphs as a directed cycle has a cyclic automorphism group. However, although conjugation is directed, if we can conjugate from one group to another we must be able to conjugate back by conjugating by the element's inverse. Having exhausted these more familiar approaches we choose to start our construction with an objective partial group with automorphism group $S_n \wr C_2$ and then add structure so that an n-cycle is the only remaining symmetry. We will describe how this construction intuitively arises and then rigorously define it and show that it forms an objective partial group with the desired automorphism group.

If one starts with two sets of subgroups $\{\, \langle t_i \rangle \mid 1 \leq i \leq n \,\}$ and $\{\, \langle u_i \rangle \mid 1 \leq i \leq n \,\}$, each isomorphic to C_2, with a further set of elements $\{\, x_i \mid 1 \leq i \leq n \,\}$, where $x_i^2 = 1$, such that we have conjugation defined by $t_i^{x_i} = u_i$, for each i, then it is easy to see that this system has at least symmetry group $S_n \wr C_2$. We can permute indices in any way and the map given by $t_i \mapsto u_i$ and $x_i \mapsto x_i$ is an involution. In order to disallow any permutation of indices we include another set of groups $\{\, \langle v_i \rangle \mid 1 \leq i \leq n \,\}$, each isomorphic to C_2, and define conjugation $u_i^{x_{i+1}} = v_i$. Thus we can now no longer arbitrarily permute indices, and the only maps permuting indices that preserve the current structure are $i \mapsto i + m$ for all i, where m in $\mathbb{N}$ and indices are considered modulo n. It is not hard to see that this forms a n-cycle. However we do still have one unwanted symmetry that means the current system has symmetry group D_{2n}. This is the map $t_i \mapsto v_i$ and $x_i \mapsto x_{i+1}$. In order to exclude this we add a final set of subgroups $\{\, \langle w_i \rangle \mid 1 \leq i \leq n \,\}$, again all isomorphic to C_2, and a final element z such that we have conjugation defined by $v_i^z = w_i$, for each i. It is easy to see that we no longer have the involution from above and we will later show that this system only has automorphisms of the form $i \mapsto i + m$.

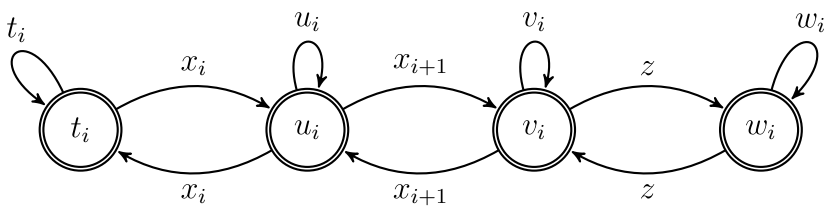

Figure 2.1: Words in D_Δ are defined as those admitted by n copies of this automaton. Each state is both a start and end state and is labelled by the element of Δ on which conjugation by a word starting at that state is defined.

Now that we have described the general idea and motivation for the construction we shall now more rigorously construct the objective partial group. First set $\langle t_i \rangle$, $\langle u_i \rangle$, $\langle v_i \rangle$ and $\langle w_i \rangle$, for $1 \leq i \leq n$, to each be isomorphic to C_2 and let $\Delta = \left\{ \langle t_i \rangle, \langle u_i \rangle, \langle v_i \rangle, \langle w_i \rangle \;\middle|\; 1 \leq i \leq n \right\}$. Define $\mathcal{L}_n$ to be the union of the sets

$$\begin{aligned} A &= \left\{ t_i,\, u_i,\, v_i,\, w_i,\, x_i,\, z \;\middle|\; 1 \leq i \leq n \right\}, \\ B &= \big\{ t_i x_i,\, u_i x_i,\, u_i x_{i+1},\, v_i x_{i+1},\, v_i z,\, w_i z,\, x_i x_{i+1},\, x_{i+1} x_i,\, t_i x_i x_{i+1},\, v_i x_{i+1} x_i,\, x_{i+1} z, \\ &\qquad z x_{i+1},\, u_i x_{i+1} z,\, w_i z x_{i+1},\, x_i x_{i+1} z,\, z x_{i+1} x_i,\, t_i x_i x_{i+1} z,\, w_i z x_{i+1} x_i \;\big|\; 1 \leq i \leq n \big\}, \end{aligned}$$

with the set $\{1\}$. Next we define the set D_Δ and the product Π as follows.

Let D_Δ be the set of all words admitted by n copies of the automaton in Figure 2.1, one for each $1 \leq i \leq n$. We define our product, Π, on this set but note that $\mathcal{L}_n \circ \mathcal{L}_n$, the set of all concatenated words consisting of two elements of $\mathcal{L}_n$ that are admitted by an automaton, is a subset of D_Δ. We begin by moving along a word w in D_Δ from left to right deleting any occurrence where a letter appears next to itself. If a pair is deleted the next two letters that now become adjacent are checked to see if they are the same before checking subsequent letters to the right. Once the end of the word is reached we have a word w' containing no adjacent copies of the same letter. We shall call w' *partially reduced*. One then checks w' again starting from the left. This time pairs of letters in the ith and $(i+1)$th position are checked and certain pairs are replaced if they occur. The pair $x_i t_i$ is replaced with $u_i x_i$, $x_i u_i$ with $t_i x_i$, $x_{i+1} u_i$ with $v_i x_{i+1}$, $x_{i+1} v_i$ with $u_i x_{i+1}$, $z v_i$ with $w_i z$ and $z w_i$ with $v_i z$. The letter in the $(i-1)$th position is then checked in case it is now the same as the ith letter, in which case both are deleted and the new pair of adjacent letters are also checked until no more can be deleted, we say c pairs were deleted. If $c < 2$ then the letter in what was the $(i+1)$th position, now the $(i+1-2c)$th position, is checked with the $(i+2-2c)$th letter and adjacent pairs are deleted if need be until no more can be deleted. One then checks back along the word from left to right seeing if any of the swaps can be applied at each letter and repeating the steps that follow. Once the final letter at the right-hand end of the word has been reached without making any further swaps the word is *completely reduced*.

We first must show that this algorithm to reduce words still gives us words admitted by one of the automaton.

Lemma 2.5.1. *The above algorithm for reducing words gives a well-defined map*

$$\Pi : D_\Delta \longrightarrow D_\Delta.$$

Furthermore, for w a word in D_Δ, w and $\Pi(w)$ both have the same set of possible start states and the same set of possible end states.

Proof. By the definition of the algorithm it is clear there is only one way to reduce a word, so Π is a well-defined map. Now note that any subwords that are just two of the same letter must start and end at the same state, so deleting these pairs means the resulting word is still admitted by an automaton. Moreover, if we delete any pairs at the start or end, the start or end states remain the same. Similarly, any one of the subwords that is substituted by another subword in the algorithm has the same set of possible start states as the word it is replaced with and the same set of possible end states. Thus by making one of these substitutions in a word we still obtain a word admitted by an automaton, and if we make a substitution at the start or end of a word the start state and end state remain the same. As each step of the reduction results in a word still in D_Δ with the same set of start states and end states as before, any reduction we have that $\Pi(w)$ is in D_Δ for all w in D_Δ and both words have the same set of start states and the same set of end states. $\square$

The set $\mathcal{L}_n \circ \mathcal{L}_n$ can be thought of as consisting of all words in D_Δ of the form $\alpha \circ \beta$ where α and β are in $\mathcal{L}_n$. We now show that the image of this set under Π is a subset of $\mathcal{L}_n$.

Lemma 2.5.2. *For any α and β in $\mathcal{L}_n$ where $\alpha \circ \beta$ is in D_Δ we have that $\Pi(\alpha \circ \beta)$ is in $\mathcal{L}_n$.*

Proof. First notice for a given $1 \leq i \leq n$ we do not have any concatenation defined between letters with this index and letters with another index with the exception of x_{i+1}, which can be concatenated with any letter with index i. We can therefore fix an i and just consider words containing letters with this index and x_{i+1} in $\mathcal{L}_n$. From here it is simply a case of concatenating all pairs of words in this subset of $\mathcal{L}_n$ which can be multiplied together based on the automaton in Figure 2.1 and then reducing these words using the algorithm given. This was done using the Magma code in Appendix A.1. $\square$

The map that we have defined using this algorithm has a further useful property on the subset $\mathcal{L}_n \circ \mathcal{L}_n \circ \mathcal{L}_n$ of D_Δ that we will require later.

Lemma 2.5.3. *The product Π is associative on $\mathcal{L}_n$; in other words for all α, β and γ in $\mathcal{L}_n$ we have $\Pi(\Pi(\alpha \circ \beta) \circ \gamma) = \Pi(\alpha \circ \Pi(\beta \circ \gamma))$.*

Proof. Again we can consider each index separately so first fix some $1 \leq i \leq n$. By Lemma 2.5.3 $\Pi(\alpha \circ \beta) \circ \gamma$ is in D_Δ if and only if $\alpha \circ \Pi(\beta \circ \gamma)$ is, and by Lemma 2.5.2 we have that both $\Pi(\Pi(\alpha \circ \beta) \circ \gamma)$ and $\Pi(\alpha \circ \Pi(\beta \circ \gamma))$ are in $\mathcal{L}_n$. Now it is simply a case of checking all triples of words in $\mathcal{L}_n$ that have the index i and x_{i+1} in them and can be multiplied together. For each of these triples we apply the multiplication in both possible ways and then checking they simplify to the same element of $\mathcal{L}_n$. This was done using the Magma code in Appendix A.1. $\square$

We can restate Π as performing a set of conjugation relations: $t_i^{x_i} = u_i$, $u_i^{x_{i+1}} = v_i$ and $v_i^z = w_i$, for each i. Finally we have $x_i^2 = z^2 = 1$, for $1 \leq i \leq n$. There is no multiplication defined between any subgroups in Δ. One can easily verify that each one of these relations is equivalent to exactly one of the switches in the definition of Π and that the deletion of pairs is equivalent to all words of length 1 having order 2. This way of describing Π is much more intuitive; however, the algorithm we have gives us a well-defined order in which to apply relations to reduce word, which these relations do not necessarily do.

Using the conjugation relations we can now recharacterise D_Δ as the set of all words that result from conjugating between groups in Δ. One should be able to see this from the automaton in Figure 2.1. The list of states visited by tracing out a word corresponds to the series of subgroups in D_Δ that defines the multiplication. We now show that any of these words can be reduced to a word in $\mathcal{L}_n$.

Lemma 2.5.4. *The image of D_Δ under the map Π is a subset of $\mathcal{L}_n$.*

Proof. Note that any word w in the set D_Δ can be thought of as the concatenation of words in $\mathcal{L}_n$ as certainly it is true that it is a concatenation of words of length 1 in $\mathcal{L}_n$. If we first move from left to right deleting any pairs of the same letter in the w then we now have a partially reduced word w' which is still the product of words of length 1 in $\mathcal{L}_n$. Write $w' = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_m$, for m the length of w' and where each α_i is a word in $\mathcal{L}_n$ of length 1. The first step of the algorithm is to check if the first two letters can be swapped, this is the same as $\Pi(\alpha_1 \circ \alpha_2)$. If they can be swapped we then delete the second letter and third letters if they are the same and then the first and now-second letter if they are the same. Note that doing just one or both of these steps will mean we have to check if we can switch the first two letters again. Thus we can either keep swapping the first two letters and deleting pairs until we have a word of length 1 – which we know must be in $\mathcal{L}_n$ – or we arrive at a pair that cannot be swapped or there is no pair deleted. So without loss of generality we assume that the second and third letter are not the same and we cannot swap α_1 and α_2. Now we have a word of length 2 at the start of w' in $\mathcal{L}_n$, by Lemma 2.5.2. Suppose now we have a word of length i at the start of w' in $\mathcal{L}_n$ and the next letter to the right is α_{i+1}. Checking from left to right along w' for switches, if we reach α_{i+1} without making any switches then we already have a reduced word and w' starts with a word of length $i+1$ in $\mathcal{L}_n$, by Lemma 2.5.2, and we are done. Therefore suppose we make a switch. This happens at the ith letter in w'. If no pairs can be deleted after this, by Lemma 2.5.2, we are again done. If the $(i-1)$th letter and the ith letter can be deleted then we obtain a word that has length less than i so by induction this will reduce to a word in $\mathcal{L}_n$. Thus, by induction, we have $\Pi(w) = \Pi(\Pi(\cdots\Pi(\Pi(\alpha_1 \circ \alpha_2) \circ \alpha_3)\cdots) \circ \alpha_m)$ and $\Pi(w)$ is in $\mathcal{L}_n$ for all w in D_Δ. $\square$

We also need to define an inversion on $\mathcal{L}_n$. For α in A or the identity set $\alpha^{-1} = \alpha$. For the remaining set we have ordered them in pairs of inverses, so the first two in each set are inverses of each other and so on. Note that α has the same set of start states as α^{-1} has end states, for all α in $\mathcal{L}_n$. It should also be easy to see that $\Pi(\alpha \circ \alpha^{-1}) = 1$ for all α in $\mathcal{L}_n$. This has also been verified by the Magma code by checking every element of $\mathcal{L}_n$ and its inverse simplify to the empty word. This naturally extends to the map

$$\cdot^{-1} : W(\mathcal{L}_n) \longrightarrow W(\mathcal{L}_n)\ ;\ (\alpha_1 \circ \cdots \circ \alpha_m)^{-1} \longmapsto \alpha_m^{-1} \circ \cdots \circ \alpha_1^{-1},$$

where each α_i is in $\mathcal{L}_n$ and m in $\mathbb{N}$ is the length of w in words in $\mathcal{L}_n$.

Proposition 2.5.5. *The pair $(\mathcal{L}_n, \Delta)$ is a finite objective partial group with product map Π and inversion $\cdot^{-1}$.*

Proof. First note we have $\mathcal{L}_n \subseteq D_\Delta \subseteq W(\mathcal{L}_n)$ and that for any word in D_Δ all of its subwords are in D_Δ as well. By Lemma 2.5.4, we have that $\Pi(D_\Delta)$ is contained in $\mathcal{L}_n$. The map Π also restricts to the identity on $\mathcal{L}_n$. This was verified using the Magma code to check that none of the elements of $\mathcal{L}_n$ can be simplified further. For any word v in D_Δ we have that v and $\Pi(v)$ both start at the same state and end at the same state, by Lemma 2.5.1. Thus, for u, v and w in D_Δ, if $u \circ v \circ w$ is in D_Δ then $u \circ \Pi(w) \circ v$ is in D_Δ as it is still accepted by an automaton. We can write each word u, v and w as a list of subwords in $\mathcal{L}_n$ of length 1, $u_1 \circ \cdots \circ u_k$, $v_1 \circ \cdots \circ v_l$ and $w_1 \circ \cdots \circ w_m$, for k, l, m in $\mathbb{N}$ the length of u, v and w respectively. As the product is associative on $\mathcal{L}_n$, by Lemma 2.5.3, we have that

$$\begin{aligned}\Pi(u \circ v \circ w) &= \Pi(\Pi(\cdots \Pi(u_1 \circ u_2) \cdots \circ u_k) \circ v_1) \cdots \circ v_l) \circ w_1) \cdots \circ w_m) \\ &= \Pi(u_1 \circ \cdots \circ u_k \circ \Pi(v_1 \circ \cdots \circ v_l) \circ w_1 \circ \cdots \circ w_m) \\ &= \Pi(u \circ \Pi(v) \circ w).\end{aligned}$$

Therefore $\Pi : D_\Delta \to \mathcal{L}_n$ is a product. Now consider w again, but this time we write $w = \alpha_1 \circ \cdots \circ \alpha_m$, where α_i is in $\mathcal{L}_n$, not necessarily length 1, and m is in $\mathbb{N}$. Thus we have that the inverse of w is $\alpha_m^{-1} \circ \cdots \circ \alpha_1^{-1}$. First note that, as the set of end states of α_m is the same as the set of start states for α_m^{-1}, we have that $w \circ w^{-1}$ is in D_Δ. Again using associativity of Π on $\mathcal{L}_n$, by Lemma 2.5.3, we have

$$\Pi(w \circ w^{-1}) = \Pi(\alpha_1 \circ \cdots \circ \alpha_m \circ \alpha_m^{-1} \circ \cdots \circ \alpha_1^{-1}),$$

which when we delete pairs of inverses, gives $\Pi(\emptyset) = 1$. Thus $(\mathcal{L}_n, D_\Delta, \Pi, \cdot^{-1})$ is a partial group. Furthermore $\mathcal{L}_n$ is finite and by construction $D = D_\Delta$ where Δ is a set of subgroups each isomorphic to C_2. Thus the second axiom of objectivity, Definition 2.2.11(O2), holds vacuously and $\mathcal{L}_n$ is a finite objective partial group. □

We therefore have that $(\mathcal{L}_n, \Delta)$ is a finite objective partial group and we will now prove that its automorphism group is C_n. First note that the sets making up $\mathcal{L}_n$ partition it by element order. The set A consists of all order 2 elements and B all elements with no defined order, in other words they cannot be multiplied with themselves as the multiplication is not defined. It is clear that any automorphism must restrict to a bijection from each of the two sets to itself.

We will now give two results which show what certain elements of $\mathcal{L}_n$ are mapped to under an automorphism.

Lemma 2.5.6. *Any automorphism of* $(\mathcal{L}_n, \Delta)$ *maps* z *to itself. Furthermore, for* $1 \leq i \leq$

n, it must map x_i to some x_j, where $1 \leq j \leq n$.

Proof. Suppose ψ is an automorphism of $\mathcal{L}_n$. From the above discussion we have that $\psi(z)$ is in A. We shall consider the number of elements in A that can be conjugated by z to show that $\psi(z)$ can only equal z for ψ to be an automorphism. It is clear that we require $\left|\left\{\ \alpha \in A \ \middle|\ \alpha^z \in D_\Delta\ \right\}\right| = \left|\left\{\ \alpha \in A \ \middle|\ \alpha^{\psi(z)} \in D_\Delta\ \right\}\right|$ for ψ to be an automorphism of $(\mathcal{L}_n, \Delta)$. There are $2n+1$ elements of A can be conjugated by z, namely v_i and w_i for each $1 \leq i \leq n$ and itself. If we now consider the other elements of A we see that for each i, only t_i, u_i, u_{i-1}, v_{i-1} and x_i can be conjugated by x_i and for each generator of a group in Δ we see that only one element of A can be conjugated by it, namely itself. Thus the only possible candidate for $\psi(z)$ is z. Furthermore $\psi(x_i)$ must be an element that can conjugate five elements of A and thus must be x_j for some $1 \leq j \leq n$. $\square$

Lemma 2.5.7. *For any $1 \leq i \leq n$, any automorphism of $(\mathcal{L}_n, \Delta)$ maps w_i to w_j, where $1 \leq j \leq n$.*

Proof. Suppose ψ is an automorphism of $\mathcal{L}_n$. From Lemma 2.5.6 we have that ψ must fix z. Consider, for some fixed i,

$$\psi(v_i) = \psi(zw_iz) = \psi(z)\psi(w_i)\psi(z) = z\psi(w_i)z,$$

so $\psi(w_i)$ must be either v_j or w_j, for arbitrary $1 \leq j \leq n$, as these are the only elements of A that can be conjugated by z, other than z itself. If $\psi(w_i) = v_j$ then $\psi(v_i) = w_j$, and

$$\psi(u_i) = \psi(x_{i+1}v_ix_{i+1}) = \psi(x_{i+1})\psi(w_i)\psi(x_{i+1}) = \psi(x_{i+1})w_j\psi(x_{i+1}).$$

From Lemma 2.5.6 we have that ψ can only map x_i to an x_k for some k. However, there is no multiplication defined between w_j and an x of any index. We therefore reach a contradiction and $\psi(w_i) = w_j$ for some j. $\square$

Now we have restricted what z, x_i and w_i can map to under an automorphism we can now prove that the automorphism group must be C_n.

