§4. 隐函数与参变量函数的求导

1. 隐函数

注：相对于显函数来讲. $y=f(x)$，$y=\ln\sqrt{\frac{x+1}{x+2}}$.

例1. $e^y = xy + \sin x$ 确定 $y=f(x)$，求 $\frac{dy}{dx}$.

定义. $y=f(x)$ 为方程 $e^y = xy+\sin x$ 确定的隐函数.

解： $e^{f(x)} = xf(x) + \sin x$.

方程两边关于自变量 x 求导： e^u. $u=f(x)$

$$e^y \cdot \frac{dy}{dx} = y + x\frac{dy}{dx} + \cos x$$

$$\frac{dy}{dx} = \frac{y+\cos x}{e^y - x}$$

例2. $x^2+y^2=1$，求 $\frac{dy}{dx}$. $\frac{d^2y}{dx^2}$.

解： $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$

$$\frac{d^2y}{dx^2} = -\frac{y-xy'}{y^2} = -\frac{x^2+y^2}{y^3}$$

例3. $y-2x = (x-y)\ln(x-y)$. 求 y''.

解： $y'-2 = (1-y')\ln(x-y) + \cancel{(x-y)}\cdot\frac{1-y'}{\cancel{x-y}}$

$$y' = 1+\frac{1}{2+\ln(x-y)}$$

$$y'' = -\frac{1}{[2+\ln(x-y)]^2}\cdot\frac{1-y'}{x-y} = \frac{1}{(x-y)[2+\ln(x-y)]^3}$$

$$y = -\frac{}{[2+\ln(x-y)]^2}\cdot\frac{}{x-y} = \frac{}{(x-y)[2+\ln(x-y)]^3}$$

例4. $y = x^x$. 求 y'.

解: 法1. (对指恒等变换) $y = e^{x\ln x}$ $\qquad x^x = e^{\ln x^x}$

$$y' = x^x(\ln x + 1)$$

法2. (化为隐求导)

$$\ln y(x) = x\ln x \to y = y(x)$$

$$\frac{1}{y}y' = 1+\ln x$$

$$\Rightarrow y' = x^x(1+\ln x)$$

练 (1) $y = x^{\sin x} = e^{\sin x\ln x}$. 求 $y' = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)$

(2) $y = x^{x^x} = e^{x^x\ln x}$. 求 $y' = x^{x^x}\left(x^x(\ln x+1)\ln x + x^{x-1}\right)$

(3) $y = 3^x\cdot x^3\cdot\sin x\cdot e^x$. 求 y'

$$\ln y = x\ln 3 + 3\ln x + \ln\sin x + x$$

$$y' = 3^x\cdots e^x\left(\ln 3 + \frac{3}{x} + \cot x + 1\right)$$

(4) $y=\dfrac{(x+1)(x+2)}{(x+3)(x+4)}$. 求 y'.

2. 参变量函数的导数

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \quad (a\le t\le b) \to y=f(x),\ 求 \frac{dy}{dx}.\ \frac{d^2y}{dx^2}.$$

$y\to u\to x$

$y=f(u),\ u=\varphi(x)$

$$\frac{dy}{dx}=\underline{\frac{dy}{du}}\cdot\frac{du}{dx}$$

解: $$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\psi'(t)}{\varphi'(t)}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{\psi'(t)}{\varphi'(t)}\right)=\boxed{\frac{\frac{d}{dt}\left(\frac{\psi'(t)}{\varphi'(t)}\right)}{\frac{dx}{dt}}}$$

$y\to t\to x$

x为自变量

$$=\underline{\frac{\psi''(t)\varphi'(t)-\psi'\varphi''}{\varphi'^{3}(t)}}$$

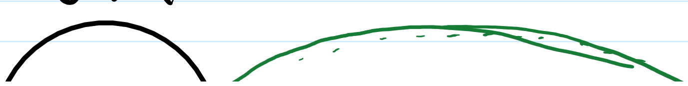
例5. $\begin{cases} x=e^t\cos t \\ y=e^t\sin t \end{cases}$. find $\left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{3}}$.

解: $$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}}=\left.\frac{e^t\sin t+e^t\cos t}{e^t\cos t-e^t\sin t}\right|_{t=\frac{\pi}{3}}=\left.\frac{\sin t+\cos t}{\cos t-\sin t}\right|_{t=\frac{\pi}{3}}$$

$$=-(2+\sqrt{3})$$

例6 $\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$ $(0\le t\le 2\pi)$. 求 $\dfrac{dx}{dy},\ \dfrac{d^2x}{dy^2}$.

摆线

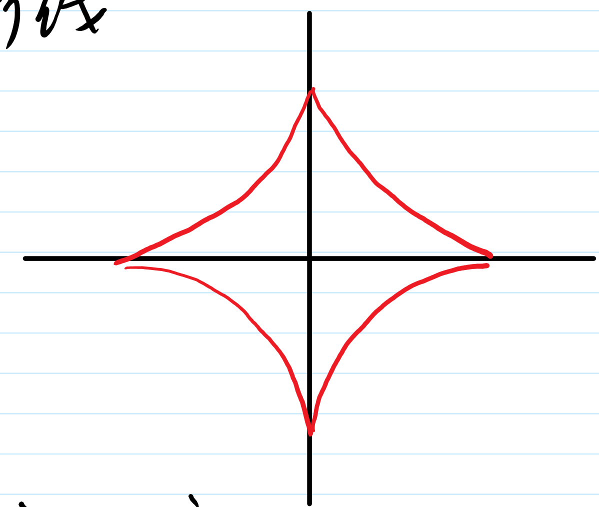


解. $\dfrac{dx}{dy} = \dfrac{a(1-\cos t)}{a\sin t} = \csc t - \cot t$

$$\frac{d^2x}{dy^2} = \frac{-\csc t\cot t + \csc^2 t}{a\sin t} = \frac{1-\cos t}{a\sin^3 t}$$

练: $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases} \to y=f(x)$. 求 y''.

星形线



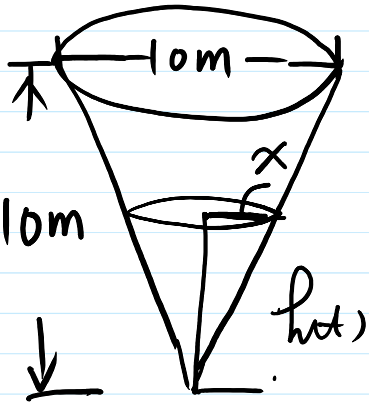
3. 相关变化率 (应用题)

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$: $\dfrac{dy}{dt}$ 与 $\dfrac{dx}{dt}$ 称为相关变化率.

例7. 向圆锥形水池内注水. 数学建模

以 9 m^3/min 速度向池内注水.

求当水深为6m的时候，水面上升的速度是多少？



1. 建模：实际问题转化为数学问题.
2. 求解：解数学问题
3. 讨论与验证.

解：$V(t)=\frac{1}{3}\pi x^2 h = \frac{1}{12}\pi h^3(t)$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2(t) \frac{dh}{dt} \qquad 9 = \frac{\pi}{4}\cdot 36 \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{\pi} \approx 0.32.$$