A predicate/state transformer semantics for Bayesian learning

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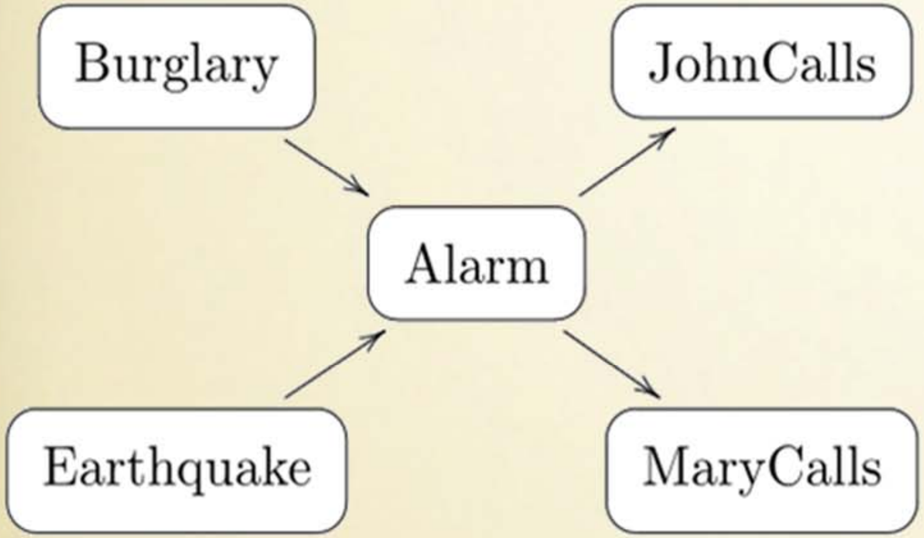
1. Introduction

Bayesian learning



$\Pr(B)$
$\frac{1}{1000}$

B	E	$\Pr(A)$
t	t	$\frac{95}{100}$
t	f	$\frac{94}{100}$
f	t	$\frac{29}{100}$
f	f	$\frac{1}{1000}$

$\Pr(E)$
$\frac{1}{500}$

A	$\Pr(J)$
t	$\frac{9}{10}$
f	$\frac{1}{20}$

A	$\Pr(M)$
t	$\frac{7}{10}$
f	$\frac{1}{100}$

P(B|M)

- **Backward inference**
  - *What is the likelihood of burglars, given that Mary called?*
- **Forward inference**
  - *if there are burglars, what is the likelihood that Mary calls?*

In this work

- An abstract approach to learning

Backward Inference	∼	**Predicate Transformer** and then **Conditioning**
Forward Inference	∼	**Conditioning** and then **State Transformer**

- Categorical framework: **effectus theory**

2. Effectuses, predicates, states

Effectuses

- An **effectus** is a category with finite coproducts (+,0) and a final object 1 satisfying a few basic requirements.

- Examples

  - $\mathbf{Set}$ classical
  - $\mathcal{K}\ell(\mathcal{D})$ discrete probabilistic
  - $\mathcal{K}\ell(\mathcal{G})$ continuous probabilistic
  - $\mathbf{vNA}^{\mathrm{op}}$ quantum

States and predicates in an effectus

Predicate

$X \to 2 = 1 + 1$

State

$1 \to X$

$$
\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathbf{Conv} \arrow[ll, bend left] \\
& \mathbf{C} \arrow[ul, "\mathrm{Pred}=\mathrm{Hom}(-,2)"] \arrow[ur, "\mathrm{Stat}=\mathrm{Hom}(1,-)"'] &
\end{tikzcd}
$$

Pred. transformer

$$\mathrm{Pred}(X) \xleftarrow{f^* = (-)\circ f} \mathrm{Pred}(Y)$$

State transformer

$$\mathrm{Stat}(X) \xrightarrow{f_* = f\circ(-)} \mathrm{Stat}(Y)$$

States and predicates in an effectus

	State $\mathbf{1} \to \boldsymbol{X}$	**Predicate** $\boldsymbol{X} \to \mathbf{2}$
Set	Element of X	Subset of X

States and predicates in an effectus

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$\mathcal{K}\ell(\mathcal{G})$	Probability measure $\omega \in \mathcal{G}(X)$	Measurable function $p : X \to [0,1]$

