The algebra of partial equivalence relations

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PROPs

``Linear''
Lawvere theory

A PROP is (just) a symmetric monoidal category with set of objects $\mathbb{N}$

Petri Nets

Signal flow graphs

Quantum processes

Syntactic PROP
freely generated by a theory (Σ, E)

$$\mathcal{T} \xrightarrow{\cong} \mathcal{S}$$

Semantic PROP

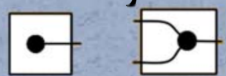
LTSs

Subspaces

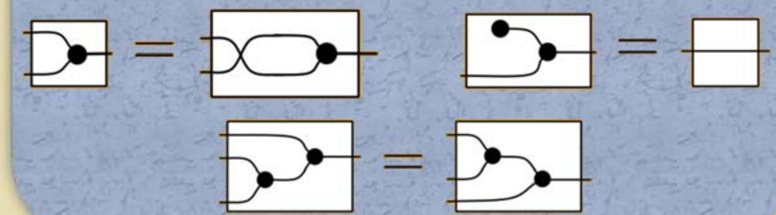
Linear maps on Hilbert spaces

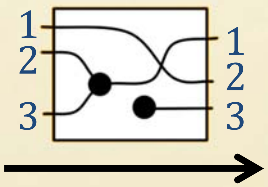
$$\text{Commutative monoids} \xrightarrow{\cong} \text{Functions}$$

Arrows are diagrams freely generated by the syntax



and quotiented by equations





Arrows $n \to m$ are functions
$\{1, \dots, n-1\} \to \{1, \dots, m-1\}$

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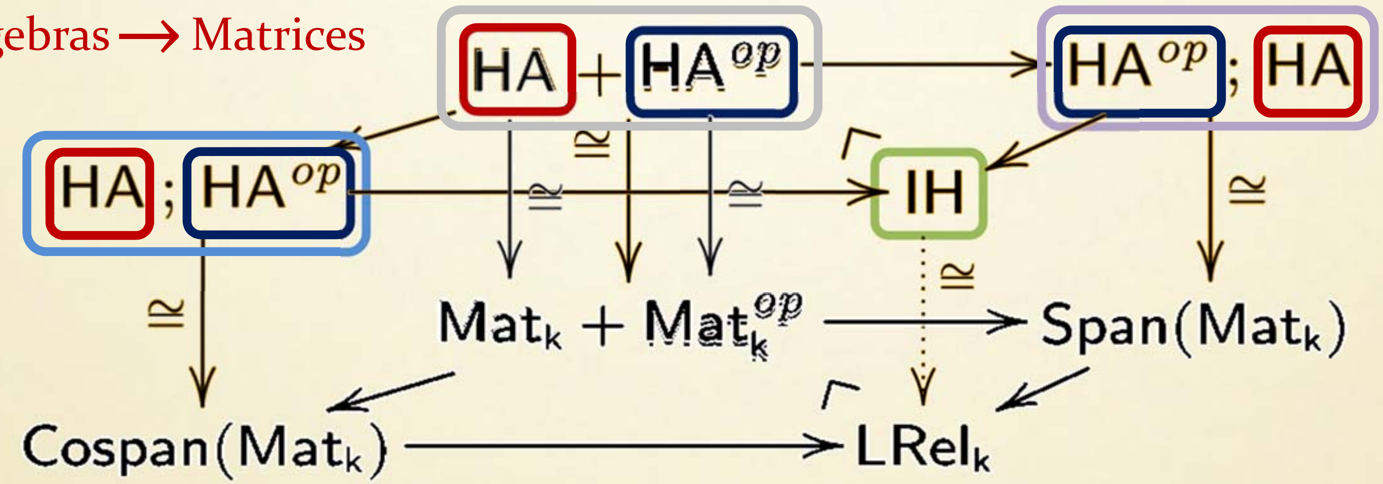
Commutative monoids	$\xrightarrow{\cong}$	Functions
Commutative comonoids	$\xrightarrow{\cong}$	Functions^{op}
Separable Frobenius algebras	$\xrightarrow{\cong}$	Cospans
Hopf algebras	$\xrightarrow{\cong}$	Matrices on comm. rings

PROPs, modularly

Interacting Hopf algebras $\xrightarrow{\cong}$ Subspaces on a field k (k-linear relations)

$\mathbf{LRel_k}$
arrows $n \to m$
are subspaces of
$k^n \times k^m$

Hopf algebras $\xrightarrow{\cong}$ Matrices



PROP sum
Disjoint union of signatures & equations.
Arrows' shape: →←→←→ ... .

