Representation Theory Epiphany

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Abstract. These lecture notes are for the Epiphany term of the Representation theory module. These notes are adapted from those written by Jack Shotton who, in turn, based his on those by Jens Funke from a previous iteration of this course. Those are based on multiple sources, especially [**1**] and [**2**]. Here is a brief rundown of a few references you could look at:

- We rely a lot on "Fulton and Harris": [**1**]. This book takes the point of view that examples should come before theory. It has lots of good exercises. It would be particularly useful for the $\mathfrak{sl}_{3,\mathbb{C}}$-theory. It doesn't discuss Lie groups (as opposed to algebras) much.
- Kosmann–Schwarzbach's book [**2**] is a relative short source, and is good for its concrete discussion of representations Lie groups.
- Lastly ones can also consult Hall, [**3**], for the theory of linear Lie groups, Lie algebras and representations. It is complete yet readable.

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CHAPTER 1

Linear Lie groups and their Lie algebras

1. Lecture 1

We fix some notation:

- k denotes either the field $\mathbb{R}$ or $\mathbb{C}$;
- $\mathfrak{gl}_{n,k} = M_n(k)$ is the vector space of all $n \times n$ matrices over k.

1.1. The exponential map.

DEFINITION 1.1. *Let $X \in \mathfrak{gl}_{n,k}$. We define*

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

This series is convergent for all $X \in \mathfrak{gl}_{n,k}$. Let $||\cdot||$ be the matrix norm

$$||X|| = \left(\sum_{i,j} |x_{ij}|^2 \right)^{1/2}.$$

This satisfies the triangle inequality and also $||XY|| \leq ||X||||Y||$ — this can be proved using Cauchy–Schwarz. Then for any $X \in \mathfrak{gl}_{n,\mathbb{C}}$ with $||X|| \leq M$, we have

$$||\exp(X)|| \leq \sum_{k=0}^{\infty} \frac{||X||^k}{k!} \leq \exp(M).$$

In particular, we see that exp is uniformly absolutely convergent on all compact subsets of $\mathfrak{gl}_{n,k}$. It follows that exp is a continuous function.

LEMMA 1.2. *For all $X, Y \in \mathfrak{gl}_{n,k}$, $s, t \in k$ and $g \in \mathrm{GL}_n(k)$, we have:*

(i) $\exp(0) = \mathrm{Id}$.
(ii) $\exp(X + Y) = \exp(X)\exp(Y)$ *if $XY = YX$. (This is NOT true in general).*
(iii) $\exp(X)$ *is invertible, with inverse* $\exp(-X)$.
(iv) $\exp(sX)\exp(tX) = \exp((s + t)X)$.
(v) $g\exp(X)g^{-1} = \exp(gXg^{-1})$.

PROOF. The first point is obvious. Let's prove (ii) from which (iii) and (iv) follow. By definition,

$$\begin{aligned}\exp(X+Y) &= \sum_{k=0}^{\infty} \frac{(X+Y)^k}{k!} \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\binom{k}{l} X^l Y^{k-l}}{k!} && \text{(using that } X \text{ and } Y \text{ commute!)} \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{X^l Y^{k-l}}{l!(k-l)!} \\ &= \left(\sum_{l=0}^{\infty} \frac{X^l}{l!}\right) \left(\sum_{j=0}^{\infty} \frac{Y^j}{j!}\right), && \text{(putting } j = k-l\text{)}\end{aligned}$$

which is equal to the right hand side. Rearranging the sums is valid by absolute convergence. Finally, (v) follows from $gX^kg^{-1} = (gXg^{-1})^k$. □

In fact the exponential map is differentiable as a function of X. For this, recall that a function $f : \mathbb{R}^N \to \mathbb{R}^M$ is differentiable at a point $p \in \mathbb{R}^N$ if there is a (necessarily unique) linear map $D_pf : \mathbb{R}^N \to \mathbb{R}^M$ such that

$$\lim_{h\to 0} \frac{||f(p+h) - f(p) - D_pf(h)||}{||h||} = 0,$$

and in this case D_pf is called the *derivative* of f at p. (This definition is independent of the choice of norms on $\mathbb{R}^N$ and $\mathbb{R}^M$).

PROPOSITION 1.3. *The exponential map is differentiable at the origin (zero matrix), and its derivative at the origin is the identity map from $\mathfrak{gl}_{n,\mathbb{C}}$ to itself.*

PROOF. In the above definition we have, $N = M = 2n^2$, $f = \exp$, $p = 0$, and we claim $D_0 \exp$ is the identity. Thus we need to show

$$\lim_{||X||\to 0} \frac{||\exp(X) - \exp(0) - X||}{||X||} = \lim_{||X||\to 0} \frac{||\exp(X) - \mathrm{Id} - X||}{||X||} = 0,$$

which follows from the definition of the exponential map. Indeed,

$$\frac{||\exp(X) - \mathrm{Id} - X||}{||X||} = \frac{||\sum_{k=2}^{\infty} \frac{X^k}{k!}||}{||X||} \leq ||X|| \cdot \sum_{k=0}^{\infty} \frac{||X||^k}{(k+2)!} < ||X|| e^{||X||},$$

which tends to zero as $||X|| \to 0$. □

REMARK 1.4. *In fact, the exponential function has derivatives to all orders at all points; this follows from the fact that it is given by power series that converge absolutely at all points and all of whose (formal) derivatives also converge absolutely at all points.*

By the inverse function theorem, it follows from the remark that

COROLLARY 1.5. *The exponential map is a local diffeomorphism at 0: there exist neighbourhoods $U_0 \subseteq \mathfrak{gl}_{n,k}$ containing 0 and $V_0 \subseteq \mathrm{GL}_n(k)$ containing* Id *such that $\exp|_{U_0}$ is a smooth homeomorphism onto V_0 with smooth inverse.*

REMARK 1.6. *In fact we can take* $V_0 = \{X \in \mathrm{GL}_n(\mathbb{C}) \,|\, ||X - \mathrm{Id}\,|| < 1\}$. *The inverse of* exp *in this neighbourhood is*

$$\log(X) = \sum_{k=0}^{\infty} (-1)^k \frac{(X - \mathrm{Id})^{k+1}}{k+1},$$

which is convergent when $||X - \mathrm{Id}\,|| < 1$.

Of course, exp *is not injective in general. For example,* $\exp(2\pi i k) = 1$ *for* $k \in \mathbb{Z}$.

1.2. Exercises.

Problem 1.

(a) Compute $\exp(X)$ for X equal to $\begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix}$, $\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$ (where $s, t \in \mathbb{R}$).

(b) Let $E_{a,b}$ be the elementary $n \times n$ matrix with 1 in the (a, b)-entry and 0 elsewhere. Compute $\exp(tE_{a,b})$ for $a \neq b$ and $a = b$.

Problem 2. Show that

$$\exp(tX)\exp(tY) = \exp\left(t(X + Y) + \frac{t^2}{2}[X, Y] + O(t^3)\right)$$

as $t \to 0$, where

$$[X, Y] = XY - YX.$$

2. Lecture 2

2.1. More results about the exponential map. For the next result about the exponential map it will be useful to know the following fact from linear algebra.

LEMMA 2.1. *Let $X \in \mathrm{GL}_n(\mathbb{C})$. Then X is conjugate to a matrix of the form DU where D is diagonal, U is upper triangular with '1's on the diagonal and D and U commute.*

PROOF. (nonexaminable) This follows from Jordan normal form. Here's a direct proof. Firstly write $\mathbb{C}^n$ as a direct sum of generalised eigenspaces for X: if λ is an eigenvalue of X then we can write the characteristic polynomial $P(T) = (T-\lambda)^a Q(T)$ where $Q(T)$ does not have λ as a root and $a \geq 1$ is an integer. Then the image of $Q(X)$ on $\mathbb{C}^n$ is the generalised eigenspace of λ. The kernel of $Q(X)$ is preserved by X and X does not have an eigenvalue equal to λ since λ is not a root of $Q(X)$, which must be the characteristic polynomial of X acting on $\ker Q(X)$. Thus

$$\operatorname{im} Q(X) \cap \ker Q(X) = \{0\}$$

and by the rank-nullity theorem

$$\mathbb{C}^n = \operatorname{im} Q(X) \oplus \ker Q(X)$$

is a decomposition of $\mathbb{C}^n$ as a direct sum of the λ generalised eigenspace and a subspace preserved by X. Repeating for each eigenvector gives the required decomposition of $\mathbb{C}^n$. This reduces the proof of the statement to the case where X has only one eigenvalue λ. In this case, we can inductively choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{C}^n$ such that, for $1 \leq i \leq n$, the image of $\mathbf{v}_i$ in $\mathbb{C}/\langle \mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\rangle$ is an eigenvector of X with eigenvalue λ. With respect to this basis, X is then diagonal with λ's on the diagonal, and we get the required decomposition with $D = \lambda\,\mathrm{Id}$. □

LEMMA 2.2. *The exponential function* $\exp : \mathfrak{gl}_{n,\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ *is surjective.*

PROOF. First prove it for all D and U in $\mathrm{GL}_n(\mathbb{C})$ as in Lemma 2.1. The case of diagonal matrices is more straightforward (Problem 4(a)) whereas for U you can use that the power series for $\log(U)$ in terms of powers of $U - \mathrm{Id}$ is actually a polynomial (Problem 4(b)).

Now consider DU. If $D = \exp(d)$ and $U = \exp(u)$ then

$$DU = \exp(d)\exp(u) = \exp(d+u)$$

because d and u commute (so long as you choose d and u carefully — Problem 4(c)).

By Lemma 2.1 we have that for any X in $\mathrm{GL}_n(\mathbb{C})$ there exists a P in $\mathrm{GL}_n(\mathbb{C})$ such that $P^{-1}XP = DU$ where D and U have the form stated in the Lemma. By Lemma 1.2(v) we have

$$X = PDUP^{-1} = P\exp(d+u)P^{-1} = \exp(P(d+u)P^{-1})$$

and thus exp is surjective. □

REMARK 2.3. *The lemma is not true over $\mathbb{R}$; as we will see, the determinant of* $\exp(X)$ *is positive for all real matrices X.*

LEMMA 2.4. *We have*

$$\det\exp(X) = \exp\operatorname{tr}(X).$$

PROOF. By Lemma 2.1 we can conjugate so that X is an upper triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, and then note that $\exp(X)$ is also upper triangular with diagonal entries $\exp(\lambda_1), \ldots, \exp(\lambda_n)$.

Thus
$$\det \exp(X) = \prod_{i=1}^{n} \exp(\lambda_i) = \exp\left(\sum_{i=1}^{n} \lambda_i\right) = \exp \operatorname{tr}(X).$$
□

The next proposition will be useful when we discuss Lie algebras of linear Lie groups.

PROPOSITION 2.5 (Lie product formula). *For $X, Y \in \mathfrak{gl}_{n,k}$ we have*
$$\exp(X+Y) = \lim_{l\to\infty}\left(\exp\left(\frac{X}{l}\right)\exp\left(\frac{Y}{l}\right)\right)^l.$$

PROOF. We have
$$\begin{aligned}\left(\exp\left(\frac{X}{l}\right)\exp\left(\frac{Y}{l}\right)\right)^l &= \left(\left(\mathrm{Id} + \frac{X}{l} + O\left(\frac{1}{l^2}\right)\right)\left(\mathrm{Id} + \frac{Y}{l} + O\left(\frac{1}{l^2}\right)\right)\right)^l \\ &= \left(\mathrm{Id} + \frac{X+Y}{l} + O\left(\frac{1}{l^2}\right)\right)^l k \\ &= \left(\exp\left(\frac{X+Y}{l} + O\left(\frac{1}{l^2}\right)\right)\right)^l \\ &= \exp\left(X + Y + O\left(\frac{1}{l}\right)\right).\end{aligned}$$
Now take the limit as $l \to \infty$. □

2.2. One-parameter subgroups.

LEMMA 2.6. *The map from $\mathbb{R}$ to $\mathrm{GL}_n(\mathbb{C})$ given by*
$$t \longmapsto \exp(tX)$$
is a differentiable group homomorphism.

We have
$$\frac{d}{dt}\exp(tX) = X\exp(tX) = \exp(tX)X.$$
In particular,
$$\left.\frac{d}{dt}\exp(tX)\right|_{t=0} = X.$$

PROOF. The given map is a group homomorphism by Lemma 1.2(iv).
By definition,
$$\exp(tX) = \sum_{k=0}^{\infty}\frac{X^k}{k!}t^k.$$
As this power series (and its termwise derivative) are uniformly convergent on any compact subset, we can compute its derivative by differentiating termwise, which gives
$$\frac{d}{dt}\exp(tX) = \sum_{k=1}^{\infty}\frac{X^k}{(k-1)!}t^{k-1} = X\exp(tX).$$
□

DEFINITION 2.7. *A* one-parameter subgroup *of* $\mathrm{GL}_n(\mathbb{C})$ *is a differentiable group homomorphism* $f : \mathbb{R} \to \mathrm{GL}_n(\mathbb{C})$*. That is, a differentiable map such that*

$$f(s+t) = f(s)f(t)$$

for all $s, t \in \mathbb{R}$*.*

The infinitesimal generator *of a one-parameter subgroup* f *is the element* $f'(0) \in \mathfrak{gl}_{n,\mathbb{C}}$*.*

The convention used here is a slight abuse of notation, $f(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ referred to in the definition but as the map defines the subgroup we just refer to the map.

REMARK 2.8. *For a one-parameter subgroup* f*, it actually suffices to require that* f *is continuous. Differentiability then comes for free.*

Indeed, if f *is continuous, the integral* $\int_0^a f(t)dt$ *exists. Moreover,*

$$f(s)\int_0^a f(t)dt = \int_0^a f(s+t)dt = \int_s^{s+a} f(t)dt.$$

The RHS is differentiable with respect to s *by the fundamental theorem of algebra. Therefore, to prove that* $f(s)$ *is differentiable, we only need to show that there is an* $a > 0$ *such that* $\int_0^a f(t)dt$ *is an invertible matrix. Now consider the function*

$$F(a) = \frac{1}{a}\int_0^a f(t)dt.$$

It is well-defined for $a \neq 0$ *and* $\lim_{a\to 0} F(a) = \mathrm{Id}$*. Hence, for* $0 < a \ll 1$*,* $F(a)$ *is invertible, and therefore so is* $aF(a) = \int_0^a f(t)dt$*.*

The following is a very important property of one-parameter subgroups: that they all come from the exponential map.

PROPOSITION 2.9. *Let* $f : \mathbb{R} \to \mathrm{GL}_n(\mathbb{C})$ *be a one-parameter subgroup with infinitesimal generator* X*.*

Then

$$f(t) = \exp(tX)$$

for all $t \in \mathbb{R}$*. That is, all one-parameter subgroups arise from the exponential function.*

PROOF. From the definition of one-parameter subgroups, we have

$$f'(t) = \lim_{s\to 0}\frac{f(s+t)-f(t)}{s} = f(t)\lim_{s\to 0}\frac{f(s)-f(0)}{s} = f(t)f'(0) = f(t)X.$$

Now consider the differential equation

$$f'(t) = f(t)X.$$

By Lemma 2.6 we have $\exp(tX)$ is a solution with the initial condition that $f(0) = \mathrm{Id}$. To show it is a unique solution suppose that $g(t)$ is also a solution. Then

$$(g(t)\exp(-tX))' = g'(t)\exp(-tX) - g(t)\exp(-tX)X = g(t)\left(X\exp(-tX) - \exp(-tX)X\right) = 0,$$

and thus $g(t)\exp(-tX) = D \in \mathrm{GL}_n(\mathbb{C})$. Applying the initial conditions we get $D = \mathrm{Id}$ and $g(t) = \exp(tX)$. □

EXAMPLE 2.10. *The map* $\mathbb{R} \to \mathrm{SO}(3) = \{g \in \mathrm{GL}_3(\mathbb{R}) \mid gg^T = \mathrm{Id}, \det g = 1\} \subseteq \mathrm{GL}_3(\mathbb{R})$ *taking* θ *to rotation by* θ *about a fixed axis is a one-parameter subgroup. Problem 5 asks you to find its infinitesimal generator.*

2.3. Exercises.

Problem 3. Let $\mathfrak{n}$ be the $\mathbb{C}$-vector space of strictly upper triangular matrices (0's on the diagonal) and let $N = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid g = \mathrm{Id} + X, X \in \mathfrak{n}\}$.

In this problem we will see that the restriction of the exponential to $\mathfrak{n}$ is a diffeomorphism onto N.

(a) Let $X \in \mathfrak{n}$. Show that $X^n = 0$.
(b) Show that $\exp(X) \in N$ for $X \in \mathfrak{n}$.
(c) Show that, for $g \in N$, the logarithm $\log(g) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(g-I)^k}{k}$ is in fact a finite sum (and hence converges).
(d) Show that $\exp|_{\mathfrak{n}}$ and $\log|_N$ are inverses of each other. *Hint: this boils down to an identity of formal power series, which you can actually deduce from the corresponding fact over* $\mathbb{R}$.

Problem 4. Using the previous question, fill in the gaps of the proof from the notes that

$$\exp : \mathfrak{gl}_{n,\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$$

is surjective.

(a) Show all matrices of the form D have a preimage.
(b) Show all matrices of the form U have a preimage.
(c) Show all matrices of the form DU have a preimage.
(d) Is the exponential map $\exp : \mathfrak{sl}_{2,\mathbb{C}} \to \mathrm{SL}_2(\mathbb{C})$ surjective? What about $\exp : \mathfrak{gl}_{2,\mathbb{R}} \to \mathrm{GL}_2^+(\mathbb{R})$?

Problem 5. Let $\mathbf{v} \in \mathbb{R}^3$ be a unit vector and let $f : \mathbb{R} \to SO(3)$ be the map with $f(\theta)$ being rotation by θ about the axis $\mathbf{v}$ (the angle is measured anticlockwise as you look along the vector from the origin).

Show that f is a one-parameter subgroup and find its infinitesimal generator in terms of $\mathbf{v}$.

3. Lecture 3

3.1. Linear Lie groups.

DEFINITION 3.1. *A (linear)* Lie group *is a closed subgroup of* $\mathrm{GL}_n(\mathbb{C})$*, for some* n*.*

Here the group is called closed if it contains the limit of any Cauchy sequence, provided that limit is invertible.

REMARK 3.2. *The usual definition of a Lie group is a smooth manifold together with a group structure such that the group operations are smooth functions. It is a theorem (Cartan's theorem, or the closed subgroup theorem) that every linear Lie group in the sense of definition 3.1 is a Lie group in this sense. Not every Lie group is a linear Lie group, but we will only be studying linear Lie groups so we will often drop the word 'linear'.*

We give various examples (note that any subgroup defined by equalities of continuous functions will be closed):

EXAMPLE 3.3.

(i) *the real general linear group* $\mathrm{GL}_n(\mathbb{R})$*: we simply impose the closed condition that all the entries of the matrix are real;*

(ii) *the (real or complex) special linear groups* $\mathrm{SL}_n(k)$*;*

(iii) *if* $\langle\cdot,\cdot\rangle$ *is a bilinear form on* $\mathbb{R}^n$ *then we obtain a linear Lie group*

$$\{g \in \mathrm{GL}_n(\mathbb{R}) \mid \langle g\mathbf{v}, g\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{w}\rangle\}.$$

There is a matrix A *such that* $\langle \mathbf{v},\mathbf{w}\rangle = \mathbf{v}^T A\mathbf{w}$ *for all* $\mathbf{v}, w$*; the bilinear form is symmetric if and only* A *is symmetric, alternating if and only if* A *is skew-symmetric (*$A^T = -A$*), and non-degenerate if and only if* $\det A$ *is non-zero. Then the group is:*

$$\{g \in GL_n(\mathbb{R}) \mid g^T A g = A\}.$$

Some special cases follow.

(iv) *The orthogonal and special orthogonal groups*

$$\mathrm{O}(n) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid gg^T = \mathrm{Id}\}$$

and $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}_n(\mathbb{R})$*;*

(v) *the unitary and special unitary groups*

$$\mathrm{U}(n) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid gg^* = \mathrm{Id}\}$$

and $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}_n(\mathbb{C})$ *(not strictly a special case of the above, but closely related);*

(vi) *the symplectic groups*

$$\mathrm{Sp}(2n,k) = \{g \in GL_{2n}(k) \mid gJg^T = J\}$$

where $J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$ *and* Id *is the* $n \times n$ *identity. This corresponds to a non-degenerate alternating bilinear form.*

(vii) *the Heisenberg group*

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x,y,z \in \mathbb{R} \right\};$$

EXAMPLE 3.4. *Non-examples are* $\mathrm{GL}_n(\mathbb{Q})$ *(this is a subgroup of* $\mathrm{GL}_n(\mathbb{C})$*, but not closed), or (if* α *is an irrational real number) the subgroup*

$$\left\{ \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{i\alpha x} \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \subseteq \mathrm{GL}_2(\mathbb{C}).$$

This is a subgroup, isomorphic — as a group — to $\mathbb{R}$*, but not closed. You should picture it as a string wound infinitely densely around a torus.*

The idea of Lie theory is to simplify the study of these groups by just studying their structure 'very close to the identity'. This crucially uses that they are groups with a *topology*. By looking at the tangent spaces of these groups at the origin, you obtain *Lie algebras*; the group operation then turns into a structure called the *Lie bracket*.

3.2. The set $\mathfrak{g}$. Given some Lie group G we can define a subset of $\mathfrak{gl}_{n,\mathbb{C}}$ related to this G.

DEFINITION 3.5. *Let* $G \subseteq \mathrm{GL}_n(\mathbb{C})$ *be a linear Lie group. We define*

$$\mathfrak{g} := \{X \in \mathfrak{gl}_{n,\mathbb{C}} \mid \exp(tX) \in G \text{ for all } t \in \mathbb{R}\}.$$

In other words, it is the set of X *such that the one-parameter subgroup infinitesimally generated by* X *is contained in the group* G.

The set $\mathfrak{g}$ can also be defined more geometrically as the *tangent space* to G at the identity; the above definition then becomes the 'exponential characterisation'. The equivalence is given by the following theorem:

THEOREM 3.6. *With* G *and* $\mathfrak{g}$ *as above, we have*

$$\mathfrak{g} = \{X \in \mathfrak{gl}_{n,\mathbb{C}} \mid X = \gamma'(0) \text{ for some differentiable map } \gamma : [-a, a] \to G, a > 0\}.$$

In other words, $\mathfrak{g}$ *is the set of all possible tangent vectors to curves in* G *passing through* Id.

PROOF. We show that $X = \gamma'(0)$ for some differentiable map $\gamma : [-a, a] \to G$ with $a > 0$ if and only if $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

If $\exp(tX) \in G$ for all $t \in \mathbb{R}$ then, by Lemma 2.6, $\gamma(t) = \exp(tX)$ is differentiable with $\gamma'(t) = X \exp(tX)$ and thus $\gamma'(0) = X$.

Now assume that there exists a differentiable map $\gamma : [-a, a] \to G$, for some $a > 0$, such that $X = \gamma'(0)$. Fix any $t \in \mathbb{R}$. As $k \to \infty$ we have an expansion

$$\gamma\left(\frac{t}{k}\right) = \mathrm{Id} + \frac{t}{k} X + O\left(\frac{1}{k^2}\right) = \exp\left(\frac{t}{k} X + O\left(\frac{1}{k^2}\right)\right)$$

of γ around 0. Then

$$\left(\gamma\left(\frac{t}{k}\right)\right)^k = \exp\left(tX + O\left(\frac{1}{k}\right)\right) \in G.$$

Since G is closed and

$$\lim_{k\to\infty} \left(\gamma\left(\frac{t}{k}\right)\right)^k = \exp(tX),$$

we conclude that $\exp(tX) \in G$. □

REMARK 3.7. *It is not true that* $\mathfrak{g} = \{X \in \mathfrak{gl}_{n,\mathbb{C}} \mid \exp(X) \in G\}$*. This is not even true for* $G = \{1\} \subseteq \mathbb{C}$*, why?*

We now collect some properties of $\mathfrak{g}$.

PROPOSITION 3.8. *Let $\mathfrak{g}$ correspond to the (linear) Lie group G. Then*

(i) $\mathfrak{g}$ is a real vector space (inside $\mathfrak{gl}_{n,\mathbb{C}}$).
(ii) If $X \in \mathfrak{g}$ and if $g \in G$, then $gXg^{-1} \in \mathfrak{g}$.
(iii) For $X, Y \in \mathfrak{g}$ we have $XY - YX \in \mathfrak{g}$.

PROOF.

(i) This is left as an exercise (Problem 6).
(ii) This follows from Lemma 1.2, $\exp(t(gXg^{-1})) = g\exp(tX)g^{-1} \in G$. Again, you could also prove this directly from the definition of $\mathfrak{g}$.
(iii) We know by part (ii) that, for $X, Y \in \mathfrak{g}$,

$$\exp(tX)Y\exp(-tX) \in \mathfrak{g}.$$

Then

$$\begin{aligned}\frac{d}{dt}\exp(tX)Y\exp(-tX)\bigg|_{t=0} &= (X\exp(tX)Y\exp(-tX) - \exp(tX)Y\exp(-tX)X)|_{t=0} \\ &= XY - YX.\end{aligned}$$

But also by definition

$$\frac{d}{dt}\exp(tX)Y\exp(-tX)\bigg|_{t=0} = \lim_{t\to 0}\frac{\exp(tX)Y\exp(-tX) - Y}{t}.$$

This is a limit of elements of the vector space $\mathfrak{g}$, which is a closed subset of $\mathfrak{gl}_{n,k}$, and so must itself be an element of $\mathfrak{g}$.

$\square$

3.3. Exercises.

Problem 6. Provide a proof of Proposition 3.8(i).

4. Lecture 4

4.1. Lie algebras. There are two ways one can introduce Lie algebras; either as an axiomatic definition or by deriving them from Lie groups (which is where they get their name). We choose the former.

DEFINITION 4.1. *A* Lie algebra $\mathfrak{g}$ *over a field* k *is a* k*-vector space together with a bilinear map (*Lie bracket*)*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

that satisfies the following properties.

(i) It is alternating: $[Y, X] = -[X, Y]$ *for all* $X, Y \in \mathfrak{g}$*.*

(ii) The Jacobi identity *holds:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$*.*

We will now proceed by investigating the motivating example (which is also the source of the Lie in the name Lie algebra).

