5.3定积分的换元法与分部积分法

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§3. 定积分的换元法与分部积分法

定理1 (换元法) f 连续. $\varphi'(x)$ 连续, $x\in[a,b]$. $\Rightarrow$

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx \xlongequal[du=\varphi'(x)dx]{u=\varphi(x)} \int_{\varphi(a)}^{\varphi(b)} f(u)\,du$$

例1 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x\,dx \quad u=\cos x \quad du=-\sin x\,dx$

$$=-\int_1^0 u^5\,du = \int_0^1 u^5\,du = \frac{1}{6}u^6\Big|_0^1 = \frac{1}{6}.$$

例2 $\int_0^4 \sqrt{2x+1}\,dx \qquad u=2x+1 \qquad du=2\,dx$

$$=\frac{1}{2}\int_1^9 \sqrt{u}\,du = \frac{1}{2}\cdot\frac{1}{1+\frac{1}{2}}u^{1+\frac{1}{2}}\Big|_1^9 = \frac{26}{3}$$

例1′ $I=-\int_0^{\frac{\pi}{2}} \cos^5 x\,d\cos x = -\frac{1}{6}\cos^6 x\Big|_0^{\frac{\pi}{2}}$

2′ $I=\frac{1}{2}\int_0^4 \sqrt{2x+1}\,d(2x+1) = \frac{1}{2}\cdot\frac{1}{1+\frac{1}{2}}(2x+1)^{\frac{3}{2}}\Big|_0^4$

例3 $\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx$

$$=\int_0^{\pi}(\sin x)^{\frac{3}{2}}|\cos x|\,dx$$

$$=\int_0^{\frac{\pi}{2}}(\sin x)^{\frac{3}{2}}\cos x\,dx-\int_{\frac{\pi}{2}}^{\pi}(\sin x)^{\frac{3}{2}}\cos x\,dx$$

$$\int_0 \cdots \int_{\frac{\pi}{2}}$$

$$= \frac{1}{1+\frac{3}{2}}\sin^{\frac{5}{2}}x\Big|_0^{\frac{\pi}{2}} - \frac{2}{5}\sin^{\frac{5}{2}}x\Big|_{\frac{\pi}{2}}^{\pi} = \frac{4}{5}$$

例4. $f(x) \in C[-a,a]$. 证明

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & 若 f(x) 是偶函数 \\ 0, & 若 f(x) 是奇函数 \end{cases}$$

分析 $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

$$= \int_0^a \underline{[f(-x)+f(x)]}\,dx$$

例5 $\int_{-1}^{1} \frac{2x^2 + x\cos x}{1+\sqrt{1-x^2}}\,dx$

$$= \int_{-1}^{1} \frac{2x^2}{1+\sqrt{1-x^2}}\,dx + \int_{-1}^{1} \frac{x\cos x}{1+\sqrt{1-x^2}}\,dx$$

$$= 4\int_0^1 \frac{x^2}{1+\sqrt{1-x^2}}\,dx$$

$$= 4\int_0^1 \left(1-\sqrt{1-x^2}\right)dx = 4\left(\int_0^1 dx - \int_0^1 \underline{\sqrt{1-x^2}}\,dx\right)$$

$$= 4\left(1-\frac{1}{4}\pi\right) = 4-\pi.$$

例6 $f(x) \in C[0,1]$ 证明

(1) $\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$

(2) $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$

并计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx.$

定理2 (分部积分法)

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

例7. $\int_0^1 \arctan x\,dx$

$$= x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

$$\int \frac{d(1-x^2)}{2\sqrt{1-x^2}}$$

例8 $\int_0^{\frac{1}{2}} \arcsin x\,dx$

$$= [x\arcsin x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{12} + \sqrt{1-x^2}\Big|_0^{\frac{1}{2}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

例9.

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & n\ \text{偶} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}, & n\ \text{奇} \end{cases}$$