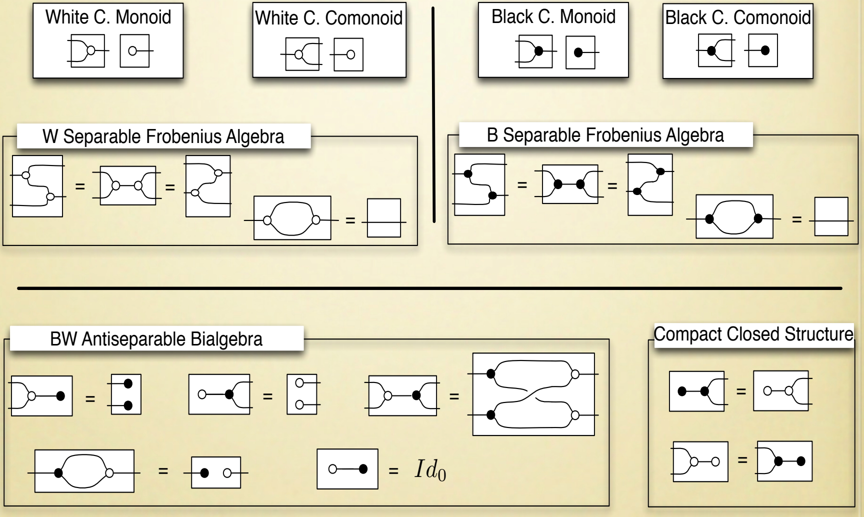
Interacting Bialgebras are Frobenius

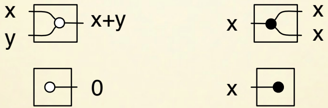
Fabio Zanasi

Joint work with Filippo Bonchi and Paweł Sobociński

The theory $\mathbb{IB}$

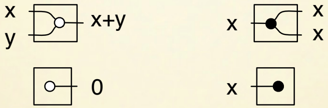


$\mathbb{Z}_2$-subspace Semantics



$\mathbb{Z}_2$-subspace Semantics

Semantics $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$



Domain of interpretation: the SMC $\mathbb{SV}$ of $\mathbb{Z}_2$-sub-vector spaces

- objects: natural numbers
- $\mathbb{SV}[n,m]$ = subspaces of $\mathbb{Z}_2^n \times \mathbb{Z}_2^m$
- relational composition
- monoidal product: direct sum

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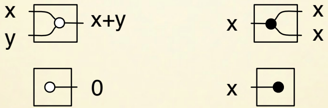

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Characterization result

$\mathbb{IB} \cong \mathbb{SV}$ and $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$ is full and faithful.

The Cube

$$
\begin{tikzcd}
 & \mathbb{AB} + \mathbb{AB}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IB}^{-w} \arrow[dl] \arrow[dd, "\cong"] \\
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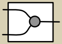
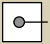
PROPs

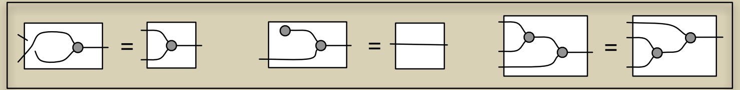
- PROPs encode algebraic theories in a symmetric monoidal setting.

A PROP is a SMC with

- objects: the natural numbers
- $n \otimes m = n + m$
- sym_n = permutations of $\overline{n}$

Example: the PROP $\mathbb{M}$ of Commutative Monoids

Arrows are freely generated by operations ,  and equations



Observations

- $\mathbb{M} \cong \mathbb{F}$ (the PROP of functions)
- Commutative comonoids: $\mathbb{C} = \mathbb{M}^{op} \cong \mathbb{F}^{op}$

Composing PROPs

- Idea: a ring = an abelian group interacting with a monoid

Build a PROP as the composite of two sub-PROPs

PROPs are monads (in a certain bicategory)

PROP composition = Distributive law between monads

S.Lack - Composing PROPs (2004)

Composing Monoids-Comonoids

A distributive law $\lambda\colon \mathbb{M};\mathbb{C} \Rightarrow \mathbb{C};\mathbb{M}$ between (white) $\mathbb{M}$ and (black) $\mathbb{C}$

