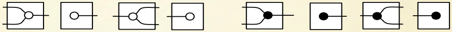
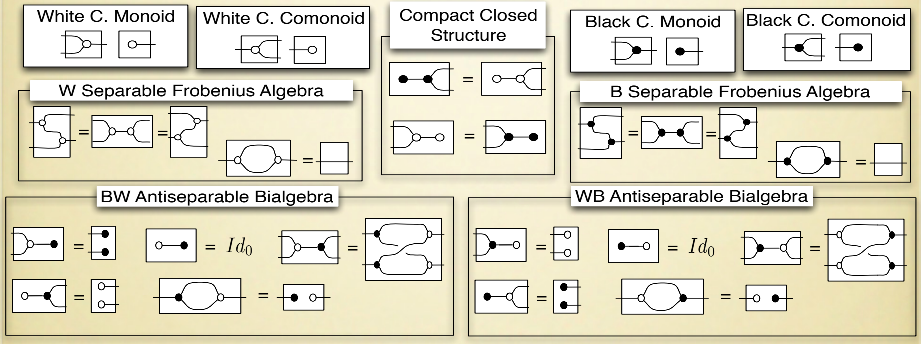
Interacting Bialgebras are Frobenius

Filippo Bonchi, Paweł Sobociński, **Fabio Zanasi**

FoSSaCS 2014

The theory $\mathbb{IB}$ - Interacting Bialgebras

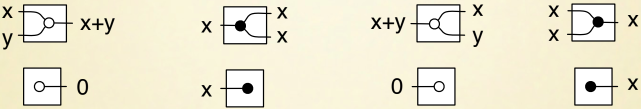
Operations : 



Theories of circuits with both Bialgebra and Frobenius Algebra structures:

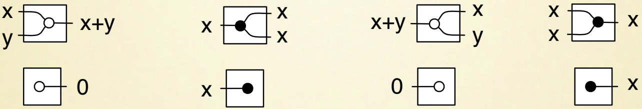
- Quantum information: ZX-calculus [Coecke & Duncan '08]
- Concurrency: algebra of stateless connectors [Bruni, Lanese, Montanari '07], algebra of Petri Nets with boundaries [Sobocinski '10].

Z_2-subspace Relational Semantics



Z_2-subspace Relational Semantics

Semantics $\mathbb{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$



Domain of interpretation: the SMC $\mathbb{SV}$ of Z_2-sub-vector spaces

- objects: natural numbers
- $\mathbb{SV}[n,m]$ = subspaces of $Z_2^n \times Z_2^m$
- relational composition
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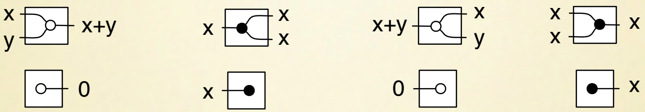

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Characterization result

$\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$ is an isomorphism.

The Cube

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\begin{tikzcd}
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PROPs

- PROPs are a “linear” variant of Lawvere Theories.

A PROP is a symmetric monoidal category where:

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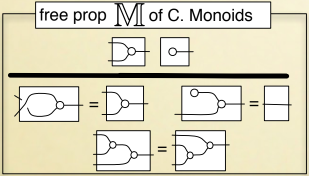

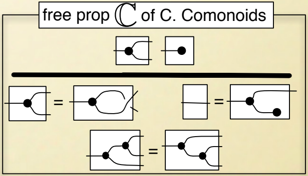

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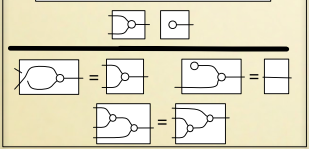

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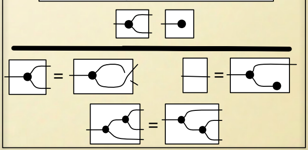

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the prop $\mathbb{F}$ of functions

