# Full Abstraction for Signal Flow Graphs

**Filippo Bonchi**, Paweł Sobociński, Fabio Zanasi

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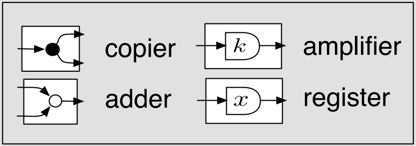
# Signal Flow Graphs

- Signal Flow Graphs are **stream processing circuits** studied in Control Theory since the 1950s.
- Constructed combining four kinds of gate

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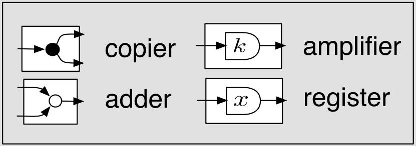


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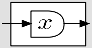

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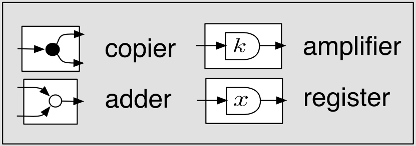


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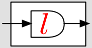

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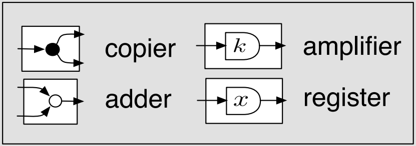


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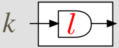

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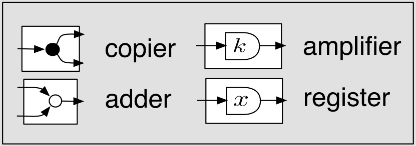


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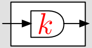


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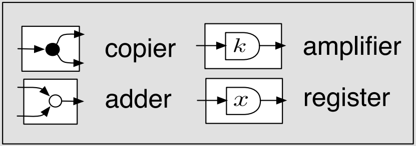


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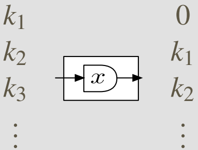

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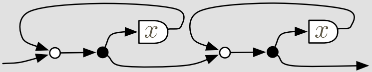
An example:



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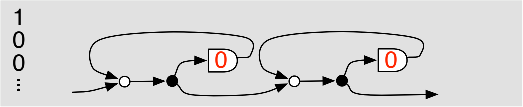

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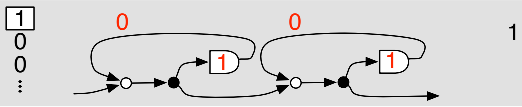

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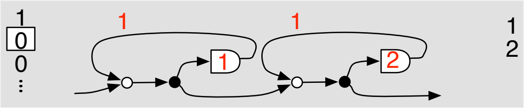

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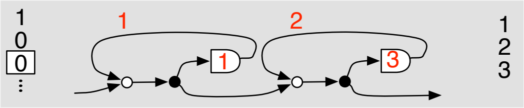

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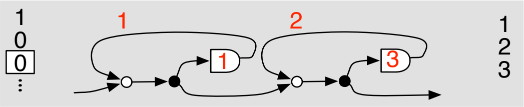


Input 1000… produces 1234….

# Signal Flow Graphs

**The orthodoxy**

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## The orthodoxy

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## In this work

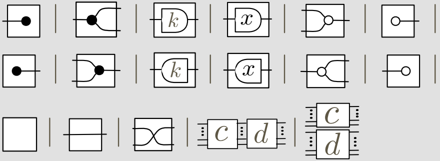
- An high-level formalism where SFGs are first-class objects:

  **the calculus of signal flow diagrams**

  - String diagrammatic (=graphical) syntax
  - Structural Operational Semantics
  - Denotational semantics
  - Sound and complete axiomatisation
  - Full Abstraction
  - Realisability

