Rewriting Modulo Symmetric Monoidal Structure

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Joint work with

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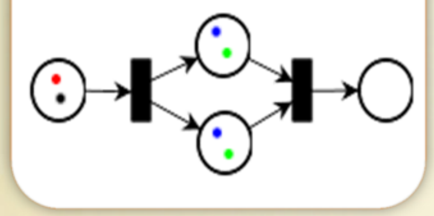
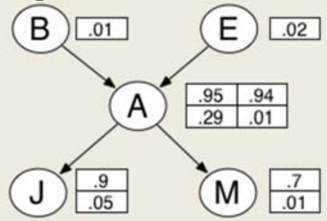
CNRS	Radboud U.	U. Pisa	U. Southampton

PROPs: algebras of network diagrams

Petri Nets

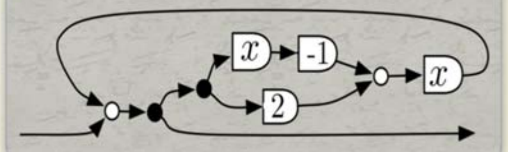


Bayesian Networks



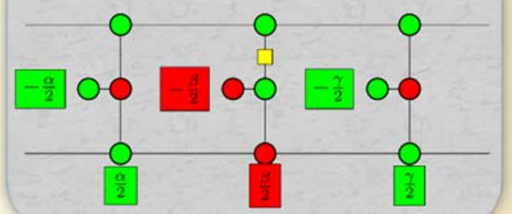
LICS'16

Signal Flow Graphs



LICS'16

Quantum Processes

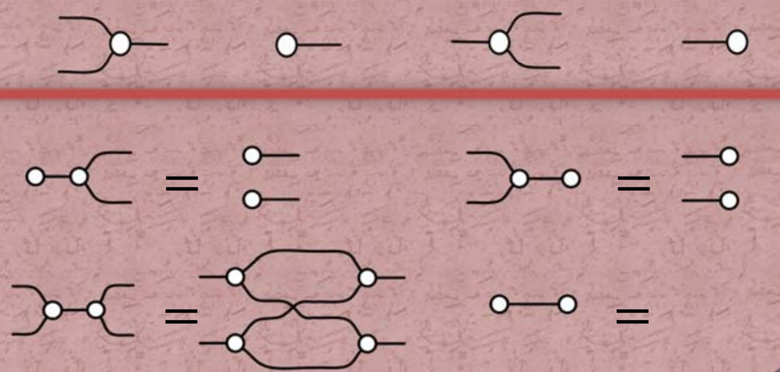


``Linear'' Lawvere theory

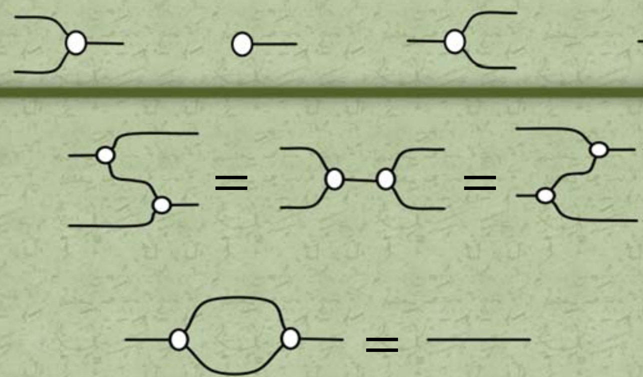
A PROP is (just) a symmetric monoidal category with set of objects $\mathbb{N}$

PROPs can be freely constructed starting from a signature Σ and equations E

PROP of Bialgebras

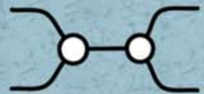
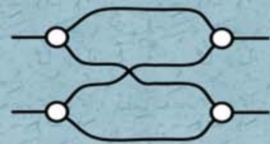


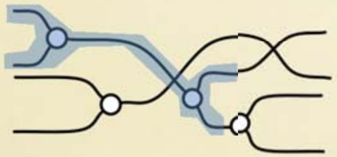
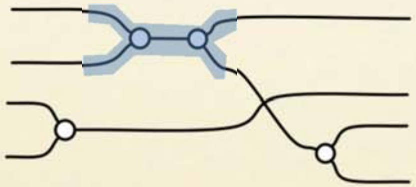
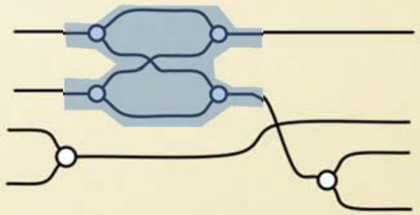
PROP of Separable Frobenius Algebras