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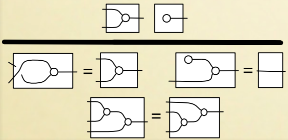

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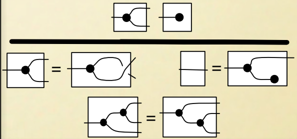


$\cong$ $\qquad$ $\cong$

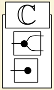
the prop $\mathbb{F}$ of functions

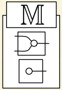
$\mathbb{F}[n, m] =$ functions $\overline{n} \to \overline{m}$

the prop $\mathbb{F}^{op}$

opposite of $\mathbb{F}$

Composing PROPs





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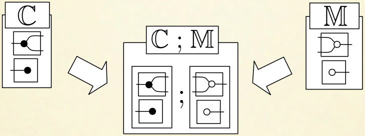

Idea: ring = abelian group + monoid + axioms describing their interaction

Composing PROPs [S.Lack, 2004]

PROPs are monads (in a certain bicategory)

PROP composition = Distributive law between monads

To define the PROP $\mathbb{C};\mathbb{M}$ we need a distributive law:

$$\lambda : \mathbb{M};\mathbb{C} \Rightarrow \mathbb{C};\mathbb{M}$$

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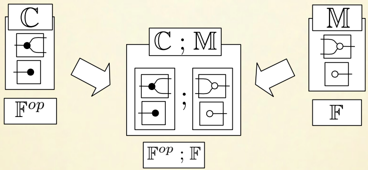

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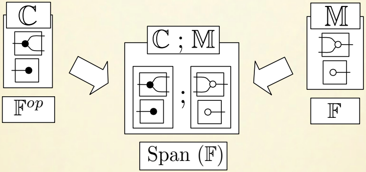

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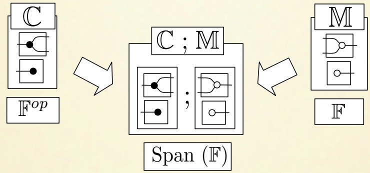

$\lambda\colon\ \mathbb{M};\mathbb{C} \Rightarrow \mathbb{C};\mathbb{M}$ defined by pullback in $\mathbb{F}$:

$$\begin{tikzcd}
 & z & \\
n \arrow[ur, "f"] & \Downarrow\lambda & m \arrow[ul, "g"'] \\
 & r \arrow[ul, "u"] \arrow[ur, "v"'] &
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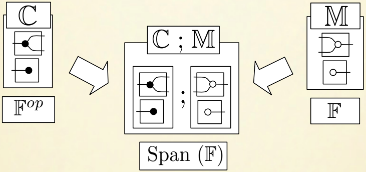

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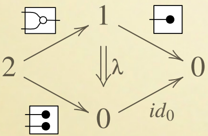

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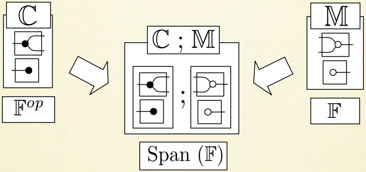

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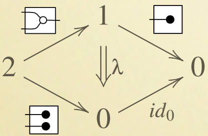

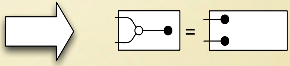

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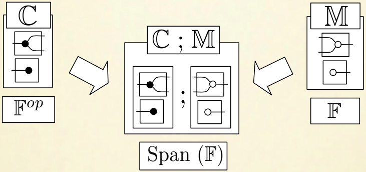

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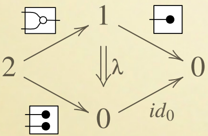

