Interacting Hopf Algebras

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Joint work with Filippo Bonchi and Paweł Sobociński

ENS DE LYON

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In a nutshell

A principal ideal domain R

$\Downarrow$

The calculus of string diagrams for subspaces over the field of fractions on R

Technology

Lack's theory for composing PROPs

In a nutshell

A principal ideal domain R

⇩

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Mat R

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$$\begin{array}{c} \mathbb{HA}_{\mathsf{R}} \\ \cong \Big\downarrow \\ \mathsf{Mat}\,\mathsf{R} \end{array}$$

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$$
\begin{tikzcd}
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\color{red}{\mathbb{IH}_{\mathsf{R}}^{b}} \arrow[ddd, "\cong"'] & & & \\
 & & & \\
 & \mathsf{Mat}\,\mathsf{R} + \mathsf{Mat}\,\mathsf{R}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{R}) \arrow[dl] \\
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\mathbb{IH}_{\mathsf{R}}^{b} \arrow[rrr] \arrow[dd, "\cong"'] & & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted, "\cong"] & \\
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 & \mathsf{Mat}\,\mathsf{R} + \mathsf{Mat}\,\mathsf{R}^{op} \arrow[rr, dashed] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{R}) \arrow[dl] \\
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Technology

Lack's theory for composing PROPs

Applications

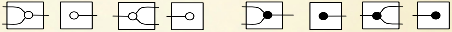
Compositional understanding of theories of string diagrams appearing in various fields (concurrency, control theory, physics, . . .).

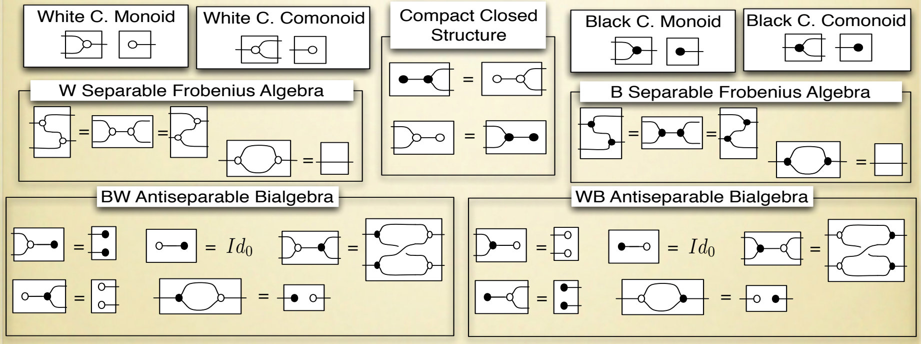
The Z_2 case

The cube for Z_2

$$
\begin{tikzcd}
& \mathbb{HA}_{Z_2}+\mathbb{HA}^{op}_{Z_2} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}^{w}_{Z_2} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}^{b}_{Z_2} \arrow[rr, crossing over] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2} & \\
& \mathrm{Mat}\,Z_2+\mathrm{Mat}\,Z_2{}^{op} \arrow[rr] \arrow[dl] & & \mathrm{Span}(\mathrm{Mat}\,Z_2) \arrow[dl] \\
\mathrm{Cospan}(\mathrm{Mat}\,Z_2) \arrow[rr] & & \mathbb{SV}_{Z_2} \arrow[from=uu, dotted, crossing over] &
\end{tikzcd}
$$

The theory $\mathbb{IH}_{Z_2}$

Operations : 

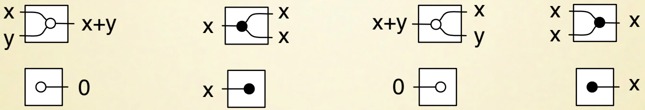


Theories of string diagrams featuring both Bialgebras and Frobenius Algebras:

- Quantum information: ZX-calculus [Coecke & Duncan ’08]
- Concurrency: algebra of stateless connectors [Bruni, Lanese, Montanari ’07], algebra of Petri Nets with boundaries [Sobocinski ’10].

Z_2-subspace Relational Semantics

Semantics $\mathcal{S} : \mathbb{IH}_{\mathsf{Z}_2} \to \mathbb{SV}_{\mathsf{Z}_2}$



Domain of interpretation: the PROP $\mathbb{SV}_{\mathsf{Z}_2}$ of Z_2-sub-vector spaces

- $\mathbb{SV}_{\mathsf{Z}_2}[n,m]$ = subspaces of $\mathsf{Z}_2^n \times \mathsf{Z}_2^m$
- relational composition

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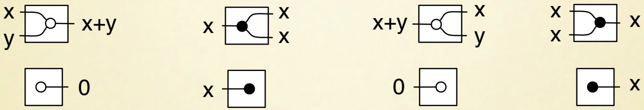

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Characterization result

$\mathcal{S} : \mathbb{IH}_{Z_2} \to \mathbb{SV}_{Z_2}$ is an isomorphism.