Rewriting in a PROP

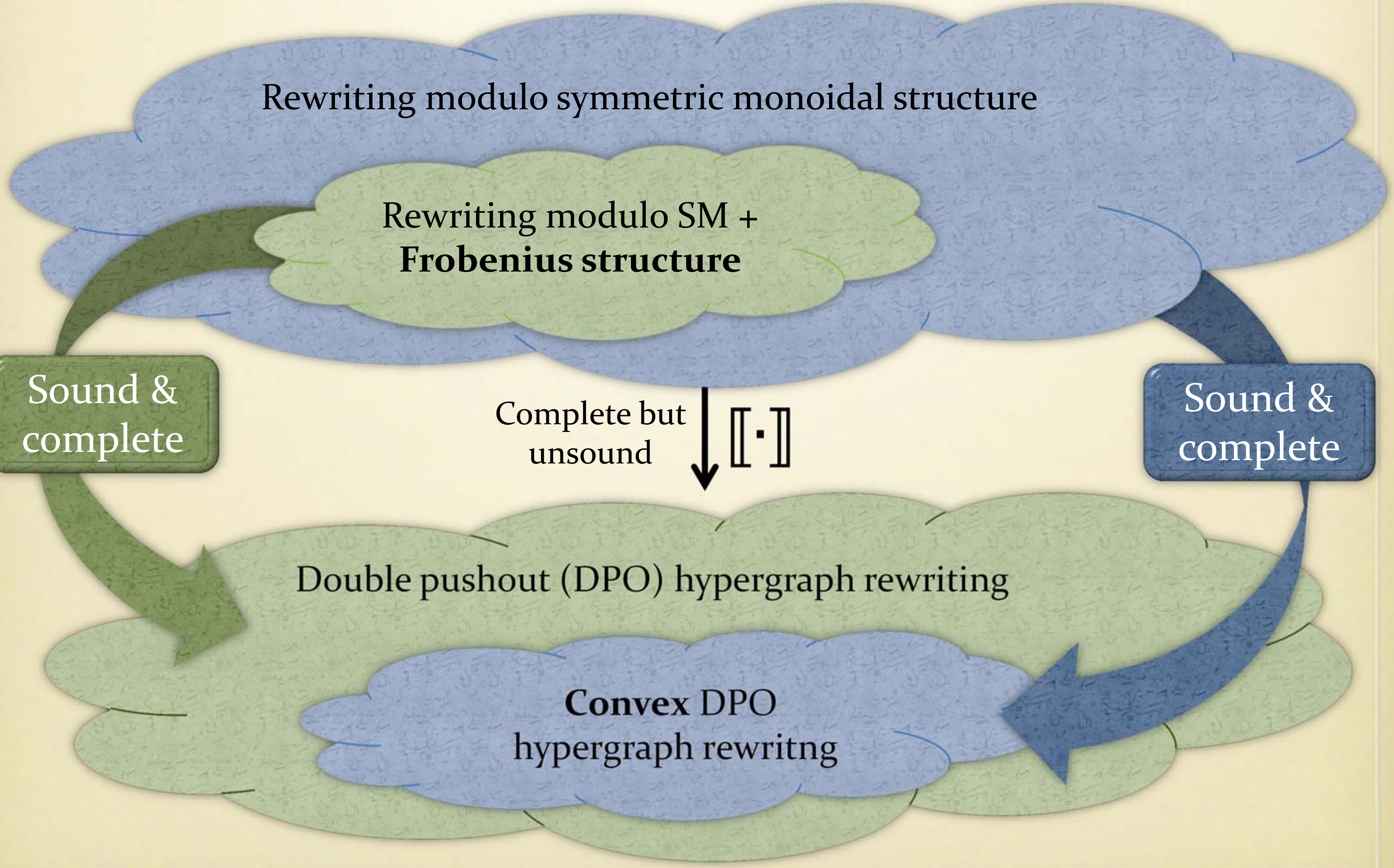
Perspective of this work:
see E as a **rewriting system** on network diagrams

(R)  $\Rightarrow$ 

 $\overset{\text{SMC}}{\approx}$  $\Rightarrow_{\text{R}}$ 

Our question
How to implement rewriting modulo symmetric monoidal structure in a simple, yet rigorous way?

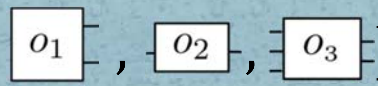
Outline

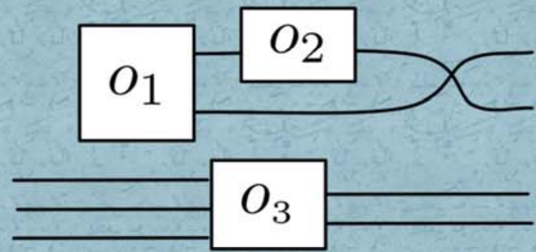

Rewriting modulo symmetric monoidal structure
Rewriting modulo SM +
Frobenius structure
Sound & complete
Complete but unsound
$[\![\cdot]\!]$
Sound & complete
Double pushout (DPO) hypergraph rewriting
Convex DPO
hypergraph rewritng

Hypergraph interpretation

PROP $Syn(\Sigma)$ of syntax

freely generated by

$\Sigma = \{$  $\}$



$\llbracket \cdot \rrbracket$

Operations from $\Sigma \sim$ hyperedges

Left/right boundary~ Cospan structure

PROP of (discrete) Cospans

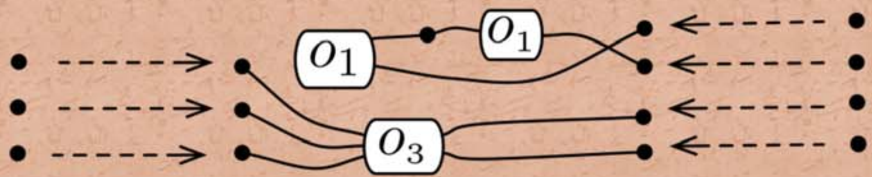
of Σ-labelled hypergraphs

$Csp(Hyp(\Sigma))$

PROP of

Σ-labelled hypergraphs

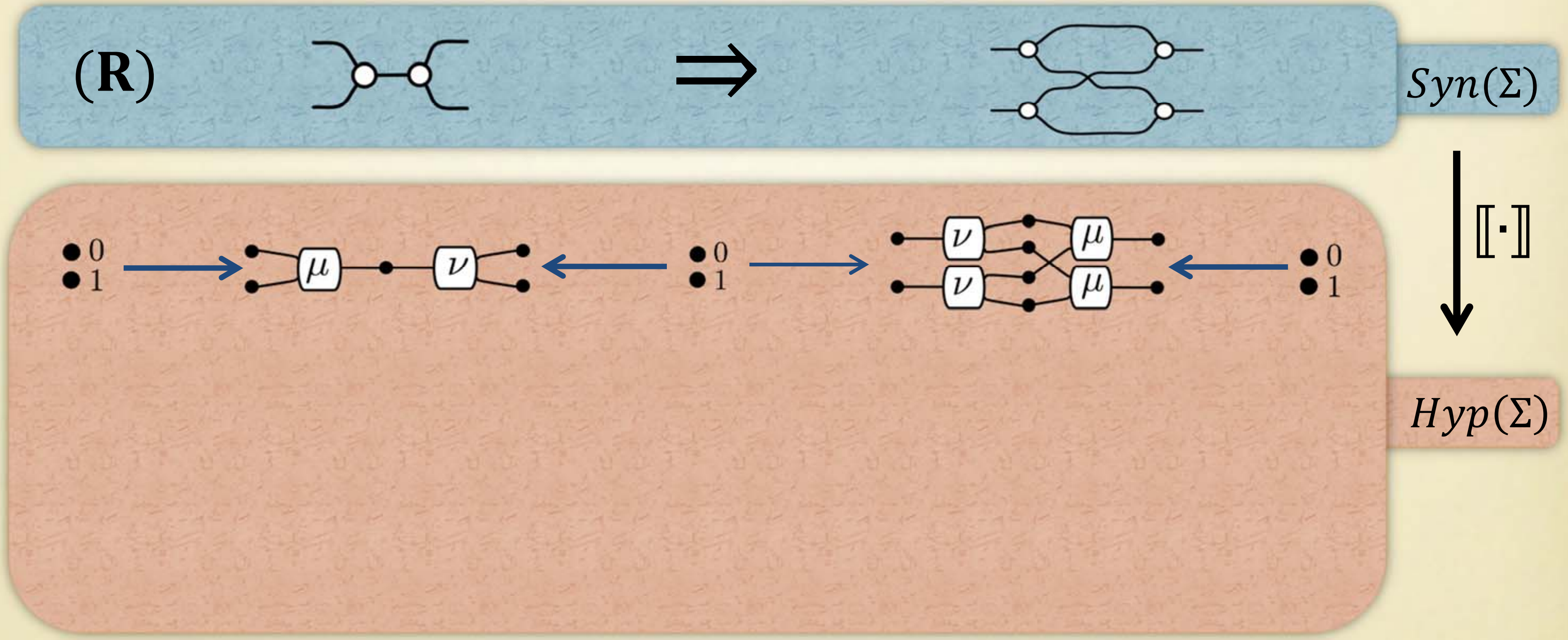
$Hyp(\Sigma)$



Proposition: $Syn(\Sigma) \xrightarrow{\llbracket \cdot \rrbracket} Csp\big(Hyp(\Sigma)\big)$ is faithful