Proposition 2.5.8. *Any automorphism of* $(\mathcal{L}_n, \Delta)$ *is of the form* $i \mapsto i + m$*, for all* i *and some* $1 \leq m \leq n$*, in other words indices are increased modulo* n*.*

Proof. Suppose ψ is a automorphism of $\mathcal{L}_n$. From Lemma 2.5.6 we have that ψ must fix z. Furthermore, from Lemma 2.5.7, we have that $\psi(w_i) = w_{i+m}$, where $1 \leq m \leq n$. We can combine these results to obtain

$$\psi(v_i) = \psi(zw_iz) = zw_{i+m}z = v_{i+m}.$$

Now consider

$$\psi(u_i) = \psi(x_{i+1}v_ix_{i+1}) = \psi(x_{i+1})v_{i+m}\psi(x_{i+1}).$$

By Lemma 2.5.7 we have that $\psi(x_{i+1}) = x_j$, for some $1 \leq j \leq n$. However the only x_j where $x_jw_{i+m}x_j$ is defined is x_{i+m+1}, so we have $\psi(x_{i+1}) = x_{i+m+1}$ and thus $\psi(u_i) = u_{i+m}$. Similarly we have

$$\psi(t_i) = \psi(x_iu_ix_i) = \psi(x_i)u_{i+m}\psi(x_i),$$

so $\psi(x_i)$ is either x_{i+m} or x_{i+m+1} however x_{i+m+1} is already mapped to so $\psi(x_i) = x_{i+m}$ and $\psi(t_i) = t_{i+m}$. We now consider

$$\psi(v_{i-1}) = \psi(x_iu_{i-1}x_i) = x_{i+m}\psi(u_{i-1})x_{i+m},$$

so $\psi(u_{i-1})$ can be either v_{i+m-1}, u_{i+m-1}, v_{i+m} or u_{i+m}. The latter two have already been mapped to so we instead suppose $\psi(u_{i-1}) = v_{i+m-1}$ and thus $\psi(v_{i-1}) = u_{i+m-1}$. But this means

$$\psi(w_{i-1}) = \psi(zv_iz) = zu_{i+m-1}z,$$

which is not defined, so we reach a contradiction. We therefore have $\psi(u_{i-1}) = u_{i+m-1}$, $\psi(v_{i-1}) = v_{i+m-1}$ and thus $\psi(w_{i-1}) = \psi(w_{i+m-1})$. Notice that $\psi(w_{i-1}) = w_{i+m-1}$ is $\psi(w_i) = w_{i+m}$ with 1 subtracted from each index. Thus repeating the same steps above will give us the same results for $i - 1$ as we have for i. The final step will give $\psi(w_{i-2}) =$

w_{i+m-2}. Thus repeating this process m times we have that for any index ψ increases that index by m. Thus there are no automorphisms of $(\mathcal{L}_n, \Delta)$ that are not of the form $i \mapsto i+m$, for all i and some $1 \leq m \leq n$. $\square$

2.5.2 A subsequent family of finite objective partial groups with cyclic automorphism group

We will now construct another family of finite objective partial groups with cyclic automorphism group and show that, for each n, our construction is not isomorphic to that in Section 2.5.1. The motivating idea for this construction was again to start with an objective partial group with automorphism group $S_n \wr C_2$ and then add structure so that an n-cycle is the only remaining symmetry. The construction is much the same as in Section 2.5.1, but here we also allow commuting between letters that are alphabetically adjacent. As one will see this results in a much larger set $\mathcal{L}_n$ and therefore showing this is an objective partial group is considerably more arduous.

First set $\langle t_i \rangle$, $\langle u_i \rangle$, $\langle v_i \rangle$ and $\langle w_i \rangle$, for $1 \leq i \leq n$, to each be isomorphic to C_2 and let $\Delta = \left\{ \langle t_i \rangle, \langle u_i \rangle, \langle v_i \rangle, \langle w_i \rangle \;\middle|\; 1 \leq i \leq n \right\}$. Next define $\mathcal{L}_n$ to be the union of the sets

$$\begin{aligned}
A = \big\{& t_i,\, u_i,\, v_i,\, w_i,\, x_i,\, z,\, t_iu_i,\, u_iv_i,\, v_iw_i,\, t_iu_ix_i,\, u_iv_ix_{i+1},\, v_iw_iz,\, t_iv_it_i,\, u_iw_iu_i,\, t_iu_iv_it_i,\\
& u_iv_iw_iu_i,\, x_iv_ix_i,\, u_ix_iv_it_ix_i,\, zu_iz,\, zu_iw_iu_iz,\, t_ix_iv_ix_i,\, t_iu_ix_iv_it_ix_i,\, w_izu_iz,\\
& w_izu_iw_iu_iz \;\big|\; 1 \leq i \leq n\big\},\\
B = \big\{& t_ix_i,\, u_ix_i,\, u_ix_{i+1},\, v_ix_{i+1},\, v_iz,\, w_iz \;\big|\; 1 \leq i \leq n\big\},\\
C = \big\{& t_iv_i,\, v_it_i,\, u_iw_i,\, w_iu_i,\, u_ix_iv_ix_i,\, x_iv_it_ix_i,\, zu_iw_iz,\, v_izu_iz \;\big|\; 1 \leq i \leq n\big\}\\
D = \big\{& t_iu_iv_i,\, u_iv_it_i,\, u_iv_iw_i,\, v_iw_iu_i,\, t_iu_ix_iv_ix_i,\, t_ix_iv_it_ix_i,\, w_izu_iw_iz,\, v_iw_izu_iz \;\big|\; 1 \leq i \leq n\big\},\\
E = \big\{& v_ix_i,\, x_iv_i,\, t_iv_ix_i,\, x_iv_it_i,\, u_ix_iv_i,\, v_it_ix_i,\, t_iv_it_ix_i,\, u_ix_iv_it_i,\, u_iv_ix_i,\, t_ix_iv_i,\, t_iu_iv_ix_i,\\
& t_iu_ix_iv_i,\, u_iv_it_ix_i,\, t_ix_iv_it_i,\, t_iu_iv_it_ix_i,\, t_iu_ix_iv_it_i,\\
& t_ix_{i+1},\, w_ix_{i+1},\, t_iv_ix_{i+1},\, u_iw_ix_{i+1},\, v_it_ix_{i+1},\, w_iu_ix_{i+1},\, t_iv_it_ix_{i+1},\, u_iw_iu_ix_{i+1},
\end{aligned}$$

$t_iu_ix_{i+1},\ v_iw_ix_{i+1},\ t_iu_iv_ix_{i+1},\ u_iv_iw_ix_{i+1},\ u_iv_it_ix_{i+1},\ v_iw_iu_ix_{i+1},\ t_iu_iv_it_ix_{i+1},$

$u_iv_iw_iu_ix_{i+1},\ u_iz,\ zu_i,\ u_iw_iz,\ zu_iw_i,\ w_iu_iz,\ v_izu_i,\ u_iw_iu_iz,\ zu_iw_iu_i,\ u_iv_iz,\ w_izu_i,$

$u_iv_iw_iz,\ w_izu_iw_i,\ v_iw_iu_iz,\ v_iw_izu_i,\ u_iv_iw_iu_iz,\ w_izu_iw_iu_i,$

$x_ix_{i+1},\ x_{i+1}x_i,\ t_ix_ix_{i+1},\ v_ix_{i+1}x_i,\ u_ix_ix_{i+1},\ w_ix_{i+1}x_i,\ t_iu_ix_ix_{i+1},\ v_iw_ix_{i+1}x_i,$

$x_iv_ix_{i+1},\ u_ix_{i+1}x_i,\ t_ix_iv_ix_{i+1},\ u_iv_ix_{i+1}x_i,\ u_ix_iv_ix_{i+1},\ u_iw_ix_{i+1}x_i,\ x_iv_it_ix_{i+1},$

$w_iu_ix_{i+1}x_i,\ t_iu_ix_iv_ix_{i+1},\ u_iv_iw_ix_{i+1}x_i,\ t_ix_iv_it_ix_{i+1},\ v_iw_iu_ix_{i+1}x_i,\ u_ix_iv_it_ix_{i+1},$

$u_iw_iu_ix_{i+1}x_i,\ t_iu_ix_iv_it_ix_{i+1},\ u_iv_iw_iu_ix_{i+1}x_i,$

$x_{i+1}z,\ zx_{i+1},\ t_ix_{i+1}z,\ v_izx_{i+1},\ u_ix_{i+1}z,\ w_izx_{i+1},\ v_ix_{i+1}z,\ zu_ix_{i+1},\ t_iu_ix_{i+1}z,$

$v_iw_izx_{i+1},\ u_iv_ix_{i+1}z,\ w_izu_ix_{i+1},\ t_iv_ix_{i+1}z,\ zu_iw_ix_{i+1},\ v_it_ix_{i+1}z,\ v_izu_ix_{i+1},$

$t_iu_iv_ix_{i+1}z,\ w_izu_iw_ix_{i+1},\ u_iv_it_ix_{i+1}z,\ v_iw_izu_ix_{i+1},\ t_iv_it_ix_{i+1}z,\ zu_iw_iu_ix_{i+1},$

$t_iu_iv_it_ix_{i+1}z,\ w_izu_iw_iu_ix_{i+1},$

$x_ix_{i+1}z,\ zx_{i+1}x_i,\ t_ix_ix_{i+1}z,\ w_izx_{i+1}x_i,\ u_ix_ix_{i+1}z,\ v_izx_{i+1}x_i,\ t_iu_ix_ix_{i+1}z,$

$v_iw_izx_{i+1}x_i,\ x_iv_ix_{i+1}z,\ zu_ix_{i+1}x_i,\ t_ix_iv_ix_{i+1}z,\ w_izu_ix_{i+1}x_i,\ u_ix_iv_ix_{i+1}z,$

$zu_iw_ix_{i+1}x_i,\ x_iv_it_ix_{i+1}z,\ v_izu_ix_{i+1}x_i,\ t_iu_ix_iv_ix_{i+1}z,\ w_izu_iw_ix_{i+1}x_i,$

$t_ix_iv_it_ix_{i+1}z,\ v_iw_izu_ix_{i+1}x_i,\ u_ix_iv_it_ix_{i+1}z,\ zu_iw_iu_ix_{i+1}x_i,\ t_iu_ix_iv_it_ix_{i+1}z,$

$w_izu_iw_iu_ix_{i+1}x_i \bigm| 1 \le i \le n\big\},$

with the set $\{1\}$, when $n \neq 3$ and

$$\begin{aligned}
A = \big\{&t_i,\ u_i,\ v_i,\ w_i,\ x_i,\ z,\ t_iu_i,\ u_iv_i,\ v_iw_i,\ t_iu_ix_i,\ u_iv_ix_{i+1},\ v_iw_iz,\ t_iv_it_i,\ t_iv_it_iv_it_i,\ v_it_iv_i,\\
&u_iw_iu_i,\ u_iw_iu_iw_iu_i,\ w_iu_iw_i,\ t_iu_iv_it_i,\ t_iu_iv_it_iv_it_i,\ u_it_iv_iv_it_i,\ u_iv_iw_iu_i,\ u_iv_iw_iu_iw_iu_i,\\
&v_iw_iu_iw_i,\ x_iv_ix_i,\ u_ix_iv_it_ix_i,\ u_ix_iv_it_iv_it_ix_i,\ x_iv_it_iv_ix_i,\ zu_iz,\ zu_iw_iu_iz,\ zu_iw_iu_iw_iu_iz,\\
&zw_iu_iw_iz,\ t_ix_iv_ix_i,\ t_iu_ix_iv_it_ix_i,\ t_iu_ix_iv_it_iv_it_ix_i,\ t_ix_iv_it_iv_ix_i,\ w_izu_iz,\\
&w_izu_iw_iu_iz,\ w_izu_iw_iu_iw_iu_iz,\ v_iw_izu_iw_iz \bigm| 1 \le i \le n\big\},\\
B = \big\{&t_ix_i,\ u_ix_i,\ u_ix_{i+1},\ v_ix_{i+1},\ v_iz,\ w_iz \bigm| 1 \le i \le n\big\},
\end{aligned}$$

$$C = \{t_iv_i,\ v_it_i,\ t_iv_it_iv_i,\ v_it_iv_it_i,\ u_iw_i,\ w_iu_i,\ u_iw_iu_iw_i,\ w_iu_iw_iu_i,\ u_ix_iv_ix_i,\ x_iv_it_ix_i,$$

$$u_ix_iv_it_iv_ix_i,\ x_iv_it_iv_it_ix_i,\ zu_iw_iz,\ v_izu_iz\, zu_iw_iu_iw_iz,\ v_izu_iw_iu_iz,\ \big|\ 1 \le i \le n\}$$

$$D = \{t_iu_iv_i,\ u_iv_it_i,\ t_iu_iv_it_iv_i,\ u_iv_it_iv_it_i,\ u_iv_iw_i, v_iw_iu_i,\ u_iv_iw_iu_iw_i,\ v_iw_iu_iw_iu_i,\ t_iu_ix_iv_ix_i,$$

$$t_ix_iv_it_ix_i,\ t_iu_ix_iv_it_iv_ix_i,\ t_ix_iv_it_iv_it_ix_i,\ w_izu_iw_iz,\ v_iw_izu_iz$$

$$w_izu_iw_iu_iw_iz,\ v_iw_izu_iw_iu_iz,\ \big|\ 1 \le i \le n\},$$

$$E = \{v_ix_i,\ x_iv_i,\ t_iv_ix_i,\ x_iv_it_i,\ t_iv_it_iv_ix_i,\ x_iv_it_iv_it_i,\ v_it_iv_it_ix_i,\ u_ix_iv_it_iv_i,\ v_it_ix_i,\ u_ix_iv_i,$$

$$t_iv_it_ix_i,\ u_ix_iv_it_i,\ t_iv_it_iv_it_ix_i,\ u_ix_iv_it_iv_it_i,\ v_it_iv_ix_i,\ x_iv_it_iv_i,\ u_iv_ix_i,\ t_ix_iv_i,$$

$$t_iu_iv_ix_i,\ t_ix_iv_it_i,\ t_iu_iv_it_iv_ix_i,\ t_ix_iv_it_iv_i,\ u_iv_it_iv_it_ix_i,\ t_iu_ix_iv_it_iv_i,\ u_iv_it_ix_i,\ t_iu_ix_iv_i,$$

$$t_iu_iv_it_ix_i,\ t_iu_ix_iv_it_i,\ t_iu_iv_it_iv_it_ix_i,\ t_iu_ix_iv_it_iv_it_i,\ u_iv_it_iv_ix_i,\ t_ix_iv_it_iv_i,$$

$$t_ix_{i+1},\ w_ix_{i+1},\ t_iv_ix_{i+1},\ u_iw_ix_{i+1},\ t_iv_it_iv_ix_{i+1},\ u_iw_iu_iw_ix_{i+1},\ v_it_iv_it_ix_{i+1},$$

$$w_iu_iw_iu_ix_{i+1},\ v_it_ix_{i+1},\ w_iu_ix_{i+1},\ t_iv_it_ix_{i+1},\ w_iu_iw_ix_{i+1},\ t_iv_it_iv_it_ix_{i+1},$$

$$u_iw_iu_iw_iu_ix_{i+1},\ v_it_iv_ix_{i+1},\ u_iw_iu_ix_{i+1},\ t_iu_ix_{i+1},\ v_iw_ix_{i+1},\ t_iu_iv_ix_{i+1},\ u_iv_iw_ix_{i+1},$$

$$t_iu_iv_it_iv_ix_{i+1},\ u_iv_iw_iu_iw_ix_{i+1},\ u_iv_it_iv_it_ix_{i+1},\ v_iw_iu_iw_iu_ix_{i+1},\ u_iv_it_ix_{i+1},\ v_iw_iu_ix_{i+1},$$

$$t_iu_iv_it_ix_{i+1},\ v_iw_iu_iw_ix_{i+1},\ t_iu_iv_it_iv_it_ix_{i+1},\ u_iv_iw_iu_iw_iu_ix_{i+1},\ u_iv_it_iv_ix_{i+1},$$

$$u_iv_iw_iu_ix_{i+1},$$

$$u_iz,\ zu_i,\ u_iw_iz,\ v_izu_i,\ u_iw_iu_iw_iz,\ v_izu_iw_iu_i,\ w_iu_iw_iu_iz,\ zu_iw_iu_iw_i,\ w_iu_iz,\ zu_iw_i,$$

$$u_iw_iu_iz,\ zu_iw_iu_i,\ u_iw_iu_iw_iu_iz,\ zu_iw_iu_iw_iu_i,\ w_iu_iw_iz,\ v_izu_iw_i,\ u_iv_iz,\ w_izu_i,$$

$$u_iv_iw_iz,\ v_iw_izu_i,\ u_iv_iw_iu_iw_iz,\ v_iw_izu_iw_iu_i,\ v_iw_iu_iw_iu_iz,\ w_izu_iw_iu_iw_i,\ v_iw_iu_iz,$$

$$w_izu_iw_i,\ u_iv_iw_iu_iz,\ w_izu_iw_iu_i,\ u_iv_iw_iu_iw_iu_iz,\ w_izu_iw_iu_iw_iu_i,\ v_iw_iu_iw_iz,$$

$$v_iw_izu_iw_i,$$

$$x_ix_{i+1},\ x_{i+1}x_i,\ t_ix_ix_{i+1},\ v_ix_{i+1}x_i,\ u_ix_ix_{i+1},\ w_ix_{i+1}x_i,\ t_iu_ix_ix_{i+1},\ v_iw_ix_{i+1}x_i,$$

$$x_iv_ix_{i+1},\ u_ix_{i+1}x_i,\ t_ix_iv_ix_{i+1},\ u_iv_ix_{i+1}x_i,\ u_ix_iv_ix_{i+1},\ u_iw_ix_{i+1}x_i,\ u_ix_iv_it_iv_ix_{i+1},$$

$$u_iw_iu_iw_ix_{i+1}x_i,\ x_iv_it_iv_it_ix_{i+1},\ w_iu_iw_iu_ix_{i+1}x_i,\ x_iv_it_ix_{i+1},\ w_iu_ix_{i+1}x_i,\ t_iu_ix_iv_ix_{i+1},$$

$$u_iv_iw_ix_{i+1}x_i,\ t_iu_ix_iv_it_iv_ix_{i+1},\ u_iv_iw_iu_iw_ix_{i+1}x_i,\ t_ix_iv_it_iv_it_ix_{i+1},\ v_iw_iu_iw_iu_ix_{i+1}x_i,$$

$$t_ix_iv_it_ix_{i+1},\ v_iw_iu_ix_{i+1}x_i,\ u_ix_iv_it_ix_{i+1},\ w_iu_iw_ix_{i+1}x_i,\ u_ix_iv_it_iv_it_ix_{i+1},$$

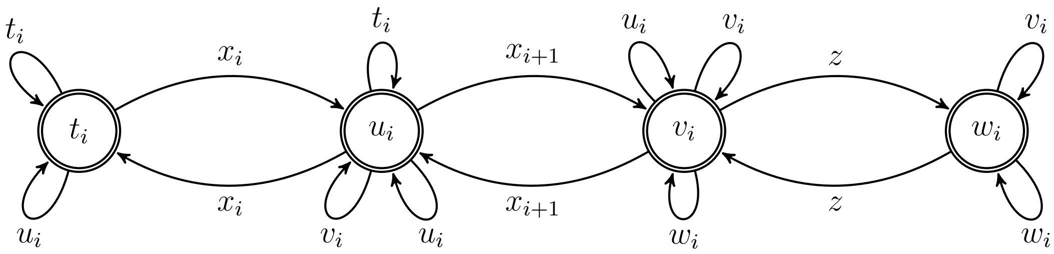

Figure 2.2: Words in D_Δ are defined as those admitted by n copies of this automaton. Each state is both a start and end state and is labelled by the element of Δ on which conjugation by a word starting at that state is defined.