$\Rightarrow$ **Equality of string diagrams can be checked by computing their subspace.**

The cube for Z_2

$$
\begin{tikzcd}
 & \mathbb{HA}_{Z_2} + \mathbb{HA}_{Z_2}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2}^{w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}_{Z_2}^{b} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2} \arrow[dd, dotted] & \\
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 & \mathsf{Mat}\, Z_2 + \mathsf{Mat}\, {Z_2}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\, Z_2) \arrow[dl] \\
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\end{tikzcd}
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The building blocks of $\mathbb{HA}_{Z_2}$

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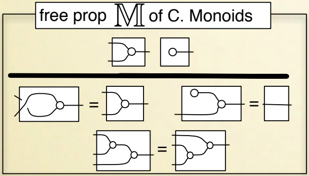
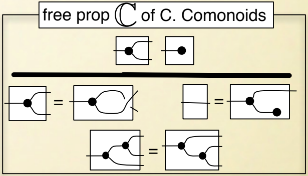
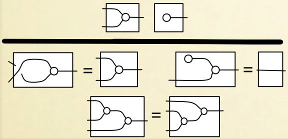


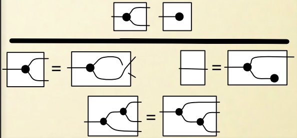


The building blocks of $\mathbb{HA}_{Z_2}$

free prop $\mathbb{M}$ of C. Monoids



free prop $\mathbb{C}$ of C. Comonoids

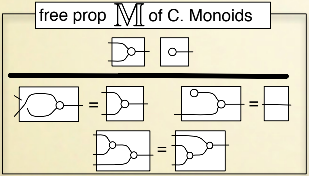


$\cong$

the prop $\mathbb{F}$ of functions

$$\mathbb{F}[n, m] = \text{functions } \overline{n} \to \overline{m}$$

The building blocks of $\mathbb{HA}_{\mathsf{Z}_2}$



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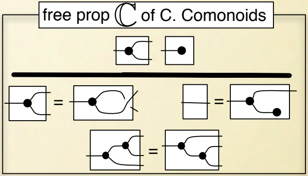
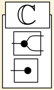


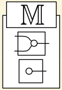
$\cong$

the prop $\mathbb{F}^{op}$

opposite of $\mathbb{F}$

Composing Monoids-Comonoids





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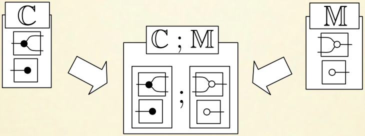

Idea: ring = abelian group + monoid + axioms describing their interaction

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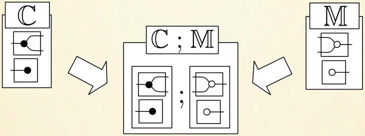

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Composing PROPs [S.Lack, 2004]

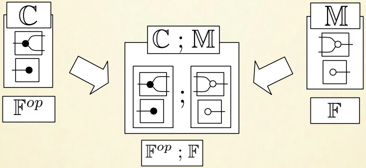
PROPs are monads (in a certain bicategory)

PROP composition = Distributive law between monads

To define the PROP $\mathbb{C}$;$\mathbb{M}$ we need a distributive law:

$$\lambda : \mathbb{M};\mathbb{C} \Rightarrow \mathbb{C};\mathbb{M}$$

Composing Monoids-Comonoids



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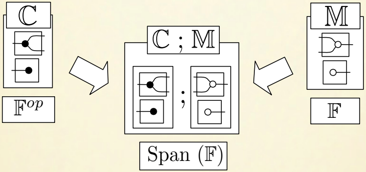

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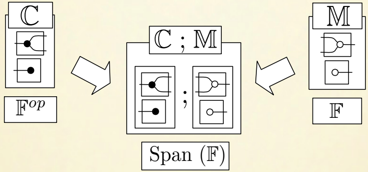
$$: \mathsf{Cospan}(\mathbb{F}) \Rightarrow \mathsf{Span}(\mathbb{F})$$

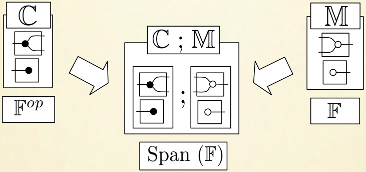
Composing Monoids-Comonoids



$\lambda\colon\ \mathbb{M};\mathbb{C} \Rightarrow \mathbb{C};\mathbb{M}$ defined by pullback in $\mathbb{F}$:

$$\begin{tikzcd}
 & z & \\
n \arrow[ur, "p"] & & m \arrow[ul, "q"'] \\
 & r \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[u, phantom, "\wedge"] &
\end{tikzcd}
\qquad\qquad
\lambda\colon\ (p,q) \mapsto (f,g)$$

Composing Monoids-Comonoids



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$$\begin{tikzcd}
 & 1 & \\
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\end{tikzcd}$$

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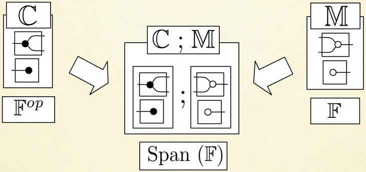

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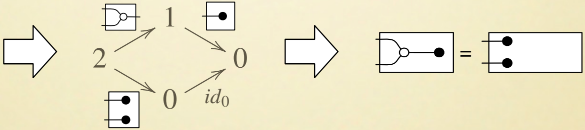

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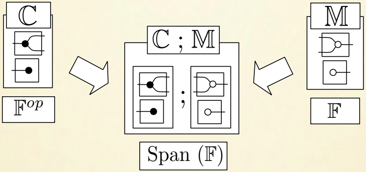

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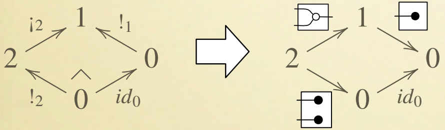

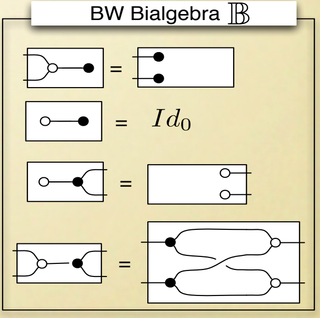

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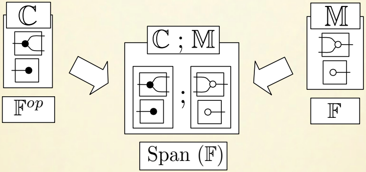

