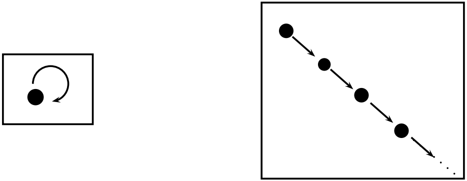
A characterization result for the alternation-free fragment of the modal μ-calculus

Alessandro Facchini, Yde Venema, **Fabio Zanasi**

LICS 2013

Motivation



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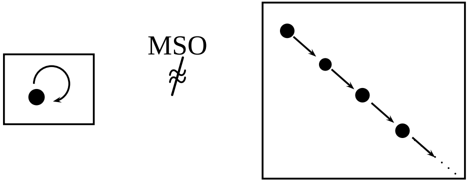
Monadic Second-Order Logic (MSO)



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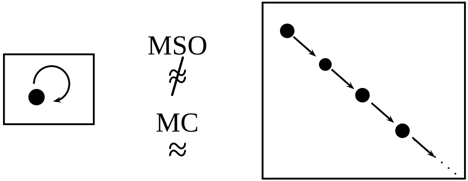
Monadic Second-Order Logic (MSO)

The Modal μ-Calculus (MC)



Janin-Walukiewicz Theorem

$$MC \quad \equiv \quad MSO/\underline{\leftrightarrow} \qquad \text{(over transition systems)}$$

The question

$$\mathcal{L}/\underline{\leftrightarrow} \quad \equiv \quad \mathcal{M} \qquad (\text{over } \mathcal{C})$$

Structure ($\mathcal{C}$)	$\mathcal{L}$	$\mathcal{M}$	Reference
TSs	FO	ML	J. van Benthem 1977
	MSO	MC	D. Janin, I. Walukiewicz 1996
	?	AFMC	–
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The alternation-free modal μ-calculus ($AFMC$)

The method

MSO $\equiv$ $\boxed{\text{MSO Automata}}$ $\qquad\qquad$ $\boxed{\text{MC Automata}}$ $\equiv$ MC

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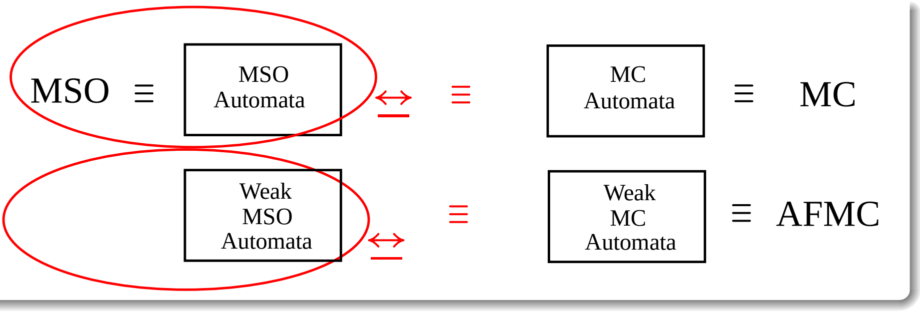
$$\text{MSO} \equiv \boxed{\begin{array}{c}\text{MSO}\\ \text{Automata}\end{array}} \;\; \underline{\leftrightarrow} \;\; \equiv \;\; \boxed{\begin{array}{c}\text{MC}\\ \text{Automata}\end{array}} \equiv \text{MC}$$

The method


MSO
≡
MSO Automata
↔
≡
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Weak MSO Automata
↔
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AFMC

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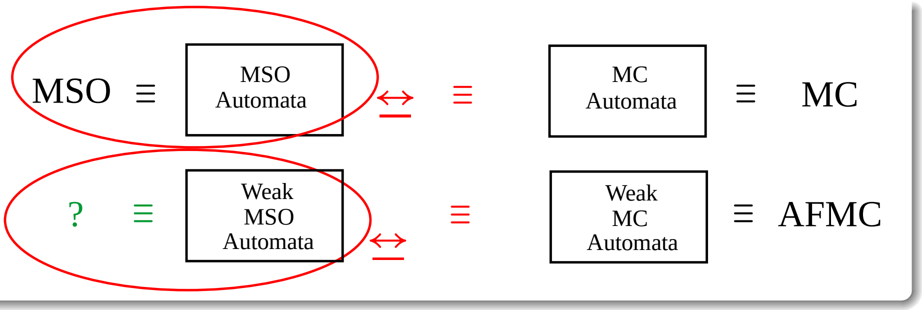
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MSO-automata

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An *MSO*-automaton is a ‘kind of’ alternating automaton working on (possibly infinite) trees.

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Acceptance conditions are determined by a **parity game**.