THEOREM 4.2. *Let* G *be a (linear) Lie group. Then* $\mathfrak{g}$ *is a real Lie algebra with Lie bracket given by*

$$[X, Y] = XY - YX.$$

PROOF. By Proposition 3.8(i), $\mathfrak{g}$ is a real vector space. By (iii) of the same result $[X, Y] \in \mathfrak{g}$ so it remains to show the bracket is bilinear and the two axioms are satisfied by it. This is left as an exercise (Problem 7). $\square$

We call $\mathfrak{g}$ the Lie algebra of G (also sometimes denoted $\mathrm{Lie}(G)$) and define the dimension of the Lie group G to be the dimension of $\mathfrak{g}$. We now compute the Lie algebras of many of the groups that we are interested in:

PROPOSITION 4.3. *The Lie algebras of* $\mathrm{GL}_n(k)$*,* $\mathrm{SL}_n(k)$*,* $\mathrm{O}(n)$*,* $\mathrm{SO}(n)$*,* $\mathrm{U}(n)$*, and* $\mathrm{SU}(n)$ *are given by*

(i) $\mathrm{Lie}(\mathrm{GL}_n(k)) = \mathfrak{gl}_{n,k}$ *with* $\dim_k \mathfrak{gl}_{n,k} = n^2$*;*

(ii) $\mathrm{Lie}(\mathrm{SL}_n(k)) := \mathfrak{sl}_{n,k} = \{X \in \mathfrak{gl}_{n,k} \mid \mathrm{tr}(X) = 0\}$*, the traceless matrices, with* $\dim_k \mathfrak{sl}_{n,k} = n^2 - 1$*;*

(iii) $\mathrm{Lie}(\mathrm{O}(n)) := \mathfrak{o}_n = \mathrm{Lie}(\mathrm{SO}(n)) := \mathfrak{so}_n = \{X \in \mathfrak{gl}_{n,\mathbb{R}} \mid X + X^T = 0\}$*, the skew symmetric real matrices, with* $\dim_{\mathbb{R}} \mathfrak{o}_n = \dim_{\mathbb{R}} \mathfrak{so}_n = \frac{n(n-1)}{2}$*;*

(iv) $\mathrm{Lie}(\mathrm{U}(n)) := \mathfrak{u}_n = \{X \in \mathfrak{gl}_{n,\mathbb{C}} \mid X + X^* = 0\}$*, the skew Hermitian matrices, with* $\dim_{\mathbb{R}} \mathfrak{u}_n = n^2$*;*

(v) $\mathrm{Lie}(\mathrm{SU}(n)) := \mathfrak{su}_n = \{X \in \mathfrak{u}_n \mid \mathrm{tr}(X) = 0\}$*, the traceless skew Hermitian matrices, with* $\dim_{\mathbb{R}} \mathfrak{su}_n = n^2 - 1$*.*

(vi) $\mathrm{Lie}(\mathrm{Sp}(2n, k)) := \mathfrak{sp}_{2n,k} = \{X \in \mathfrak{gl}_{2n,k} \mid X^T J + JX = 0\}$ *where* J *is the matrix of the alternating bilinear form.*

PROOF.

(i) This obvious for $k = \mathbb{C}$ and left as an exercise for $k = \mathbb{R}$.

(ii) First suppose that $\mathrm{tr}(X) = 0$. Then $\det \exp(tX) = \exp \mathrm{tr}(tX) = 1$ so $X \in \mathfrak{sl}_{n,k}$. Conversely, if $X \in \mathfrak{sl}_{n,K}$ then $1 = \det \exp(tX) = \exp(t\,\mathrm{tr}(X))$ for all t; differentiating at $t = 0$ gives $\mathrm{tr}(X) = 0$ as required.

(iii) We need to find all X such that

$$\exp(tX)\exp(tX)^T = \exp(tX)\exp(tX^T) = I \tag{4.4}$$

for all t. Taking the derivative for both sides with respect to t, we obtain

$$X\exp(tX)\exp(tX^T) + \exp(tX)\exp(X^T)X^T = 0.$$

Evaluating at $t = 0$, we get

$$X + X^T = 0.$$

Conversely, if $X + X^T = 0$, then clearly equation (4.4) holds because

$$\exp(tX)^T = \exp(tX^T) = \exp(-tX) = \exp(tX)^{-1}.$$

For the dimension, notice that X satisfying $X = -X^T$ is determined by its upper triangular part and that the diagonal entries must be all zeros; as their are $\frac{n(n-1)}{2}$ entries strictly above the diagonal, that is the dimension of $\mathfrak{o}_n$. Since these matrices already have trace zero, we see that $\mathfrak{so}_n = \mathfrak{o}_n$.

The unitary and symplectic Lie algebras can be computed in a similar way. This is an exercise (Problem 8). $\square$

We now give an example of a Lie algebra that does not initially appear to be of the form $\mathrm{Lie}(G)$ for some Lie group G.

Example 4.5. *Let $\mathfrak{g} = \mathbb{R}^3$ and let $[\mathbf{v}, \mathbf{w}] = \mathbf{v} \times \mathbf{w}$. Then this is a Lie algebra (just check the axioms).*

In fact, $\mathfrak{g} \cong \mathfrak{so}_3$. To see this, send the vector $\mathbf{v}$ to the infinitesimal generator of the one parameter subgroup of $\mathrm{SO}(3)$ given by 'rotating around the axis $\mathbf{v}$ at speed $|\mathbf{v}|$'.

Example 4.6. *Let V be a k-vector space. Then $\mathrm{End}(V)$, the space of linear maps fro V to itself, is a k-Lie algebra with*

$$[f, g] = f \circ g - g \circ f,$$

for all $f, g \in \mathrm{End}(V)$.

Definition 4.7. *If $\mathfrak{g}$ is a Lie algebra, then a* <u>Lie subalgebra</u> *is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ that is closed under the Lie bracket.*

Proposition 4.8. *If $G \subseteq \mathrm{GL}_n(k)$ is a Lie group, then $\mathfrak{g}$ is a (real) Lie subalgebra of $\mathfrak{gl}_{n,k}$.*

Proof. By Proposition 3.8(i) and (iii) we have $\mathfrak{g} \subseteq \mathfrak{gl}_{n,k}$ and $\mathfrak{g}$ is closed under the Lie bracket. $\square$

Definition 4.9. *A Lie algebra $\mathfrak{g}$ is called* <u>abelian</u> *if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.*

Definition 4.10. *The* <u>centre</u> *of a Lie algebra $\mathfrak{g}$ is*

$$Z(\mathfrak{g}) = \{Z \in \mathfrak{g} \mid [Z, X] = 0 \text{ for all } X \in \mathfrak{g}\}.$$

Lemma 4.11. *The centre of a Lie algebra is an abelian Lie subalgebra.*

Proof. This is Exercise 9. $\square$

Definition 4.12. *A* <u>complex-linear Lie group</u> *is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ whose Lie algebra is a complex subspace of $\mathfrak{gl}_{n,\mathbb{C}}$ (as opposed to just a real subspace).*

Note that $\mathfrak{u}_n$ and $\mathfrak{su}_n$ are only real Lie algebras and correspondingly $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are only real Lie groups, even though they consist of complex matrices. On the other hand, $\mathfrak{gl}_{n,\mathbb{C}}$ and $\mathfrak{sl}_{n,\mathbb{C}}$ are complex Lie algebras, so $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ are complex Lie groups.

A *complex Lie algebra* is a $\mathbb{C}$-vector space with a $\mathbb{C}$-bilinear Lie bracket satisfying the same axioms as for a Lie algebra. Thus one can show the Lie algebra of a complex Lie group is a complex Lie algebra (since the Lie bracket on $\mathfrak{gl}_{n,\mathbb{C}}$ is clearly $\mathbb{C}$-bilinear).

4.2. Exercises.

Problem 7. Complete the proof of Theorem 4.2.

Problem 8. Complete the proof of Proposition 4.3, i.e. prove;
(a) part (i),
(b) part (iv),
(c) part (v),
(d) part (vi).

Problem 9. Prove Lemma 4.11.

Problem 10. Let $I_{p,q} = \begin{pmatrix} \mathrm{Id}_p & \\ & -\mathrm{Id}_q \end{pmatrix}$, where Id_k denotes the identity matrix of size k. Let $n = p + q$. Let

$$\mathrm{O}(p,q) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g I_{p,q} g^T = I_{p,q}\}$$

be the orthogonal group of signature (p,q). Let $\mathrm{SO}(p,q) = \mathrm{O}(p,q) \cap \mathrm{SL}_n(\mathbb{R})$. We let $\mathfrak{o}_{p,q}$ and $\mathfrak{so}_{p,q}$ be their Lie algebras.

Show that the Lie algebra $\mathfrak{o}_{p,q}$ is given by

$$\mathfrak{o}(p,q) = \{X \in M_n(\mathbb{R}) \mid X I_{p,q} + I_{p,q} X^T = 0\}$$

and that $\mathfrak{so}_{p,q} = \mathfrak{o}_{p,q}$.

5. Lecture 5

5.1. Lie group and Lie algebra homomorphisms.

DEFINITION 5.1. *A* Lie group homomorphism *$\phi : G \to G'$ between two linear Lie groups G and G' is a continuous group homomorphism.*

An isomorphism is a bijective Lie group homomorphism whose inverse is also continuous.

REMARK 5.2. *In fact, a continuous homomorphism between linear Lie groups is automatically a smooth map of smooth manifolds, and if it is bijective then the inverse is automatically continuous.*

DEFINITION 5.3. *A* homomorphism *$\varphi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras is a k-linear map such that*

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$$

for all $X, Y \in \mathfrak{g}$.

An isomorphism *is an invertible homomorphism.*

DEFINITION 5.4. *Let $\phi : G \to G'$ be a Lie group homomorphism. Define the* derivative *(or* differential *or* derived homomorphism*)*

$$D\phi : \mathfrak{g} \longrightarrow \mathfrak{g}'$$

by

$$D\phi(X) = \frac{d}{dt}\phi(\exp(tX))\bigg|_{t=0}$$

for $X \in \mathfrak{g}$. [1]

REMARK 5.5. *In fact, $D\phi$ is the derivative of ϕ at the identity in the sense of smooth manifolds; recall that $\mathfrak{g}$ and $\mathfrak{g}'$ are the tangent spaces to G and G' at the identity.*

THEOREM 5.6. *Let $\phi : G \to G'$ be a Lie group homomorphism with derivative $D\phi$. Then*

(i) The following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "D\phi"] \arrow[d, "\exp"'] & \mathfrak{g}' \arrow[d, "\exp"] \\ G \arrow[r, "\phi"'] & G'. \end{tikzcd}$$

That is, for $X \in \mathfrak{g}$ we have

$$\phi(\exp(X)) = \exp(D\phi(X)).$$

(ii) For all $g \in G, X \in \mathfrak{g}$,

$$D\phi(gXg^{-1}) = \phi(g)D\phi(X)\phi(g)^{-1}.$$

(iii) The map $D\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism.

PROOF.

[1]We can justify taking the derivative by appealing to Remark 2.8.

(i) Consider the one parameter subgroup $f : \mathbb{R} \to G'$ defined by $f(t) = \phi(\exp(tX))$. By construction, $f'(0) = D\phi(X)$. By Proposition 2.9, one parameter subgroups are determined by their derivative at 0, so that we must have

$$\phi(\exp(tX)) = f(t) = \exp(tD\phi(X)).$$

(ii) We have

$$\begin{aligned}
D\phi(gXg^{-1}) &= \frac{d}{dt}\phi(\exp(tgXg^{-1}))\Big|_{t=0} \\
&= \frac{d}{dt}\phi(g\exp(tX)g^{-1})\Big|_{t=0} \\
&= \frac{d}{dt}\phi(g)\phi(\exp(tX))\phi(g^{-1})\Big|_{t=0} \\
&= \phi(g)\ \frac{d}{dt}\phi(\exp(tX))\Big|_{t=0}\ \phi(g^{-1}) \\
&= \phi(g)D\phi(X)\phi(g)^{-1},
\end{aligned}$$

as claimed.

(iii) To show that $D\phi$ is a Lie algebra homomorphism, we need to show that $D\phi$ is $\mathbb{R}$-linear and $D\phi([X,Y]) = [D\phi(X), D\phi(Y)]$.

So let $X, Y \in \mathfrak{g}$ and $s \in \mathbb{R}$. By definition,

$$D\phi(sX) = \frac{d}{dt}\phi(\exp(tsX))\Big|_{t=0}.$$

If we now set $\mu = st$, we can rewrite this as:

$$\begin{aligned}
\frac{d}{dt}\phi(\exp(tsX))\Big|_{t=0} &= s\ \frac{d}{d\mu}\phi(\exp(\mu X))\Big|_{\mu=0} \\
&= sD\phi(X).
\end{aligned}$$

So $D\phi$ commutes with scalar multiplication. For additivity, we have

$$D\phi(X+Y) = \frac{d}{dt}\phi(\exp(t(X+Y)))\Big|_{t=0}.$$

On the other hand, by Corollary 2.5 and using part (i)

$$\begin{aligned}
\phi(\exp(t(X+Y))) &= \lim_{k\to\infty} \phi\left(\left(\exp\left(\frac{t}{k}X\right)\exp\left(\frac{t}{k}Y\right)\right)^k\right) \\
&= \lim_{k\to\infty} \left(\phi\left(\exp\left(\frac{t}{k}X\right)\right)\phi\left(\exp\left(\frac{t}{k}Y\right)\right)\right)^k \\
&= \lim_{k\to\infty} \left(\exp\left(\frac{t}{k}D\phi(X)\right)\exp\left(\frac{t}{k}D\phi(Y)\right)\right)^k \\
&= \exp(t(D\phi(X) + D\phi(Y))).
\end{aligned}$$

Taking the derivative at $t = 0$, we conclude that

$$D\phi(X+Y) = D\phi(X) + D\phi(Y),$$

showing additivity.

Finally we show that $D\phi$ respects the Lie bracket. Let $X, Y \in \mathfrak{g}$. By parts (i) and (ii) we have

$$\begin{aligned} D\phi\left(\exp(-tY)X\exp(tY)\right) &= \phi(\exp(-tY))D\phi(X)\phi(\exp(tY)) \\ &= \exp(-tD\phi(Y))D\phi(X)\exp(tD\phi(Y)). \end{aligned}$$

We then take the derivative for both sides at $t = 0$. The derivative of the RHS is $[D\phi(X), D\phi(Y)]$, and the derivative of the LHS is $D\phi([X, Y])$ (as $D\phi$ is linear).

$\square$

DEFINITION 5.7. *Suppose that G and G' are complex Lie groups and $\phi : G \to G'$ is a homomorphism. Then ϕ is* <u>holomorphic</u> *if $D\phi : \mathfrak{g} \to \mathfrak{g}'$ is $\mathbb{C}$-linear.*

(This implies that ϕ is a holomorphic map of complex manifolds.)

EXAMPLE 5.8. *The map* $\det : \mathrm{GL}_{2,\mathbb{C}} \to \mathrm{GL}_{1,\mathbb{C}}$ *is holomorphic.*

Back to the real case. We have shown that the differential $\phi \mapsto D\phi$ gives a map

$$D : \mathrm{Hom}(G, G') \longrightarrow \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}').$$

This raises two natural questions:

1. Is the map injective? Does the derivative $D\phi$ uniquely determine the Lie group homomorphism ϕ?
2. Is the map surjective? Or in other words, does every Lie algebra homomorphism φ 'exponentiate' (or 'lift') to a Lie group homomorphism ϕ such that $D\phi = \varphi$? We say 'exponentiate' since if yes, then ϕ would need to satisfy $\phi(\exp(X)) = \exp(\varphi(X))$. So this gives a formula for ϕ, at least on the image of exp. The question is whether this is well-defined (the exponential map is neither injective nor surjective in general) and whether this defines a homomorphism.

The answer to these questions is actually of topological nature, which we discuss in the next subsection.

5.2. Exercises.

Problem 11.

(a) Show that the Lie algebras $\mathfrak{so}_3$ and $\mathfrak{su}_2$ are isomorphic. (Later on, we will see a conceptual reason for this).

Hint: it is enough to find a basis for $\mathfrak{so}_3$ and a basis for $\mathfrak{su}_2$ which satisfy the 'same' Lie bracket relations. Try using the basis of $\mathfrak{so}_3$ consisting of infinitesimal generators for rotations around the axes, and a basis for $\mathfrak{su}_2$ related to the quaternions.

(b) Show that the Lie algebras $\mathfrak{so}_{2,1}$ and $\mathfrak{sl}_{2,\mathbb{R}}$ are isomorphic.

(c) Show that the Lie algebras $\mathfrak{so}_{3,1}$ and $\mathfrak{sl}_{2,\mathbb{C}}$ are isomorphic (as real Lie algebras).

Problem 12. Show that:

(a) If $X \in \mathfrak{sp}_{2n,k}$, then $\mathrm{tr}(X) = 0$.

(b) If $g \in \mathrm{Sp}(2n, \mathbb{R})$, then $\det(g) = \pm 1$.

6. Lecture 6

6.1. Topological properties: answer to question (1). While we only defined linear Lie groups to be closed subgroups of $\mathrm{GL}_n(\mathbb{C})$, in fact they have much nicer topological properties than arbitrary closed subsets (which can be pretty wild, like the Cantor set).

THEOREM 6.1. *(closed subgroup theorem) Let $G \le \mathrm{GL}_n(\mathbb{C})$ be a closed subgroup. Then G is actually a smoothly embedded submanifold: for every $g \in G$ there is an open set $g \in U \subseteq \mathrm{GL}_n(\mathbb{C})$, an open subset $0 \in V \subseteq \mathfrak{gl}_{n,\mathbb{C}}$, and a diffeomorphism $\phi : V \to U$ such that $\phi(0) = g$ and $\phi(\mathfrak{g} \cap V) = G \cap U$. See Figure 1.*

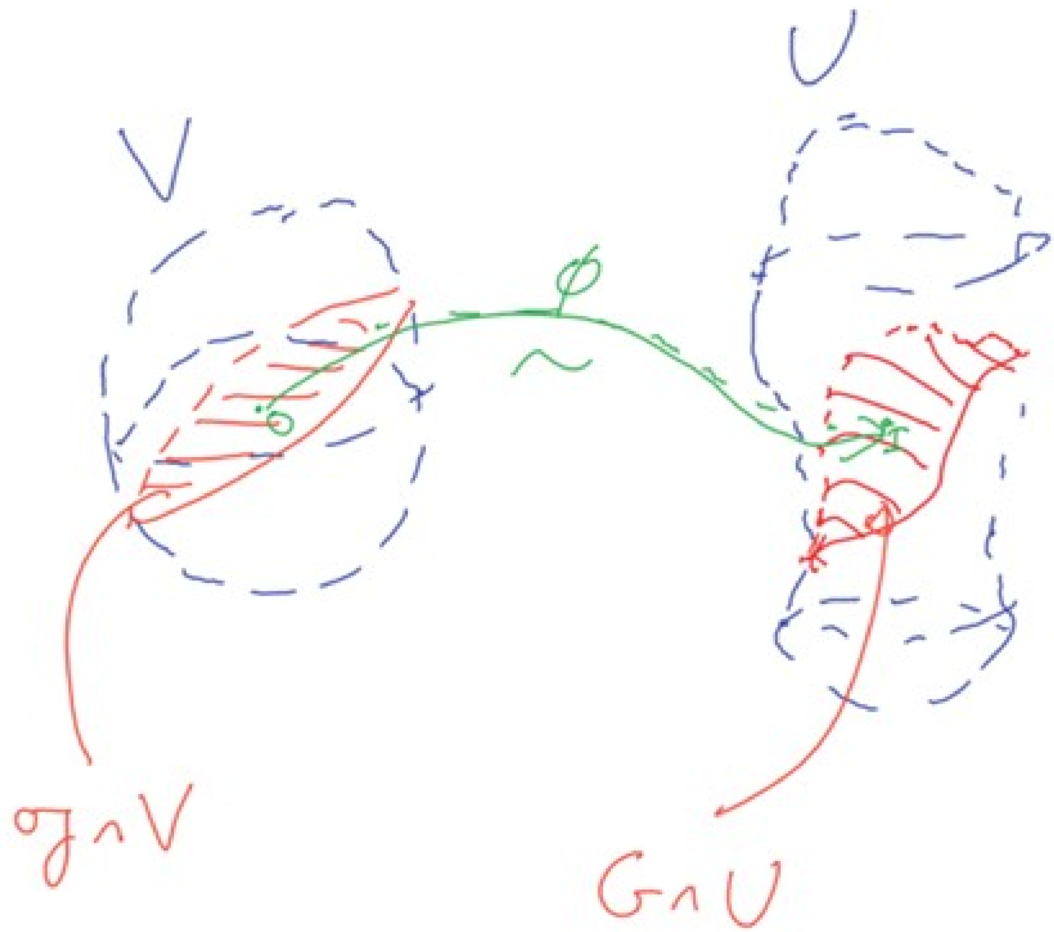

FIGURE 1. Chart ϕ in closed subgroup theorem.

PROOF. (Sketch, non-examinable) Since, for any g, the map 'multiply by g' is smooth, it suffices to prove this when g is the identity element. In this case, one can show that, for a sufficiently small neighbourhood V of 0, the exponential map satisfies $\exp(V \cap \mathfrak{g}) = \exp(V) \cap G$ — the tricky point is to show that, if $g \in G$ is sufficiently close to the identity, then $\log(g) \in \mathfrak{g}$. $\square$

If you don't want to take this theorem on faith, then feel free to include its conclusion as part of the definition of a linear Lie group (in all our examples, it would be straightforward to verify).

Definition 6.2. *We say that G is* connected *if, for every $x, y \in G$, there is a continuous function $\gamma : [0,1] \to G$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

For those of you taking courses in topology, this is actually the definition of *path-*connected; however, it follows from the closed subgroup theorem that Lie groups are locally path-connected, and being path-connected is equivalent to being connected for such spaces.

Let G be a linear Lie group and let G^0 be the set of all $g \in G$ such that there is a continuous path $\gamma : [0,1] \to G$ with $\gamma(0) = \mathrm{Id}$ and $\gamma(1) = g$.

Proposition 6.3. *The subset G^0 is a normal subgroup of G.*

Proof. Let $g, h \in G^0$, and let γ_1, γ_2 be paths with $\gamma_i(0) = \mathrm{Id}$ and $\gamma_1(1) = g, \gamma_2(1) = h$.

Then define a path from Id to gh by following γ_1 and then $g\gamma_2$. Concretely, define $\gamma : [0,1] \to G$ by

$$\gamma(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ g\gamma_2(2t) & 1/2 \le t \le 1 \end{cases}$$

and observe that this is a continuous path from Id to gh. This shows G^0 is closed under multiplication.

The identity and inverse axioms, and the normality, are left as exercises (Problem 13). $\square$

Proposition 6.4. *The subgroup G^0 is an open and closed subset of G.*

Proof. Firstly, if $H \subseteq G$ is an open subgroup then

$$G \setminus H = \bigcup_{g \in G \setminus H} gH$$

is a union of open subsets, so H is also closed.

To show G^0 is open, it suffices to show that it contains an open subset U containing the identity, as then gU is an open subset containing g, for all $g \in G$. If V is a sufficiently small open ball around $0 \in \mathfrak{g}$ then by Theorem 6.1 $\exp(V)$ is an open subset around $\mathrm{Id} \in G$, and $\exp(V)$ is path-connected since V is. Thus $V \subseteq G^0$ as required. $\square$

It follows from this result that the quotient topology on G/G^0 is discrete. [2]

It is clear that G is connected if and only if $G = G^0$.

Proposition 6.5. *If $X \in \mathfrak{g}$, then $\exp(X) \in G^0$.*

Proof. Note that for any $X \in \mathfrak{g}$, the image $\{\exp(tX) \,|\, t \in \mathbb{R}\}$ defines a curve in G containing the identity $\mathrm{Id} \in G$. This curve is in the connected component of the identity. So $\exp(tX) \in G^0$ for all t. $\square$

Theorem 6.6. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Then the subgroup generated by $\exp(\mathfrak{g})$ is G^0.*

In particular, if G is connected, then each element of G is a (non-unique) product of a finite number of exponentials.

[2] If you don't know the definition of the quotient topology, please ignore this.

Proof. Since $\exp(\mathfrak{g})$ contains an open neighbourhood of the identity (by Theorem 6.1), it follows that $\exp(\mathfrak{g})$ generates an open subgroup of G^0, which is then necessarily closed. But since G^0 is connected it has no proper non-empty open and closed subsets.[3] □

As a corollary we immediately obtain the answer to the first question above.

Proposition 6.7. *Let G be a connected (linear) Lie group and let $\phi : G \to G'$ be a Lie group homomorphism. Then the differential $D\phi : \mathfrak{g} \to \mathfrak{g}'$ uniquely determines ϕ.*

Proof. Since $\phi(\exp(X)) = \exp(D\phi(X))$, the values $D\phi(X)$ determine ϕ on the subgroup generated by the $\exp(X)$, which is exactly $G^0 = G$. □

Example 6.8. *Any finite group G can be embedded in $\mathrm{GL}_n(\mathbb{C})$ for some n, and so regarded as a linear Lie group. Its Lie algebra is the zero vector space, so the derivative of a homomorphism $G \to H$ is always zero. In other words, the Lie algebra knows nothing in this case.*

Example 6.9. *Recall that on the orthogonal group $\mathrm{O}(n)$, the determinant (which is a continuous map!) takes the values $\{\pm 1\}$. Hence $\mathrm{O}(n)$ is not connected; in fact, it is not too hard to show that $\mathrm{SO}(n)$ is the connected component of the identity. (This is related to $\mathfrak{so}_n = \mathfrak{o}_n$; that is, the condition $X = -{}^tX$ automatically implies that X has trace zero and hence that $\exp(X)$ has determinant 1.)*

Correspondingly, the determinant $\det$ on $\mathrm{O}(n)$ has zero differential, as it is constant on an open neighbourhood of the identity. This means that the differential on $\mathrm{O}(n)$ cannot distinguish the determinant from the trivial map ($g \mapsto 1 \in \mathbb{R}^\times$).

Proposition 6.10. *The group $\mathrm{SL}_n(\mathbb{R})$ is connected.*

Proof. (sketch, non-examinable)

(a) Use Gram–Schmidt orthogonalisation to show that $SL_n(\mathbb{R}) = SO(n)N_+$ where N_+ is the group of upper triangular matrices with positive diagonal entries.
(b) Show that $SO(n)$ is connected (see previous exercise) and N_+ is connected.
(c) Deduce that $\mathrm{SL}_n(\mathbb{R})$ is connected.

□

Remark 6.11. *There is an alternative proof: show that $\mathrm{SL}_n(\mathbb{R})$ is generated by elementary matrices, and then connect every elementary matrix to the identity.*

Among the Lie groups related to this course, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{SO}(n)$, and $\mathrm{Sp}(2n)$ are connected, while $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{O}(n)$ are not connected, with their connected components being $\mathrm{GL}_n^+(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \det g > 0\}$ and $\mathrm{SO}(n)$ respectively.

6.2. Exercises.

Problem 13. Complete the proof of Proposition 6.3, i.e. that if G is a Lie group and G^0 is the connected component of the identity, then G^0 is a normal subgroup.

Problem 14.

(a) Give a direct proof that $\mathrm{SO}(3)$ is connected, by constructing a path from an arbitrary element of $\mathrm{SO}(3)$ to the identity. *Hint: every element of* $\mathrm{SO}(3)$ *is rotation by some angle about some axis.*

[3]This argument uses the basic fact that a path-connected topological space is connected.

(b) Prove by induction on n that $SO(n)$ is connected for all $n \geq 1$.

Problem 15. Show that, if G is a connected (linear) Lie group with Lie algebra $\mathfrak{g}$, then G is abelian if and only if $\mathfrak{g}$ is (see Definition 4.9). *Hint: for the converse, consider the adjoint map* $G \to GL(\mathfrak{g})$.