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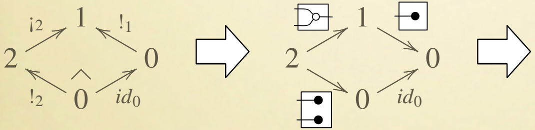
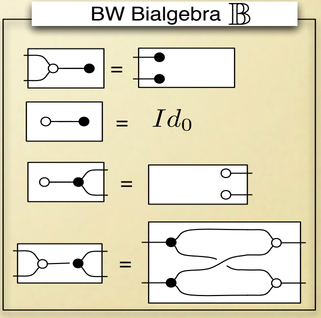
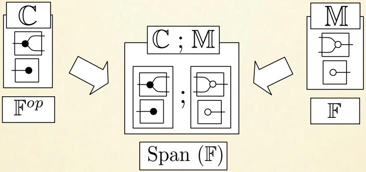




$\mathbb{B}$ as composed PROP

- $\mathbb{B} \cong \mathbb{C};\mathbb{M}$
- $\mathbb{B} \cong$ Span$(\mathbb{F})$

Composing Monoids-Comonoids



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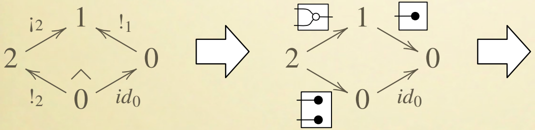

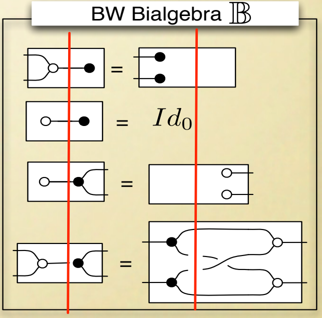

$\mathbb{B}$ as composed PROP

- $\mathbb{B} \cong \mathbb{C};\mathbb{M}$
- $\mathbb{B} \cong \mathsf{Span}(\mathbb{F})$
- factorisation for $\mathbb{B}$-circuits

The cube for Z_2

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 & \mathsf{Mat}\, Z_2 + \mathsf{Mat}\, Z_2{}^{op} \arrow[dl] \arrow[rr] & & \mathsf{Span}(\mathsf{Mat}\, Z_2) \arrow[dl] \\
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$$

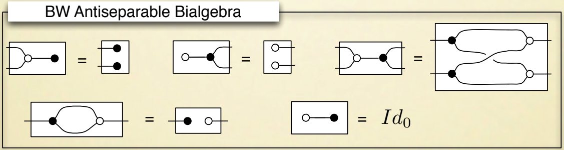
The theory of Z_2-matrices

The PROP $\mathbb{B}$ of bialgebras characterises spans:

$$\mathbb{B} \cong \mathsf{Span}(\mathbb{F}) \cong \mathsf{Mat}\,\mathbb{N}$$

The PROP of antiseparable bialgebras characterises Z_2-matrices:

$$\mathbb{HA}_{Z_2} \cong \mathsf{Mat}\,Z_2 \qquad \text{(Y. Lafont, A. Burroni 1992-95)}$$



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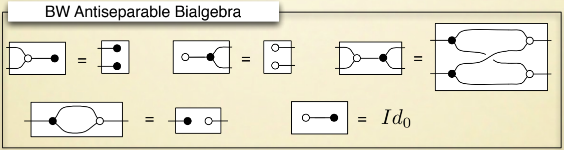

The matrix encoding of a string diagram of $\mathbb{HA}_{Z_2}$:

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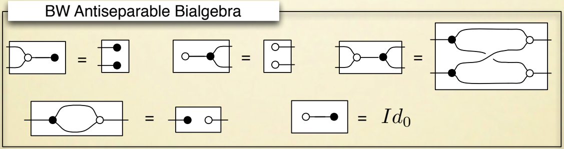

The matrix encoding of a string diagram of $\mathbb{HA}_{Z_2}$:

The theory of Z_2-matrices

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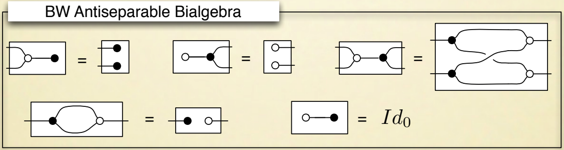

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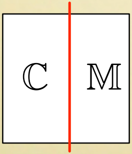

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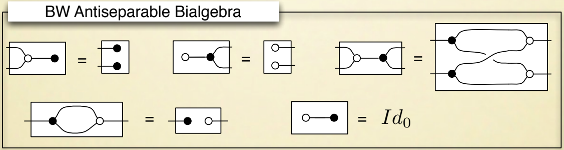

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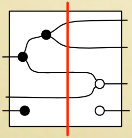

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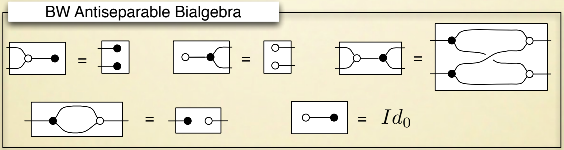

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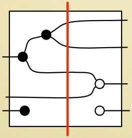


