1.6极限存在准则，两个重要极限

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定理1（夹逼定理 Sandwich theorem）

(1) 数列：$\{x_n\}$, $\{y_n\}$, $\{z_n\}$

$$\left.\begin{aligned} &(\text{i})\ y_n \le x_n \le z_n,\ \forall n \in N. \\ &(\text{ii})\ \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = a \end{aligned}\right\} \Rightarrow \lim_{n\to\infty} x_n = a.$$

证：$\lim\limits_{n\to\infty} y_n = a \Leftrightarrow \forall \varepsilon > 0,\ \exists N_1 > 0,\ n > N_1,\ |y_n - a| < \varepsilon$

$$a-\varepsilon < y_n < a+\varepsilon \quad *$$

$\lim\limits_{n\to\infty} z_n = a \Leftrightarrow$ 对上述 $\varepsilon > 0$, $\exists N_2 > 0$, $n > N_2$, $|z_n - a| < \varepsilon$

$$a-\varepsilon < z_n < a+\varepsilon \quad *$$

$\therefore a-\varepsilon < y_n \le x_n \le z_n < a+\varepsilon$, $N = \max(N_1, N_2)$

即 $|x_n - a| < \varepsilon$, $N = \max(N_1, N_2)$

所以 $\lim\limits_{n\to\infty} x_n = a$.

例1. $\lim\limits_{n\to\infty} n\left(\dfrac{1}{n^2+\pi} + \dfrac{1}{n^2+2\pi} + \cdots + \dfrac{1}{n^2+n\pi}\right) = 1.$

$$1 \leftarrow \underbrace{n\cdot\frac{n}{n^2+n\pi}}_{y_n} \le x_n \le \underbrace{n\cdot\frac{n}{n^2+\pi}}_{z_n} \le \underbrace{n\cdot\frac{n}{n^2}}_{z_n} \to 1$$

(2) 函数：$f(x)$, $g(x)$, $h(x)$

$$\left.\begin{aligned} &(\text{i})\ g(x) \le f(x) \le h(x),\ x \in \mathring{U}(x_0)\ (|x| > X > 0) \\ &(\text{ii})\ \lim_{\substack{x\to x_0 \\ (x\to\infty)}} g(x) = \lim_{\substack{x\to x_0 \\ (x\to\infty)}} h(x) = A \end{aligned}\right\} \Rightarrow \lim_{\substack{x\to x_0 \\ (x\to\infty)}} f(x) = A.$$

重要极限1.

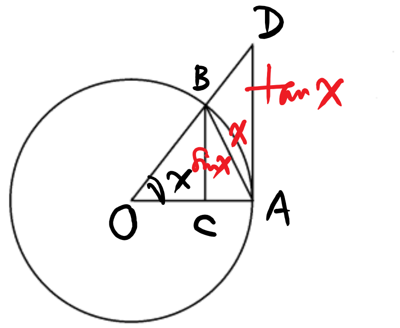
$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

注 … $\sin\square$ … $\sin\square$ … $\square = 0$

注：$\lim\limits_{\square\to 0}\dfrac{\sin\square}{\square}=1$ 或 $\lim\limits_{x\to a}\dfrac{\sin\square}{\square}=1$. 若 $\lim\limits_{x\to a}\square=0$

证明：作单位圆.

$0<x<\frac{\pi}{2}$.



$S_{\triangle AOB}<S_{扇AOB}<S_{\triangle AOD}$

$\frac{1}{2}\sin x<\frac{1}{2}x<\frac{1}{2}\tan x$.

即 $\cos x<\dfrac{\sin x}{x}<1$

而 $\lim\limits_{x\to 0^+}\cos x=1$ ∴ $\lim\limits_{x\to 0^+}\dfrac{\sin x}{x}=1$ ✓

同理. $\lim\limits_{x\to 0^-}\dfrac{\sin x}{x}=1$

∴ $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.

例2. 求极限.

