The algebra of partial equivalence relations

Fabio Zanasi

Radboud University Nijmegen, The Netherlands

Abstract

Recent work by the author with Bonchi and Sobociński shows how PROPs of linear relations (subspaces) can be presented by generators and equations via a "cube construction", based on letting very simple structures interact according to PROP operations of sum, fibered sum and composition via a distributive law. This paper shows how the same construction can be used in a cartesian setting to obtain presentations by generators and equations for the PROP of equivalence relations and of partial equivalence relations.

Keywords: PROP, distributive law, string diagram, partial equivalence relation, Frobenius algebra

1 Introduction

PROPs (**pro**duct and **p**ermutation categories [21]) are symmetric monoidal categories with objects the natural numbers. In the last two decades, they have become increasingly popular as an environment where to study diverse computational models in a compositional, resource sensitive fashion. To make a few examples, they have recently featured in algebraic approaches to Petri nets [7,26], bigraphs [8], quantum processes [11] and signal flow graphs [2,4,1].

PROPs can be used to specify both the *syntax* and the *semantics* of systems. A "syntactic" PROP $\mathcal{T}$ is generated starting from a *symmetric monoidal theory* (Σ, E), which intuitively is an algebraic specification for operations with multiple inputs and outputs; arrows of $\mathcal{T}$ are freely constructed by composition of operations in the signature Σ, and then quotiented by the equations in E. On the other hand, a "semantic" PROP S is specified with a direct definition of its arrows, typically in terms of some mathematical object of interest. A full completeness result is a precise correspondence between these two perspectives, in the form of an isomorphism

$$\mathcal{T} \xrightarrow{\cong} \mathsf{S}. \tag{1}$$

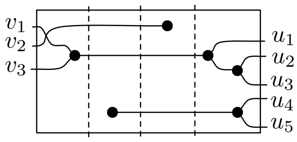
In this situation, we say that (Σ, E) *presents* S. Examples of (1) are ubiquitous and play a foundational role in most of the aforementioned research threads. For instance, the theory of commutative monoids presents the PROP of functions; the theory of Hopf algebras presents the PROP of integer matrices; the theory of Frobenius algebras presents the PROP of 2-Dimensional cobordisms.

In recent years, increasingly more elaborated examples have been tackled using *modular* reasoning principles. An illustrative case is the theory of interacting Hopf algebras IH, which characterises the PROP $\mathsf{LRel}_{\mathsf{k}}$ of k-linear relations [5]. This result inspired recent investigation in the foundations of the ZX-calculus [3,14] and in categorical control theory [2,4,1]. What is most interesting for our purposes is that the isomorphism $\mathsf{IH} \cong \mathsf{LRel}_{\mathsf{k}}$ can be obtained as a universal arrow through a "cube" construction, based on seeing the two PROPs as the result of the interaction of simpler theories by means of operations of sum, fibered sum and composition. This modular account is a valuable source of information about the structural properties of the theories of interest: for instance, it shows that $\mathsf{LRel}_{\mathsf{k}}$ is the result of combining PROPs of spans and of cospans of linear maps, and the equations of IH essentially describe this interaction.

The central idea of this work is to show how the same cube construction can be used to characterise other PROPs of relations: whereas [5] focuses on the *linear* case, we shall study the *cartesian* case, both total and partial. In the total case, we construct a modular characterisation for the PROP ER of *equivalence relations* starting from PROPs of spans and cospans of (injective) functions, see (5) below. This will show an isomorphism between ER and the PROP IFr freely generated by a quotient of the theory of special Frobenius algebras [9], which plays a foundational role in many recent works [23,2,1,11].

$$\mathsf{IFr} \xrightarrow{\cong} \mathsf{ER} \tag{2}$$

To give an idea of how the isomorphism (2) works, an arrow of IFr, for which we shall use the 2-dimensional representation as a string diagram, as on the left below, shall represent an equivalence relation on the sets of variables associated with its left and right ports, as on the right below. Two variables are in the same equivalence class if they are linked in the graphical representation.



$\mapsto$ an equivalence relation with classes $[v_2], [v_1, v_3, u_1, u_2, u_3], [u_4, u_5]$.

The dotted lines hint at the fact that, as a result of our modular perspective, any diagram of IFr will enjoy a factorisation in terms of simpler theories, whose interaction is what the axioms of IFr describe.

Building on this result, we will shift to the *partial* case. First, we use PROP composition to construct a presentation PMn (**p**artial commutative **mon**oids) for the PROP PF of partial functions. Then, we will show that the PROP PER of *partial equivalence relations* (PERs) [1] arises as the result of merging PROPs of cospans of partial functions and of spans of injective functions, see (10) below. As for the case of ER, an isomorphism arises from this modular account: it will relate PER and the syntactic PROP IPFr, yet another variation of the theory of special Frobenius algebras.

$$\mathsf{IPFr} \xrightarrow{\cong} \mathsf{PER}$$

[1] Recall that a relation on a set X is a PER if it is symmetric and transitive — equivalently, if it is an equivalence relation on a subset $Y \subseteq X$.

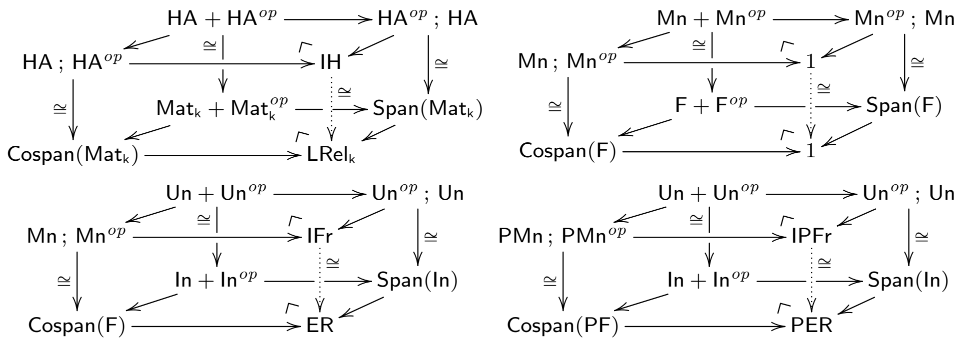


Figure 1. An overview of the various cube constructions considered in this paper. From the top-left corner: the linear case, yielding a characterisation for the PROP LRel_k of k-linear relations (see [5]); the degenerate cartesian case, collapsing to the terminal PROP (Remark 4.11); the (non-degenerate) cartesian case, yielding a characterisation for the PROP ER of equivalence relations (Theorem 4.2); the partial cartesian case, yielding a characterisation for the PROP PER of partial equivalence relations (Theorem 5.4). In the main text we shall write PROPs of spans and cospans in factorised form to emphasise their provenance from distributive laws, e.g. Span(F) as F^{op} ; F ans Cospan(F) as F ; F^{op}.

In a nutshell, the diagrammatic rendition of partial equivalence relations given by IPFr enhances the total case by integrating connectors , for partiality.

Related work. The use of partial equivalence relations in program semantics dates back to the seminal work of Scott [24]. They have been used extensively in the semantics of higher order λ-calculi (e.g., [17,28]) and, more recently, of quantum computations (e.g., [18,15]). Note that in most of these applications PERs are the objects of the category of interest, whereas in the PROP PER they are the arrows, with relational composition, and only defined on *finite* domains. In fact, our emphasis is on the modular techniques to characterise PER (and their applicability to similar families of structures) rather than on the use of PERs in semantics.

Algebraic presentations for categories of equivalence relations have been studied in the last two decades by a few authors. A characterisation for ER in terms of Frobenius structures is given in [13], with a proof based on finding a normal form for string diagrams. The same result appears in a recent manuscript [12], which is based, like our work, on treating equivalence relations as jointly-epi cospans. This idea, as well as its algebraic implications, is studied in the earlier paper [6] as part of a taxonomy of span/cospan categories over **Set**.

The present work is part of the author's PhD thesis [29], defended in October 2015. Differently from the aforementioned papers, our approach focuses on a modular reconstruction of ER: its presentation is built from the interaction of very simple algebraic theories, by the use of PROP operations. In particular, Lack's technique for composing PROPs [19] is pivotal. Also, we extend our methodology to the analysis of partial functions and partial equivalence relations, in a way that to the best of our knowledge did not appear before in the literature.

It is also worth mentioning that there is a pleasant symmetry between the analysis of equivalence relations and (plain) relations. Whereas the former are jointly-epic cospans and are modeled by separable Frobenius algebras with an additional axiom from the theory of bialgebras, the latter are jointly-mono spans and are modeled by bialgebras with the addition of an axiom from the theory of separable Frobenius algebras [20]. Interestingly, the combination of the two theories in their entirety collapses to the terminal PROP, see Remark 4.11 below. [2]

Synopsis. In §2 we recall the basics of the theory of PROPs. § 3 introduces the PROP operations of sum, fibered sum and (iterated) composition, with the example of partial functions (Ex. 3.3). § 4 constructs the cube (5) necessary for the characterisation of equivalence relations (Th. 4.2). § 5 completes the picture with the characterisation (10) of partial equivalence relations (Th. 5.4).

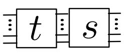
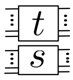
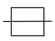
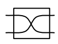
Prerequisites and notation. We assume familiarity with basic category theory (see e.g. [22]) and the definition of symmetric strict monoidal category [22,25] (often abbreviated as SMC). We write $f\,;g\colon a \to c$ for composition of $f\colon a \to b$ and $g\colon b \to c$ in a category $\mathbb{C}$. It will be sometimes convenient to indicate an arrow $f\colon a \to b$ of $\mathbb{C}$ as $x \xrightarrow{f\in\mathbb{C}} y$ or also $\xrightarrow{\in\mathbb{C}}$, if names are immaterial. For $\mathbb{C}$ an SMC, $\oplus$ is its monoidal product, with unit object I, and $\sigma_{a,b}\colon a \oplus b \to b \oplus a$ is the symmetry associated with $a, b \in \mathbb{C}$. We write $\overline{0}$ for $\emptyset$ and $\overline{n+1}$ for $\{1, \ldots, n, n+1\}$.

2 PROPs

Our exposition is founded on PROPs (**pro**duct and **p**ermutation categories [21]).

Definition 2.1 A *PROP* is a symmetric strict monoidal category with objects the natural numbers, where $\oplus$ on objects is addition. PROPs form a category **PROP** with morphisms the identity-on-objects symmetric strict monoidal functors.