$u_iw_iu_iw_iu_ix_{i+1}x_i,\ x_iv_it_iv_ix_{i+1},\ u_iw_iu_ix_{i+1}x_i,\ t_iu_ix_iv_it_ix_{i+1},\ v_iw_iu_iw_ix_{i+1}x_i,$

$t_iu_ix_iv_it_iv_it_ix_{i+1},\ u_iv_iw_iu_iw_iu_ix_{i+1}x_i,\ t_ix_iv_it_iv_ix_{i+1}, u_iv_iw_iu_ix_{i+1}x_i,$

$x_{i+1}z,\ zx_{i+1},\ t_ix_{i+1}z,\ v_izx_{i+1},\ u_ix_{i+1}z,\ w_izx_{i+1},\ v_ix_{i+1}z,\ zu_ix_{i+1},\ t_iu_ix_{i+1}z,$

$v_iw_izx_{i+1},\ u_iv_ix_{i+1}z,\ w_izu_ix_{i+1},\ t_iv_ix_{i+1}z,\ zu_iw_ix_{i+1},\ t_iv_it_iv_ix_{i+1}z,\ zu_iw_iu_iw_ix_{i+1},$

$v_it_iv_it_ix_{i+1}z,\ v_izu_iw_iu_ix_{i+1},\ v_it_ix_{i+1}z,\ zw_iu_ix_{i+1},\ t_iu_iv_ix_{i+1}z,\ w_izu_iw_ix_{i+1},$

$t_iu_iv_it_iv_ix_{i+1}z,\ w_izu_iw_iu_iw_ix_{i+1},\ u_iv_it_iv_it_ix_{i+1}z,\ v_iw_izu_iw_iu_ix_{i+1},$

$u_iv_it_ix_{i+1}z,\ v_iw_izu_ix_{i+1},\ t_iv_it_ix_{i+1}z,\ v_izu_iw_ix_{i+1},\ t_iv_it_iv_it_ix_{i+1}z,\ zu_iw_iu_iw_iu_ix_{i+1},$

$v_it_iv_ix_{i+1}z,\ zu_iw_iu_ix_{i+1},\ t_iu_iv_it_ix_{i+1}z,\ v_iw_izu_iw_ix_{i+1},\ t_iu_iv_it_iv_it_ix_{i+1}z,$

$w_izu_iw_iu_iw_iu_ix_{i+1},\ u_iv_it_iv_ix_{i+1}z,\ w_izu_iw_iu_ix_{i+1},$

$x_ix_{i+1}z,\ zx_{i+1}x_i,\ t_ix_ix_{i+1}z,\ w_izx_{i+1}x_i,\ u_ix_ix_{i+1}z,\ v_izx_{i+1}x_i,\ t_iu_ix_ix_{i+1}z,$

$v_iw_izx_{i+1}x_i,\ x_iv_ix_{i+1}z,\ zu_ix_{i+1}x_i,\ t_ix_iv_ix_{i+1}z,\ w_izu_ix_{i+1}x_i,\ u_ix_iv_ix_{i+1}z,$

$zu_iw_ix_{i+1}x_i,\ u_ix_iv_it_iv_ix_{i+1}z,\ zu_iw_iu_iw_ix_{i+1}x_i,\ x_iv_it_iv_it_ix_{i+1}z,\ v_izu_iw_iu_ix_{i+1}x_i,$

$x_iv_it_ix_{i+1}z,\ v_izu_ix_{i+1}x_i,\ t_iu_ix_iv_ix_{i+1}z,\ w_izu_iw_ix_{i+1}x_i,\ t_iu_ix_iv_it_iv_ix_{i+1}z,$

$w_izu_iw_iu_iw_ix_{i+1}x_i,\ t_ix_iv_it_iv_it_ix_{i+1}z,\ v_iw_izu_iw_iu_ix_{i+1}x_i,\ t_ix_iv_it_ix_{i+1}z,$

$v_iw_izw_ix_{i+1}x_i,\ u_ix_iv_it_ix_{i+1}z,\ v_izu_iw_ix_{i+1}x_i,\ u_ix_iv_it_iv_it_ix_{i+1}z,\ zu_iw_iu_iw_iu_ix_{i+1}x_i,$

$x_iv_it_iv_ix_{i+1}z,\ zu_iw_iu_ix_{i+1}x_i,\ t_iu_ix_iv_it_ix_{i+1}z,\ v_iw_izu_iw_ix_{i+1}x_i,\ t_iu_ix_iv_it_iv_it_ix_{i+1}z,$

$w_izu_iw_iu_iw_iu_ix_{i+1}x_i,\ t_ix_iv_it_iv_ix_{i+1}z,\ w_izu_iw_iu_ix_{i+1}x_i \;\big|\; 1 \leq i \leq n\big\},$

with the set $\{1\}$, when $n \neq 5$. Next we define the set D_Δ and the product Π as follows.

Let D_Δ be the set of all words admitted by n copies of the automaton in Figure 2.2, one for each $1 \leq i \leq n$. We then define our product, Π. We begin by moving along a word w in D_Δ from left to right deleting any occurrence where a letter appears next to itself. If a pair is deleted the next two letters that now become adjacent are checked to see if they are the same before checking subsequent letters to the right. Once the end of the word is reached we have a word w' containing no adjacent copies of the same letter. We shall call w' *partially reduced*. One then checks w' again starting from the left. This time pairs of letters in the ith and $(i+1)$th position are checked and certain pairs are replaced if they occur. The pair x_it_i is replaced with u_ix_i, x_iu_i with t_ix_i, $x_{i+1}t_i$ with w_ix_{i+1}, $x_{i+1}u_i$ with v_ix_{i+1}, $x_{i+1}v_i$ with u_ix_{i+1}, $x_{i+1}w_i$ with t_ix_{i+1}, zv_i with w_iz, zw_i with v_iz, u_it_i with t_iu_i, v_iu_i with u_iv_i and w_iv_i with v_iw_i. The letter in the $(i-1)$th position is then checked in case it is now the same as the ith letter, in which case both are deleted and the new pair of adjacent letters are also checked until no more can be deleted, we say c pairs were deleted. If $c < 2$ then the letter in what was the $(i+1)$th position, now the $(i+1-2c)$th position, is checked with the $(i+2-2c)$th letter and adjacent pairs are deleted if need be until no more can be deleted. If the ith and $(i+1)$th letters cannot be replaced then the triple of letters starting at the ith is checked. The triple zu_iv_i is replaced with w_izu_i, $x_iv_iu_i$ with $t_ix_iv_i$, $v_it_iu_i$ with $u_iv_it_i$, $w_iu_iv_i$ with $v_iw_iu_i$ and in the case when $q = 3$, $v_it_iv_i$ with $t_iv_it_i$ and $w_iu_iw_i$ with $u_iw_iu_i$. The same process is then applied as after a pair is switched, the $(i-1)$th and ith are checked and if they are the same they are deleted and the new adjacent pair are checked until no more pairs are deleted. If $c < 3$ then the letters that were in the $(i+2)$th and $(i+3)$th position before any deleting, now the $(i+2-2c)$th and $(i+3-2c)$th position, are checked and pairs deleted until no more can be removed. Lastly, only in the $q = 5$ case, if no swaps have been made the letters in the ith to $(i+4)$th position are checked and $w_iu_iw_iu_iw_i$ is replaced with $u_iw_iu_iw_iu_i$ and $v_it_iv_it_iv_i$ with $t_iv_it_iv_it_i$. Again the next step is to check the $(i-1)$th and ith letter and delete adjacent pairs until none can be deleted, then, if $c < 5$ check the $(i+5-2c)$th and $(i+6-2c)$th letters and delete any pairs necessary.

From here one moves left along the word checking for first two-letter swaps, then three letter swaps then, for $q = 5$, five letter swaps at each letter to the left of the $(i - c)$th. One then checks back along the word from left to right seeing if any of the swaps can be applied at each letter and repeating the steps that follow. Once the final letter at the right-hand end of the word has been reached without making any further swaps the word is *completely reduced.*

We first must show that this algorithm to reduce words still gives us words admitted by one of the automaton.

Lemma 2.5.9. *The above algorithm for reducing words gives a well-defined map*

$$\Pi : D_\Delta \longrightarrow D_\Delta.$$

Furthermore, for w a word in D_Δ, w and $\Pi(w)$ both have the same set of possible start states and the same set of possible end states.

Proof. By the definition of the algorithm it is clear there is only one way to reduce a word, so Π is a well-defined map. Now note that any subword that is just two of the same letter must start and end at the same state, so deleting these pairs means the resulting word is still admitted by an automaton. Moreover, if we delete any pairs at the start or end, the start or end states remain the same. Similarly any one of the subwords that is substituted by another subword in the algorithm has the same set of possible start states as the word it is replaced by and the same set of possible end states. Thus by making one of these substitutions in a word we still obtain a word admitted by an automaton and if we make a substitution at the start or end of a word the start state and end state remain the same. As each step of the reduction results in a words still in D_Δ with the same set of start states and end states as before any reduction we have that $\Pi(w)$ is in D_Δ for all w in D_Δ and both words have the same set of start states and the same set of end states. $\square$

The set $\mathcal{L}_n \circ \mathcal{L}_n$ can be thought of as consisting of all words in D_Δ of the form $\alpha \circ \beta$ where α and β are in $\mathcal{L}_n$. We now show that the image of this set under Π is a subset of

$\mathcal{L}_n$.

Lemma 2.5.10. *For any α and β in $\mathcal{L}_n$ where $\alpha \circ \beta$ is in D_Δ we have that $\Pi(\alpha \circ \beta)$ is in $\mathcal{L}_n$.*

Proof. First notice for a given $1 \leq i \leq n$ we do not have any concatenation defined between letters with this index and ones with another index with the exception of x_{i+1}, which can be concatenated with any letter with index i. We can therefore fix an i and consider just words containing letters with this index and x_{i+1} in $\mathcal{L}_n$. From here it is simply a case of concatenating all pairs of words in this subset of $\mathcal{L}_n$ which can be multiplied together based on the automaton in Figure 2.2 and then reducing these words using the algorithm given. This was done using the Magma code in Appendix A.2. □

The map that we have defined using this algorithm has a further useful property on the subset $\mathcal{L}_n \circ \mathcal{L}_n \circ \mathcal{L}_n$ of D_Δ.

Lemma 2.5.11. *The product Π is associative on $\mathcal{L}_n$; in other words for all α, β and γ in $\mathcal{L}_n$ we have $\Pi(\Pi(\alpha \circ \beta) \circ \gamma) = \Pi(\alpha \circ \Pi(\beta \circ \gamma))$.*

Proof. Again we can consider each index separately, so first fix some $1 \leq i \leq n$. By Lemma 2.5.11, $\Pi(\alpha \circ \beta) \circ \gamma$ is in D_Δ if and only if $\alpha \circ \Pi(\beta \circ \gamma)$ is, and by Lemma 2.5.10 we have that both $\Pi(\Pi(\alpha \circ \beta) \circ \gamma)$ and $\Pi(\alpha \circ \Pi(\beta \circ \gamma))$ are in $\mathcal{L}_n$. Now it is simply a case of checking all triples of words in $\mathcal{L}_n$ that have the index i and x_{i+1} in them and can be multiplied together. For each of these triples we apply the multiplication in both possible ways and then checking they simplify to the same element of $\mathcal{L}_n$. This was done using the Magma code in Appendix A.2. □

We can restate Π as performing a set of conjugation relations: $t_i^{u_i} = t_i$, $t_i^{v_i} = v_i t_i v_i$, $t_i^{x_i} = u_i$, $t_i^{x_{i+1}} = w_i$, $u_i^{w_i} = w_i u_i w_i$, $u_i^{x_{i+1}} = v_i$, $u_i^z = z u_i z$, $v_i^{w_i} = v_i$, $v_i^{x_i} = x_i v_i x_i$ and $v_i^z = w_i$, for each i. Finally we have $x_i^2 = z^2 = (t_i v_i)^q = (u_i w_i)^q = 1$, for $1 \leq i \leq n$ and q in $\{3, 5\}$ with $q \neq n$. There is no multiplication between t_i and w_i or any subgroups in Δ with a different index. We do however have that alphabetically adjacent pairs of the same index in Δ commute, which is where this construction differs from the one in Section 2.5.1. One can easily verify that each one of these relations is equivalent to exactly one of the switches in the definition of Π and that the deletion of pairs is equivalent to all words of length 1 having order 2. Again this way of describing Π is much more intuitive; however the algorithm we have gives us a well-defined order in which to apply relations to reduce words, which these relations do not necessarily do.

Using the conjugation relations we can now recharacterise D_Δ as the set of all words that result from conjugating between groups in Δ. One should be able to see this from the automaton in Figure 2.2. The list of states visited by tracing out a word corresponds to the series of subgroups in D_Δ that defines the multiplication. We now show that any of these words can be reduced to a word in $\mathcal{L}_n$.

Lemma 2.5.12. *The image of D_Δ under the map Π is a subset of $\mathcal{L}_n$.*

Proof. Note that any word, w in the set D_Δ can be thought of as the concatenation of words in $\mathcal{L}_n$, certainly it is true that it is a concatenation of words of length 1 in $\mathcal{L}_n$. If we first move from left to right deleting any pairs of the same letter in the w then we now have a partially reduced word w' which is still the product of words of length 1 in $\mathcal{L}_n$. If we write $w' = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_m$, for m the length of w' and where each α_i is a word in $\mathcal{L}_n$ of length 1. The first step of the algorithm is to check if the first two letters can be swapped, this is the same as $\Pi(\alpha_1 \circ \alpha_2)$. If they can be swapped we then delete the second letter and third letter if they are the same and then the first and now second letter if they are the same. Note that doing just one or both of these steps will mean we have to check if we can switch the first two letters again. Thus we can either keep swapping the first two letters and deleting pairs until we have a word of length 1, which we know must be in $\mathcal{L}_n$, or we reach a pair that cannot be swapped or there is no pair deleted. So without loss of generality we assume that either the second and third letter are not the same or we cannot swap α_1 and α_2. Now we have a word of length 2 at the start of w' in $\mathcal{L}_n$, by Lemma 2.5.10. Suppose now we have a word of length i at the start of w' in $\mathcal{L}_n$ and the next letter to the right is α_{i+1}. Checking from left to right along

w' for switches, if we reach the α_{i+1} without making any switches then we already have a reduced word and w' starts with a word of length $i+1$ in $\mathcal{L}_n$, by Lemma 2.5.10, and we are done. Therefore suppose we make a switch. This either happens at the $(i-3)$th letter in w', only in the case $q=5$ and $i>3$, the $(i-1)$th letter in w', when $i>1$, and the ith letter in w'. If no pairs can be deleted after this, by Lemma 2.5.10 we are again done. If the first letter of the swap and the letter to the left of that can be deleted then we obtain a word that has length less than i so by induction this will reduce to a word in $\mathcal{L}_n$. Similarly if the ith and $(i+1)$th letters are the same and we delete them, and any subsequent pairs that are the same then we obtain a word shorter than i as well. Thus, by induction, we have $\Pi(w) = \Pi(\Pi(\cdots\Pi(\Pi(\alpha_1 \circ \alpha_2) \circ \alpha_3) \cdots) \circ \alpha_m)$ and $\Pi(w)$ is in $\mathcal{L}_n$ for all w in D_Δ. $\square$

We also need to define an inversion on $\mathcal{L}_n$. For α in A or the identity set $\alpha^{-1} = \alpha$. For the remaining sets we have ordered them in pairs of inverses, so the first two in each set are inverse of each other and so on. Note that α has the same set of start states as α^{-1} has end states, for all α in $\mathcal{L}_n$. It should also be easy to see that $\Pi(\alpha \circ \alpha^{-1}) = 1$ for all α in $\mathcal{L}_n$. This has also been verified by the Magma code by checking every element of $\mathcal{L}_n$ and its inverse simplify to the empty word. This naturally extends to the map

$$\cdot^{-1} : W(\mathcal{L}_n) \longrightarrow W(\mathcal{L}_n)\ ;\ (\alpha_1 \circ \cdots \circ \alpha_m)^{-1} \longmapsto \alpha_m^{-1} \circ \cdots \circ \alpha_1^{-1},$$

where each α_i is in $\mathcal{L}_n$ and m in $\mathbb{N}$ is the length of w in words in $\mathcal{L}_n$.

Proposition 2.5.13. *The pair $(\mathcal{L}_n, \Delta)$ is a finite objective partial group with product map Π and inversion $\cdot^{-1}$.*

Proof. First note we have $\mathcal{L}_n \subseteq D_\Delta \subseteq W(\mathcal{L}_n)$ and that for any word in D_Δ all of its subwords are in D_Δ as well. By Lemma 2.5.12, we have that $\Pi(D_\Delta)$ is contained in $\mathcal{L}_n$. The map Π also restricts to the identity on $\mathcal{L}_n$ as verified by the Magma code to check no elements of $\mathcal{L}_n$ can be further simplified. For any word v in D_Δ we have that v and $\Pi(v)$ both start at the same state and end at the same state, by Lemma 2.5.9. Thus, for

u, v and w in D_Δ, if $u \circ v \circ w$ is in D_Δ then $u \circ \Pi(w) \circ v$ is in D_Δ as it is still accepted by an automaton. We can write each word u, v and w as a list of subwords in $\mathcal{L}_n$ of length 1, $u_1 \circ \cdots \circ u_k$, $v_1 \circ \cdots \circ v_l$ and $w_1 \circ \cdots \circ w_m$, for k, l, m in $\mathbb{N}$ the lengths of u, v and w respectively. As the product is associative on $\mathcal{L}_n$, by Lemma 2.5.11, we have that

$$\begin{aligned}\Pi(u \circ v \circ w) &= \Pi(\Pi(\cdots \Pi(u_1 \circ u_2) \cdots \circ u_k) \circ v_1) \cdots \circ v_l) \circ w_1) \cdots \circ w_m) \\ &= \Pi(u_1 \circ \cdots \circ u_k \circ \Pi(v_1 \circ \cdots \circ v_l) \circ w_1 \circ \cdots \circ w_m) \\ &= \Pi(u \circ \Pi(v) \circ w).\end{aligned}$$

Therefore $\Pi : D_\Delta \to \mathcal{L}_n$ is a product. Now consider w again, but this time we write $w = \alpha_1 \circ \cdots \circ \alpha_m$, where α_i is in $\mathcal{L}_n$, not necessarily length 1, and m is in $\mathbb{N}$. Thus we have that the inverse of w is $\alpha_m^{-1} \circ \cdots \circ \alpha_1^{-1}$. First note that, as the set of end states of α_m is the same as the set of start states for α_m^{-1}, we have that $w \circ w^{-1}$ is in D_Δ. Again using associativity of Π on $\mathcal{L}_n$ by Lemma 2.5.11, we have

$$\Pi(w \circ w^{-1}) = \Pi(\alpha_1 \circ \cdots \circ \alpha_m \circ \alpha_m^{-1} \circ \cdots \circ \alpha_1^{-1}),$$

which when we delete pairs of inverses gives $\Pi(\emptyset) = 1$. Thus $(\mathcal{L}_n, D_\Delta, \Pi, \cdot^{-1})$ is a partial group. Furthermore $\mathcal{L}_n$ is finite and by construction $D = D_\Delta$ where Δ is a set of subgroups each isomorphic to C_2. Thus the second axiom of objectivity holds vacuously and $\mathcal{L}_n$ is a finite objective partial group. $\square$

We therefore have that $(\mathcal{L}_n, \Delta)$ is a finite objective partial group and we will now prove that its automorphism group is C_n. First note that the sets making up $\mathcal{L}_n$ partition it by element order. The set A consists of all order 2 elements, B all order 4 elements, C all order q elements, D all order $2q$ elements and E all elements with no defined order, in other words they cannot be multiplied with themselves as the multiplication is not defined. It is clear that any automorphism must restrict to a bijection from each set to itself.