PROP composition
Quotient of the sum by a distributive law λ: ←→ ⟶ →←
Arrows' shape: →←.

PROP fibered sum
Sum modulo some common structure.
Arrows' shape: both →← and ←→.

PROP composition
Quotient of the sum by a distributive law λ: →← ⟶ ←→
Arrows' shape: ←→ .

Question of this work

- Is the cube construction a more general phenomenon?

$$\begin{tikzcd}
& \mathsf{HA} + \mathsf{HA}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"] & & \mathsf{HA}^{op} \,;\, \mathsf{HA} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{HA} \,;\, \mathsf{HA}^{op} \arrow[rr] \arrow[dd, "\cong"] & & \mathsf{IH} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat_k} + \mathsf{Mat_k}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat_k}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat_k}) \arrow[rr] & & \mathsf{LRel_k} &
\end{tikzcd}$$

"Functions" $\rightarrow$

$\longleftarrow\longrightarrow$ $\qquad$ "Relations" $\leftrightarrow$ $\qquad$ $\longrightarrow\longleftarrow$

- Outcome: modular characterisation of other relational PROPs
  - A presentation for equivalence relations

$$\mathrm{IFR} \xrightarrow{\cong} \mathrm{ER}$$

  - A presentation for partial equivalence relations

$$\mathrm{IPFR} \xrightarrow{\cong} \mathrm{PER}$$

Overview

$$
\begin{tikzcd}
& \mathsf{HA} + \mathsf{HA}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathsf{HA}^{op} \,;\, \mathsf{HA} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{HA} \,;\, \mathsf{HA}^{op} \arrow[rr] \arrow[dd, "\cong"'] & & \mathsf{IH} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat_k} + \mathsf{Mat_k}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat_k}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat_k}) \arrow[rr] & & \mathsf{LRel_k} \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd}
$$

The linear case

$$
\begin{tikzcd}
& \mathsf{Mn} + \mathsf{Mn}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathsf{Mn}^{op} \,;\, \mathsf{Mn} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{Mn} \,;\, \mathsf{Mn}^{op} \arrow[rr] \arrow[dd, "\cong"'] & & 1 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dd, dotted, "\cong"] & \\
& \mathsf{F} + \mathsf{F}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{F}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{F}) \arrow[rr] & & 1 \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd}
$$

The cartesian case

$$
\begin{tikzcd}
& \mathsf{Un} + \mathsf{Un}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathsf{Un}^{op} \,;\, \mathsf{Un} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{Mn} \,;\, \mathsf{Mn}^{op} \arrow[rr] \arrow[dd, "\cong"'] & & \mathsf{IFr} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dd, dotted, "\cong"] & \\
& \mathsf{In} + \mathsf{In}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{In}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{F}) \arrow[rr] & & \mathsf{ER} \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd}
$$

The cartesian case, take II

$$
\begin{tikzcd}
& \mathsf{Un} + \mathsf{Un}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathsf{Un}^{op} \,;\, \mathsf{Un} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{PMn} \,;\, \mathsf{PMn}^{op} \arrow[rr] \arrow[dd, "\cong"'] & & \mathsf{IPFr} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dd, dotted, "\cong"] & \\
& \mathsf{In} + \mathsf{In}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{In}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{PF}) \arrow[rr] & & \mathsf{PER} \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd}
$$