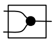
A typical way of constructing a PROP is starting from a *symmetric monoidal theory* (SMT): it is a pair (Σ, E), where Σ is a signature of *generators* $o\colon n \to m$ with *arity* n and *coarity* m. The set of Σ-*terms* is obtained by composing generators in Σ, the unit $id\colon 1 \to 1$ and the symmetry $\sigma_{1,1}\colon 2 \to 2$ with ; and $\oplus$. That means, given Σ-terms $t\colon k \to l$, $u\colon l \to m$, $v\colon m \to n$, one constructs new Σ-terms $t\,;u\colon k \to m$ and $t \oplus v\colon k + n \to l + n$. The set E of *equations* contains pairs $(t, t'\colon n \to m)$ of Σ-terms with the same arity and coarity.

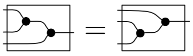
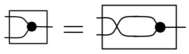
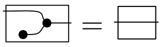
There is a natural graphical representation for Σ-terms using the formalism of string diagrams [25]. A Σ-term $n \to m$ is pictured as a box with n ports on the left and m ports on the right. Composition $t\,; s$ is rendered graphically as  and $t \oplus s$ as . The symmetric monoidal structure is generated from , representing $id_1\colon 1 \to 1$, , representing $id_0\colon 0 \to 0$, and , representing $\sigma_{1,1}\colon 2 \to 2$.

An SMT (Σ, E) *freely generates* a PROP $\mathcal{T}$ by letting arrows $n \to m$ in $\mathcal{T}$ be Σ-terms modulo E. We say that (Σ, E) is a *presentation* of a PROP S when $\mathsf{S} \cong \mathcal{T}$. When $\Sigma' \subseteq \Sigma$ and $E' \subseteq E$, there is an evident inclusion PROP morphism from the PROP $\mathcal{T}'$ generated by (Σ', E') to the one $\mathcal{T}$ generated by (Σ, E), for which henceforth we reserve notation $\mathcal{T}' \hookrightarrow \mathcal{T}$.

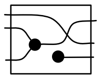
[2] This observation is also relevant for algebraic approaches to quantum processes, see e.g. [16, Th. 5.6].

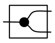
Example 2.2

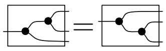
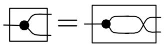
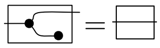
(a) In the SMT (Σ_M, E_M) of *commutative monoids*, Σ_M contains a multiplication  : $2 \to 1$ and a unit  : $0 \to 1$. Equations E_M assert associativity (M1), commutativity (M2) and unitality (M3).

 (M1)  (M2)  (M3)

(Σ_M, E_M) presents the PROP F whose arrows $n \to m$ are total functions from $\overline{n}$ to $\overline{m}$, with $\overline{n} = \{1, \ldots, n\}$. Writing Mn for the PROP freely generated by (Σ_M, E_M), the isomorphism $\mathsf{Mn} \cong \mathsf{F}$ is defined by interpreting string diagrams as graphs of functions. For instance, the diagram on the right represents the function $\overline{3} \to \overline{3}$ mapping 1 on the left to 2 on the right and 2, 3 on the left to 1 on the right.

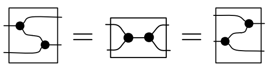
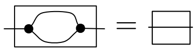


(b) The SMT (Σ_C, E_C) of *cocommutative comonoids* is based on a comultiplication  : $1 \to 2$ and a counit  : $1 \to 0$. E_C is the following set of equations.

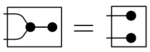
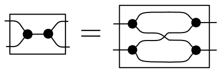
 (C1)  (C2)  (C3)

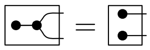
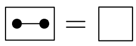
We write Cm for the PROP freely generated by (Σ_C, E_C). There is an evident isomorphism $\mathsf{Cm} \cong \mathsf{Mn}^{op}$ given by "vertical rotation" of string diagrams. Therefore, (Σ_C, E_C) presents F^{op}.

(c) The PROP Fr of *special Frobenius algebras* [9] is generated by the theory $(\Sigma_M \uplus \Sigma_C, E_M \uplus E_C \uplus F)$, where F is the following set of equations.

 (F1)  (F2)

(d) The PROP B of (commutative/cocommutative) *bialgebras* is generated by the theory $(\Sigma_M \uplus \Sigma_C, E_M \uplus E_C \uplus B)$, where B is the following set of equations.

 (B1)  (B3)

 (B2)  (B4)

Remark 2.3 The assertion that (Σ_M, E_M) is the SMT of *commutative monoids*—and similarly for other SMTs appearing in our exposition—can be made precise by establishing a correspondence between commutative monoids in a symmetric monoidal category $\mathbb{C}$ and objects $F(1)$ identified by symmetric monoidal functors $F : \mathsf{Mn} \to \mathbb{C}$, often called *models* or algebras of Mn. As models are not central in our work, we refer the reader to [19] for more information.

3 PROP operations

The following table summarises three operations on given PROPs $\mathcal{T}_1$ and $\mathcal{T}_2$. Supposing that they are presented by SMTs (Σ_1, E_1) and (Σ_2, E_2) respectively, the second column describes a presentation for the PROP resulting from the operation.

	PROPs	**SMTs**	**Reference**
Sum	$\mathcal{T}_1 + \mathcal{T}_2$	signature: $\Sigma_1 \uplus \Sigma_2$ equations: $E_1 \uplus E_2$	see e.g. [29, §2.3].
Fibered sum over $\mathcal{T}_3$	$\mathcal{T}$ defined by $\begin{tikzcd} \mathcal{T}_3 \arrow[r, hook] \arrow[d, hook] & \mathcal{T}_1 \arrow[d, hook] \\ \mathcal{T}_2 \arrow[r, hook] & \mathcal{T} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$	sig.: $(\Sigma_1 \uplus \Sigma_2)_{\equiv_{\Sigma_3}}$ eq.: $(E_1 \uplus E_2)_{\equiv_{E_3}}$	see e.g. [29, §2.5].
Composition via λ	$\mathcal{T}_1 \,;\, \mathcal{T}_2$ defined by $\lambda\colon \mathcal{T}_2 \,;\, \mathcal{T}_1 \to \mathcal{T}_1 \,;\, \mathcal{T}_2$.	sig.: $\Sigma_1 \uplus \Sigma_2$ eq.: $E_1 \uplus E_2 \uplus E_\lambda$	introduced in [19], see also [29, §2.4].

We now illustrate the three operations. The simplest, the sum, just combines the two theories without adding any interaction.

The fibered sum mimics a kind of construction typical in algebra, from geometric gluing constructions of topological spaces to amalgamated free products of groups. The idea is to identify some structure $\mathcal{T}_3$ that is in common between the two theories. In all applications, the assumption is that $\Sigma_3 \subseteq \Sigma 1 \cap \Sigma_2$ and $E_3 \subseteq E_1 \cap E_2$: the quotient $\equiv_{\Sigma_3}$ identifies $o_1 \in \Sigma_1$ and $o_2 \in \Sigma_2$ when $o_1 = o_2$ is in Σ_3, and $\equiv_{E_3}$ acts similarly on equations. On PROPs, this operation amounts to pushing out the inclusion morphisms $\mathcal{T}_1 \hookleftarrow \mathcal{T}_3 \hookrightarrow \mathcal{T}_2$ from the PROP $\mathcal{T}_3$ freely generated by (Σ_3, E_3).

The composition enhances the sum with compatibility conditions between $\mathcal{T}_1$ and $\mathcal{T}_2$. Also this operation mimics a standard pattern in algebra: e.g. a ring is given by a monoid and an abelian group, subject to equations that ensure that the former distributes over the latter. Formally, the operation $\mathcal{T}_1 \,;\, \mathcal{T}_2$ is defined in [19] by understanding PROPs $\mathcal{T}_1$, $\mathcal{T}_2$ as monads in a certain bicategory [27], and then compose them via a distributive law $\lambda\colon \mathcal{T}_2 \,;\, \mathcal{T}_1 \to \mathcal{T}_1 \,;\, \mathcal{T}_2$. The resulting monad $\mathcal{T}_1 \,;\, \mathcal{T}_2$ is also a PROP, enjoying a presentation as the quotient of $\mathcal{T}_1 + \mathcal{T}_2$ by the equations E_λ encoded by the distributive law. The set E_λ is simply the graph of λ, which can be seen as a set of directed equations $(\xrightarrow{\in \mathcal{T}_2}\xrightarrow{\in \mathcal{T}_1}) \approx (\xrightarrow{\in \mathcal{T}_1}\xrightarrow{\in \mathcal{T}_2})$ telling how arrows of $\mathcal{T}_2$ distribute over arrows of $\mathcal{T}_1$. In fortunate cases, like the examples below, it is possible to present E_λ by a simpler, or even finite, set of equations, thus giving a sensible axiomatisation of the compatibility conditions expressed by λ.

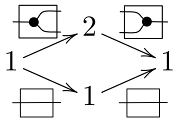
Example 3.1

(a) The PROP F of functions can be described as the composite $\mathsf{Su} \,;\, \mathsf{In}$, where Su and In are respectively the PROP of surjective and of injective functions [19]. The witnessing distributive law $\lambda\colon \mathsf{In} \,;\, \mathsf{Su} \to \mathsf{Su} \,;\, \mathsf{In}$ maps a function $\xrightarrow{\in \mathsf{In}}\xrightarrow{\in \mathsf{Su}}$ to its epi-mono factorisation $\xrightarrow{\in \mathsf{Su}}\xrightarrow{\in \mathsf{In}}$.

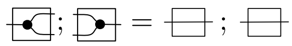
In more syntactic terms, using the isomorphism $\mathsf{F} \cong \mathsf{Mn}$, this result says that Mn is the composite $\mathsf{Mu} \,;\, \mathsf{Un}$, where $\mathsf{Mu} \cong \mathsf{Su}$ is the PROP freely generated by the SMT $(\{\}, \{(M1), (M2)\})$ and $\mathsf{Un} \cong \mathsf{In}$ by the SMT $(\{\}, \emptyset)$. The distributive law explains the origin of equation (M3) of Mn, which indeed describes how to move the generator of Un past the one of Mu.