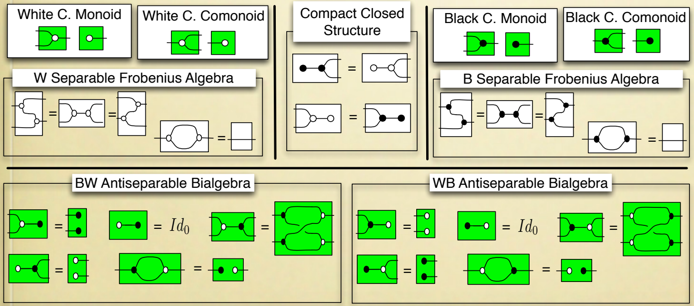
$$\mapsto \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The cube for Z_2

$$
\begin{tikzcd}
 & \mathbb{HA}_{Z_2} + \mathbb{HA}^{op}_{Z_2} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}^{w}_{Z_2} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}^{b}_{Z_2} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2} \arrow[dd, dotted] & \\
 & \mathrm{Mat}\, Z_2 + \mathrm{Mat}\, {Z_2}^{op} \arrow[rr] \arrow[dl] & & \mathrm{Span}(\mathrm{Mat}\, Z_2) \arrow[dl] \\
\mathrm{Cospan}(\mathrm{Mat}\, Z_2) \arrow[rr] & & \mathbb{SV}_{Z_2} &
\end{tikzcd}
$$

The cube for Z_2

$$\begin{tikzcd}
 & \color{red}{\mathbb{HA}_{Z_2} + \mathbb{HA}_{Z_2}^{op}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2}^{w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}_{Z_2}^{b} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2} \arrow[dd, dotted] & \\
 & \color{red}{\mathsf{Mat}\,Z_2 + \mathsf{Mat}\,Z_2{}^{op}} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,Z_2) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,Z_2) \arrow[rr] & & \mathbb{SV}_{Z_2} &
\end{tikzcd}$$



$\mathbb{HA}_{Z_2} + \mathbb{HA}_{Z_2}^{op}$ $\sim$ black-white interaction.

The cube for Z_2

$$\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{Z}_2} + \mathbb{HA}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}^{w}_{\mathsf{Z}_2} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}^{b}_{\mathsf{Z}_2} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{Z}_2} \arrow[dd, dotted] & \\
 & \mathsf{Mat}\,\mathsf{Z}_2 + \mathsf{Mat}\,\mathsf{Z}_2{}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[dl] \\
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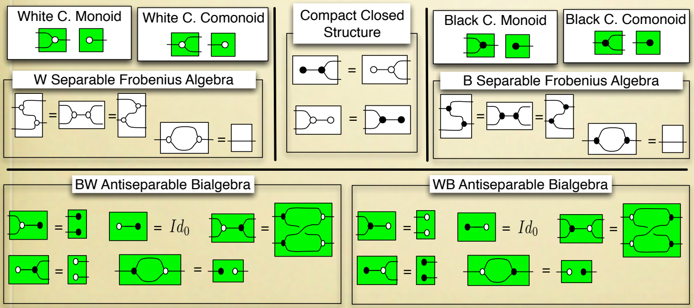
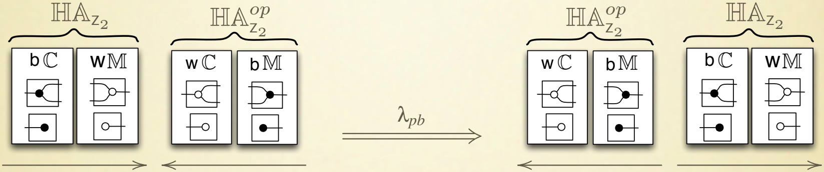


Composing $\mathbb{HA}_{\mathsf{Z}_2}$, $\mathbb{HA}^{op}_{\mathsf{Z}_2}$: black-black & white-white interaction.

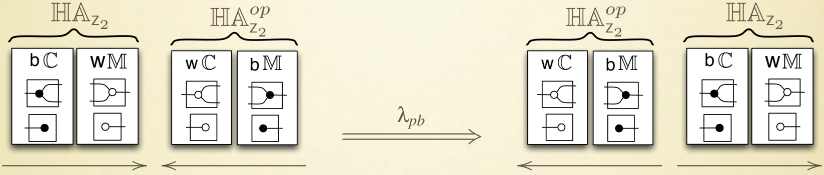
Composing $\mathbb{HA}_{\mathsf{Z}_2}$ and $\mathbb{HA}_{\mathsf{Z}_2}^{op}$

Construct the PROP $\mathbb{HA}_{\mathsf{Z}_2}^{op} ; \mathbb{HA}_{\mathsf{Z}_2}$ by pullback:

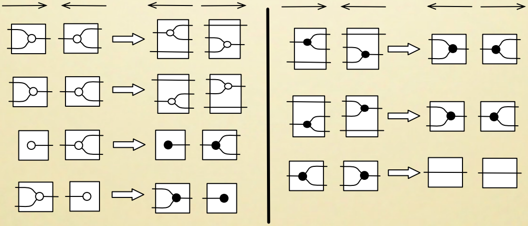


Composing $\mathbb{HA}_{\mathsf{Z}_2}$ and $\mathbb{HA}^{op}_{\mathsf{Z}_2}$

Construct the PROP $\mathbb{HA}^{op}_{\mathsf{Z}_2} ; \mathbb{HA}_{\mathsf{Z}_2}$ by pullback:



Read (in $\mathsf{Mat}\,\mathsf{Z}_2$) the equations of $\mathbb{HA}^{op}_{\mathsf{Z}_2} ; \mathbb{HA}_{\mathsf{Z}_2}$ out of pullback squares:



Interacting Bialgebras are Frobenius!

How are these axioms enough?

Correctness

- Each axiom is read off by some pullback square.

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- Each axiom is read off by some pullback square.

Completeness

- All the equations arising by pullback squares are derivable by the axioms.