DPO hypergraph rewriting

Hyp(Σ) is an *adhesive category* (Lack & Sobocinski) and thus adapted to double-pushout rewriting.



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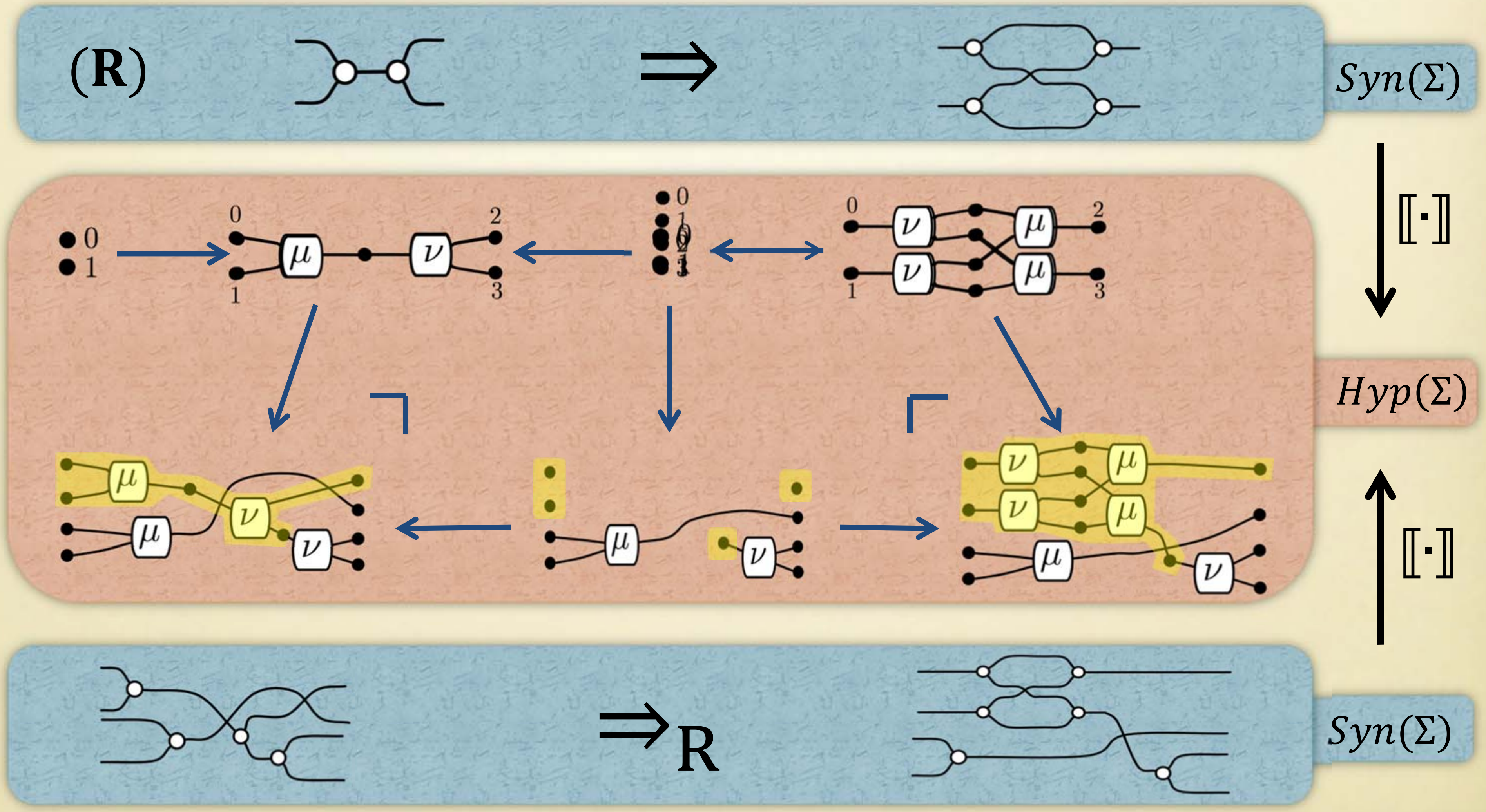
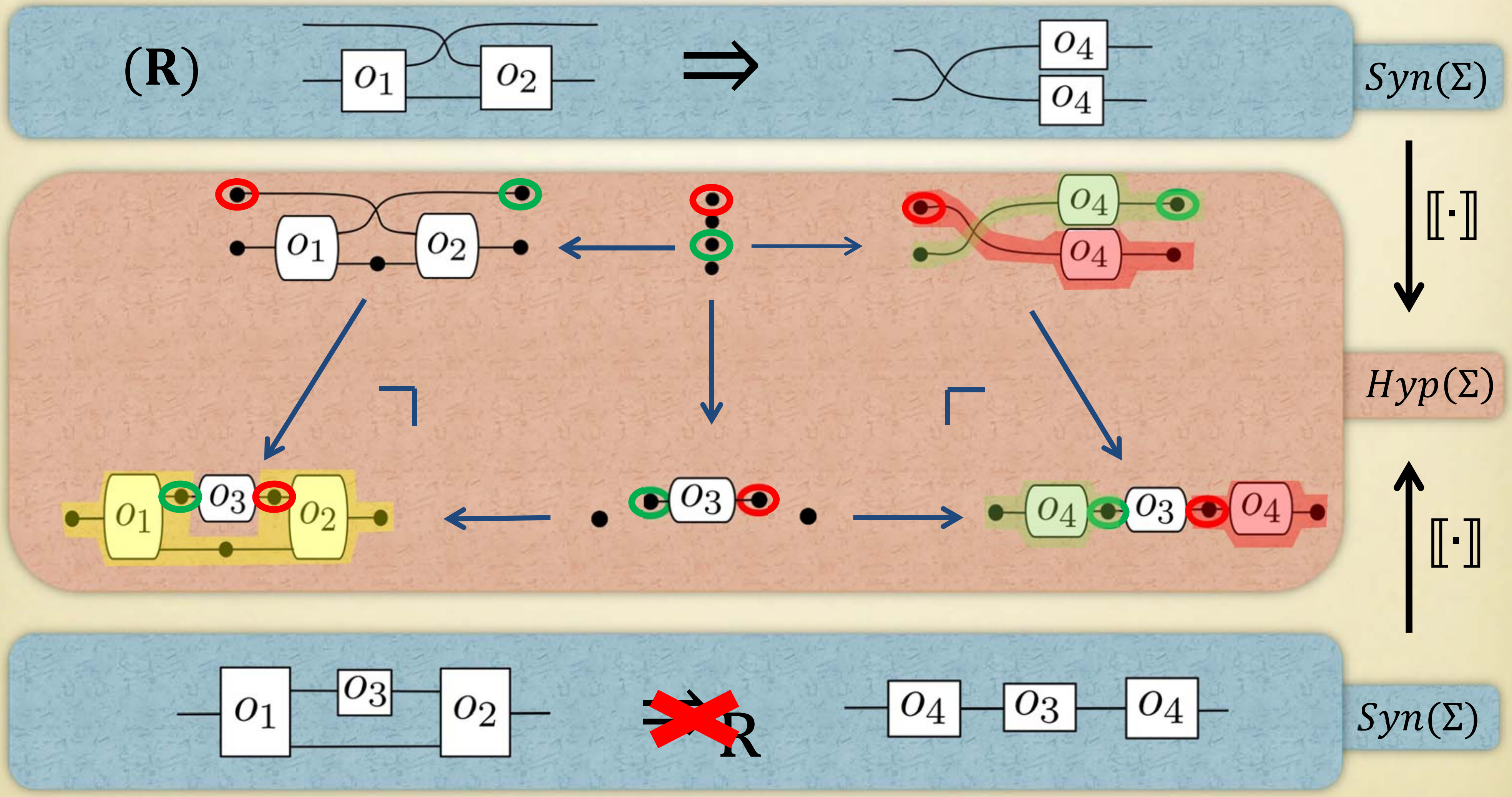