(b) There is a distributive law $\lambda\colon \mathsf{F}^{op} \,;\, \mathsf{F} \to \mathsf{F} \,;\, \mathsf{F}^{op}$ mapping a pair $\xrightarrow{\in \mathsf{F}^{op}}\xrightarrow{\in \mathsf{F}}$, i.e. a span $\xleftarrow{\in \mathsf{F}}\xrightarrow{\in \mathsf{F}}$, to (a choice of) its pushout *cospan* $\xrightarrow{\in \mathsf{F}}\xleftarrow{\in \mathsf{F}}$, i.e. a pair $\xrightarrow{\in \mathsf{F}}\xrightarrow{\in \mathsf{F}^{op}}$ [19]. Because $\mathsf{Mn} \cong \mathsf{F}$ and $\mathsf{Cm} \cong \mathsf{F}^{op}$, this yields a composite PROP $\mathsf{Mn}\,;\mathsf{Cm}$, presented as $\mathsf{Mn} + \mathsf{Cm}$ modulo the equations arising from the distributive law. By definition of λ, such equations can be read from pushout squares in F. For instance:

$$\begin{tikzcd}[row sep=small] & 2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[dr] & \wedge & 1 \arrow[dl] \\ & 1 & \end{tikzcd} \quad \Longrightarrow$$



yields



where the second diagram is obtained from the pullback by applying the isomorphisms $\mathsf{F} \cong \mathsf{Mn}$ and $\mathsf{F}^{op} \cong \mathsf{Cm}$. In fact, Lack [19] shows that in order to present λ it suffices to check three pushout squares, corresponding to equations (F1)-(F2). Therefore, $\mathsf{Mn}\,;\mathsf{Cm}$ is isomorphic to Fr (Example 2.2), and both have a concrete description in terms of *cospans*, i.e. the arrows of $\mathsf{F}\,;\mathsf{F}^{op}$.

(c) Dually, there exists a distributive law $\lambda\colon \mathsf{F}\,;\mathsf{F}^{op} \to \mathsf{F}^{op}\,;\mathsf{F}$, defined by pullback in F [19], which yields the PROP $\mathsf{F}^{op}\,;\mathsf{F}$ of *spans*. All the equations arising by this distributive law can be proven from (B1)-(B4), yielding $\mathsf{F}^{op}\,;\mathsf{F} \cong \mathsf{B}$.

Composing distributive laws

For our developments it is useful to generalise PROP composition to the case when there are more than two theories interacting with each other. The following result, a variation of a theorem by Cheng [10], is proven in [29, §2.4.6].

Proposition 3.2 *Let $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{G}$ be PROPs presented by SMTs $(\Sigma_{\mathcal{F}}, E_{\mathcal{F}})$, $(\Sigma_{\mathcal{H}}, E_{\mathcal{H}})$ and $(\Sigma_{\mathcal{G}}, E_{\mathcal{G}})$ respectively. Suppose there are distributive laws*

$$\lambda\colon \mathcal{H}\,;\mathcal{F} \to \mathcal{F}\,;\mathcal{H} \qquad \chi\colon \mathcal{H}\,;\mathcal{G} \to \mathcal{G}\,;\mathcal{H} \qquad \psi\colon \mathcal{G}\,;\mathcal{F} \to \mathcal{F}\,;\mathcal{G}$$

satisfying the following "Yang-Baxter" equation:

$$\begin{tikzcd}[column sep=small] & \mathcal{H}\,;\mathcal{F}\,;\mathcal{G} \arrow[r, "\lambda\mathcal{G}"] & \mathcal{F}\,;\mathcal{H}\,;\mathcal{G} \arrow[dr, "\mathcal{F}\chi"] & \\ \mathcal{H}\,;\mathcal{G}\,;\mathcal{F} \arrow[ur, "\mathcal{H}\psi"] \arrow[dr, "\chi\mathcal{F}"'] & & & \mathcal{F}\,;\mathcal{G}\,;\mathcal{H} \\ & \mathcal{G}\,;\mathcal{H}\,;\mathcal{F} \arrow[r, "\mathcal{G}\lambda"] & \mathcal{G}\,;\mathcal{F}\,;\mathcal{H} \arrow[ur, "\psi\mathcal{H}"'] & \end{tikzcd} \tag{3}$$

then the following two are distributive laws:

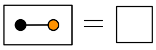
$$\left(\mathcal{H}\,;\mathcal{F}\,;\mathcal{G} \xrightarrow{\lambda\mathcal{G}} \mathcal{F}\,;\mathcal{H}\,;\mathcal{G} \xrightarrow{\mathcal{F}\chi} \mathcal{F}\,;\mathcal{G}\,;\mathcal{H}\right) \quad \left(\mathcal{G}\,;\mathcal{H}\,;\mathcal{F} \xrightarrow{\mathcal{G}\lambda} \mathcal{G}\,;\mathcal{F}\,;\mathcal{H} \xrightarrow{\psi\mathcal{H}} \mathcal{F}\,;\mathcal{G}\,;\mathcal{H}\right)$$

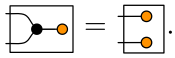
yielding the same PROP $\mathcal{F}\,;\mathcal{G}\,;\mathcal{H}$. Furthermore, call E_λ, E_χ and E_ψ the sets of equations encoding the three laws. Then $\mathcal{F}\,;\mathcal{G}\,;\mathcal{H}$ is presented by the signature $\Sigma_{\mathcal{F}} \uplus \Sigma_{\mathcal{H}} \uplus \Sigma_{\mathcal{G}}$ and equations $E_{\mathcal{F}} \uplus E_{\mathcal{H}} \uplus E_{\mathcal{G}} \uplus E_\lambda \uplus E_\chi \uplus E_\psi$.

Example 3.3 We show how the PROP PF of *partial* function can be presented modularly using iterated distributive laws. First, we introduce a new PROP Cu, generated by the signature {} and no equations: modulo the different colouring, it is just Un^{op}. Following the recipe of Proposition 3.2, we now combine Cu, Un and Mu via three distributive laws:

$$\lambda\colon \mathsf{Un}\,;\mathsf{Cu} \to \mathsf{Cu}\,;\mathsf{Un} \qquad \chi\colon \mathsf{Un}\,;\mathsf{Mu} \to \mathsf{Mu}\,;\mathsf{Un} \qquad \psi\colon \mathsf{Mu}\,;\mathsf{Cu} \to \mathsf{Cu}\,;\mathsf{Mu}$$

Using the isomorphisms $\mathsf{Un} \cong \mathsf{In}$, $\mathsf{Cu} \cong \mathsf{In}^{op}$ and $\mathsf{Mu} \cong \mathsf{Su}$, we can define χ by epi-mono factorisation as in Example 3.1(a); therefore, the resulting PROP $\mathsf{Mu}\,;\mathsf{Un}$ is $\mathsf{Mu}+\mathsf{Un}$ quotiented by (M3). Because pullbacks in F preserve both monos and epis, we define λ and ψ by pullback in F. It is readily seen that λ and ψ are presented, respectively, by the first and the second equation below:

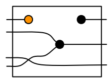
 (P1)

 (P2)

Also, λ, χ and ψ verify the Yang-Baxter equation (3) and thus Proposition 3.2 yields a PROP $\mathsf{Cu}\,;\mathsf{Mu}\,;\mathsf{Un}$ presented as the quotient of $\mathsf{Cu}+\mathsf{Mu}+\mathsf{Un}$ by (M3), (P1) and (P2). By analogy with the total case $\mathsf{Mn} \cong \mathsf{Mu}\,;\mathsf{Un}$, we shall use PMn (partial commutative monoids) as a shorthand for $\mathsf{Cu}\,;\mathsf{Mu}\,;\mathsf{Un}$.

We now claim that $\mathsf{PMn} \cong \mathsf{PF}$. To see this, observe that partial functions $n \xrightarrow{f\in\mathsf{PF}} m$ are in bijective correspondence with spans $n \xleftarrow{i\in\mathsf{In}} z \xrightarrow{f\in\mathsf{F}} m$: the injection i tells on which elements $\overline{z}$ of $\overline{n}$ the function f is defined. Since $\mathsf{In}^{op} \cong \mathsf{Cu}$ and $\mathsf{F} \cong \mathsf{Mn} \cong \mathsf{Mu}\,;\mathsf{Un}$, this correspondence yields the desired isomorphism $\mathsf{PF} \cong \mathsf{In}^{op}\,;\mathsf{F} \cong \mathsf{Cu}\,;\mathsf{Mu}\,;\mathsf{Un} \cong \mathsf{PMn}$.

As a last remark, note that the factorisation property of PMn allows to interpret any arrow of this PROP as the graph of a partial function, where  indicates partiality. For instance, the diagram on the right represents the function $\overline{4} \to \overline{3}$ undefined on 1 and mapping $2, 4$ to 2 and 3 to 3.



4 A presentation of equivalence relations

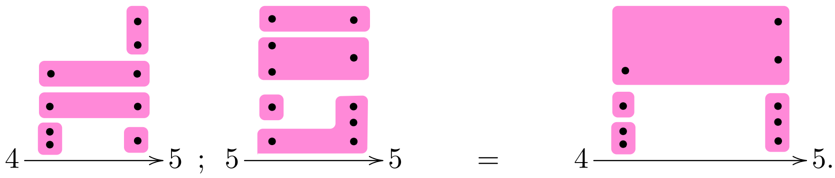
This section builds modularly a presentation for the PROP ER of equivalence relations, using the operations introduced in § 3. In defining ER, we use the following notation: $\lfloor e \rfloor$ is the symmetric and transitive closure of a relation e and $d_{\restriction Y}$ is the restriction of an equivalence relation d on a set X to a subset $Y \subseteq X$.

Definition 4.1 Let ER be the PROP whose arrows $n \to m$ are the equivalence relations on $\overline{n} \uplus \overline{m}$. Given $e_1 \colon n \to z$ and $e_2 \colon z \to m$, the composite $e_1\,; e_2 \colon n \to m$ is defined in steps as follows.

$$\begin{aligned} e_1 * e_2 &:= \{(v,w) \mid \exists u.\ (v,u) \in e_1 \wedge (u,w) \in e_2\} \\ e_1 \diamond e_2 &:= e_1 \cup e_2 \cup \lfloor e_1 * e_2 \rfloor \\ e_1\,; e_2 &:= e_1 \diamond e_{2\restriction \overline{n}\uplus\overline{m}} \end{aligned}$$

The monoidal product $e_1 \oplus e_2$ is given by disjoint union of e_1 and e_2.