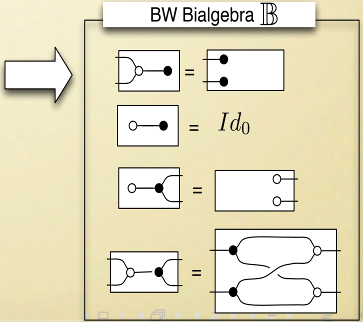

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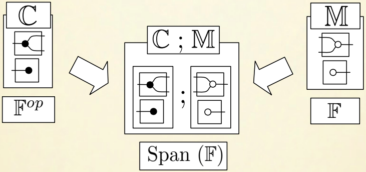

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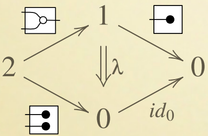

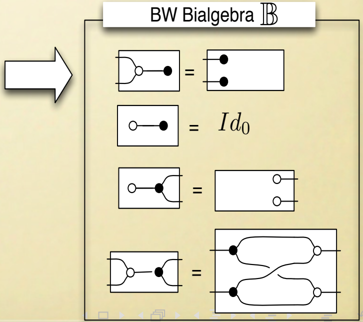

$\mathbb{B}$ as composed PROP

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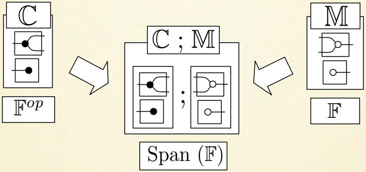

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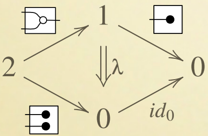

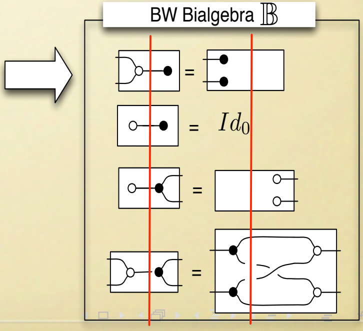

$\mathbb{B}$ as composed PROP

- $\mathbb{B} = \mathbb{C};\mathbb{M}$
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- factorisation for $\mathbb{B}$-circuits

The Cube

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The theory $\mathbb{AB}$ of Z_2-matrices

The PROP $\mathbb{B}$ of bialgebras characterizes spans:

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The PROP $\mathbb{AB}$ of antiseparable bialgebras characterizes Z_2-matrices:

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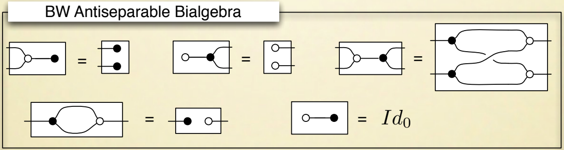

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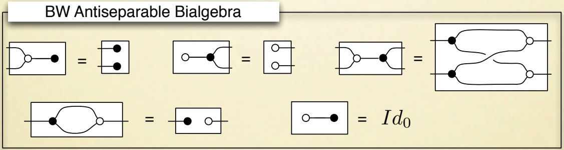

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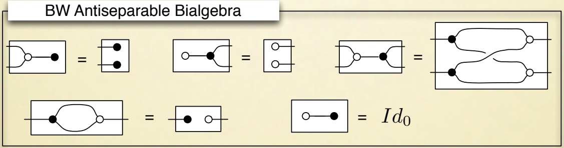

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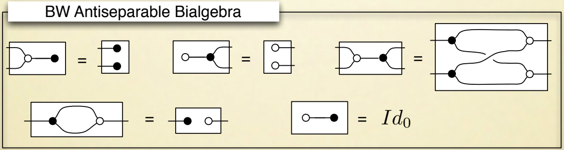

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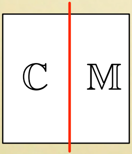

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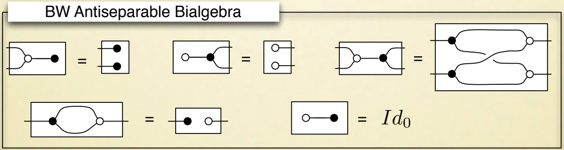