States and predicates in an effectus

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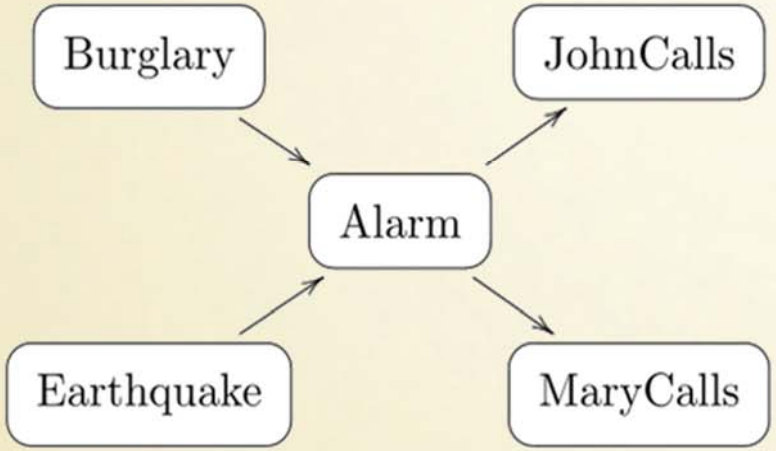
$\mathcal{K}\ell(\mathcal{G})$	Probability measure $\omega \in \mathcal{G}(X)$	Measurable function $p : X \to [0,1]$
$\mathbf{vNA}^{\mathrm{op}}$	State $\omega : X \to \mathbb{C}$ (in $\mathbf{vNA}$)	Positive unital map $\omega : \mathbb{C}^2 \to X$ (= effect $\omega \in \{e \mid 0 \leq e \leq 1\}$)

3. Learning in an effectus

The discrete probability case



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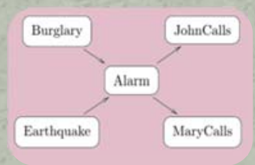
$f =$ 

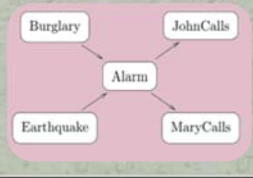
$$\{B, B^{\perp}, E, E^{\perp}\} = X \longrightarrow Y = \{J, J^{\perp}, M, M^{\perp}\}$$

$\mathcal{K}\ell(\mathcal{D})$

Backward inference

What is the likelihood of burglars, given that Mary called?

State ω

$f =$ 

predicate p

$$1 \longrightarrow X \longrightarrow Y \longrightarrow 2$$

$$\frac{1}{1000}|B\rangle + \frac{999}{1000}|B^{\perp}\rangle$$

$$\begin{aligned} |J\,M\rangle &\mapsto 1 \\ |J\,M^{\perp}\rangle &\mapsto 0 \\ |J^{\perp}\,M\rangle &\mapsto 1 \\ |J^{\perp}\,M^{\perp}\rangle &\mapsto 0 \end{aligned}$$

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$\mathcal{K}\ell(\mathcal{D})$

Backward inference

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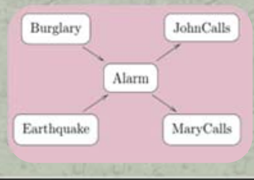
$f =$ 

predicate p

$$\begin{tikzcd} 1 \arrow[r, "\omega"] \arrow[r, bend right, "\omega|_{f^*(p)}"'] & X \arrow[r, "f"] \arrow[rr, bend right, "f^*(p)"'] & Y \arrow[r, "p"] & 2 \end{tikzcd}$$

Predicate transformer

Conditioning

$$\omega|_{f^*(p)} \coloneqq \sum_{x \in X} \frac{\omega(x) \cdot f^*(p)(x)}{\omega \models f^*(p)} |x\rangle$$

$$\omega|_{f^*(p)} \coloneqq norm\left(1 \xrightarrow{\omega} X \xrightarrow{asrt_{f^*(p)}} X + 1\right)$$