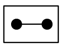
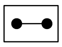
In words, for composition one first defines an equivalence relation $e_1 \diamond e_2$ on $\overline{n}\uplus\overline{z}\uplus\overline{m}$ by gluing together equivalence classes of e_1 and e_2 along common witnesses in $\overline{z}$, then obtains $e_1 ; e_2$ by restricting to elements of $\overline{n} \uplus \overline{m}$. Here is an example:



Our approach in characterising ER stems from the observation that cospans can be interpreted as "redundant" equivalence relations. This becomes particularly neat when representing cospans as string diagrams via the characterisation $\mathsf{Fr} \cong \mathsf{F} ; \mathsf{F}^{op}$ (Example 3.1(b)), as below.

(4)

The dotted line emphasizes the fact that Fr factorises as $\mathsf{Mn} ; \mathsf{Cm}$. Both string diagrams in (4) define an equivalence relation e on $\overline{5} \uplus \overline{7}$ by letting $(v, w) \in e$ if the port associated with v and the one associated with w are linked in the graphical representation. For instance, $1, 2 \in \overline{5}$ on the left boundary are in the same equivalence class as $1, 2, 3 \in \overline{7}$ on the right boundary, whereas $5 \in \overline{5}$ and $4 \in \overline{7}$ are the only members of their equivalence class.

Observe that the two representations of e in (4) only differ for the sub-diagram , which indeed does not play any role in the interpretation and stands for an "empty" equivalence class. Equation (B4) will be employed to express the redundancy of . Let us call IFr (**i**rredundant **Fr**obenius algebras) the PROP defined as the quotient of Fr by (B4). Our discussion leads to the following claim.

Theorem 4.2 $\mathsf{IFr} \cong \mathsf{ER}$.

The isomorphism of Theorem 4.2 shall arise as a universal arrow in the following "cube" diagram in **PROP**, provided that the top and bottom square are pushouts.

$$\begin{tikzcd}
 & \mathsf{Un} + \mathsf{Cu} \arrow[dl, hook] \arrow[rr, hook] \arrow[dd, "\cong"] & & \mathsf{Cu} ; \mathsf{Un} \arrow[dl, hook] \arrow[dd, "\cong"] \\
\mathsf{Fr} \arrow[rr, hook, two heads] \arrow[dd, "\cong"'] & & \mathsf{IFr} \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{In} + \mathsf{In}^{op} \arrow[dl, "{[\iota_1,\iota_2]}"'] \arrow[rr, "{[\kappa_1,\kappa_2]}"] & & \mathsf{In}^{op} ; \mathsf{In} \arrow[dl, "\Upsilon"] \\
\mathsf{F} ; \mathsf{F}^{op} \arrow[rr, "\Pi"'] & & \mathsf{ER} &
\end{tikzcd} \tag{5}$$

First we explain the PROP morphisms in (5). Those of the top face are defined by inclusion of the corresponding SMTs and the rear vertical isomorphisms have been introduced in Examples 3.1-3.3. Thus we focus on the bottom face.

Definition 4.3

- morphisms $\kappa_1 \colon \mathsf{In} \to \mathsf{In}^{op} ; \mathsf{In}$, $\kappa_2 \colon \mathsf{In}^{op} \to \mathsf{In}^{op} ; \mathsf{In}$, $\iota_1 \colon \mathsf{In} \to \mathsf{F} ; \mathsf{F}^{op}$ and $\iota_2 \colon \mathsf{In}^{op} \to \mathsf{F} ; \mathsf{F}^{op}$ are given by

$$\kappa_1(n \xrightarrow{f} m) = (n \xleftarrow{id} n \xrightarrow{f} m) \qquad \kappa_2(n \xrightarrow{f} m) = (n \xleftarrow{f} m \xrightarrow{id} m)$$
$$\iota_1(n \xrightarrow{f} m) = (n \xrightarrow{f} m \xleftarrow{id} m) \qquad \iota_2(n \xrightarrow{f} m) = (n \xrightarrow{id} n \xleftarrow{f} m).$$

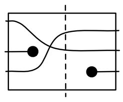
- $\Pi\colon \mathsf{F} \mathbin{;} \mathsf{F}^{op} \to \mathsf{ER}$ is defined on a cospan $n \xrightarrow{p} z \xleftarrow{q} m$ by

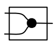
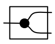
$$(v,w) \in \Pi(n \xleftarrow{f} z \xrightarrow{g} m) \qquad \text{iff} \qquad \begin{cases} p(v) = q(w) & \text{if } v \in \overline{n}, w \in \overline{m} \\ q(v) = p(w) & \text{if } v \in \overline{m}, w \in \overline{n} \\ p(v) = p(w) & \text{if } v, w \in \overline{n} \\ q(v) = q(w) & \text{if } v, w \in \overline{m}. \end{cases} \tag{6}$$

- $\Upsilon\colon \mathsf{In}^{op} \mathbin{;} \mathsf{In} \to \mathsf{ER}$ is defined on a span $n \xleftarrow{f \in \mathsf{In}} z \xrightarrow{g \in \mathsf{In}} m$ as the reflexive and symmetric closure of $\{(v,w) \mid f^{-1}(v) = g^{-1}(w)\}$.

It is lengthy but conceptually simple to verify that Π and Υ are indeed functorial assignments — details are reported in [29, Appendix A].

Informally, Π implements the idea of interpreting a cospan as an equivalence relation. For Υ, the key observation is that spans of injective functions can also be seen as equivalence relations. Once again, the graphical representation of an arrow of $\mathsf{In}^{op} \mathbin{;} \mathsf{In}$ as a string diagram in $\mathsf{Cu} \mathbin{;} \mathsf{Un}$ can help visualising this fact. A factorised arrow of $\mathsf{Cu} \mathbin{;} \mathsf{Un}$ as on the right can be interpreted as the equivalence relation associating 1 on the left boundary with 2 on the right boundary, 3 on the left with 1 on the right and letting 2 on the left, 3 on the right be the only representatives of their equivalence class.



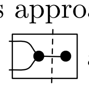
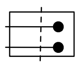
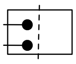
Note that this interpretation would not work the same way for spans of non-injective functions, as their graphical representation in $\mathsf{F}^{op} \mathbin{;} \mathsf{F}$ may involve  and  — more on this in Remark 4.11.

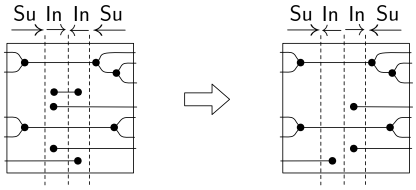
As explained above, Theorem 4.2 will follow from the following two lemmas.

Lemma 4.4 *The top face of* (5) *is a pushout.*

Proof The PROP $\mathsf{Cu} \mathbin{;} \mathsf{Un}$ is defined as in Example 3.3, by pullback in In, whence it is presented as the quotient of $\mathsf{Un} + \mathsf{Cu}$ by (B4). Therefore, by definition, the SMT of IFr consists of the SMTs for Fr and $\mathsf{Cu} \mathbin{;} \mathsf{Un}$, modulo the identification of generators and equations of $\mathsf{Un} + \mathsf{Cu}$. This is the situation described by the fibered sum operation of § 3, which implies the statement of the lemma. □

Lemma 4.5 *The bottom face of* (5) *is a pushout.*

We will get to the proof of Lemma 4.5 in steps. First, we need an understanding of when two cospans are identified by Π. (4) gives us a lead: two cospans represent the same equivalence relation precisely when they are the same modulo (B4). Now, since (B4) arises by a distributive law $\mathsf{F} \mathbin{;} \mathsf{F}^{op} \to \mathsf{F}^{op} \mathbin{;} \mathsf{F}$ defined by pullback in F (Example 3.1(b)), one could be tempted of claiming that Π identifies two cospans precisely when they have the same pullback. However, this approach identifies too much. A counterexample is given by cospans represented by  and , which have the same pullback  but express different partitions of $\overline{2}$. The correct approach is subtler: since we only need to rewrite $\boxed{\bullet\!-\!\bullet}$ as $\boxed{}$, it suffices to pull back the region of the cospan where all sub-diagrams of shape $\boxed{\bullet\!-\!\bullet}$ lie. Formally, we decompose a cospan $\xrightarrow{\in \mathsf{F}}\xleftarrow{\in \mathsf{F}}$ as $\xrightarrow{\in \mathsf{Su}}\xrightarrow{\in \mathsf{In}}\xleftarrow{\in \mathsf{In}}\xleftarrow{\in \mathsf{Su}}$ using the factorisation $\mathsf{F} \cong \mathsf{Su}\,;\mathsf{In}$ (Example 3.1(a)), and then pull back the middle cospan $\xrightarrow{\in \mathsf{In}}\xleftarrow{\in \mathsf{In}}$. This removes all sub-diagrams of shape $\boxed{\bullet\!-\!\bullet}$, as in the following riproposition of (4).



(7)

We crystallise our approach with the following definition.

Definition 4.6 We say that two cospans $n \xrightarrow{p_1 \in \mathsf{F}} z \xleftarrow{q_1 \in \mathsf{F}} m$ and $n \xrightarrow{p_2 \in \mathsf{F}} r \xleftarrow{q_2 \in \mathsf{F}} m$ are *equal modulo-zeros* if there is an epi-mono factorisation $\xrightarrow{e^1_p \in \mathsf{Su}}\xrightarrow{m^1_p \in \mathsf{In}}\xleftarrow{m^1_q \in \mathsf{In}}\xleftarrow{e^1_q \in \mathsf{Su}}$ of $\xrightarrow{p_1}\xleftarrow{q_1}$, and one $\xrightarrow{e^2_p \in \mathsf{Su}}\xrightarrow{m^2_p \in \mathsf{In}}\xleftarrow{m^2_q \in \mathsf{In}}\xleftarrow{e^2_q \in \mathsf{Su}}$ of $\xrightarrow{p_2}\xleftarrow{q_2}$ such that $\xrightarrow{m^1_p}\xleftarrow{m^1_q}$ and $\xrightarrow{m^2_p}\xleftarrow{m^2_q}$ have the same pullback and $e^1_p = e^2_p$, $e^1_q = e^2_q$.

