On Creative Definitions in Leśniewski's Ontology

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Definitions in Mereology

The objects a and b are identical

$$\forall ab(a \approx b \equiv a \mathrel{\varepsilon} b \wedge b \mathrel{\varepsilon} a)$$

The object a is external to the object b

$$\forall ab(a \mathrel{\varepsilon} \mathsf{Ex}(b) \equiv a \mathrel{\varepsilon} a \wedge \neg\exists c(c \mathrel{\varepsilon} \mathsf{ingr}(b) \wedge c \mathrel{\varepsilon} \mathsf{ingr}(a)))$$

The object a is the class of objects b

$$\forall ab(a \mathrel{\varepsilon} \mathsf{Kl}(b) \equiv a \mathrel{\varepsilon} a \wedge \forall c(a \mathrel{\varepsilon} \mathsf{Ex}(c) \equiv \forall r(r \mathrel{\varepsilon} b \rightarrow r \mathrel{\varepsilon} \mathsf{Ex}(c))))$$

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What is the status of these definitions?

A comparison with *Principia Mathematica*

> *[T]he definitions are no part of our subject, but are, strictly speaking, mere typographical conveniences. Practically, of course, if we introduced no definitions, our formulae would very soon become so lengthy as to be unmanageable; but theoretically, all definitions are superfluous.*
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> (Russell & Whitehead, *PM, I, 1910)*

"Outside" *PM*:

$$\alpha \supset \beta \ =_{\text{def}} \ \neg\alpha \vee \beta$$

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"Outside" *PM*:

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A derivation in *PM*:

$$\text{Axioms}, \sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$$

Vocabulary:

$\boxed{\text{Primitive Symbols } (\vee, \neg, \ldots)}$ (<u>no</u> $\supset$)

Definitions in Mereology

A development of Mereology:

Axioms

Vocabulary at stage 0 : $\boxed{\text{Primitives} \begin{cases} \varepsilon : S/NN \\ \text{ingr}(-) : N/N \\ \ldots \end{cases}}$

Definitions in Mereology

A development of Mereology:

$$\text{Axioms}, \tau_1, \tau_2, \ldots, \tau_n$$

Vocabulary at stage n : $\boxed{\text{Primitives} \begin{cases} \varepsilon : S/NN \\ \text{ingr}(-) : N/N \\ \ldots \end{cases}}$

Definitions in Mereology

A development of Mereology:

$$\text{Axioms}, \tau_1, \tau_2, \ldots, \tau_n, \text{Def}_{\approx}$$

Vocabulary at stage $n+1$:

$$\boxed{\begin{array}{c} \text{Primitives} \begin{cases} \varepsilon : S/NN \\ \text{ingr}(-) : N/N \\ \ldots \end{cases} \\ - \approx - : S/NN \end{array}}$$

$$\forall ab(a \approx b \equiv a \mathrel{\varepsilon} b \wedge b \mathrel{\varepsilon} a) \qquad (\text{Def}_{\approx})$$

Definitions in Mereology

A development of Mereology:

$$\text{Axioms}, \tau_1, \tau_2, \ldots, \tau_n, \text{Def}_{\approx}, \tau_{n+2}, \tau_{n+3}, \ldots, \tau_m$$

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A development of Mereology:

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Vocabulary at stage $m+1$:

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$$\forall ab(a \approx b \equiv a \;\varepsilon\; b \wedge b \;\varepsilon\; a) \tag{$\text{Def}_{\approx}$}$$

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The logical systems of Leśniewski

Mereology

Ontology

$\epsilon : S/NN$

Protothetics

$\equiv : S/SS$

Definitions in *Ontology*

Ontology

Axiomatization of *Ontology*

- The axiom A_p of *Protothetic*
- The axiom A_o, fixing the intuitive meaning of $\varepsilon : S/NN$

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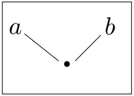
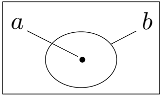
$a \ \varepsilon \ b$ means: either  or 

Ontology

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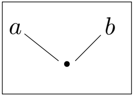
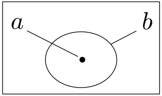


or



Inference rules

- Usual rules of axiomatic systems (substitution, detachment, . . .)
- (Laws of extensionality)
- Rules for introducing definitions

Introduction Rules for Definitions

Definition of a constant $\psi : S/c_1 \ldots c_n$

$$\forall b_1 \ldots b_n \, (\underbrace{\psi(b_1, \ldots, b_n)}_{\text{Definiendum}} \equiv \underbrace{\varphi(b_1, \ldots, b_n))}_{\text{Definiens}}$$