Backward Inference

$\frac{1}{1000}$

$\approx \frac{1}{18}$

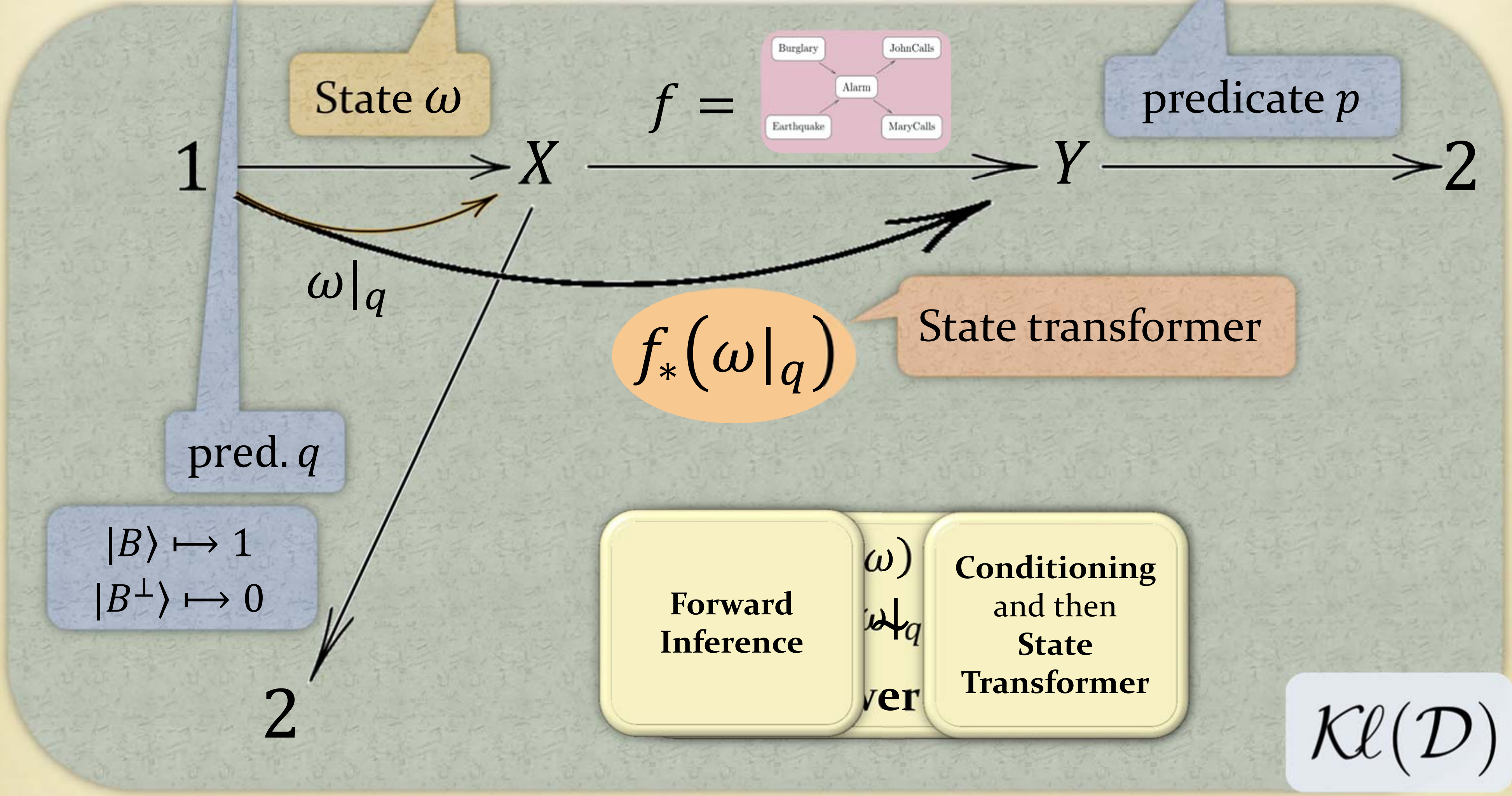
r is

Predicate Transformer and then **Conditioning**

$\mathcal{K}\ell(\mathcal{D})$

Forward inference

if there are burglars, what is the likelihood that Mary calls?



The continuous probability case

Aim: infer the age (in the interval 0-100 AD) of a tomb at an archeological site.
Evidences: three kinds of objects, with associated predicates p_0, p_1, p_2.