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defined by pullback in $\mathbb{F}$:

$$\begin{tikzcd}
 & z \arrow[dd, phantom, "\Downarrow\lambda"] & \\
n \arrow[ur, "f"] & & m \arrow[ul, "g^\star"'] \\
 & z' \arrow[ul, "u^\star"] \arrow[ur, "v"'] &
\end{tikzcd}$$

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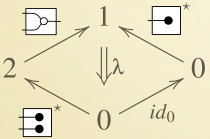

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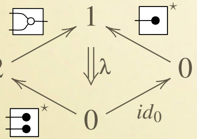

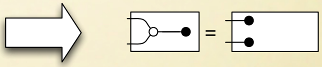

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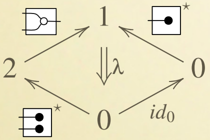

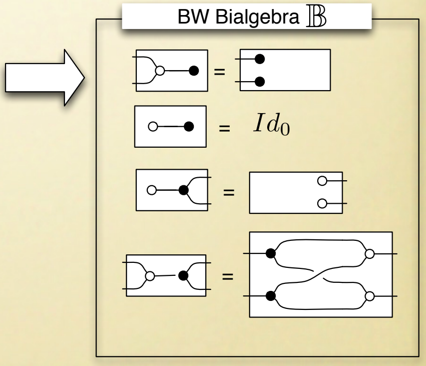

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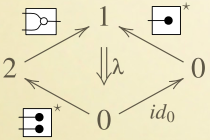

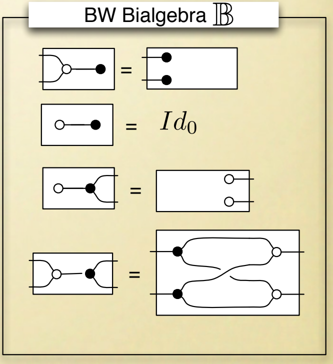

Characterization Result (Lack)

- $\mathbb{B} = \mathbb{C};\mathbb{M}$
- complete for semantics $\mathsf{Span}(\mathbb{F})$

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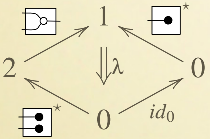

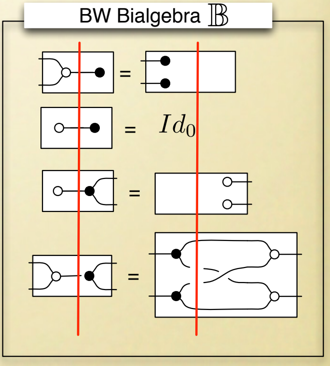

Characterization Result (Lack)

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- factorisation for $\mathbb{B}$-nets

The Cube

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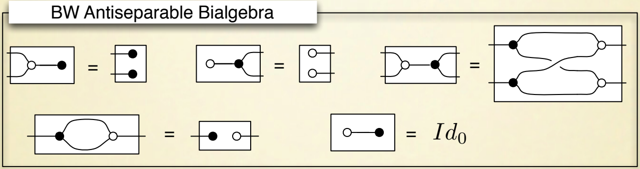
Antiseparable Bialgebra $\mathbb{AB}$

$\mathbb{AB}$ = BW Bialgebra + Antiseparability axiom



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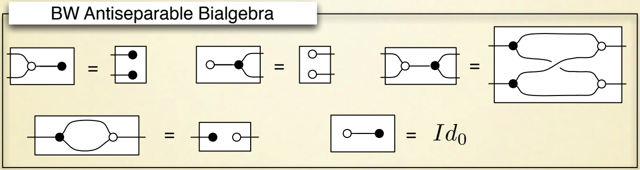

There is a 1-1 correspondence between $\mathbb{AB}$-nets and $\mathbb{Z}_2$-matrices.