DPO rewriting is unsound

DPO hypergraph rewriting is *complete* but generally not *sound*.



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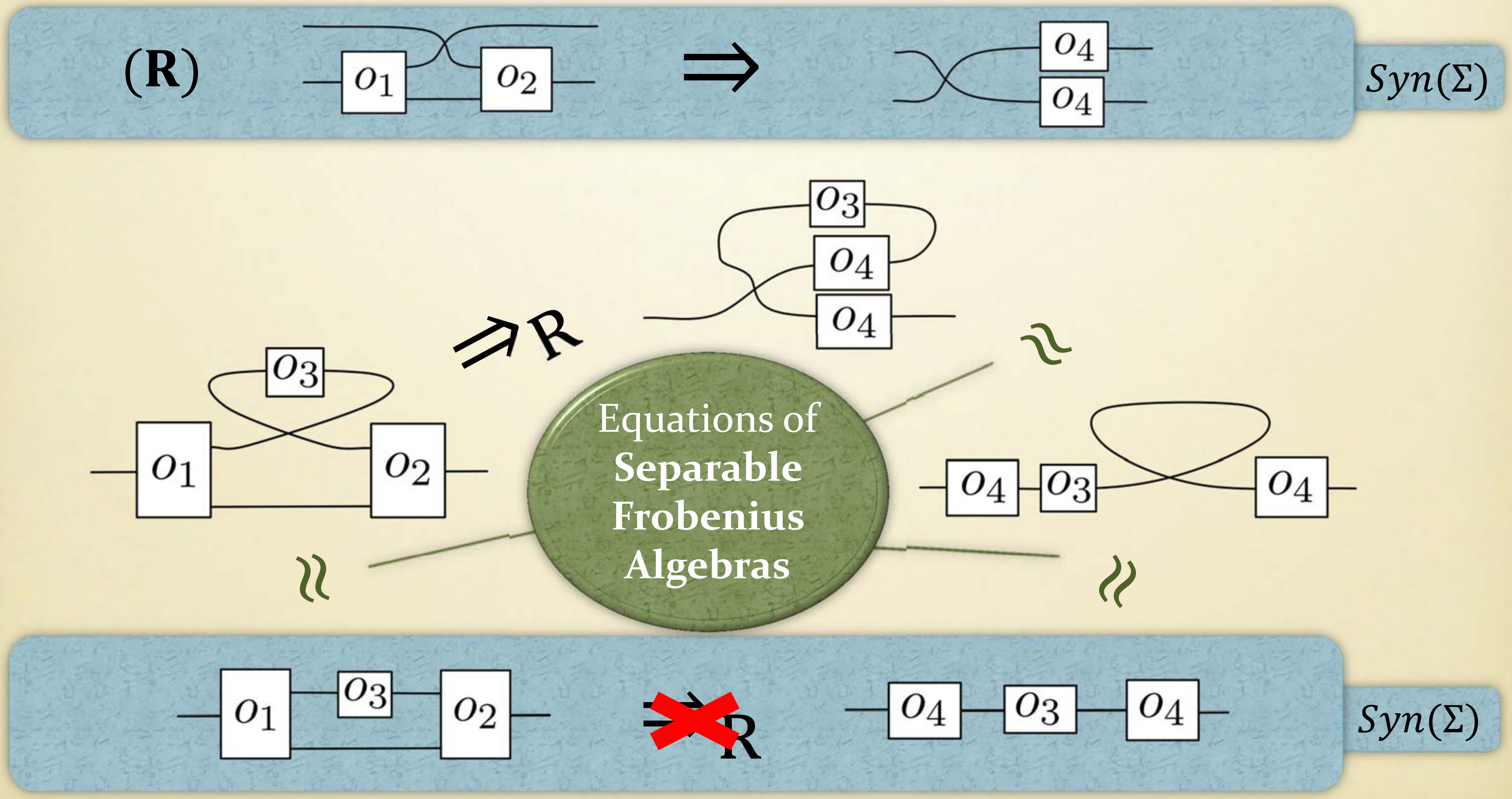
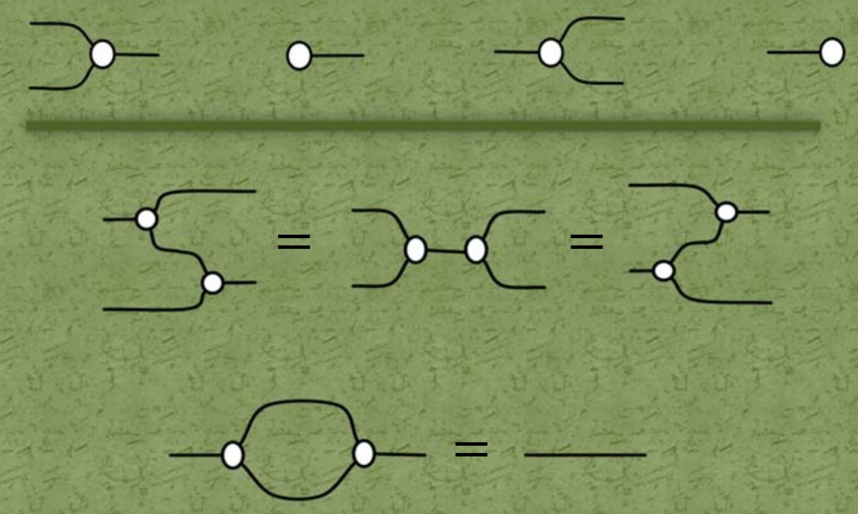


