4.2换元积分法

2017年11月29日 10:20

§4.2. 换元积分法

定理1 (第一换元法, "凑"微分法)

$f(u)$ 在 I 上连续. $u=\varphi(x)$ 可导. 则

$$\int f(\varphi(x))\varphi'(x)\,dx \overset{u=\varphi(x)}{\underset{du=\varphi'(x)dx}{=}} \left[\int f(u)\,du\right]_{u=\varphi(x)}$$

例1. $\int 2\cos 2x\,dx$ 令 $u=2x$, 则 $du=2dx$

$= \int \cos 2x\, d(2x) = \sin 2x + C$

$= \int \cos u\,du = \sin u + C = \sin(2x) + C.$

例2 $\int x^3\cos(x^4+2)\,dx = \frac14\int \cos(x^4+2)\,d(x^4+2) = \frac14\sin(x^4+2)+C$

$= \int \frac14\cos u\,du = \frac14\sin u + C = \frac14\sin(x^4+2)+C$

例3 $\int 2x e^{x^2}dx$ $u=x^2$, $du=2x\,dx$

$= \int e^u\,du = e^{x^2}+C$ $\int e^u du$

$\int x e^{x^2}dx = \frac12\int \frac{2x}{2} e^{x^2} = \frac12\int e^{x^2}d(x^2)$

$= \frac12 e^{x^2}+C$

$\int f(u)\,du = F(u)+C$

$\int f(\varphi(x))\,d\varphi(x) = F(\varphi(x))+C$

$= \int f(\varphi(x))\varphi'(x)\,dx$

4. $\int \frac{1}{2x+3}dx = \frac12\int \frac{d(2x+3)}{2x+3} = \frac12\ln|2x+3|+C$

5. $\int \sqrt{2x+1}\,dx$

$= \frac12\int \sqrt{2x+1}\,d(2x+1)$ $\int x^\alpha dx$

$$=\frac{1}{2}\int \sqrt{2x+1}\, d(2x+1)$$

$$= \frac{1}{2}\cdot \frac{1}{1+\frac{1}{2}}(2x+1)^{1+\frac{1}{2}}+C$$

$$= \frac{1}{3}(2x+1)^{\frac{3}{2}}+C$$

$$\int x^{\alpha}\, dx = \frac{1}{1+\alpha}x^{1+\alpha}+C$$

6. $\int \frac{x}{\sqrt{1-4x^2}}dx$

$$=-\frac{1}{8}\int \frac{d(1-4x^2)}{\sqrt{1-4x^2}}$$

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u}+C$$

$$= -\frac{1}{4}\sqrt{1-4x^2}+C$$

7 $\int \tan x\, dx$

$$= \int \frac{\sin x}{\cos x}dx$$

$$=-\int \frac{d\cos x}{\cos x}$$

$$=-\ln|\cos x|+C$$

8. $\int \cot x\, dx$

$$= \ln|\sin x|+C$$

8. $\int \frac{dx}{a^2+x^2}$

$$= \frac{1}{a^2}\int \frac{dx}{1+(\frac{x}{a})^2} = \frac{1}{a}\int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a}\arctan\frac{x}{a}+C$$

9 $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C$

10 $\int \sin^3 x\, dx = \int \sin^2 x \sin x dx = -\int(1-\cos^2 x)\, d\cos x$

$$= -\cos x+\frac{1}{3}\cos^3 x+C$$

11 $\int \sin^2 x\cos^5 x dx = \int \sin^2 x(1-\sin^2 x)^2\, d\sin x$

12 $\int \cos^2 x\, dx = \frac{1}{2}\int(1+\cos 2x)dx = \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C$

13 $\int \cos^4 x\, dx = \int\left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$

13 $\int \cos^4 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int \left(1+2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$

注: $\int \sin^m x \cos^n x\, dx$

(i) m / n 奇: $m = 2k+1$.

$I = -\int (1-\cos^2 x)^k \cos^n x\, d\cos x \xlongequal{u=\cos x} -\int (1-u^2)^k u^n du$

(ii) m、n 偶: 只能降次. 别无他法.

定理2 (第二换元法, 反主故换元法)

$F(t)$ 是 $f(\varphi(t))\varphi'(t)$ 的一个原函数. $F'(t) = f(\varphi(t))\varphi'(t)$

$$\int f(x)dx \xlongequal[dx=\varphi'(t)dt]{x=\varphi(t)} \int f(\varphi(t))\varphi'(t)dt = \left[F(t)+C\right]_{t=\varphi^{-1}(x)}$$

注: 三角换元.

表达式	代换	依据
$f(\sqrt{a^2-x^2})$	$x = a\sin t$, $\sqrt{a^2-x^2} = a\cos t$	$\sin^2 t + \cos^2 t = 1$
$f(\sqrt{a^2+x^2})$	$x = a\tan t$, $\sqrt{a^2+x^2} = a\sec t$	$1+\tan^2 t = \sec^2 t$
$f(\sqrt{x^2-a^2})$	$x = a\sec t$, $\sqrt{x^2-a^2} = a\tan t$	$\sec^2 t - 1 = \tan^2 t$

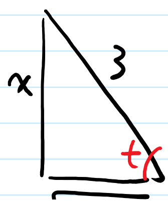
$-\frac{\pi}{2} < t < \frac{\pi}{2}$.

