4.4有理函数的积分

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§4. 有理函数的积分.

1. 有理函数的积分.

(1) $\int$ 假分式 $dx = \int$ 多项式 $dx + \int$ 真分式 dx

(2) $\int$ 真分式 dx: 分子次数 < 分母次数.

定理: 任一多项式在实数范围内都可以分解成若干个一次因式和二次不可约因式的乘积.

$(x-1)(x-2)^2(x^2+x+2)(x^2+2x+3)^3$ $\quad x^2+x+1$

$(x-a)^m$ — 一次因式 $\quad 1+x^3$

x^2+px+q — 二次因式.

例1 $\dfrac{x+3}{x^2-5x+6}$ 部分分式分解

解. $\dfrac{x+3}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3} = \dfrac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

$\Rightarrow x+3 = A(x-3)+B(x-2) = (A+B)x-3A-2B$

$\Rightarrow \begin{cases} A+B=1 \\ -3A-2B=3 \end{cases} \Rightarrow \begin{cases} A=-5 \\ B=6 \end{cases}$

$\therefore \dfrac{x+3}{x^2-5x+6} = \dfrac{-5}{x-2} + \dfrac{6}{x-3}$

令 $x=3$. 则 $B=6$

$x=2$ $\quad A=-5$

例2. $\dfrac{1}{x(x-1)^2} = \dfrac{1}{x} + \dfrac{-1}{x-1} + \dfrac{1}{(x-1)^2}$

解: $\dfrac{1}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$$= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2} \qquad 1 = A(x-1)^2 + Bx(x-1) + Cx$$

$x=0$: $1 = A$

$x=1$: $1 = C$

$B = -1$

例3. $$\frac{1}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2} = \frac{\frac{4}{5}}{1+2x} + \frac{-\frac{2}{5}x + \frac{1}{5}}{1+x^2}$$

$$\frac{1}{(1+2x)^2(1+x^2)^3} = \frac{A}{1+2x} + \frac{B}{(1+2x)^2} + \frac{C_1x+D_1}{1+x^2} + \frac{C_2x+D_2}{(1+x^2)^2} + \frac{C_3x+D_3}{(1+x^2)^3}$$

例4. 求不定积分

(1) $$\int \frac{x+3}{x^2-5x+6}\,dx = \int \left(\frac{-5}{x-2} + \frac{6}{x-3}\right)dx = -5\ln|x-2| + 6\ln|x-3| + C$$

(2) $$\int \frac{1}{x(x-1)^2}\,dx = \int \left[\frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}\right]dx$$
$$= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$$

注: (i) $$\int \frac{dx}{x-a} = \ln|x-a| + C$$

(ii) $$\int \frac{dx}{(x-a)^\alpha} = \frac{1}{1-\alpha}(x-a)^{1-\alpha} + C \quad (\alpha \neq 1)$$

(3) $$\int \frac{dx}{(1+2x)(1+x^2)} = \int \left[\frac{\frac{4}{5}}{1+2x} + \frac{-\frac{2}{5}x + \frac{1}{5}}{1+x^2}\right]dx$$
$$= \frac{2}{5}\ln|1+2x| - \frac{1}{5}\int \frac{2x-1}{1+x^2}\,dx$$
$$= \frac{2}{5}\ln|1+2x| - \frac{1}{5}\ln(1+x^2) + \frac{1}{5}\arctan x + C$$

(4) $$\int \frac{x-2}{x^2+2x+3}\,dx \qquad a^2 \pm x^2$$

$$= \frac{1}{2}\int \frac{(2x+2)}{x^2+2x+3}\,dx - \int \frac{3}{x^2+2x+3}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+2)}{x^2+2x+3}dx - \int \frac{3}{x^2+2x+3}dx$$

$$= \frac{1}{2}\int \frac{d(x^2+2x+3)}{x^2+2x+3} - 3\int \frac{1}{(x+1)^2+2}dx$$

$$= \frac{1}{2}\ln(x^2+2x+3) - \frac{3}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$$

(5) $\int \frac{x^2}{1+x^6}dx = \frac{1}{3}\int \frac{dx^3}{1+(x^3)^2} = \frac{1}{3}\arctan x^3 + C$

注：化部分分式拆解.

(6) $\int \frac{x^7}{(1-x^2)^5}dx$ 　　倒代换

2. 三角函数有理式的积分

$$\int f(\sin x, \cos x)dx$$

解法： $t = \tan\frac{x}{2}$， $x = 2\arctan t$ 　$dx = \frac{2}{1+t^2}dt$

$\sin x = \frac{2t}{1+t^2}$， $\cos x = \frac{1-t^2}{1+t^2}$， （万能公式）

$$I = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\cdot\frac{2}{1+t^2}dt$$

例5. $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$

$$= \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2}{1+t^2}dt$$

$$= \frac{1}{2}\int \left(t+2+\frac{1}{t}\right)dt$$

$$=\frac{1}{2}\left(\ln|t| + 2t + \frac{1}{2}t^2\right) + C$$

$$=\frac{1}{2}\left(\ln\left|\tan\frac{x}{2}\right| + 2\tan\frac{x}{2} + \frac{1}{2}\tan^2\frac{x}{2}\right) + C$$

例6 $\int \frac{\cos x}{1+\sin x}\,dx$

$$=\int \frac{d(\sin x+1)}{1+\sin x}$$

$$=\ln|1+\sin x| + C$$

注： $\int R(\sin x)\cos x\,dx = \int R(\sin x)\,d\sin x$

$\int R(\tan x)\sec^2 x\,dx = \int R(\tan x)\,d\tan x$

3. 简单无理函数的积分

$f(\sqrt{ax+b}), f(\sqrt[3]{ax+b})$: $t=\sqrt{ax+b}$. $t=\sqrt[3]{ax+b}$

$f\left(\sqrt[n]{\frac{ax+b}{cx+d}}\right)$: $t=\sqrt[n]{\frac{ax+b}{cx+d}}$

例7 $\int \frac{1}{1+\sqrt{x}}\,dx$ $t=\sqrt{x}$. $x=t^2$. $dx=2t\,dt$

$$=\int \frac{1}{1+t}\cdot 2t\,dt = 2\int\left(1-\frac{1}{1+t}\right)dt$$

$$=2\left(t-\ln|1+t|\right)+C$$

$$=2\left(\sqrt{x}-\ln|1+\sqrt{x}|\right)+C$$

例8 $\int e^{\sqrt[3]{x}}\,dx$ $t=\sqrt[3]{x}$. $x=t^3$. $dx=3t^2\,dt$

$$=3\int t^2 e^t\,dt$$

$$= 3\int t^2 \, d e^t$$

$$= 3\left(t^2 e^t - 2\int t e^t \, dt\right)$$

$$= 3\left(t^2 e^t - 2\int t \, d e^t\right)$$

$$= 3\left(t^2 e^t - 2t e^t + 2 e^t\right) + C$$

例9 $\int \frac{1}{x}\sqrt{\frac{1+x}{x}}\, dx$ $\quad t = \sqrt{\frac{1+x}{x}} \quad x = \frac{1}{t^2-1}$

$$= \int t(t^2-1)\cdot\left[-\frac{2t}{(t^2-1)^2}\right] dt = -2\int \frac{t^2}{t^2-1}\, dt$$

$$= -2\int\left(1 + \frac{1}{t^2-1}\right) dt$$

$$= -2\left(t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right) + C$$

$$= -2\sqrt{\frac{1+x}{x}} - \ln\left|\frac{\sqrt{1+x}-\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}\right| + C$$