Remark 4.7 It may be insightful to remark that two cospans are equal modulo-zeros precisely when they are in the equivalence relation generated by

$$\left(n \xrightarrow{p} z \xleftarrow{q} m\right) \sim \left(n \xrightarrow{p} z \xrightarrow{h} z' \xleftarrow{h} z \xleftarrow{q} m\right), \text{ where } h \text{ is an injection.}$$

The idea is that $z \xrightarrow{h} z' \xleftarrow{h} z$ plays a role akin to a repeated use of equation (B4) in the diagrammatic language: it deflates the codomain of $[p, q]\colon n + m \to z$ so as to "make it surjective".

Our proof of Lemma 4.5 relies on showing that Π equalizes two cospans precisely when they are equal modulo-zeros. As a preliminary step, we need to establish some properties holding for any Γ, Δ and $\mathbb{X}$ making the following diagram commute.

$$\begin{tikzcd}
 & \mathsf{In} + \mathsf{In}^{op} \arrow[r, "{[\kappa_1, \kappa_2]}"] \arrow[dl, "{[\iota_1, \iota_2]}"'] & \mathsf{In}^{op}\,;\mathsf{In} \arrow[dl, "\Gamma"] \\
\mathsf{F}\,;\mathsf{F}^{op} \arrow[r, "\Delta"'] & \mathbb{X} &
\end{tikzcd} \tag{8}$$

Lemma 4.8 *Given a PROP $\mathbb{X}$ and a commutative diagram (8), the following hold.*

(i) If $\xrightarrow{p}\xleftarrow{q}$ is a cospan in In with pullback (in In) $\xleftarrow{f}\xrightarrow{g}$, then $\Gamma(\xleftarrow{f}\xrightarrow{g}) = \Delta(\xrightarrow{p}\xleftarrow{q})$.

(ii) If $\xleftarrow{p_1}\xrightarrow{q_1}$ and $\xleftarrow{p_2}\xrightarrow{q_2}$ are cospans in In with the same pullback then $\Delta(\xrightarrow{p_1}\xleftarrow{q_1}) = \Delta(\xrightarrow{p_2}\xleftarrow{q_2})$.

(iii) If $\xrightarrow{p_1}\xleftarrow{q_1}$ and $\xrightarrow{p_2}\xleftarrow{q_2}$ are equal modulo-zeros then $\Delta(\xrightarrow{p_1}\xleftarrow{q_1}) = \Delta(\xrightarrow{p_2}\xleftarrow{q_2})$.

(iv) If $\xleftarrow{f}\xrightarrow{g}$ is a span in In with pushout (in F) $\xrightarrow{p}\xleftarrow{q}$, then $\Gamma(\xleftarrow{f}\xrightarrow{g}) = \Delta(\xrightarrow{p}\xleftarrow{q})$.

Proof

(i) We have that $\Delta(\xrightarrow{p}\xleftarrow{q}) = \Delta(\iota_1 p\,; \iota_2 q) = \Delta\iota_1 p\,; \Delta\iota_2 q = \Gamma\kappa_1 p\,; \Gamma\kappa_2 q = \Gamma(\kappa_1 p\,; \kappa_2 q) = \Gamma(\xleftarrow{f}\xrightarrow{g})$.

(ii) Let $\xleftarrow{f}\xrightarrow{g}$ be the pullback of both $\xrightarrow{p_1}\xleftarrow{q_1}$ and $\xrightarrow{p_2}\xleftarrow{q_2}$. By (i) $\Gamma(\xleftarrow{f}\xrightarrow{g}) = \Delta(\xrightarrow{p_1}\xleftarrow{q_1})$ and $\Gamma(\xleftarrow{f}\xrightarrow{g}) = \Delta(\xrightarrow{p_2}\xleftarrow{q_2})$. The statement follows.
(iii) By assumption $n \xrightarrow{p_1} z \xleftarrow{q_1} m$ and $n \xrightarrow{p_2} r \xleftarrow{q_2} m$ have epi-mono factorisations $n \xrightarrow{e_p}\xrightarrow{m_p^1} z \xleftarrow{m_q^1}\xleftarrow{e_q} m$ and $n \xrightarrow{e_p}\xrightarrow{m_p^2} r \xleftarrow{m_q^2}\xleftarrow{e_q} m$ respectively, where $\xrightarrow{m_p^1}\xleftarrow{m_q^1}$ and $\xrightarrow{m_p^2}\xleftarrow{m_q^2}$ have the same pullback. Then:

$$\begin{aligned}\Delta(\xrightarrow{p_1}\xleftarrow{q_1}) &= \Delta(\xrightarrow{e_p}\xrightarrow{m_p^1}\xleftarrow{m_q^1}\xleftarrow{e_q}) = \Delta(\xrightarrow{e_p}\xleftarrow{id}) \mathbin{;} \Delta(\xrightarrow{m_p^1}\xleftarrow{m_q^1}) \mathbin{;} \Delta(\xrightarrow{id}\xleftarrow{e_q}) \\ &\overset{\text{(ii)}}{=} \Delta(\xrightarrow{e_p^2}\xleftarrow{id}) \mathbin{;} \Delta(\xrightarrow{m_p^2}\xleftarrow{m_q^2}) \mathbin{;} \Delta(\xrightarrow{id}\xleftarrow{e_q}) = \Delta(\xrightarrow{e_p}\xrightarrow{m_p^2}\xleftarrow{m_q^2}\xleftarrow{e_q}) = \Delta(\xrightarrow{p_2}\xleftarrow{q_2}).\end{aligned}$$

(iv) Analogous to (i). $\square$

Lemma 4.8 states that any commutative diagram (8) equalizes all cospans that are equal modulo-zeros. In our cube (5), also the converse statement holds.

Lemma 4.9 *The following are equivalent*

(a) $n \xrightarrow{p_1} z \xleftarrow{q_1} m$ *and* $n \xrightarrow{p_2} r \xleftarrow{q_2} m$ *are equal modulo zeros.*

(b) $\Pi(\xrightarrow{p_1}\xleftarrow{q_1}) = \Pi(\xrightarrow{p_2}\xleftarrow{q_2})$.

Proof Since bottom face of (5) commutes (see Lemma A.1 in the Appendix), Lemma 4.8 yield the direction (a) $\Rightarrow$ (b). For the converse direction, a routine check shows that the definition of Π enforces the two cospans to have epi-mono factorisations with the desired properties. For details, see Appendix A. $\square$

We now have all the ingredients to show that the bottom face of (5) is a pushout.

Proof of Lemma 4.5 Commutativity is given by Lemma A.1, thus it remains to show the universal property. Suppose that we have a commutative diagram as in (8). It suffices to show that there exists a PROP morphism $\Theta\colon \mathsf{ER} \to \mathbb{X}$ with $\Theta\Upsilon = \Gamma$ and $\Theta\Pi = \Delta$ – uniqueness is automatic by fullness of Π (Lemma A.2).

Given an equivalence relation $e\colon n \to m$, there exist a cospan $\xrightarrow{p}\xleftarrow{q}$ such that $\Pi(\xrightarrow{p}\xleftarrow{q}) = e$. We let $\Theta(e) = \Delta(\xrightarrow{p}\xleftarrow{q})$. This is well-defined: if $\xrightarrow{p'}\xleftarrow{q'}$ is another cospan such that $\Pi(\xrightarrow{p'}\xleftarrow{q'}) = e$ then Lemma 4.9 says that $\xrightarrow{p}\xleftarrow{q}$ and $\xrightarrow{p'}\xleftarrow{q'}$ are equal modulo-zeros and thus, by Lemma 4.8, $\Delta(\xrightarrow{p}\xleftarrow{q}) = \Delta(\xrightarrow{p'}\xleftarrow{q'})$. This argument also shows that, generally, $\Theta\Pi = \Delta$. Finally, Θ preserves composition:

$$\begin{aligned}\Theta(e \mathbin{;} e') &= \Theta(\Pi(\xrightarrow{p}\xleftarrow{q}) \mathbin{;} \Pi(\xrightarrow{p'}\xleftarrow{q'})) = \Theta(\Pi((\xrightarrow{p}\xleftarrow{q}) \mathbin{;} (\xrightarrow{p'}\xleftarrow{q'}))) \\ &= \Delta((\xrightarrow{p}\xleftarrow{q}) \mathbin{;} (\xrightarrow{p'}\xleftarrow{q'})) = \Delta(\xrightarrow{p}\xleftarrow{q}) \mathbin{;} \Delta(\xrightarrow{p'}\xleftarrow{q'}) = \Theta(e) \mathbin{;} \Theta(e').\end{aligned}$$

We conclude by showing $\Theta\Upsilon = \Gamma$: given a span $\xleftarrow{f}\xrightarrow{g}$ in In, let $\xrightarrow{p}\xleftarrow{q}$ be its pushout span in F. By Lemma 4.8.(iv), $\Gamma(\xleftarrow{f}\xrightarrow{g}) = \Delta(\xrightarrow{p}\xleftarrow{q}) = \Theta\Pi(\xrightarrow{p}\xleftarrow{q}) = \Theta\Upsilon(\xleftarrow{f}\xrightarrow{g})$. $\square$

We can now conclude the characterisation of ER.

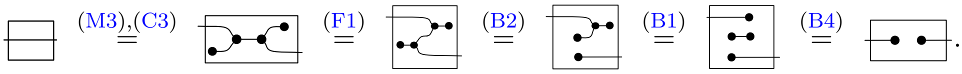
Proof of Theorem 4.2 The top and the bottom face of (5) are pushouts by Lemma 4.4 and 4.5. This yields a unique PROP morphism $\mathsf{IFr} \to \mathsf{ER}$ making the diagram commute. Since the other vertical arrows in (5) are isomorphisms, then $\mathsf{IFr} \to \mathsf{ER}$ is also an isomorphism. □

Remark 4.10 As hinted by the rightmost diagram in (7), one can give an alternative characterisation of ER as the composite PROP $\mathsf{Su} \, ; \mathsf{In}^{op} \, ; \mathsf{In} \, ; \mathsf{Su}$. This would rely on defining the appropriate distributive laws and combine them using Proposition 3.2: the resulting equations are precisely those of IFr. Then, showing that factorised arrows of $\mathsf{Su} \, ; \mathsf{In}^{op} \, ; \mathsf{In} \, ; \mathsf{Su}$ are in bijective correspondence with equivalence relations in ER completes the proof that $\mathsf{IFr} \cong \mathsf{ER}$. In our exposition we preferred to use the "cube" construction (5), as it applies also to linear and partial functions (*cf.* § 6). Also, it yields the isomorphism $\mathsf{IFr} \cong \mathsf{ER}$ as a universal arrow.