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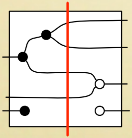

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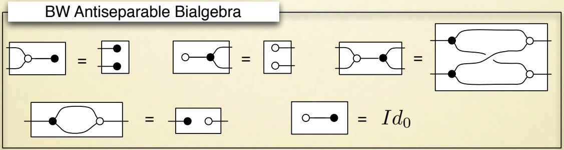

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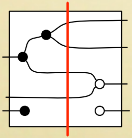

$$\mapsto \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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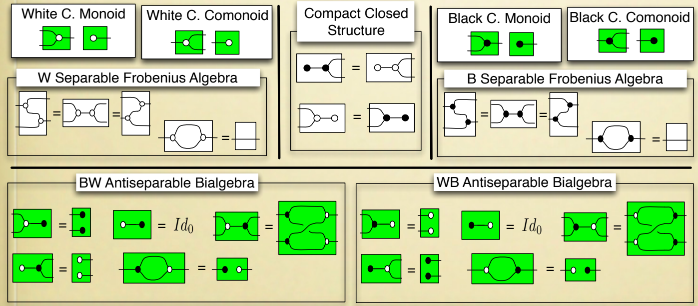

$\mathbb{AB}+\mathbb{AB}^{op} \sim$ black-white interaction.

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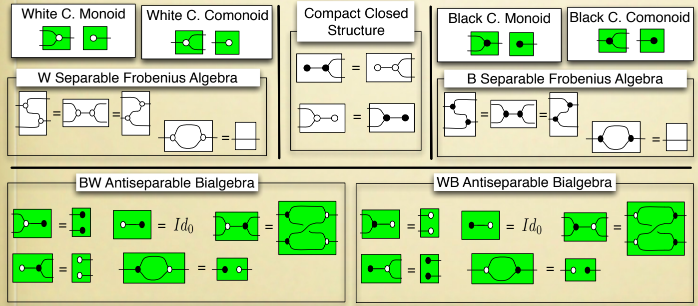
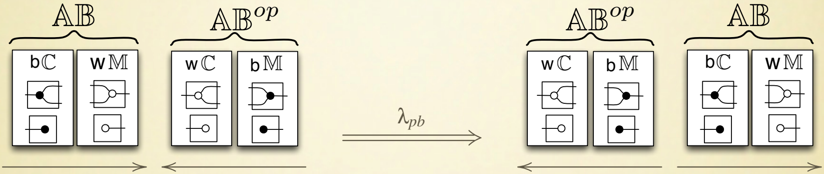


Composing $\mathbb{AB}$, $\mathbb{AB}^{op}$: black-black & white-white interaction.

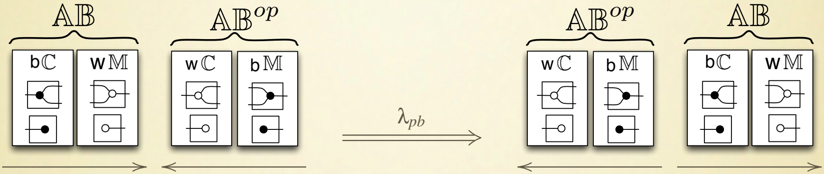
Composing $\mathbb{AB}$ and $\mathbb{AB}^{op}$

Construct the PROP $\mathbb{AB}^{op};\mathbb{AB}$ by pullback:

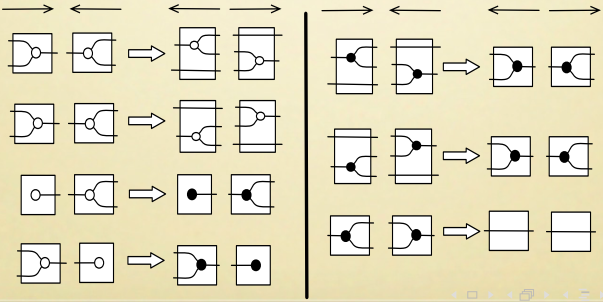


Composing $\mathbb{AB}$ and $\mathbb{AB}^{op}$

Construct the PROP $\mathbb{AB}^{op};\mathbb{AB}$ by pullback:

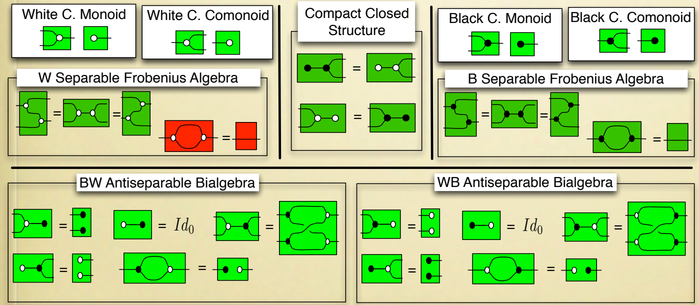


Read (in Mat Z_2) the equations of $\mathbb{AB}^{op};\mathbb{AB}$ out of pullback diagrams:



The Cube

$$\begin{tikzcd}
 & \mathbb{AB}+\mathbb{AB}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"] & & \mathbb{IB}^{-w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IB}^{-b} \arrow[rr] \arrow[dd, "\cong"] & & \mathbb{IB} \arrow[dd, dotted] & \\
 & \mathsf{Mat}\,\mathsf{Z}_2+\mathsf{Mat}\,\mathsf{Z}_2{}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[rr] & & \mathbb{SV} &
\end{tikzcd}$$



$\mathbb{AB}^{op};\mathbb{AB}$

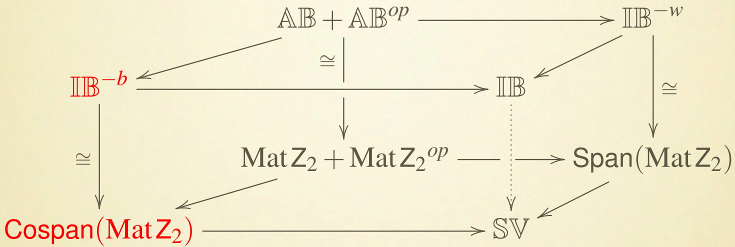
$=$

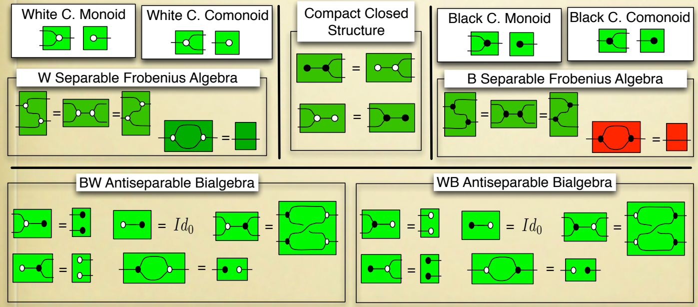
$\mathbb{IB}$ minus White Separability

$=$

$\mathbb{IB}^{-w}$

The Cube





$\mathbb{AB}\ ;\ \mathbb{AB}^{op}$

$=$

$\mathbb{IB}^{-b}$ ($\mathbb{IB}$ minus Black Sep.)

$=$

"photographic negative" of $\mathbb{IB}^{-w}$

The Cube

$$
\begin{tikzcd}
 & \mathbb{AB}+\mathbb{AB}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IB}^{-w} \arrow[dl, "\text{W Sep}"] \arrow[dd, "\cong"] \\
\mathbb{IB}^{-b} \arrow[rr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & \mathbb{IB} \arrow[dd, dotted] & \\
 & \mathrm{Mat}\,\mathsf{Z}_2+\mathrm{Mat}\,\mathsf{Z}_2{}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathrm{Mat}\,\mathsf{Z}_2) \arrow[dl] \\
\mathsf{Cospan}(\mathrm{Mat}\,\mathsf{Z}_2) \arrow[rr] & & \mathbb{SV} &
\end{tikzcd}
$$