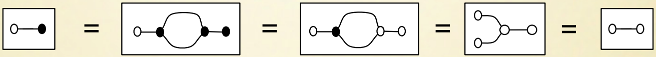
  $\Rightarrow$ Pullbacks in $\mathsf{Mat}\,\mathsf{Z}_2$ are constructed essentially by computing kernels of matrices.

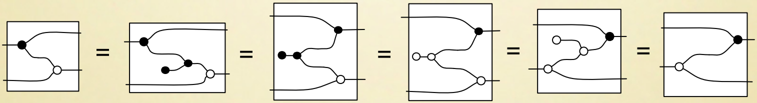
  $\Rightarrow$ The linear algebraic calculations yielding the kernel can be mimicked at the syntactic level (using the equational theory).

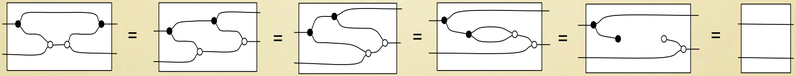
  $\Rightarrow$ Graphical linear algebra!

A glance at the equational theory of $\mathsf{Span}(\mathsf{Mat}\,\mathsf{Z}_2)$

Id_0 =

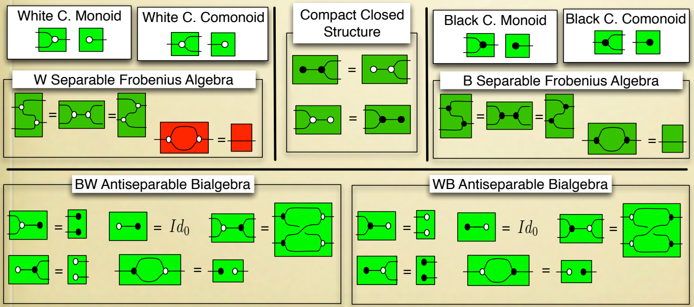






The cube for Z_2

$$\begin{tikzcd}
 & & \mathbb{HA}_{Z_2} + \mathbb{HA}_{Z_2}^{op} \arrow[rr] \arrow[dll] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2}^{w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}_{Z_2}^{b} \arrow[rrr] \arrow[dd, "\cong"'] & & & \mathbb{IH}_{Z_2} \arrow[dd, dotted] & \\
 & & \mathsf{Mat}\,Z_2 + \mathsf{Mat}\,Z_2{}^{op} \arrow[rr] \arrow[dll] & & \mathsf{Span}(\mathsf{Mat}\,Z_2) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,Z_2) \arrow[rrr] & & & \mathbb{SV}_{Z_2} &
\end{tikzcd}$$



$$\mathbb{HA}_{Z_2}^{op} ; \mathbb{HA}_{Z_2}$$

$$=$$

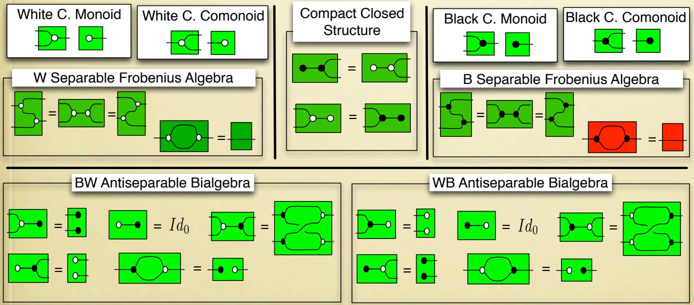
$\mathbb{IH}_{Z_2}$ minus White Separability

$$=$$

$$\mathbb{IH}_{Z_2}^{w}$$

The cube for Z_2

$$
\begin{tikzcd}
& \mathbb{HA}_{Z_2} + \mathbb{HA}^{op}_{Z_2} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}^{w}_{Z_2} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}^{b}_{Z_2} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2} \arrow[dd, dotted] & \\
& \mathsf{Mat}\,Z_2 + \mathsf{Mat}\,Z_2{}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,Z_2) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,Z_2) \arrow[rr] & & \mathbb{SV}_{Z_2} &
\end{tikzcd}
$$



$\mathbb{HA}_{Z_2}\ ;\ \mathbb{HA}^{op}_{Z_2}$

=

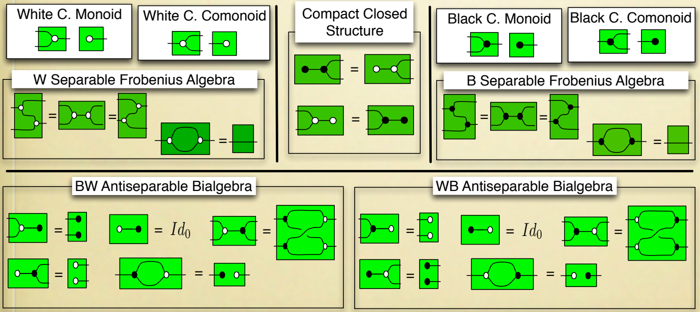
$\mathbb{IH}^{b}_{Z_2}$ ($\mathbb{IH}_{Z_2}$ minus Black Sep.)

