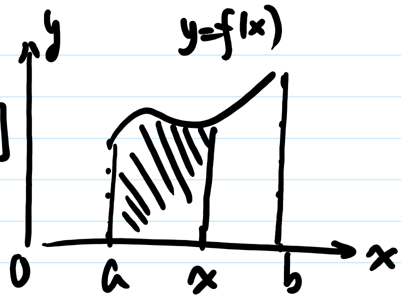
§2 微积分基本公式

1. 积分限函数：$f(x)$ 在 $[a,b]$ 上连续，$\forall x\in[a,b]$



$$\Phi(x)=\int_a^x f(t)\,dt=\int_a^x f(u)\,du$$

$$\Phi(x)=\int_a^x f(x)\,dx \qquad \int_a^b f(x)\,dx=\int_a^b f(t)\,dt$$

$\begin{cases}\text{自变量}\\\text{积分变量}\end{cases}$

定理1. 设 $f(x)\in C[a,b]$，$\Phi(x)=\int_a^x f(t)\,dt$，则

(1) $\Phi'(x)=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$；

(2) $\Phi(x)$ 是 $f(x)$ 的一个原函数。

分析：
$$\Phi'(x)=\lim_{\Delta x\to 0}\frac{\Phi(x+\Delta x)-\Phi(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x}f(u)\,du-\int_a^x f(u)\,du}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x}f(t)\,dt}{\Delta x}=\lim_{\xi\to x}\frac{f(\xi)\Delta x}{\Delta x}=f(x)$$

注：推广：

(i) $\Phi(x)=\int_a^{b(x)}f(t)\,dt \Rightarrow \Phi'(x)=f(b(x))\,b'(x)$

(ii) $\Phi(x)=\int_{a(x)}^{b}f(t)\,dt \Rightarrow \Phi'(x)=-f(a(x))\,a'(x)$

(iii) $\Phi(x)=\int_{a(x)}^{b(x)}f(t)\,dt \rightarrow \Phi'(x)=f(b(x))\,b'(x)-f(a(x))\,a'(x)$

例1. 求导数

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(1) $\Phi(x)=\int_0^x \sqrt{1+t^2}\,dt$. 求 $\Phi'(x)=\sqrt{1+x^2}$.

(2) $\Phi(x)=\int_a^{x^4} \sec t\,dt$. 求 $\Phi'(x)=4x^3\sec x^4 \ldots$

例2. 求极限 $\lim\limits_{x\to 0}\dfrac{\int_{\cos x}^{1} e^{-t^2}dt}{x^2}$　　$f(a(x))\,a'(x)$

解. $I=\lim\limits_{x\to 0}\dfrac{-e^{-\cos^2 x}(-\sin x)}{2x}=\dfrac{1}{2e}$

定理2 (牛顿-莱布尼茨公式) $f(x)\in C[a,b]\Rightarrow$

$$\int_a^b f(x)dx = F(b)-F(a) = F(x)\Big|_a^b=\Big[\int f(x)dx\Big]_a^b$$

其中 $F(x)$ 是 $f(x)$ 的一个原函数.

分析:

$$\left.\begin{aligned}F(x) &= \Phi(x)+C\\ F(b) &= \Phi(b)+C\\ F(a) &= \Phi(a)+C\end{aligned}\right\}\xrightarrow{\text{相减}}\int_a^b f(x)\,dx=F(b)-F(a)$$

例3. $\int_0^1 x^2\,dx=\frac{1}{3}x^3\Big|_0^1=\frac{1}{3}-0=\frac{1}{3}$

4 $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{4n^2-1^2}}+\cdots+\dfrac{1}{\sqrt{4n^2-n^2}}\right)=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{1}{\sqrt{4-(\frac{i}{n})^2}}\dfrac{1}{n}$

$=\int_0^1\dfrac{1}{\sqrt{4-x^2}}dx=\arcsin\dfrac{x}{2}\Big|_0^1=\dfrac{\pi}{6}$

5 $\int_0^{\frac{\pi}{2}}(2\cos x+\sin x-1)\,dx$

$=(2\sin x-\cos x-x)\Big|_0^{\frac{\pi}{2}}=(2-\frac{\pi}{2})-(-1)=3-\frac{\pi}{2}.$

6. $\int^{2}\max(x,x^2)\,dx$

6. $\displaystyle\int_{-2}^{2} \max(x, x^2)\, dx$

$$= \int_{-2}^{0} x^2\, dx + \int_{0}^{1} x\, dx + \int_{1}^{2} x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_{-2}^{0} + \frac{1}{2}x^3 \Big|_{0}^{1} + \frac{1}{3}x^3 \Big|_{1}^{2} = \frac{11}{2}$$