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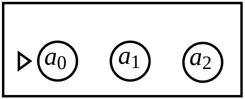
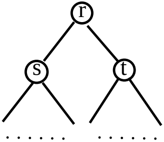
Automaton



Input Tree



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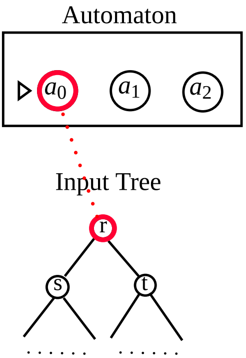

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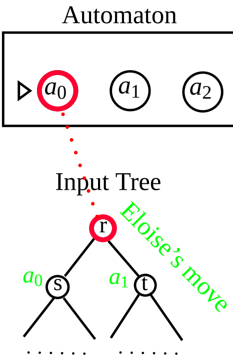

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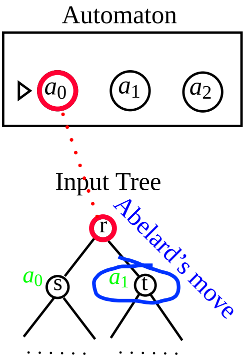

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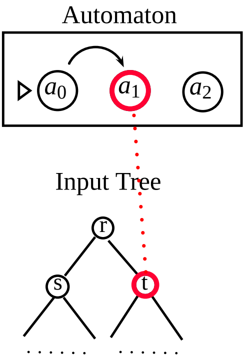

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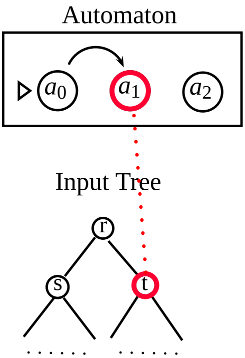

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Parities encountered in the match:
$\Omega(a_0) = 2$, $\Omega(a_1) = 3$, $\ldots$.

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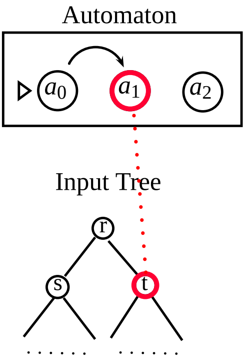

Acceptance conditions are determined by a **parity game**.

Parities encountered in the match:
$\Omega(a_0) = 2$, $\Omega(a_1) = 3$, ...

Eloise (Abelard) wins the match if the smallest parity occurring infinitely often is even (odd).

Weak *MSO*-automata

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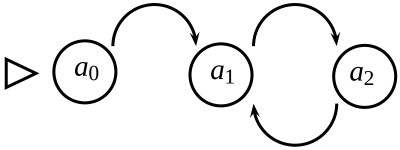
An *MSO*-automaton is weak if in each strongly connected component all states have the same parity.

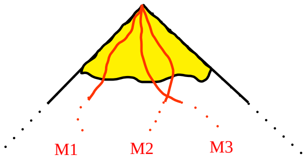


Condition: $\Omega(a_1) = \Omega(a_2)$

Exactly one parity occur infinitely often in infinite matches.

In which sense weak *MSO*-automata are weak?

Input tree

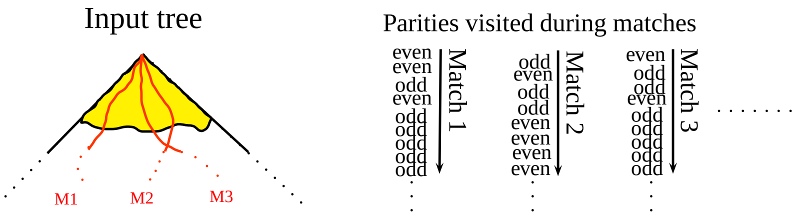


Parities visited during matches

Match 1	Match 2	Match 3
even	odd	even
even	even	odd
odd	odd	odd
even	odd	even
odd	even	odd
odd	even	odd
odd	even	odd
odd	even	odd
odd		odd
⋮	⋮	⋮

. . . .

In which sense weak *MSO*-automata are weak?



The key intuition

The amount of ‘complex’ information that a weak *MSO*-automaton can process is limited on the vertical dimension of trees.

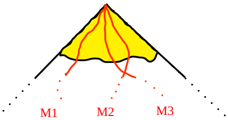
Example

The language of trees where on each path there are only finitely many nodes labeled with p is not recognized by any weak *MSO*-automaton.

A logic for weak MSO-automata

Well-founded monadic second-order logic ($WFMSO$)

$\mathbb{T} \vDash \exists X.\varphi$ *iff* $\mathbb{T}[X \mapsto S] \vDash \varphi$ for some set S of nodes in a well-founded subtree of $\mathbb{T}$.