=

"photographic negative" of $\mathbb{IH}^{w}_{Z_2}$

The cube for Z_2

$$
\begin{tikzcd}
& \mathbb{HA}_{Z_2}+\mathbb{HA}_{Z_2}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}^{w}_{Z_2} \arrow[dl, "\text{W Sep}"] \arrow[dd, "\cong"] \\
\mathbb{IH}^{b}_{Z_2} \arrow[rr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & \mathbb{IH}_{Z_2} \arrow[dd, dotted] & \\
& \mathsf{Mat}\,Z_2+\mathsf{Mat}\,Z_2{}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,Z_2) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,Z_2) \arrow[rr] & & \mathbb{SV}_{Z_2} &
\end{tikzcd}
$$



The cube for Z_2

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\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{Z}_2} + \mathbb{HA}_{\mathsf{Z}_2}^{op} \arrow[dl] \arrow[rrr] \arrow[dd, "\cong"'] & & & \mathbb{IH}_{\mathsf{Z}_2}^{w} \arrow[dl, "\text{W Sep}"] \arrow[dd, "\cong"] \\
\mathbb{IH}_{\mathsf{Z}_2}^{b} \arrow[rrr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & & \mathbb{IH}_{\mathsf{Z}_2} \arrow[dd, dotted] & \\
 & \mathsf{Mat}\,\mathsf{Z}_2 + \mathsf{Mat}\,\mathsf{Z}_2{}^{op} \arrow[dl] \arrow[rrr] & & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[dl, "\Phi"] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{Z}_2) \arrow[rrr, "\Psi"'] & & & \mathbb{SV}_{\mathsf{Z}_2} &
\end{tikzcd}
$$

$$
\begin{aligned}
\Psi(n \xrightarrow{A} z \xleftarrow{B} m) &= \{ (\mathbf{x},\mathbf{y}) \mid \mathbf{x} \in \mathsf{Z}_2^n,\, \mathbf{y} \in \mathsf{Z}_2^m,\ A\mathbf{x} = B\mathbf{y} \} \\
\Phi(n \xleftarrow{A} z \xrightarrow{B} m) &= \{ (\mathbf{x},\mathbf{y}) \mid \mathbf{x} \in \mathsf{Z}_2^n,\, \mathbf{y} \in \mathsf{Z}_2^m,\ \exists \mathbf{z} \in \mathsf{Z}_2^z.\ A\mathbf{z} = \mathbf{x} \wedge B\mathbf{z} = \mathbf{y} \}
\end{aligned}
$$

The cube for Z_2

$$\begin{tikzcd}
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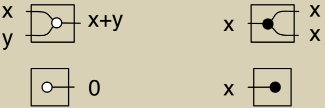
\mathbb{IH}_{Z_2}^{b} \arrow[rr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & \color{red}{\mathbb{IH}_{Z_2}} \arrow[dd, dotted] & \\
 & \mathsf{Mat}\,Z_2 + \mathsf{Mat}\,Z_2{}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,Z_2) \arrow[dl, "\Phi"] \\
\mathsf{Cospan}(\mathsf{Mat}\,Z_2) \arrow[rr, "\Psi"'] & & \color{red}{\mathbb{SV}_{Z_2}} &
\end{tikzcd}$$

- $\mathbb{IH}_{Z_2}$ and $\mathbb{SV}_{Z_2}$ are pushout objects.

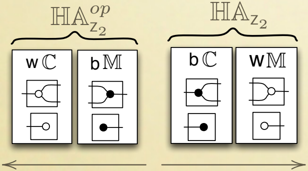
- Unique arrow $\mathcal{S} : \mathbb{IH}_{Z_2} \xrightarrow{\cong} \mathbb{SV}_{Z_2}$



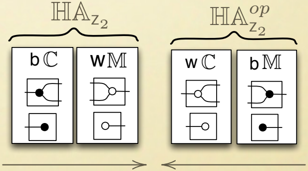
Benefits

- Cube construction revealing the modular structure of $\mathbb{IH}_{\mathsf{Z}_2}$.
- Functorial semantics $\mathcal{S}_{\mathbb{IH}_{\mathsf{Z}_2}} : \mathbb{IH}_{\mathsf{Z}_2} \to \mathbb{SV}_{\mathsf{Z}_2}$.
- Factorisation properties of $\mathbb{IH}_{\mathsf{Z}_2}$

Factorisation of $\mathbb{IH}^{w}_{\mathsf{Z}_2}$
(span)



Factorisation of $\mathbb{IH}^{b}_{\mathsf{Z}_2}$
(cospan)



The general case

The cube for an arbitrary PID

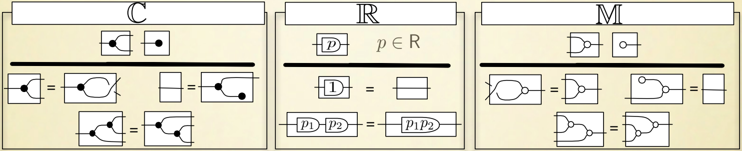
$$
\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{R}} + \mathbb{HA}_{\mathsf{R}}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}}^{w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}_{\mathsf{R}}^{b} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted] & \\
 & \mathsf{Mat}\,\mathsf{R} + \mathsf{Mat}\,\mathsf{R}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{R}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{R}) \arrow[rr] & & \mathbb{SV}_{\mathsf{R}} &
\end{tikzcd}
$$

The cube for an arbitrary PID

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{R}} + \mathbb{HA}_{\mathsf{R}}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}}^{w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}_{\mathsf{R}}^{b} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted] & \\
& \mathsf{Mat}\,\mathsf{R} + \mathsf{Mat}\,\mathsf{R}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{R}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{R}) \arrow[rr] & & \mathbb{SV}_{\mathsf{R}} &
\end{tikzcd}
$$

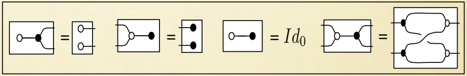
Modular construction of $\mathbb{HA}_{\mathsf{R}}$