What goes wrong if G is not connected?

Solution: see Proposition 10.7.

7. Lecture 7

7.1. Topological properties: answer to question (2). We now turn to the second question, whether every Lie algebra homomorphism exponentiates to a Lie group homomorphism. In the light of what we have seen, it is sensible to restrict to the case of connected Lie groups. However, even with this restriction, the answer is in general no, as the next example shows!

EXAMPLE 7.1. *The linear Lie groups* $\mathrm{GL}_1^+(\mathbb{R}) = \mathbb{R}_{>0}$ *and* $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ *both have Lie algebra* $\mathbb{R}$ *with trivial Lie bracket; in the second case we get the subspace* $i\mathbb{R} \subseteq \mathbb{C}$ *of* $\mathfrak{gl}_{2,\mathbb{C}}$ *and identify it with* $\mathbb{R}$ *by dividing by* i.

The Lie algebra homomorphisms $\mathbb{R} \to \mathbb{R}$ *are all of the form* $\phi_a : t \mapsto at$ *for some* $a \in \mathbb{R}$. *We consider which of these exponentiate to homomorphisms of Lie groups.*

(i) The map ϕ_a *always exponentiates to a map* $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$, *specifically the map*

$$x \longmapsto e^{ax}.$$

(ii) The map ϕ_a *always exponentiates to a map* $\mathbb{R}_{>0} \to U(1)$, *specifically the map*

$$x \longmapsto e^{iax}.$$

(iii) The map ϕ_a *never exponentiates to a map* $U(1) \to \mathbb{R}_{>0}$ *if* $a \neq 0$. *If it did, the map would have to send*

$$e^{ix} \longmapsto e^{ax},$$

and setting $x = 2\pi$ *gives* $a = 0$.

(iv) The map ϕ_a *exponentiates to a map* $U(1) \to U(1)$ *if and only if* $a \in \mathbb{Z}$, *in which case the map is*

$$z \longmapsto z^a.$$

Indeed, the map would have to be

$$e^{ix} \longrightarrow e^{iax}$$

and setting $x = 2\pi$ *shows that* $a \in \mathbb{Z}$, *when the map is as claimed.*

The key difference between $\mathbb{R}_{>0}$ and $U(1)$ is that the former is simply connected while the latter is not (it has fundamental group $\mathbb{Z}$). We explain this a bit further.

DEFINITION 7.2. *A topological space* X *is* simply connected *if it is path-connected and if every continuous map from the unit circle to* X *can be extended to a continuous map from the unit disc to* X.

In other words if it is path-connected and if every loop can be continuous shrunk to a single point. In topology, the failure of a space to be simply connected is measured by the 'fundamental group' $\pi_1(X)$: X is simply-connected if and only if $\pi_1(X)$ is trivial.

THEOREM 7.3. *Let* G *be a simply connected (linear) Lie group. Let* G' *be any other (linear) Lie group. Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be their Lie algebras. Then every homomorphism* $\mathfrak{g} \to \mathfrak{g}'$ *exponentiates to a unique homomorphism* $G \to G'$.

Hence we have a 1-1 correspondence

$$\{\mathrm{Hom}(G, G')\} \quad \longleftrightarrow \quad \{\mathrm{Hom}(\mathfrak{g}, \mathfrak{g}')\}.$$

Proof. This is beyond the scope of this course. Note in the above example $\mathrm{GL}_1^+(\mathbb{R})$ is simply connected while the circle group $U(1)$ is not. □

One can show that $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{SU}(n)$ are simply connected. Here is a small table showing our connected groups and their fundamental groups.

G	$\pi_1(G)$
$\mathrm{GL}_n(\mathbb{C})$	$\mathbb{Z}$
$\mathrm{SL}_n(\mathbb{C})$	1
$\mathrm{SL}_2(\mathbb{R})$	$\mathbb{Z}$
$\mathrm{SL}_n(\mathbb{R})$, $n \geq 3$	C_2
$\mathrm{SO}(2)$	$\mathbb{Z}$
$\mathrm{SO}(n)$, $n \geq 3$	C_2
$\mathrm{U}(n)$	$\mathbb{Z}$
$\mathrm{SU}(n)$	1
$\mathrm{Sp}(2n)$	Z

Remark 7.4. *It is not an accident that the fundamental groups of* $\mathrm{SL}_n(\mathbb{R})$ *and* $\mathrm{SO}(n)$ *are isomorphic — Gram–Schmidt orthogonalisation, as used in the proof of Proposition 6.10, shows that* $\mathrm{SL}_n(\mathbb{R})$ *and* $\mathrm{SO}(n)$ *are homotopy equivalent. A similar remark applies to* $\mathrm{SL}_n(\mathbb{C})$ *and* $\mathrm{SU}(n)$*.*

If G is not connected, or its identity component is not simply connected, we can work in the following way.

- There exists a 'universal cover' $\widetilde{G}$ of G^0 which is simply connected, and also has the structure of a Lie group (not necessarily linear, unfortunately). There is a surjective group homomorphism $\pi : \widetilde{G} \to G^0$ with discrete kernel $Z \cong \pi_1(G^0)$, so that $G^0 \cong \widetilde{G}/Z$.
- The kernel Z of π is isomorphic to the fundamental group $\pi_1(G^0)$.
- Homomorphisms out of G^0 are in 1-1 correspondence with homomorphisms out of $\widetilde{G}$ which are trivial on Z.
- The Lie algebras of G, G^0 and $\widetilde{G}$ coincide (more precisely, the maps $\pi : \widetilde{G} \to G^0$ and $\iota : G^0 \to G$ induce isomorphisms of Lie algebras).
- In general G/G^0 can be an arbitrary finite group! For this reason, it is common to restrict attention to connected Lie groups.

The diagram looks as follows:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\exp"] \arrow[d, equal] & \widetilde{G} \arrow[d, "\pi"] & \\
\mathfrak{g} \arrow[r, "\exp"'] & G^0 \arrow[r, "\iota"] & G.
\end{tikzcd}$$

Example 7.5. *The group* $U(1)$ *is not simply connected. Here the universal cover is* $(\mathbb{R}, +)$ *(this is a linear Lie group because it is isomorphic to the upper triangular* 2×2 *matrices with* 1*s on the diagonal — a similar argument shows that any vector space (with*

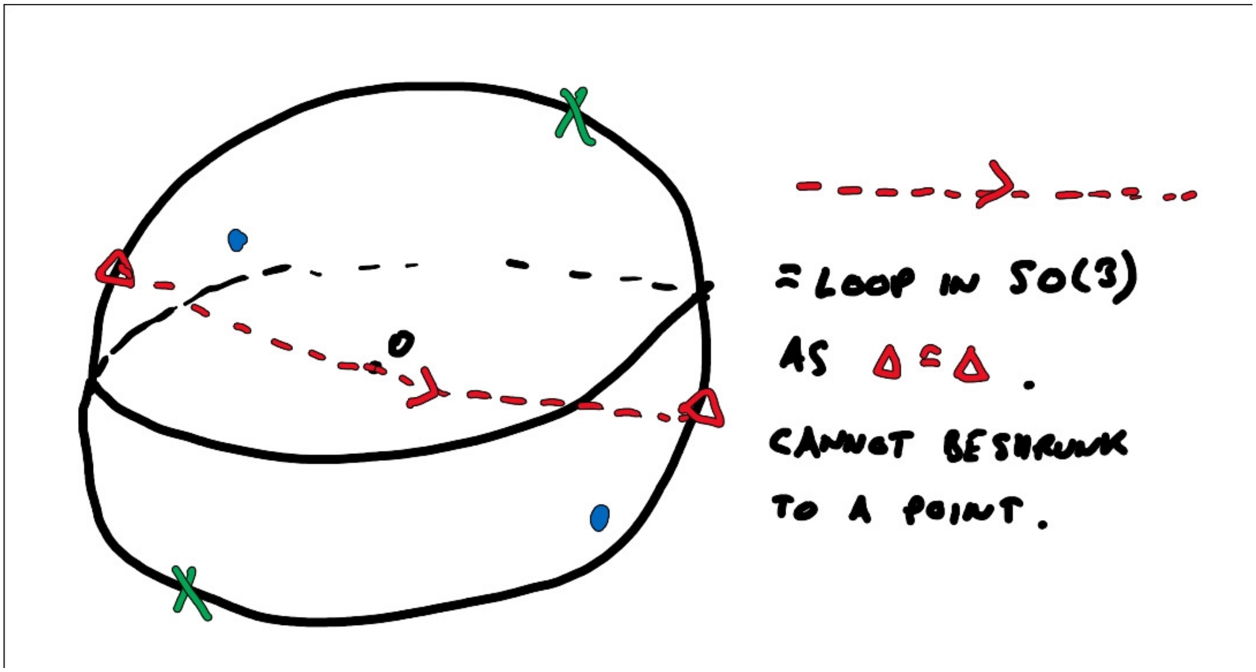

FIGURE 2. Picture of SO(3)

addition) is a Lie group). The map $\widetilde{G} \to G^0$ is then

$$\begin{aligned} \pi : \mathbb{R} &\longrightarrow U(1) \\ x &\longmapsto e^{2\pi i x} \end{aligned}$$

and we see that the kernel of π is $\mathbb{Z}$, which is indeed the fundamental group $\pi_1(U(1))$.

7.2. The example of $SU(2)$ and $SO(3)$. We illustrate the previous section with the example of SO(3). According to the table of fundamental groups above, the fundamental group of SO(3) is C_2. We can visualise this as follows: An element of SO(3) is rotation by some angle $\theta \in [0, \pi]$ about some (oriented) axis. We can represent this as a vector in $\mathbb{R}^3$ of length θ in the direction of the axis. Elements of SO(3) then correspond to points in the closed ball in $\mathbb{R}^3$ of radius 2π. However, rotation by π about the axis $\mathbf{v}$ is the same as rotation by π about the axis $-\pi$, and so we must identify diametrically opposite points on the boundary of this ball.

Now, the straight line in this three-dimensional sphere from a point on the boundary to its diametrically opposite point is a *loop* in SO(3) since the endpoints represent the same rotation. You can convince yourself that this loop cannot be shrunk to a point (proving it rigorously requires some topology). However, if you go around the loop twice, then that can (!) be shrunk to a point. The idea is to move one copy of the loop out to the boundary, then use the 'opposite point' identification to move it to the other side, when you get a normal loop inside the ball which may be shrunk. See Figure 3.

A nice physical illustration of this is provided by the "Dirac belt trick"; here is a video of this demonstrated with long hair!

According to the general picture of the previous section, there should be a Lie group homomorphism $\tilde{G} \to \mathrm{SO}(3)$ whose kernel has order 2 and such that $\tilde{G}$ is simply connected, and it turns out that we can take $\tilde{G} = \mathrm{SU}(2)$. So we study this group for a bit.

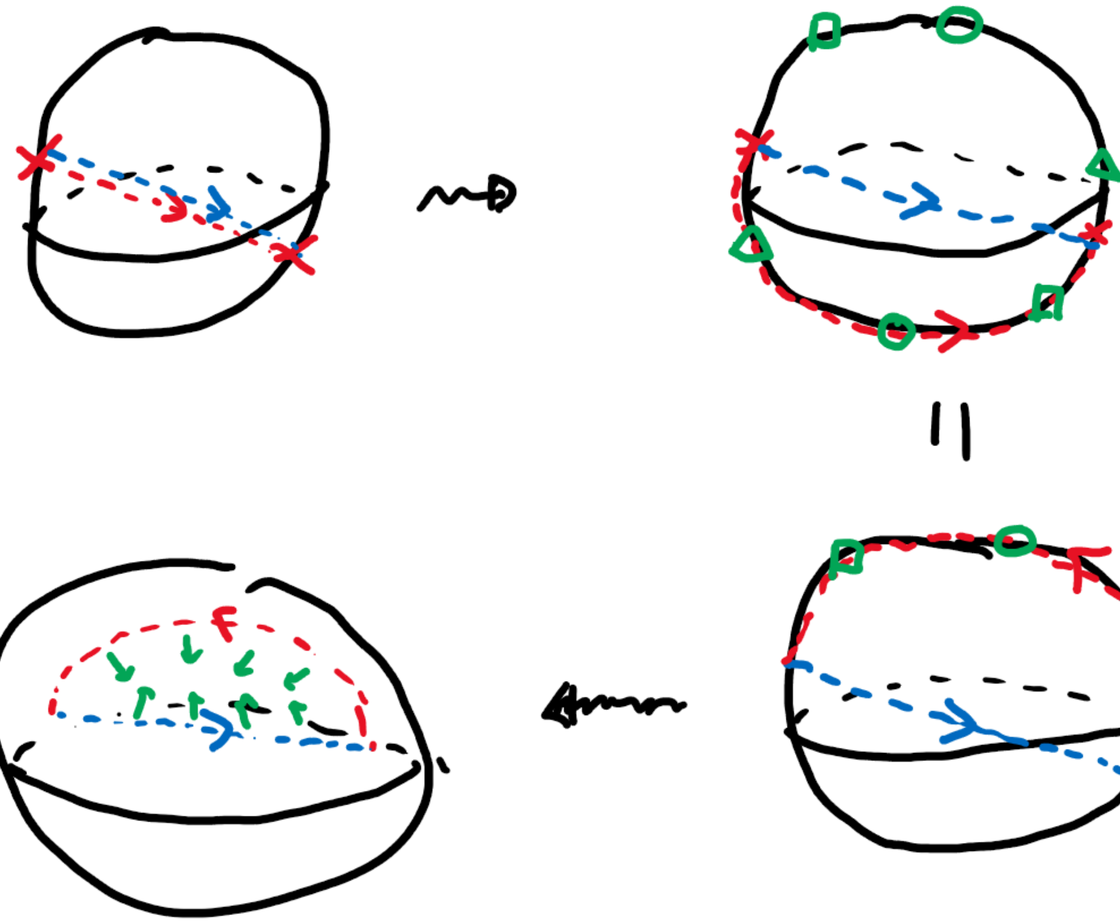

FIGURE 3. Contracting twice a loop

Firstly, one can show (see problem 17) that every element of SU(2) has the form

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$$

for $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. It follows that SU(2) is diffeomorphic to the unit sphere S^3 in $\mathbb{R}^4$, which is simply connected.

We would like to write down a homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Recall Example 3.3(ii), if we take an inner product space V then all $g \in \mathrm{GL}_n(k)$ such that $\langle g\mathbf{v}, g\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$, for all $\mathbf{v}, \mathbf{w} \in V$, forms a Lie group. We can also write this as all $g \in \mathrm{GL}_n(k)$ such that $g^T A g = A$, where A is the matrix of the inner product, with respect to some basis. If we pick an orthonormal basis with respect to this inner product we get the definition of $\mathrm{SO}(n)$.

Thus, to find our homomorphism, we want to find a three-dimensional real vector space V, equipped with an inner product (i.e. a positive definite, symmetric, bilinear form) that is preserved by the action of SU(2). If we write down an orthonormal basis for V, then the matrix of the action of each element of SU(2) on V, with respect to this basis, will be an element of SO(3) giving the required homomorphism.

Where can we find V? From SU(2) itself! We take $V = \mathfrak{su}_2$, a three-dimensional vector space (by problem 8), and let SU(2) act on V via conjugation. We just need an inner product, and we may define one as follows: for $X, Y \in \mathfrak{su}_2$, let

$$\langle X, Y\rangle := -\operatorname{tr}(XY).$$

The proof this is an inner product is Problem 18. We therefore obtain a homomorphism $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ once we fix an orthonormal basis of $\mathfrak{su}_2$ with respect to the inner product. The basis elements may be taken to be the matrices

$$\frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}.$$

It is an exercise to check that it has the right kernel, i.e. $\ker \pi = C_2$.

7.3. Exercises.

Problem 16. Check that the Lie algebra of SO(2) is also isomorphic to $\mathbb{R}$. Write down an isomorphism $U(1) \to \mathrm{SO}(2)$; what is the identification of Lie algebras it induces?

Problem 17. Show that a general element of SU(2) may be written

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a}\end{pmatrix}$$

for $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$.

Deduce that SU(2) is diffeomorphic to the three-sphere $S^3 = \{v \in \mathbb{R}^4 \mid |v| = 1\}$.

In other words, write down a smooth bijection $\mathrm{SU}(2) \to S^3$ *with smooth inverse. Don't worry about checking that the maps are smooth, just write them down. The result of this problem implies that* SU(2) *is simply-connected.*

Problem 18.

(a) Show that $\langle\cdot,\cdot\rangle$ is a symmetric, positive definite bilinear form on $\mathfrak{su}_2$.
(b) Show that it is preserved by the action of SU(2), i.e. that

$$\left\langle gXg^{-1}, gYg^{-1}\right\rangle = \langle X, Y\rangle$$

for all $g \in \mathrm{SU}(2)$.

Problem 19. Write down explicitly the image of $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a}\end{pmatrix}$ under the homomorphism π.

Problem 20. Show that, if $g \in \mathrm{SU}(2)$ satisfies $gXg^{-1} = X$ for all $X \in \mathfrak{su}_2$, then $g = \pm\,\mathrm{Id}$. Deduce that

$$\ker(\pi) = \{\pm\,\mathrm{Id}\} \cong C_2.$$

Problem 21. If $\mathfrak{g}$ is a Lie algebra, let $\mathfrak{z}$ be its centre:

$$\mathfrak{z} = \{Z \in \mathfrak{g} \mid [X, Z] = 0,\ \forall X \in \mathfrak{g}.\}.$$

Suppose that G is a connected Lie group with centre Z and Lie algebra $\mathfrak{g}$ with centre $\mathfrak{z}$.

Prove that $\mathfrak{z}$ is the Lie algebra of Z.

CHAPTER 2

Representations of Lie groups and Lie algebras - generalities

8. Lecture 8

8.1. Basics.

Definition 8.1. *A* finite-dimensional (complex) representation *(ρ, V) of a Lie group G is a Lie group homomorphism*

$$\rho : G \longrightarrow \mathrm{GL}(V),$$

where V is a finite-dimensional complex vector space.

Remark 8.2. *Infinite dimensional representations are important, but subtle. In general one must equip V with some topology and add topological conditions to all notions which follow. For instance, one might take V to be a Hilbert space.*

An example of such a representation arises naturally if you attempt to generalise the regular representation! One must take V to be something like the space of square-integrable functions on the group, rather than the space of arbitrary functions, to get a pleasant theory.

If ρ is a finite-dimensional representation of G as above, then we can take its derivative:

$$D\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = \mathrm{End}(V),$$

mapping from the Lie algebra $\mathfrak{g}$ of G to the space of endomorphisms of V. Note that

$$\mathrm{End}(V)$$

is a Lie algebra with bracket

$$[S, T] = ST - TS.$$

The map $D\rho$ is a Lie algebra homomorphism, by Theorem 5.6(iii). Often we write, abusively, ρ instead of $D\rho$.

Note that choosing an isomorphism $V \cong \mathbb{C}^n$ induces isomorphisms $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{gl}(V) \cong \mathfrak{gl}_{n,\mathbb{C}}$.

Definition 8.3. *A* (complex) representation *(ρ, V) of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism*

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V),$$

where V is a complex vector space. That is,

(i) ρ is $\mathbb{R}$-linear;
(ii) $\rho([X, Y]) = [\rho(X), \rho(Y)]$.

Additionally, we say (ρ, V) is complex-linear *if $\mathfrak{g}$ is a complex Lie algebra and ρ is $\mathbb{C}$-linear.*

By our above discussion the differential of a Lie group representation is a Lie algebra representation.

REMARK 8.4. *Warning! It is not the case that, if ρ is a Lie algebra representation, then*

$$\rho(XY) = \rho(X)\rho(Y).$$

Indeed, in general XY need not be an element of the Lie algebra at all, and even if it is the displayed equation will not usually hold.

The notions of *G-homomorphism* (or *$\mathfrak{g}$-homomorphism*, or *intertwiner*), *isomorphism*, *subrepresentation*, and *irreducible* representation stay the same as for finite groups. For example, a $\mathfrak{g}$-homomorphism from (ρ, V) to (ρ', V') is a linear map $\phi : V \to V'$ such that

$$\phi(\rho(X)\mathbf{v}) = \rho'(X)\phi(\mathbf{v})$$

for all $\mathbf{v} \in V$ and $X \in \mathfrak{g}$.

DEFINITION 8.5. *A* $\mathbb{C}$-linear *representation of $\mathfrak{g}$ is a complex representation ρ of $\mathfrak{g}$ such that*

$$\rho(\lambda X) = \lambda\rho(X)$$

for all $\lambda \in \mathbb{C}$ and $X \in \mathfrak{g}$.

If G is a complex Lie group, then a holomorphic representation of G is a complex representation whose derivative is $\mathbb{C}$-linear; equivalently, the map $G \to \mathrm{GL}(V)$ is holomorphic.

THEOREM 8.6. *Let G be a Lie group, $\mathfrak{g}$ be a Lie algebra.*

(i) (Schur's Lemma) If V_1 and V_2 are irreducible finite-dimensional representations of G (or $\mathfrak{g}$), then

$$\dim \mathrm{Hom}_{G\ (or\ \mathfrak{g})}(V_1, V_2) = \begin{cases} 1 & \text{if } V_1 \cong V_2 \\ 0 & \text{otherwise.} \end{cases}$$

If $V_1 = V_2$, then any G- (or $\mathfrak{g}$-)homomorphism $T : V \to V$ is scalar.

(ii) Any irreducible finite-dimensional representation of an abelian *Lie group or Lie algebra is one-dimensional.*

(iii) If (ρ, V) is an irreducible finite-dimensional representation of G (or $\mathfrak{g}$) and Z (or $\mathfrak{z}$) is the center of G (or $\mathfrak{g}$) then there is a homomorphism $\chi : Z \to \mathbb{C}^\times$ (or $\chi : Z \to \mathbb{C}$) such that

$$\rho(z)\mathbf{v} = \chi(z)\mathbf{v}$$

for all $z \in Z$ (or $\mathfrak{z}$) and $\mathbf{v} \in V$. We call this the central character.

PROOF. The proofs are all the same as in the finite group case! □

PROPOSITION 8.7. *Let (ρ, V) be a finite-dimensional representation of a Lie group G. Let $D\rho$ be its derivative.*

(i) If $W \subseteq V$ is invariant under $\rho(G)$, then W is invariant under $D\rho(\mathfrak{g})$.

(ii) If $(D\rho, V)$ is irreducible, then (ρ, V) is irreducible.

(iii) If ρ is unitary, that is, there is a basis for V such that $\rho(g) \in \mathrm{U}(n)$ for all $g \in G$, then $D\rho$ is skew Hermitian, that is, $D\rho(X) \in \mathfrak{u}_n$ for all $X \in \mathfrak{g}$ (using the same basis for V).

(iv) Let (ρ', V') be another finite-dimensional representation of G. If $(\rho, V) \cong (\rho', V')$, then $(D\rho, V) \cong (D\rho', V')$.

If G is connected, then the converses to these statements hold.

So, for connected Lie groups, we can test irreducibility and isomorphism at the level of Lie algebras.

Proof. For (i), we know that $\rho(\exp(tX))(\mathbf{w}) \in W$ for any $X \in \mathfrak{g}$ and $\mathbf{w} \in W$. Taking the derivative at $t = 0$, it follows that $D\rho(X)(\mathbf{w}) \in W$ as required. Part (ii) follows from (i).

For (iii), if ρ is unitary, then after choosing a basis appropriately it is a Lie group homomorphism $\rho : G \to U(n)$. The derived homomorphism therefore lands in the Lie algebra $\mathfrak{u}(n)$ of $\mathrm{U}(n)$.

For part (iv), let T be a G-isomorphism, so that in particular,

$$T\rho(\exp(tX))T^{-1} = \rho'(\exp(tX))$$

for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. Taking the derivative at $t = 0$ gives

$$TD\rho(X)T^{-1} = D\rho'(X)$$

so that T is a $\mathfrak{g}$-isomorphism as required.

If G is connected, then G is generated by $\exp(\mathfrak{g})$. Hence all proofs above can be reversed. For example, for (i), suppose that W is preserved by $D\rho(\mathfrak{g})$. If $\mathbf{w} \in W$ and $X \in \mathfrak{g}$, then

$$\rho(\exp(X))\mathbf{w} = \exp(D\rho(X))\mathbf{w} = \sum_{n=0}^{\infty} \frac{(D\rho(X))^n}{n!}\mathbf{w} \in W$$

as W is preserved by $D\rho(X)$ and also closed. Since every $g \in G$ can be written as a finite product of $\exp(X_i)$ for $X_i \in \mathfrak{g}$, we see that W is preserved by $\rho(G)$ as required. The converse of (iii) and (iv) are left as exercises. $\square$

8.2. Exercises.

Problem 22. Let G be connected. Prove the converse of Proposition 8.7(iii) and (iv).

9. Lecture 9

9.1. Standard constructions for representations. We give a list of various constructions with representations of Lie groups, and the analogous constructions for their derivatives.

DEFINITION 9.1. *The* standard representation *of a linear Lie group $G \subseteq \mathrm{GL}_n(\mathbb{C})$ comes from its action on $\mathbb{C}^n$ so for $\mathbf{v} \in \mathbb{C}^n$*

$$\rho(g)\mathbf{v} = g\mathbf{v}$$
$$D\rho(X)\mathbf{v} = X\mathbf{v}.$$

DEFINITION 9.2. *The* direct sum *of representations (ρ_1, V_1), (ρ_2, V_2) of G is $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$ with derivative*

$$D(\rho_1 \oplus \rho_2) = D\rho_1 \oplus D\rho_2.$$

DEFINITION 9.3. *The* determinant representation *of $G \subseteq \mathrm{GL}_n(\mathbb{C})$ is $\det : G \to \mathbb{C}^*$ which sends g to $\det(g)$. We have*

$$D\det(X) = \mathrm{tr}(X),$$

which follows from $\det\exp(tX) = e^{t\,\mathrm{tr}(X)}$.

DEFINITION 9.4. *If (ρ, V) is a representation of G, the* dual representation *(ρ^*, V^*) of (ρ, V) is defined by*

$$\left(\rho^*(g)(\lambda)\right)(\mathbf{v}) = \lambda\left(\rho(g^{-1})(\mathbf{v})\right),$$

for $\lambda \in V^*$ a linear functional on V and $g \in G$. It has derivative

$$D\rho^*(X)(\lambda)(\mathbf{v}) = -\lambda(D\rho(X)\mathbf{v}).$$

Given a basis of V, then the matrix of ρ^* with respect to the dual basis is

$$\rho^*(g) = (\rho(g)^{-1})^T,$$

which differentiates to

$$(D\rho^*)(X) = -D\rho(X)^T.$$

We can take tensor/symmetric/alternating products of more than one factor.