(1) $\lim\limits_{x\to 0}\dfrac{\tan 3x}{x}=\lim\limits_{x\to 0}\underbrace{\dfrac{\sin 3x}{3x}}_{\to 1}\cdot\underbrace{\dfrac{3}{\cos 3x}}_{\to 3}=1\cdot 3=3$

(2) $\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}\overset{t=\arcsin x}{\underset{x=\sin t}{=\!=\!=}}\lim\limits_{x\to 0}\dfrac{t}{\sin t}=1$ ✓

(3) $\lim\limits_{x\to\pi}\dfrac{\sin x}{x-\pi}$

(4) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\frac{1}{2}x^2}=1$ ✓

(5) 设圆内接正n边形面积为

$A_n=nR^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$.

证明：$\lim\limits_{n\to\infty} A_n = \pi R^2$.

Th2 （单调有界数列必有极限）

单调数列：$x_1 \le x_2 \le x_3 \le \cdots \le x_n \le \cdots$

$x_1 \ge x_2 \ge x_3 \ge \cdots \ge x_n \ge \cdots$

严格单调数列：$x_1 > x_2 > x_3 > \cdots > x_n > \cdots$

$x_1 < x_2 < x_3 < \cdots < x_4 < \cdots$

单调递增有上界之数列必有极限；

单调递减有下界之数列必有极限.

例3. $\{x_n\}$：$x_1=\sqrt{2}$，$\underset{\downarrow\, a}{x_n} = \sqrt{2+\underset{\downarrow\, a}{x_{n-1}}}$.（证明 $\lim\limits_{n\to\infty} x_n$ 存在）并求之.

证明：(i) $\{x_n\}$ 有上界.

$x_1=\sqrt{2}<2$！ 假设 $n=k$ 时 $x_k<2$. 当 $n=k+1$ 时

$x_{k+1}=\sqrt{2+x_k} < \sqrt{2+2}=2$.

$\therefore$ $\{x_n\}$ 有上界.

(ii) $\{x_n\}$ 单增. $x_{n+1} > x_n$

$x_{n+1}^2 - x_n^2 = 2+x_n-x_n^2 = (2-x_n)(1+x_n) > 0$

$\therefore$ $\{x_n\}$ 单增

$\therefore$ $\lim\limits_{n\to\infty} x_n$ 存在.

设 $\lim\limits_{n\to\infty} x_n = a$，则 $a=\sqrt{2+a} \Rightarrow a=2$.（舍负）

重要极限2.

$$\lim_{x\to 0}\underline{(1+x)}^{\frac{1}{x}} = e \qquad \text{幂指函数.}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\underline{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e} \quad \#$$

注：(i) $\lim (1+\square)^{\frac{1}{\square}} = e$

注：(i) $\lim\limits_{\square\to 0}(1+\square)^{\frac{1}{\square}}=e$

$\lim\limits_{x\to a}(1+\square)^{\frac{1}{\square}}=e$，若 $\lim\limits_{x\to a}\square=0$.

(ii) $\left(1+\frac{1}{n}\right)^n<\left(1+\frac{1}{n+1}\right)^{n+1}<e$

例4. 求极限

(1^∞)

(1) $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x=\lim\limits_{x\to\infty}\left[\left(1+\frac{2}{x}\right)^{\frac{x}{2}}\right]^{\frac{2}{x}\cdot x}=e^2$

(2) $\lim\limits_{x\to 0}(1-x)^{\frac{1}{x}}=\lim\limits_{x\to 0}\left[\left(1+(-x)\right)^{\left(-\frac{1}{x}\right)}\right]^{-1}=e^{-1}$

(3) $\lim\limits_{x\to\infty}\left(\frac{x+4}{x+2}\right)^x=\lim\limits_{x\to\infty}\left[\left(1+\frac{2}{x+2}\right)^{\frac{x+2}{2}}\right]^{\frac{2x}{x+2}}=e^2$

(4) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=\lim\limits_{n\to\infty}\frac{1}{x}\ln(1+x)=\lim\limits_{\cancel{x}\to\infty}\ln(1+x)^{\frac{1}{x}}=1$.

(5) $\lim\limits_{x\to 0}\frac{a^x-1}{x}$

(6) $\lim\limits_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x$