- $\mathbb{C}\,;\mathbb{R}\,;\mathbb{M}$ is the composite of three PROPs

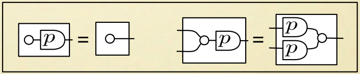


formed (equivalently, by $\lambda_{\mathbb{R}}\,;\mathbb{C}\sigma$ or $\mathbb{R}\lambda\,;\tau_{\mathbb{M}}$) via distributive laws

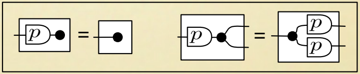
  - $\lambda\colon\ \mathbb{M}\,;\mathbb{C} \Rightarrow \mathbb{C}\,;\mathbb{M}$



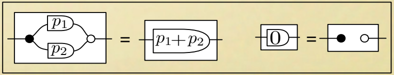
  - $\sigma\colon\ \mathbb{M}\,;\mathbb{R} \Rightarrow \mathbb{R}\,;\mathbb{M}$



  - $\tau\colon\ \mathbb{R}\,;\mathbb{C} \Rightarrow \mathbb{C}\,;\mathbb{R}$



- $\mathbb{HA}_{\mathsf{R}}$ is $\mathbb{C}\,;\mathbb{R}\,;\mathbb{M}$ quotiented by

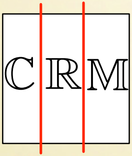


The theory of R-matrices

- Interpretation of a string diagram of $\mathbb{HA}_{\mathsf{R}}$ as an R-matrix

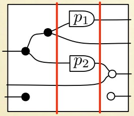
The theory of R-matrices

- Interpretation of a string diagram of $\mathbb{HA}_{\mathsf{R}}$ as an R-matrix



The theory of R-matrices

- Interpretation of a string diagram of $\mathbb{HA}_{\mathsf{R}}$ as an R-matrix



The theory of R-matrices

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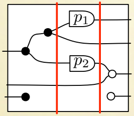
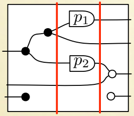


$$\mapsto \qquad \begin{pmatrix} p_1 & 0 & 0 \\ 1 & 0 & 0 \\ p_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The theory of R-matrices

- Interpretation of a string diagram of $\mathbb{HA}_{\mathsf{R}}$ as an R-matrix



$$\mapsto \begin{pmatrix} p_1 & 0 & 0 \\ 1 & 0 & 0 \\ p_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Characterisation result

$$\mathbb{HA}_{\mathsf{R}} \cong \mathsf{Mat}\,\mathsf{R}$$

(Y. Lafont 2003 - over fields)

The cube for an arbitrary PID

$$
\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{R}} + \mathbb{HA}_{\mathsf{R}}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}}^{w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}_{\mathsf{R}}^{b} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted] & \\
 & \mathsf{Mat}\,\mathsf{R} + \mathsf{Mat}\,\mathsf{R}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{R}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{R}) \arrow[rr] & & \mathbb{SV}_{\mathsf{R}} &
\end{tikzcd}
$$

The cube for an arbitrary PID

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{R}} + \mathbb{HA}_{\mathsf{R}}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}}^{w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}_{\mathsf{R}}^{b} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted] & \\
& \mathsf{Mat}\,\mathsf{R} + \mathsf{Mat}\,\mathsf{R}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{R}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{R}) \arrow[rr] & & \mathbb{SV}_{\mathsf{R}} &
\end{tikzcd}
$$

The bottom face

$$
\begin{tikzcd}
& \mathsf{Mat\,R} + \mathsf{Mat\,R}^{op} \arrow[dl] \arrow[rr] & & \mathsf{Span}(\mathsf{Mat\,R}) \arrow[dl, "\Phi"] \\
\mathsf{Cospan}(\mathsf{Mat\,R}) \arrow[rr, "\Psi"'] & & \mathbb{SV}_{\mathsf{R}} &
\end{tikzcd}
$$

$$
\begin{aligned}
\Psi(n \xrightarrow{A} z \xleftarrow{B} m) &= \{\, (\mathbf{x},\mathbf{y}) \mid \mathbf{x} \in \mathsf{k}^n, \mathbf{y} \in \mathsf{k}^m,\ A\mathbf{x} = B\mathbf{y} \,\} \\
\Phi(n \xleftarrow{A} z \xrightarrow{B} m) &= \{\, (\mathbf{x},\mathbf{y}) \mid \mathbf{x} \in \mathsf{k}^n, \mathbf{y} \in \mathsf{k}^m, \exists \mathbf{z} \in \mathsf{k}^z.\, A\mathbf{z} = \mathbf{x} \wedge B\mathbf{z} = \mathbf{y} \,\}
\end{aligned}
$$

The bottom face

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- **Why R needs to be a PID?**
  - R is a PID iff submodules of free R-modules are free.
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The bottom face

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\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{R}) \arrow[rr, "\Psi"'] & & \mathbb{SV}_{\mathsf{R}}
\end{tikzcd}$$

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- **Why subspaces over the field of fractions k of R?**
  - Ψ and Φ mimick the Set-like construction of pullbacks and pushouts.
  - Functoriality of Ψ relies on the fact that k is a field and the category of (free) k-modules has Set-like pushouts.