We will now give a series of three results which show what certain elements of $\mathcal{L}_n$ are mapped to under an automorphism. These are analogous to Lemmas 2.5.6 and 2.5.7 in the previous section.

Lemma 2.5.14. *Any automorphism of* $(\mathcal{L}_n, \Delta)$ *maps* z *to itself.*

Proof. Suppose ψ is an automorphism of $\mathcal{L}_n$. From the above discussion we have that $\psi(z)$ is in A. We shall consider centralisers of elements in A to show that $\psi(z)$ can only equal z for ψ to be an automorphism. It is clear that we must have $|\mathrm{C}_{\mathcal{L}_n}(z)\,| = |\mathrm{C}_{\mathcal{L}_n}(\psi(z))\,|$. It should also be clear that the number of elements of any given order in each centraliser must be the same. From the table in Figure 2.3 we see that the only order 2 element that is centralised by exactly $2n+1$ order 2 elements is z, so $\psi(z) = z$. $\square$

Lemma 2.5.15. *For any* $1 \leq i \leq n$*, any automorphism of* $(\mathcal{L}_n, \Delta)$ *maps* x_i *to* x_j*, where* $1 \leq j \leq n$*.*

Proof. First fix some index $1 \leq i \leq n$. We proceed in a similar fashion to the proof of Lemma 2.5.14. Suppose ψ is an automorphism of $\mathcal{L}_n$. We consider the candidate elements in A that commute with the same number of elements of A as x_i does. From the table in Figure 2.3 we see that the only elements in A with the same-sized centraliser in A as x_i are $t_j u_j$, $u_j v_j$ and $v_j w_j$, for any $1 \leq j \leq n$. Now consider the set B. No elements in B commute with x_i, but $t_j u_j$, $u_j v_i$ and $v_j w_j$ commute with $t_j x_j$, $u_j x_{j+1}$ and $v_j z$ respectively, for each j. Thus $\psi(x_i)$ can only be x_j, for some $1 \leq j \leq n$. $\square$

Lemma 2.5.16. *For* $1 \leq i \leq n$*, any automorphism of* $(\mathcal{L}_n, \Delta)$ *maps* w_i *to* w_j*, where* $1 \leq j \leq n$*.*

Proof. Suppose ψ is an automorphism of $\mathcal{L}_n$. From Lemma 2.5.14 we have that ψ must fix z. Consider, for some fixed i, $\psi(zw_i) = z\psi(w_i)$. As zw_i is an element of order 4 we have that $\psi(zw_i)$ must be in B, so $\psi(w_i)$ is either v_j or w_j, where $1 \leq j \leq n$ is arbitrary. If $\psi(w_i) = v_j$ then we have that

$$\psi(v_i) = \psi(zw_i z) = \psi(z)\psi(w_i)\psi(z) = zv_j z = w_j.$$

$\alpha \in A$	$\mathrm{C}_A(\alpha)$	$\lvert \mathrm{C}_A(\alpha) \rvert$
t_i	$t_i,\, u_i,\, t_iu_i,\, x_iv_ix_i,\, u_ix_i(t_iv_i)^kx_i,\, t_ix_iv_ix_i,\, t_iu_ix_i(t_iv_i)^kx_i \,\big\vert\, 2 \leq k \leq q-1$	$2q+1$
u_i	$t_i,\, u_i,\, v_i,\, t_iu_i,\, u_iv_i,\, t_i(t_iv_i)^k,\, (t_iu_i(t_iv_i)^k \,\big\vert\, 2 \leq k \leq q-1$	$2q+1$
v_i	$u_i,\, v_i,\, w_i,\, u_iv_i,\, v_iw_i,\, u_i(u_iw_i)^k, u_iv_i(u_iw_i)^k \,\big\vert\, 2 \leq k \leq q-1$	$2q+1$
w_i	$v_i,\, w_i,\, v_iw_i,\, zu_iz,\, zu_i(u_iw_i)^kz,\, w_izu_iz,\, w_izu_i(u_iw_i)^kz \,\big\vert\, 2 \leq k \leq q-1$	$2q+1$
x_i	$x_i,\, t_iu_i,\, u_{i-1}v_{i-1},\, t_iu_ix_i,\, u_{i-1}v_{i-1}x_i$	5
z	$v_iw_i,\, z,\, v_iw_iz \,\big\vert\, 1 \leq i \leq n$	$2n+1$
t_iu_i	$t_i,\, u_i,\, x_i,\, t_iu_i,\, t_iu_ix_i$	5
u_iv_i	$u_i,\, v_i,\, x_{i+1},\, u_iv_i,\, u_iv_ix_{i+1}$	5
v_iw_i	$v_i,\, w_i,\, v_iw_i,\, z,\, v_iw_iz$	5
$t_iu_ix_i$	$x_i,\, t_iu_i,\, t_iu_ix_i$	3
$u_iv_ix_{i+1}$	$x_{i+1},\, u_iv_i,\, u_iv_ix_{i+1}$	3
v_iw_iz	$z,\, v_iw_i,\, v_iw_iz$	3
$t_i(t_iv_i)^k$	$u_i,\, t_i(t_iv_i)^k,\, t_iu_i(t_iv_i)^k$	3
$u_i(u_iw_i)^k$	$v_i,\, u_i(u_iw_i)^k,\, u_iv_i(u_iw_i)^k$	3
$t_iu_i(t_iv_i)^k$	$u_i,\, t_i(t_iv_i)^k,\, t_iu_i(t_iv_i)^k$	3
$u_iv_i(u_iw_i)^k$	$v_i,\, u_i(u_iw_i)^k,\, u_iv_i(u_iw_i)^k$	3
$x_iv_ix_i$	$t_i,\, x_iv_ix_i,\, t_ix_iv_ix_i$	3
$u_ix_i(t_iv_i)^kx_i$	$t_i,\, u_ix_i(t_iv_i)^kx_i,\, t_iu_ix_i(t_iv_i)^kx_i$	3
zu_iz	$w_i,\, zu_iz,\, w_izu_iz$	3
$zu_i(u_iw_i)^kz$	$w_i,\, zu_i(u_iw_i)^kz,\, w_izu_i(u_iw_i)^kz$	3
$t_ix_iv_ix_i$	$t_i,\, x_iv_ix_i,\, t_ix_iv_ix_i$	3
$t_iu_ix_i(t_iv_i)^kx_i$	$t_i,\, u_ix_i(t_iv_i)^kx_i,\, t_iu_ix_i(t_iv_i)^kx_i$	3
w_izu_iz	$w_i,\, zu_iz,\, w_izu_iz$	3
$w_izu_i(u_iw_i)^kz$	$w_i,\, zu_i(u_iw_i)^kz,\, w_izu_i(u_iw_i)^kz$	3

Figure 2.3: The centralisers in A of each element of A, where $1 \leq i \leq n$ and $2 \leq k \leq q-1$ in the left-hand column and the indices are the same in the second column unless stated otherwise.

Next consider $\psi(x_i v_i x_i) = \psi(x_i) w_j \psi(x_i)$. From Lemma 2.5.15 we have that ψ can only map x_i to an x_k, for some k. The only index where the above relation is defined is $j+1$. So $\psi(x_i) = x_{j+1}$ and $\psi(x_i v_i x_i) = x_{j+1} w_j x_{j+1} = t_j$. However this gives us a contradiction as t_j and $x_i v_i x_i$ do not commute with the same number of elements in A, as shown in the table in Figure 2.3. Thus $\psi(w_i)$ cannot be v_j for any $1 \leq j \leq n$ so we have that $\psi(w_i) = w_j$ for some j. $\square$

Now we have restricted what z, x_i and w_i can map to under an automorphism we can now prove that the automorphism group must be C_n.

Proposition 2.5.17. *Any automorphism of $(\mathcal{L}_n, \Delta)$ is of the form $i \mapsto i+m$, for all i and some $1 \leq m \leq n$, in other words indices are increased modulo n.*

Proof. Suppose ψ is an automorphism of $\mathcal{L}_n$. From Lemma 2.5.14 we have that ψ must fix z. Furthermore, from Lemma 2.5.16, we have that $\psi(w_i) = w_{i+m}$, where $1 \leq m \leq n$. We can combine these results to obtain

$$\psi(v_i) = \psi(z w_i z) = z w_{i+m} z = v_{i+m}.$$

Now consider

$$\psi(t_i) = \psi(x_{i+1} w_i x_{i+1}) = \psi(x_{i+1}) w_{i+m} \psi(x_{i+1}).$$

By Lemma 2.5.16 we have that $\psi(x_{i+1}) = x_j$, for some $1 \leq j \leq n$. However the only x_j where $x_j w_{i+m} x_j$ is defined is x_{i+m+1}, so we have $\psi(x_{i+1}) = x_{i+m+1}$ and thus $\psi(t_i) = t_{i+m}$. Thus it is a simple case of using the relations to compute the rest. We have

$$\psi(u_i) = \psi(x_{i+1} v_i x_{i+1}) = x_{i+m+1} v_{i+m} x_{i+m+1} = u_{i+m}.$$

To compute $\psi(x_i)$ consider

$$\psi(t_i) = \psi(x_i u_i x_i) = \psi(x_i) u_{i+m} \psi(x_i) = t_{i+m}.$$

Hence the only x_j that satisfies $x_j u_{i+m} x_j = t_{i+m}$ is x_{i+m}, so $\psi(x_i) = x_{i+m}$. Now consider

$$\psi(t_{i-1}) = \psi(x_i w_{i-1} x_i) = x_{i+m}\psi(w_{i-1})x_{i+m}.$$

By Lemma 2.5.16, again, we have that $\psi(w_{i-1}) = w_j$ for some $1 \leq j \leq n$ and the only w_j where the above conjugation is defined is w_{i+m-1}. Thus $\psi(w_{i-1}) = w_{i+m-1}$ and also $\psi(t_{i-1}) = t_{i+m-1}$. Notice that $\psi(w_{i-1}) = w_{i+m-1}$ is $\psi(w_i) = w_{i+m}$ with 1 subtracted from each index. Thus repeating the same steps above will give us the same results for $i-1$ as we have for i. The final step will give $\psi(w_{i-2}) = w_{i+m-2}$. Thus repeating this process n times we see that for any index ψ increases that index by m. Thus there are no automorphisms of $(\mathcal{L}_n, \Delta)$ that are not of the form $i \mapsto i+m$, for all i and some $1 \leq m \leq n$. $\square$

We have now constructed, for each $n \geq 3$, a finite objective partial group and we note that each example is not isomorphic to the examples in Section 2.5.1. This can been seen by simply considering the order of the sets $\mathcal{L}_n$ in each section.

2.5.3 Infinite objective partial groups

We will now, given any simply connected graph Γ, construct an objective partial group with the same automorphism group as Γ. Unfortunately this objective partial group will almost always be infinite.

Let V be the vertex set of Γ and let E be the edge set. If x and y are in V with an edge between them then we label that edge as e_{xy} in E. Note that this convention is different from that previously used to label elements of E. Also note that because Γ is undirected we identify e_{yx} in E with e_{xy}. To each v in V we associate a copy of C_2 such that $v^2 = 1$. We also do the same with each edge in E, in other words $e_{xy}^2 = 1$ for all x and y in V connected by an edge. Let Δ_Γ be the set of all copies of C_2 associated to vertices in V. For ease we will abuse notation somewhat and identify v in V with the group associated to it so we will often think of having $\Delta_\Gamma = V$. We will now define conjugation of our elements of Δ_Γ. Certainly they can each be conjugated by themselves, but we also define $x^{e_{xy}} = y$ for all x and y in V connected by an edge.

We can characterise the set $\mathbf{D}_{\Delta,\Gamma}$ by constructing the following automaton from Γ. We allow each vertex in Γ to be both a start and end state labelled in the same way as the vertex, and add a directed edge from each vertex to itself, again labelled with the vertex label. We then replace each edge in Γ by a pair of directed edges, one going in each direction, both labelled by the edge they replace. Thus $\mathbf{D}_{\Delta,\Gamma}$ is the set of words admitted by this automaton. An example automaton can be seen in Figure 2.4, in this case for the graph K_3, the complete graph on three vertices. It is easy to see that this characterisation of $\mathbf{D}_{\Delta,\Gamma}$ is correct; any walk in the automaton corresponds to a chain of elements in Δ_Γ, given by the chain of states that the walk passes through. When an edge occurs in a path on the automaton this is the same as conjugating from one vertex to another, and when a vertex occurs this is conjugating from one vertex to itself. Note that any element w in $\mathbf{D}_{\Delta,\Gamma}$ corresponds to a walk in Γ given by just the list of edges in w. We will often refer to this, as well as the set of vertices in w.

We can now define Π and $\mathcal{L}_\Gamma$. Given w in $\mathbf{D}_{\Delta,\Gamma}$ we define $\Pi(w)$ as follows. If there are an odd number of vertices in w then $\Pi(w)$ starts with the vertex that w starts at, and otherwise there is no vertex present in $\Pi(w)$. For the edges in $\Pi(w)$, we consider the walk that w takes and remove any edges that are traversed in both directions consecutively.

More precisely we define the algorithm for reducing a word in $\mathbf{D}_{\Delta,\Gamma}$ as follows. If w is of length 1 it remains unchanged. Otherwise we replace the first two letters of w as follows: $e_{xy}y \mapsto xe_{xy}$, xx and $e_{xy}e_{xy}$ are replaced by the empty word and xe_{xy} and $e_{xy}e_{yz}$ remain the same, where x, y and z are arbitrary vertices of Γ. If the first two letters of w are changed then this step is repeated until the resulting word is either length 1, or the first two letters are the same upon applying the step. One then considers the next letter to the right. If it is a vertex and the first letter of the word at this stage is a vertex then remove both from the word. If the first letter of the word at this stage is not a vertex then remove the vertex and add a vertex at the start of the word corresponding to the start state of the word. Otherwise, if the next letter to the right is an edge and it is the same edge as the letter to the left of it delete both letters otherwise leave them. One now considers the next letter to the right of this and repeat. Note that, because letters are deleted, this next letter might now be the second letter in w, in which case you return to the step where you consider just the first two letters. Otherwise you repeat the above process. Once the last letter on the right-hand end of the word has been checked and any deleting done the word is considered fully reduced..

Example 2.5.18. If $w = e_{xy}y^3e_{yz}e_{xz}x^2e_{xy}e_{yz}z^2e_{yz}$ is a word admitted by the automaton in Figure 2.4, then $\Pi(w) = xe_{xy}e_{yz}e_{xz}e_{xy}$. If w had one more, or fewer, occurrence of a vertex anywhere in it then we would have $\Pi(w) = e_{xy}e_{yz}e_{xz}e_{xy}$ instead.

Lemma 2.5.19. *If w is in $\mathbf{D}_{\Delta,\Gamma}$ then w and $\Pi(w)$ have the same start and end states.*

Proof. At each step the algorithm corresponding to Π can change w in two ways; either change where vertices occur in the word, or remove two copies of the same edge. Vertices in w map from one state to itself so by adding or removing a vertex from w one either adds another occurrence of the same vertex to the list of states w passes through or removes one of a pair of the same vertex from the list of states. If this happens either at the start or end of w it is clear that the start or end states do not change. If we remove a repeat of the same edge, say e_{xy}, then the states w passes through go either from $\ldots, x, y, x, \ldots$ to $\ldots, x, \ldots$ or $\ldots, y, x, y, \ldots$ to $\ldots, y, \ldots$. In both cases it is clear that if this happens at either end of w the start and end states stay the same. As $\Pi(w)$ is obtained by applying both of these steps a finite number of times we have that both w and $\Pi(w)$ have the same start and end state. $\square$

We define $\mathcal{L}_\Gamma$ to be the image of $\mathbf{D}_{\Delta,\Gamma}$ under Π, in other words all walks on Γ with no doubling back on themselves either with or without the vertex of the start state at the front of the word. We can think of the walk in Γ corresponding to $\Pi(w)$ as the shortest walk on Γ that is homotopy equivalent with fixed endpoints to the walk corresponding to w, under the topology on Γ induced by the standard distance metric.

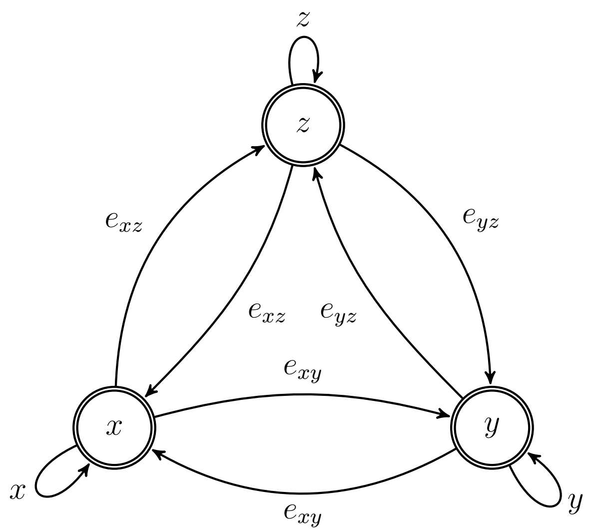

Figure 2.4: The automaton associated to K_3, the complete graph on three vertices, with vertices labelled x, y and z.

The algorithmic definition of this product will be useful for some purposes but we also wish to characterise it in another way. To do this we define a map π on pairs in $\mathcal{L}_\Gamma$ where the first word ends at the same state that the second starts at. Let $v_1^{\alpha_1}p_1$ and $v_2^{\alpha_2}p_2$ be arbitrary elements of $\mathcal{L}_\Gamma$, so α_1 and α_2 are either 0 or 1, p_1 is either an empty walk or of the form $e_1 \dots e_k$ and p_2 is either an empty walk or of the form $e_{k+1} \dots e_{k+l}$, for k and l in $\mathbb{N}$. If the end state of $v_1^{\alpha_1}p_1$ is the same as the start state for $v_2^{\alpha_2}p_2$, and for $\gamma \geq 0$ the largest such integer where $e_{k-\gamma} \dots e_k = e_{k+\gamma+1} \dots e_{k+1}$, define the map $\pi : \mathbf{D}_{\Delta,\Gamma}|_{\mathcal{L}_\Gamma \circ \mathcal{L}_\Gamma} \to \mathcal{L}_\Gamma$ as follows

$$\pi(v_1^{\alpha_1}p_1 \circ v_2^{\alpha_2}p_2) = \begin{cases} v_1^{\beta} e_{k+|p_1|+1} \dots e_{k+l} & \text{if } \gamma = |p_1| - 1 \text{ and } \gamma \neq |p_2| - 1, \\ v_1^{\beta} e_1 \dots e_{k-|p_2|} & \text{if } \gamma = |p_2| - 1 \text{ and } \gamma \neq |p_1| - 1, \\ v_1^{\beta} & \text{if } \gamma = |p_2| - 1 = |p_1| - 1, \\ v_1^{\beta} e_1 \dots e_{k-\gamma-1} e_{k+\gamma+2} \dots e_{k+l} & \text{if } \gamma \neq |p_2| - 1 \text{ and } \gamma \neq |p_1| - 1, \\ v_1^{\beta} p_1 p_2 & \text{if no such } \gamma \text{ exists,} \end{cases}$$

where $\beta = \alpha_1 + \alpha_2 \mod 2$ and $|p_1|$ denotes the length of p_1. Again here we use $\mathcal{L}_\Gamma \circ \mathcal{L}_\Gamma$ to denote the set of all words in $\mathcal{D}_{\Delta,\Gamma}$ that are the concatenation of two elements of $\mathcal{L}_\Gamma$.

We now need to show that this map agrees with our other product where they are both defined.