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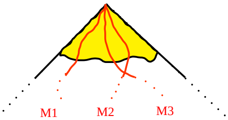

Characterization Theorem

$$\text{WFMSO} \equiv \boxed{\begin{array}{c}\text{Weak}\\ \text{MSO}\\ \text{Automata}\end{array}}$$

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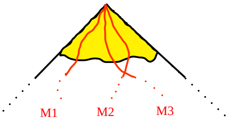

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Simulation Theorem $\boxed{\mathbb{A}} \Leftrightarrow \boxed{\mathbb{A}^\sharp}$

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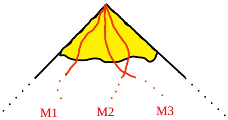

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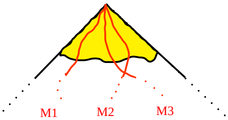

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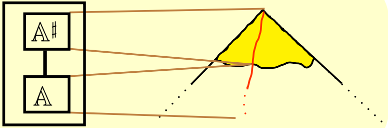

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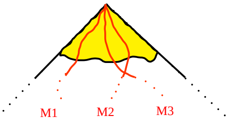

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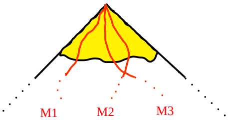

Characterization Theorem

$$WFMSO/\underline{\leftrightarrow} \equiv AFMC$$

Comparing with *WMSO*

Weak monadic second order logic (*WMSO*)

$$\mathbb{T} \models \exists X.\varphi \quad \textit{iff} \quad \mathbb{T}[X \mapsto S] \models \varphi \text{ for some set } S \text{ of nodes}$$
$$\text{in a finite subtree of } \mathbb{T}.$$

On finitely branching trees

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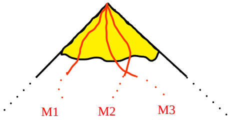

On finitely branching trees

$$WMSO \quad \equiv \quad WFMSO$$

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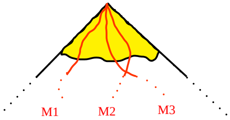


On finitely branching trees

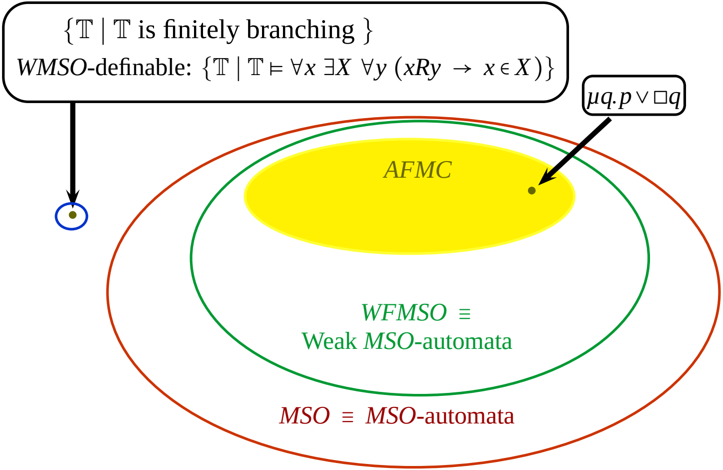
$$WMSO \equiv WFMSO$$

Different ways of being weak

WFMSO bound on the vertical dimension of trees

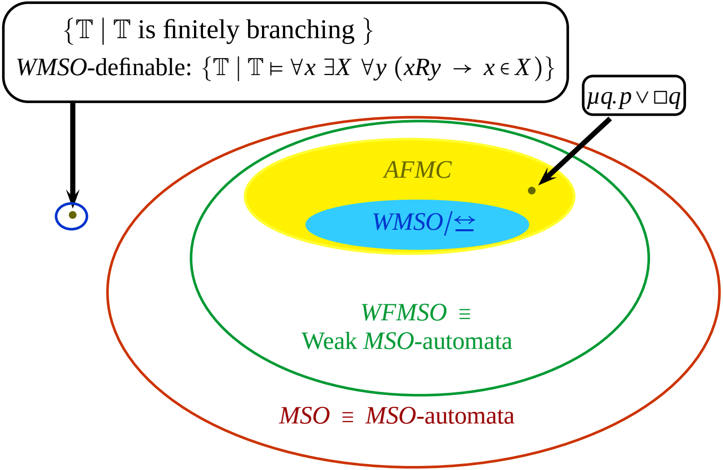
WMSO bound on the horizontal and vertical dimension of trees

The overall picture



On trees: $WMSO \mid\mid WFMSO$

The overall picture



Conjecture: $WMSO/\underline{\leftrightarrow} \subsetneq AFMC$