§3.2 洛必达法则

1. 定义：未定式 $\lim \dfrac{f(x)}{g(x)}$

$$\lim f(x)=\lim g(x)=0 \quad \left(\frac{0}{0}\right)$$

$$\lim f(x)=\lim g(x)=\infty \quad \left(\frac{\infty}{\infty}\right)$$

2. 定理：

(1) $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} g(x)=0$

(2) $f(x),\ g(x)\in D(\mathring{U}(x_0))$, $g'(x)\neq 0$

(3) $\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}=A$ (或 ∞)

$$\Rightarrow \lim_{x\to x_0}\frac{f(x)}{g(x)}=A \text{ (或 } \infty)$$

注：(i) 当 $\lim f(x)=\lim g(x)=\infty$ 时，洛必达法则成立

(ii) 其它过程，仍然成立

$x\to x_0^+$, $x\to x_0^-$, $x\to+\infty$, $x\to-\infty$, $x\to\infty$

(iii) $\lim\dfrac{f(x)}{g(x)}=\lim\dfrac{f'(x)}{g'(x)}=\lim\dfrac{f''(x)}{g''(x)}$ 反复使用.

例1. $\lim\limits_{x\to0}\dfrac{\tan x}{x}=\lim\limits_{x\to0}\dfrac{\sec^2x}{1}=1$

2. $\lim\limits_{x\to0}\dfrac{\tan x-x}{x-\sin x}\overset{\frac{0}{0}}{=}\lim\limits_{x\to0}\dfrac{\sec^2x-1}{1-\cos x}=\lim\limits_{x\to0}\dfrac{2\sec^2x\tan x}{\sin x}=2$

$$=\lim_{x\to0}\frac{1-\cos^2x}{\cos^2x\,(1-\cos x)}$$

1

$$= \lim_{x\to 0}(1+\cos x) = 2$$

例 3. 求 $\lim\limits_{h\to 0}\dfrac{f(x+h)+f(x-h)-2f(x)}{h^2}$ $= f''(x)$

(1) $f''(x)$ 存在； (2) $f''(x)$ 连续.

解：(1)
$$I = \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}$$
$$= \lim_{h\to 0}\frac{[f'(x+h)-f'(x)]-[f'(x-h)-f'(x)]}{2h} = \frac{1}{2}[f''(x)+f''(x)] = f''(x)$$

(2)
$$I = \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}$$
$$= \frac{f''(x)+f''(x)}{2} = f''(x)$$

$$\lim_{h\to 0} f''(x+h) = f''(x)$$

例4. $f''(x)$ 连续，$f(0)=0$. $g(x)=\begin{cases}\dfrac{f(x)}{x}, & x\neq 0\\ f'(0), & x=0\end{cases}$.

(1) 求 $g'(x)$；

(2) 证明 $g'(x)$ 在 $x=0$ 处连续.

解. (1) 当 $x\neq 0$ 时，$g'(x) = \dfrac{xf'(x)-f(x)}{x^2}$

求导法则直接计算

只能用导数定义！

当 $x=0$ 时，
$$g'(0) = \lim_{x\to 0}\frac{g(x)-g(0)}{x-0}$$
$$= \lim_{x\to 0}\frac{\frac{f(x)}{x}-f'(0)}{x}$$
$$= \lim_{x\to 0}\frac{f(x)-xf'(0)}{x^2}$$

$\lim\limits_{x\to 0} f(x) = 0$

$\lim\limits_{x\to 0} f(x) = f(0) = 0$

$$\lim_{x\to 0} \qquad x^2$$

$$\begin{cases} = \frac{1}{2}\lim_{x\to 0} \dfrac{f'(x)-f'(0)}{x-0} \\ = \frac{1}{2} f''(0). \end{cases}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{f''(x)}{1}$$

$\lim_{x\to 0} f(x) = f(0) = 0$

$\left[\lim_{x\to 0} f'(x) = f'(0)\right]$

$$F(x) = g'(x) = \begin{cases} \dfrac{xf'(x)-f(x)}{x^2}, & x \neq 0 \\ \frac{1}{2} f''(0), & x = 0 \end{cases}$$

证明：　只需证明 $\lim_{x\to 0} g'(x) = g'(0)$

$$\Leftrightarrow \lim_{x\to 0} \frac{xf'(x)-f(x)}{x^2} = \frac{1}{2} f''(0).$$

3. 其它形式的未定式

(1) $0\cdot\infty$ 型 $\rightarrow \dfrac{0}{\frac{1}{\infty}}$ 或 $\dfrac{\infty}{\frac{1}{0}}$

$\left(\frac{0}{0}\right)$　$\left(\frac{\infty}{\infty}\right)$

(2) $\infty_1-\infty_2$ 型 $\rightarrow \dfrac{1}{\frac{1}{\infty_1}} - \dfrac{1}{\frac{1}{\infty_2}} \overset{\text{通分}}{=} \dfrac{\frac{1}{\infty_2}-\frac{1}{\infty_1}}{\frac{1}{\infty_1}\cdot\frac{1}{\infty_2}}$ $\left(\frac{0}{0}\right)$

(3) $1^\infty = e^{\infty\cdot\ln 1} = e^{\underline{0\cdot\infty}}$

$0^0 = e^{0\cdot\ln 0} = e^{\underline{0\cdot\infty}}$

$\infty^0 = e^{0\cdot\ln\infty} = e^{\underline{0\cdot\infty}}$

例5. 求极限

(1) $\lim_{x\to 0^+} \sin x \ln x$　　(2) $\lim_{x\to +\infty} \dfrac{x^n}{e^x}$

(3) $\lim\limits_{x\to 0^+} x^{\sin x}$

(4) $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

(5) $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$

(6) $\lim\limits_{x\to 0}\left(\dfrac{\sin^2 x}{x^4}-\dfrac{\tan x}{x^3}\right)$

(7) $\lim\limits_{x\to +\infty} x^{\frac{1}{x}}$