Characterization result (Lafont): $\mathbb{AB} \cong \mathrm{Mat}\,\mathbb{Z}_2$

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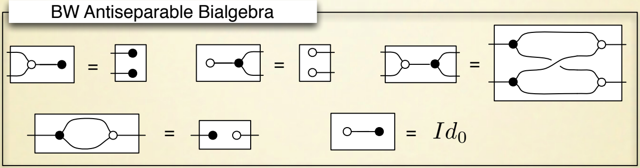

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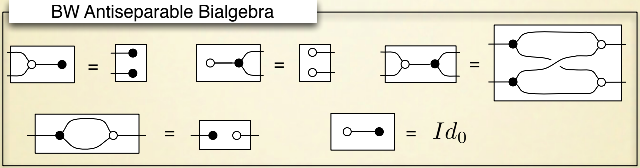

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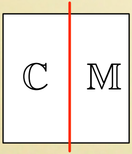

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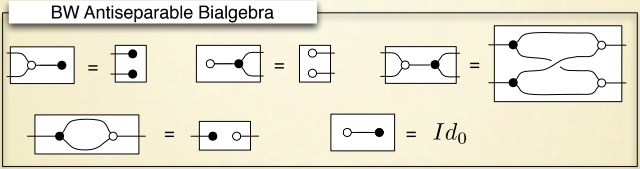

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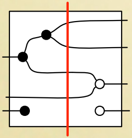

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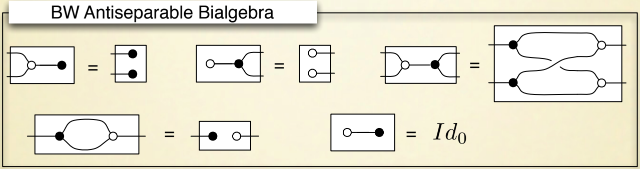

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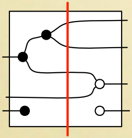

$$\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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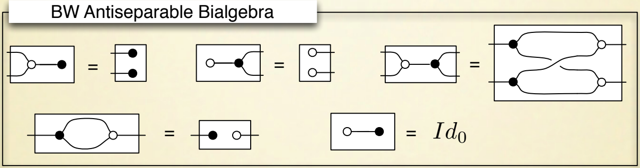

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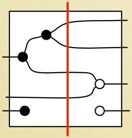

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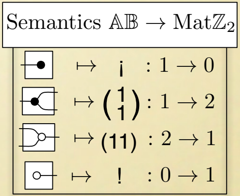

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The Cube

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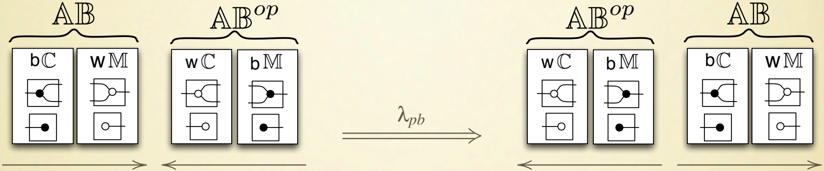
Composing $\mathbb{AB}$ and $\mathbb{AB}^{op}$

- $\mathbb{AB} \sim$ interaction of black $\mathbb{C}$ and white $\mathbb{M}$
- $\mathbb{AB}^{op} \sim$ interaction of white $\mathbb{C}$ and black $\mathbb{M}$
- Composing $\mathbb{AB}$ and $\mathbb{AB}^{op}$: we make the two white and the two black (co)monoids interact