DEFINITION 9.5. *Suppose (ρ_i, V_i) are representations of G. We form the* tensor product

$$V_1 \otimes V_2 \otimes \ldots \otimes V_l.$$

It is generated by symbols $\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_l$ subject to the multilinear relations, i.e.

$$\mathbf{v}_1 \otimes \ldots \otimes (\lambda\mathbf{v}_i + \mu\mathbf{v}_i') \otimes \ldots \otimes \mathbf{v}_l = \lambda(\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_i \otimes \ldots \otimes \mathbf{v}_l) + \mu(\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_i' \otimes \ldots \otimes \mathbf{v}_l),$$

for each V_i. One has

$$\dim\left(V_1 \otimes V_2 \otimes \ldots \otimes V_l\right) = \prod_{i=1}^{l} \dim V_i.$$

The action of G is

$$(\rho_1 \otimes \ldots \otimes \rho_l)(g)(\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_l) = \rho_1(g)\mathbf{v}_1 \otimes \ldots \otimes \rho_l(g)\mathbf{v}_l.$$

for $g \in G$, $\mathbf{v}_i \in V_i$. We also write

$$V^{\otimes l} = V \otimes \ldots \otimes V.$$

LEMMA 9.6. *Suppose $(\rho_1 \oplus \ldots \oplus \rho_n, V_1 \oplus \ldots \oplus V_n)$ is a tensor product of representations of G. Then the derivative of this representation is given by*

$$\begin{aligned} D(\rho_1 \otimes \ldots \otimes \rho_l)(X) = & D\rho_1(X) \otimes \mathrm{Id}_{V_2} \otimes \ldots \otimes \mathrm{Id}_{V_l} \\ & + \mathrm{Id}_{V_1} \otimes D\rho_2(X) \ldots \otimes \mathrm{Id}_{V_l} \\ & + \ldots \\ & + \mathrm{Id}_{V_1} \otimes \mathrm{Id}_{V_2} \ldots \otimes D\rho_l(X), \end{aligned}$$

for all $X \in \mathfrak{g}$.

PROOF. Suppose (π, V), (ρ, W) are representations of G. Let $X \in \mathfrak{g}$ then

$$\begin{aligned} D(\pi \otimes \rho)(X) &= \frac{d}{dt}\pi(\exp(tX)) \otimes \rho(\exp(tX))\bigg|_{t=0} \\ &= \frac{d}{dt}\exp(D\pi(tX)) \otimes \exp(D\rho(tX))\bigg|_{t=0} \\ &= \frac{d}{dt}(\mathrm{Id}_V + tD\pi(X) + O(t^2)) \otimes (\mathrm{Id}_W + tD\rho(X) + O(t^2))\bigg|_{t=0} \\ &= \frac{d}{dt}\mathrm{Id}_V \otimes \mathrm{Id}_W + t(D\pi(X) \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes D\rho(X)) + O(t^2)\bigg|_{t=0} \\ &= D\pi(X) \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes D\rho(X), \end{aligned}$$

as required. We can then apply an inductive argument to show this hold for all tensor products. ☐

DEFINITION 9.7. *Let (ρ, V) be a representation of G. The lth* <u>symmetric product</u> *is the space $\mathrm{Sym}^l(V)$ generated by symbols $\mathbf{v}_1 \ldots \mathbf{v}_l$ with linearity in each of the $\mathbf{v}_i$s and any permutation of the vectors giving the same element.*

We have

$$\dim \mathrm{Sym}^l(V) = \binom{n+l-1}{l} = \binom{n+l-1}{n-1}$$

where $\dim V = n$. Indeed, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a basis for V then a basis for $\mathrm{Sym}^l(V)$ is

$$\{\mathbf{e}_{i_1} \ldots \mathbf{e}_{i_l} \,|\, 1 \leq i_1 \leq i_2 \leq \ldots \leq i_l \leq n\}$$

from which finding the dimension is a simple counting problem.

The action of G is

$$\mathrm{Sym}^l \rho(g)(\mathbf{v}_1 \ldots \mathbf{v}_l) = (\rho(g)\mathbf{v}_1) \ldots (\rho(g)\mathbf{v}_l),$$

for all $g \in G$, and all $\mathbf{v}_1 \ldots \mathbf{v}_l \in \mathrm{Sym}^l(V)$.

LEMMA 9.8. *Suppose $(\mathrm{Sym}^l \rho, \mathrm{Sym}^l(V))$ is the lth symmetric product of a representation of G. Then the derivative of this representation is given by*

$$D\,\mathrm{Sym}^l \rho(X)(\mathbf{v}_1 \ldots \mathbf{v}_l) = (D\rho(X)\mathbf{v}_1)\mathbf{v}_2 \ldots \mathbf{v}_l + \ldots + \mathbf{v}_1 \ldots \mathbf{v}_{l-1}(D\rho(X)\mathbf{v}_l),$$

for all $X \in \mathfrak{g}$, and all $\mathbf{v}_1 \ldots \mathbf{v}_l \in \mathrm{Sym}^l(V)$.

PROOF. Exercise. ☐

Definition 9.9. *The lth alternating product is the space $\bigwedge^l(V)$ generated by symbols $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_l$ with linearity in each entry and having the alternating property: for any permutation $\sigma \in S_l$, we have*

$$\mathbf{v}_{\sigma(1)} \wedge \ldots \wedge \mathbf{v}_{\sigma(l)} = \operatorname{sgn}(\sigma)(\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_l).$$

In particular, switching the places of two components reverses the sign, while $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_l = 0$ if two of the vectors coincide (more generally, if they are linearly dependent). Thus if $l > \dim V$ we have $\bigwedge^l V = 0$.

We have

$$\dim \bigwedge^l V = \binom{n}{l}$$

where $\dim V = n$. Indeed, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are a basis for V then a basis for $\bigwedge^l(V)$ is

$$\{\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l} \mid 1 \leq i_1 < i_2 < \ldots < i_l \leq n\}$$

from which finding the dimension is a simple counting problem. In particular, $\bigwedge^n \mathbb{C}^n$ is one-dimensional generated by $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n$.

The representation on $\bigwedge^l(V)$ is

$$\bigwedge^l \rho(g)(\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_l) = \rho(g)\mathbf{v}_1 \wedge \ldots \wedge \rho(g)\mathbf{v}_l,$$

for all $g \in G$, and all $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_l \in \bigwedge^l(V)$.

Lemma 9.10. *Suppose $(\bigwedge^l \rho, \bigwedge^l(V))$ is the lth alternating product of a representation of G. Then the derivative of this representation is given by*

$$D \bigwedge^l \rho(X)(\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_l) = D\rho(X)\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_l + \ldots + \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_{l-1} \wedge D\rho(X)\mathbf{v}_l.$$

for all $g \in G$, and all $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_l \in \bigwedge^l(V)$.

Proof. Exercise. □

Remark 9.11. *We defined tensor products (and so on) of representations of Lie groups and then differentiated them. We could also directly make these definitions with Lie algebras. For instance, if $\mathfrak{g}$ is a Lie algebra and (π, V) is a representation of $\mathfrak{g}$, we define the symmetric square representation on $\mathrm{Sym}^2(V)$ by*

$$\pi(X)(\mathbf{v}\mathbf{w}) = (\pi(X)\mathbf{v})\mathbf{w} + \mathbf{v}(\pi(X)\mathbf{w}).$$

Remark 9.12. *We can construct representations as vector spaces of functions on topological spaces with actions of G. If G acts on a set X, then it also acts on the vector space of functions $X \to \mathbb{C}$ by $(g \cdot f)(x) = f(g^{-1}x)$. Usually this will be infinite dimensional, and so out of the scope of our course, but sometimes we can impose conditions allowing us to handle it. For example, $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n$, and hence on the space of polynomial functions in n variables. Imposing a further restriction — to homogeneous polynomials of some fixed degree — gives a finite-dimensional representation. The derivative must be calculated on a case-by-case basis.*

9.2. Exercises.

Problem 23. Prove Lemmas 9.8 and 9.10.

Problem 24.

(a) If (ρ, V) is an irreducible finite-dimensional complex representation of $\mathfrak{g}$ and $\mathfrak{z}$ is the centre of $\mathfrak{g}$ (see problem 21), show that there is a linear map $\alpha : \mathfrak{z} \to \mathbb{C}$ such that $\rho(Z)\mathbf{v} = \alpha(Z)\mathbf{v}$ for all $Z \in \mathfrak{z}$.

(b) For $\mathfrak{g} = \mathrm{GL}_{n,\mathbb{C}}$, find $\mathfrak{z}$. Find α when $V = \bigwedge^k \mathbb{C}^n$, where $\mathbb{C}^n$ is the standard representation and $1 \leq k \leq n$. *These representations are in fact irreducible, though we haven't proved that yet; you can just directly show that α exists.*

Problem 25. Let $V = \mathbb{C}^2$ be the standard representation of $\mathrm{GL}_2(\mathbb{C})$.

(a) Show that $\bigwedge^2(V) \cong \det$ as Lie group representations.

(b) Show that $\bigwedge^2(V) \cong \mathrm{tr}$ as representations of $\mathfrak{gl}_2(\mathbb{C})$. (You could just 'take the derivative' of part (a), but please do it directly instead.)

(c) Find an explicit homomorphism $\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{C})$ corresponding to $\mathrm{Sym}^2(V)$.

10. Lecture 10

10.1. The adjoint representation. Let G be a linear Lie group and $\mathfrak{g}$ be its Lie algebra. By Proposition 3.8(ii), gYg^{-1} is in $\mathfrak{g}$ so $Y \mapsto gYg^{-1}$ is in $\mathrm{GL}(\mathfrak{g})$, for all $g \in G$ and $Y \in \mathfrak{g}$.

Definition 10.1. *The Lie group homomorphism*

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g}) \ ; \ g \longmapsto \mathrm{Ad}_g,$$

where

$$\mathrm{Ad}_g(Y) = gYg^{-1}$$

for $g \in G$ and $Y \in \mathfrak{g}$ is the adjoint representation *of G.*

By convention we write Ad_g for $\mathrm{Ad}(g)$.

Remark 10.2. *An alternative definition, that works for general Lie groups, is to consider the conjugation by g map $h \mapsto ghg^{-1}$ and then take the derivative:*

$$\mathrm{Ad}_g(Y) = \frac{d}{dt} g \exp(tY) g^{-1}|_{t=0}.$$

The derivative of Ad, denoted by ad, is called the *adjoint representation* of the Lie algebra $\mathfrak{g}$. Thus

$$\mathrm{ad} = D\,\mathrm{Ad}\,.$$

Again, we write ad_X for $\mathrm{ad}(X)$, so we have

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g}) \ ; \ X \longmapsto \mathrm{ad}_X\,.$$

By Theorem 5.6 we have the formula

$$\mathrm{Ad}_{\exp(tX)} = \exp(t\,\mathrm{ad}_X)$$

as elements of $\mathrm{GL}(\mathfrak{g})$.

Theorem 10.3. *Let G and $\mathfrak{g}$ be as above and let $X, Y \in \mathfrak{g}$. Then*

(i)

$$\mathrm{ad}_X(Y) = [X, Y] = XY - YX.$$

(ii) The map ad *is a Lie algebra homomorphism, so that*

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y].$$

Thus, for all $Z \in \mathfrak{g}$,

$$\mathrm{ad}_{[X,Y]}(Z) = [\mathrm{ad}_X, \mathrm{ad}_Y](Z) = \mathrm{ad}_X(\mathrm{ad}_Y(Z)) - \mathrm{ad}_Y(\mathrm{ad}_X(Z)).$$

Proof.

(i) Since $\mathrm{Ad}_{\exp(tX)}(Y) = \exp(tX)Y\exp(-tX)$, taking the differential at $t = 0$, we get

$$\mathrm{ad}_X(Y) = XY - YX = [X, Y].$$

(ii) The map ad is a Lie algebra homomorphism because it is the differential of a Lie group homomorphism, Theorem 5.6(iii).

□

REMARK 10.4. *This explains the origin of the Jacobi identity:*

$$\begin{aligned}[\mathrm{ad}_X, \mathrm{ad}_Y](Z) &= \mathrm{ad}_X(\mathrm{ad}_Y(Z)) - \mathrm{ad}_Y(\mathrm{ad}_X(Z)) \\ &= [X,[Y,Z]] - [Y,[X,Z]] \\ &= [X,[Y,Z]] + [Y,[Z,X]],\end{aligned}$$

while

$$\begin{aligned}\mathrm{ad}_{[X,Y]}(Z) &= [[X,Y],Z] \\ &= -[Z,[X,Y]].\end{aligned}$$

Equating these gives the Jacobi identity.

REMARK 10.5. *The first formula,* $\mathrm{ad}_X(Y) = [X,Y]$, *could have been used to define the adjoint representation for any Lie algebra, without reference to Lie groups.*

REMARK 10.6. *Warning! It is very easy to misinterpret some of the formulas concerning the adjoint representation. For example,* $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$ *does* not *mean that*

$$\mathrm{ad}_{[X,Y]}(Z) = [\mathrm{ad}_X(Z), \mathrm{ad}_Y(Z)],$$

but (as already noted and proved) that

$$\mathrm{ad}_{[X,Y]}(Z) = [\mathrm{ad}_X, \mathrm{ad}_Y](Z) = \mathrm{ad}_X(\mathrm{ad}_Y(Z)) - \mathrm{ad}_Y(\mathrm{ad}_X(Z)).$$

Similarly, $\mathrm{Ad}_{\exp(tX)} = \exp(t\,\mathrm{ad}_X)$ *does* not *mean that* $\exp(tX)Y\exp(-tX)$ *is equal to* $\exp(t[X,Y])$ *but rather is the identity*

$$\begin{aligned}\exp(tX)Y\exp(-tX) &= \exp(t\,\mathrm{ad}_X)(Y) \\ &= \sum_{k=0}^{\infty} \frac{t^k(\mathrm{ad}_X)^k}{k!}(Y) \\ &= \sum_{k=0}^{\infty} \frac{[X,[X,\ldots,[X,Y]]\ldots]}{k!}t^k.\end{aligned}$$

PROPOSITION 10.7. *If G is abelian, so is $\mathfrak{g}$. If, moreover, G is connected, then the converse holds.*

PROOF. Suppose that G is abelian. Then, for all $g \in G$ and $Y \in \mathfrak{g}$,

$$g\exp(tY)g^{-1} = \exp(tY).$$

Taking the derivative at $t = 0$ we see that $gYg^{-1} = Y$. Thus Ad is trivial. Differentiating, we see ad is trivial, so $\mathrm{ad}_X = 0$ for all X. Thus $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$ as required.

Conversely, suppose G is connected and $\mathfrak{g}$ is abelian. Then ad is trivial and, since G is connected, Ad is trivial. Thus $gYg^{-1} = Y$ for all $g \in G, Y \in \mathfrak{g}$. Thus $g\exp(Y)g^{-1} = \exp(Y)$ for all $g \in G$ and all $Y \in \mathfrak{g}$. Since $\exp(\mathfrak{g})$ generates G, we see that G is commutative. □

10.2. Exercises.

Problem 26. Prove that, for $X, Y \in \mathfrak{gl}_n$,

$$(\mathrm{ad}_X)^m(Y) = [X,[X,\ldots,[X,Y]\ldots]] = \sum_{i=0}^{m} \binom{m}{k} X^k Y(-X)^{m-k}.$$

Hence give a direct proof that $\exp(\mathrm{ad}_X) = \mathrm{Ad}_{\exp(X)}$.

11. Lecture 11

11.1. Representations of U(1) and Maschke's theorem. We now discuss the representation theory of the unitary group $\mathrm{U}(1) = \{e^{it} \,|\, t \in \mathbb{R}\}$, which is isomorphic to SO(2), the circle group. Its Lie algebra is $i\mathbb{R} \subseteq \mathbb{C}$ with trivial Lie bracket, which is isomorphic to $\mathbb{R}$.

PROPOSITION 11.1. *All irreducible finite-dimensional representations of* $\mathrm{U}(1) = \{z \in \mathbb{C} \,|\, |z| = 1\}$ *are one-dimensional. They are given by*

$$z = e^{it} \longmapsto e^{int} = z^n$$

for $n \in \mathbb{Z}$.

PROOF. By Schur's lemma, Theorem 8.6(i), and the fact that U(1) is abelian, all irreducible finite-dimensional representations of U(1) are one-dimensional, so are homomorphisms $\mathrm{U}(1) \to \mathbb{C}^\times$. Since U(1) is connected, such a homomorphism is determined by the derivative $\mathfrak{u}_1 \to \mathfrak{gl}_1(\mathbb{C}) = \mathbb{C}$, which has the form $it \mapsto \lambda t$ for some $\lambda \in \mathbb{C}$. As in Example 7.1, this exponentiates to a map $\mathrm{U}(1) \to \mathbb{C}^\times$ if and only if $\lambda = in$ for some $n \in \mathbb{Z}$, giving the homomorphism $z \mapsto z^n$. □

THEOREM 11.2.

(i) All finite dimensional representations of U(1) *are unitary.*
(ii) All finite dimensional representations of U(1) *are completely reducible, that is, decompose into irreducible representations.*

PROOF.

(i) This is Problem 27.
(ii) This follows from (i) as in the proof of Maschke's theorem for finite groups. We will sketch another proof. Let (ρ, V) be a finite dimensional representation of U(1). Consider its differential $D\rho : \mathfrak{u}_1 = i\mathbb{R} \to \mathfrak{gl}_{n,\mathbb{C}}$. Let $A = D\rho(2\pi i)$. By the proof of Lemma 2.2, we may write $A = D + N$ for some strictly upper triangular matrix N and diagonal matrix D such that D and N commute. Then

$$\exp(A) = \exp(D\rho(2\pi i)) = \rho(e^{2\pi i}) = \rho(1) = \mathrm{Id}\,.$$

On the other hand,

$$\exp(A) = \exp(D + N) = \exp(D)\exp(N),$$

since D and N commute. As D is diagonal $\exp(D)$ is diagonal, but $\exp(N) = \exp(D)^{-1}$ and as N is nilpotent we must have $N = 0$. It follows that $\exp(N) = \exp(D) = \mathrm{Id}$ and A is diagonal. As every other element of $\mathfrak{u}_1$ is a scalar multiple of $2\pi i$ we have $\rho(it) = \lambda A$, for all $t \in \mathbb{R}$, where $\lambda \in \mathbb{C}$. Thus $\rho(it)$ is always diagonal and thus reducible to 1 dimensional representations.

□

REMARK 11.3. *Complete irreducibility does* not *hold for representations of a general Lie group. For example, the standard representation of*

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$$

does not decompose into a direct sum of two one-dimensional invariant subspaces (otherwise we would diagonalise $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$*, which is impossible). Furthermore, it is not unitary (as unitary matrices are diagonalisable).*

Some of this actually generalises substantially:

THEOREM 11.4. *(Maschke's theorem for compact Lie groups) Let G be a* compact *Lie group.*

(i) Any finite-dimensional representation of G is unitarisable.
(ii) (complete reducibility) Any finite-dimensional representation of G is a direct sum of irreducible representations.

PROOF. The proof of the first part uses the same idea as for finite groups. Let (ρ, V) be a finite-dimensional representation and take $(\ ,\ )$ to be any inner product on V. Then define

$$\langle \mathbf{v}, \mathbf{w} \rangle = \int_{g \in G} (\rho(g)\mathbf{v}, \rho(g)\mathbf{w}) dg.$$

This is also an inner product, and

$$\begin{aligned} \langle \rho(h)\mathbf{v}, \rho(h)\mathbf{w} \rangle &= \int_{g \in G} (\rho(gh)\mathbf{v}, \rho(gh)\mathbf{w}) dg \\ &= \int_{k \in G} (\rho(k)\mathbf{v}, \rho(k)\mathbf{w}) d(kh^{-1}) && \text{(putting } k = gh\text{)} \\ &= \int_{k \in G} (\rho(k)\mathbf{v}, \rho(k)\mathbf{w}) dk && \text{(since } dk = d(kh^{-1})\text{)} \\ &= \langle \mathbf{v}, \mathbf{w} \rangle . \end{aligned}$$

The challenge here is to show that there is an appropriate notion of $\int_{g \in G} f(g) dg$ for which the step "$dk = d(kh^{-1})$" is valid — this goes by the name of 'existence of Haar measure'. For $U(1)$ you can do it by hand, see problem 27.

The second part is proved exactly as for finite groups in Michaelmas (Theorem 5.12). Let (ρ, V) be a finite-dimensional representation and let W be a subrepresentation. Let $\langle\ ,\ \rangle$ be the G-invariant inner product on V guaranteed by part (i). Then the orthogonal complement $W^{\perp}$ is also a subrepresentation, and $V = W \oplus W^{\perp}$. Iterating, we obtain that V is a direct sum of irreducible representations.

$\square$

11.2. Exercises.

Problem 27. Consider $G = \mathrm{U}(1)$.

(a) For φ a continuous function on G, we define its integral

$$\int_G \varphi(g) dg = \tfrac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) dt.$$

Note that $\int_g 1 dg = 1$. Show that

$$\int_G \varphi(hg) dg = \int_G \varphi(gh) dg = \int_G \varphi(g) dg$$

for any $h \in G$.

(b) Let (V, ρ) be a finite dimensional representation of G and let $(\, , \,)$ be any inner product on V. Define a new inner product by

$$(\mathbf{v}, \mathbf{w})_\rho = \int_G (\rho(g)\mathbf{v}, \rho(g)\mathbf{w}) dg.$$

Show that $(\, , \,)_\rho$ is a G-invariant inner product on V.

(c) Conclude that every finite-dimensional representation of $\mathrm{U}(1)$ is completely reducible. (Compare this to the proof of Theorem 5.12 from Michaelmas).

Problem 28. Consider the orthogonal group $\mathrm{O}(2)$.

(a) Show that $\mathrm{SO}(2)$ has index 2 in $\mathrm{O}(2)$. Deduce that every element in $\mathrm{O}(2)$ can be uniquely written as r_θ or $r_\theta s$ with $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and r_θ the matrix for rotation by θ. Show that

$$s r_\theta = r_{-\theta} s.$$

(b) Mimic the method we used for dihedral groups to classify all irreducible finite-dimensional representations of $\mathrm{O}(2)$.

Problem 29. Let V be the space of functions on $\mathbb{C}^2$ that are polynomials in the coordinates x and y. Consider the (left) action of $\mathrm{GL}_2(\mathbb{C})$ on V given by

$$(g\varphi)(\mathbf{v}) = \varphi(g^{-1}v)$$

(here, think of $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$ as a column vector).

Compute the derived action for the "standard" basis of $\mathfrak{sl}_2(\mathbb{C})$ given by $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *You should get something involving the partial derivatives* $\frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial y}$.

CHAPTER 3

Representations of $\mathfrak{sl}_{2,\mathbb{C}}$

In this section we discuss the finite-dimensional representation theory of the Lie algebra $\mathfrak{sl}_{2,\mathbb{C}}$. We then use that to study the representation theory of $\mathrm{SL}_2(\mathbb{C})$, $\mathfrak{sl}_{2,\mathbb{R}}$, $\mathrm{SL}_2(\mathbb{R})$, and $\mathrm{SU}(2)$, which are all closely related.

12. Lecture 12

12.1. Weights. We fix the following basis of $\mathfrak{sl}_{2,\mathbb{C}}$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These satisfy the following commutation relations, which are fundamental (check them!):

$$[H, X] = 2X,$$
$$[H, Y] = -2Y,$$

and

$$[X, Y] = H.$$

We will decompose representations of $\mathfrak{sl}_{2,\mathbb{C}}$ into their eigenspaces for the action of H. The elements X and Y will then move vectors between these eigenspaces, and this will let us analyse the representation theory of $\mathfrak{sl}_{2,\mathbb{C}}$.

Since $\mathrm{SL}_2(\mathbb{C})$ is simply connected, we have

PROPOSITION 12.1. *Every finite-dimensional representation of $\mathfrak{sl}_{2,\mathbb{C}}$ is the derivative of a unique representation of $\mathrm{SL}_2(\mathbb{C})$.*

Note that we have not proved this. However, we will use the result freely in what follows. It is possible to give purely algebraic proofs of all the results for which we use the previous proposition, but it is more complicated.

PROPOSITION 12.2. *Let (ρ, V) be a finite-dimensional complex-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Then $\rho(H)$ is diagonalisable with integer eigenvalues.*

PROOF. By the previous proposition ρ is the derivative of a representation $\tilde{\rho}$ of $\mathrm{SL}_2(\mathbb{C})$. We can identify $\mathrm{U}(1)$ as a subgroup of $\mathrm{SL}_2(\mathbb{C})$ by the following map:

$$f : e^{it} \longmapsto \begin{pmatrix} e^{it} & \\ & e^{-it} \end{pmatrix}.$$

By Maschke's Theorem for U(1), Theorem 11.2(ii), $\tilde{\rho}\left(\begin{pmatrix} e^{it} & \\ & e^{-it}\end{pmatrix}\right)$ on V can be diagonalised. Taking the derivative, we see that $\rho\left(\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\right)$ can be diagonalised and hence so can $\rho(H) = -i\rho\left(\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\right)$.

In fact, the classification of irreducible representations of U(1), Proposition 11.1, shows that $\rho(iH)$ has eigenvalues in $i\mathbb{Z}$ and so $\rho(H)$ has eigenvalues in $\mathbb{Z}$. □

Remark 12.3. *The proof of the proposition is a instance of* Weyl's unitary trick. *We turned the action of H, which infinitesimally generates a non-compact one-parameter subgroup of* $\mathrm{SL}_2(\mathbb{C})$, *into the action of the compact group* U(1) *infinitesimally generated by iH. The action of this compact subgroup can be diagonalised.*

The proposition does not hold for an arbitrary representation of the one-dimensional Lie algebra $\mathfrak{h} = \mathbb{C}H$ generated by H. Namely, the map $zH \mapsto \begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}$ cannot be diagonalised. It is implicitly the interaction of H with the other generators X and Y which makes the proposition work.

Let (ρ, V) be a finite-dimensional complex-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. By Proposition 12.2 we get a decomposition

$$V = \bigoplus_{\alpha} V_\alpha \tag{12.4}$$

where each V_α is the eigenspace for $\rho(H)$ with eigenvalue $\alpha \in \mathbb{C}$:

$$V_\alpha = \{\mathbf{v} \in V \mid \rho(H)\mathbf{v} = \alpha\mathbf{v}\}.$$

Definition 12.5.

(i) *Each $\alpha \in \mathbb{C}$ occurring in equation (12.4) is called a* weight *(more precisely, an H-weight) for the representation ρ.*
(ii) *Each V_α is called a* weight space *for ρ.*
(iii) *The non-zero vectors in V_α are called* weight vectors *for ρ.*

Note that the weights corresponding to V form a *multiset*, in other words a set with repeated elements. We say that a weight has *multiplicity* n if it appears in the multiset n times (in general, the multiplicity of a weight is the dimension of the weight space).

Example 12.6. *The set of weights of the zero representation is empty, while the trivial representation has a single weight,* 0.

Example 12.7. *Let $V = \mathbb{C}^2$ be the standard representation. Write $\mathbf{e}_1, \mathbf{e}_{-1}$ for the standard basis. Then $H\mathbf{e}_1 = \mathbf{e}_1$ and $H\mathbf{e}_{-1} = -\mathbf{e}_{-1}$. Thus the set of weights of V is $\{\pm 1\}$.*

Example 12.8. *We consider the adjoint representation* ad *of $\mathfrak{g} = \mathfrak{sl}_{2,\mathbb{C}}$ on itself. By the commutation relations, we see directly that* ad_H *has eigenvalues* 0 *($[H,H] = 0$),* 2 *($[H,X] = 2X$), and* -2 *($[H,Y] = -2Y$), so the set of weights is*

$$\{-2, 0, 2\}.$$

The non-zero weights 2 *and* -2 *are called the* roots *of $\mathfrak{sl}_{2,\mathbb{C}}$ and their weight spaces are the* root spaces $\mathfrak{g}_2$ *and* $\mathfrak{g}_{-2}$. *The weight vectors are called* root vectors.