Lemma 2.5.20. *If w_1 and w_2 are arbitrary elements of $\mathcal{L}_\Gamma$ such that $w_1 \circ w_2$ is in $\mathbf{D}_{\Delta,\Gamma}$ then $\Pi(w_1 \circ w_2) = \pi(w_1 \circ w_2)$.*

Proof. We characterise elements of $\mathcal{L}_\Gamma$ as before, in other words let $w_1 = v_1^{\alpha_1} p_1$ and $w_2 = v_2^{\alpha_2} p_2$ are arbitrary elements of $\mathcal{L}_\Gamma$, so α_1 and α_2 are either 0 or 1, p_1 is either an empty walk or of the form $e_1 \dots e_k$ and p_2 is either an empty walk or of the form $e_{k+1} \dots e_{k+l}$, for k and l in $\mathbb{N}$. First note that if w_2 is empty then it is clear both maps are just constant on w_1. If w_1 is empty then $v_1 = v_2$ and again both definitions agree. We therefore assume both words are non-empty.

If $w_1 = v_1$, then if $\alpha_2 = 0$ both maps are just constant so agree. If $\alpha_2 = 1$ then Π deletes both vertices at the start of $w_1 \circ w_2$ and leaves the rest constant so $\Pi(w_1 \circ w_2) = p_2$ which agrees with π, as p_1 is empty and $\alpha_1 + \alpha_2 = 2$. Now let $w_1 = e_1$, in other words $k = 1$ and $\alpha_1 = 0$. If $\alpha_2 = 0$ and $e_1 = e_2$ then Π deletes this pair and $\Pi(w_1 \circ w_2) = e_3 \dots e_{l+1}$, which agrees with π, where here $\gamma = k - 1$. If $\alpha_2 = 0$ and $e_1 \neq e_2$ then Π leaves this constant and $\Pi(w_1 \circ w_2) = e_1 p_2$, which agrees with π as no γ exists. If $\alpha_2 = 1$ then Π replaces the first two letters of $w_1 \circ w_2$ with $v_1 e_1$ and either $e_1 = e_2$ and $\Pi(w_1 \circ w_2) = v_1 e_3 \dots e_{l+1}$ or $\Pi(w_1 \circ w_2) = v_1 e_1 p_2$. In both cases this agrees with π, the first we have $\gamma = k - 1$ and the second we have γ does not exist.

Now let w_1 have length greater than or equal to 2. If we suppose $\alpha_2 = 0$ and e_k is different from e_{k+1} then Π leaves this constant and $\Pi(w_1 \circ w_2) = v_1^{\alpha_1} p_1 p_2$ which agrees with π as γ is not defined. If we instead suppose $e_k = e_{k+1}$, then Π deletes this pair and any subsequent pair that are the same. If we suppose it deletes another γ pairs then

$$\Pi(w_1 \circ w_2) = \begin{cases} v_1^{\alpha_1} e_{2k+1} \dots e_{k+l} & \text{if } \gamma = k-1 \text{ and } \gamma \neq l-1, \\ v_1^{\alpha_1} e_1 \dots e_{k-l} & \text{if } \gamma = l-1 \text{ and } \gamma \neq k-1, \\ v_1^{\alpha_1} & \text{if } \gamma = l-1 = k-1, \\ v_1^{\alpha_1} e_1 \dots e_{k-\gamma-1} e_{k+\gamma+2} \dots e_{k+l} & \text{if } \gamma \neq l-1 \text{ and } \gamma \neq k-1, \end{cases}$$

which agrees with π in each case. If $\alpha_2 = 1$ then the first step of applying Π deletes v_2 and, if $\alpha_1 = 0$, adding v_1 at the start of the word otherwise, if $\alpha_1 = 1$, removing v_1 from the start of the word. Thus the power of v_1 at the start of the word is $\beta = \alpha_1 + \alpha_2$ mod 2. Now the next step of Π is to check e_k against e_{k+1} and we note that we arrive at the same case we considered above, however this time v_1 is to the power of β not α_1. Thus Π and π agree in this case and we have $\Pi(w_1 \circ w_2) = \pi(w_1 \circ w_2)$ for all w_1 and w_2 where $w_1 \circ w_2$ are in $\mathbf{D}_{\Delta,\Gamma}$. $\square$

This alternative characterisation of Π will be very useful in showing that Π is associative where defined. In order to do this we need to define the inverse map $\cdot^{-1}$, first on $\mathcal{L}_\Gamma$. For each vertex or edge in $\mathcal{L}_\Gamma$ we have already defined them to be self-inverse. Now consider some walk w in $\mathcal{L}_\Gamma$ with at least two edges in it and without a vertex at the start of it. We can write $w = e_1 \dots e_m$, where each e_i is in E and m is the length of w in edges. Define $w^{-1} = e_m \dots e_1$, in other words the walk traced in reverse. If w now starts with a vertex, so is of the form $w = ve_1 \dots e_m$, then we define $w^{-1} = \Pi(e_m \dots e_1 v)$. One can see that this is the walk w traced in reverse with the suitable vertex at the start. We then extend this map to Δ_Γ in the natural way.

Lemma 2.5.21. *If w_1, w_2 and w_3 are in $\mathbf{D}_{\Delta,\Gamma}$ such that $w_1 \circ w_2 \circ w_3$ is also in $\mathbf{D}_{\Delta,\Gamma}$ then $\pi(\pi(w_1 \circ w_2) \circ w_3) = \pi(w_1 \circ \pi(w_2 \circ w_3))$.*

Proof. The proof of this is a matter of considering all cases. Write $w_1 = v_1^{\alpha_1} p_1$, $w_2 = v_2^{\alpha_2} p_2$ and $w_3 = v_3^{\alpha_3} p_3$ for arbitrary elements of $\mathcal{L}_\Gamma$, so α_1, α_2 and α_3 are either 0 or 1, p_1 is either an empty walk or of the form $e_1 \dots e_k$, p_2 is either an empty walk or of the form $e_{k+1} \dots e_{k+l}$, and p_3 is either an empty walk or of the form $e_{k+l+1} \dots e_{k+l+m}$ for k, l and m in $\mathbb{N}$. Note that, by Lemma 2.5.20, $\Pi(w_1 \circ w_2) = \pi(w_1 \circ w_2)$ and, by Lemma 2.5.19, we have that $\Pi(w_1 \circ w_2)$ and $w_1 \circ w_2$ have the same start and end state so if $w_1 \circ w_2 \circ w_3$ is in $\mathbf{D}_{\Delta,\Gamma}$ then so is $\pi(w_1 \circ w_2) \circ w_3$. As $\pi(w_1 \circ w_2)$ is in $\mathcal{L}_\Gamma$ by definition of π then $\pi(\pi(w_1 \circ w_2) \circ w_3)$ is well-defined. A similar argument gives us that $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is also well-defined.

Let $\beta_1 = \alpha_1 + \alpha_2 \mod 2$, $\beta_2 = \alpha_2 + \alpha_3 \mod 2$ and $\beta_3 = \alpha_1 + \alpha_2 + \alpha_3 \mod 2$. Also let $\gamma_1 + 1$ denote the overlap between p_1 and p_2, $\delta_1 + 1$ denote the overlap between the walk corresponding to $\pi(w_1 \circ w_2)$ and p_3, $\delta_2 + 1$ denote the overlap between p_2 and p_3 and $\gamma_2 + 1$ denote the overlap between p_1 and the walk corresponding to $\pi(w_2 \circ w_3)$.

We now compute all cases for $\pi(\pi(w_1 \circ w_2) \circ w_3)$. If $\gamma_1 = k - 1$ and $\gamma_1 \neq l - 1$ then we have $\pi(\pi(w_1 \circ w_2) \circ w_3) = \pi(v_1^{\beta_1} e_{2k+1} \dots e_{k+l} \circ w_3)$ is equal to

$$\begin{cases} v_1^{\beta_3} e_{2l+1} \dots e_{k+l+m} & \text{if } \delta_1 = -k + l - 1 \text{ and } \delta_1 \neq m - 1, \quad (2.5.1a) \\ v_1^{\beta_3} e_{2k+1} \dots e_{k+l-m} & \text{if } \delta_1 \neq -k + l - 1 \text{ and } \delta_1 = m - 1, \quad (2.5.1b) \\ v_1^{\beta_3} & \text{if } \delta_1 = -k + l - 1 = m - 1, \quad (2.5.1c) \\ v_1^{\beta_3} e_{2k+1} \dots e_{k+l-\delta_1-1} e_{k+l+\delta_1+2} \dots e_{k+l+m} & \text{if } \delta_1 \neq -k + l - 1 \text{ and } \delta_1 \neq m - 1, \quad (2.5.1d) \\ v_1^{\beta_3} e_{2k+1} \dots e_{k+l} p_3 & \text{if no such } \delta_1 \text{ exists.} \quad (2.5.1e) \end{cases}$$

If $\gamma_1 \neq k - 1$ and $\gamma_1 = l - 1$, then $\pi(\pi(w_1 \circ w_2) \circ w_3) = \pi(v_1^{\beta_1} e_1 \dots e_{k-l} \circ w_3))$ is equal to

$$\begin{cases} v_1^{\beta_3} e_{2k+1} \dots e_{k+l+m} & \text{if } \delta_1 = k - l - 1 \text{ and } \delta_1 \neq m - 1, \quad (2.5.2a) \\ v_1^{\beta_3} e_1 \dots e_{k-l-m} & \text{if } \delta_1 \neq k - l - 1 \text{ and } \delta_1 = m - 1, \quad (2.5.2b) \\ v_1^{\beta_3} & \text{if } \delta_1 = k - l - 1 = m - 1, \quad (2.5.2c) \\ v_1^{\beta_3} e_1 \dots e_{k-l-\delta_1-1} e_{k+l+\delta_1+2} \dots e_{k+l+m} & \text{if } \delta_1 \neq k - l - 1 \text{ and } \delta_1 \neq m - 1, \quad (2.5.2d) \\ v_1^{\beta_3} e_1 \dots e_{k-l} p_3 & \text{if no such } \delta_1 \text{ exists.} \quad (2.5.2e) \end{cases}$$

If $\gamma_1 = k-1 = l-1$ then

$$\pi(\pi(w_1 \circ w_2) \circ w_3) = \pi(v_1^{\beta_1} \circ w_3) = v_1^{\beta_3} p_3. \tag{2.5.3}$$

If $\gamma_1 \neq k-1$ and $\gamma_1 \neq l-1$, then $\pi(\pi(w_1 \circ w_2) \circ w_3) = \pi(v_1^{\beta_1} e_1 \dots e_{k-\gamma_1-1} e_{k+\gamma_1+2} \dots e_{k+l} \circ w_3)$ is equal to

$$\begin{cases}
v_1^{\beta_3} e_{2k+2l-2\gamma_1-1} \cdots e_{k+l+m} & \text{if } \delta_1 = k+l-2\gamma_1-3 \text{ and } \delta_1 \neq m-1, & (2.5.4a)\\
v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-1} e_{k+\gamma_1+2} \cdots e_{k+l-m} & \text{if } \delta_1 \neq k+l-2\gamma_1-3,\ \delta_1 = m-1 \text{ and } \delta_1 < l-\gamma_1-2, & (2.5.4b)\\
v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-1} & \text{if } \delta_1 \neq k+l-2\gamma_1-3,\ \delta_1 = m-1 \text{ and } \delta_1 = l-\gamma_1-2, & (2.5.4c)\\
v_1^{\beta_3} e_1 \dots e_{k+l-m-2\gamma_1-2} & \text{if } \delta_1 \neq k+l-2\gamma_1-3,\ \delta_1 = m-1 \text{ and } \delta_1 > l-\gamma_1-2, & (2.5.4d)\\
v_1^{\beta_3} & \text{if } \delta_1 = k+l-2\gamma_1-3 = m-1, & (2.5.4e)\\
v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-1} e_{k+\gamma_1+2} \cdots e_{k+l-\delta_1-1} e_{k+l+\delta_1+2} \cdots e_{k+l+m} & \text{if } \delta_1 \neq k+l-2\gamma_1-3,\ \delta_1 \neq m-1 \text{ and } \delta_1 < l-\gamma_1-2, & (2.5.4f)\\
v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-1} e_{k+2l-\gamma_1} \cdots e_{k+l+m} & \text{if } \delta_1 \neq k+l-2\gamma_1-3,\ \delta_1 \neq m-1 \text{ and } \delta_1 = l-\gamma_1-2, & (2.5.4g)\\
v_1^{\beta_3} e_1 \dots e_{k-l-2\gamma_1-\delta_1-1} e_{k+l+\delta_1+1} \cdots e_{k+l+m} & \text{if } \delta_1 \neq k+l-2\gamma_1-3,\ \delta_1 \neq m-1 \text{ and } \delta_1 > l-\gamma_1-2, & (2.5.4h)\\
v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-1} e_{k+\gamma_1+2} \cdots e_{k+l} p_3 & \text{if no such } \delta_1 \text{ exists.} & (2.5.4i)
\end{cases}$$

If no such γ_1 exists, then $\pi(\pi(w_1 \circ w_2) \circ w_3) = \pi(v_1^{\beta_1} p_1 p_2 \circ w_3)$ is equal to

$$\begin{cases}
v_1^{\beta_3} e_{2k+2l+1} \dots e_{k+l+m} & \text{if } \delta_1 \neq m-1 \text{ and } \delta_1 = k+l-1, & (2.5.5a)\\
v_1^{\beta_3} e_1 \dots e_{k+l-m} & \text{if } \delta_1 = m-1 \text{ and } \delta_1 \neq k+l-1, & (2.5.5b)\\
v_1^{\beta_3} & \text{if } \delta_1 = k+l-1 = m-1, & (2.5.5c)\\
v_1^{\beta_3} e_1 \dots e_{k+l-\delta_1-1} e_{k+l+\delta_1+2} \dots e_{k+l+m} & \text{if } \delta_1 \neq l+k-1 \text{ and } \delta_1 \neq m-1, & (2.5.5d)\\
v_1^{\beta_3} p_1 p_2 p_3 & \text{if no such } \delta_1 \text{ exists.} & (2.5.5e)
\end{cases}$$

We now go through each of the cases for $\pi(\pi(w_1 \circ w_2) \circ w_3)$ and show which cases of $\pi(w_1 \circ \pi(w_2 \circ w_3))$ it pairs with.

First consider (2.5.1a) and notice that in this case we have $k > l$ and $m > l-k$. We also have that p_1 is equal to the last k edges of p_2^{-1} and the first $l-k$ edges of p_3 are equal to the first $l-k$ edges of p_2^{-1}. If $\delta_2 = l-1 \neq m-1$ then the first l edges of p_3 are equal to p_2^{-1}, of which the last k are equal to p_1. Therefore we have $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is given by

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+2l+1} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_k e_{k+2l+1} \dots e_{k+l+m} = v_1^{\beta_3} e_{2l+1} \dots e_{k+l+m}.$$

If we now suppose $\delta_2 \neq l-1$ and $\delta_2 = m-1$, then the last $k-l+m$ edges of p_3 are the same as the first $k-l+m$ edges of p_2^{-1}, and hence the first $k-l+m$ edges of p_1, as $k+m > l$. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is equal to

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_{k-l+m} e_{k+2l+1} \dots e_{k+l+m} = v_1^{\beta_3} e_{2l+1} \dots e_{k+l+m}.$$

By our assumption we have that $k+m > l$ so we do not have the case $\delta_2 = l-1 = m-1$. Instead consider when $\delta_2 \neq l-1$ and $\delta_2 \neq m-1$. This implies the first δ_2+1 edges of p_3 are the same as the first δ_2+1 edges of p_2^{-1} and the first δ_2+1 edges of p_1. By our assumption of $\delta_1 = l-k-1$ we have that $\delta_2 \geq l-k-1$ so first suppose $\delta_2 = l-k-1$. It is clear that $\gamma_2 = k-1$ and the second application of the product will cancel all of p_1 and all that remains of p_2, leaving $v_1^{\beta_3} e_{2l+1} \dots e_{k+l+m}$ as required. If $\delta_2 > l-k-1$ then $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is equal to

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_{\delta_2+k-l+1} e_{k+l+\delta_2+2} \dots e_{k+l+m},$$

which after relabelling edges is $v_1^{\beta_3} e_{2l+1} \dots e_{k+l+m}$. Lastly from our assumption it is not possible for δ_2 to not exist. We have therefore considered all cases that correspond to (2.5.1a). The proof proceeds in a similar manner for each of the remaining 24 cases, however some of these cases are considerably more concise.

Consider (2.5.1b). In this case it is clear that δ_2 is also equal to $m-1$ and γ_2 is $k-1$. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_{2k+1} \dots e_{k+l-m}$.

In (2.5.1c) we have a similar situation to (2.5.1b). As δ_1 is $m-1$ then δ_2 must also be $m-1 = -k+l-1$ and hence $\gamma_2 = \gamma_1$. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m})$ and both remaining paths cancel with each other leaving $v_1^{\beta_3}$.

Case (2.5.1d) is also relatively straightforward. The cancellations from both times we apply π do not meet so we again have that $\gamma_1 = \gamma_2$ and $\delta_1 = \delta_2$. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3))$

is given by

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-\delta-1} e_{k+l+\delta+2} \dots e_{k+l+m}) = v_1^{\beta_3} e_{2k+1} \dots e_{k+l-\delta-1} e_{k+l+\delta+2} \dots e_{k+l+m}.$$

In (2.5.1e) there is no δ_1 and hence no δ_2. We therefore must have $\gamma_1 = k-1 = \gamma_2$ and thus

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} p_2 p_3) = v_1^{\beta_3} e_{2k+1} \dots e_{k+l+m}.$$

We now consider (2.5.2a), (2.5.2b), (2.5.2c) and (2.5.2d) together. In each the last l edges of p_1 are equal to p_2^{-1} and δ_1 is defined, so at least the first edge of p_3 is the same as the $(k-l)$th edge of p_1. In each of these cases if δ_2 is defined then this means the first edge of p_3 is equal to the last edge of p_2. However, as these edges are the same as the $(k-l)$th and $(k-l+1)$th edges of p_1 this gives a contradiction as w_1 is assumed to be in $\mathcal{L}_\Gamma$ so therefore p_1 does not contain any pairs of the same edge. We therefore have δ_2 not defined for each of these cases, in other words $\pi(w_2 \circ w_3) = v_2^{\delta_2} p_2 p_3$. Also note that in each we have $\gamma_2 = \gamma_1 + \delta_1$ and hence associativity holds for each of these cases.

Now consider (2.5.2e). Here we simply have that the last l edges of p_1 are the same as p_2^{-1}. If we suppose $\delta_2 = l-1 \neq m-1$ then the first l edges of p_3 are equal to p_2^{-1} and thus in turn equal to the last l of p_1. The second application of π gives no cancellation as this would mean w_1 and w_3 would not be in $\mathcal{L}_\Gamma$. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is equal to

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+2l+1} \dots e_{k+l+m}) = v_1^{\beta_3} p_1 e_{k+2l+1} \dots e_{k+l+m} = v_1^{\beta_3} e_1 \dots e_{k-l} p_3.$$

Now suppose $\delta_2 = m-1 \neq l-1$, then p_3 is equal to the first m edges of p_2^{-1} which itself is equal to the $(k-l)$th to $(k-l+m)$th edges of p_1. We therefore have

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_{k-l+m} = v_1^{\beta_3} e_1 \dots e_{k-l} p_3.$$

Now, if $\delta_2 = l-1 = m-1$ then $p_3 = p_2^{-1}$ and both of these are equal to the last l edges of p_1, so $p_1 = e_1 \dots e_{k-l} p_3$ and $\pi(w_1 \circ \pi(w_2 \circ w_3)) = v_1^{\beta_3} e_1 \dots e_{k-l} p_3$. Suppose $\delta_2 \neq l-1$

and $\delta_2 \neq m-1$. Then the first δ_2+1 edges of p_3 are the same as the first δ_2+1 edges of p_2^{-1}, and thus the $(k-l)$th to $(k-l+\delta_2+1)$th edges of p_1. Therefore $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is given by

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_{k-l-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m},$$

which after relabelling is $v_1^{\beta_3} e_1 \dots e_{k-l} p_3$. Lastly if no δ_2 exists then it is clear that we have $\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} p_2 p_3) = v_1^{\beta_3} e_1 \dots e_{k-l} p_3$.