Composing $\mathbb{AB}$ and $\mathbb{AB}^{op}$

Construct the PROP $\mathbb{AB}^{op};\mathbb{AB} = \mathsf{Span}(\mathbb{AB})$



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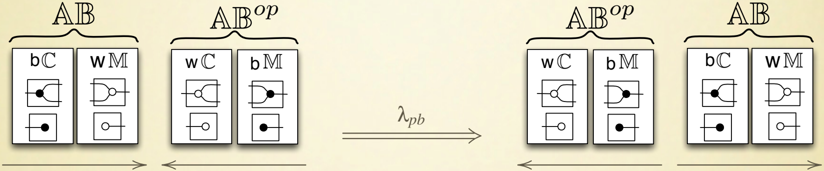
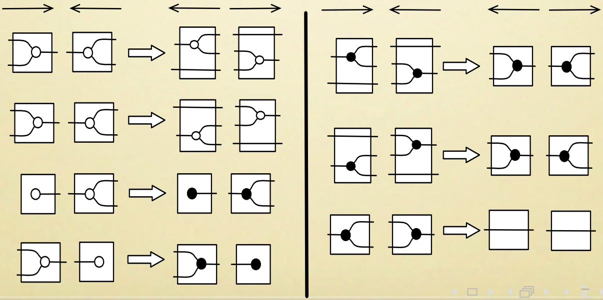
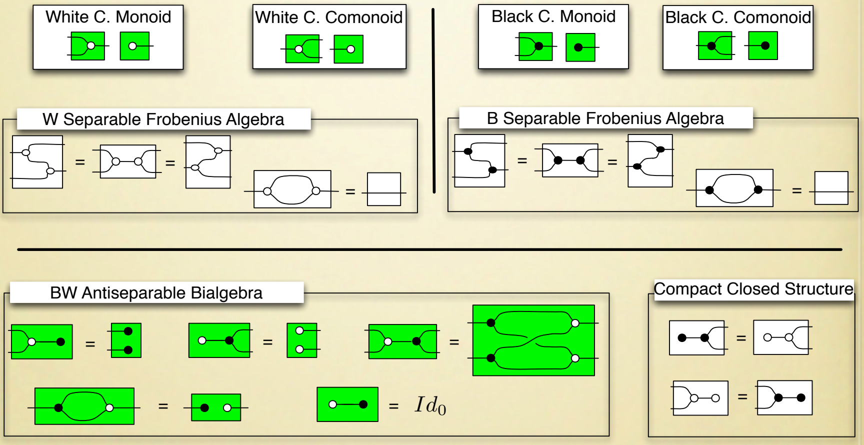


Calculate (in $\mathrm{Mat}\,\mathbb{Z}_2$) the equations of $\mathsf{Span}(\mathbb{AB})$ out of pullbacks:



Comparing Span($\mathbb{AB}$) and $\mathbb{IB}$



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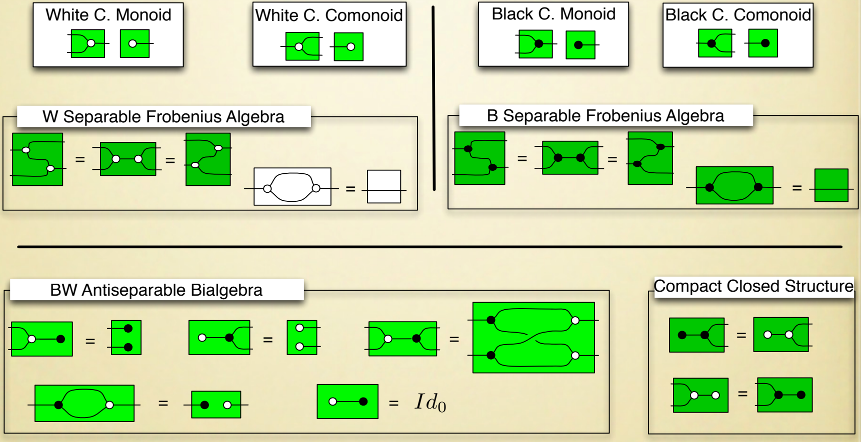
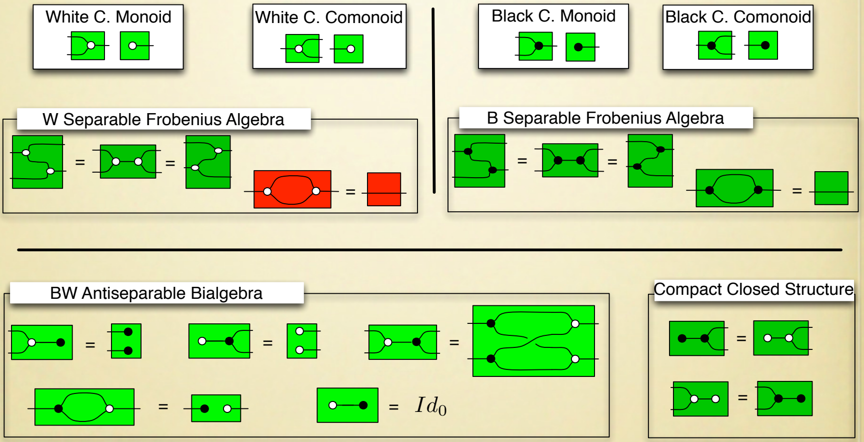


