5.4广义积分

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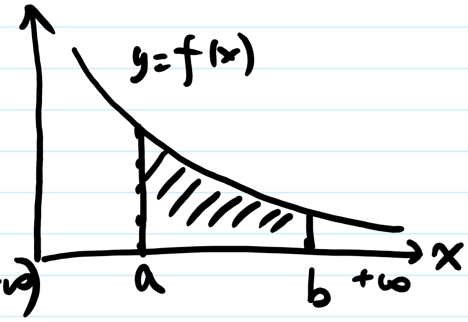
§4. 反常积分

$f(x)\begin{cases}有界\\ [a,b]\end{cases}$

1. 无穷限的反常积分.

定义: (1) $f(x)\in C[a,+\infty)$. 取 $b>a$. 若

$$\lim_{b\to+\infty}\int_a^b f(x)\,dx$$

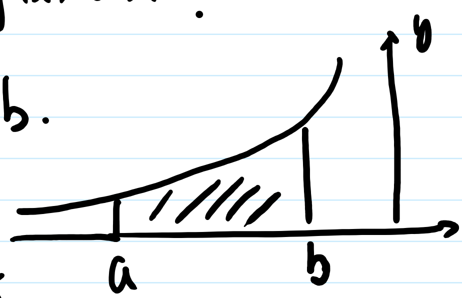


存在, 则该极限值称为 $f(x)$ 在 $[a,+\infty)$ 上的反常积分, 或称反常积分收敛,

记作

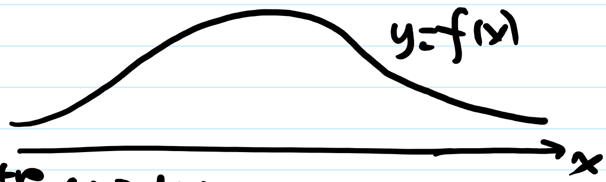
$$\int_a^{+\infty} f(x)\,dx=\lim_{b\to+\infty}\int_a^b f(x)\,dx.$$

(2) $f(x)\in C(-\infty,b]$, 取 $a<b$.



$$\int_{-\infty}^{b} f(x)\,dx=\lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

(3) $f(x)\in C(-\infty,+\infty)$



$$\int_{-\infty}^{+\infty} f(x)\,dx=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{+\infty} f(x)\,dx$$

例1. $I=\int_1^{+\infty}\frac{1}{x^2}\,dx=-\frac{1}{x}\Big|_1^{+\infty}=1$

解: $\because$ $\int_1^b\frac{1}{x^2}\,dx=-\frac{1}{x}\Big|_1^b=1-\frac{1}{b}$.

$\therefore$ $I=\lim_{b\to+\infty}\left(1-\frac{1}{b}\right)=1$.

$$\therefore \quad I = \lim_{b \to +\infty} \left(1 - \frac{1}{b}\right) = 1.$$

2. $$I = \int_{-\infty}^{0} x e^{x}\, dx = \left(x e^{x} - e^{x}\right)\Big|_{-\infty}^{0} = -1$$

补：$$\because \int_a^0 x e^x\, dx = \int_a^0 x\, de^x = x e^x \Big|_a^0 - \int_a^0 e^x\, dx$$

$$= -a e^{a} - 1 + e^{a}$$

$$\therefore \quad I = \lim_{a \to -\infty} \left(-a e^{a} - 1 + e^{a}\right) = -1.$$

注：(1) 假设 $F'(x) = f(x)$，

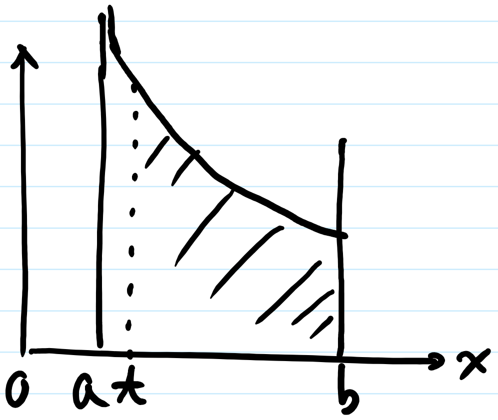
$$\int_a^{+\infty} f(x)\, dx = F(x)\Big|_a^{+\infty} = \underline{\underline{F(+\infty)}} - F(a)$$

$$\int_{-\infty}^{b} f(x)\, dx = F(x)\Big|_{-\infty}^{b} = F(b) - F(-\infty)$$

$$\int_{-\infty}^{+\infty} f(x)\, dx = F(x)\Big|_{-\infty}^{+\infty} = F(+\infty) - F(-\infty).$$

(2) $$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx = \arctan x \Big|_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

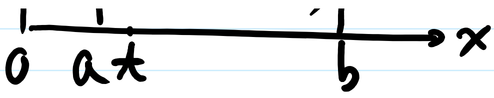
$$\int_{-\infty}^{+\infty} \frac{x}{1+x^2}\, dx = \underbrace{\int_{-\infty}^{0}}_{\text{发}} + \underbrace{\int_{0}^{+\infty}}_{\text{发}} \quad \text{发散}$$



2. 无界函数的反常积分.

定义2.(1) $f(x) \in C(a, b]$，在 a 的右邻域内无界. 若极限

$$\lim_{t \to a^+} \int_t^b f(x)\, dx$$



$$\lim_{t\to a^+}\int_t^b f(x)\,dx$$

存在，则该极限称为 $f(x)$ 在 $(a,b]$ 上的反常积分，记作

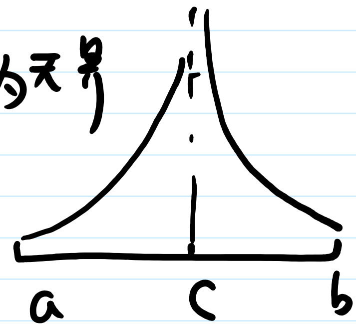
$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx.$$

(2) $f(x)\in C[a,b)$，在 b 的左邻域内无界.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.$$

(3) $f(x)\in C[a,b]/\{c\}$，在 c 的邻域内无界

$$\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



瑕点

例3. $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

解：$I = 2\sqrt{x-2}\,\Big|_2^5 = 2\sqrt{3}$

例4. $\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx$

$$= \arcsin\frac{x}{a}\Big|_0^a = \frac{\pi}{2}$$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x\Big|_0^{\frac{1}{2}} = \frac{\pi}{6}$$

例5 $\int^1 \frac{1}{x}\,dx = \ln|x|\,\Big|^1 = 0$ (×)

例5 $\int_{-1}^{1} \frac{1}{x} dx = \ln|x| \Big|_{-1}^{1} = 0$ (X)

论 $\int_{-1}^{1} \frac{1}{x} dx = \int_{-1}^{0} \frac{1}{x} dx + \int_{0}^{1} \frac{1}{x} dx$

而 $\int_{0}^{1} \frac{1}{x} dx = \ln x \Big|_{0}^{1} = \infty$

$\therefore$ $\int_{-1}^{1} \frac{1}{x} dx$ 发散.