Consider now (2.5.3), in other words we have $p_1 = p_2^{-1}$. If $\delta_2 = l-1 \neq m-1$ then the first l edges of p_3 are the same as p_2^{-1} and therefore p_1. Hence

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+2l+1} \dots e_{k+l+m}) = v_1^{\beta_3} p_3.$$

Note no cancellation happens in the second application of the product as this would imply w_3 was not in $\mathcal{L}_\Gamma$. If $\delta_2 = m-1 \neq l-1$ then p_3 is the same as the first m edges of p_2^{-1} and hence the first m edges of p_1. Thus

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_m = v_1^{\beta_3} p_3.$$

Again no cancellation happens in the second application of the product as this would imply w_3 was not in $\mathcal{L}_\Gamma$. If $\delta_2 = l-1 = m-1$ then $p_3 = p_2^{-1}$, which in turn is equal to p_1, and hence $\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2}) = v_1^{\beta_3} p_3$. If δ_2 is equal to neither $l-1$ or $m-1$ then the first δ_2+1 edges of p_3 are equal to the first δ_2+1 edges of p_2, and hence the first δ_2+1 edges of p_1. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is given by

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_{k-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m},$$

which is simply $v_1^{\beta_3} p_3$ after relabelling. No cancellation happens on the second application of π for the same reason as above. Lastly we have that if no δ_2 exists then we must have

$\gamma_2 = \gamma_1$ and thus $\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} p_2 p_3) = v_1^{\beta_3} p_3$.

Now consider (2.5.4a), (2.5.4d), (2.5.4e) and (2.5.4h). We will reduce each of these using the following argument. In each the last $\gamma_1 + 1$ edges of p_1 are the same as the last $\gamma_1 + 1$ edges of p_2^{-1} and the $(k - \gamma_1 - 1)$th edge of p_1 concatenated with the last $l - \gamma_1 - 1$ edges of p_2 are the same as the last $l - \gamma_1$ edges of p_3^{-1}. In each of these cases this implies that we have $\delta_2 \geq l - \gamma_1 - 1$. If $\delta_2 + 1 > l - \gamma_1 - 1$, then the $(l - \gamma_1)$th edge of p_3 is the same as the $(\gamma_1 + 1)$th edge of p_2, which is also the same as the $(k - \gamma_1)$th edge of p_1. But we have that the $(l - \gamma_1)$th edge of p_3 is also the same as the $(k - \gamma_1 - 1)$th edge of p_1, so p_1 has two copies of the same edge adjacent to each other and thus w_1 is not in $\mathcal{L}_\Gamma$. This gives a contradiction so we conclude that $\delta_2 + 1 = l - \gamma_1 - 1$ in each of the four cases. If we now consider just (2.5.4a) with $\delta_2 + 1 = l - \gamma_1 - 1$ it is clear that we will have $\gamma_2 = k - 1$. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is equal to

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m}) = v_1^{\beta_3} e_{2k+2\delta_2+3} \dots e_{k+l+m},$$

which is $v_1^{\beta_3} e_{2k+2l-2\gamma_1-1} \dots e_{k+l+m}$ with relabelling. Similarly (2.5.4d) with $\delta_2 + 1 = l - \gamma_1 - 1$ means that $\gamma_2 = -l + m - 2\delta_2 - 3$ and $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is given by

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_{k-l-m+2\delta_2+2},$$

which, when edges are suitably relabelled, is $v_1^{\beta_3} e_1 \dots e_{k+l-m-2\gamma_1-2}$. If we consider (2.5.4e) with $\delta_2 = l - \gamma_1 - 2$ then $\gamma_2 = m - l + 2\gamma_1 - 1$, which one should see is also equal to $k - 1$ and thus we have

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m}) = v_1^{\beta_3}.$$

The last case we consider with $\delta_2 = l - \gamma_1 - 2$ is (2.5.4h). Note that this implies that

$\gamma_2 = \gamma_1 + \delta_1 + 1$ and we have $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is equal to

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-\delta_2-1} e_{k+l+\delta_2+2} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_{k-\gamma_2-1} e_{k+\gamma_2+2\delta_2+2} \dots e_{k+l+m},$$

or $v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-\delta_1-2} e_{k+2l-\gamma_1+\delta_1+1} \dots e_{k+l+m}$ after relabelling edges.

Next we consider cases (2.5.4b) and (2.5.4c) together. In each note that as $\delta_1 = m-1$ and we must have $\delta_2 \geq \delta_1$ so we have $\delta_2 = m-1$. We therefore also have that $\gamma_1 = \gamma_2$ in both cases, and

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+l+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-1} e_{k+\gamma_1+2} \dots e_{k+l-m}$$

and

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+l+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-2},$$

respectively.

Case (2.5.4f) is relatively straightforward. It is clear that δ_2 must equal δ_1 and thus also $\gamma_2 = \gamma_1$, and hence the two regions where cancellation occurs do not overlap. Therefore the order does not matter.

Now consider (2.5.4g) and notice that we have $\gamma_1 + \delta_1 + 2 = l$. We also have that the last $\gamma_1 + 1$ edges of p_1 are the same as the last γ_1 edges of p_2^{-1} and the first $l - \gamma_1 - 1$ edges of p_3 are the same as the first $l - \gamma_1 - 1$ edges of p_2^{-1}. Again we must have $\delta_2 \geq \delta_1$, so we consider first the case when $\delta_2 = l - 1 \neq m - 1$. This implies the first l edges of p_3 are the same as p_2^{-1} and hence the last γ_1 edges of p_1 are the same as the $(l - \gamma_1)$th to lth edges of p_3. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is equal to

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+2l+1} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_k e_{k+2l+1} \dots e_{k+l+m},$$

which is $v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-1} e_{k+2l+\gamma_1} \dots e_{k+l+m}$ after relabelling. If we now let $\delta_2 = m - 1 \neq l - 1$ then p_3 is equal to the first m edges of p_2^{-1} and thus the $(l - \gamma_1)$th to the mth edges of p_3 are the same as the $(k - \gamma_1)$th to the $(k - l + m)$th edges of p_1. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3))$

is given by

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_{k-l+m} = v_1^{\beta_3} e_1 \dots e_{k-\gamma_1-1} e_{k+2l-\gamma_1} \dots e_{k+l+m}.$$

If we have $\delta_2 = l-1 = m-1$ then $p_3 = p_2^{-1}$, so the last γ_1+1 edges of p_3 are the same as the last γ_1+1 of p_1. Hence $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_k = v_1^{\beta_3} e_1 \dots e_{k-\delta_1-1} e_{k+2l-\gamma_1} \dots e_{k+l+m}.$$

When $\delta_2 \neq l-1$ and $\delta_2 \neq m-1$ then we have two further cases: $\delta_2 = \delta_1$ and $\delta_2 > \delta_1$. If $\delta_2 = \delta_1$ then clearly we also have $\gamma_2 = \gamma_1$ and the result follows trivially. If $\delta_2 > \delta_1$ then the first δ_2+1 edges of p_3 are the same as the first δ_2+1 edges of p_2^{-1} and thus the $(k-\gamma_1)$th to $(k-l+\delta_2+1)$th edges of p_1 are the same as the $(k+2l-\gamma_1)$th to the $(k+l+\delta_2+1)$th of p_3. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3))$ is given by

$$\pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_{k-l\delta_2+1} e_{k+l+\delta_2+2} \dots e_{k+l+m},$$

which is $v_1^{\beta_3} e_1 \dots e_{k-\delta_1-1} e_{k+2l-\gamma_1} \dots e_{k+l+m}$ after relabelling.

Case (2.5.4i) is straightforward; as there is no δ_1 there must also be no δ_2 and thus $\gamma_2 = \gamma_1$ as well. The result then follows trivially.

In case (2.5.5a) we have $\delta_1 = k+l-1$ so we must have $\delta_2 = l-1$ and $\gamma_2 = k-1$. Thus $\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+2l+1} \dots e_{k+l+m}) = v_1^{\beta_3} e_{2k+2l+1} \dots e_{k+l+m}$.

Consider now (2.5.5b). As $\delta_1 = m-1$ we must have either $\delta_2 = m-1$ or $\delta_2 \neq m-1$. If $\delta_2 = m-1$ then γ_2 is not defined and we have

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+1} \dots e_{k+l-m}) = v_1^{\beta_3} e_1 \dots e_{k+l-m}.$$

If $\delta_2 \neq m-1$ then $\delta_2 = l-1$ and the last $m-l$ edges of p_1 are the same as the last $m-l$

edges of p_3^{-1}. Thus $\gamma_2 = m - l - 1$ and we have

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+2l+1} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_{k+l-m}.$$

Case (2.5.5c) is similar to the previous case, but as $\delta_1 = m - 1 = k + l - 1$ we have that $\delta_2 = l - 1$ and $\gamma_2 = k - 1 = m - l - 1$. Thus

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+2l+1} \dots e_{k+l+m}) = v_1^{\beta_3}.$$

Case (2.5.5d) is the final case that is not completely trivial. As δ_1 can be greater than $l - 1$ we have either that $\delta_2 = \delta_1$ or $\delta_1 > l - 1$. In the case where $\delta_2 = \delta_1$ it is clear that γ_2 must not be defined and the result follows. When $\delta_1 > l - 1$ then we have $\delta_2 = l - 1$ and thus $\gamma_2 = \delta_1 - l$ and

$$\pi(w_1 \circ \pi(w_2 \circ w_3)) = \pi(w_1 \circ v_2^{\beta_2} e_{k+2l+1} \dots e_{k+l+m}) = v_1^{\beta_3} e_1 \dots e_{k-l+\delta_1-1} e_{k+l+\delta_1+2} \dots e_{k+l+m}.$$

Lastly we consider (2.5.5e) which is trivial as γ_1 and δ_1 not being defined means γ_2 and δ_2 are not either and we are done. □

We now have enough to show that what we have constructed is an objective partial group.

Proposition 2.5.22. *The pair* $(\mathcal{L}_\Gamma, \Delta_\Gamma)$ *is an objective partial group with product map* Π *and inversion* $\cdot^{-1}$.

Proof. It is easy to see that $\mathcal{L}_\Gamma$ is contained in $\mathbf{D}_{\Delta,\Gamma}$ as $\mathbf{D}_{\Delta,\Gamma}$ contains all walks on Γ and all walks with a vertex at the start of them. It is also clear that if w is in $\mathbf{D}_{\Delta,\Gamma}$ then all subwords of w are in $\mathbf{D}_{\Delta,\Gamma}$; we can remove vertices or edges from the end of a word admitted by the automaton and still obtain a word admitted by the automaton as all states are start and end states.

It is clear by definition of $\mathcal{L}_\Gamma$ that Π is the identity on this set. All walks p of just edges in $\mathcal{L}_\Gamma$ by definition do not double back on themselves, so $\Pi(p) = p$. Moreover if we have a vertex followed by a walk, vp, then we have an odd number of vertices, so $\Pi(vp)$ starts with a vertex. Hence $\Pi(vp) = vp$.

Suppose $u \circ v \circ w$ is a word in $\mathbf{D}_{\Delta,\Gamma}$. By Lemma 2.5.19 the walks in the automaton corresponding to v and $\Pi(v)$ have the same start and end states so if u ends at the same state that v starts at then it ends at the same state that $\Pi(v)$ starts at. Similarly if w starts at the same state that v ends at then it must start at the same state that $\Pi(v)$ ends at. Hence $u \circ \Pi(v) \circ w$ is a word in $\mathbf{D}_{\Delta,\Gamma}$. We then invoke an argument used several times already in this text, namely in Propositions 2.4.3, 2.4.11 and 2.5.5. We can write each word u, v and w as a list of subwords in $\mathcal{L}_n$ of length 1, $u_1 \circ \cdots \circ u_k$, $v_1 \circ \cdots \circ v_l$ and $w_1 \circ \cdots \circ w_m$, for k, l, m in $\mathbb{N}$ the length of u, v and w respectively. The map π is associative on $\mathcal{L}_n$, by Lemma 2.5.21, and we have that Π and π agree on pairs in $\mathcal{L}_\Gamma$ by Lemma 2.5.20. Thus

$$\begin{aligned}
\Pi(u \circ v \circ w) &= \pi(\pi(\cdots \pi(u_1 \circ u_2) \cdots \circ u_k) \circ v_1) \cdots \circ v_l) \circ w_1) \cdots \circ w_m), \\
&= \pi(\pi(\Pi(u_1 \circ \cdots \circ u_k) \circ \Pi(v_1 \circ \cdots \circ v_l)) \circ \Pi(w_1 \circ \cdots \circ w_m)), \\
&= \Pi(u_1 \circ \cdots \circ u_k \circ \Pi(v_1 \circ \cdots \circ v_l) \circ w_1 \circ \cdots \circ w_m), \\
&= \Pi(u \circ \Pi(v) \circ w).
\end{aligned}$$

Hence $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$.

If w is a word admitted by the automaton associated to Γ then, as every path between states can be retraced, it is easy to see that w^{-1} is also admitted by the automaton. As w^{-1} ends at the same state that w starts at it is clear that $w^{-1} \circ w$ is admitted by the automaton and thus is in $\mathbf{D}_{\Delta,\Gamma}$. Certainly $w^{-1} \circ w$ contains an even number of vertices, as both w^{-1} and w contain the same number, so $\Pi(w^{-1} \circ w)$ contains no vertices. Furthermore as the walk on the automaton that corresponds to w^{-1} is by definition w traced in reverse it is clear that the walk corresponding to $w^{-1} \circ w$ doubles back on itself entirely. Thus $\Pi(w^{-1} \circ w) = 1$.

We therefore have $(\mathcal{L}_\Gamma, \Delta_\Gamma)$ is a partial group and as, by our construction, $\mathbf{D}_{\Delta,\Gamma} = \mathbf{D}_\Delta$ it is also objective. Hence $(\mathcal{L}_\Gamma, \Delta_\Gamma)$ is an objective partial group. $\square$

Theorem 2.5.23. *Let Γ be a simply connected graph and let $(\mathcal{L}_\Gamma, \Delta_\Gamma)$ be constructed from Γ. Then $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{L}_\Gamma)$.*

Proof. If ψ is in $\mathrm{Aut}(\Gamma)$ then ψ is a bijection on V and the set of walks of any length on Γ so induces a bijection on $\mathcal{L}_\Gamma$. Furthermore, as it is a bijection on both these sets it induces a bijection on $\mathbf{D}_{\Delta,\Gamma}$. Let $v_1^{\alpha_1}p_1$ and $v_2^{\alpha_2}p_2$ be arbitrary elements of $\mathcal{L}_\Gamma$ such that $v_1^{\alpha_1}p_1 \circ v_2^{\alpha_2}p_2$ is in $\mathbf{D}_{\Delta,\Gamma}$. Then it is easy to see that $\psi(\pi(v_1^{\alpha_1}p_1 \circ v_2^{\alpha_2}p_2)) = \pi(\psi(v_1^{\alpha_1}p_1 \circ v_2^{\alpha_2}p_2))$ and thus ψ must commute with Π on all of $\mathbf{D}_{\Delta,\mathcal{L}}$. Therefore ψ is an automorphism of $(\mathcal{L}_\Gamma, \Delta_\Gamma)$.

If ϕ is in $\mathrm{Aut}(\mathcal{L}_\Gamma)$ then ϕ is a bijection on $\mathcal{L}_\Gamma$ and certainly must be a bijection on the set of all order 2 elements in this set, namely the set $V \cup E$. However, from this set only elements of V can be conjugated by an element other than themselves, they can also be conjugated by edges that connect to them. Thus ϕ is a bijection on V and therefore also on E. Hence ϕ is in $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{L}_\Gamma)$. $\square$

Theorem 2.5.24. *The categories $\mathfrak{O}\mathcal{P}art$ and $\mathcal{O}\mathcal{P}art$ are universal.*

Proof. First note that by Lemma 2.2.24 isomorphisms in both categories are the same. By the infinite version of Frucht's theorem, [40] and [18, Theorem 7], we have that for any group G there exists a simply connected undirected graph Γ such that $\mathrm{Aut}(\Gamma) \cong G$. Theorem 2.5.23 gives us that $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{L}_\Gamma)$ so $\mathrm{Aut}(\mathcal{L}_\Gamma) \cong G$ and as G was an arbitrary group then both categories of objective partial groups are universal. $\square$

2.6 Localities

We now move on to consider localities, in other words objective partial groups where there is a maximal p-subgroup S in Δ such that every p-subgroup of $\mathcal{L}$ is a subgroup of

S, and Δ is closed upon taking overgroups. In the hierarchy of definitions that Chermak gives this is the next step in adding structure from an objective partial group.

2.6.1 Automorphisms of localities

We will show here that the additional structure of a locality already means localities have too much structure to have automorphism groups that are cyclic of odd-order. In order to do this we first recall a basic result of group theory.

Lemma 2.6.1. *If G is a finite group with* $\mathrm{Inn}(G)$ *cyclic then G is abelian.*

Proof. Suppose $\mathrm{Inn}(G) \cong G/\mathrm{Z}(G)$ is cyclic. We can write any g in G as $x^n h$ for $\langle\, x \,\rangle$ a representative of $\mathrm{Inn}(G)$ in G and h a element of $\mathrm{Z}(G)$. Thus

$$g_1 g_2 = x^n h_1 x^m h_2 = x^{n+m} h_1 h_2 = x^m h_2 x^n h_1 = g_2 g_1,$$

for g_1 and g_2 arbitrary elements of G. Hence G is abelian. □

We now prove a couple of useful lemmas about automorphisms of localities.

Lemma 2.6.2. *If $(\mathcal{L}, \Delta, S)$ is a locality then every automorphism in* $\mathrm{Inn}(S)$ *extends uniquely to an automorphism in* $\mathrm{Inn}(\mathcal{L})$*. Furthermore there is a subgroup of* $\mathrm{Inn}(\mathcal{L})$ *isomorphic to* $\mathrm{Inn}(S)$*.*

Proof. Let γ be an element of S such that conjugation by γ is a non-trivial element of $\mathrm{Inn}(S)$. Let $g_1 \dots g_n$ be a word in $\mathbf{D}_\Delta$, so there exists a list of subgroups of S, $X_0, \dots, X_n$, such that $X_i^{g_{i+1}} = X_{i+1}$. As γ is in S there exists a Y_{-1} in Δ such that $X_0^\gamma = Y_{-1}$ and a Y_n in Δ such that $X_n^\gamma = Y_n$. Thus $(g_1 \dots g_n)^\gamma = \gamma^{-1} g_1 \dots g_n \gamma$ is in $\mathbf{D}_\Delta$ by the list of subgroups $Y_{-1}, X_0, \dots, X_n, Y_n$. By the definition of Π we have that

$$\Pi((g_1 \dots g_n)^\gamma) = \Pi(\gamma^{-1} g_1 \dots g_n \gamma) = \Pi(\gamma^{-1} \Pi(g_1 \dots g_n) \gamma),$$

so conjugation by γ commutes with Π. Thus conjugation by γ is a homomorphism of partial groups and as γ was arbitrary so is all of $\mathrm{Inn}(S)$. Conjugation by γ^{-1} is therefore also a homomorphism of partial groups and it is clear that composing the two morphisms in either way gives the identity on $\mathcal{L}$. Thus conjugation by γ is an automorphism of $\mathcal{L}$.