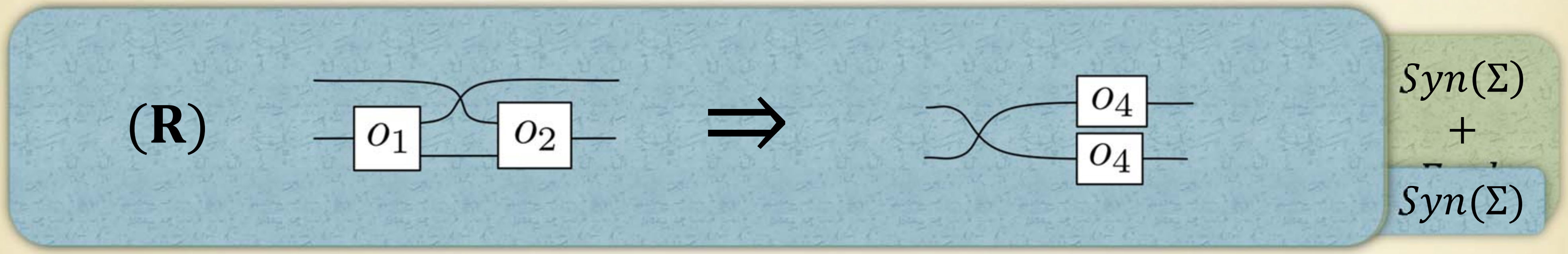
Frobenius makes DPO rewriting sound

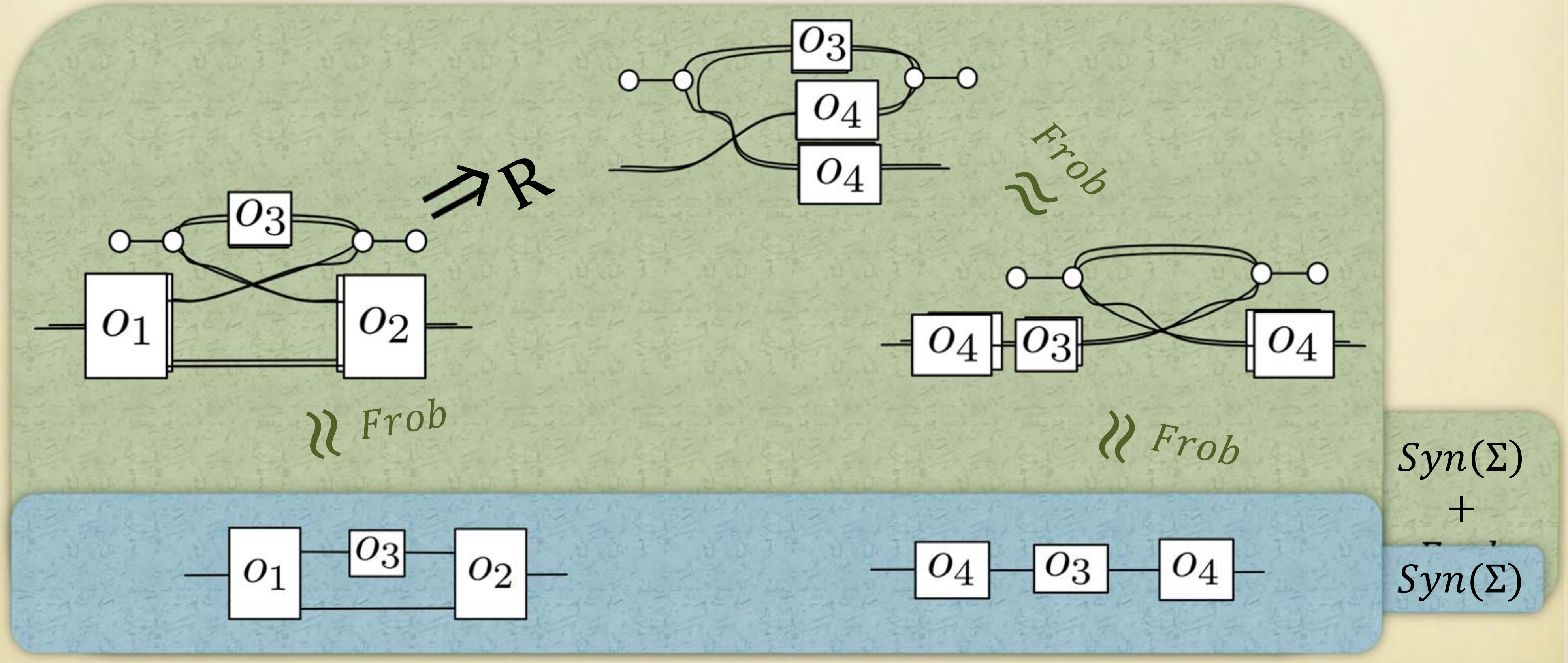
Theorem I

DPO hypergraph rewriting is *sound and complete* for symmetric monoidal categories *with a chosen separable Frobenius structure*.