The cube for an arbitrary PID

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{R}} + \mathbb{HA}_{\mathsf{R}}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}}^{w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}_{\mathsf{R}}^{b} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted] & \\
& \mathsf{Mat}\,\mathsf{R} + \mathsf{Mat}\,\mathsf{R}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{R}) \arrow[dl] \\
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\mathbb{IH}_{\mathsf{R}}^{b} \arrow[rr, crossing over] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} & \\
& \mathsf{Mat}\,\mathsf{R} + \mathsf{Mat}\,\mathsf{R}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{R}) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{R}) \arrow[rr] & & \mathbb{SV}_{\mathsf{R}} \arrow[from=uu, dotted, crossing over] &
\end{tikzcd}
$$

The top face

$$\begin{tikzcd}
& \mathbb{HA}_{\mathsf{R}} + \mathbb{HA}_{\mathsf{R}}^{op} \arrow[rr] \arrow[dl] & & \mathbb{IH}_{\mathsf{R}}^{w} \arrow[dl] \\
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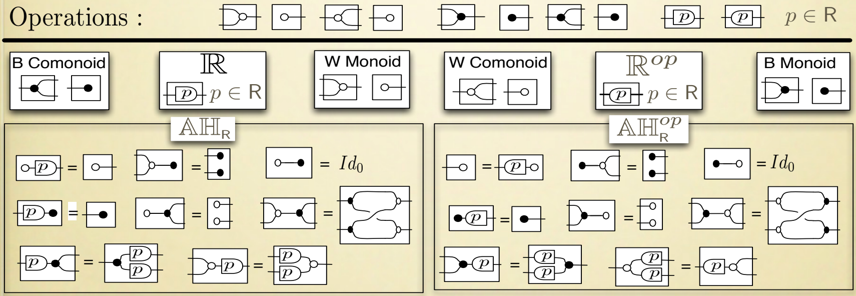
\end{tikzcd}$$

The top face

$$\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{R}} + \mathbb{HA}_{\mathsf{R}}^{op} \arrow[r] \arrow[dl] & \mathbb{IH}_{\mathsf{R}}^{w} \arrow[d] \\
\mathbb{IH}_{\mathsf{R}}^{b} \arrow[rr] & & \mathbb{IH}_{\mathsf{R}}
\end{tikzcd}$$

Operations :



The top face

$$\begin{tikzcd}
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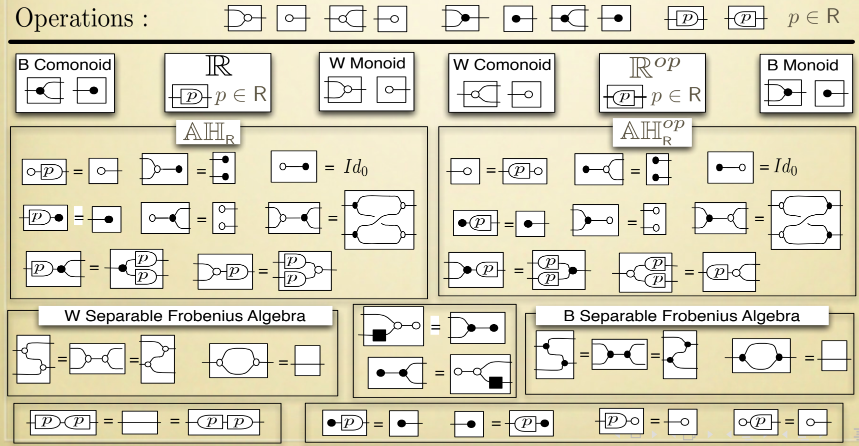
\end{tikzcd}$$



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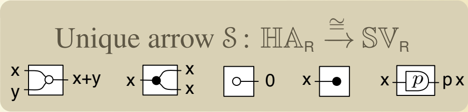
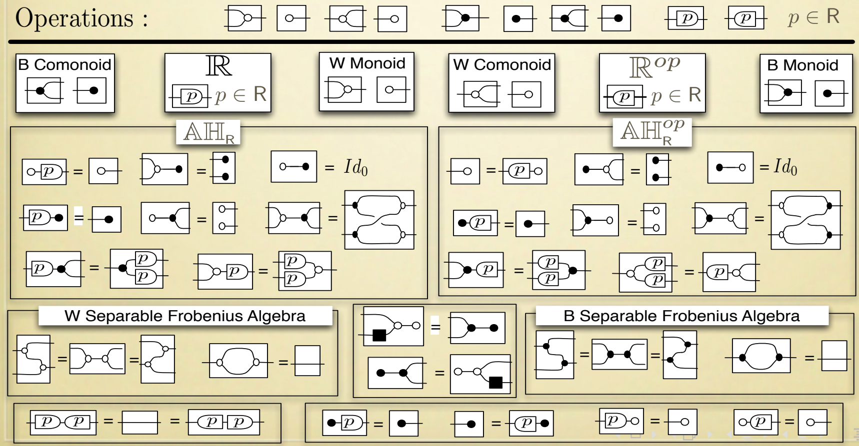
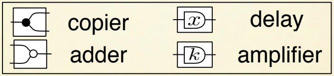




Further directions

- For k a field, $\mathbb{IH}_{\mathsf{k}[X]}$ can be thought as a theory of *stateful* connectors.



$\Rightarrow$ We can characterise the sub-PROP of $\mathbb{IH}_{\mathsf{k}[X]}$ whose string diagrams are signal flow-graphs.

$\Rightarrow$ Study implementability of string diagrams as circuits.

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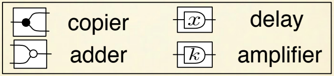

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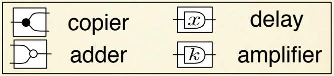

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- Modular approach to the algebra of graphs (*cf.* Fiore & Campos) and of Petri nets.

References

- The general case

  Bonchi, Sobocinski, Z. *Interacting Hopf Algebras*

- The $\mathbb{Z}_2$ case

  Bonchi, Sobocinski, Z. *Interacting Bialgebras are Frobenius* (FoSSaCS'14)

- The polynomial case

  Bonchi, Sobocinski, Z. *A categorical semantics of signal flow graphs*