Thus we have the root space decomposition

$$\begin{aligned}\mathfrak{sl}_{2,\mathbb{C}} &= \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} \\ &= \langle H \rangle \oplus \langle X \rangle \oplus \langle Y \rangle .\end{aligned}$$

EXAMPLE 12.9. *We consider* $\mathbb{C}^2 \otimes \mathbb{C}^2$ *where* $\mathbb{C}^2$ *is the standard representation. Then*

$$H(\mathbf{e}_1 \otimes \mathbf{e}_1) = (H\mathbf{e}_1) \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes H\mathbf{e}_1 = 2\mathbf{e}_1 \otimes \mathbf{e}_1$$

and similarly

$$H(\mathbf{e}_1 \otimes \mathbf{e}_{-1}) = H(\mathbf{e}_{-1} \otimes \mathbf{e}_1) = 0, H(\mathbf{e}_{-1} \otimes \mathbf{e}_{-1}) = -2\mathbf{e}_{-1} \otimes \mathbf{e}_{-1}$$

so that the weights are $\{-2, 0, 0, 2\}$.

EXAMPLE 12.10. *Take* $V = \mathrm{Sym}^k(\mathbb{C}^2)$. *As* $\{\mathbf{e}_1, \mathbf{e}_{-1}\}$ *is a basis of the standard representation, a set of basis vectors for* $\mathrm{Sym}^k(\mathbb{C}^2)$ *is*

$$\{\mathbf{e}_1^a \mathbf{e}_{-1}^{k-a} \,|\, 0 \leq a \leq k\}.$$

Given some arbitrary $\mathbf{e}_1^a \mathbf{e}_{-1}^{k-a}$ *in the basis we calculate*

$$\begin{aligned}H(\mathbf{e}_1^a \mathbf{e}_{-1}^{k-a}) &= a(H\mathbf{e}_1)\mathbf{e}_1^{a-1}\mathbf{e}_{-1}^{k-a} + (k-a)(H\mathbf{e}_{-1})\mathbf{e}_1^a \mathbf{e}_{-1}^{k-a-1} \\ &= (2a-k)\mathbf{e}_1^a \mathbf{e}_{-1}^{k-a}.\end{aligned}$$

Thus the weights are:

$$\{-k, 2-k, 4-k, \ldots, k-4, k-2, k\}.$$

We will soon see an explanation for this pattern.

PROPOSITION 12.11. *If* V, W *are representations of* $\mathfrak{sl}_{2,\mathbb{C}}$ *with weights* $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_n$ *respectively, then:*

(i)

$$\begin{aligned}\{\textit{weights of } V \otimes W\} &= \{\textit{all sums of pairs of weights from } V \textit{ and } W\} \\ &= \{\alpha_i + \beta_j \,|\, 1 \leq i \leq n, 1 \leq j \leq m\}.\end{aligned}$$

(ii)

$$\begin{aligned}\{\textit{weights of } \mathrm{Sym}^k(V)\} &= \{\textit{sums of unordered } k\textit{-tuples of weights of } V\} \\ &= \{\alpha_{i_1} + \ldots + \alpha_{i_k} \,|\, 1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n\}.\end{aligned}$$

(iii)

$$\begin{aligned}\{\textit{weights of } \textstyle\bigwedge^k(V)\} &= \{\textit{sums of unordered `distinct' } k\textit{-tuples of weights of } V\} \\ &= \{\alpha_{i_1} + \ldots + \alpha_{i_k} \,|\, 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}.\end{aligned}$$

PROOF. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of weight vectors of V such that $\mathbf{v}_i$ has weight α_i, and let $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be similar for W with weights β_i. Then

$$\begin{aligned}H(\mathbf{v}_i \otimes \mathbf{w}_j) &= (H\mathbf{v}_i \otimes \mathbf{w}_j) + \mathbf{v}_i \otimes (H\mathbf{w}_j) \\ &= \alpha_i \mathbf{v}_i \otimes \mathbf{w}_j + \mathbf{v}_i \otimes \beta_j \mathbf{w}_j \\ &= (\alpha_i + \beta_j)(\mathbf{v}_i \otimes \mathbf{w}_j).\end{aligned}$$

So $\{\mathbf{v}_i \otimes \mathbf{w}_j\}$ is a basis of $V \otimes W$ with the given weights. Parts (ii) and (iii) are left as an exercise (Problem 30). $\square$

Example 12.12. *We should illustrate what is meant by 'distinct': it is 'distinct' as elements of the multiset. Suppose that the weights of V are $\{2, 0, 0, -2\}$. Then to obtain the weights of $\bigwedge^2(V)$ we add together unordered, distinct, pairs of these in every possible way, getting:*

$$\{2+0, 2+0, 2+-2, 0+0, 0+-2, 0+-2\} = \{2, 2, 0, 0, -2, -2\}.$$

12.2. Exercises.

Problem 30. Prove the rest of Proposition 12.11.

13. Lecture 13

13.1. Highest weights. The following is our first version of the *fundamental weight calculation.*

LEMMA 13.1. *Let (ρ, V) be a complex-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Let α be a weight of V and let $\mathbf{v} \in V_\alpha$. Then*

$$X\mathbf{v} \in V_{\alpha+2}$$

and

$$Y\mathbf{v} \in V_{\alpha-2}.$$

Thus we have three maps:

$$\begin{aligned} H &: V_\alpha \longrightarrow V_\alpha \\ X &: V_\alpha \longrightarrow V_{\alpha+2} \\ Y &: V_\alpha \longrightarrow V_{\alpha-2}. \end{aligned}$$

PROOF. We have, for $\mathbf{v} \in V_\alpha$,

$$\begin{aligned} \rho(H)\rho(X)\mathbf{v} &= [\rho(H), \rho(X)]\mathbf{v} + \rho(X)\rho(H)\mathbf{v} \\ &= \rho([H, X])\mathbf{v} + \rho(X)\rho(H)\mathbf{v} \\ &= 2\rho(X)\mathbf{v} + \alpha\rho(X)\mathbf{v} \\ &= (\alpha + 2)\rho(X)\mathbf{v}. \end{aligned}$$

So $\rho(X)\mathbf{v} \in V_{\alpha+2}$ as required.

The claim about the action of Y is proved similarly. □

DEFINITION 13.2. *A vector $\mathbf{v} \in V_\alpha$ is a* highest weight vector *if it is a weight vector and if*

$$X\mathbf{v} = 0.$$

In this case we call the weight of $\mathbf{v}$ a highest weight.

LEMMA 13.3. *Any finite-dimensional complex-linear representation (ρ, V) of $\mathfrak{sl}_{2,\mathbb{C}}$ has a highest weight vector.*

PROOF. Indeed, let α be the numerically greatest weight of V (there must be one, as V is finite-dimensional) and let $\mathbf{v}$ be a weight vector of weight α. Then Xv has weight $\alpha + 2$ by the fundamental weight calculation, so must be zero as α was maximal. □

EXAMPLE 13.4. *Let $V = \mathbb{C}^2 \otimes \mathbb{C}^2$. Then the highest weight vectors are $\mathbf{e}_1 \otimes \mathbf{e}_1$ and $\mathbf{e}_1 \otimes \mathbf{e}_{-1} - \mathbf{e}_{-1} \otimes \mathbf{e}_1$.*

These are easily checked to be highest weight vectors — the first is killed by X since $X\mathbf{e}_1 = 0$, the second becomes

$$\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_1 = 0.$$

It is left to you to check that there are no further highest weight vectors.

13.2. Classification of irreducible representations of $\mathfrak{sl}_{2,\mathbb{C}}$. The key point here is that a highest weight vector must have non-negative integer weight n, and it then generates an irreducible representation of dimension $n+1$ whose isomorphism class is determined by n and which has a very natural basis of weight vectors.

Let (ρ, V) be a complex-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$.

LEMMA 13.5. *Suppose that V has a highest weight vector $\mathbf{v}$ of weight n. Then the subspace W spanned by the vectors*

$$\mathbf{v}, Y\mathbf{v}, Y^2\mathbf{v} = Y(Y\mathbf{v}), \ldots$$

is an $\mathfrak{sl}_{2,\mathbb{C}}$-invariant subspace of V.

Moreover, for $n \geq 0$, the dimension of W is $n+1$, with basis $\mathbf{v}, Y\mathbf{v}, \ldots, Y^n\mathbf{v}$, where $Y^{n+1}\mathbf{v} = 0$.

PROOF. Let W be the span of the $Y^k\mathbf{v}$. Since the $Y^k\mathbf{v}$ are weight vectors, their span is H-invariant. It is clearly also Y-invariant. So we only need to check the invariance under the X-action. We claim that for all $m \geq 1$,

$$XY^m\mathbf{v} = m(n-m+1)Y^{m-1}\mathbf{v}. \tag{13.6}$$

The proof is by induction. The case $m = 1$ is:

$$XY\mathbf{v} = ([X,Y] + YX)\mathbf{v} = H\mathbf{v} + Y(X\mathbf{v}) = H\mathbf{v} = n\mathbf{v}.$$

If the formula holds for m, then

$$\begin{aligned} XY^{m+1}\mathbf{v} &= ([X,Y] + YX)Y^m\mathbf{v} \\ &= HY^m\mathbf{v} + Y(XY^m\mathbf{v}) \\ &= (n-2m)Y^m\mathbf{v} + m(n-m+1)Y^m\mathbf{v} && \text{(induction hypothesis)} \\ &= (m+1)(n-m)Y^m\mathbf{v} \end{aligned}$$

as required.

If n is not a nonnegative integer, then $m(n-m+1) \neq 0$ for all m. So

$$Y^{m-1}\mathbf{v} \neq 0 \implies XY^m\mathbf{v} \neq 0 \implies Y^m\mathbf{v} \neq 0,$$

whence $Y^m\mathbf{v} \neq 0$ for all m. As these are weight vectors with distinct weights, they are linearly independent and so span an infinite dimensional subspace.

If $Y^i\mathbf{v} = 0$ for some $0 < i \leq n$, then

$$0 = XY^i\mathbf{v} = i(n-i+1)Y^{i-1}\mathbf{v}$$

and so $Y^{i-1}\mathbf{v} = 0$. Repeating gives that $\mathbf{v} = Y^0\mathbf{v} = 0$, a contradiction.

Now,

$$XY^{n+1}\mathbf{v} = (n+1)(n-n)\mathbf{v} = 0$$

and so $Y^{n+1}\mathbf{v}$ is either zero or a highest weight vector of weight $-(n+2) < 0$. We have already seen that the second possibility cannot happen, so $Y^{n+1}\mathbf{v} = 0$.

Thus W is spanned by the (non-zero) weight vectors $\mathbf{v}, Y\mathbf{v}, \ldots, Y^n\mathbf{v}$ with distinct weights, which are therefore linearly independent and so a basis for W. $\square$

REMARK 13.7. *If we didn't assume that V was finite-dimensional, then the first part of the previous lemma would still be true.*

COROLLARY 13.8. *In the situation of the previous lemma, W is irreducible.*

PROOF. Suppose that $W' \subseteq W$ is a non-zero subrepresentation. Then it has a highest weight vector, which must be (proportional to) $Y^i\mathbf{v}$ for some $0 \leq i \leq n$. But then

$$XY^i\mathbf{v} = i(n-i+1)Y^{i-1}\mathbf{v} \neq 0$$

if $i > 0$ and so $i = 0$, meaning that $\mathbf{v} \in W'$. But then $Y^i\mathbf{v} \in W'$ for all i, so $W' = W$ as required. $\square$

The weights of W are illustrated in Figure 13.2.

THEOREM 13.9. *Suppose V is an irreducible finite-dimensional complex-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Then:*

(i) There is a unique (up to scalar) highest weight vector, with highest weight $n \geq 0$.
(ii) The weights of V are $n, n-2, \ldots, 2-n, -n$.
(iii) All weight spaces V_α are one-dimensional (we say that V is 'multiplicity free').
(iv) The dimension of V is $n+1$.
(v) For every $n \geq 0$, the unique irreducible complex-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$ with highest weight n is $\mathrm{Sym}^n(\mathbb{C}^2)$ *(up to isomorphism).*

PROOF. Let V be an irreducible finite-dimensional complex-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Let $\mathbf{v} \in V$ be a highest weight vector. By Lemma 13.5 its weight is a nonnegative integer n, and by Corollary 13.8 the vectors $\mathbf{v}, Y\mathbf{v}, \ldots, Y^n\mathbf{v}$ span an irreducible subrepresentation of V of dimension $n+1$, which must therefore be the whole of V. The claims (i) to (iv) follow immediately.

Moreover, the actions of X, Y and H on this basis are given by explicit matrices depending only on n, so the isomorphism class of V is determined by n. It remains only to show that a representation with highest weight n exists for all $n \geq 0$. Consider $\mathrm{Sym}^n(\mathbb{C}^2)$. The vector $\mathbf{e}_1^n$ is a highest weight vector of weight n, so we are done. In fact, in this case $Y^i\mathbf{e}_1^n$ is proportional to $\mathbf{e}_1^{n-i}\mathbf{e}_{-1}^i$, so the irreducible representation generated by $\mathbf{e}_1^n$ is in fact the whole of $\mathrm{Sym}^n(\mathbb{C}^2)$. $\square$

In fact, if (ρ, V) is the irreducible representation with highest weight n, highest weight vector $\mathbf{v}$, then the matrices of H, Y, and X with respect to the basis

$$\{\mathbf{v}, Y(\mathbf{v}), \ldots, Y^n(\mathbf{v})\}$$

are respectively

$$\rho(H) = \begin{pmatrix} n & & & \\ & n-2 & & \\ & & \ddots & \\ & & & -n \end{pmatrix},$$

$$\rho(Y) = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

and

$$\rho(X) = \begin{pmatrix} 0 & n & & & & \\ & 0 & 2(n-1) & & & \\ & & 0 & 3(n-2) & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & n \\ & & & & & 0 \end{pmatrix}.$$

THEOREM 13.10. *Every finite-dimensional irreducible complex-linear representation of* $\mathrm{SL}_2(\mathbb{C})$ *or* $\mathfrak{sl}_{2,\mathbb{C}}$ *is isomorphic to* $\mathrm{Sym}^n(\mathbb{C}^2)$*, the symmetric product of the standard representation.*

PROOF. We already proved this for $\mathfrak{sl}_{2,\mathbb{C}}$, Theorem 13.9(v). If V is a representation of $\mathrm{SL}_2(\mathbb{C})$, then its derivative will be isomorphic to $\mathrm{Sym}^n(\mathbb{C}^2)$. Since $\mathrm{SL}_2(\mathbb{C})$ is connected, this implies that $V \cong \mathrm{Sym}^n(\mathbb{C}^2)$. $\square$

13.3. Exercises.

Problem 31. Let V, W be representations of $\mathfrak{sl}_{2,\mathbb{C}}$. Let $\mathbf{v}$ and $\mathbf{w}$ be two weight vectors of V and W respectively with respective weights α and β. Show that

$$\mathbf{v} \otimes \mathbf{w} \in V \otimes W$$

is a weight vector with weight $\alpha + \beta$, and that if $\mathbf{v}$ and $\mathbf{w}$ are highest weight vectors then so is $\mathbf{v} \otimes \mathbf{w}$.

Problem 32. If $\lambda \in \mathbb{C}$, show that there is a (possibly infinite dimensional!) representation of $\mathfrak{sl}_{2,\mathbb{C}}$ with highest weight λ.

Problem 33. Consider $V = \mathrm{Sym}^n(\mathbb{C}^2)$, the irreducible representation of highest weight n of $\mathfrak{sl}_{2,\mathbb{C}}$. Decompose the following representations into irreducibles, and find highest weight vectors for the irreducible constituents:

(a) $\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbb{C}^2))$;
(b) $\bigwedge^2 \left(\mathrm{Sym}^2(\mathbb{C}^2)\right)$;
(c) $\mathrm{Sym}^3(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2)$;
(d) $\mathrm{Sym}^3 \left(\mathrm{Sym}^2(\mathbb{C}^2)\right)$.

For the third example, find bases for the irreducible subrepresentations.

Problem 34. Let (π, V) be a finite-dimensional representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Consider the *Casimir* element[1]

$$\mathcal{C} = \pi(X)\pi(Y) + \pi(Y)\pi(X) + \frac{1}{2}\pi(H)^2.$$

(a) Show $\mathcal{C}$ commutes with the action of $\mathfrak{sl}_{2,\mathbb{C}}$. Conclude that if V is irreducible then $\mathcal{C}$ acts as a scalar.
(b) What is the scalar for $V = \mathrm{Sym}^n(\mathbb{C}^2)$, the irreducible representation of highest weight n?
(c) Compute the action of $\mathcal{C}$ on the space V of polynomial functions ϕ on $\mathbb{C}^2$, with action the derivative of $(g\phi)(\mathbf{v}) = \phi(g^{-1}v)$ (see problem 29).

[1]Conventions differ; it might be more usual to call $1 + 2\mathcal{C}$ the Casimir.

14. Lecture 14

14.1. Real forms. Here we will use our understanding of the representation theory of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}(2)$ to tell us about the representation theory of $\mathfrak{sl}_{2,\mathbb{C}}$, and this will then be used to prove complete reducibility.

DEFINITION 14.1. *A real form of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that every element Z of $\mathfrak{g}$ can be written uniquely as $X + iY$ for $X, Y \in \mathfrak{h}$.*

For dimension reasons, necessary and sufficient conditions are that $\mathfrak{h} \cap i\mathfrak{h} = 0$ and $\dim_{\mathbb{R}} \mathfrak{h} = \dim_{\mathbb{C}} \mathfrak{g}$.

EXAMPLE 14.2. *The Lie algebra $\mathfrak{sl}_{n,\mathbb{R}}$ is a real form of $\mathfrak{sl}_{n,\mathbb{C}}$.*

EXAMPLE 14.3. *The Lie algebra $\mathfrak{su}_n$ is a real form of $\mathfrak{sl}_{n,\mathbb{C}}$. Indeed,* $\dim_{\mathbb{R}} \mathfrak{su}_n = n^2 - 1 = \dim_{\mathbb{C}} \mathfrak{sl}_{n,\mathbb{C}}$ *and if $X, iX \in \mathfrak{su}_n$ then*

$$iX = -(iX)^* = iX^* = -iX$$

so $X = 0$.

Explicitly, we may write $A \in \mathfrak{sl}_{n,\mathbb{C}}$ as $X + iY$ with

$$X = \frac{1}{2}(A - A^*)$$

and

$$Y = \frac{-i}{2}(A + A^*)$$

in $\mathfrak{su}_n$.

EXAMPLE 14.4. *Recall that*

$$\mathfrak{so}_n = \{A \in \mathfrak{sl}_{n,\mathbb{R}} \mid A + A^T = 0\}.$$

Let

$$\mathfrak{so}_{n,\mathbb{C}} = \{A \in \mathfrak{sl}_{n,\mathbb{C}} \mid A + A^T = 0\}.$$

Since the defining equation $A + A^T = 0$ can be checked separately on the real and imaginary parts of A,

$$\mathfrak{so}_{n,\mathbb{C}} = \{B + iC \mid B, C \in \mathfrak{so}_n\}$$

and so $\mathfrak{so}_n$ is a real form of $\mathfrak{so}_{n,\mathbb{C}}$.

PROPOSITION 14.5. *Let $\mathfrak{h}$ be a real form of $\mathfrak{g}$. There is a one-to-one correspondence between representations of $\mathfrak{h}$ and complex-linear representations of $\mathfrak{g}$ under which irreducible representations correspond to irreducible representations.*

PROOF. Given a $\mathbb{C}$-linear representation of $\mathfrak{g}$, it is a representation of $\mathfrak{h}$ by restriction. Conversely, if (ρ, V) is a representation of $\mathfrak{h}$, then it extends to a unique $\mathbb{C}$-linear representation of $\mathfrak{g}$ given by the formula (forced by $\mathbb{C}$-linearity)

$$\rho(X + iY)\mathbf{v} = \rho(X)\mathbf{v} + i\rho(Y)\mathbf{v}.$$

It is an exercise to check this preserves the Lie bracket. The second statement is left as an exercise (Problem 36). □

As a corollary we immediately obtain

THEOREM 14.6. *The representation theories of* $\mathfrak{sl}_{n,\mathbb{R}}$ *and* $\mathfrak{su}_n$ *are 'the same' as the complex-linear representation theory of* $\mathfrak{sl}_{n,\mathbb{C}}$*. All finite-dimensional irreducible representations of* $\mathfrak{sl}_{2,\mathbb{R}}$, $\mathrm{SL}_2(\mathbb{R})$, $\mathfrak{su}_2$, *or* $\mathrm{SU}(2)$ *are of the form* $\mathrm{Sym}^n(\mathbb{C}^2)$*.*

PROOF. The claims about Lie algebras follow from Proposition 14.5. By Theorem 13.10, every (finite-dimensional) irreducible representation of $\mathfrak{sl}_{2,\mathbb{C}}$, and thus $\mathfrak{sl}_{2,\mathbb{R}}$, is of the form $\mathrm{Sym}^n(\mathbb{C}^2)$, and these clearly exponentiate to representations of $\mathrm{SL}_2(\mathbb{R})$, despite this not being a simply connected group! Similarly for $\mathrm{SU}(2)$ (which is simply connected). Since $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}(2)$ are connected, every representation of them is determined by its derivative, so we have a complete list of the irreducible representations. $\square$

14.2. Decomposing representations.

THEOREM 14.7. *Let V be any finite-dimensional complex-linear representation of* $\mathfrak{sl}_{n,\mathbb{C}}$*. Then V is completely reducible, that is, splits into a direct sum of irreducible representations.*

PROOF. Let V be a finite-dimensional complex-linear representation of $\mathfrak{sl}_{n,\mathbb{C}}$ and let $W \subseteq V$ be a subrepresentation. Then W is an $\mathfrak{su}_n$-subrepresentation, by Theorem 14.6. As $\mathrm{SU}(n)$ is simply-connected, V and W exponentiate to representations of $\mathrm{SU}(n)$. Since $\mathrm{SU}(n)$ is compact, by Maschke's theorem (Theorem 11.4(ii)) there is a complementary $\mathrm{SU}(n)$-subrepresentation W' with

$$V = W \oplus W'.$$

Then W' is a $\mathfrak{su}_n$-subrepresentation, and so a $\mathbb{C}$-linear $\mathfrak{sl}_{n,\mathbb{C}}$-subrepresentation, so that

$$V = W \oplus W'$$

as representations of $\mathfrak{sl}_{n,\mathbb{C}}$. Complete reducibility follows. $\square$

The argument in this theorem is called *Weyl's unitary trick*. For a similar application of this idea, see the proof of Proposition 12.2.

COROLLARY 14.8. *Two finite-dimensional complex-linear representations of* $\mathfrak{sl}_{2,\mathbb{C}}$ *are isomorphic if and only if they have the same multiset of weights.*

PROOF. Let (π, V) and (ρ, W) be finite-dimensional complex-linear representations of $\mathfrak{sl}_{2,\mathbb{C}}$. We can decompose V into irreducibles by looking at the weights. Firstly, look at the maximal weight k of V. Then there must be a weight vector $\mathbf{v}$ of weight k, which is necessarily a highest weight vector, and so V must contain a copy of $\mathrm{Sym}^k(\mathbb{C}^2)$ — namely, the subspace spanned by $\{\mathbf{v}, Y\mathbf{v}, \ldots, Y^k\mathbf{v}\}$. By Theorem 14.7 we have

$$V \cong \mathrm{Sym}^k(\mathbb{C}^2) \oplus V'.$$

The weights of V' are then obtained by removing the weights of $\mathrm{Sym}^k(\mathbb{C}^2)$ from the weights of V, and we repeat the process, noting that it terminates as V is finite dimensional. If we then do the same for W we see that they decompose into the same sum of copies of $\mathrm{Sym}^k(\mathbb{C}^2)$ if and only if they have the same multiset of weights. $\square$

14.3. Exercises.

Problem 35. Show that the real Lie algebras $\mathfrak{sl}_{2,\mathbb{R}}$ and $\mathfrak{su}_2$ are *not* isomorphic. *Hint: consider the adjoint action of an arbitrary element of* $\mathfrak{su}_2$*.*

Problem 36. Finish the proof of Proposition 14.5.

Problem 37. We have that $\mathrm{Sym}^n(\mathbb{C}^2)$ is the irreducible representation of $\mathrm{SU}(2)$ of dimension $n+1$. Let χ_n be its character. Every conjugacy class of $\mathrm{SU}(2)$ contains an element of the form

$$\exp(itH) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

Show that

$$\chi_n(\exp(itH)) = \frac{\sin((n+1)t)}{\sin(t)}.$$

Problem 38. Let (ρ, V) be an irreducible representation of $\mathfrak{gl}_{2,\mathbb{C}}$, and let $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Show that $\rho(Z)$ is as a scalar.
(b) Show that the restriction of V to $\mathfrak{sl}_{2,\mathbb{C}}$ is irreducible.
(c) Show that for every $\lambda \in \mathbb{C}$ and integer $n \geq 0$, there is a unique irreducible representation of $\mathfrak{gl}_{2,\mathbb{C}}$ of dimension $n+1$ with $\rho(Z) = \lambda\,\mathrm{Id}$.
(d) Which of these are derivatives of representations of $\mathrm{GL}_2(\mathbb{C})$? Hence classify the finite dimensional holomorphic irreducible representations of $\mathrm{GL}_2(\mathbb{C})$.

Problem 39. Let V be a finite-dimensional representation of $\mathfrak{sl}_{2,\mathbb{C}}$.

(a) What are the weights of the dual representation V^*?
(b) Deduce that $V \cong V^*$.

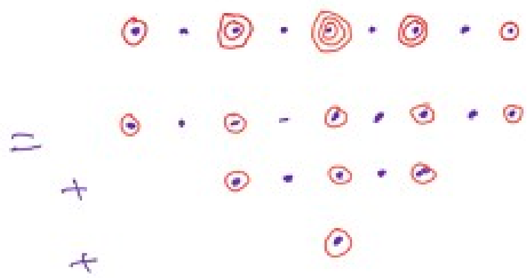

FIGURE 1. Decomposing $\mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2)$.