Consider $g_1 \dots g_n$ and $X_0, \dots, X_n$ again and note that for each X_i we have a Y_{i-1} such that $X_i^\gamma = Y_{i-1}$ because γ is in S. Thus $g_1^\gamma \dots g_n^\lambda = \gamma^{-1} g_1 \gamma \dots \gamma^{-1} g_n \gamma$ is in $\mathbf{D}_\Delta$ by $Y_{-1}, X_0, Y_0, \dots, X_n, Y_n$. Furthermore, again by the definition of Π, we have

$$\Pi(\gamma^{-1} g_1 \gamma\gamma^{-1} g_2 \dots \gamma\gamma^{-1} g_n \gamma) = \Pi(\gamma^{-1} g_1 \Pi(\gamma\gamma^{-1}) g_2 \dots \Pi(\gamma\gamma^{-1}) g_n \gamma) = \Pi(\gamma^{-1} g_1 \dots g_n \gamma),$$

hence conjugation by γ is an inner automorphism of $\mathcal{L}$. It is clear that the group structure of $\mathrm{Inn}(S)$ is preserved in extending to $\mathcal{L}$ so there is a subgroup of $\mathrm{Inn}(\mathcal{L})$ isomorphic to $\mathrm{Inn}(S)$. $\square$

A version of this result exists for fusion systems but not in the language of localities. We now have enough to prove the main result of this section.

Proposition 2.6.3. *If $(\mathcal{L}, \Delta, S)$ is a locality then $\mathrm{Aut}(\mathcal{L})$ is not isomorphic to C_n for any $n > 1$ odd.*

Proof. By Lemma 2.2.15, we have that if S abelian then $\mathcal{L} = \mathrm{N}_{\mathcal{L}}(S)$ and $\mathcal{L}$ is a group. Thus, by Theorem 2.3.4, $\mathrm{Aut}(\mathcal{L})$ cannot be a non-trivial odd-order cyclic group. We therefore assume S is not abelian. But in this case $\mathrm{Inn}(S)$ is non-trivial. Furthermore, by Lemma 2.6.1, S being not abelian means $\mathrm{Inn}(S)$ is not cyclic. Thus, by Lemma 2.6.2, we have $\mathrm{Inn}(S)$ is a subgroup of $\mathrm{Inn}(\mathcal{L})$ which in turn is a subgroup of $\mathrm{Aut}(\mathcal{L})$, by Proposition 2.2.29(ii). Thus $\mathrm{Aut}(\mathcal{L})$ cannot be cyclic. We therefore have that given any locality $\mathrm{Aut}(\mathcal{L})$ is not isomorphic to C_n for any $n > 1$ odd. $\square$

Corollary 2.6.4. *The category $\mathcal{F}in\mathcal{L}oc$ is not finite universal.*

Proof. By Proposition 2.6.3, we have that there does not exist a locality with automorphism group C_n for $n > 1$ odd. Thus the category is not finitely universal. $\square$

Note that this result also clearly implies that one cannot construct a proper locality with automorphism group C_n, for $n > 1$ odd. Thus, in Chermak's structural hierarchy, localities are the natural place to stop when looking for universality.

APPENDIX A

CODE

A.1 Code pertaining to 2.5.1

Each of the following arrays of strings is an element of $\mathcal{L}$ where the first string is the element, the second is the set of potential start states of the element in Δ, when considered as a walk on the automaton in Figure 2.1 and the third is the set of potential end states of the element in Δ, when considered as a walk on the automaton in Figure 2.1. Note that as elements of different index cannot be multiplied together, with the exception of x_{i+1} and elements of index i, we consider some fixed index i and relabel x_{i+1} to y.

```
A:={["t","t","t"],["u","u","u"],["v","v", "v"],["w","w","w"],
["x","tu","ut"],["y","uv","vu"],["z","vw","wv"};
B:={["tx","t","u"],["ux","u","t"],["uy","u","v"],["vy","v","u"],
["vz","v","w"],["wz","w","v"],["xy","t","v"],["txy","t","v"],
["yx","v","t"],["vyx","v","t"],["yz","u","w"],["uyz","u","w"],
["zy","w","u"],["wzy","w","u"],["xyz","t","w"],["txyz","t","w"],
["zyx","w","t"],["wzyx","w","t"]};
```

The following function concatenates two elements if they can be multiplied and removes any double letters that appear as a result of the concatenation.

```
function Prod(X,Y)
Ans:=["","",""];
```

```
for i in {1..#X[3]} do
for j in {1..#Y[2]} do
if (X[3][i] eq Y[2][j]) then
while (Min(#X[1],#Y[1]) gt 0 and X[1][#X[1]] eq Y[1][1]) do
X[1]:=Prune(X[1]);
if (1 eq #Y[1]) then Y[1]:="";
else Y[1]:=Substring(Y[1], 2, #Y[1]-1);
end if;
end while;
Ans[1]:= X[1] cat Y[1];
Ans[2]:=Ans[2] cat X[2][i];
Ans[3]:=Ans[3] cat Y[3][j];
end if;
end for;
end for;
if (#Ans[1] gt 0) then return Ans;
else return 0;
end if;
end function;
```

The following function checks to see if the first two letters can be swapped and swaps if they can, and then simplifies.

```
function Flip(X)
FLIP:=[["xt","ux"],["xu","tx"],["yu","vy"],["yv","uy"],
["zv","wz"],["zw","vz"]];
for i in [1..#FLIP] do
if (#X[1] gt 1 and Substring(X[1], 1, 2) eq FLIP[i][1]) then
if (#X[1] eq 2) then X[1]:=FLIP[i][2];
else
Y:=FLIP[i][2];
```

```
Z:=Substring(X[1], 3, #X[1]-2);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
end for;
return X;
end function;
```

The following function checks to see if the nth and $n+1$th letters can be swapped and swaps if they can, and then simplifies.

```
function Flipn(X,n)
c:=0;
FLIP:=[["xt","ux"],["xu","tx"],["yu","vy"],["yv","uy"],
["zv","wz"],["zw","vz"]];
for i in [1..#FLIP] do
if (#X[1] gt n and Substring(X[1], n, 2) eq FLIP[i][1]) then
if (#X[1] eq n+1) then X[1]:=Substring(X[1], 1, n-1) cat FLIP[i][2];
else X[1]:=Substring(X[1], 1, n-1) cat FLIP[i][2] cat
Substring(X[1], n+2, #X[1]-1-n);
end if;
Y:= Substring(X[1], 1, n-1);
Z:=Substring(X[1], n, #X[1]-n+1);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
c:=c+1;
```

```
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
if (c lt 2 and #Z gt 2-c) then
A:=Y cat Substring(Z, 1, 2-c);
B:=Substring(Z, 3-c, #Z-2+c);
while (Min(#A,#B) gt 0 and A[#A] eq B[1]) do
A:=Prune(A);
if (1 eq #B) then B:="";
else B:=Substring(B, 2, #B-1);
end if;
end while;
X[1]:= A cat B;
return Append(X,IntegerToString(c));
end if;
end if;
end for;
return Append(X,IntegerToString(c));
end function;
```

The following function moves along a word and applies Flipn each time until it swaps letters and then moves back down applying Flipn.

```
function Flips(X)
Y:=Flip(X);
i:=2;
while i lt #Y[1] do
e:=0;
```

```
if (i gt 1 and not Flipn(Y,i)[1] eq Y[1]) then
Z:=Y;
Y:=Flipn(Z,i)[[1,2,3]];
c:=StringToInteger(Flipn(Z,i)[4]);
j:=i;
d:=c;
e:=c;
while j-d gt 2 do
Z:=Y;
Y:=Flipn(Z,j-1-d)[[1,2,3]];
d:=StringToInteger(Flipn(Z,j-1-d)[4]);
e:=e+d;
j:=j-1-d;
end while;
Y:=Flip(Y);
end if;
i:=i+1-2*e;
end while;
return Y;
end function;
```

The following multiplies all elements of SET together that can be and simplifies, then checks if the resulting element is in SET and only prints it if not.

```
SET:=A join B;
a:=0;
ANS:={};
for X in SET do
for Y in SET do
if (Type(Prod(X,Y)) eq SeqEnum and not Prod(X,Y) in SET) then
if (#Prod(X,Y)[1] gt 0 and not Prod(X,Y) in ANS then ANS:=ANS join
```

```
{Prod(X,Y)};
a:=a+1;
end if;
end if;
end for;
end for;
a;
b:=0;
ANS2:={};
for i in [1..2] do
ANS2:=ANS;
ANS:={};
for X in ANS2 do
if (#Flips(X)[1] gt 0 and not Flips(X) in ANS2 then ANS:= ANS join
{Flips(X)};
end if;
end for;
end for;
ANS;
```

The following checks associativity on all triples and only prints triples that are not associative.

```
ANS:={};
for X in SET do
for Y in SET do
for Z in SET do
A:=0;
B:=0;
C:=0;
D:=0;
```

```
if (Type(Prod(X,Y)) eq SeqEnum  and Type(Prod(Y,Z)) eq SeqEnum) then
if (#Prod(X,Y)[1] gt 0  or #Prod(Y,Z)[1] gt 0) then
A:=Prod(X,Y);
B:=Prod(Y,Z);
for i in [1..2] do
if (#A[1] gt 2 and Flips(A) ne A) then
A:=Flips(A);
end if;
end for;
for i in [1..2] do
if (#B[1] gt 2 and Flips(B) ne B) then
B:=Flips(B);
end if;
end for;
if (Type(Prod(A,Z)) eq SeqEnum and Type(Prod(X,B)) eq SeqEnum) then
if (#Flip(Prod(A,Z))[1] gt 0  or #Flip(Prod(X,B))[1] gt 0) then
C:=Prod(A,Z);
D:=Prod(X,B);
for i in [1..2] do
if (#C[1] gt 1 and Flips(C) ne C) then
C:=Flips(C);
end if;
end for;
for i in [1..2] do
if (#D[1] gt 1 and Flips(D) ne D) then
D:=Flips(D);
end if;
end for;
if (C[1] ne D[1]) then
ANS:= ANS join {[C,D]};
```

```
end if;
end if;
end if;
end if;
end if;
end for;
end for;
end for;
ANS;
```

A.2 Code pertaining to 2.5.2

Sets A, B, C, D and E for $q = 3$. Each of the following arrays of strings is an element where the first string is the element, the second is the set of potential start states of the element in Δ, when considered as a walk on the automaton in Figure 2.1 and the third is the set of potential end states of the element in Δ, when considered as a walk on the automaton in Figure 2.3. Again we consider with respect to a fixed index i and relabel x_{i+1} to y.

```
A:={["t","tu","tu"],["u","tuv","tuv"],["v","uvw", "uvw"],
["w","vw","vw"],["x","tu","ut"],["y","uv","vu"],["z","vw","wv"],
["tu","tu","tu"],["uv","uv","uv"],["vw","vw","vw"],
["tux","tu","ut"],["uvy","uv","vu"],["vwz","vw","wv"],
["xvx","t","t"],["zuz","w","w"],["txvx","t","t"],["wzuz","w","w"],
["tvt","u","u"],["uwu","v","v"],,["tuvt","u","u"],["uvwu","v","v"],
["uxvtx","t","t"],["zuwuz","w","w"],["tuxvtx","t","t"],
["wzuwuz","w","w"]};
B:={["tx","tu","ut"],["ux","tu","ut"],["uy","uv","vu"],
["vy","uv","vu"],["vz","vw","wv"],["wz","vw","wv"]};
```

```
C:={["tv","u","u"],["vt","u","u"],["uw","v","v"],["wu","v","v"],
["xvtx","t","t"],["uxvx","t","t"],["zuwz","w","w"],
["vzuz","w","w"]};
D:={["tuv","u","u"],["uvt","u","u"],["uvw","v","v"],
["vwu","v","v"],["txvtx","t","t"],["tuxvx","t","t"],
["wzuwz","w","w"],["vwzuz","w","w"]};
E:={["vx","u","t"],["tvx","u","t"],["vtx","u","t"],
["tvtx","u","t"],["uvx","u","t"],["tuvx","u","t"],["uvtx","u","t"],
["tuvtx","u","t"],["xv","t","u"],["uxv","t","u"],["xvt","t","u"],
["uxvt","t","u"],["txv","t","u"],["tuxv","t","u"],["txvt","t","u"],
["tuxvt","t","u"],["ty","u","v"],["tvy","u","v"],["vty","u","v"],
["tvty","u","v"],["tuy","u","v"],["tuvy","u","v"],["uvty","u","v"],
["tuvty","u","v"],["wy","v","u"],["uwy","v","u"],["wuy","v","u"],
["uwuy","v","u"],["vwy","v","u"],["uvwy","v","u"],["vwuy","v","u"],
["uvwuy","v","u"],["uz","v","w"],["uwz","v","w"],["wuz","v","w"],
["uwuz","v","w"],["uvz","v","w"],["uvwz","v","w"],["vwuz","v","w"],
["uvwuz","v","w"],["zu","w","v"],["zuw","w","v"],["vzu","w","v"],
["zuwu","w","v"],["wzu","w","v"],["wzuw","w","v"],["vwzu","w","v"],
["wzuwu","w","v"],["xy","t","v"],["txy","t","v"],["uxy","t","v"],
["tuxy","t","v"],["xvy","t","v"],["txvy","t","v"],["uxvy","t","v"],
["xvty","t","v"],["tuxvy","t","v"],["txvty","t","v"],
["uxvty","t","v"],["tuxvty","t","v"],["yx","v","t"],
["vyx","v","t"],["wyx","v","t"],["vwyx","v","t"],["uyx","v","t"],
["uvyx","v","t"],["uwyx","v","t"],["wuyx","v","t"],
["uvwyx","v","t"],["vwuyx","v","t"],["uwuyx","v","t"],
["uvwuyx","v","t"],["yz","u","w"],["tyz","u","w"],["uyz","u","w"],
["vyz","u","w"],["tuyz","u","w"],["uvyz","u","w"],["tvyz","u","w"],
["vtyz","u","w"],["tuvyz","u","w"],["uvtyz","u","w"],
["tvtyz","u","w"],["tuvtyz","u","w"],["zy","w","u"],
["vzy","w","u"],["wzy","w","u"],["zuy","w","u"],["vwzy","w","u"],
```

```
["wzuy","w","u"],["zuwy","w","u"],["vzuy","w","u"],
["wzuwy","w","u"],["vwzuy","w","u"],["zuwuy","w","u"],
["wzuwuy","w","u"],["xyz","t","w"],["txyz","t","w"],
["uxyz","t","w"],["tuxyz","t","w"],["xvyz","t","w"],
["txvyz","t","w"],["uxvyz","t","w"],["xvtyz","t","w"],
["tuxvyz","t","w"],["txvtyz","t","w"],["uxvtyz","t","w"],
["tuxvtyz","t","w"],["zyx","w","t"],["wzyx","w","t"],
["vzyx","w","t"],["vwzyx","w","t"],["zuyx","w","t"],
["wzuyx","w","t"],["zuwyx","w","t"],["vzuyx","w","t"],
["wzuwyx","w","t"],["vwzuyx","w","t"],["zuwuyx","w","t"],
["wzuwuyx","w","t"]}
```

The following function concatenates two elements if they can be multiplied and removes any double letters that appear as a result of the concatenation. This function is also used in the $q = 5$ case.

```
function Prod(X,Y)
Ans:=["","",""];
for i in {1..#X[3]} do
for j in {1..#Y[2]} do
if (X[3][i] eq Y[2][j]) then
while (Min(#X[1],#Y[1]) gt 0 and X[1][#X[1]] eq Y[1][1]) do
X[1]:=Prune(X[1]);
if (1 eq #Y[1]) then Y[1]:="";
else Y[1]:=Substring(Y[1], 2, #Y[1]-1);
end if;
end while;
Ans[1]:= X[1] cat Y[1];
Ans[2]:=Ans[2] cat X[2][i];
Ans[3]:=Ans[3] cat Y[3][j];
end if;
```

```
end for;
end for;
if (#Ans[1] gt 0) then return Ans;
else return 0;
end if;
end function;
```

The following function checks to see if the first two letters can be swapped and swaps if they can, and then simplifies. It then checks if the first three letters can be substituted and if they can it substitutes and simplifies and repeated letters. It then checks for two remaining subwords and replaces these if they occur.

```
function Flip(X)
FLIP:=[["xt","ux"],["xu","tx"],["yt","wy"],["yu","vy"],["yv","uy"],
["yw","ty"],["zv","wz"],["zw","vz"],["ut","tu"],["vu","uv"],
["wv","vw"]];
FLOP:=[["zuv","wzu"],["xvu","txv"],["vtv","tvt"],["wuw","uwu"],
["vtu","uvt"],["wuv","vwu"]];
for i in [1..#FLIP] do
if (#X[1] gt 1 and Substring(X[1], 1, 2) eq FLIP[i][1]) then
if (#X[1] eq 2) then X[1]:=FLIP[i][2];
else Y:=FLIP[i][2];
Z:=Substring(X[1], 3, #X[1]-2);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
```

```
end if;
end for;
for i in [1..#FLOP] do
if (#X[1] gt 2 and Substring(X[1], 1, 3) eq FLOP[i][1]) then
if (#X[1] eq 3) then X[1]:=FLOP[i][2];
else X[1]:=FLOP[i][2] cat Substring(X[1], 4, #X[1]-3);
if (X[1][3] eq X[1][4]) then
if (#X[1] eq 4) then X[1]:=Substring(X[1], 1, 2);
else X[1]:= Substring(X[1], 1, 2) cat Substring(X[1], 5, #X[1]-4);
end if;
end if;
if (#X[1] gt 2 and X[1][2] eq X[1][3]) then
if (#X[1] eq 3) then X[1]:=X[1][1];
else X[1]:= X[1][1] cat Substring(X[1], 4, #X[1]-3);
end if;
end if;
if (#X[1] gt 1 and X[1][1] eq X[1][2]) then
if (#X[1] eq 2) then X[1]:="";
else X[1]:=Substring(X[1], 3, #X[1]-2);
end if;
end if;
end if;
end if;
end for;
if (#X[1] gt 3 and Substring(X[1], 1, 4) eq "vzuw") then
if (#X[1] eq 4) then X[1]:="zuwu";
else Y:="zuwu";
Z:=Substring(X[1], 5, #X[1]-4);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
```

```
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
if (#X[1] gt 4 and Substring(X[1], 1, 5) eq "vwzuw") then
if (#X[1] eq 5) then X[1]:="wzuwu";
else Y:="wzuwu";
Z:=Substring(X[1], 6, #X[1]-5);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
return X;
end function;
```

The following function checks to see if the nth and $n+1$th letters can be swapped and swaps if they can and simplifies. It then checks if the nth to $n+2$th letters can be substituted and if they can it substitutes and simplifies any repeated letters.

```
function Flipn(X,n)
c:=0;
FLIP:=[["xt","ux"],["xu","tx"],["yt","wy"],["yu","vy"],["yv","uy"],
["yw","ty"],["zv","wz"],["zw","vz"],["ut","tu"],["vu","uv"],
```

```
["wv","vw"]];
FLOP:=[["zuv","wzu"],["xvu","txv"],["vtv","tvt"],["wuw","uwu"],
["vtu","uvt"],["wuv","vwu"]];
for i in [1..#FLIP] do
if (#X[1] gt n and Substring(X[1], n, 2) eq FLIP[i][1]) then
if (#X[1] eq n+1) then X[1]:=Substring(X[1], 1, n-1) cat FLIP[i][2];
else X[1]:=Substring(X[1], 1, n-1) cat FLIP[i][2] cat
Substring(X[1],n+2, #X[1]-1-n);
end if;
Y:= Substring(X[1], 1, n-1);
Z:=Substring(X[1], n, #X[1]-n+1);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
c:=c+1;
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
if (c lt 2 and #Z gt 2-c) then
A:=Y cat Substring(Z, 1, 2-c);
B:=Substring(Z, 3-c, #Z-2+c);
while (Min(#A,#B) gt 0 and A[#A] eq B[1]) do
A:=Prune(A);
if (1 eq #B) then B:="";
else B:=Substring(B, 2, #B-1);
end if;
end while;
X[1]:= A cat B;
return Append(X,IntegerToString(c));
```

```
end if;
end if;
end for;
for i in [1..#FLOP] do
if (#X[1] gt n+1 and Substring(X[1], n, 3) eq FLOP[i][1]) then
if (#X[1] eq n+2) then X[1]:=Substring(X[1], 1, n-1) cat FLOP[i][2];
else X[1]:=Substring(X[1], 1, n-1) cat FLOP[i][2] cat
Substring(X[1], n+3, #X[1]-2-n);
end if;
Y:= Substring(X[1], 1, n-1);
Z:=Substring(X[1], n, #X[1]-n+1);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
c:=c+1;
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
if (c lt 3 and #Z gt 3-c) then
A:=Y cat Substring(Z, 1, 3-c);
B:=Substring(Z, 4-c, #Z-3+c);
while (Min(#A,#B) gt 0 and A[#A] eq B[1]) do
A:=Prune(A);
if (1 eq #B) then B:="";
else B:=Substring(B, 2, #B-1);
end if;
end while;
X[1]:= A cat B;
return Append(X,IntegerToString(c));
```

```
end if;
end if;
end for;
return Append(X,IntegerToString(c));
end function;
```