Remark 4.11 Our construction merges the theory of cospans of functions and of spans of injective functions to form the theory of equivalence relations. One may wonder what happens with a more symmetric approach, namely if we consider spans of arbitrary functions. Mimicking the cube construction (5) would result in the following diagram in **PROP**, where the top and the bottom face are pushouts.

$$\begin{tikzcd}
& \mathsf{Mn}+\mathsf{Cm} \arrow[dl] \arrow[rr, hook] \arrow[dd, "\cong"'] & & \mathsf{B} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{Fr} \arrow[rr] \arrow[dd, "\cong"'] & & \mathcal{T} \arrow[dd, dotted] & \\
& \mathsf{F}+\mathsf{F}^{op} \arrow[dl] \arrow[rr] & & \mathsf{F}^{op} \, ; \mathsf{F} \arrow[dl] \\
\mathsf{F} \, ; \mathsf{F}^{op} \arrow[rr] & & \cdot &
\end{tikzcd} \tag{9}$$

The SMT for $\mathcal{T}$ includes the SMTs for Fr and B, allowing us to prove



This derivation trivialises the theory, as it implies that any two arrows of the same type are equal. Thus $\mathcal{T}$, as well as the pushout object of the bottom face in (9), is the *terminal object* in **PROP**: for any PROP S there is a unique morphism that maps any arrow $n \xrightarrow{\in \mathsf{S}} m$ into the unique arrow with that source and target in $\mathcal{T}$.

5 A presentation of partial equivalence relations

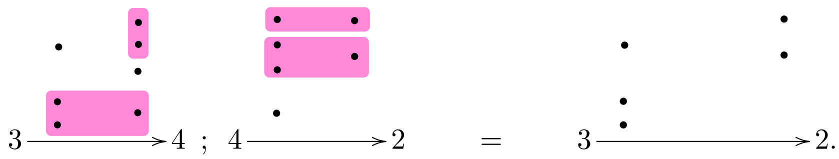
Building on the results of the previous section, we shall now characterise the PROP PER of *partial equivalence relations* (PERs) via another cube construction. In defining PER, we write $\mathrm{dom}(e)$ for the set $Y \subseteq X$ of elements on which a partial equivalence relation e on X is defined. Also, we reuse the operation $- \diamond -$ introduced in defining ER (Definition 4.1).

Definition 5.1 Let PER be the PROP with arrows $n \to m$ partial equivalence relations on $\overline{n} \uplus \overline{m}$. Given $e_1 \colon n \to z$, $e_2 \colon z \to m$, the composite $e_1 \, ; e_2$ is defined by

$$\begin{aligned}
\Omega_{(e_1,e_2)} &:= \{u \in \overline{n} \uplus \overline{m} \mid \forall w \in \overline{z}.\ (u,w) \in e_1 \diamond e_2 \Rightarrow w \in \mathrm{dom}(e_1) \cap \mathrm{dom}(e_2)\} \\
e_1 \, ; e_2 &:= e_1 \diamond e_2{}_{\restriction \Omega_{(e_1,e_2)}}.
\end{aligned}$$

The monoidal product $e_1 \oplus e_2$ is given by disjoint union.

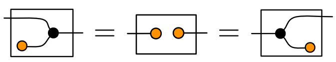
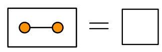
In words, composition in PER is defined as in ER, but $e_1 \, ; e_2$ is left undefined on elements that, while gluing e_1 and e_2 into $e_1 \diamond e_2$, fall into the same equivalence class as an element of $\overline{z}$ on which either e_1 or e_2 is undefined. Here is an example in which the composite $e_1 \,;\, e_2$ turns out to be everywhere undefined:



We now discuss what SMT will present PER. As we did for equivalence relations, we first establish some preliminary intuition on the diagrammatic rendition of PERs. For functions, partiality was captured graphically by incorporating an additional generator (Example 3.3). The strategy for PERs is analogous: for the elements on which a PER e is defined, the diagrammatic description is the same given for equivalence relations in (4); the elements on which e is undefined will correspond instead to ports where we plug in (if on the left) or (if on the right).

Therefore, the string diagrammatic theory for PERs will involve Fr expanded with generators , , subject to suitable compatibility conditions. This plan concretises into the PROP of "partial" special Frobenius algebras, whose definition relies on the PROP PMn discussed in Example 3.3.

Definition 5.2 The PROP PFr is defined as $\mathsf{PMn} + \mathsf{PMn}^{op}$ quotiented by equations (F1), (F2) and the following two.

 (PFR1)  (PFR2)

Intuitively, (PFR1) (together with (P1) and (P2) from PMn and their counterparts in PMn^{op}) is the algebraic rendition of the "cancellation property" that we observed in the composition of partial equivalence relations.

As a partial version of Fr, we expect PFr to characterise cospans of partial functions. To phrase this statement, note that PF is equivalently described as the coslice category $1/\mathsf{F}$ (that is, the skeletal category of pointed finite sets and functions) and thus has pushouts inherited from F. We can then form the PROP $\mathsf{PF} \,;\, \mathsf{PF}^{op}$ of cospans in PF via a distributive law $\mathsf{PF}^{op} \,;\, \mathsf{PF} \to \mathsf{PF} \,;\, \mathsf{PF}^{op}$ defined by pushout, analogously to the case of functions (Example 3.1(b)).

Proposition 5.3 $\mathsf{PFr} \cong \mathsf{PF} \,;\, \mathsf{PF}^{op}$.

Proof For soundness of PFr, one simply needs to check that (PFR1) and (PFR2) can be read off pushout squares in PF, analogously to Example 3.1(c). Conversely, completeness amounts to show that any equation that can be read off pushout squares in PF is provable in PFr. The key insight is that any such pushout can be decomposed into simpler pushout squares only involving the generators of PFr. Thus it suffices to check that the interaction of generators is covered by the axioms of PFr. We leave further details for Appendix A. $\square$

Now that we have an algebraic theory of cospans of partial functions, we can approach PERs by removing redundancy. Let us call IPFr (**i**rredundant **p**artial **Fr**obenius algebras) the quotient of PFr by (B4).

Theorem 5.4 $\mathsf{IPFr} \cong \mathsf{PER}$.

We proceed analogously to the case of equivalence relations. The isomorphism of Theorem 5.4 arises as a universal arrow in the following diagram in **PROP**, provided that the top and the bottom face are pushouts.

$$
\begin{tikzcd}
 & \mathsf{Un}+\mathsf{Cu} \arrow[dl, hook] \arrow[rr, hook] \arrow[dd, "\cong"'] & & \mathsf{Fr} \arrow[dl, hook] \arrow[rr, hook] \arrow[dd, "\cong"'] & & \mathsf{PFr} \arrow[dl, hook] \arrow[dd, "\cong"] \\
\mathsf{Cu}\,;\mathsf{Un} \arrow[rr, hook] \arrow[dd, "\cong"'] & & \mathsf{IFr} \arrow[rr, hook] \arrow[dd, "\cong"] & & \mathsf{IPFr} \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{In}+\mathsf{In}^{op} \arrow[dl, "{[\kappa_1,\kappa_2]}"'] \arrow[rr, "{[\iota_1,\iota_2]}"] & & \mathsf{F}\,;\mathsf{F}^{op} \arrow[dl, "\Pi"'] \arrow[rr, "\Lambda"] & & \mathsf{PF}\,;\mathsf{PF}^{op} \arrow[dl, "\Pi'"'] \\
\mathsf{In}^{op}\,;\mathsf{In} \arrow[rr, "\Upsilon"'] & & \mathsf{ER} \arrow[rr, "\Xi"'] & & \mathsf{PER} &
\end{tikzcd}
\tag{10}
$$

The leftmost cube is just (5). We now specify Λ, Ξ and Π'.

- For Λ, recall that there is a functor $R\colon \mathsf{PF} \to \mathsf{F}$ which maps $\overline{n}$ to $\overline{n+1}$ and $f\colon \overline{n} \to \overline{m}$ to the function $\overline{n+1} \to \overline{m+1}$ sending to $\star \in \overline{1}$ the elements on which f is undefined. Now, R has a left adjoint $L\colon \mathsf{F} \to \mathsf{PF}$: the obvious embedding of functions into partial functions. We define Λ as the embedding of $\mathsf{F}\,;\mathsf{F}^{op}$ into $\mathsf{PF}\,;\mathsf{PF}^{op}$ induced by L. This is a functorial assignment because left adjoints preserve pushouts.
- Similarly, we let Ξ be the obvious embedding of ER into PER. This assignment is functorial because composition in PER behaves as composition in ER on PERs that are totally defined.
- The PROP morphism $\Pi'\colon \mathsf{PF}\,;\mathsf{PF}^{op} \to \mathsf{PER}$ is the extension of $\Pi\colon \mathsf{F}\,;\mathsf{F}^{op} \to \mathsf{ER}$ to partial functions, defined by the same clause (6). Note that the generality of PER is necessary: the value e of Π' on a cospan $\xrightarrow{p}\xleftarrow{q}$ in PF is possibly not a reflexive relation, since p and q may be undefined on some elements of $\overline{n}$, $\overline{m}$.

Proof of Theorem 5.4 The leftmost top and bottom squares of (10) have been proven to be pushouts in Lemmas 4.4 and 4.5. The rightmost top square is readily seen to be a pushout by definition of the SMTs involved, similarly to the proof of Lemma 4.4. It thus remains to show that the rightmost bottom square is also a pushout. It clearly commutes by definition of Π, Π', Ξ and Λ. To complete the proof, because Λ is an embedding, it suffices to check that $\Xi(e) = \Pi'(\xrightarrow{p}\xleftarrow{q})$ precisely when there exist $\xrightarrow{p'}\xleftarrow{q'}$ in $\mathsf{F}\,;\mathsf{F}^{op}$ such that $e = \Pi(\xrightarrow{p'}\xleftarrow{q'})$ and $\Lambda(\xrightarrow{p'}\xleftarrow{q'}) = \xrightarrow{p}\xleftarrow{q}$. We leave the (simple) details to Appendix A.

Finally, since the top and the bottom face of (10) are pushouts and the vertical arrows are isomorphisms, the universal arrow $\mathsf{IPFr} \to \mathsf{PER}$ is also an isomorphism. $\square$

6 Conclusions

Our work combines PROPs of spans and cospans of functions to give an algebraic characterisation for PROPs of equivalence relations. What we find most striking is that the same "cube" pattern leads to similar results in the total and partial cartesian case, explored here, and in the linear case, investigated in [5]. It seems that we are scratching the surface of a more general construction, which needs some further insights to be better understood — as we saw, it collapses with spans of non-injective functions (Remark 4.11). We leave this investigation for future work.