Comparing Span$(\mathbb{AB})$ and $\mathbb{IB}$



Span$(\mathbb{AB}) = \mathbb{IB}$ minus White Separability $= \mathbb{IB}^{-w}$

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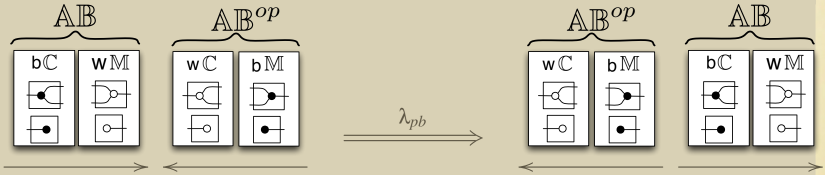
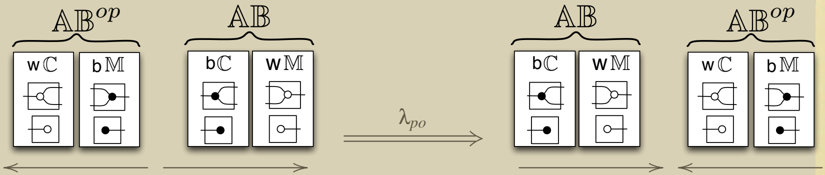
\end{tikzcd}$$

Composing $\mathbb{AB}$ and $\mathbb{AB}^{op}$

Span($\mathbb{AB}$)

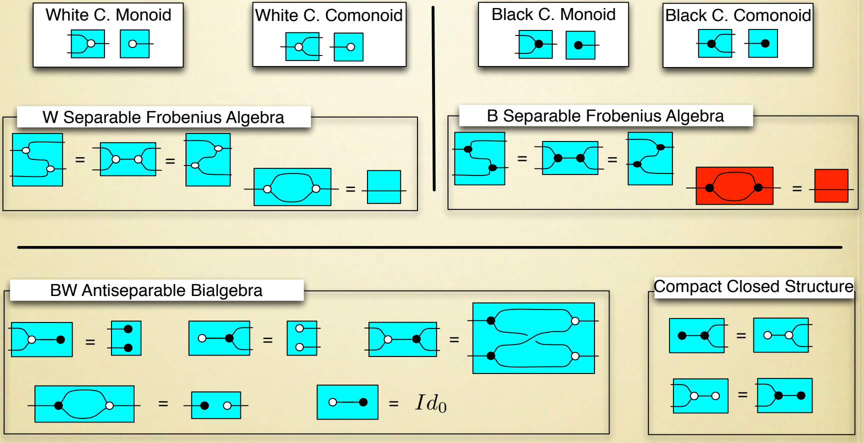


Cospan($\mathbb{AB}$)



Cospan($\mathbb{AB}$) is the "photographic negative" of Span($\mathbb{AB}$)

Comparing Cospan($\mathbb{AB}$) and $\mathbb{IB}$



Cospan($\mathbb{AB}$) = $\mathbb{IB}$ minus Black Separability = $\mathbb{IB}^{-b}$

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 & & \mathbb{AB} + \mathbb{AB}^{op} \arrow[rr] \arrow[dll] \arrow[dd, "\cong"'] & & \mathbb{IB}^{-w} \arrow[dl, "\text{W Sep}"] \arrow[dd, "\cong"] \\
\mathbb{IB}^{-b} \arrow[rrr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & & \mathbb{IB} \arrow[dd, dotted] & \\
 & & \mathrm{Mat}\,\mathbb{Z}_2 + \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \arrow[rr] \arrow[dll] & & \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[dl] \\
\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[rrr] & & & \mathbb{SV} &
\end{tikzcd}
$$

The Cube

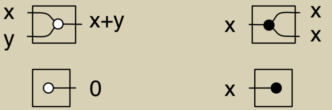
$$
\begin{tikzcd}
& \mathbb{AB} + \mathbb{AB}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IB}^{-w} \arrow[dl, "\text{W Sep}"] \arrow[dd, "\cong"] \\
\mathbb{IB}^{-b} \arrow[rr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & \mathbb{IB} \arrow[dd, dotted] & \\
& \mathsf{Mat}\,\mathbb{Z}_2 + \mathsf{Mat}\,\mathbb{Z}_2{}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathbb{Z}_2) \arrow[dl, "\text{Span}\mapsto\text{Rel}"] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathbb{Z}_2) \arrow[rr, "\text{Cospan}\mapsto\text{Rel}"'] & & \mathbb{SV} &
\end{tikzcd}
$$