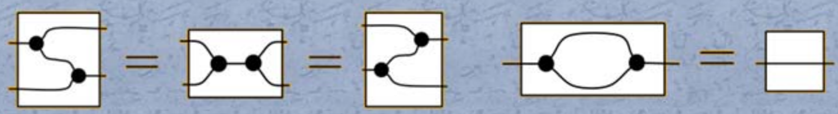
The partial cartesian case

The cartesian case

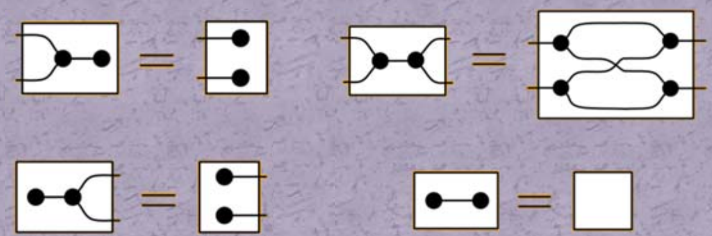
Commutative monoids $\xrightarrow{\cong}$ Functions

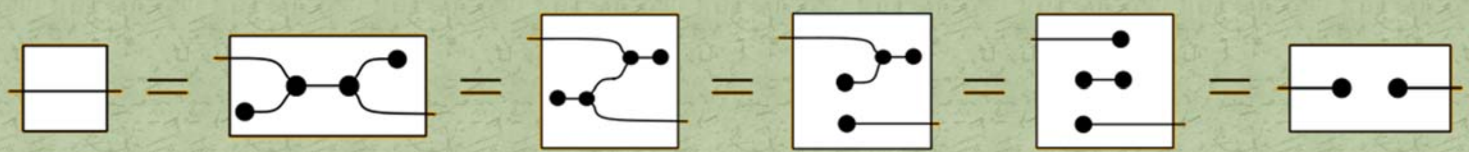
$$
\begin{tikzcd}
 & \mathsf{Mn} + \mathsf{Mn}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathsf{Mn}^{op} \,;\, \mathsf{Mn} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{Mn} \,;\, \mathsf{Mn}^{op} \arrow[rr] \arrow[dd, "\cong"'] & & 1 \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{F} + \mathsf{F}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{F}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{F}) \arrow[rr] & & 1 &
\end{tikzcd}
$$

Separable Frobenius algebras



Bialgebras



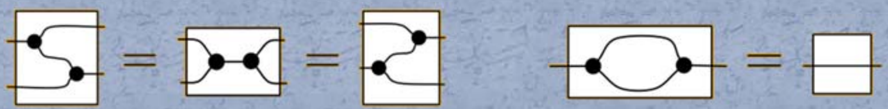


The cartesian case, take II

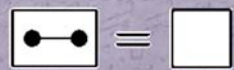
Theory of the unit $\xrightarrow{\cong}$ Injective Functions

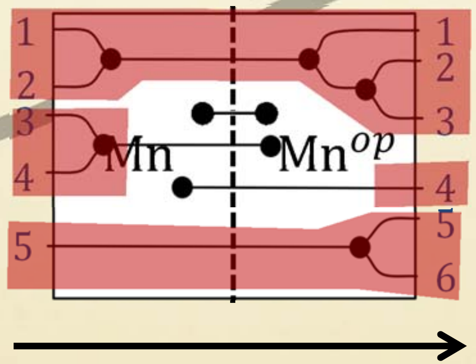
$$
\begin{tikzcd}
 & \mathsf{Un} + \mathsf{Un}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"] & & \mathsf{Un}^{op} \,;\, \mathsf{Un} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{Mn} \,;\, \mathsf{Mn}^{op} \arrow[rr] \arrow[dd, "\cong"] & & \mathsf{IFr} \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{In} + \mathsf{In}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{In}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{F}) \arrow[rr] & & \mathsf{ER} &
\end{tikzcd}
$$

