§3 Taylor 公式 (27′)

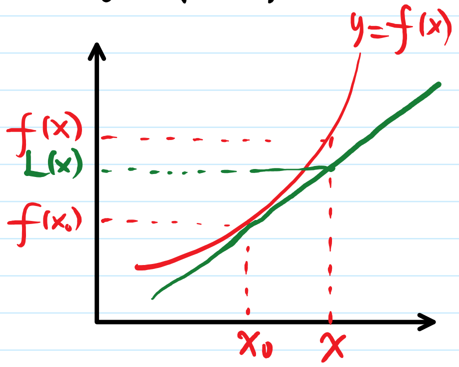
1685 - 1731 [illegible]

1. 思想

$f(x)\to A \Leftrightarrow f(x)=A+\alpha$

$$\lim_{x\to x_0} f(x)=f(x_0) \Rightarrow f(x)\approx f(x_0)$$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0) \Rightarrow f(x)\approx f(x_0)+f'(x_0)(x-x_0)=L(x)$$



思想：通过提高多项式次数来提高近似 $f(x)$ 的精度.

问题：找到一个多项式 $P_n(x)$，使 $f(x)\approx P_n(x)$，且 $R_n(x)=f(x)-P_n(x)$ 可以估计.

2. 推导 $P_n(x)$.

设 $P_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots+a_n(x-x_0)^n$ —n次多项式.

$P_n'(x)=a_1+\cdots(x-x_0)$

分析：

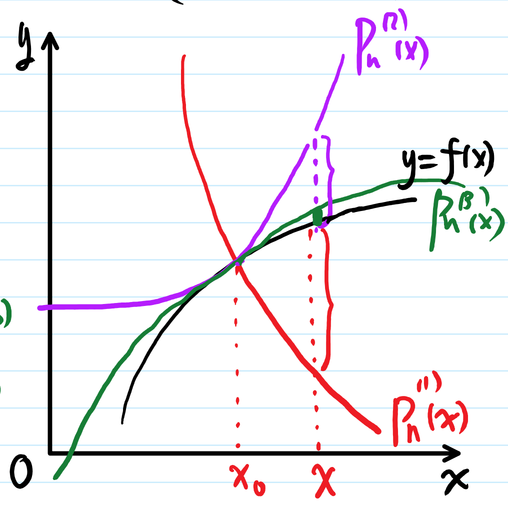
x_0: $P_n(x)$与$f(x)$相交： $P_n(x_0)=f(x_0)$

$P_n(x)$与$f(x)$相切： $P_n'(x_0)=f'(x_0)$

$P_n(x)$与$f(x)$有相同的弯曲方向. $P_n''(x_0)=f''(x_0)$

$\vdots$

$P_n^{(n)}(x_0)=f^{(n)}(x_0)$



$$\Rightarrow \begin{cases} a_0=f(x_0)\\ a_1=f'(x_0)\\ 2!a_2=f''(x_0)\\ \vdots\\ n!a_n=f^{(n)}(x_0)\end{cases} \Rightarrow \begin{cases} a_0=f(x_0)\\ a_1=f'(x_0)\\ a_2=\frac{1}{2!}f''(x_0)\\ \vdots\\ a_n=\frac{1}{n!}f^{(n)}(x_0).\end{cases}$$

$$\therefore P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

— n次 Taylor 多项式

$R_n(x) = f(x) - P_n(x)$ — 余项.

泰勒公式：设 $f(x)$ 具有直到 $n+1$ 阶的导数，则 $f(x)$ 可表示成一个 n 次多项式 $P_n(x)$ 和一个余项 $R_n(x)$ 之和，

即

$$f(x) = P_n(x) + R_n(x),$$

其中 $P_n(x) = \sum_{i=0}^{n} \frac{1}{i!} f^{(i)}(x_0)(x-x_0)^i$ — n次泰勒多项式

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}$$ — 拉格朗日余项

注：(1) $R_n(x) = o((x-x_0)^n)$ — 皮亚诺余项

(2) $|R_n(x)| = \left|\frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}\right| \le \frac{M}{(n+1)!} |x-x_0|^{n+1} \to 0$ $\quad x \to x_0$

(3) $x_0 = 0$：麦克劳林公式

$$\boxed{f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i + o(x^n)}$$

(4) $n=0$：Lagrange's MVT

$$f(x) = f(x_0) + f'(\xi)(x-x_0)$$

例1. 写出 Maclaurin 公式

(1) $y = e^x$.

