# §5. 极限运算法则.

## 1. 无穷小的运算法则.

**Th1** $\lim \alpha(x)=0$, $\lim \beta(x)=0 \Rightarrow \lim(\alpha(x)+\beta(x))=0$.

(有限个无穷小的和仍为无穷小).

注: 无限个无穷小的和为无穷小. (X)

如: $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right)$

$=\lim\limits_{n\to\infty}\frac{1}{n^2}\cdot\frac{n(1+n)}{2}=\frac{1}{2}$.

**Th2** $\lim \alpha(x)=0$, $|u(x)|\le M \Rightarrow \lim u(x)\cdot\alpha(x)=0$.

Eg1. $\lim\limits_{x\to 0} x\sin\frac{1}{x}=\lim\limits_{x\to+\infty}\frac{1}{x}\arctan x=0$.

(有界函数 × 无穷小 = 无穷小)

推论1. $C\cdot\alpha(x)\to 0$

2. $\alpha(x)\beta(x)\to 0$

(有限个无穷小的乘积仍为无穷小)

注: 无限个 (X)

Eg2. $x^2\sin x\to 0$. $x\to 0$

## 2. 极限的四则运算.

**Th3** 若 $\lim f(x) = A$, $\lim g(x) = B$, 则

(i) $\lim\left(f(x) \pm g(x)\right) = \lim f(x) \pm \lim g(x) = A \pm B$.

(ii) $\lim\left(f(x) \cdot g(x)\right) = \lim f(x) \cdot \lim g(x) = A \cdot B$.

(iii) $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{A}{B}$, 其中 $B \neq 0$.

(iv) $\lim f^n(x) = \left(\lim f(x)\right)^n = A^n$.

(v) $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)} = \sqrt[n]{A}$.

**Th4.** (数列极限的运算法则)

若 $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$, 则四则运算成立.

**Th5** (复合函数的极限运算).

$y = f(u)$, $u = \varphi(x)$ : $y = f(\varphi(x))$. $\lim\limits_{x\to x_0} \varphi(x) = a$.

$(u \neq a)$. 则 $\lim\limits_{u\to a} f(u) = A$.

$$\lim_{x\to x_0} f(\varphi(x)) = \lim_{u\to a} f(u) = A.$$

**Eg3.** $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$ 恒等变形：约分. 通分.

$= \lim\limits_{x\to 2} \dfrac{(x-2)(x+2)}{x-2}$

$= \lim\limits_{x\to 2} (x+2)$

$= 4$

**Eg4.** $\lim\limits_{x\to 1} \left(\dfrac{1}{1-x} - \dfrac{3}{1-x^3}\right) = -1$

$= \lim\limits_{x\to 1} \dfrac{1+x+x^2-3}{(1-x)(1+x+x^2)} = \lim\limits_{x\to 1} \dfrac{x^2+x-2}{(1-x)(1+x+x^2)} = \lim\limits_{x\to 1} \dfrac{(x+2)(x-1)}{(1-x)(1+x+x^2)}$

$= \quad x^4 + x^3 + x$

E5. $\lim\limits_{x\to+\infty}\dfrac{x^4+x^3+x}{2x^4+1}=\dfrac{1}{2}$

$\lim\limits_{x\to+\infty}\dfrac{x^3+x}{2x^4+1}=0$

$\lim\limits_{x\to+\infty}\dfrac{x^4+x^3+x}{2x^3+1}=+\infty$

注: $$\lim_{x\to\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}=\begin{cases}\dfrac{a_n}{b_m}, & n=m.\\ 0, & n<m.\\ \infty, & n>m.\end{cases}$$

练习: 证明: $\lim\limits_{x\to+\infty}\dfrac{[x]}{x}=1$. $[x]$- 取整函数.