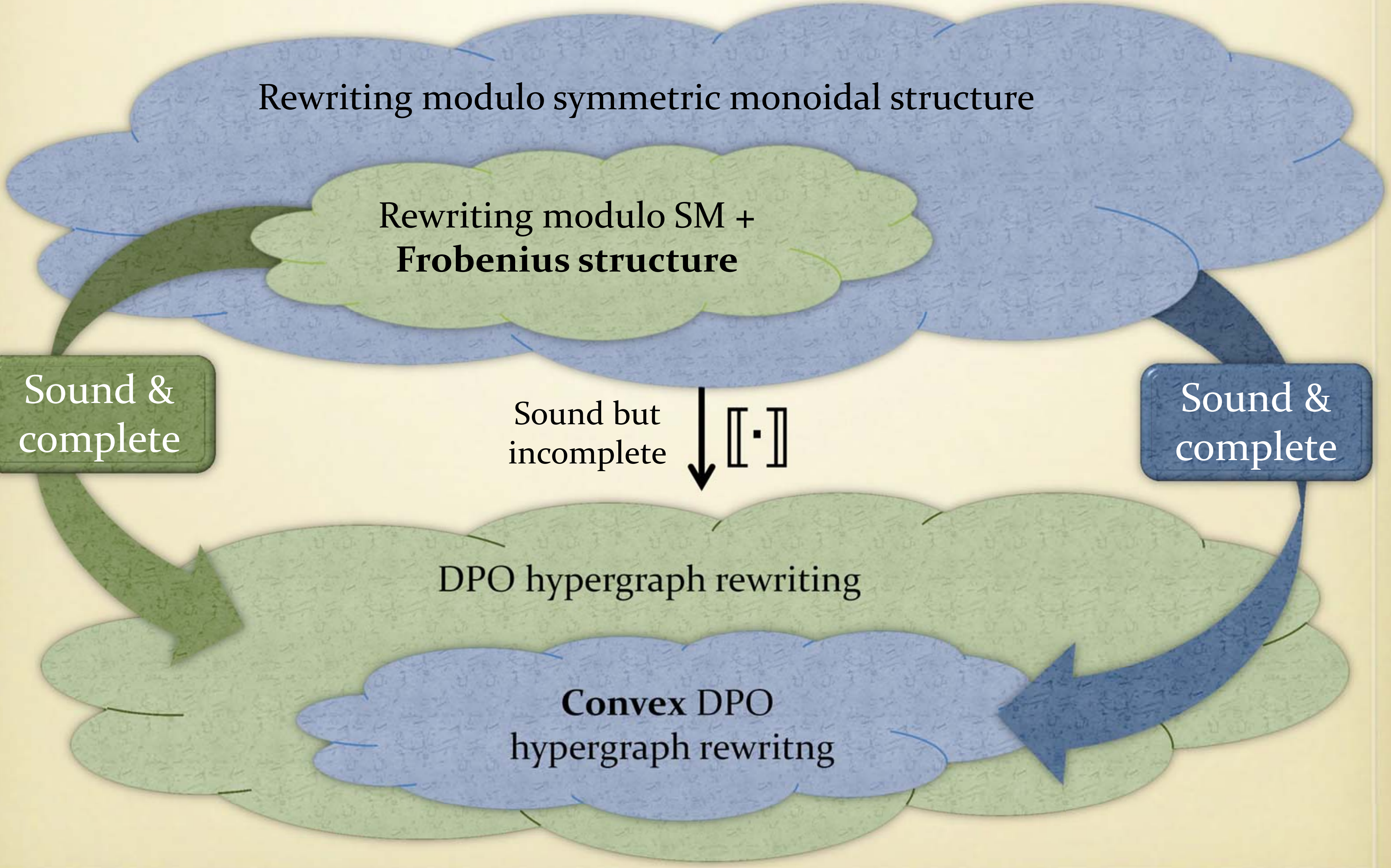


Frobenius makes DPO rewriting sound

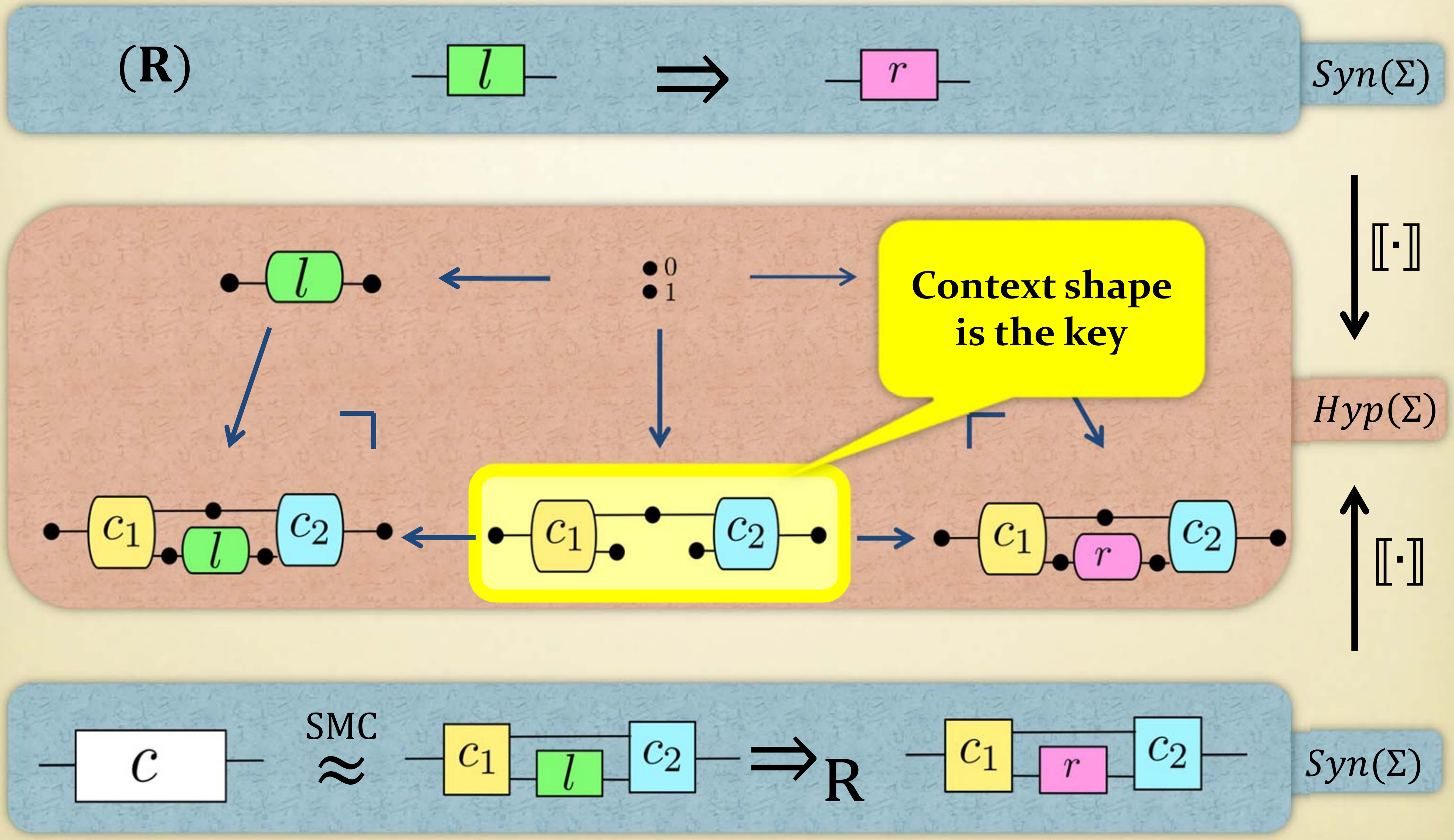




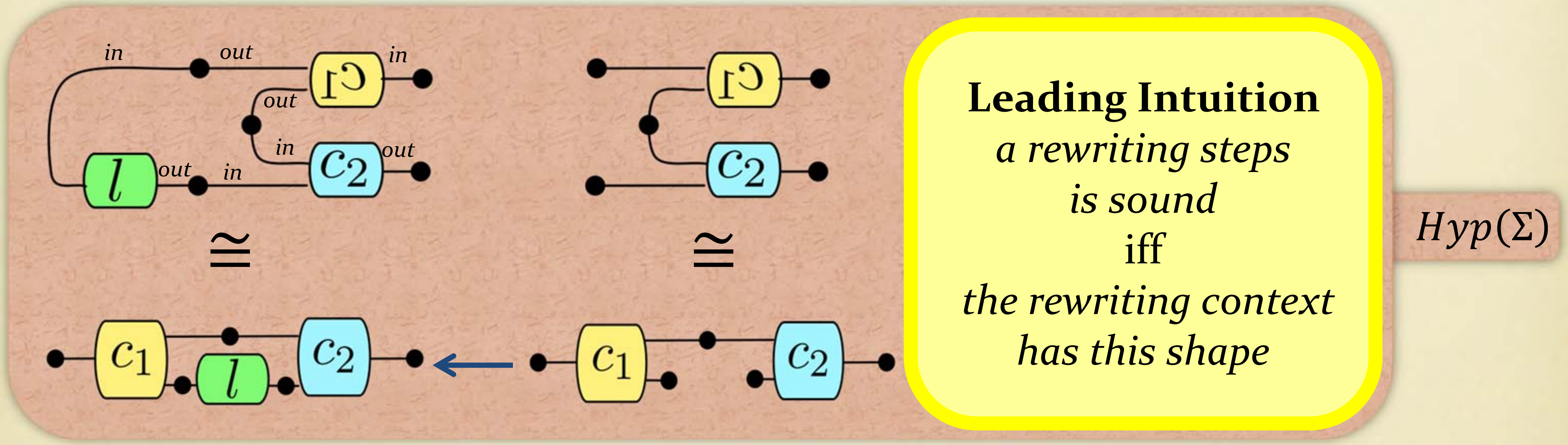
