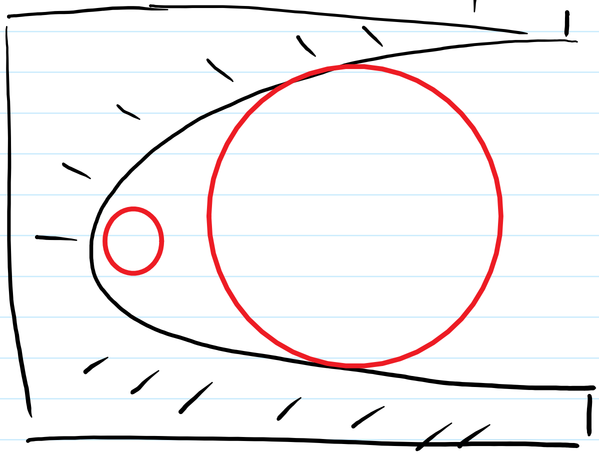
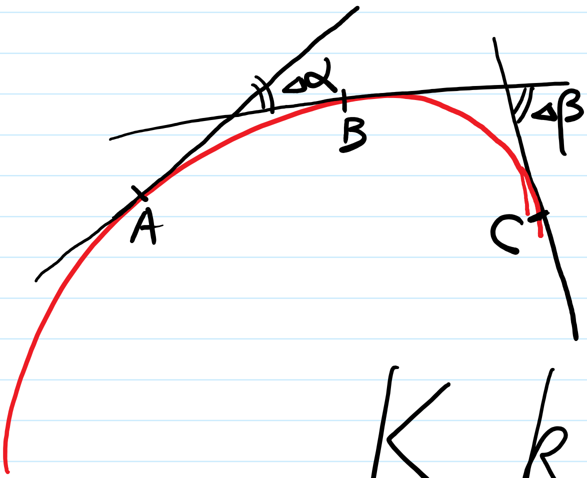
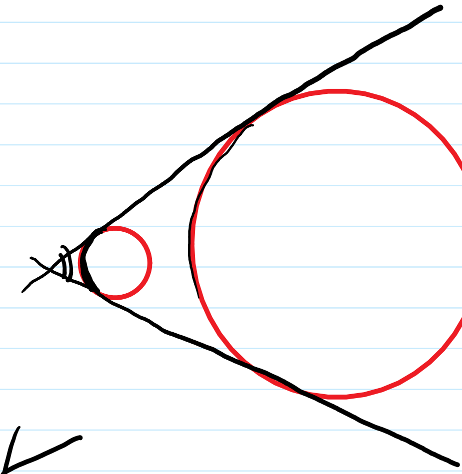
3.6曲率

2017年11月13日 8:31

§6. 曲率 — 弯曲程度







K k Kappa

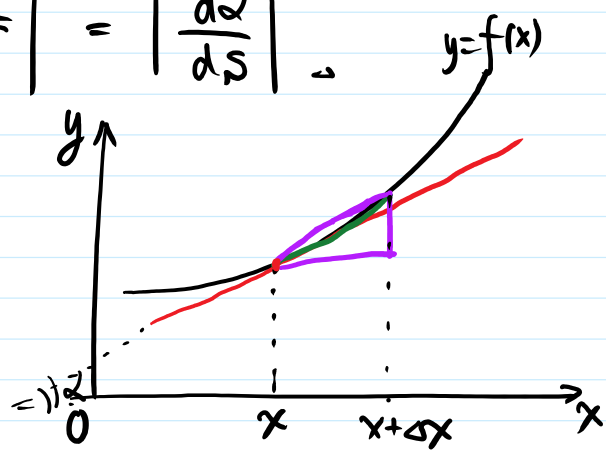
1. 定义:

平均曲率 $\bar{K} = \left|\frac{\Delta\alpha}{\Delta s}\right|$

曲率: $K = \lim\limits_{\Delta s\to 0}\bar{K} = \lim\limits_{\Delta s\to 0}\left|\frac{\Delta\alpha}{\Delta s}\right|$

$= \left|\lim\limits_{\Delta s\to 0}\frac{\Delta\alpha}{\Delta s}\right| = \left|\frac{d\alpha}{ds}\right|$ 。

2. 计算



切线斜率 $k = \tan\alpha = f'(x) = y'$.

$\Rightarrow \alpha = \arctan y'$

$\Rightarrow d\alpha = \frac{y''}{1+y'^2}\,dx$

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弧微分 $\underline{ds = \sqrt{1+y'^2}\,dx}$ 弧长是 x 的增函数.

$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$$

$$\left(\frac{\Delta s}{\Delta x}\right)^2 \approx 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$ 取极限

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \pm\sqrt{1+y'^2}$$

$$ds = \pm\sqrt{1+y'^2}\,dx$$

$$\therefore \boxed{\text{曲率 } K = \left|\frac{\frac{y''}{1+y'^2}}{\sqrt{1+y'^2}}\right| = \frac{|y''|}{(1+y'^2)^{3/2}}.}$$

例1. 求曲率: (1) $y = kx + b$.

(2) $x^2 + y^2 = R^2$.

解: (1) $y' = k$, $y'' = 0 \Rightarrow K = 0$

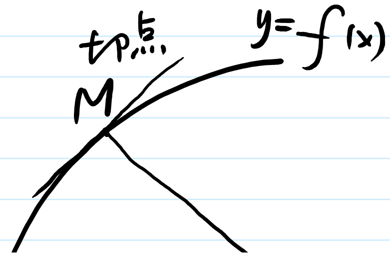
(2) $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$

$$y'' = -\frac{y - xy'}{y^2} = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{x^2+y^2}{y^3} = -\frac{R^2}{y^3}$$

$$\Rightarrow K = \frac{\frac{R^2}{y^3}}{\left(1+\frac{x^2}{y^2}\right)^{3/2}} = \frac{1}{R}.$$

3. 曲率圆.

曲率半径: $R = \frac{1}{K}$.



曲率半径：$R = \frac{1}{K}$.

曲率中心：D：$|DM| = R$

曲率圆：以D为圆心．R为半径的圆