Definition of a constant $\gamma : N/c_1 \ldots c_n$

$$\forall a \, b_1 \ldots b_n \, \underbrace{(a \, \varepsilon \, \gamma(b_1, \ldots, b_n)}_{\text{Definiendum}} \equiv \underbrace{a \, \varepsilon \, a \wedge \varphi(a, b_1, \ldots, b_n))}_{\text{Definiens}}$$

Examples

Everything named by a is also named by b

$$\forall a(a \subseteq b \equiv \forall c(c \mathrel{\varepsilon} a \rightarrow c \mathrel{\varepsilon} b))$$

The individual named by a has also names b and c

$$\forall a(a \mathrel{\varepsilon} b \cap c \equiv a \mathrel{\varepsilon} a \wedge a \mathrel{\varepsilon} b \wedge a \mathrel{\varepsilon} c)$$

The name a has no extension

$$\forall a(a \mathrel{\varepsilon} \bigwedge \equiv a \mathrel{\varepsilon} a \wedge \neg(a \mathrel{\varepsilon} a)) \qquad (D_{\bigwedge})$$

Creative Definitions

Each definition enriches the **vocabulary** of (the development of) the system in which it is introduced.

$$A_p, A_o, \tau_1, \ldots, \tau_n, \underbrace{\mathsf{Def}_\lambda, \tau_{n+2}, \ldots, \tau_k}_{\lambda \text{ is in the vocabulary}}$$

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Some definitions also increase the **deductive power** of the system.

$$\{A_p, A_o, \tau_1, \ldots, \tau_n, \mathsf{Def}_\lambda\} \vdash_{ont} \tau_k \qquad \text{but}$$
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Def_λ is **creative** for τ_k in the development

$$A_p, A_o, \tau_1, \ldots, \tau_n.$$

$D_{\bigwedge}$ is creative

$$\frac{\forall a(a\,\varepsilon\,\bigwedge \equiv a\,\varepsilon\,a \wedge \neg(a\,\varepsilon\,a))}{}$$

$D_\wedge$ is creative

$$\frac{\forall a(a \mathrel{\varepsilon} \wedge \equiv a \mathrel{\varepsilon} a \wedge \neg(a \mathrel{\varepsilon} a))}{\exists d\ \forall a(a \mathrel{\varepsilon} d \equiv a \mathrel{\varepsilon} a \wedge \neg(a \mathrel{\varepsilon} a))}\ Int.\exists \qquad (P_\wedge)$$

Claim

$D_\wedge$ is creative for $P_\wedge$ in the developement A_p, A_o.

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Proof

Interpret *Ontology* in *Protothetic* :

$$\begin{array}{ccc} & \star & \\ N & \Rightarrow & S \\ \epsilon\colon S/NN & \Rightarrow & \equiv\colon S/SS \end{array}$$

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$\vdash_{ont} A_p, A_o$ and $\vdash_{ont} P_\wedge$ but $\vdash_{prot} A^\star_p, A^\star_o$ and $\vdash_{prot} \neg P^\star_\wedge$

Exploring Creativity in *Ontology*

The boundaries of *Ontology*

Question - Leśniewski(?), Sobociński, Rickey (1975)

How many creative definitions can be introduced in *Ontology* ?

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Theorem 1

There are arbitrarily many developments of *Ontology* that contain a creative definition, and all of those definitions are different.

Proof of Theorem 1

An arbitrarily long sequence of definitions:

$$(\forall a)(a \mathrel{\varepsilon} 2^{+} \equiv (a \mathrel{\varepsilon} a \wedge (\exists bc)(a \mathrel{\varepsilon} b \wedge a \mathrel{\varepsilon} c \\ \wedge \neg(\forall z)(z \mathrel{\varepsilon} b \equiv z \mathrel{\varepsilon} c)))) \tag{D_2}$$

$$(\forall a)(a \mathrel{\varepsilon} 3^{+} \equiv (a \mathrel{\varepsilon} a \wedge (\exists bcd)(a \mathrel{\varepsilon} b \wedge a \mathrel{\varepsilon} c \wedge a \mathrel{\varepsilon} d \\ \wedge \neg(\forall z)(z \mathrel{\varepsilon} b \equiv z \mathrel{\varepsilon} c)))) \\ \wedge \neg(\forall z)(z \mathrel{\varepsilon} b \equiv z \mathrel{\varepsilon} d)))) \\ \wedge \neg(\forall z)(z \mathrel{\varepsilon} c \equiv z \mathrel{\varepsilon} d)))) \tag{D_3}$$

$$(\forall a)(a \mathrel{\varepsilon} 4^{+} \equiv (a \mathrel{\varepsilon} a \wedge (\exists bcde) \ \ldots \tag{D_4}$$

a is n^{+} iff the individual a has at least n distinct names.