解：$y^{(n)}\big|_{x=0} = e^x\big|_{x=0} = 1$

$$\therefore \boxed{e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)}$$

(2) $y = \sin x$

解: $(\sin x)^{(n)}\Big|_{x=0} = \sin\left(x + n\cdot\frac{\pi}{2}\right)\Big|_{x=0} = \sin\left(n\cdot\frac{\pi}{2}\right)$.

$y^{(0)} = 0$

$y' = 1$

$y'' = 0$

$y''' = -1$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + o(x^{2n})$$

$$= \sum_{i=1}^{n} (-1)^{i-1}\frac{x^{2i-1}}{(2i-1)!} + o(x^{2n})$$

(3) $y = \cos x$.

$$\cos x = (\sin x)' = \sum_{i=1}^{n} (-1)^{i-1}\frac{x^{2i-2}}{(2i-2)!} + o(x^{2n+1})$$

$$\Downarrow$$

$$= \sum_{i=0}^{n-1} (-1)^{i}\frac{x^{2i}}{(2i)!} + o(x^{2n-1})$$

例2. 求极限 (用 Taylor 公式)

(1) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1-\sin x^2}{x^4}$ (2) $\lim\limits_{x\to 0}\dfrac{\sin x - x\cos x}{\sin^3 x} = \lim\limits_{x\to 0}\dfrac{\sin x - x\cos x}{x^3}$

解. $e^{x^2} = 1 + x^2 + \dfrac{x^4}{2!} + o(x^4)$

$\sin x^2 = x^2 - \dfrac{x^6}{3!} + o(x^6)$

$$\therefore I = \lim_{x\to 0}\frac{\left(1 + x^2 + \frac{1}{2}x^4\right) - 1 - x^2 + o(x^4)}{x^4}$$

$$= \lim_{x\to 0}\left(\frac{1}{2} + \frac{o(x^4)}{x^4}\right)$$

$$= \frac{1}{2}$$

例3. $f(x)$ 二阶连续导数, $f(0) = f(1) = 0$, $\min\limits_{0<x<1} f(x) = -1$.

证明：$\max_{0<x<1} f''(x) \geqslant 8$.

分析：(1) 泰勒公式

(2) x_0 = 0, 1, 最小值点.

证：由 $\min_{0<x<1} f(x) = -1$. 设 x_0 s.t. $f(x_0)=-1$, $f'(x_0)=0$

$$\because f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2!}(x-x_0)^2$$

ξ 介于 x 与 x_0.

$$\therefore \begin{cases} f(0) = f(x_0) + f'(x_0)(-x_0) + \frac{f''(\xi_1)}{2}(-x_0)^2. & 0<\xi_1<x_0 \\ f(1) = f(x_0) + f'(x_0)(1-x_0) + \frac{f''(\xi_2)}{2}(1-x_0)^2, & x_0<\xi_2<1. \end{cases}$$

$$\text{有} \begin{cases} 0 = -1 + \frac{f''(\xi_1)}{2}x_0^2 \\ 0 = -1 + \frac{f''(\xi_2)}{2}(1-x_0)^2 \end{cases}$$

$$\Rightarrow \begin{cases} f''(\xi_1) = \frac{2}{x_0^2} \geqslant 8, & 0<x_0\leqslant\frac{1}{2} \\ f''(\xi_2) = \frac{2}{(1-x_0)^2} \geqslant 8, & \frac{1}{2}\leqslant x_0<1 \end{cases}$$

[取 $f''(\xi) = \max(f''(\xi_1).\ f''(\xi_2))$]

$$\therefore \max_{0<\xi<1} f''(\xi) \geqslant 8.$$