$(0, \frac{\pi}{2})$

$(\pi, \frac{3\pi}{2})$

例14 $\int \frac{\sqrt{9-x^2}}{x^2} dx$ 令 $x = 3\sin t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$. 则 $dx = 3\cos t\, dt$

$= \int \frac{3\cos t}{9\sin^2 t} \cdot 3\cos t\, dt = \int \cot^2 t\, dt$



$= \int (\csc^2 t - 1)\, dt = -\cot t - t + C$

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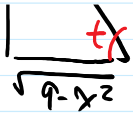
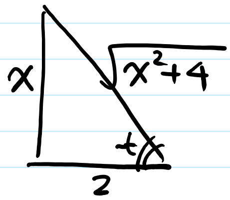


$$= -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + C$$

例15 $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$　　令 $x = 2\tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$，则 $dx = 2\sec^2 t\, dt$



$$= \int \frac{2\sec^2 t}{4\tan^2 t \cdot 2\sec t}\, dt$$

$$= \frac{1}{4}\int \frac{\cos t}{\sin^2 t}\, dt = \frac{1}{4}\int \frac{d\sin t}{\sin^2 t} = -\frac{1}{4}\cdot\frac{1}{\sin t} + C$$

$$= -\frac{1}{4}\cdot\frac{\sqrt{x^2+4}}{x} + C$$

例16 $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$

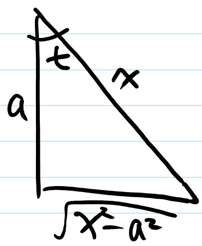
$$= \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{x^2+1}} = \sqrt{1+x^2} + C$$

练习：(1) $\displaystyle\int \frac{x^2+1-1}{\sqrt{1+x^2}}\,dx$　　(2) $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$　　$x^2 \cdot x$

例17 $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$ $(a>0)$　　a^2

解：(1) $x > a$：　$x = a\sec t$, $0 < t < \frac{\pi}{2}$.　$dx = a\sec t\tan t\, dt$

$$\therefore \text{原式} = \int \frac{a\cdot\sec t\tan t}{a\tan t}\, dt$$



$$= \int \sec t\, dt$$

$$= \ln(\sec t + \tan t) + C$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + C \qquad \frac{a}{x} = \cos t$$

$$= \ln\left(x + \sqrt{x^2-a^2}\right) + C_1, \quad \cancel{C_1 = C - \ln a}$$

(2) $x < -a$：$u = -x > a$，$dx = -du$

$$I = -\int \frac{du}{\sqrt{u^2-a^2}} = -\ln\left(u + \sqrt{u^2-a^2}\right) + C$$

$= -\ln(\ x\ \ \ \ \ ) + C$ (ttrrzle)

$$= -\ln\left(-x+\sqrt{x^2-a^2}\right)+C \quad ([illegible])$$

$$= \ln\left(\underline{-x-\sqrt{x^2-a^2}}\right)+C$$

$$\therefore\ I = \ln\left|x+\sqrt{x^2-a^2}\right|+C$$

例18 $\displaystyle\int \frac{dx}{x(x^7+2)}$ 分母次数 > 分子次数：倒代换 $x=\frac{1}{t}$，$dx=-\frac{1}{t^2}dt$

解法1. $\displaystyle I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\left(\frac{1}{t^7}+2\right)}$

$$= -\int \frac{t^6}{1+2t^7}dt = -\frac{1}{14}\int \frac{d(1+2t^7)}{1+2t^7} = -\frac{1}{14}\ln\left|1+2t^7\right|+C$$

$$= -\frac{1}{14}\ln\left|1+\frac{2}{x^7}\right|+C$$

解法2. $\displaystyle\int \frac{x^6\,dx}{x^7(x^7+2)} = \frac{1}{7}\int \frac{d(x^7)}{x^7(x^7+2)} \overset{x^7=u}{=\!=} \frac{1}{7}\int \frac{du}{u(u+2)}$

$$= \frac{1}{14}\int\left(\frac{1}{u}-\frac{1}{u+2}\right)du$$

$$= \frac{1}{14}\ln\left|\frac{u}{u+2}\right|+C = \frac{1}{14}\ln\left|\frac{x^7}{x^7+2}\right|+C$$

练习: $\displaystyle\int \frac{1}{x^4\sqrt{1+x^2}}dx. \quad (x>0)$ $\quad -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}+\sqrt{1+\frac{1}{x^2}}+C$

例19. $\displaystyle\int \frac{1}{1+\sqrt{x}}dx.$ $\quad t=\sqrt{x}$，$x=t^2$，$dx=2t\,dt$

例20 $\displaystyle\int \frac{1}{1+e^x}dx$ $\quad t=e^x$，$x=\ln t$，$dx=\frac{1}{t}dt$