The Cube

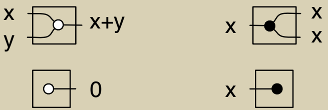
$$
\begin{tikzcd}
 & & \mathbb{AB} + \mathbb{AB}^{op} \arrow[dll] \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IB}^{-w} \arrow[dl, "\text{W Sep}"] \arrow[dd, "\cong"] \\
\mathbb{IB}^{-b} \arrow[rrr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & & \mathbb{IB} \arrow[dd, dotted] & \\
 & & \mathsf{Mat}\,\mathsf{Z}_2 + \mathsf{Mat}\,\mathsf{Z}_2{}^{op} \arrow[dll] \arrow[rr] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[dl, "\text{Span}\mapsto\text{Rel}"] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[rrr, "\text{Cospan}\mapsto\text{Rel}"'] & & & \mathbb{SV} &
\end{tikzcd}
$$

The Cube

$$
\begin{tikzcd}
 & & \mathbb{AB} + \mathbb{AB}^{op} \arrow[dll] \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IB}^{-w} \arrow[dl, "\text{W Sep}"'] \arrow[dd, "\cong"] \\
\mathbb{IB}^{-b} \arrow[rrr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & & {\color{red}\mathbb{IB}} \arrow[dd, dotted] & \\
 & & \mathsf{Mat}\,\mathsf{Z}_2 + \mathsf{Mat}\,\mathsf{Z}_2{}^{op} \arrow[dll] \arrow[rr] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[dl, "\text{Span}\mapsto\text{Rel}"'] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[rrr, "\text{Cospan}\mapsto\text{Rel}"'] & & & {\color{red}\mathbb{SV}} &
\end{tikzcd}
$$

- $\mathbb{IB}$ and $\mathbb{SV}$ are pushout objects.

- Unique arrow $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$



Discussion

Results

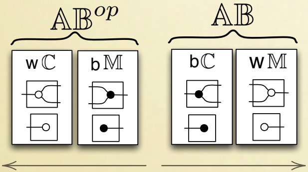
- Cube construction revealing the modular structure of $\mathbb{IB}$.
- Completeness for the semantics $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$.
- Factorisation properties of $\mathbb{IB}$.

Discussion

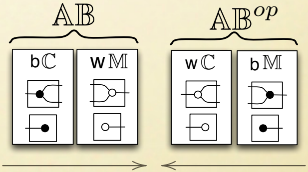
Results

- Cube construction revealing the modular structure of $\mathbb{IB}$.
- Completeness for the semantics $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$.
- Factorisation properties of $\mathbb{IB}$.

Factorisation of $\mathbb{IB}^{-w}$



Factorisation of $\mathbb{IB}^{-b}$



Discussion

Results

- Cube construction revealing the modular structure of $\mathbb{IB}$.
- Completeness for the semantics $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$.
- Factorisation properties of $\mathbb{IB}$.

Work in progress:

Discussion

Results

- Cube construction revealing the modular structure of $\mathbb{IB}$.
- Completeness for the semantics $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$.
- Factorisation properties of $\mathbb{IB}$.

Work in progress:

- The cube for an arbitrary Principal Ideal Domain in place of Z_2.

Discussion

Results

- Cube construction revealing the modular structure of $\mathbb{IB}$.
- Completeness for the semantics $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$.
- Factorisation properties of $\mathbb{IB}$.

Work in progress:

- The cube for an arbitrary Principal Ideal Domain in place of Z_2.
- Other applications:
  - Quantum information: full ZX-calculus, partial equivalence relations
  - Concurrency: Algebra of stateless/stateful connectors, Petri nets with boundaries
  - Electrical circuits, Signal Flow Graphs, Stream Calculus