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References

[1] Baez, J. C. and J. Erbele, *Categories in control*, CoRR **abs/1405.6881** (2014).

[2] Bonchi, F., P. Sobociński and F. Zanasi, *A categorical semantics of signal flow graphs*, in: *Concurrency Theory - 25th International Conference, CONCUR 2014. Proceedings*, 2014, pp. 435–450.

[3] Bonchi, F., P. Sobociński and F. Zanasi, *Interacting bialgebras are Frobenius*, in: *Foundations of Software Science and Computation Structures - 17th International Conference, FOSSACS 2014*, 2014, pp. 351–365.

[4] Bonchi, F., P. Sobociński and F. Zanasi, *Full abstraction for signal flow graphs*, in: *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2015*, 2015, pp. 515–526.

[5] Bonchi, F., P. Sobociński and F. Zanasi, *Interacting Hopf algebras*, Journal of Pure and Applied Algebra (2016), to appear.
URL http://arxiv.org/abs/1403.7048

[6] Bruni, R. and F. Gadducci, *Some algebraic laws for spans (and their connections with multi-relations)*, in: *RelMiS 2001* (2001).

[7] Bruni, R., H. C. Melgratti and U. Montanari, *A connector algebra for P/T nets interactions*, in: *CONCUR '11* (2011), pp. 312–326.

[8] Bruni, R., U. Montanari, G. D. Plotkin and D. Terreni, *On hierarchical graphs: Reconciling bigraphs, gs-monoidal theories and gs-graphs*, Fundam. Inform. **134** (2014), pp. 287–317.

[9] Carboni, A. and R. F. C. Walters, *Cartesian bicategories I*, J Pure Appl Algebra **49** (1987), pp. 11–32.

[10] Cheng, E., *Iterated distributive laws.*, Math. Proc. Camb. Philos. Soc. **150** (2011), pp. 459–487.

[11] Coecke, B. and R. Duncan, *Interacting quantum observables: categorical algebra and diagrammatics*, New Journal of Physics **13** (2011), p. 043016.

[12] Coya, B. and B. Fong, *Corelations are the prop for extraspecial commutative Frobenius monoids*, CoRR **abs/1601.02307** (2016).

[13] Došen, K. and Z. Petri, *Syntax for split preorders*, Annals of Pure and Applied Logic **164** (2013), pp. 443–481.

[14] Duncan, R. and K. Dunne, *Interacting Frobenius algebras are Hopf*, CoRR **abs/1601.04964** (2016).

[15] Hasuo, I. and N. Hoshino, *Semantics of higher-order quantum computation via geometry of interaction*, in: *Proceedings of the 26th IEEE Symposium on Logic in Computer Science, LICS*, 2011, pp. 237–246.

[16] Heunen, C. and J. Vicary, *Lectures on categorical quantum mechanics* (2012).
URL www.cs.ox.ac.uk/files/4551/cqm-notes.pdf

[17] Jacobs, B., "Categorical Logic and Type Theory," Number 141 in Studies in Logic and the Foundations of Mathematics, North Holland, Amsterdam, 1999.

[18] Jacobs, B. and J. Mandemaker, *Coreflections in algebraic quantum logic*, Foundations of Physics **42** (2012), pp. 932–958.

[19] Lack, S., *Composing PROPs*, Theor App Categories **13** (2004), pp. 147–163.

[20] Lafont, Y., *Equational reasoning with 2-dimensional diagrams*, in: H. Comon and J.-P. Jounnaud, editors, *Term Rewriting*, Lecture Notes in Computer Science **909**, Springer Berlin Heidelberg, 1995 pp. 170–195.

[21] Mac Lane, S., *Categorical algebra*, B Am Math Soc **71** (1965), pp. 40–106.

[22] Mac Lane, S., “Categories for the Working Mathematician,” Springer, 1998.

[23] Rosebrugh, R., N. Sabadini and R. Walters, *Generic commutative separable algebras and cospans of graphs*, Theory and applications of categories **15** (2005), pp. 164–177.

[24] Scott, D., *Data types as lattices*, in: G. H. Müller, A. Oberschelp and K. Potthoff, editors, *Proceedings of the International Summer Institute and Logic Colloquium, Kiel 1974* (1975), pp. 579–651.

[25] Selinger, P., *A survey of graphical languages for monoidal categories*, Springer Lecture Notes in Physics **13** (2011), pp. 289–355.

[26] Sobociński, P., *Nets, relations and linking diagrams*, in: *Algebra and Coalgebra in Computer Science - 5th International Conference, CALCO 2013*, 2013, pp. 282–298.

[27] Street, R., *The formal theory of monads*, J Pure Appl Algebra **2** (2002), pp. 243–265.

[28] Streicher, T., “Semantics of type theory : correctness, completeness, and independence results,” Progress in theoretical computer science, Birkhäuser, Boston, 1991.

[29] Zanasi, F., “Interacting Hopf Algebras: the theory of linear systems,” Ph.D. thesis, Ecole Normale Supérieure de Lyon (2015).

A Omitted Proofs

The following lemma is used in § 4.

Lemma A.1 *The bottom face of* (5) *commutes.*

Proof It suffices to show that it commutes on the two injections into $\mathsf{In}+\mathsf{In}^{op}$, that means, for any $f\colon n \to m$ in In, $\Upsilon(\xleftarrow{id}\xrightarrow{f}) = \Pi(\xrightarrow{f}\xleftarrow{id})$ and $\Upsilon(\xleftarrow{f}\xrightarrow{id}) = \Pi(\xrightarrow{id}\xleftarrow{f})$. These statements are clearly symmetric, so it is enough to check one:

$$\begin{array}{rcl} \Upsilon(\xleftarrow{id}\xrightarrow{f}) & = & \{(v,w) \mid v = f^{-1}(w) \vee w = f^{-1}(v) \vee v = w\} \\ & = & \{(v,w) \mid f(v) = w \vee f(w) = v \vee v = w\} \\ & \overset{f \in \mathsf{In}}{=} & \{(v,w) \mid f(v) = w \vee f(w) = v \vee f(v) = f(w)\} = \Pi(\xrightarrow{f}\xleftarrow{id}). \end{array}$$

□

Proof of Lemma 4.9 We complete the proof in the main text by showing that (b) ⇒ (a). For this purpose, it is useful to first verify the following properties:

(i) for all $u, u' \in \overline{n}$, $p_1(u) = p_1(u')$ if and only if $p_2(u) = p_2(u')$

(ii) for all $v, v' \in \overline{m}$, $q_1(v) = q_1(v')$ if and only if $q_2(v) = q_2(v')$

(iii) for all $u \in \overline{n}$, $v \in \overline{m}$, $p_1(u) = q_1(v)$ if and only if $p_2(u) = q_2(v)$

(iv) Let $p_1[\overline{n}]$ be the number of elements of $\overline{n}$ that are in the image of p_1, and similarly for $p_2[\overline{n}]$. Then $p_1[\overline{n}] = p_2[\overline{n}]$.

(v) $q_1[\overline{n}] = q_2[\overline{n}]$.

For statement (i), observe that, by definition of Π, for any two elements $u, u' \in \overline{n}$ the pair (u,u') is in $\Pi(\xrightarrow{p_1}\xleftarrow{q_1})$ if and only if $p_1(u) = p_1(u')$. Similarly, $(u,u') \in \Pi(\xrightarrow{p_2}\xleftarrow{q_2})$ if and only if $p_2(u) = p_2(u')$. Since by assumption $\Pi(\xrightarrow{p_1}\xleftarrow{q_1}) = \Pi(\xrightarrow{p_2}\xleftarrow{q_2})$, we obtain (i). A symmetric reasoning yields (ii). The argument for statement (iii) is analogous: for $i \in \{1,2\}$ and $u \in \overline{n}$, $v \in \overline{m}$, by definition of Π, $(u,v) \in \Pi(\xrightarrow{p_i}\xleftarrow{q_i})$ if and only if $p_i(u) = q_i(v)$. Since $\Pi(\xrightarrow{p_1}\xleftarrow{q_1}) = \Pi(\xrightarrow{p_2}\xleftarrow{q_2})$, we obtain (iii). Statement (iv) is an immediate consequence of (i), and (v) of (ii).

Now, by virtue of properties (i)-(v), it should be clear that we can define epi-mono factorisations $n \xrightarrow{e_p^1}\xrightarrow{m_p^1} z \xleftarrow{m_q^1}\xleftarrow{e_q^1} m$ and $n \xrightarrow{e_p^2}\xrightarrow{m_p^2} r \xleftarrow{m_q^2}\xleftarrow{e_q^2} m$ of $n \xrightarrow{p_1} z \xleftarrow{q_1} m$ and $n \xrightarrow{p_2} r \xleftarrow{q_2} m$ respectively, with the following properties.

(vi) e_p^1 and e_p^2 are the same function, with source n and target $p_1[\overline{n}] = p_2[\overline{n}]$. Also e_q^1 and e_q^2 are the same function, with source m and target $q_1[\overline{m}] = q_2[\overline{m}]$.

(vii) For all $u \in p_1[\overline{n}] = p_2[\overline{n}]$ and $v \in q_1[\overline{n}] = q_2[\overline{n}]$, $m_p^1(u) = m_q^1(v)$ iff $m_p^2(u) = m_q^2(v)$.

It remains to prove that $\xrightarrow{m_p^1}\xleftarrow{m_q^1}$ and $\xrightarrow{m_p^2}\xleftarrow{m_q^2}$ have the same pullback. For this purpose, let the following be pullback squares in In:

$$\begin{tikzcd} h_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] & q_1[\overline{n}] \arrow[d, "m_q^1"] \\ p_1[\overline{n}] \arrow[r, "m_p^1"'] & z \end{tikzcd} \qquad \begin{tikzcd} h_2 \arrow[r, "f_2"] \arrow[d, "g_2"'] & q_1[\overline{n}] \arrow[d, "m_q^2"] \\ p_1[\overline{n}] \arrow[r, "m_p^2"'] & r \end{tikzcd}$$

By the way pullbacks are computed in In (i.e., in F), using (vii) we can conclude that $m^1_p g_2 = m^1_q f_2$ and $m^2_p g_1 = m^2_q f_1$. By universal property of pullbacks, this implies that the spans $\xleftarrow{g_1}\xrightarrow{f_1}$ and $\xleftarrow{g_2}\xrightarrow{f_2}$ are isomorphic. □

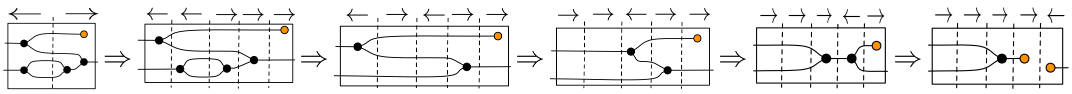
The following observation is used in the proof of Lemma 4.5.