Where we are, so far



How does a sound DPO rewriting look like?

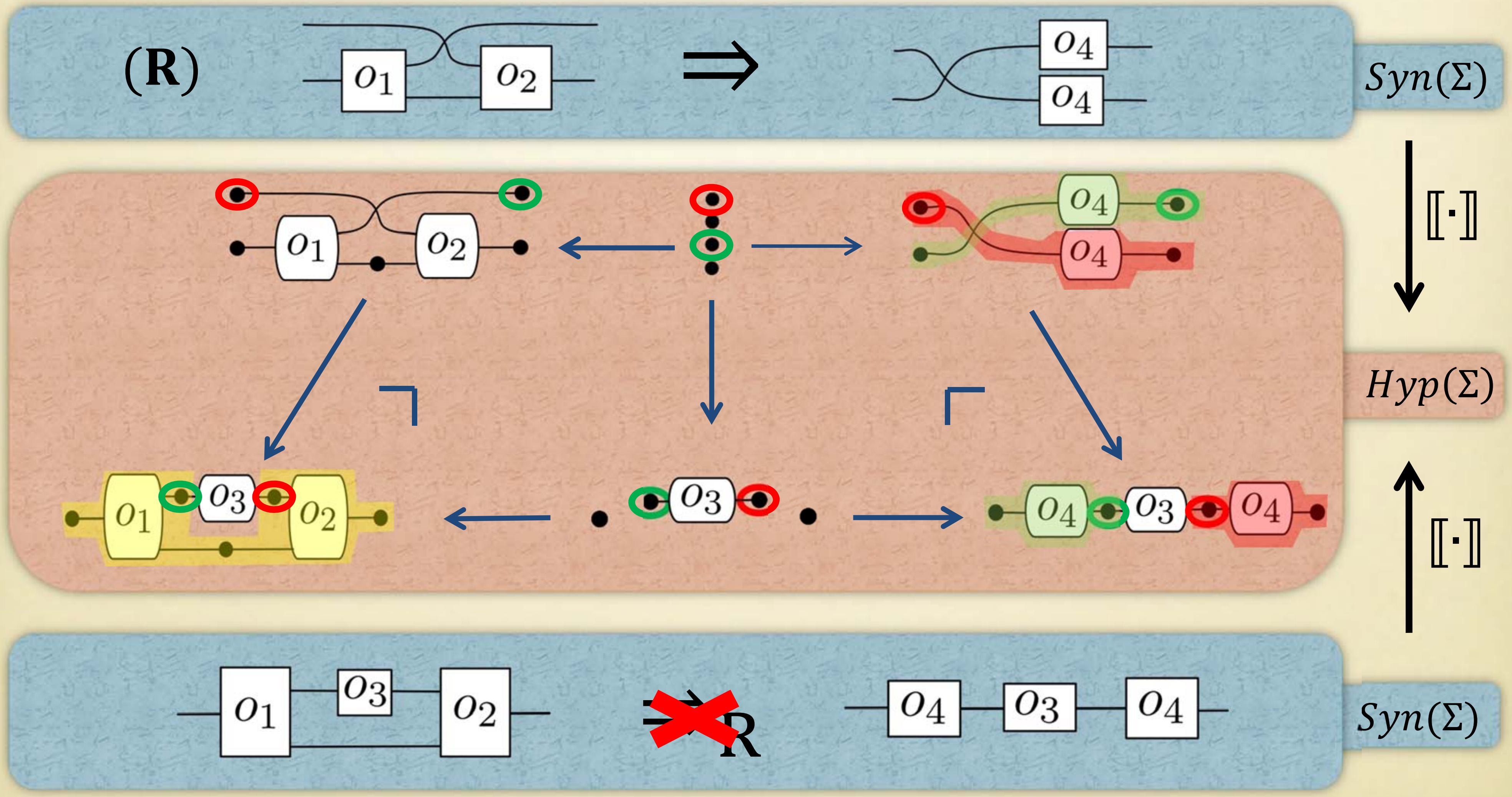


How does a sound DPO rewriting look like?