15. Lecture 15

15.1. Decomposing $\mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2)$.

EXAMPLE 15.1. *Let $\mathbb{C}^2$ be the standard representation of $\mathrm{SL}_2(\mathbb{C})$ with weight basis $\mathbf{e}_1$, $\mathbf{e}_{-1}$. Consider $V = \mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2)$. Let $\mathbf{v}_2 = \mathbf{e}_1^2$, $\mathbf{v}_0 = \mathbf{e}_1\mathbf{e}_{-1}$ and $\mathbf{v}_{-2} = \mathbf{e}_{-1}^2$ be weight vectors in $\mathrm{Sym}^2(\mathbb{C}^2)$ corresponding to the weights 2, 0 and -2. Then the weights of V are*

- 4, *multiplicity one, weight vector:* $\mathbf{v}_2 \otimes \mathbf{v}_2$.
- 2, *multiplicity two, weight space:* $\langle \mathbf{v}_2 \otimes \mathbf{v}_0, \mathbf{v}_0 \otimes \mathbf{v}_2 \rangle$.
- 0, *multiplicity three, weight space:* $\langle \mathbf{v}_2 \otimes \mathbf{v}_{-2}, \mathbf{v}_0 \otimes \mathbf{v}_0, \mathbf{v}_{-2} \otimes \mathbf{v}_2 \rangle$.
- -2, *multiplicity two, weight space:* $\langle \mathbf{v}_0 \otimes \mathbf{v}_{-2}, \mathbf{v}_{-2} \otimes \mathbf{v}_0 \rangle$.
- -4, *multiplicity one, weight vector:* $\langle \mathbf{v}_{-2} \otimes \mathbf{v}_{-2} \rangle$.

The weights of $\mathrm{Sym}^2(\mathbb{C}^2)$ are $\{-2, 0, 2\}$ and so the weights of $\mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2)$ are

$$\{-2, 0, 2\} + \{-2, 0, 2\} = \{-4, -2, -2, 0, 0, 0, 2, 2, 4\}.$$

This is the same as the multiset of weights of

$$\mathrm{Sym}^4(\mathbb{C}^2) \oplus \mathrm{Sym}^2(\mathbb{C}^2) \oplus \mathbb{C}$$

and so this is the required decomposition into irreducibles.

*We can go further, and decompose V into irreducible *sub*representations. This means finding irreducible subrepresentations of V such that V is their direct sum (not just isomorphic to their sum).*

The copy of $\mathrm{Sym}^4(\mathbb{C}^2)$ in V has highest weight vector $\mathbf{v}_2 \otimes \mathbf{v}_2$. We can find a basis by repeatedly acting with Y on it (writing $\propto$ for 'equal up to a non-zero scalar'):

$$\begin{aligned}
Y(\mathbf{v}_2 \otimes \mathbf{v}_2) &= 2(\mathbf{v}_2 \otimes \mathbf{v}_0 + \mathbf{v}_0 \otimes \mathbf{v}_2) \propto \mathbf{v}_2 \otimes \mathbf{v}_0 + \mathbf{v}_0 \otimes \mathbf{v}_2 \\
Y^2(\mathbf{v}_2 \otimes \mathbf{v}_2) &\propto \mathbf{v}_2 \otimes \mathbf{v}_{-2} + 4\mathbf{v}_0 \otimes \mathbf{v}_0 + \mathbf{v}_{-2} \otimes \mathbf{v}_2 \\
Y^3(\mathbf{v}_2 \otimes \mathbf{v}_2) &\propto 6(\mathbf{v}_0 \otimes \mathbf{v}_{-2} + \mathbf{v}_{-2} \otimes \mathbf{v}_0) \propto \mathbf{v}_0 \otimes \mathbf{v}_{-2} \otimes \mathbf{v}_{-2} \otimes \mathbf{v}_0 \\
Y^4(\mathbf{v}_2 \otimes \mathbf{v}_2) &\propto \mathbf{v}_{-2} \otimes \mathbf{v}_{-2}.
\end{aligned}$$

These vectors are a basis for the copy of $\mathrm{Sym}^4(\mathbb{C}^2)$ in V.

Next, we find the copy of $\mathrm{Sym}^2(\mathbb{C}^2)$ in V. We start by looking for a highest weight vector of weight 2:

$$\mathbf{v}_2 \otimes \mathbf{v}_0 - \mathbf{v}_0 \otimes \mathbf{v}_2$$

does the trick. Hitting this with Y gives $\mathbf{v}_2 \otimes \mathbf{v}_{-2} - \mathbf{v}_{-2} \otimes \mathbf{v}_2$, and doing so again gives $\mathbf{v}_{-2} \otimes \mathbf{v}_0 - \mathbf{v}_0 \otimes \mathbf{v}_{-2}$ (up to scalar). These vectors are a basis for the copy of $\mathrm{Sym}^2(\mathbb{C}^2)$ in V.

Finally, we find the trivial representation $\mathbb{C}$ in V. We need only find a weight vector of weight 0 which is killed by X, and

$$\mathbf{v}_2 \otimes \mathbf{v}_{-2} - 2\mathbf{v}_0 \otimes \mathbf{v}_0 + \mathbf{v}_{-2} \otimes \mathbf{v}_2$$

does the job. This vector spans a copy of the trivial representation.

15.2. Classification of irreducible $SO(3)$ representations. We have already seen that $\mathfrak{su}_2$ and $\mathfrak{so}_3$ are isomorphic (Problem 11). We therefore have:

THEOREM 15.2. *There is an irreducible two-dimensional representation V of $\mathfrak{so}_3$ such that the irreducible complex representations of $\mathfrak{so}_3$ are exactly $\mathrm{Sym}^n(V)$ for $n \geq 0$.*

PROOF. By Theorem 14.6, all irreducible representations of $\mathfrak{su}_2$ are of the form $\mathrm{Sym}^n(\mathbb{C}^2)$ for $n \geq 0$. By the isomorphism from Problem 11, $\mathfrak{so}_3 \xrightarrow{\sim} \mathfrak{su}_2 \subseteq \mathfrak{gl}_{2,\mathbb{C}}$, we have that all irreducible representations of $\mathfrak{so}_3$ are isomorphic to something of this form. □

We want to know which of these representations exponentiate to an irreducible representation of SO(3). For this, we revisit the connection with SU(2).

Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $\mathfrak{su}_2$ defined by

$$\langle X, Y \rangle = -\operatorname{tr}(XY).$$

LEMMA 15.3. *The form $\langle \cdot, \cdot \rangle$ is a positive definite bilinear form preserved by the adjoint action of* SU(2).

PROOF. This is Problem 18 where we note that $\mathrm{Ad}_g X = gXg^{-1}$. □

COROLLARY 15.4. *There is an isomorphism* $\mathrm{SU}(2)/\{\pm \mathrm{Id}\} \cong \mathrm{SO}(3)$.

PROOF. Choosing an orthonormal basis for $\mathfrak{su}_2$ with respect to $\langle \cdot, \cdot \rangle$ identifies the set of linear maps $\mathfrak{su}_2 \to \mathfrak{su}_2$ preserving the inner product with SO(3). We therefore have a homomorphism

$$\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$$

whose kernel is $\{g \in \mathrm{SU}(2) \mid \mathrm{Ad}_g = \mathrm{Id}\} = \{\pm \mathrm{Id}\}$. The derivative of this homomorphism is injective, otherwise there would be a one-parameter subgroup in the kernel of the group homomorphism, and since the Lie algebras have the same dimension we get an isomorphism of Lie algebras. The group homomorphism is therefore surjective, since exponentials of elements of $\mathfrak{so}_3$ generate the group SO(3). □

REMARK 15.5. *Let $\mathcal{I}, \mathcal{J}, \mathcal{K}$ be the elements of $\mathfrak{su}_2$ given by*

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

respectively. Up to scalar, these are an orthonormal basis for $\mathfrak{su}_2$. Since the derivative of the above homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ *is the adjoint map, computed with respect to this* *basis, we see that the induced isomorphism $\mathfrak{su}_2 \to \mathfrak{so}_3$ takes:*

$$\mathcal{I} \longmapsto J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{J} \longmapsto J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\mathcal{K} \longmapsto J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It will be useful to know where this isomorphism takes the elements H, X, Y of $\mathfrak{sl}_{2,\mathbb{C}} = \mathfrak{su}_{2,\mathbb{C}}$. For example, as $H = -2i\mathcal{K}$, we see that it goes to $-2iJ_z$. Or, for the lowering operator Y, we have

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -(\mathcal{I} + i\mathcal{J}) \longmapsto -(J_x + iJ_y)$$

and similarly $X \mapsto J_x - iJ_y$.

Corollary 15.6. *For each $\ell \geq 0$, there is a unique irreducible representation $V^{(\ell)}$ of* SO(3) *of dimension $2\ell + 1$ with derivative isomorphic to the representation $\mathrm{Sym}^{2\ell}(V)$ of $\mathfrak{so}_3$. This gives the complete list of irreducible representations of* SO(3) *up to isomorphism.*

Proof. We simply have to work out which irreducible representations $\mathrm{Sym}^k(V)$ of $\mathfrak{so}_3$ exponentiate to a representation of SO(3). Since we have $\mathrm{SU}(2)/\{\pm \mathrm{Id}\} \cong \mathrm{SO}(3)$ and each $\mathrm{Sym}^k(V)$ exponentiates to a unique representation of SU(2) — which we also call $\mathrm{Sym}^k(V)$ — this is equivalent to asking for which k the centre $\{\pm \mathrm{Id}\}$ of SU(2) acts trivially on $\mathrm{Sym}^k(V)$. But we see that $-\mathrm{Id}$ acts as $(-1)^k$, so the answer is for even k only.

Thus $\mathrm{Sym}^k(V)$ exponentiates to a representation of SO(3) if, and only if, $k = 2\ell$ is even, and we obtain the result. ☐

We can consider the weights of these representations. Under the isomorphism

$$\mathfrak{su}_{2,\mathbb{C}} = \mathfrak{sl}_{2,\mathbb{C}} \longrightarrow \mathfrak{so}_{3,\mathbb{C}}$$

the element $H = -2i\mathcal{K}$ maps to $-2iJ_z$. Since the H-weights of $V^{(\ell)}$ are $-2\ell, -2(\ell - 1), \ldots, 2(\ell - 1), 2\ell$, we must divide these by $-2i$ to find the weights of J_z acting on $V^{(\ell)}$:

$$-i\ell, -i(\ell - 1), \ldots, i(\ell - 1), i\ell.$$

Example 15.7. *Consider the standard three-dimensional representation of* SO(3) *on $\mathbb{C}^3$. The weights of J_z are simply its eigenvalues as a 3×3 matrix, which are $-i, 0, i$. We see that this representation is isomorphic to $V^{(1)}$.*

15.3. Exercises.

Problem 40.

(a) For $a \geq b$ integers, decompose the representation $\mathrm{Sym}^a \mathbb{C}^2 \otimes \mathrm{Sym}^b \mathbb{C}^2$ of $\mathfrak{sl}_{2,\mathbb{C}}$ into irreducibles. (This is known as the **Clebsch–Gordan formula**).
(b) (+) Can you find a general expression for the highest weight vectors for the irreducible subrepresentations? What about for the weight bases?

16. Lecture 16

16.1. Harmonic functions. We can use our understanding of the representation theory of SO(3) to shed light on the classical theory of spherical harmonics.

We let $\mathcal{P}^\ell$ be the subspace of $\mathbb{C}[x, y, z]$ consisting of homogeneous polynomials of degree ℓ. This has an action of SO(3) given by

$$g \cdot f(\mathbf{x}) = f(g^T \mathbf{x}),$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a vector in $\mathbb{C}^3$. We therefore get a representation of SO(3).

LEMMA 16.1. *The action of $A = (a_{ij}) \in \mathfrak{so}_3$ on $\mathcal{P}^\ell$ is given by*

$$\sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_j},$$

where $(x, y, z) = (x_1, x_2, x_3)$, and thus the elements J_x, J_y, and J_z of $\mathfrak{so}_3$ act according to the following formulae:

$$J_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$
$$J_y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$
$$J_z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

PROOF. Exercise (Problem 41). □

It will be useful in what follows to recall the folowing result.

LEMMA 16.2 (Euler's homogeneous function theorem). *If f is a homogeneous polynomial of degree ℓ, then*

$$x\frac{\partial}{\partial x} f + y\frac{\partial}{\partial y} f + z\frac{\partial}{\partial z} f = \ell f.$$

PROOF. Omitted (thus non-examinable) but straightforward. □

The representation $\mathcal{P}^\ell$ is not irreducible. Let $r^2 \in \mathbb{C}[x, y, z]$ be the polynomial

$$r^2 = x^2 + y^2 + z^2.$$

Note that r^2 is invariant under the action of SO(3), SO(3) is just rotation about an axis so the length of $g^T\mathbf{x}$ is the same as the length of $\mathbf{x}$.

LEMMA 16.3. *The map $\mathcal{P}^\ell \to \mathcal{P}^{\ell+2}$ defined by*

$$f \longmapsto r^2 f$$

is an injective homomorphism of SO(3)*-representations.*

PROOF. We have, for $g \in \mathrm{SO}(3)$,

$$g(r^2 f) = g(r^2)g(f) = r^2 g(f),$$

thus r^2 is an SO(3)-homomorphism. It is injective as we have unique factorisation in $\mathbb{C}[x, y, z]$. □

Next, we consider the Laplace operator Δ given by

$$f \longmapsto \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

LEMMA 16.4. *The Laplace operator $\Delta : \mathcal{P}^\ell \to \mathcal{P}^{\ell-2}$ is an* SO(3)*-homomorphism.*

PROOF. We must show that, for $g = (g_{ij}) \in \mathrm{SO}(3)$,

$$g \cdot (\Delta f) = \Delta(g \cdot f).$$

We have

$$\frac{\partial}{\partial x_i}(g \cdot f)(\mathbf{x}) = \sum_j g_{ij} \frac{\partial f}{\partial x_j}(g^T \mathbf{x})$$

and so

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}(g \cdot f)(\mathbf{x}) = \sum_{j,k} g_{ij} g_{ik} \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}(g^T \mathbf{x}).$$

We sum over i, for fixed j and k:

$$\sum_i g_{ij} g_{ik} = \delta_{jk}$$

since g is orthogonal. We therefore obtain

$$\sum_i \frac{\partial^2}{\partial x_i^2}(g \cdot f)(\mathbf{x}) = \sum_j \frac{\partial^2}{\partial x_j^2}(f)(g^T \mathbf{x}),$$

and therefore

$$\Delta(g \cdot f) = g \cdot \Delta(f).$$

□

An element $f \in \mathcal{P}^\ell$ is *harmonic* if $\Delta f = 0$. Since $\dim(\mathcal{P}^\ell) > \dim(\mathcal{P}^{\ell-2})$, harmonic polynomials must exist. We write $\mathcal{H}^\ell \subseteq \mathcal{P}^\ell$ for the space of harmonic polynomials. Note that as $\mathcal{H}^\ell$ is the kernel of a SO(3)-homomorphism it is an SO(3) representation.

LEMMA 16.5. *On $\mathcal{P}^\ell$, we have*

$$r^2 \Delta = J_x^2 + J_y^2 + J_z^2 + \ell^2 + \ell.$$

PROOF. This is Problem 42. □

It follows that $r^2\Delta$ is also an $\mathfrak{so}_3$ homomorphism and furthermore, it preserves each irreducible $\mathfrak{so}_3$-subrepresentation of $\mathcal{P}^\ell$ (since J_x, J_y, J_z do) and thus each SO(3)-subrepresentation. Furthermore, by Schur's lemma it must act on each irreducible subrepresentation as a scalar. We determine that scalar.

LEMMA 16.6. *Suppose that $V \subseteq \mathcal{P}^\ell$ is an irreducible subrepresentation with highest weight ik. Then, for all $f \in V$,*

$$(r^2\Delta)(f) = (\ell^2 + \ell - k^2 - k) f = (\ell - k)(\ell + k + 1) f.$$

Proof. Since $r^2\Delta$ is a $\mathfrak{so}_3$-homomorphism and V is irreducible, by Schur's lemma it acts as a scalar on V. It therefore suffices to compute the action on a highest weight vector $\mathbf{v} \in V$. So $J_z\mathbf{v} = ik\mathbf{v}$ and $(J_x - iJ_y)\mathbf{v} = 0$. It follows that $J_z^2\mathbf{v} = -k^2\mathbf{v}$ and, as

$$\begin{aligned} J_x^2 + J_y^2 &= (J_x + iJ_y)(J_x - iJ_y) + i[J_x, J_y] \\ &= (J_x + iJ_y)(J_x - iJ_y) + iJ_z, \end{aligned}$$

we have $(J_x^2 + J_y^2)\mathbf{v} = 0\mathbf{v} - k\mathbf{v}$. Applying Lemma 16.5 gives the result. □

Theorem 16.7. *For every $\ell \geq 0$,*

$$\mathcal{P}^\ell = \mathcal{H}^\ell \oplus r^2\mathcal{P}^{\ell-2}.$$

The space $\mathcal{H}^\ell$ is the irreducible highest weight representation of $\mathrm{SO}(3)$ *of dimension $2\ell+1$, and the space $\mathcal{P}^\ell$, as an* $\mathrm{SO}(3)$*-representation, the direct sum of representations of weights $i\ell, i(\ell-2), \ldots$, each occurring with multiplicity one.*

Proof. We use induction on ℓ. The case $\ell = 0$ is clear (we just have the trivial representation). For $\ell = 1$ we only have degree 1 polynomials so again $\mathcal{P}^1 = \mathcal{H}^1$. As $\mathcal{H}^1$ is the kernel of $r^2\Delta$, $\mathcal{H}^1$ is a sum of copies of the representation of highest weight i. As this representation has dimension 3 it must be irreducible of highest weight i.

Suppose true for $\ell - 1$ with $\ell \geq 2$. As $\mathcal{H}^\ell$ is the kernel of $r^2\Delta$, the space $\mathcal{H}^\ell$ is the sum of all the copies inside $\mathcal{P}^\ell$ of the irreducible representation with with highest weight $i\ell$, by Lemma 16.6. Since $\mathcal{P}^{\ell-2}$ does not contain this irreducible representation, by the inductive hypothesis, we have

$$\mathcal{H}^\ell \cap r^2\mathcal{P}^{\ell-2} = \{0\}.$$

Since, $\mathcal{H}^\ell \neq 0$ as already discussed, its dimension is a positive multiple of $2\ell+1$. However, its dimension is at most

$$\dim \mathcal{P}^\ell - \dim \mathcal{P}^{\ell-2} = \binom{\ell+2}{2} - \binom{\ell}{2} = 2\ell + 1.$$

It follows that $\mathcal{H}^\ell$ is irreducible, and that we have

$$\mathcal{H}^\ell \oplus r^2\mathcal{P}^{\ell-2} = \mathcal{P}^\ell.$$

The statement about the decomposition into irreducibles follows. □

The proof of this theorem shows that

$$\mathcal{P}^\ell = \mathcal{H}^\ell \oplus r^2\mathcal{H}^{\ell-2} \oplus r^4\mathcal{H}^{\ell-4} \ldots$$

so that every polynomial has a unique decomposition as a sum of harmonic polynomials multiplied by powers of r^2.

We can go further and give nice bases for the $\mathcal{H}^\ell$ by taking weight vectors for J_z. First, we have

Lemma 16.8. *The function $(x - iy)^\ell \in \mathcal{P}^\ell$ is a highest weight vector of weight $i\ell$.*

Proof. This is Problem 43(a). □

We then obtain a weight basis by repeatedly applying the lowering operator

$$J_x + iJ_y = (ix - y)\frac{\partial}{\partial z} + z\left(\frac{\partial}{\partial y} - i\frac{\partial}{\partial x}\right).$$

The functions thus obtained are known as 'spherical harmonics' (at least, up to normalisation), and give a particularly nice basis for the space of functions on the sphere $S^2 \subseteq \mathbb{R}^3$. The decomposition of a function into spherical harmonics is analogous to the Fourier decomposition of a function on the unit circle.

EXAMPLE 16.9. *If $\ell = 1$, then $\mathcal{P}^\ell = \mathcal{H}^\ell$, and the weight vectors are*

$$x + iy, z, x - iy.$$

If $\ell = 2$, a basis of $\mathcal{H}^\ell$ made up of weight vectors is

$$(x - iy)^2, z(x - iy), x^2 + y^2 - 2z^2, z(x + iy), (x + iy)^2.$$

16.2. Exercises.

Problem 41. Prove that the action of $A = (a_{ij}) \in \mathfrak{so}_3$ is given by

$$\sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_j}$$

by considering

$$\frac{d}{dt} f(\exp(tA^T)\mathbf{x}) = \frac{d}{dt} f((\mathrm{Id} + tA^T)\mathbf{x})$$

at $t = 0$ (where we rewrite x, y, z as x_1, x_2, x_3). Thus prove Lemma 16.1.

Problem 42.

(a) Verify the formula

$$r^2 \Delta = J_x^2 + J_y^2 + J_z^2 + \ell^2 + \ell$$

as operators on $\mathcal{P}^\ell$.

(b) Find the image of the Casimir element from problem 34 under our isomorphism $\mathfrak{sl}_{2,\mathbb{C}} \to \mathfrak{so}_{3,\mathbb{C}}$, and compare to part (a).

Problem 43. Let $\ell \geq 1$.

(a) Verify that $(x - iy)^\ell$ is a highest weight vector in $\mathcal{H}^\ell$.

(b) By applying the lowering operator, find weight vectors of weights $i(\ell - 1)$ and $i(\ell - 2)$.

(c) Find a basis of weight vectors in $\mathcal{H}^\ell$ when $\ell = 1$ and $\ell = 2$ (i.e. verify Example 16.9).

Problem 44.

(a) Prove that, for $f \in \mathcal{P}^\ell$,

$$\Delta(r^2 f) = r^2 \Delta(f) + 2(2\ell + 3) f.$$

(b) Find a similar formula for

$$\Delta(r^{2k} f) - r^{2k} \Delta(f).$$

(c) Use this to give another proof that

$$\mathcal{H}^\ell \cap r^2 \mathcal{P}^{\ell - 2} = \{0\}.$$

(Hint: if f is in the intersection, let $f = r^{2k} g$, g not divisible by r^2).

Problem 45.

(a) Let V be the standard — three-dimensional — representation of $\mathfrak{so}_3$. Find a basis of weight vectors for $\mathrm{Sym}^2(V)$, and decompose it into irreducible subrepresentations.
(b) Let $\mathcal{H}^2$ be the five-dimensional representation of SO(3). Decompose $\mathcal{H}^2 \otimes \mathcal{H}^2$ into irreducible representations.

CHAPTER 4

Representations of $\mathfrak{sl}_{3,\mathbb{C}}$

We now discuss the finite-dimensional complex-linear representation theory of $\mathfrak{sl}_{3,\mathbb{C}}$. This then tells us everything about the holomorphic representations of $\mathrm{SL}_3(\mathbb{C})$ and, via the machinery of real forms, tells us about representations of $\mathrm{SU}(3)$ and $\mathrm{SL}_3(\mathbb{R})$.

Much of what we say, particularly at first, will generalise in a fairly obvious way to $\mathfrak{sl}_{n,\mathbb{C}}$.

In this section, all representations of $\mathfrak{sl}_{3,\mathbb{C}}$ will be assumed to be complex-linear.

17. Lecture 17

17.1. The Lie algebra $\mathfrak{sl}_{3,\mathbb{C}}$. We study the Lie algebra

$$\mathfrak{g} = \mathfrak{sl}_{3,\mathbb{C}} = \{X \in \mathfrak{gl}_{3,\mathbb{C}} \mid \mathrm{tr}(X) = 0\},$$

of traceless 3×3 matrices. It has dimension 8. We first need to find the analogue of the standard basis H, X, Y of $\mathfrak{sl}_{2,\mathbb{C}}$. We denote by E_{ij} the matrix with a 1 in row i and column j , and 0 elsewhere. Then $E_{ij} \in \mathfrak{sl}_{3,\mathbb{C}}$ if and only if $i \neq j$. The analogue of H will be the entire subalgebra of diagonal matrices.

Proposition 17.1. *The set $\mathfrak{h} \subseteq \mathfrak{g}$, given by*

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \;\middle|\; a_1 + a_2 + a_3 = 0, \right\}.$$

is an abelian subalgebra.

Proof. It is straightforward to see this is a subalgebra as it is closed under scalar multiplication, addition and the Lie bracket. It is abelian as diagonal matrices commute with each other. $\square$

We call $\mathfrak{h}$ the (standard) Cartan subalgebra of $\mathfrak{g}$. We pick as a basis of $\mathfrak{h}$ the elements

$$H_{12} = E_{11} - E_{22} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

and

$$H_{23} = E_{22} - E_{33} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix},$$

and also define $H_{13} = E_{11} - E_{33} = H_{12} + H_{23}$.

Next we consider the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$, seeking eigenvectors and eigenvalues, i.e.

$$\left[\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}, E_{ij} \right] = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} E_{ij} - E_{ij} \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = (a_i - a_j) E_{ij}.$$

Thus $\{E_{ij} \mid i \neq j\} \cup \{H_{12}, H_{23}\}$ is a basis of simultaneous eigenvectors in $\mathfrak{sl}_{3,\mathbb{C}}$ for the adjoint action of $\mathfrak{h}$.

17.2. Weights. Suppose that (ρ, V) is a finite-dimensional complex-linear representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Suppose that $\mathbf{v} \in V$ is an eigenvector for all $\rho(H)$, $H \in \mathfrak{h}$ (called a *simultaneous* eigenvector). Then, for each $H \in \mathfrak{h}$, there is an $\alpha(H) \in \mathbb{C}$ such that $\rho(H)\mathbf{v} = \alpha(H)\mathbf{v}$. Since ρ is complex-linear, α is a complex-linear map $\mathfrak{h} \to \mathbb{C}$. In other words, α is an element of the dual space $\mathfrak{h}^*$. This motivates the following definition:

DEFINITION 17.2. *Suppose that (ρ, V) is a representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Then a* <u>weight vector</u> *in V is $\mathbf{v} \in V$ such that there is $\alpha \in \mathfrak{h}^*$ (the* <u>weight</u>*) with:*

$$\rho(H)\mathbf{v} = \alpha(H)\mathbf{v}$$

for all $H \in \mathfrak{h}$.

The <u>weight space</u> *of α is*

$$V_\alpha = \{\mathbf{v} \in V \mid \rho(H)\mathbf{v} = \alpha(H)\mathbf{v} \text{ for all } H \in \mathfrak{h}\}.$$

On $\mathfrak{h}$ we have some 'obvious' maps $L_i \in \mathfrak{h}^*$ given by

$$L_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i.$$

These span $\mathfrak{h}^*$, subject to the relation[1]

$$L_1 + L_2 + L_3 = 0.$$

We compute the weights of some particular representations.

EXAMPLE 17.3. *If $V = \mathbb{C}^3$ is the standard representation of $\mathfrak{sl}_{3,\mathbb{C}}$ (for which $\rho(A) = A$ for all $A \in \mathfrak{sl}_{3,\mathbb{C}}$), then the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are all weight vectors:*

$$\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \mathbf{e}_i = a_i \mathbf{e}_i$$

from which we see that $H\mathbf{e}_i = L_i(H)\mathbf{e}_i$ for all $H \in \mathfrak{h}$. Hence $\mathbf{e}_i$ is a weight vector with weight L_i.

EXAMPLE 17.4. *If $V = (\mathbb{C}^3)^*$ is the* dual *of the standard representation then it has a basis $\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*$ defined by*

$$\mathbf{e}_i^*(\mathbf{e}_j) = \delta_{ij}.$$

One can show that $\mathbf{e}_i^*$ is a weight vector of weight $-L_i$, so the weights are $-L_1, -L_2, -L_3$. See problem 48.