Separable Frobenius algebras



Distributivity of over





Irredundant Separable Frobenius Algebras

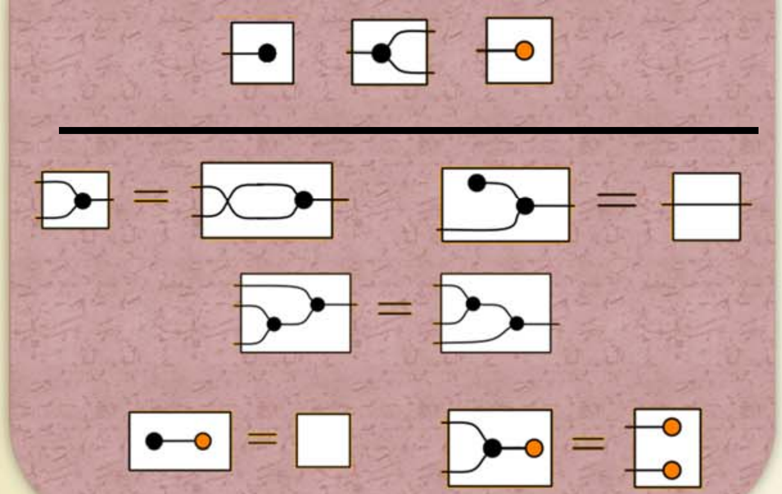
ER has arrows $n \to m$ equivalence relations on $\{1, \dots, n-1\} \uplus \{1, \dots, m-1\}$

The partial cartesian case

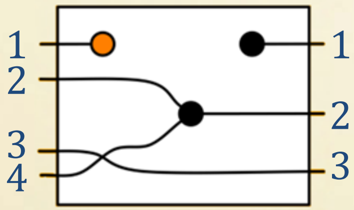
$$
\begin{tikzcd}
 & \mathsf{Un} + \mathsf{Un}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathsf{Un}^{op} \,;\, \mathsf{Un} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{PMn} \,;\, \mathsf{PMn}^{op} \arrow[rr] \arrow[dd, "\cong"'] & & \mathsf{IPFr} \arrow[dd, dotted, "\cong"] \arrow[ul, phantom, very near start, "\ulcorner"] & \\
 & \mathsf{In} + \mathsf{In}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{In}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{PF}) \arrow[rr] & & \mathsf{PER} \arrow[ul, phantom, very near start, "\ulcorner"] &
\end{tikzcd}
$$

The partial cartesian case

Partial commutative monoids



$$\begin{tikzcd}
& & \mathsf{Un}+\mathsf{Un}^{op} \arrow[dl] \arrow[dd, "\cong"'] \arrow[rr] & & \mathsf{Un}^{op}\,;\mathsf{Un} \arrow[dl] \arrow[dd, "\cong"] \\
& \boxed{\mathsf{PMn}}\,;\mathsf{PMn}^{op} \arrow[dl, "\cong"'] \arrow[dd, "\cong"] \arrow[rr] & & \mathsf{IPFr} \arrow[dd, dotted, "\cong"] & \\
\boxed{\mathsf{PF}} & & \mathsf{In}+\mathsf{In}^{op} \arrow[dl] \arrow[rr] & & \mathsf{Span}(\mathsf{In}) \arrow[dl] \\
& \mathsf{Cospan}(\mathsf{PF}) \arrow[rr] & & \mathsf{PER} &
\end{tikzcd}$$



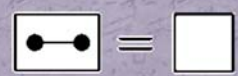
PF has arrows $n \to m$
partial functions
$\{1, \ldots, n-1\} \to \{1, \ldots, m-1\}$

The partial cartesian case

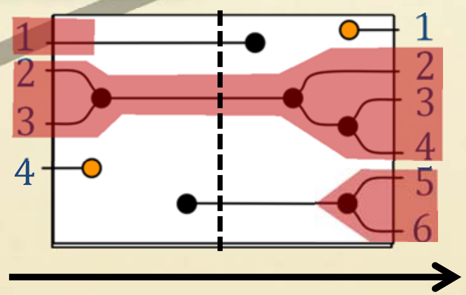
$$\begin{tikzcd}
& \mathsf{Un} + \mathsf{Un}^{op} \arrow[dl] \arrow[rr] \arrow[dd, "\cong"] & & \mathsf{Un}^{op} \,;\, \mathsf{Un} \arrow[dl] \arrow[dd, "\cong"] \\
\mathsf{PMn} \,;\, \mathsf{PMn}^{op} \arrow[rr] \arrow[dd, "\cong"'] & & \mathsf{IPFr} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{In} + \mathsf{In}^{op} \arrow[dl] \arrow[rr] & & \mathsf{Span}(\mathsf{In}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{PF}) \arrow[rr] & & \mathsf{PER} &
\end{tikzcd}$$

