2.3高阶导数

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1. 背景.

$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$$

2. 定义:

$$f''(x) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x} \; . \qquad = 二阶导数$$

$$\vdots$$

$$f^{(4)}(x) = (f'''(x))' \qquad =$$

$$\vdots$$

$$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)' = \frac{d^n y}{dx^n} = \frac{d^n f}{dx^n} = y^{(n)}$$

3. 计算.

方法一: 数学归纳法 — 定义.

例1. $y = e^x$. 求 $y^{(n)} = e^x$.

2. $y = \sin x$. 求 $y^{(n)} = \sin\left(x + n\cdot\frac{\pi}{2}\right)$

解: $y' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$

$$f^{(0)}(x) = f(x)$$

$$y'' = -\sin x = \sin\left(x + 2\cdot\frac{\pi}{2}\right)$$

$$y''' = -\cos x = \sin\left(x + 3\cdot\frac{\pi}{2}\right)$$

$$y^{(4)} = \sin x = \sin\left(x + 4\cdot\frac{\pi}{2}\right)$$

3. $(\cos x)^{(n)} = \cos\left(x + n\cdot\frac{\pi}{2}\right)$ $\qquad \cos\left(x - \frac{\pi}{2}\right)$

练: 1. $(\sin ax)^{(n)} = a^n \sin\left(ax + n\cdot\frac{\pi}{2}\right)$ ✱

$(\sin u)^{(n)}$ $\qquad u = ax$

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2. $(\ln(1+x))^{(n)} = (-1)^{n-1}(n-1)!(1+x)^{-n}$ $\quad 0! = 1$

$\left(\frac{1}{1+x}\right)^{(n)} = (-1)^{n} n! (1+x)^{-(n+1)}$ $\quad 1! = 1$

方法二.运算法则—公式法

定理 1. $[f(x) \pm g(x)]^{(n)} = f^{(n)}(x) \pm g^{(n)}(x)$;

2. (莱布尼兹公式)

$$[f(x)g(x)]^{(n)} = \sum_{i=0}^{n} C_n^i f^{(i)}(x) g^{(n-i)}(x)$$

$$(a+b)^n = \underline{\sum_{i=1}^{n} C_n^i a^i b^{n-i}}$$

例4. $y = e^x \cos x$. 求 y'''.

解: 法1. (定义)

法2. (公式)

$$y''' = C_3^0 (e^x)^{(0)} (\cos x)''' + C_3^1 (e^x)'(\cos x)'' + C_3^2 (e^x)''(\cos x)' + C_3^3 (e^x)'''(\cos x)^{(0)}.$$

$$= -2e^x[\sin x + \cos x]$$

5. $y = \underline{x^2} \sin x$. 求 $y^{(80)}$.

解: $(x^2)' = 2x$, $(x^2)'' = 2$.

$(\sin x)^{(80)} = \sin x$

$(\sin x)^{(79)} = -\cos x$

$(\sin x)^{(78)} = -\sin x$

$$(\sin x)^{(78)} = -\sin x$$

$$\therefore\ y^{(80)} = x^2\sin x + 80\cdot 2x\cdot(-\cos x) + \underline{C_{80}^2}\cdot 2\cdot(-\sin x)$$

$$= x^2\sin x - 160x\cos x - 6320\sin x.$$

6. $y = \dfrac{1}{x^2-5x+6}$. 求 $y^{(n)}$.

解： 部分分式分解： $y = \dfrac{1}{x-3} - \dfrac{1}{x-2}$

7. 验证 $y = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$ 满足：

$$y'' - \lambda^2 y = 0 \quad (C_1, C_2 \text{ 为常数})$$