The Cube

$$
\begin{tikzcd}
 & & \mathbb{AB}+\mathbb{AB}^{op} \arrow[rr] \arrow[dll] \arrow[dd, "\cong"'] & & \mathbb{IB}^{-w} \arrow[dl, "\text{W Sep}"'] \arrow[dd, "\cong"] \\
\mathbb{IB}^{-b} \arrow[rrr, "\text{B Sep}"'] \arrow[dd, "\cong"'] & & & \mathbb{IB} \arrow[dd, dotted] & \\
 & & \mathrm{Mat}\,\mathbb{Z}_2+\mathrm{Mat}\,\mathbb{Z}_2{}^{op} \arrow[rr] \arrow[dll] & & \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[dl, "\text{Span}\mapsto\text{Rel}"] \\
\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[rrr, "\text{Cospan}\mapsto\text{Rel}"'] & & & \mathbb{SV} &
\end{tikzcd}
$$

- $\mathbb{IB}$ and $\mathbb{SV}$ are pushout objects.

- Unique arrow $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$



Discussion

Results

- Cube construction revealing the modular structure of $\mathbb{IB}$.
- Completeness for the semantics $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$.
- Factorisation properties of $\mathbb{IB}$.

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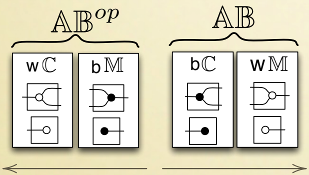
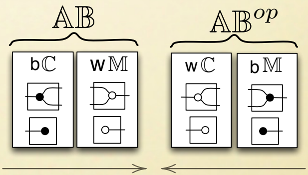
Factorisation of $\mathbb{IB}^{-w}$



Factorisation of $\mathbb{IB}^{-b}$



Discussion

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- Cube construction revealing the modular structure of $\mathbb{IB}$.
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Future Work

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- Cube construction revealing the modular structure of $\mathbb{IB}$.
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- Factorisation properties of $\mathbb{IB}$.

Future Work

- Cubes are everywhere.

$$
\begin{tikzcd}
 & \mathbb{M}+\mathbb{C} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{B} \arrow[dl] \arrow[dd, "\cong"] \\
\mathrm{SF{\scriptsize ROB}} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{B} \uplus \mathrm{SF{\scriptsize ROB}} \arrow[dd, dotted] & \\
 & \mathbb{F}+\mathbb{F}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathbb{F}) \arrow[dl] \\
\mathsf{Cospan}(\mathbb{F}) \arrow[rr] & & \mathrm{E{\scriptsize Q}R{\scriptsize EL}} &
\end{tikzcd}
$$

Discussion

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$$
\begin{tikzcd}
& \mathbb{M}+\mathbb{C} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{B} \arrow[dl] \arrow[dd, "\cong"] \\
\textsc{SFrob} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{B} \uplus \textsc{SFrob} \arrow[dd, dotted] & \\
& \mathbb{F}+\mathbb{F}^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathbb{F}) \arrow[dl] \\
\mathsf{Cospan}(\mathbb{F}) \arrow[rr] & & \textsc{EqRel} &
\end{tikzcd}
$$

- Explore the syntactic PROP of $\mathsf{Mat}\,\mathbb{R}$, where $\mathbb{R}$ is a field/ring.

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$$
\begin{tikzcd}
 & \mathbb{M}+\mathbb{C} \arrow[ld] \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{B} \arrow[ld] \arrow[dd, "\cong"] \\
\textsc{SFrob} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{B} \uplus \textsc{SFrob} \arrow[dd, dotted] & \\
 & \mathbb{F}+\mathbb{F}^{op} \arrow[rr] \arrow[ld] & & \mathsf{Span}(\mathbb{F}) \arrow[ld] \\
\mathsf{Cospan}(\mathbb{F}) \arrow[rr] & & \textsc{EqRel} &
\end{tikzcd}
$$

- Explore the syntactic PROP of Mat $\mathbb{R}$, where $\mathbb{R}$ is a field/ring.
- Other directions: full ZX-calculus, Algebra of stateless connectors, Petri Calculus, Stream Calculus.