Leading Intuition
a rewriting steps
is sound
iff
the rewriting context
has this shape

Back to the soundness counterexample



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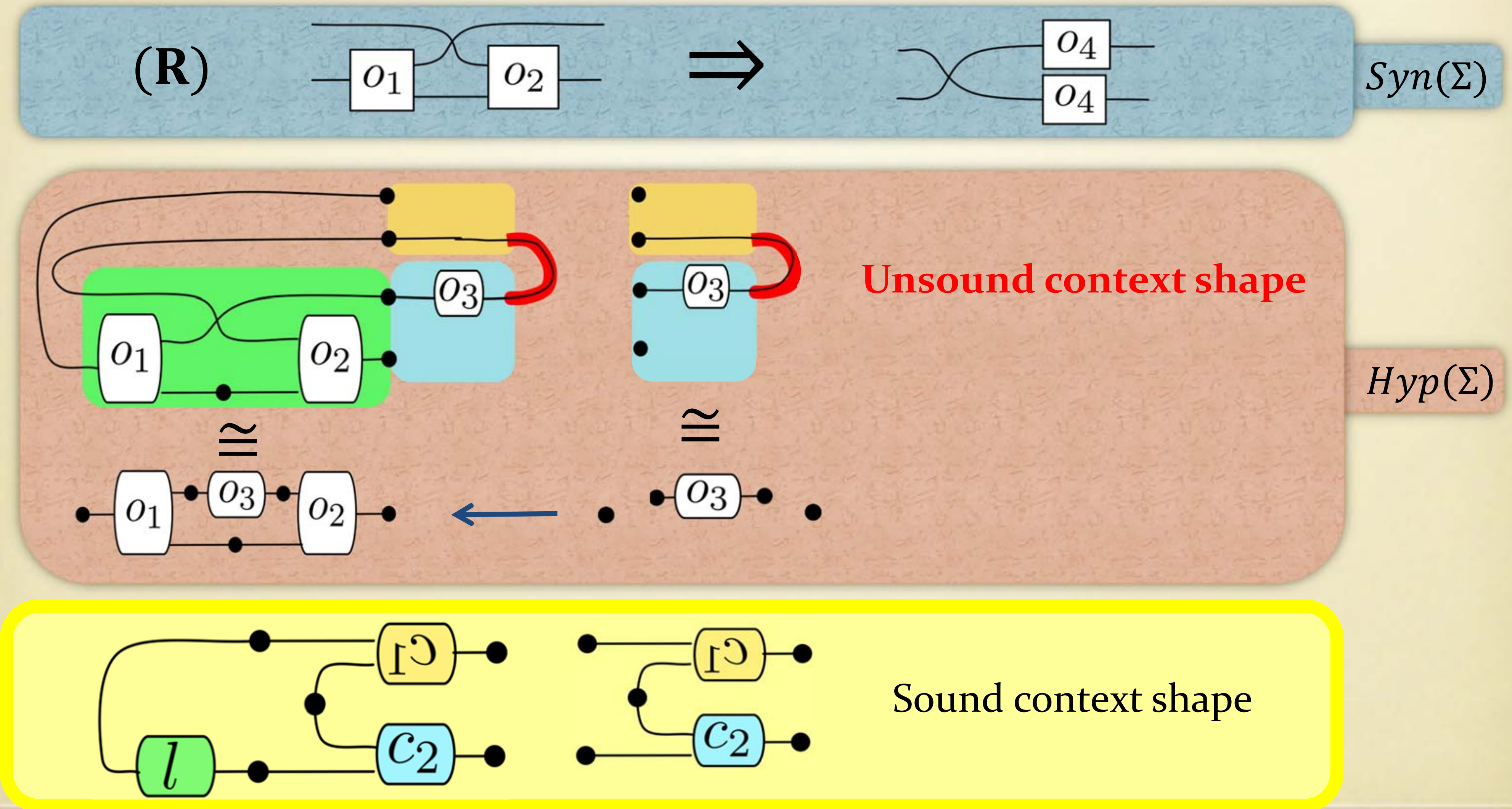
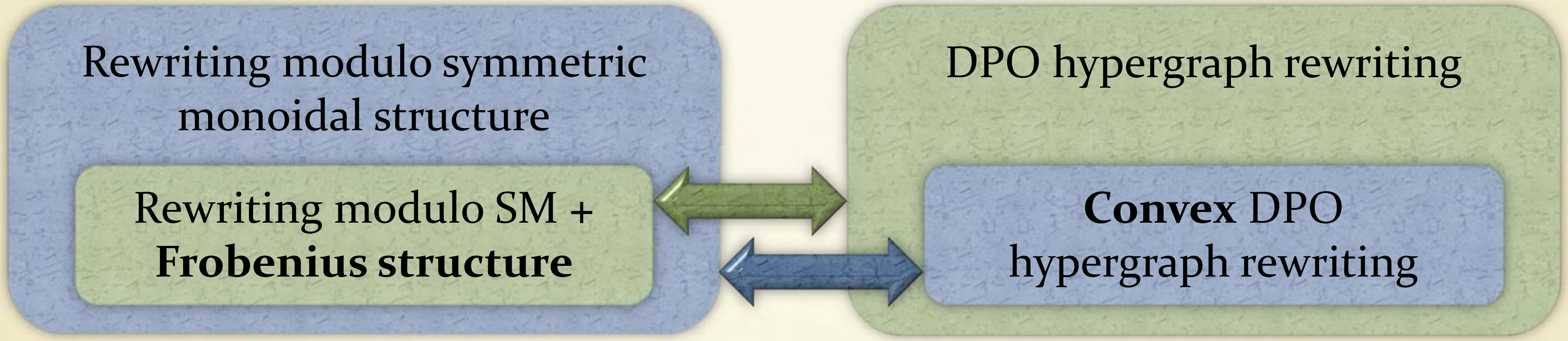


Convex DPO rewriting is sound

Theorem II
Convex DPO hypergraph rewriting is *sound and complete* for symmetric monoidal categories.

Discussion



- Ongoing and future work
  - More examples
    - Frobenius structures are commonplace in algebras of circuit diagrams.
  - Study of critical pairs, confluence, termination.
  - Relationship with equational theories generated by distributive laws of PROPs.