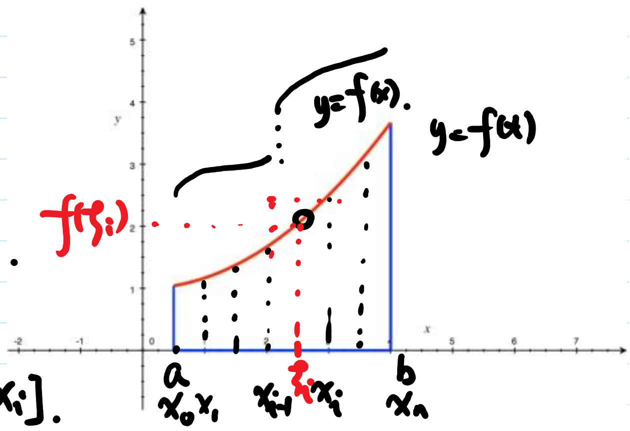
第五章 定积分

§1. 概念与性质

1. 定义.(曲边梯形的面积)

$f(x)$在$[a,b]$上有界.

在$[a,b]$内插入若干个分点$a=x_0, x_1, \cdots, x_{n-1}, x_n=b$. 将$[a,b]$分割成$n$个小区间$\Delta x_i = x_i - x_{i-1}$, $\forall \xi_i \in [x_{i-1}, x_i]$.



若极限 $\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$, $\lambda = \max\limits_{1\le i\le n}\{\Delta x_i\}$

存在,则该极限值称为$f(x)$在$[a,b]$上的定积分.

记作

$$\int_a^b f(x)\,dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i .$$

注:(i) 定积分:分割.乘积.求和.取极限.

(ii) 定积分是数.与积分变量无关.即

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

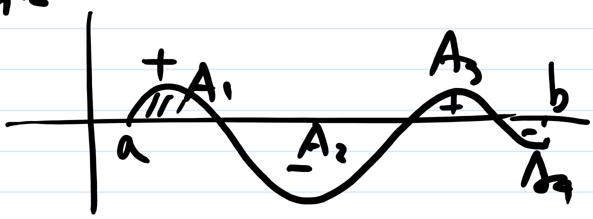
(iii) 两个"任意":任意分法,ξ_i任取法

(iv) $f(x)$在$[a,b]$上连续,则$\int_a^b f(x)\,dx$存在.

或 $f(x)$在$[a,b]$上可积.

(v) 几何意义:面积之代数和

$$\int_a^b f(x) = A_1 - A_2 + A_3 - A_4$$



例1. 应用定义计算 $\int_0^1 x^2\,dx$.

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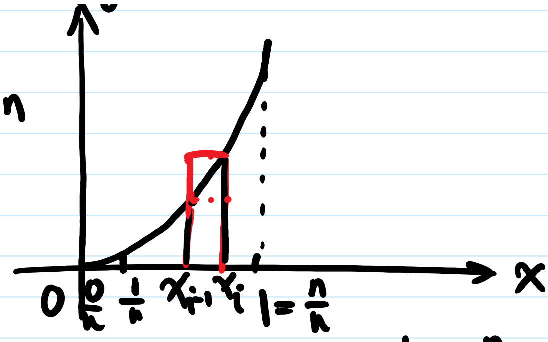
解: n 等分 $[0,1]$. $\Delta x_i = \frac{1}{n}$, $i=1,2,\cdots,n$



$\xi_i = \frac{i}{n}$

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{i=1}^{n} i^2\right)$$

$$\frac{1}{n},\ \frac{n}{n} \to 0,\ 1$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{3}.$$

例2. $$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1^2}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right)=\frac{\pi}{6}$$

注: $$\int_0^1 f(x)\,dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$$

解. $$I = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{4-\left(\frac{i}{n}\right)^2}}\,\frac{1}{n} = \int_0^1\frac{1}{\sqrt{4-x^2}}\,dx$$

例3. $f(x)\in C[0,1]$, $f(x)>0$. 证明:

$$\lim_{n\to\infty}\sqrt[n]{f\left(\frac{1}{n}\right)f\left(\frac{2}{n}\right)\cdots f\left(\frac{n}{n}\right)} = e^{\int_0^1 \ln f(x)\,dx}.$$

2. 性质

规定: $\int_a^a f(x)\,dx = 0$, $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

定理1. (线性性)

(i) $\int_a^b [f(x)\pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

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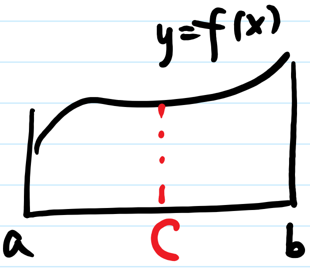
(ii) $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

(iii) $\int_a^b 1\,dx = \int_a^b dx = b-a$



定理2 (可加性)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

定理3 (不等式性质)

(i) $f(x)\ge 0$, $a\le x\le b \Rightarrow \int_a^b f(x)\,dx \ge 0$.

(ii) $f(x)\ge g(x)$, $a\le x\le b \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

(iii) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

定理4 (估值定理) $m\le f(x)\le M$. $a\le x\le b$

$$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

定理5 (积分中值定理). $f(x)$ 在 $[a,b]$ 上连续,

$\Rightarrow$ 至少 $\exists\ \xi\in[a,b]$, 使得

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

例4. 估计 $\int_0^1 e^{-x^2}\,dx$

解: $\frac{1}{e} \le e^{-x^2} \le 1$

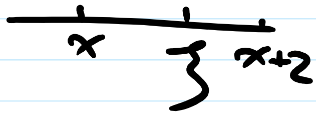
$\therefore \quad \frac{1}{e} \le I \le 1$.

例5. $f(x)$ 可导, $\lim\limits_{x\to+\infty} f(x) = 1$. 求

$$\lim_{x\to+\infty} \int_x^{x+2} t\sin\frac{3}{t}\cdot f(t)\,dt.$$

解: $\because \int_x^{x+2} t\sin\frac{3}{t}\cdot f(t)\,dt \xlongequal{\exists \xi\in[x,x+2]} 2\xi\sin\frac{3}{\xi}\cdot f(\xi)$

$\therefore I = \lim\limits_{\xi\to+\infty} 2\xi\sin\frac{3}{\xi}\cdot f(\xi)$



$= 6\lim\limits_{\xi\to+\infty} f(\xi)$

$= 6$.