EXAMPLE 17.5. *Let $V = \mathrm{Sym}^2(\mathbb{C}^3)$ be the symmetric square of the standard representation. The rules for calculating the weights of V are the same as for $\mathfrak{sl}_{2,\mathbb{C}}$ — so, for the symmetric square, we add all unordered pairs of weights of $\mathbb{C}^3$. For details see section 18.3 The weights of $\mathbb{C}^3$ are $\{L_1, L_2, L_3\}$ and so the weights of $\mathrm{Sym}^2(\mathbb{C}^3)$ are*

$$\{2L_1, 2L_2, 2L_3, L_1 + L_2, L_2 + L_3, L_1 + L_3\}.$$

Note that, if we wanted, we could also write $L_1 + L_2 = -L_3$ etc.

[1]More precisely, $\mathfrak{h}^*$ is isomorphic to the quotient of the three dimensional vector space with basis $\{L_1, L_2, L_3\}$ by the subspace spanned by $L_1 + L_2 + L_3$.

Example 17.6. *Let $V = \mathfrak{g}$ with the adjoint representation defined by $\rho(X)Y = [X, Y]$. As already observed, we have*

$$[H, E_{ij}] = (L_i - L_j)(H)E_{ij},$$

for $H \in \mathfrak{h}$ and $i \neq j$, while $[H, H'] = 0$ for $H, H' \in \mathfrak{h}$. Thus the weights of the adjoint representation are $L_i - L_j$ $(i \neq j)$ and 0. The weight space for 0 is $\mathfrak{h}$, which has dimension two with basis H_{12} and H_{23}; we say that the weight 0 has multiplicity *two in V. We obtain:*

Weight	*Weight space basis*		
$L_1 - L_2$	E_{12}		*Simple roots*
$L_2 - L_3$	E_{23}	*Positive roots*	
$L_1 - L_3$	E_{13}		
$L_2 - L_1$	E_{21}		
$L_3 - L_2$	E_{32}	*Negative roots*	
$L_3 - L_1$	E_{31}		
0	H_{12}, H_{23}		

Definition 17.7. *A* root *of $\mathfrak{g} = \mathfrak{sl}_{3,\mathbb{C}}$ is a non-zero weight of the adjoint representation. A* root vector *is a weight vector of a root, and a* root space *is the weight space of a root.*

In other words, a root α with root vector $0 \neq E \in \mathfrak{g}$ is a non-zero element $\alpha \in \mathfrak{h}^*$ such that

$$[H, E] = \alpha(H)E.$$

We write

$$\Phi = \{\pm(L_1 - L_2), \pm(L_2 - L_3), \pm(L_1 - L_3)\}$$

for the set of roots of $\mathfrak{sl}_{3,\mathbb{C}}$. Out of these, we call $\Phi^+ = \{L_1 - L_2, L_2 - L_3, L_1 - L_3\}$ the *positive roots* and $\Phi^- = \{L_2 - L_1, L_3 - L_2, L_3 - L_1\}$ the *negative roots*. We write $\Delta = \{L_1 - L_2, L_2 - L_3\}$; these are the *simple roots*. Note that $L_1 - L_3$ is the sum of the two simple roots. We will sometimes write α_{ij} for the root $L_i - L_j$.

Finally, we have the *root space* or *Cartan* decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where the $\mathfrak{g}_\alpha$ are the root spaces, which are all one-dimensional.

17.3. Exercises.

Problem 46. Verify that

$$\left[\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, E_{ij}\right] = (a_i - a_j)E_{ij}$$

and

$$[E_{12}, E_{23}] = E_{13}.$$

Problem 47. Work through all the theory in Section 17.1 for the case of $\mathfrak{sl}_{2,\mathbb{C}}$. What are the roots and root spaces? What is the relation between the weights (as linear maps on $\mathfrak{h}$) and between the weights defined in section 3?

Problem 48. Let $V = (\mathbb{C}^3)^*$ be the dual of the standard representation, with basis $\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*$ dual to the standard basis.

(a) Show that the $\mathbf{e}_i^*$ are weight vectors with weights $-L_i$.

(b) Find the action of each E_{ij} on $\mathbf{e}_3^*$ and deduce that $\mathbf{e}_3^*$ is a highest weight vector with weight $-L_3$.

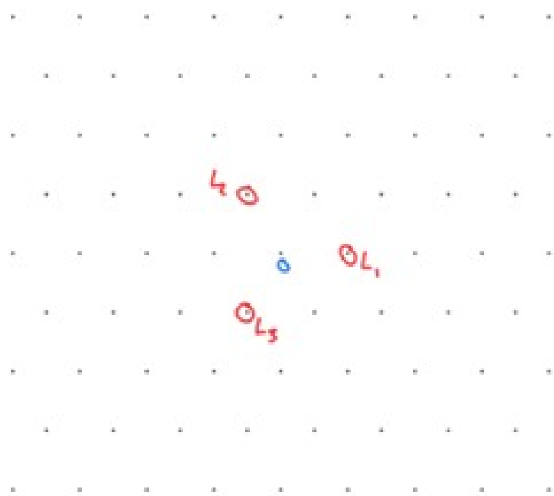

Figure 1. Weights for $\mathbb{C}^3$.

18. Lecture 18

18.1. Visualising weights. Shortly we will prove that, if (ρ, V) is a finite dimensional representation of $\mathfrak{sl}_{3,\mathbb{C}}$, then its weights are integer linear combinations of the L_i. In other words, they lie in the *weight lattice*

$$\Lambda_W = \{a_1 L_1 + a_2 L_2 + a_3 L_3 \,|\, a_1, a_2, a_3 \in \mathbb{Z}\}.$$

We want to visualise this in a way that treats L_1, L_2, L_3 symmetrically. Noticing that they sum to zero, we regard them them as the position vectors of the vertices an equilateral triangle with unit side length, centred on the origin. The weight lattice Λ_W is then the set of vertices of equilateral triangles tiling the plane. For any representation (ρ, V), its *weight diagram* is then obtained by circling the weights that occur in that representation.

Example 18.1. *We draw the weight diagram for the standard representation, in Figure 1.*

Example 18.2. *We draw the weight diagram for the adjoint representation in Figure 2. Note that in this case the dimension of the weight space for the weight* 0 *is two. We say the weight has* multiplicity *two, and indicate this on the weight diagram by circling the weight twice. If the multiplicity was much higher, we would need another method (like writing the multiplicity next to the circle as a number).*

18.2. Representations and weights. Firstly, we recall from above that any finite-dimensional representation of $\mathfrak{sl}_{3,\mathbb{C}}$ is completely reducible. This is Theorem 14.7 from the previous section, that we proved using the unitary trick.

Theorem 18.3. *Let (ρ, V) be a finite-dimensional representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Then:*

(i) There is a basis of V consisting of weight vectors.

(ii) Every weight of V is in the weight lattice Λ_W.

We may combine these two into the single equality

$$V = \bigoplus_{\alpha \in \Lambda_W} V_\alpha.$$

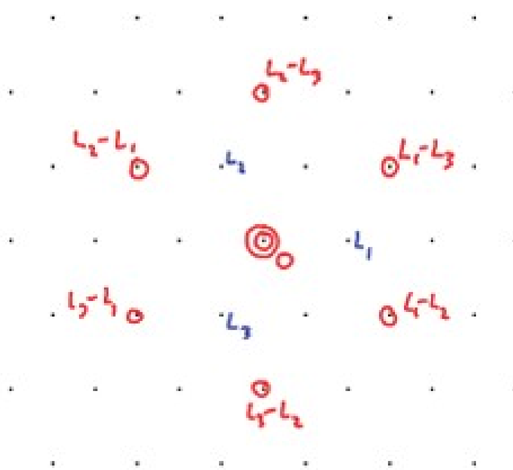

FIGURE 2. Weights for the adjoint representation.

PROOF. Consider the embedding $\iota_{12} : \mathfrak{sl}_{2,\mathbb{C}} \to \mathfrak{sl}_{3,\mathbb{C}}$ embedding a 2×2 matrix into the 'top left' of a 3×3 matrix:

$$\iota_{12} : \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \longmapsto \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is a Lie algebra homomorphism, and $\rho \circ \iota_{12}$ is a representation of $\mathfrak{sl}_{2,\mathbb{C}}$. We know from the $\mathfrak{sl}_{2,\mathbb{C}}$ theory, Proposition 12.2, that

$$(\rho \circ \iota_{12})(H) = \rho \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} = \rho(H_{12})$$

is diagonalisable with integer eigenvalues. If $\mathbf{v} \in V$ is a weight vector with weight $a_1 L_1 + a_2 L_2 + a_3 L_3$, then it is an eigenvector for $\rho(\iota_{12}(H))$ with eigenvalue $a_1 - a_2$. Thus $a_1 - a_2$ is an integer.

Now, there is another embedding ι_{23} putting a 2×2 matrix in the 'bottom right' corner. The same argument then shows that $\rho(H_{23})$ is diagonalisable with integer eigenvalues, which shows that $a_2 - a_3$ is an integer for every weight. Thus, by Problem 49, every weight is in the weight lattice.

Moreover, $\rho(H_{12})$ and $\rho(H_{23})$ are diagonalisable, and they commute with each other since H_{12} and H_{23} commute and ρ is a Lie algebra homomorphism. A theorem from linear algebra states that commuting, diagonalisable matrices are simultaneously diagonalisable. It follows that there is a basis of V consisting of simultaneous eigenvectors for $\rho(H_{12})$ and $\rho(H_{23})$. Since H_{12} and H_{23} span $\mathfrak{h}$, this is a basis of weight vectors. $\square$

Remark 18.4. *There is a third homomorphism*

$$\iota_{13} : \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & -a \end{pmatrix}.$$

We have $\iota_{13}(H) = H_{13}$.

Note that, for $i < j$, $\iota_{ij}(X) = E_{ij}$ and $\iota_{ij}(Y) = E_{ji}$, so E_{ij} and E_{ji} will play the role of raising and lowering operators.

Remark 18.5. *We could also prove Theorem 18.3 by exponentiating ρ to a representation of $\mathrm{SL}_3(\mathbb{C})$ and considering the action of the subgroup of diagonal matrices with entries in $U(1)$, which is compact (isomorphic to $U(1)$). Compare this with the proof of the statement that the weights of representations of $\mathfrak{sl}_{2,\mathbb{C}}$ are integers (Proposition 12.2).*

18.3. Tensor constructions. We record how the various linear algebra constructions we know about interact with the theory of weights. If (ρ, V) is a representation of $\mathfrak{g}$ then we consider its weights as a *multiset*

$$\{\alpha_1, \ldots, \alpha_n\}$$

where $n = \dim V$ and each $\alpha_i \in \mathfrak{h}^*$ is written in this list $\dim V_\alpha$, the multiplicity of α_i, times.

Suppose that (σ, W) is another representation of $\mathfrak{g}$ with multiset of weights

$$\{\beta_1, \ldots, \beta_m\}.$$

Proposition 18.6. *Suppose that V, W, α_i, β_j are as above. Then:*

(i) The weights of V^* are $\{-\alpha_1, \ldots, -\alpha_n\}$.

(ii) The weights of $V \otimes W$ are

$$\{\alpha_i + \beta_j \,|\, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

(iii) The weights of $\mathrm{Sym}^k(V)$ are

$$\{\alpha_{i_1} + \ldots + \alpha_{i_k} \,|\, 1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n\}.$$

(iv) The weights of $\bigwedge^k(V)$ are

$$\{\alpha_{i_1} + \ldots + \alpha_{i_k} \,|\, 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}.$$

Proof. This is similar to the proof of Proposition 12.11 in the $\mathfrak{sl}_{2,\mathbb{C}}$ case and thus is left as an exercise (Problem 51). $\square$

18.4. Exercises.

Problem 49. Show that $a_1L_1 + a_2L_2 + a_3L_3 \in \Lambda_W$ if and only if $a_1 - a_2, a_2 - a_3 \in \mathbb{Z}$. Must the a_i be integers?

Problem 50. The *root lattice* $\Lambda_R \subseteq \Lambda_W$ is the subgroup of the weight lattice generated by the roots.

(a) Draw a picture showing the root lattice inside the weight lattice.
(b) Show that Λ_R has index three in Λ_W (i.e. the quotient Λ_W/Λ_R has order three).
(c) What would the root lattice and weight lattice be for $\mathfrak{sl}_{2,\mathbb{C}}$? What is the index in this case?
(d) Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Show that any two weights of V differ by an element of the root lattice.

Problem 51. Prove Proposition 18.6

Problem 52. Find the weights of $\mathrm{Sym}^3(\mathbb{C}^3)$ and draw the weight diagram.

Problem 53. Using weights, or otherwise, show that

$$\mathbb{C}^3 \otimes (\mathbb{C}^3)^* \cong \mathbb{C} \oplus \mathfrak{sl}_{3,\mathbb{C}}$$

where $\mathbb{C}$ is the trivial representation and $\mathfrak{sl}_{3,\mathbb{C}}$ is the adjoint representation.

19. Lecture 19

19.1. Highest weights. We now develop the theory of highest weights, analogous to that for $\mathfrak{sl}_{2,\mathbb{C}}$. We first carry out the fundamental weight calculation (the analogue of Lemma 13.1).

LEMMA 19.1 (Fundamental Weight Calculation). *Let (ρ, V) be a representation of $\mathfrak{sl}_{3,\mathbb{C}}$ and let $\mathbf{v} \in V_\beta$ be a weight vector with weight $\beta \in \mathfrak{h}^*$. Let $\alpha \in \mathfrak{h}^*$ be a root and let $X_\alpha \in \mathfrak{g}_\alpha$ be a root vector. Then*

$$X_\alpha(\mathbf{v}) \in V_{\alpha+\beta}.$$

Thus we obtain a map

$$X_\alpha : V_\beta \longrightarrow V_{\alpha+\beta}.$$

PROOF. Let $H \in \mathfrak{h}$. Then

$$H(X_\alpha(\mathbf{v})) = ([H, X_\alpha] + X_\alpha H)(\mathbf{v}) = \alpha(H)X_\alpha(\mathbf{v}) + X_\alpha(\beta(H)\mathbf{v}) = (\alpha + \beta)(H)X_\alpha(\mathbf{v}).$$

□

EXAMPLE 19.2. *We work this out for the adjoint representation. Recall that, for $i \neq j$, we have the root $\alpha_{ij} = L_i - L_j$ with root vector E_{ij}. The above calculation shows that, if α and β are roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Here are some examples:*

(i) If $\alpha = \alpha_{12}$, $\beta = \alpha_{13}$, then $\alpha + \beta$ is not a root so $\mathfrak{g}_{\alpha+\beta} = 0$. Thus $[E_{12}, E_{23}] = 0$ (which could also be checked directly).

(ii) If $\alpha = -\beta$ then we get

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}.$$

(iii) If $\alpha = \alpha_{12}$, $\beta = \alpha_{23}$, then $\alpha + \beta = \alpha_{13}$ and we get

$$[E_{12}, E_{23}] \in \mathfrak{g}_{\alpha_{13}} = \langle E_{13} \rangle .$$

In fact, you can check that $[E_{12}, E_{23}] = E_{13}$.

COROLLARY 19.3. *Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Then the weights occurring in V all differ by integral linear combinations of the roots of $\mathfrak{sl}_{3,\mathbb{C}}$, that is, by integral linear combinations of $L_i - L_j$.*

PROOF. Let α be any weight of V. Then the weights obtained by (repeatedly) applying elements of $\mathfrak{sl}_{3,\mathbb{C}}$ differ by integral linear combinations of the roots. On the other hand this also gives an invariant subspace, which by irreducibility is all of V. □

With regard to the weight diagram, we observe that the positive root vectors E_{12}, E_{23}, E_{13} move in the 'north-east' direction while the negative root vectors move in the 'south-west' direction (roughly speaking). See Figure 3.

DEFINITION 19.4. *Let (ρ, V) be a representation of $\mathfrak{sl}_{3,\mathbb{C}}$. A* <u>highest weight vector</u> *in V is a vector $\mathbf{v} \in V$ such that:*

(i) $\mathbf{v}$ is a weight vector; and

(ii) $\rho(E_{12})\mathbf{v} = \rho(E_{23})\mathbf{v} = 0$.

The weight of $\mathbf{v}$ is then a <u>highest weight</u> *for V.*

REMARK 19.5. *Since $[E_{12}, E_{23}] = E_{13}$, it follows also that $E_{13}\mathbf{v} = 0$ for $\mathbf{v}$ a highest weight vector. So all positive root vectors send $\mathbf{v}$ to 0.*

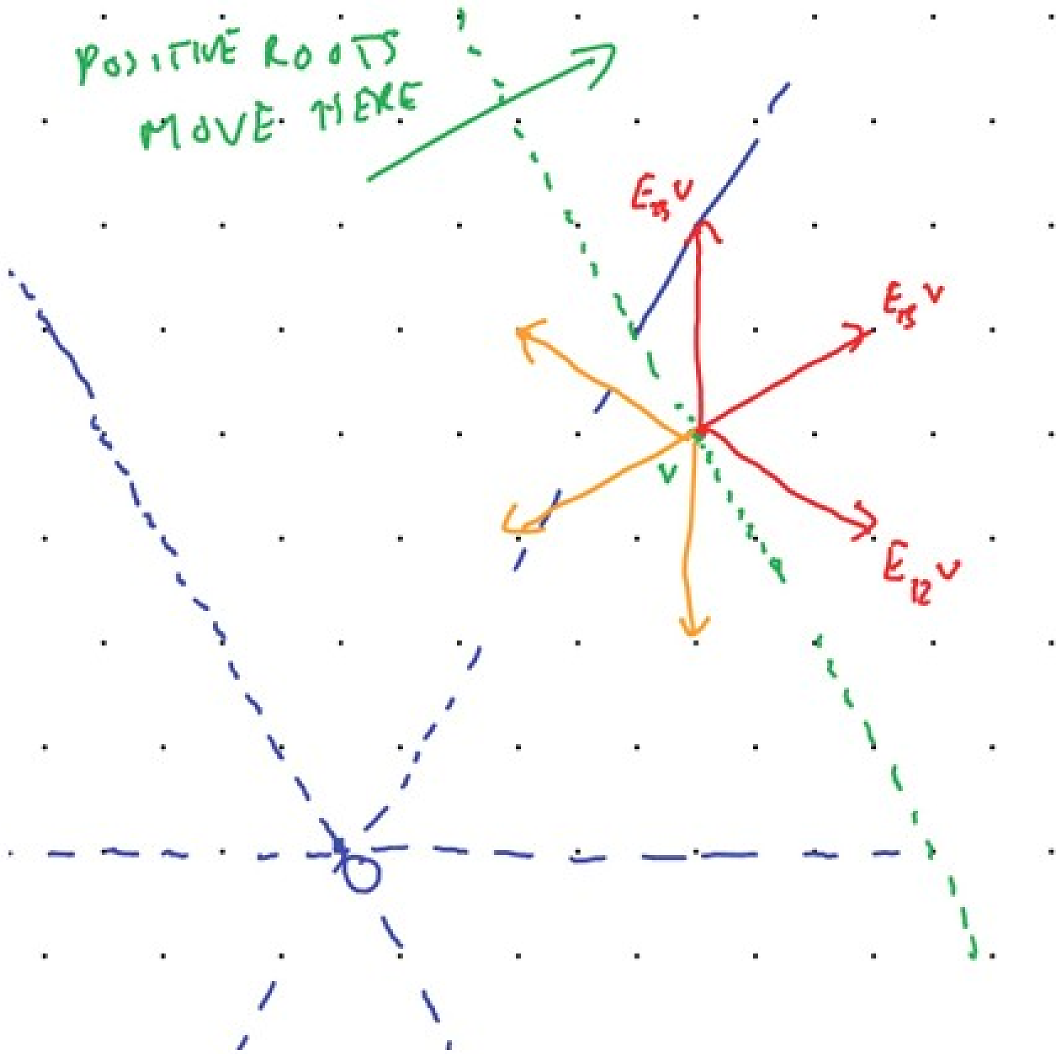

Figure 3. Effect of roots

Example 19.6.

(i) The standard representation $\mathbb{C}^3$ has highest weight L_1 with highest weight vector $\mathbf{e}_1$.
(ii) The dual $(\mathbb{C}^3)^*$ has highest weight $-L_3$ with highest weight vector $\mathbf{e}_3^*$.
(iii) The adjoint representation has highest weight $L_1 - L_3$ with highest weight vector E_{13}.
(iv) The symmetric square $\mathrm{Sym}^2(\mathbb{C}^3)$ has highest weight $2L_1$ with highest weight vector $\mathbf{e}_1^2$.

Lemma 19.7. *Let (ρ, V) be a finite-dimensional representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Then V has a highest weight vector.*

Proof. For a weight $\alpha = a_1L_1 + a_2L_2 + a_3L_3$, define $l(\alpha) = a_1 - a_3$. Of all the finitely many weights of V, choose a weight α such that $l(\alpha)$ is maximal.

Let $\mathbf{v}$ be a weight vector with this weight. Then $\rho(E_{12})\mathbf{v}$, if non-zero, has weight $\alpha + L_1 - L_2$ and

$$l(\alpha + L_1 - L_2) = l(\alpha) + l(L_1 - L_2) = l(\alpha) + 1 > l(\alpha).$$

This is not a weight of V by maximality of $l(\alpha)$. Thus $\rho(E_{12})\mathbf{v} = 0$. Similarly $\rho(E_{23})\mathbf{v}$, if non-zero, has weight $\alpha + L_2 - L_3$ and $l(\alpha + L_2 - L_3) = l(\alpha) + 1$, so $\rho(E_{23})\mathbf{v} = 0$. $\square$

19.2. Weyl symmetry. Let s_1, s_2, and s_3 be, respectively, reflections in the lines through L_1, L_2, and L_3. Then any two of these (say s_1 and s_3) generate the *Weyl group* W, which is the group of symmetries of the triangle with vertices L_1, L_2, L_3. So we have $W \cong D_3 \cong S_3$. Note that W acts on the plane in a way that preserves the weight lattice. See Figure 4.

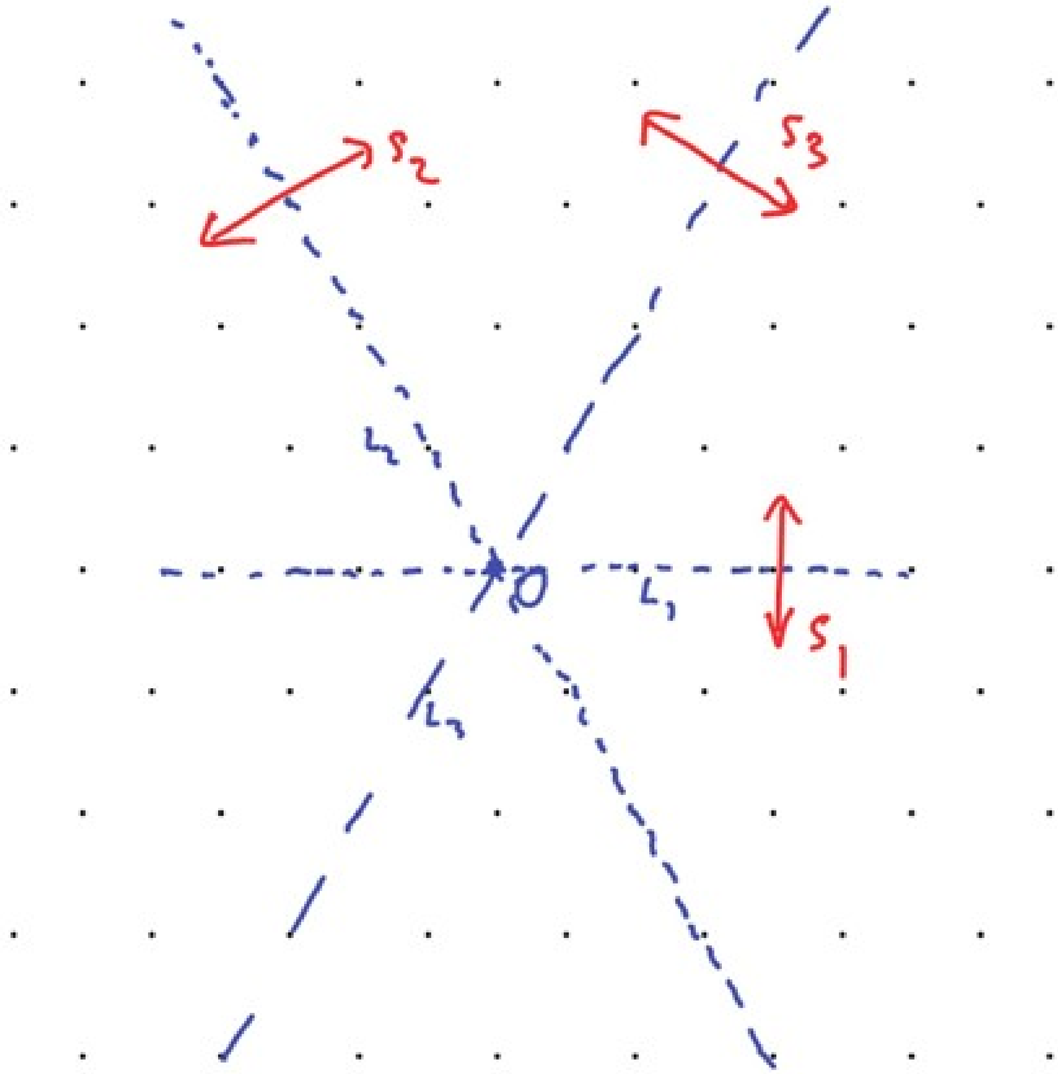

FIGURE 4. Simple reflections

THEOREM 19.8. *Let (ρ, V) be a finite-dimensional representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Then the weights of V are symmetric with respect to the action of the Weyl group.*

PROOF. We will prove they are symmetric with respect to s_3 by using the inclusion

$$\iota_{12} : \mathfrak{sl}_{2,\mathbb{C}} \to \mathfrak{sl}_{3,\mathbb{C}}$$

that puts a 2×2 matrix in the top left corner of a 3×3 matrix. We consider the restriction of V to $\mathfrak{sl}_{2,\mathbb{C}}$.

Note that if $\mathbf{v}\in V$ is a weight vector with weight $aL_1 - bL_3$, then

$$\rho(\iota_{12}(H))\mathbf{v} = \rho\left(\begin{pmatrix}1 & & \\ & -1 & \\ & & 0\end{pmatrix}\right)\mathbf{v} = a\mathbf{v}.$$

Thus $\mathbf{v}$ is an $\mathfrak{sl}_{2,\mathbb{C}}$-weight vector with weight a. Note that $\iota_{12}(X) = E_{12}$, so an $\mathfrak{sl}_{2,\mathbb{C}}$-weight vector in V is an $\mathfrak{sl}_{2,\mathbb{C}}$-highest weight vector if it is sent to 0 under the action of E_{12}.