Proof of Theorem 1

For each natural number n:

$$\vdash_{prot} \neg P_n^\star$$

Each of the following developments of Ontology contains a creative definition:

$$\begin{gathered} A_p, A_o, D_2, P_2 \\ A_p, A_o, D_3, P_3 \\ A_p, A_o, D_4, P_4 \\ \vdots \end{gathered}$$

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Theorem 2

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$$A_p, A_o, D_3, P_3, D_4, P_4, \ldots$$

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Problem

How to prove: D_{n+1} is creative for P_{n+1} in the development $A_p, A_o, \ldots, D_n, P_n$?

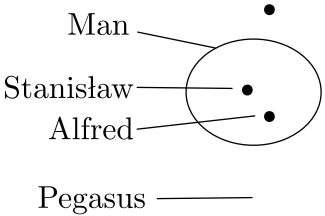
$$\vdash_{prot} \neg P_n^\star \qquad\qquad \vdash_{prot} \neg P_{n+1}^\star$$

Natural Models

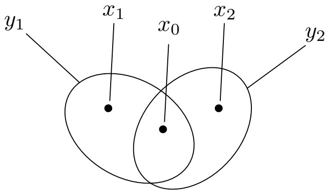
An interpretation for (elementary) *Ontology*

$$\mathbb{M} = < \mathcal{N}, \mathcal{D}, \varepsilon^{\star}, \sim >$$

$$v : (\lambda : N) \mapsto (n \in \mathcal{N})$$



D_4 is creative

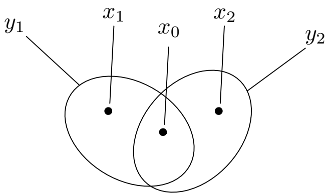


$\mathbb{M}_3$

$(\mathbb{M}_3, v) \vDash A_o \wedge D_3 \wedge P_3 \qquad v(3^+) = x_0$

$(\mathbb{M}_3, v) \nvDash P_4$

D_4 is creative



$\mathbb{M}_3$

$(\mathbb{M}_3, v) \models A_o \wedge D_3 \wedge P_3$

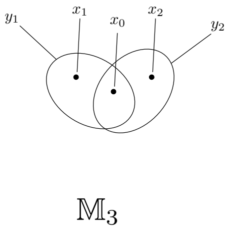
$(\mathbb{M}_3, v) \not\models P_4$

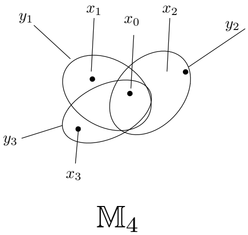
$v(3^+) = x_0$

D_4 is creative for P_4 in

A_p, A_o, D_3, P_3.

A sequence of models





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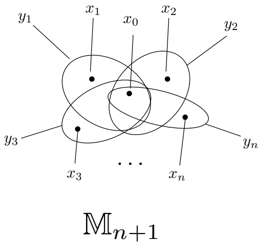

For each $n > 2$, D_{n+1} is creative for P_{n+1} in $A_p, A_o, D_3, P_3, \dots, D_n, P_n$.

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There is a development of *Ontology* that contains arbitrarily many creative definitions.

$$A_p, A_o, D_3, P_3, D_4, P_4, \ldots$$

Afterthoughts

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Creativity in Elementary *Ontology*

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Creativity in higher-order *Ontology*

- The addition of a Comprehension Principle neutralizes creativity in *Ontology* (Stachniak 1981, *cf.* Urbaniak 2008):

$$\exists\psi\ \forall b_1 \ldots b_n (\psi(b_1 \ldots b_n) \equiv \varphi(b_1 \ldots b_n))$$

$$\exists\gamma\ \forall a\ b_1 \ldots b_n (a\ \varepsilon\ \gamma(a\ b_1 \ldots b_n) \equiv a\ \varepsilon\ a \wedge \varphi(a\ b_1 \ldots b_n)).$$

- Introducing a definition in Leśniewski’s systems is an abstraction procedure (*cf.* Joray 2007).

Questions

Investigate further at what conditions a definition is creative.

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Creative definitions in higher-order Ontology.

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More?

References

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For Theorem 1 and Theorem 2:

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