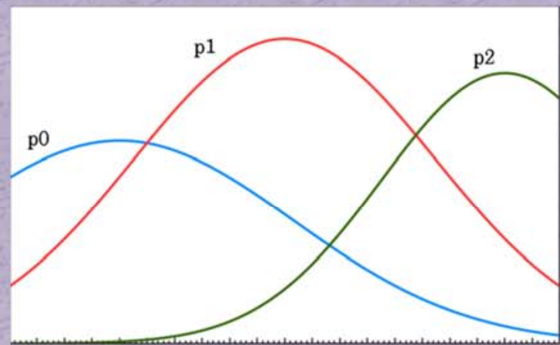
State ω

$$1 \longrightarrow [0,100] \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} 2$$

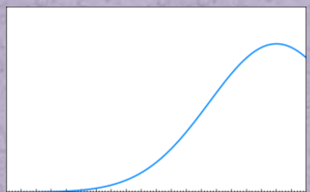
Predicates
p_0, p_1, p_2

Uniform measure
$x \mapsto 1/100$

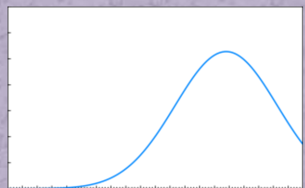
Prob. density functions
$p_i : [0,100] \to [0,1] \cong \mathcal{G}(2)$



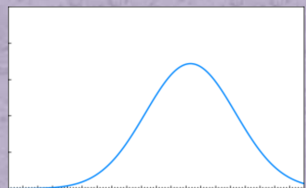
Finding an object of one of the three kinds updates ω by backward learning:



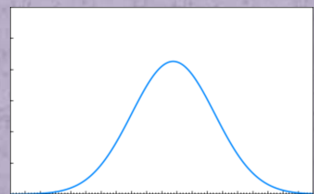
$\omega|_{p_2}$



$(\omega|_{p_2})|_{p_1}$



$((\omega|_{p_2})|_{p_1})|_{p_0}$



$(((\omega|_{p_2})|_{p_1})|_{p_0})|_{p_0}$

$\mathcal{K}\ell(\mathcal{G})$

The quantum case

A backward inference situation

$$1 \xrightarrow{\text{State } \omega} \mathcal{B} \xrightarrow{\quad f \quad} \mathcal{A} \xrightarrow{\text{Predicate } q} 2$$

$\mathbf{vNA}^{\mathrm{op}}$

The quantum case

A backward inference situation

$$\mathbb{C} \xrightarrow{\text{State } \omega} \mathcal{B} \xrightarrow{f} \mathcal{A} \xrightarrow{\text{Predicate } q} \mathbb{C}^2$$

$\mathbf{vNA}^{\mathrm{op}}$

The quantum case

A backward inference situation

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\text{State } \omega"] & \mathcal{B} \arrow[l, bend left, "\omega|_{f^*(q)}"] \arrow[r, "f"] & \mathcal{A} \arrow[r, "\text{Predicate } q"] & \mathbb{C}^2 \end{tikzcd}$$

$$b \;\mapsto\; \frac{\omega\left(\sqrt{f^*(q)} \cdot b \cdot \sqrt{f^*(q)}\right)}{\omega \models f^*(q)}$$

$$\begin{tikzcd} \mathbb{C} & M_2 \arrow[l, "\text{State } \omega"'] \end{tikzcd}$$

Probabilistic $\subseteq$ Quantum

$$\omega\colon \begin{bmatrix} x & y \\ v & w \end{bmatrix} \mapsto \frac{1}{1000}x + \frac{999}{1000}w$$

Probabilistic $\not\supseteq$ Quantum

$$\omega\colon \begin{bmatrix} x & y \\ v & w \end{bmatrix} \mapsto \frac{1}{2}(x - y - v + w)$$

vNA

Conclusions

- Generalisation of Bayesian learning to Effectus theory

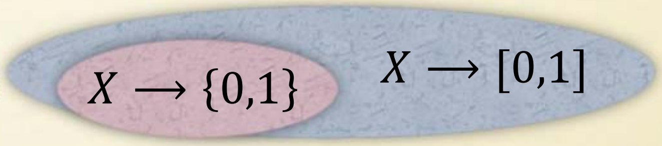
Backward Inference

$$\begin{tikzcd} 1 \arrow[r, "\omega"] \arrow[r, bend right, "\omega|_{f^*(p)}"'] & X \arrow[r, "f"] & Y \arrow[d, "p"'] \\ & & 2 \end{tikzcd}$$

Forward Inference

$$\begin{tikzcd} 1 \arrow[r, "\omega"] \arrow[rr, bend right, "f_*(\omega|_q)"] & X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ & 2 & \end{tikzcd}$$

  - probabilistic case:
    - now allows for non-sharp predicates



    - states and predicates may concern any part of a Bayesian network
  - quantum case is largely *terra incognita*
- Importance of state/predicate transformers for learning
  - revision of the primitives of Bayesian inference theory
    - influence, evidence, d-separation, analysis of counterfactuals, ...