The kernel of $\rho(E_{12})$ on V is preserved by $\mathfrak{h}$ (check this, Problem 54) and so has a basis made up of $\mathfrak{sl}_{3,\mathbb{C}}$-weight vectors $\mathbf{v}_1,\ldots,\mathbf{v}_r$. These are then a maximal set of linearly independent highest weight vectors for $\mathfrak{sl}_{2,\mathbb{C}}$ and in particular, if V_i is the $\mathfrak{sl}_{2,\mathbb{C}}$-representation generated by $\mathbf{v}_i$ then, as an $\mathfrak{sl}_{2,\mathbb{C}}$-representation,

$$V = \bigoplus_{i=1}^{r} V_i.$$

Fix i; it suffices to show that V_i has a basis of $\mathfrak{sl}_{3,\mathbb{C}}$-weight vectors whose weights are preserved by s_3. Let $\mathbf{v}_i$ have weight $aL_1 - bL_3$. It follows from the $\mathfrak{sl}_{2,\mathbb{C}}$-theory that $a\geq 0$ and — remembering that $\iota_{12}(Y) = E_{21}$ — that V_i has a basis

$$\mathbf{v}, E_{21}\mathbf{v},\ldots,E_{21}^{a}\mathbf{v}.$$

By the FWC (Lemma 19.1) we see that these are $\mathfrak{sl}_{3,\mathbb{C}}$ weight vectors with $\mathfrak{sl}_{3,\mathbb{C}}$ weights

$$aL_1 - bL_3, (a-1)L_1 + L_2 - bL_3, \ldots, L_1 + (a-1)L_2 - bL_3, aL_2 - bL_3$$

which are symmetrical under s_3 (this reflection swaps L_1 and L_2), as required. This argument is illustrated in Figure 5.

Invariance with respect to the other reflections is proved similarly using the other inclusions ι_{ij} of $\mathfrak{sl}_{2,\mathbb{C}}$ in $\mathfrak{sl}_{3,\mathbb{C}}$. □

Corollary 19.9. *Every highest weight is of the form $aL_1 - bL_3$ for $a,b\geq 0$ integers.*

Proof. Indeed, in the course of the proof of Theorem 19.8 we showed that if $aL_1 - bL_3$ was a highest weight, then $a\geq 0$. A similar argument shows that $b\geq 0$. □

Definition 19.10. *The region*

$$\{aL_1 - bL_3 \mid a,b\in\mathbb{R}_{\geq 0}\}$$

is called the dominant Weyl chamber *and weights inside it (including the boundary) are* dominant weights.

19.3. Exercises.

Problem 54. Show that the kernel of E_{12} is preserved by $\mathfrak{h}$.

Problem 55. Let (ρ, V) is a representation of $\mathfrak{sl}_{3,\mathbb{C}}$. As $\mathrm{SL}_3(\mathbb{C})$ is simply-connected ρ exponentiates to a representation, $\tilde{\rho}$, of $\mathrm{SL}_3(\mathbb{C})$. Let

$$\sigma_3 = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\in\mathrm{SL}_3(\mathbb{C}).$$

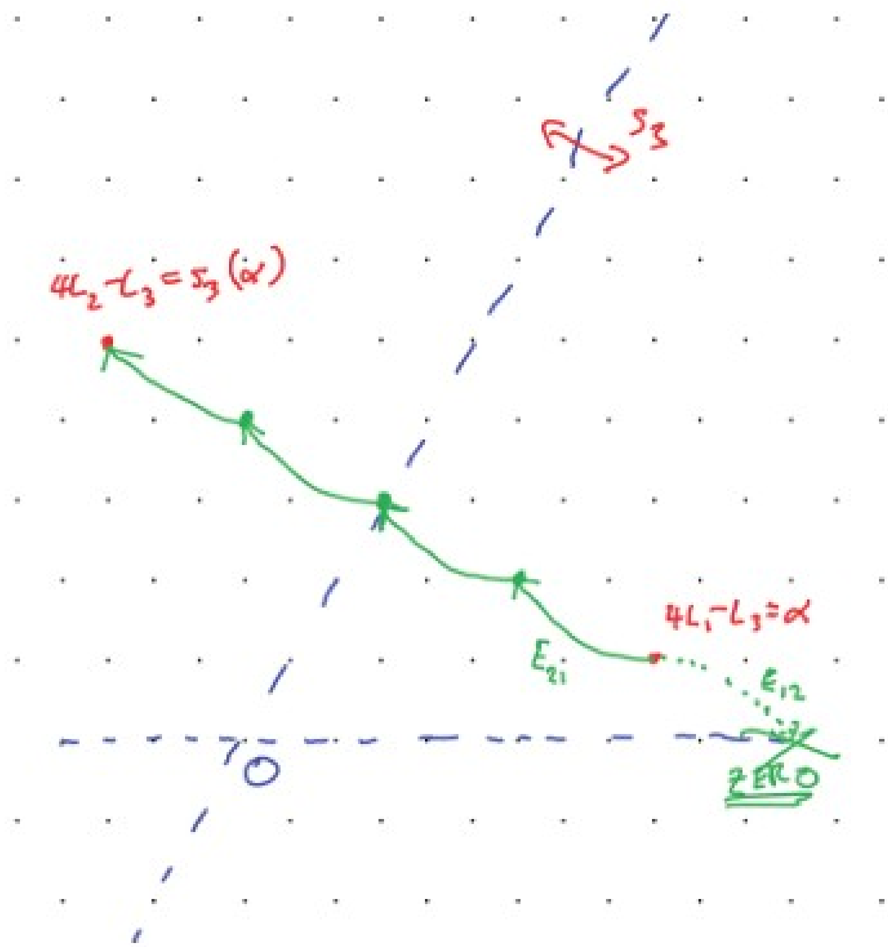

FIGURE 5. Proof of Weyl symmetry

(i) Show that, for every weight α, $\tilde{\rho}(\sigma_3)$ is an isomorphism

$$V_\alpha \to V_{s_3\alpha}.$$

Here $s_3(a_1L_1 + a_2L_2 + a_3L_3) = a_1L_2 + a_2L_1 + a_3L_3$.

(ii) Give another proof of Theorem 19.8.

Problem 56. Let $a, b \geq 0$ be integers. Check that

$$\mathbf{e}_1^a \otimes (\mathbf{e}_3^*)^b \in \mathrm{Sym}^a(\mathbb{C}^3) \otimes \mathrm{Sym}^b((\mathbb{C}^3)^*)$$

is a highest weight vector with weight $aL_1 - bL_3$.

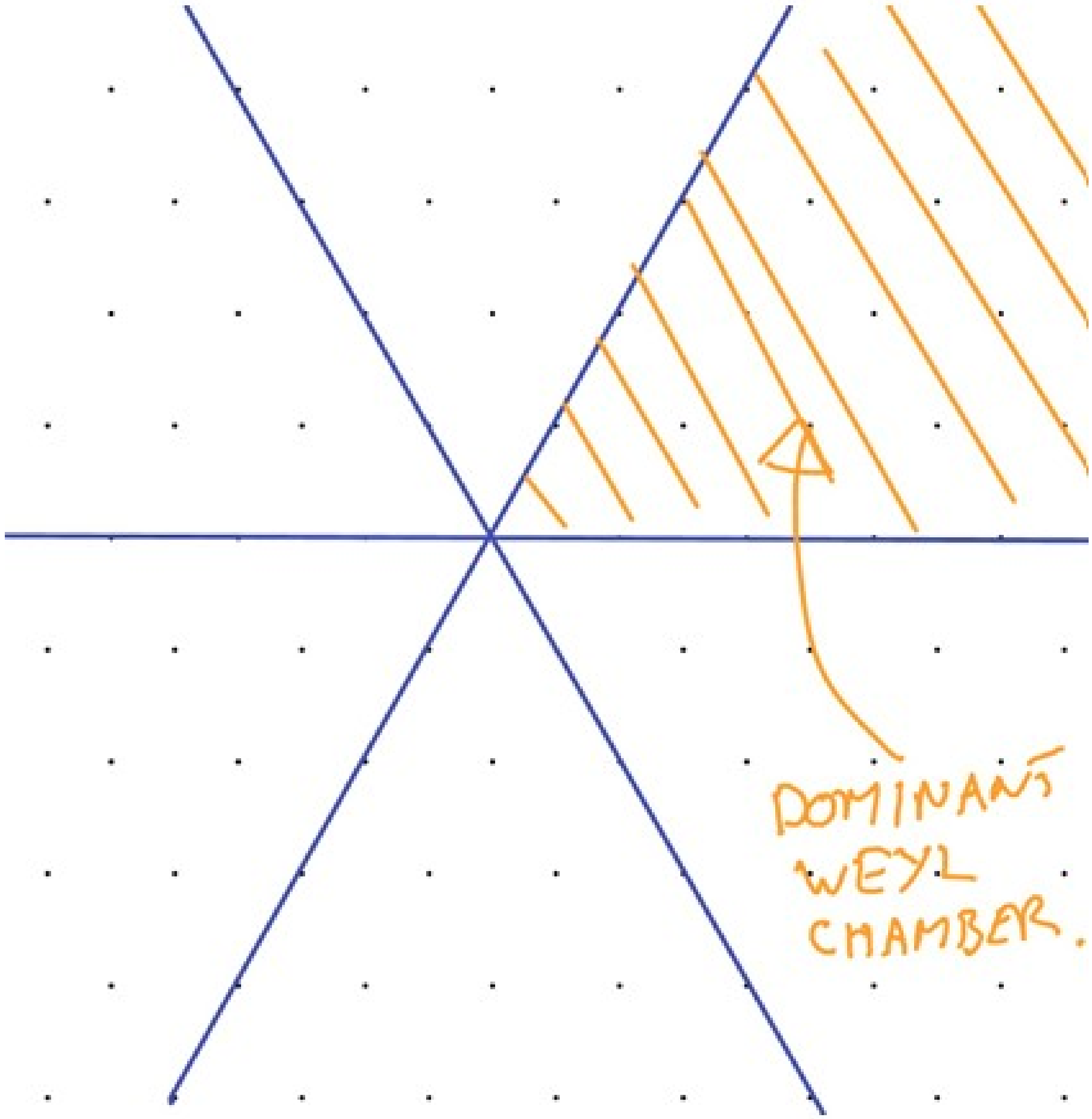

Figure 6. Dominant Weyl chamber

20. Lecture 20

20.1. Irreducible representations of $\mathfrak{sl}_{3,\mathbb{C}}$. We are now in a position to state the main theorem of $\mathfrak{sl}_{3,\mathbb{C}}$-theory.

Theorem 20.1. *For every pair a, b of non-negative integers, there is a unique (up to isomorphism) irreducible finite-dimensional representation $V^{(a,b)}$ of $\mathfrak{sl}_{3,\mathbb{C}}$ with a highest weight vector of weight $aL_1 - bL_3$.*

Note that the highest weights occurring in the theorem are exactly the dominant elements of the weight lattice.

Since every irreducible finite-dimensional representation does have a highest weight, necessarily dominant, every irreducible representation is isomorphic to $V^{(a,b)}$ for some integers $a, b \geq 0$.

EXAMPLE 20.2. *We have already seen some examples:*

(i) *The standard representation* $\mathbb{C}^3$ *is irreducible with highest weight* L_1*, therefore*

$$V^{(1,0)} = \mathbb{C}^3.$$

(ii) *The dual to the standard representation is irreducible with highest weight* $-L_3$*, and so*

$$V^{(0,1)} = (\mathbb{C}^3)^*.$$

(iii) *The adjoint representation* $\mathfrak{g}$ *is irreducible with highest weight* $L_1 - L_3$*, and so*

$$V^{(1,1)} = \mathfrak{g}.$$

(iv) *The symmetric square* $\mathrm{Sym}^2(\mathbb{C}^3)$ *has highest weight* $2L_1$ *with highest weight vector* $\mathbf{e}_1^2$*.*

20.2. Proof of theorem 20.1.

LEMMA 20.3. *Let* (ρ, V) *be a finite-dimensional representation of* $\mathfrak{sl}_{3,\mathbb{C}}$*. Let* $\mathbf{v} \in V$ *be a highest weight vector of weight* α*, and let*

$$W = \{\rho(Y_n)\rho(Y_{n-1})\ldots\rho(Y_1)\mathbf{v} \,|\, n \geq 0, Y_i \in \{E_{21}, E_{32}\}\}.$$

Then

(i) W *is a subrepresentation of* V

(ii) $W_\alpha = \langle \mathbf{v} \rangle$ *i.e.* $\mathbf{v}$ *is the unique weight vector in* W *of weight* α*, up to scaling.*

(iii) W *is irreducible.*

PROOF.

(i) Let

$$W_n = \{\rho(Y_m)\rho(Y_{m-1})\ldots\rho(Y_1)\mathbf{v} \,|\, n \geq m \geq 0, Y_i \in \{E_{21}, E_{32}\}\}.$$

Then

$$W = \bigcup_{n=0}^{\infty} W_n.$$

Firstly, it is clear that $\rho(E_{21})$ and $\rho(E_{32})$ take W_n to W_{n+1}, and so preserve W. Since

$$\rho(E_{31}) = \rho([E_{32}, E_{21}]) = [\rho(E_{32}), \rho(E_{21})]$$

we see that $\rho(E_{31})$ also preserves W.

Secondly, every $\rho(Y_m)\rho(Y_{m-1})\ldots\rho(Y_1)\mathbf{v}$ is a weight vector (by the fundamental weight calculation, Lemma 19.1) and so an eigenvector for all $\rho(H)$, $H \in \mathfrak{h}$. Thus $\rho(H)$ preserves each W_n (and hence also W).

Finally, we show that $\rho(E_{12})$ preserves W_n. A similar proof then applies for $\rho(E_{23})$, and then $\rho(E_{13})$ preserves W by the same argument as for $\rho(E_{31})$. We prove the statement for $\rho(E_{12})$ by induction on n.

For $n = 0$, $W_0 = \langle \mathbf{v} \rangle$. Since $\mathbf{v}$ is a highest weight vector, $\rho(E_{12})\mathbf{v} = 0$ and so $\rho(E_{12})(W_0) \subseteq W_0$.

Suppose that the claim is true for n. Consider $\mathbf{w} = \rho(Y_{n+1})\ldots\rho(Y_1)\mathbf{v} \in W_{n+1}$ with $Y_i \in \{E_{21}, E_{32}\}$. We must show that $\rho(E_{12})\mathbf{w} \in W_{n+1}$. Suppose first that $Y_{n+1} = E_{21}$.

Then, as $[E_{12}, E_{21}] = H_{12} \in \mathfrak{h}$, we have

$$\begin{aligned}
\rho(E_{12})\mathbf{w} &= \rho(E_{12})\rho(E_{21})\rho(Y_n)\dots\rho(Y_1)\mathbf{v} \\
&= \rho(E_{21})\rho(E_{12})\rho(Y_n)\dots\rho(Y_1)\mathbf{v} + \rho(H_{12})\rho(Y_n)\dots\rho(Y_1)\mathbf{v} \\
&\in \rho(E_{21})\rho(E_{12})W_n + \rho(H_{12})W_n \\
&\subseteq \rho(E_{21})W_n + W_n
\end{aligned}$$

by the induction hypothesis and the fact that W_n is preserved by $\mathfrak{h}$

$$\begin{aligned}
&\subseteq W_{n+1} + W_n \\
&= W_{n+1},
\end{aligned}$$

as required. The proof in the case $Y_{n+1} = E_{32}$ is similar, using that $[E_{12}, E_{32}] = 0$.

(ii) Note that if β the weight of $\rho(Y_n)\rho(Y_{m-1})\dots\rho(Y_1)\mathbf{v}$, with n and Y_i as in the lemma, then a calculation using the fundamental weight calculation, as in the proof of lemma 19.7, shows that

$$l(\beta) = l(\alpha) - n$$

and so $\beta \neq \alpha$ if $n > 0$. Since these vectors span W, $\mathbf{v}$ is the unique (up to scalar) weight vector in W of weight α.

(iii) Suppose that W is reducible. By complete reducibility (Theorem 14.7) we have

$$W = U \oplus U'$$

for U, U' non-zero proper subrepresentations of W. We must have $\mathbf{v} = \mathbf{u} + \mathbf{u}'$ for unique $\mathbf{u} \in U$, $\mathbf{u}' \in U'$. The unicity implies that $\mathbf{u}$ and $\mathbf{u}'$ are both weight vectors of weight α so, by part (ii), either $\mathbf{u} = 0$ or $\mathbf{u}' = 0$. Without loss of generality let $\mathbf{v} = \mathbf{u} \in U$. But then all $\rho(Y_n)\dots\rho(Y_1)\mathbf{v} \in U$ as U is a subrepresentation, so $W = U$ contradicting that U is a proper subrepresentation.

□

Remark 20.4. *It follows that W as in Lemma 20.3 is actually the subrepresentation generated by $\mathbf{v}$, that is, the span of all vectors obtained by applying arbitrary elements of $\mathfrak{sl}_{3,\mathbb{C}}$ some number of times. The content of the lemma is then that it suffices to apply only E_{21} and E_{32}.*

Proof of Theorem 20.1. First we show the existence. Let $a, b \in \mathbb{Z}_{\geq 0}$. Consider

$$V = \mathrm{Sym}^a(\mathbb{C}^3) \otimes \mathrm{Sym}^b((\mathbb{C}^3)^*).$$

This has a highest weight vector $\mathbf{v} = \mathbf{e}_1^a \otimes (\mathbf{e}_3^*)^b$ of weight $aL_1 - bL_3$. Let W be the representation generated by $\mathbf{v}$. Then W is irreducible by 20.3(iii), and has a highest weight vector $\mathbf{v}$ of weight $aL_1 - bL_3$. Thus we can take $V^{(a,b)} = W$.

Next we show the uniqueness. Suppose that V, W are two irreducible representations with highest weight vectors $\mathbf{v}$ and $\mathbf{w}$, respectively, of weight $aL_1 - bL_3$. Let $U \subseteq V \oplus W$ be the representation generated by $\mathbf{u} = (\mathbf{v}, \mathbf{w})$. Then U is irreducible by 20.3(iii). The projection $V \oplus W \to V$ sending $(\mathbf{v}', \mathbf{w}')$ to $\mathbf{v}'$ restricts to a homomorphism $U \to V$ which sends $\mathbf{u}$ to $\mathbf{v}$. This is therefore a non-zero homomorphism between irreducible representations, and so must be an isomorphism. Thus $U \cong V$. Similarly $U \cong W$, and so $V \cong W$ as required. □

In fact, it is possible to give an explicit description of the irreducible representations.

THEOREM 20.5 (non-examinable). *Let $a, b \geq 0$ and $V = \mathbb{C}^3$. Define*

$$\phi : \mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V^*) \to \mathrm{Sym}^{a-1}(V) \otimes \mathrm{Sym}^{b-1}(V^*)$$

to be the map

$$(\mathbf{v}_1 \dots \mathbf{v}_a) \otimes (\lambda_1 \dots \lambda_b) \longmapsto \sum_{i=1}^{a} \sum_{j=1}^{b} \lambda_j(\mathbf{v}_i)(\mathbf{v}_1 \dots \hat{\mathbf{v}}_i \dots \mathbf{v}_a) \otimes (\lambda_1 \dots \hat{\lambda}_j \dots \lambda_b).$$

Then ϕ is a surjective $\mathfrak{sl}_{3,\mathbb{C}}$-homomorphism, and its kernel is the irreducible representation with highest weight $aL_1 - bL_3$.

PROOF. (non-examinable) This is Problem 60. □

20.3. Exercises.

Problem 57. Show that, if V is a finite-dimensional representation of $\mathfrak{sl}_{3,\mathbb{C}}$ with a unique highest weight vector (up to scalar multiplication), then V is necessarily irreducible.

Deduce that the standard representation, its dual, and the adjoint representation are irreducible.

Problem 58.

(a) Find the weights of $\mathrm{Sym}^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$ and draw the weight diagram.

(b) Show that

$$\mathbf{e}_1^2 \otimes \mathbf{e}_1^* + \mathbf{e}_1\mathbf{e}_2 \otimes \mathbf{e}_2^* + \mathbf{e}_1\mathbf{e}_3 \otimes \mathbf{e}_3^* \in \mathrm{Sym}^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$$

is a highest weight vector with weight L_1.

(c) Let $\mathbf{v} = \mathbf{e}_1^2 \otimes \mathbf{e}_3^*$. Calculate $E_{32}E_{21}v$ and $E_{21}E_{32}v$ and show that they are linearly independent.

(d) Show that

$$\mathrm{Sym}^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^* \cong V^{(2,1)} \oplus \mathbb{C}^3$$

and find the weight diagram for $V^{(2,1)}$.

Problem 59. (harder!)

The aim of this problem is to show that, for $n \geq 0$,

$$V^{(n,0)} = \mathrm{Sym}^n(\mathbb{C}^3).$$

It suffices to show that $\mathrm{Sym}^n(\mathbb{C}^3)$ is irreducible with highest weight nL_1.

(a) Show that $\mathrm{Sym}^n(\mathbb{C}^3)$ has a basis of weight vectors

$$\{\mathbf{e}_1^a\mathbf{e}_2^b\mathbf{e}_3^c \,|\, a, b, c \geq 0, a + b + c = n\}$$

and that these have distinct weights (so, every weight has multiplicity one).

(b) Show that $\mathbf{e}_1^n$ is the unique highest weight vector in $\mathrm{Sym}^n(\mathbb{C}^3)$, up to scalar multiplication.

(c) Deduce that $\mathrm{Sym}^n(\mathbb{C}^3)$ is an irreducible representation with highest weight nL_1. See problem 57.

Problem 60. (monster!)(non-examinable) Let $V = \mathbb{C}^3$, let $W = V^*$, and let $a, b > 0$. For $\mathbf{v} \in V, \mathbf{w} \in W$, define $(\mathbf{v}, \mathbf{w}) = w(\mathbf{v})$.

Let

$$\phi : \mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(W) \longrightarrow \mathrm{Sym}^{a-1}(V) \otimes \mathrm{Sym}^{b-1}(W)$$

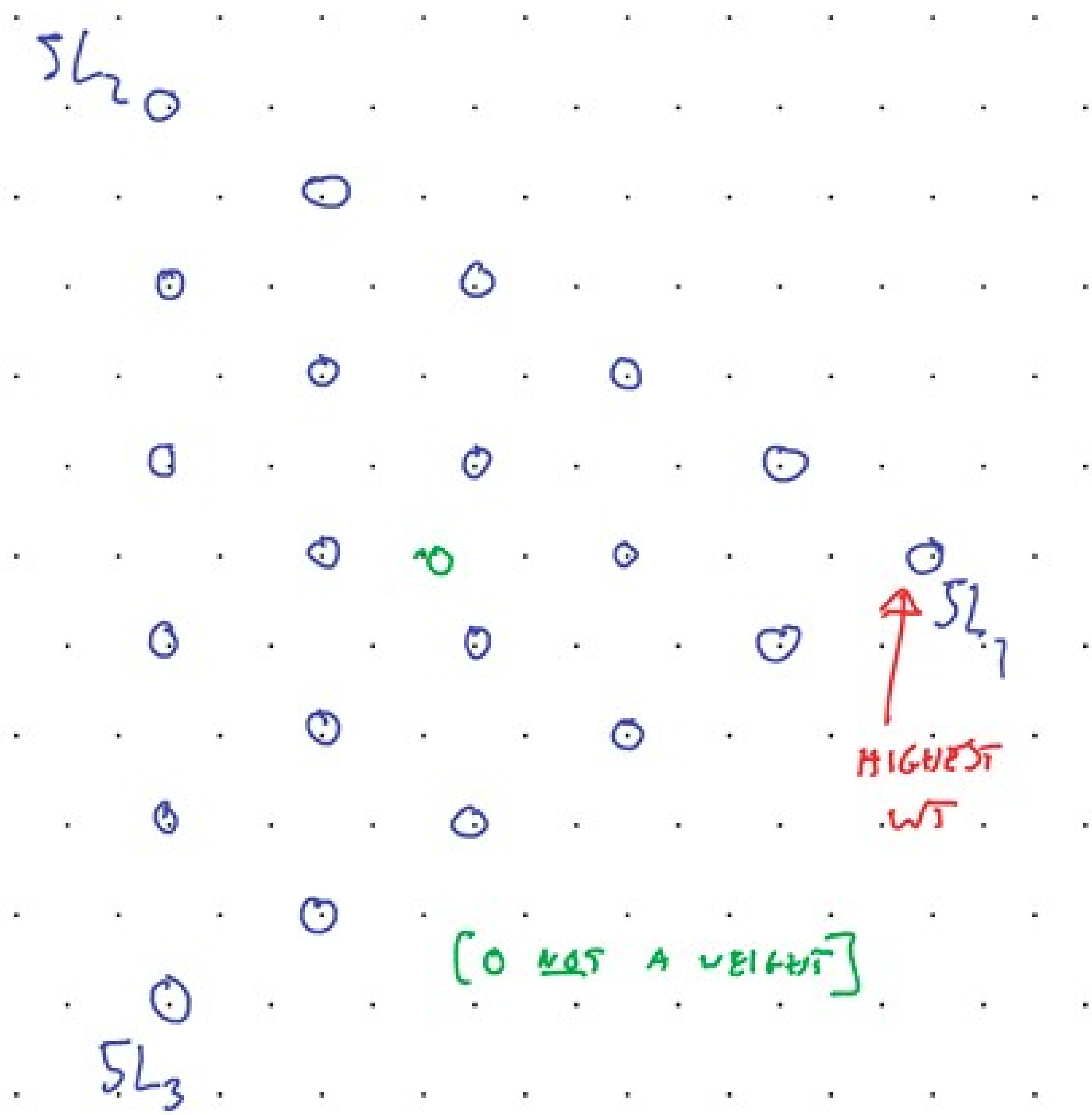

FIGURE 7. Weights for $\mathrm{Sym}^n(\mathbb{C}^3)$.

be defined by

$$\phi((\mathbf{v}_1 \dots \mathbf{v}_a) \otimes (\mathbf{w}_1 \dots \mathbf{w}_a)) = \sum_{i=1}^{a} \sum_{j=1}^{b} (\mathbf{v}_i, \mathbf{w}_j)(\mathbf{v}_1 \dots \hat{\mathbf{v}}_i \dots \mathbf{v}_a) \otimes \mathbf{w}_1 \dots \hat{\mathbf{w}}_j \dots \mathbf{w}_b$$

where $\hat{\mathbf{v}}_i$ means $\mathbf{v}_i$ is omitted (and similarly for $\hat{\mathbf{w}}_j$).

(a) Show that ϕ is an $\mathfrak{sl}_{3,\mathbb{C}}$-homomorphism.
(b) Show that $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(W)$ has a unique highest weight vector of weight $(a-i)L_1 - (b-i)L_3$ for each $0 \le i \le \min(a,b)$, and no other highest weight vectors.
(c) Show that the highest weight vector from the previous part is in $\ker(\phi)$ if and only if $i = 0$.
(d) Deduce that $\ker(\phi) \cong V^{(a,b)}$ is the irreducible representation of highest weight $aL_1 - bL_3$.

(e) Show that ϕ is surjective, and hence decompose $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V^*)$ into irreducibles.
(f) Find the dimension of $V^{(a,b)}$. Find its weights.

This problem is hard! For a solution, see Fulton and Harris, section 13.2, but watch out for the unjustified 'clearly' just before Claim 13.4.

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