The following function moves along a word and applies Flipn each time until it does flip and then moves back down applying Flipn.

```
function Flips(X)
Y:=Flip(X);
i:=2;
while i lt #Y[1] do
e:=0;
if (i gt 1 and not Flipn(Y,i)[1] eq Y[1]) then
Z:=Y;
Y:=Flipn(Z,i)[[1,2,3]];
c:=StringToInteger(Flipn(Z,i)[4]);
j:=i;
d:=c;
e:=c;
while j-d gt 2 do
Z:=Y;
Y:=Flipn(Z,j-1-d)[[1,2,3]];
d:=StringToInteger(Flipn(Z,j-1-d)[4]);
e:=e+d;
j:=j-1-d;
end while;
Y:=Flip(Y);
end if;
i:=i+1-2*e;
```

```
end while;
return Y;
end function;
```

The following multiplies all elements of SET together that can be and simplifies, then checks if the resulting element is in SET and only prints it if not. This is also the same for the $q = 5$ case using the appropriate functions.

```
SET:=A join B join C join D join E;
a:=0;
ANS:={};
for X in SET do
for Y in SET do
if (Type(Prod(X,Y)) eq SeqEnum and not Prod(X,Y) in SET) then
if (#Prod(X,Y)[1] gt 0 and not Prod(X,Y) in ANS then ANS:=ANS join
{Prod(X,Y)};
a:=a+1;
end if;
end if;
end for;
end for;
a;
b:=0;
ANS2:={};
for i in [1..2] do
ANS2:=ANS;
ANS:={};
for X in ANS2 do
if (#Flips(X)[1] gt 0 and not Flips(X) in ANS2 then ANS:= ANS join
{Flips(X)};
end if;
```

```
end for;
end for;
ANS;
```

The following checks associativity on all triples and only prints triples that are not associative. This is also the same for the $q = 5$ case using the appropriate functions.

```
ANS:={};
for X in SET do
for Y in SET do
for Z in SET do
A:=0;
B:=0;
C:=0;
D:=0;
if (Type(Prod(X,Y)) eq SeqEnum  and Type(Prod(Y,Z)) eq SeqEnum) then
if (#Prod(X,Y)[1] gt 0  or #Prod(Y,Z)[1] gt 0) then
A:=Prod(X,Y);
B:=Prod(Y,Z);
for i in [1..2] do
if (#A[1] gt 2 and Flips(A) ne A) then
A:=Flips(A);
end if;
end for;
for i in [1..2] do
if (#B[1] gt 2 and Flips(B) ne B) then
B:=Flips(B);
end if;
end for;
if (Type(Prod(A,Z)) eq SeqEnum and Type(Prod(X,B)) eq SeqEnum) then
if (#Flip(Prod(A,Z))[1] gt 0  or #Flip(Prod(X,B))[1] gt 0) then
```

```
C:=Prod(A,Z);
D:=Prod(X,B);
for i in [1..2] do
if (#C[1] gt 1 and Flips(C) ne C) then
C:=Flips(C);
end if;
end for;
for i in [1..2] do
if (#D[1] gt 1 and Flips(D) ne D) then
D:=Flips(D);
end if;
end for;
if (C[1] ne D[1]) then
ANS:= ANS join {[C,D]};
end if;
end if;
end if;
end if;
end if;
end for;
end for;
end for;
ANS;
```

Sets A, B, C, D and E for $q = 5$. What follows this is a version of each of the functions given above but for $q = 5$. Each function therefore includes a further set of subwords that can be substituted.

```
A:={["t","tu","tu"],["u","tuv","tuv"],["v","uvw", "uvw"],
["w","vw","vw"],["x","tu","ut"],["y","uv","vu"],["z","vw","wv"],
["tu","tu","tu"],["uv","uv","uv"],["vw","vw","vw"],
```

```
["tux","tu","ut"],["uvy","uv","vu"],["vwz","vw","wv"],
["xvx","t","t"],["zuz","w","w"],["txvx","t","t"],["wzuz","w","w"],
["tvt","u","u"],["tvtvt","u","u"],["vtv","u","u"],["uwu","v","v"],
["uwuwu","v","v"],["wuw","v","v"],["tuvt","u","u"],
["tuvtvt","u","u"],["uvtv","u","u"],["uvwu","v","v"],
["uvwuwu","v","v"],["vwuw","v","v"],["uxvtx","t","t"],
["uxvtvtx","t","t"],["xvtvx","t","t"],["zuwuz","w","w"],
["zuwuwuz","w","w"],["vzuwz","w","w"],["tuxvtx","t","t"],
["tuxvtvtx","t","t"],["txvtvx","t","t"],["wzuwuz","w","w"],
["wzuwuuwz","w","w"],["vwzuwz","w","w"]};
B:={["tx","tu","ut"],["ux","tu","ut"],["uy","uv","vu"],
["vy","uv","vu"],["vz","vw","wv"],["wz","vw","wv"]};
C:={["tv","u","u"],["tvtv","u","u"],["vtvt","u","u"],["vt","u","u"],
["uw","v","v"],["uwuw","v","v"],["wuwu","v","v"],["wu","v","v"],
["xvtx","t","t"],["xvtvtx","t","t"],["uxvtvx","t","t"],
["uxvx","t","t"],["zuwz","w","w"],["zuwuwz","w","w"],
["vzuwuz","w","w"],["vzuz","w","w"]};
D:={["tuv","u","u"],["tuvtv","u","u"],["uvtvt","u","u"],
["uvt","u","u"],["uvw","v","v"],["uvwuw","v","v"],["vwuwu","v","v"],
["vwu","v","v"],["txvtx","t","t"],["txvtvtx","t","t"],
["tuxvtvx","t","t"],["tuxvx","t","t"],["wzuwz","w","w"],
["wzuwuwz","w","w"],["vwzuwuz","w","w"],["vwzuz","w","w"]};
E:={["vx","u","t"],["tvx","u","t"],["tvtvx","u","t"],
["vtvtx","u","t"],["vtx","u","t"],["tvtx","u","t"],
["tvtvtx","u","t"],["vtvx","u","t"],["uvx","u","t"],
["tuvx","u","t"],["tuvtvx","u","t"],["uvtvtx","u","t"],
["uvtx","u","t"],["tuvtx","u","t"],["tuvtvtx","u","t"],
["uvtvx","u","t"],["xv","t","u"],["uxv","t","u"],["uxvtv","t","u"],
["xvtvt","t","u"],["xvt","t","u"],["uxvt","t","u"],
["uxvtvt","t","u"],["xvtv","t","u"],["txv","t","u"],
```

```
["tuxv","t","u"],["tuxvtv","t","u"],["txvtvt","t","u"],
["txvt","t","u"],["tuxvt","t","u"],["tuxvtvt","t","u"],
["txvtv","t","u"],["ty","u","v"],["tvy","u","v"],["tvtvy","u","v"],
["vtvty","u","v"],["vty","u","v"],["tvty","u","v"],
["tvtvty","u","v"],["vtvy","u","v"],["tuy","u","v"],
["tuvy","u","v"],["tuvtvy","u","v"],["uvtvty","u","v"],
["uvty","u","v"],["tuvty","u","v"],["tuvtvty","u","v"],
["uvtvy","u","v"],["wy","v","u"],["uwy","v","u"],["uwuwy","v","u"],
["wuwuy","v","u"],["wuy","v","u"],["uwuy","v","u"],
["uwuwuy","v","u"],["wuwy","v","u"],["vwy","v","u"],
["uvwy","v","u"],["uvwuwy","v","u"],["vwuwuy","v","u"],
["vwuy","v","u"],["uvwuy","v","u"],["uvwuwuy","v","u"],
["vwuwy","v","u"],["uz","v","w"],["uwz","v","w"],["uwuwz","v","w"],
["wuwuz","v","w"],["wuz","v","w"],["uwuz","v","w"],
["uwuwuz","v","w"],["wuwz","v","w"],["uvz","v","w"],
["uvwz","v","w"],["uvwuwz","v","w"],["vwuwuz","v","w"],
["vwuz","v","w"],["uvwuz","v","w"],["uvwuwuz","v","w"],
["vwuwz","v","w"],["zu","w","v"],["zuw","w","v"],["zuwuw","w","v"],
["vzuwu","w","v"],["vzu","w","v"],["zuwu","w","v"],
["zuwuwu","w","v"],["vzuw","w","v"],["wzu","w","v"],
["wzuw","w","v"],["wzuwuw","w","v"],["vwzuwu","w","v"],
["vwzu","w","v"],["wzuwu","w","v"],["wzuwuwu","w","v"],
["vwzuw","w","v"],["xy","t","v"],["txy","t","v"],["uxy","t","v"],
["tuxy","t","v"],["xvy","t","v"],["txvy","t","v"],["uxvy","t","v"],
["uxvtvy","t","v"],["xvtvty","t","v"],["xvty","t","v"],
["tuxvy","t","v"],["tuxvtvy","t","v"],["txvtvty","t","v"],
["txvty","t","v"],["uxvty","t","v"],["uxvtvty","t","v"],
["xvtvy","t","v"],["tuxvty","t","v"],["tuxvtvty","t","v"],
["txvtvy","t","v"],["yx","v","t"],["vyx","v","t"],["wyx","v","t"],
["vwyx","v","t"],["uyx","v","t"],["uvyx","v","t"],["uwyx","v","t"],
```

```
["uwuwyx","v","t"],["wuwuyx","v","t"],["wuyx","v","t"],
["uvwyx","v","t"],["uvwuwyx","v","t"],["vwuwuyx","v","t"],
["vwuyx","v","t"],["uwuyx","v","t"],["uwuwuyx","v","t"],
["wuwyx","v","t"],["uvwuyx","v","t"],["uvwuwuyx","v","t"],
["vwuwyx","v","t"],["yz","u","w"],["tyz","u","w"],["uyz","u","w"],
["vyz","u","w"],["tuyz","u","w"],["uvyz","u","w"],["tvyz","u","w"],
["tvtvyz","u","w"],["vtvtyz","u","w"],["vtyz","u","w"],
["tuvyz","u","w"],["tuvtvyz","u","w"],["uvtvtyz","u","w"],
["uvtyz","u","w"],["tvtyz","u","w"],["tvtvtyz","u","w"],
["vtvyz","u","w"],["tuvtyz","u","w"],["tuvtvtyz","u","w"],
["uvtvyz","u","w"],["zy","w","u"],["vzy","w","u"],["wzy","w","u"],
["zuy","w","u"],["vwzy","w","u"],["wzuy","w","u"],["zuwy","w","u"],
["zuwuwy","w","u"],["vzuwuy","w","u"],["vzuy","w","u"],
["wzuwy","w","u"],["wzuwuwy","w","u"],["vwzuwuy","w","u"],
["vwzuy","w","u"],["zuwuy","w","u"],["zuwuwuy","w","u"],
["vzuwy","w","u"],["wzuwuy","w","u"],["wzuwuwuy","w","u"],
["vwzuwy","w","u"],["xyz","t","w"],["txyz","t","w"],
["uxyz","t","w"],["tuxyz","t","w"],["xvyz","t","w"],
["txvyz","t","w"],["uxvyz","t","w"],["uxvtvyz","t","w"],
["xvtvtyz","t","w"],["xvtyz","t","w"],["tuxvyz","t","w"],
["tuxvtvyz","t","w"],["txvtvtyz","t","w"],["txvtyz","t","w"],
["uxvtyz","t","w"],["uxvtvtyz","t","w"],["xvtvyz","t","w"],
["tuxvtyz","t","w"],["tuxvtvtyz","t","w"],["txvtvyz","t","w"],
["zyx","w","t"],["wzyx","w","t"],["vzyx","w","t"],
["vwzyx","w","t"],["zuyx","w","t"],["wzuyx","w","t"],
["zuwyx","w","t"],["zuwuwyx","w","t"],["vzuwuyx","w","t"],
["vzuyx","w","t"],["wzuwyx","w","t"],["wzuwuwyx","w","t"],
["vwzuwuyx","w","t"],["vwzuyx","w","t"],["zuwuyx","w","t"],
["zuwuwuyx","w","t"],["vzuwyx","w","t"],["wzuwuyx","w","t"],
["wzuwuwuyx","w","t"],["vwzuwyx","w","t"]}
```

```
SET:=A join B join C join D join E;
```

```
function Flip(X)
FLIP:=[["xt","ux"],["xu","tx"],["yt","wy"],["yu","vy"],["yv","uy"],
["yw","ty"],["zv","wz"],["zw","vz"],["ut","tu"],["vu","uv"],
["wv","vw"]];
FLOP:=[["zuv","wzu"],["xvu","txv"],["vtu","uvt"],["wuv","vwu"]];
FLAP:=[["wuwuw","uwuwu"],["vtvtv","tvtvt"]];
for i in [1..#FLIP] do
if (#X[1] gt 1 and Substring(X[1], 1, 2) eq FLIP[i][1]) then
if (#X[1] eq 2) then X[1]:=FLIP[i][2];
else
Y:=FLIP[i][2];
Z:=Substring(X[1], 3, #X[1]-2);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
end for;
for i in [1..#FLOP] do
if (#X[1] gt 2 and Substring(X[1], 1, 3) eq FLOP[i][1]) then
if (#X[1] eq 3) then X[1]:=FLOP[i][2];
else
Y:=FLOP[i][2];
Z:=Substring(X[1], 4, #X[1]-3);
```

```
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
end for;
for i in [1..#FLAP] do
if (#X[1] gt 4 and Substring(X[1], 1, 5) eq FLAP[i][1]) then
if (#X[1] eq 5) then X[1]:=FLAP[i][2];
else
Y:=FLAP[i][2];
Z:=Substring(X[1], 6, #X[1]-5);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
end for;
for i in [1..#FLAP] do
if (#X[1] gt 4 and Substring(X[1], 1, 5) eq FLAP[i][1]) then
if (#X[1] eq 5) then X[1]:=FLAP[i][2];
else
```

```
Y:=FLAP[i][2];
Z:=Substring(X[1], 6, #X[1]-5);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
end for;
if (#X[1] gt 5 and Substring(X[1], 1, 6) eq "vzuwuw") then
if (#X[1] eq 6) then X[1]:="zuwuwu";
else Y:="zuwuwu";
Z:=Substring(X[1], 7, #X[1]-6);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
if (#X[1] gt 6 and Substring(X[1], 1, 7) eq "vwzuwuw") then
if (#X[1] eq 7) then X[1]:="wzuwuwu";
else Y:="wzuwuwu";
Z:=Substring(X[1], 8, #X[1]-7);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
```

```
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
end if;
end if;
return X;
end function;
```

```
function Flipn(X,n)
c:=0;
FLIP:=[["xt","ux"],["xu","tx"],["yt","wy"],["yu","vy"],["yv","uy"],
["yw","ty"],["zv","wz"],["zw","vz"],["ut","tu"],["vu","uv"],
["wv","vw"]];
FLOP:=[["zuv","wzu"],["xvu","txv"],["vtu","uvt"],["wuv","vwu"]];
FLAP:=[["wuwuw","uwuwu"],["vtvtv","tvtvt"]];
for i in [1..#FLIP] do
if (#X[1] gt n and Substring(X[1], n, 2) eq FLIP[i][1]) then
if (#X[1] eq n+1) then X[1]:=Substring(X[1], 1, n-1) cat FLIP[i][2];
else X[1]:=Substring(X[1], 1, n-1) cat FLIP[i][2] cat
Substring(X[1], n+2, #X[1]-1-n);
end if;
Y:= Substring(X[1], 1, n-1);
Z:=Substring(X[1], n, #X[1]-n+1);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
c:=c+1;
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
```

```
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
if (c lt 2 and #Z gt 2-c) then
A:=Y cat Substring(Z, 1, 2-c);
B:=Substring(Z, 3-c, #Z-2+c);
while (Min(#A,#B) gt 0 and A[#A] eq B[1]) do
A:=Prune(A);
if (1 eq #B) then B:="";
else B:=Substring(B, 2, #B-1);
end if;
end while;
X[1]:= A cat B;
return Append(X,IntegerToString(c));
end if;
end if;
end for;
for i in [1..#FLOP] do
if (#X[1] gt n+1 and Substring(X[1], n, 3) eq FLOP[i][1]) then
if (#X[1] eq n+2) then X[1]:=Substring(X[1], 1, n-1) cat FLOP[i][2];
else X[1]:=Substring(X[1], 1, n-1) cat FLOP[i][2] cat
Substring(X[1], n+3, #X[1]-2-n);
end if;
Y:= Substring(X[1], 1, n-1);
Z:=Substring(X[1], n, #X[1]-n+1);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
c:=c+1;
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
```

```
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
if (c lt 3 and #Z gt 3-c) then
A:=Y cat Substring(Z, 1, 3-c);
B:=Substring(Z, 4-c, #Z-3+c);
while (Min(#A,#B) gt 0 and A[#A] eq B[1]) do
A:=Prune(A);
if (1 eq #B) then B:="";
else B:=Substring(B, 2, #B-1);
end if;
end while;
X[1]:= A cat B;
return Append(X,IntegerToString(c));
end if;
end if;
end for;
for i in [1..#FLAP] do
if (#X[1] gt n+3 and Substring(X[1], n, 5) eq FLAP[i][1]) then
if (#X[1] eq n+4) then X[1]:=Substring(X[1], 1, n-1) cat FLAP[i][2];
else X[1]:=Substring(X[1], 1, n-1) cat FLAP[i][2] cat
Substring(X[1], n+5, #X[1]-4-n);
end if;
Y:= Substring(X[1], 1, n-1);
Z:=Substring(X[1], n, #X[1]-n+3);
while (Min(#Y,#Z) gt 0 and Y[#Y] eq Z[1]) do
c:=c+1;
Y:=Prune(Y);
if (1 eq #Z) then Z:="";
```

```
else Z:=Substring(Z, 2, #Z-1);
end if;
end while;
X[1]:=Y cat Z;
if (c lt 5 and #Z gt 5-c) then
A:=Y cat Substring(Z, 1, 5-c);
B:=Substring(Z, 6-c, #Z-5+c);
while (Min(#A,#B) gt 0 and A[#A] eq B[1]) do
A:=Prune(A);
if (1 eq #B) then B:="";
else B:=Substring(B, 2, #B-1);
end if;
end while;
X[1]:= A cat B;
return Append(X,IntegerToString(c));
end if;
end if;
end for;
return Append(X,IntegerToString(c));
end function;
```

```
function Flips(X)
Y:=Flip(X);
i:=2;
while i lt #Y[1] do
e:=0;
if (i gt 1 and not Flipn(Y,i)[1] eq Y[1]) then
Z:=Y;
Y:=Flipn(Z,i)[[1,2,3]];
c:=StringToInteger(Flipn(Z,i)[4]);
```

```
j:=i;
d:=c;
e:=c;
while j-d gt 2 do
Z:=Y;
Y:=Flipn(Z,j-1-d)[[1,2,3]];
d:=StringToInteger(Flipn(Z,j-1-d)[4]);
e:=e+d;
j:=j-1-d;
end while;
Y:=Flip(Y);
end if;
i:=i+1-2*e;
end while;
return Y;
end function;
```

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