Partial Separable Frobenius algebras

Distributivity of over



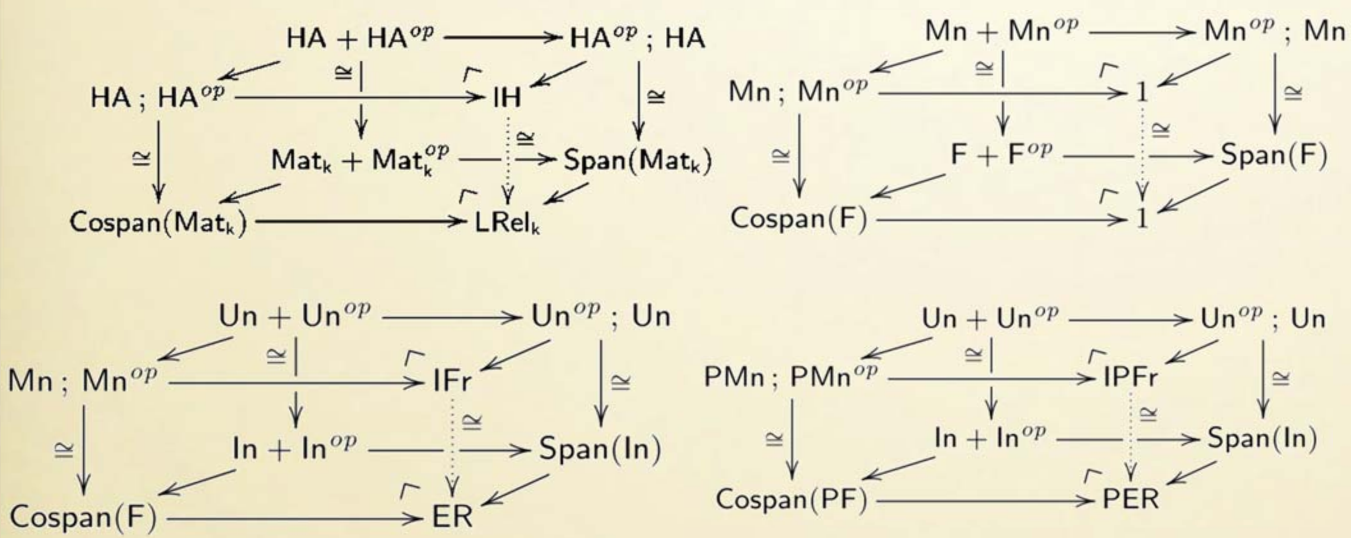
Irredundant
Partial Separable
Frobenius Algebras

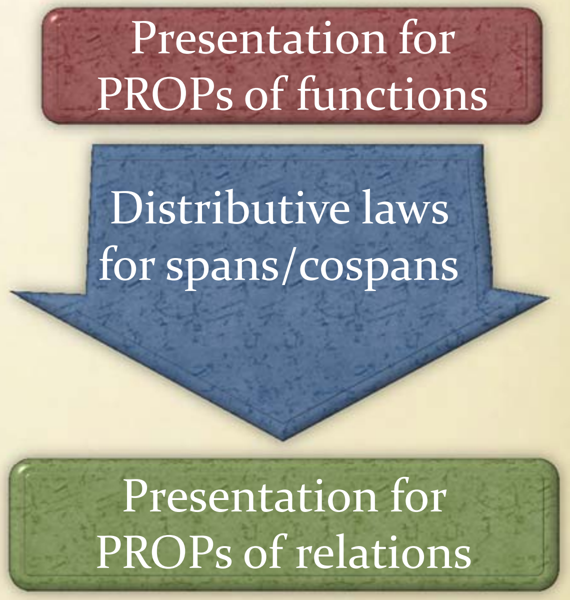


PER has arrows $n \to m$
partial eq. relations on
$\{1, \dots, n-1\} \uplus \{1, \dots, m-1\}$

Conclusions

- A case study for axiomatisation by PROP operations: ER and PER.
- Cube constructions are ubiquitous





- The general cube recipe remains an open question.