Lemma A.2 $\Pi \colon \mathsf{F}\,;\mathsf{F}^{op} \to \mathsf{ER}$ *is full.*

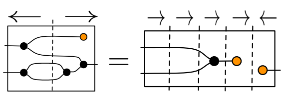
Proof Let $c_1, \ldots, c_k$ be the equivalence classes of an equivalence relation e on $\overline{n} \uplus \overline{m}$. We define a cospan $n \xrightarrow{p} k \xleftarrow{q} m$ by letting p map $v \in \overline{n}$ to the equivalence class c_i to which v belongs, and symmetrically for q on values $w \in \overline{m}$. It is routine to check that $\Pi(\xrightarrow{p}\xleftarrow{q}) = e$. □

Next, we give more details on the proof of Proposition 5.3. The hard part is to check that the equation associated with any pushout diagram in PF is provable by the equations of PFr. The key observation is that we can confine ourselves to just pushouts involving the generators of PMn.

Before making this formal, we illustrate the idea of the argument with the following example. The leftmost diagram below is a diagram representing a span $\xleftarrow{f}\xrightarrow{g}$ (left), which we transform into a cospan (right) pushing out $\xleftarrow{f}\xrightarrow{g}$, only using equations of PFr.



The steps are as follows. First, we expand $\xleftarrow{f}$ and $\xrightarrow{g}$ as $\xleftarrow{f_1}\xleftarrow{f_2}$ and $\xrightarrow{g_1}\xrightarrow{g_2}\xrightarrow{g_3}$ respectively, in such a way that each f_i and g_i contains at most one generator of PF and PF^{op}. In the next steps, we proceed pushing out spans $\xleftarrow{f_i}\xrightarrow{g_j}$ whenever possible: graphically, this amounts to apply valid equations of PFr of a very simple kind, namely those describing the interaction of a single (or no) generator of PF^{op} with one (or none) of PF. Note that pushing out spans of this form always gives back a cospan $\xrightarrow{p}\xleftarrow{q}$ with p, q containing at most one generator, meaning that the procedure can be applied again until no more spans appear. The resulting diagram (the rightmost above) is the pushout of the leftmost one by pasting properties of pushouts. Therefore, we just proved that the equation



arising by the distributive law $\mathsf{PF}^{op}\,;\mathsf{PF} \to \mathsf{PF}\,;\mathsf{PF}^{op}$ is provable in PFr.

We now formalise the argument sketched above. Let us call *atom* any diagram of PMn of shape $\xrightarrow{f}\xrightarrow{b}\xrightarrow{g}$, where f and g consist of components and composed together via $\oplus$ or $;$, and b is either or a generator of PMn. The following lemma establishes that PFr is complete for pushouts involving atoms.

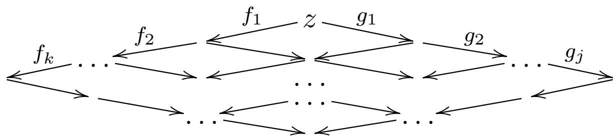
Lemma A.3 *Let* $\xleftarrow{f}\xrightarrow{g}$ *be a span in* PF *where* f *and* g *are in the image (under the isomorphism* $\mathsf{PMn} \cong \mathsf{PF}$*) of atoms and suppose that the following is a pushout* *square.*

$$\begin{tikzcd}[row sep=small] & r \arrow[dl, "f"'] \arrow[dr, "g"] & \\ m \arrow[dr, "p"'] & & n \arrow[dl, "q"] \\ & z & \end{tikzcd} \tag{A.1}$$

Then (i) p and q are also in the image of atoms and (ii) the associated equation is provable in PFr.

Proof The two points are proved by case analysis on all the possible choices of generators of PMn and $(\mathsf{PMn})^{op}$. □

Proof of Proposition 5.3 Fix any pushout square (A.1) in PF and pick expansions $f = f_1 \,;\, \dots \,;\, f_k$ and $g = g_1 \,;\, \dots \,;\, g_j$, with each f_i and g_i in the image of an atom. We can calculate the pushout above by tiling pushouts of atoms as follows:



(A.2)

Point (i) of Lemma A.3 guarantees that each inner square only involves arrows in the image of some atom and Point (ii) ensures that all the associated equations are provable in PFr. It follows that also the equation associated with the outer pushout (A.2) is provable. □

We complete the proof sketch of Theorem 5.4 given in the main text. The following is the key lemma.

Lemma A.4 *Let $e \in \mathsf{ER}[n,m]$ and $\xrightarrow{p}\xleftarrow{q} \in \mathsf{PF}\,;\,\mathsf{PF}^{op}$. The following are equivalent.*

(i) $\Xi(e) = \Pi'(\xrightarrow{p}\xleftarrow{q})$.

(ii) There are cospans $\xrightarrow{p_1}\xleftarrow{q_1}, \dots, \xrightarrow{p_k}\xleftarrow{q_k}$ *in* $\mathsf{F}\,;\,\mathsf{F}^{op}[n,m]$ *such that*

$$\begin{aligned} e &= \Pi(\xrightarrow{p_1}\xleftarrow{q_1}) \\ \Lambda(\xrightarrow{p_1}\xleftarrow{q_1}) &= \Lambda(\xrightarrow{p_2}\xleftarrow{q_2}) \\ \Pi(\xrightarrow{p_2}\xleftarrow{q_2}) &= \Pi(\xrightarrow{p_3}\xleftarrow{q_3}) \\ &\cdots\cdots \\ \Lambda(\xrightarrow{p_k}\xleftarrow{q_k}) &= \xrightarrow{p}\xleftarrow{q} . \end{aligned}$$

Proof First we observe that, because Λ is an embedding, $\Lambda(\xrightarrow{p_i}\xleftarrow{q_i}) = \Lambda(\xrightarrow{p_{i+1}}\xleftarrow{q_{i+1}})$ implies $\xrightarrow{p_i}\xleftarrow{q_i} = \xrightarrow{p_{i+1}}\xleftarrow{q_{i+1}}$. It follows that (ii) is equivalent to the statement that (iii) there exist $\xrightarrow{p'}\xleftarrow{q'} \in \mathsf{F}\,;\,\mathsf{F}^{op}[n,m]$ such that $e = \Pi(\xrightarrow{p'}\xleftarrow{q'})$ and $\Lambda(\xrightarrow{p'}\xleftarrow{q'}) = \xrightarrow{p}\xleftarrow{q}$.

It is very easy to show that (iii) implies (i):

$$\Xi(e) \overset{\text{(iii)}}{=} \Xi\Pi(\xrightarrow{p'}\xleftarrow{q'}) \overset{\text{comm. of (10)}}{=} \Pi'\Lambda(\xrightarrow{p'}\xleftarrow{q'}) \overset{\text{(iii)}}{=} \Pi'(\xrightarrow{p}\xleftarrow{q}).$$

For the converse direction, suppose that we can show (*) the existence of $\xrightarrow{p'}\xleftarrow{q'} \in \mathsf{F}\,;\,\mathsf{F}^{op}[n,m]$ such that $\xrightarrow{p'}\xleftarrow{q'} = \Lambda(\xrightarrow{p}\xleftarrow{q})$. Then the following derivation gives statement (iii):

$$\Xi(e) \overset{\text{(i)}}{=} \Pi'(\xrightarrow{p}\xleftarrow{q}) \overset{(*)}{=} \Pi'\Lambda(\xrightarrow{p'}\xleftarrow{q'}) \overset{\text{comm. of (10)}}{=} \Xi\Pi(\xrightarrow{p'}\xleftarrow{q'}).$$

Indeed, because Ξ is an embedding, the derivation above implies that $e = \Pi(\xrightarrow{p'}\xleftarrow{q'})$. Therefore it suffices to show (*). For this purpose, we just need to prove that both $n \xrightarrow{p\in\mathsf{PF}} z$ and $m \xrightarrow{q\in\mathsf{PF}} z$ are *total* functions. Let u be an element of n: since $\Pi'(\xrightarrow{p}\xleftarrow{q}) = \Xi(e)$ and Ξ embeds equivalence relations into PERs, then $\Pi'(\xrightarrow{p}\xleftarrow{q})$ is in fact an equivalence relation, meaning that u belongs to some equivalence class of the partition induced by $\Pi'(\xrightarrow{p}\xleftarrow{q})$. It follows by definition of Π' that $p\colon n \to z$ is defined on u. With a similar argument, one can show that $q\colon m \to z$ is defined on all elements of m and thus both p and q are total functions. This implies that $\xrightarrow{p}\xleftarrow{q}$ is in the image of the embedding Λ. $\square$

Proof of Theorem 5.4 In order to complete the proof of the main text, it remains to show that the leftmost bottom face of (10) is a pushout. First, recall that pushouts in **PROP** can be calculated as in **Cat**. In particular, (10) involves categories all with the same objects and identity-on-objects functors. This means that the pushout object is the quotient of ER and $\mathsf{F}\,;\mathsf{F}^{op}$ along the equivalence relation generated by

$$\{(e, \xrightarrow{p}\xleftarrow{q}) \mid \text{there is } \xrightarrow{p'}\xleftarrow{q'} \text{ such that } \Pi(\xrightarrow{p'}\xleftarrow{q'}) = e \text{ and } \Lambda(\xrightarrow{p'}\xleftarrow{q'}) = \xrightarrow{p}\xleftarrow{q}\}. \quad \text{(A.3)}$$

Lemma A.4 proves that Π' and Ξ map $n \xrightarrow{e\in\mathsf{ER}} m$ and $n \xrightarrow{p}\xleftarrow{q} m$ to the same arrow exactly when they are in the equivalence relation described above. This means that PER indeed quotients by (A.3) and thus is the desired pushout object. $\square$