# The Calculus of SF Diagrams

Circuit diagrams of Circ are generated by the grammar

$c, d ::=$ 

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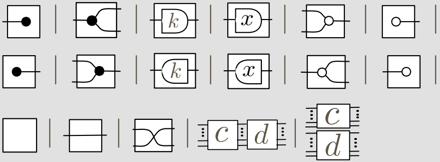
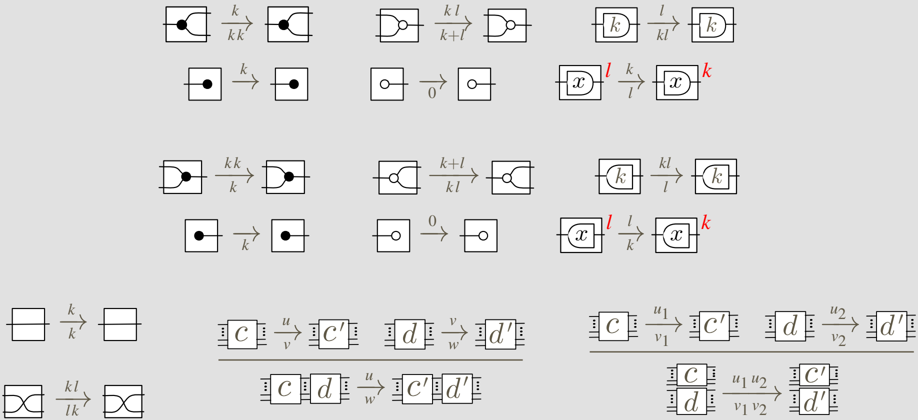
$c, d ::=$ 

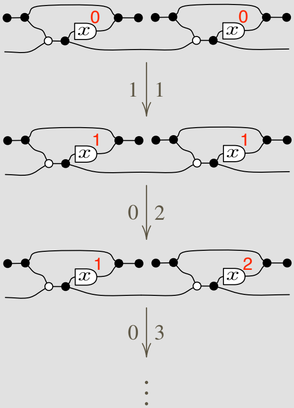

We can represent (orthodox) signal flow graphs as circuit diagrams:

# Structural Operational Semantics



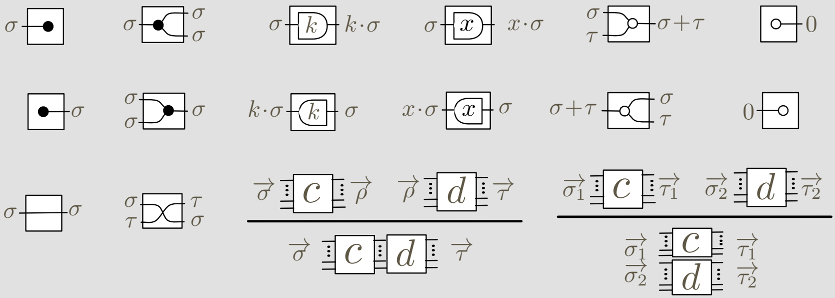
The operational semantics $\langle c \rangle$ is the set of all traces starting from an initial state for $c$ (i.e. one where all the registers are labeled with $0$).

# Example



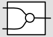
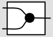
# Denotational Semantics

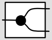
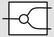
The semantics $[\![\cdot]\!]$ maps a circuit to a linear relation between stream vectors



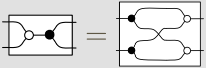
# Axiomatisation of $[\![\cdot]\!]$

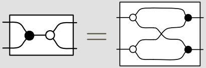
The equational theory of *interacting Hopf algebras* ($\mathbb{IH}$):

– { ,  } and { ,  } form two commutative monoids.

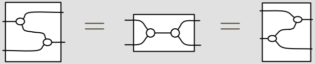
– { ,  } and { ,  } form two commutative comonoids.

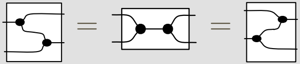
– monoid-comonoid pairs of different colors form Hopf algebras.



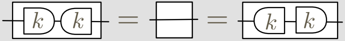


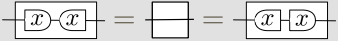
– monoid-comonoid pairs of the same color form Frobenius algebras.





– scalars and delays have formal inverses.




**Soundness and Completeness**

$$[\![c]\!] = [\![d]\!] \iff c \stackrel{\mathbb{IH}}{=} d$$

# Full Abstraction

**Theorem (?)**

For any $c$ and $d$ in $\mathsf{Circ}$

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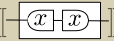
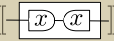
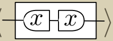
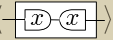
Not true in general.
The denotational semantics is *coarser* than the operational semantics.

# Full Abstraction

## A counterexample

$[\![$  $]\!] = [\![$  $]\!] = [\![$  $]\!]$

$\langle$  $\rangle \subsetneq \langle$  $\rangle \subsetneq \langle$  $\rangle$

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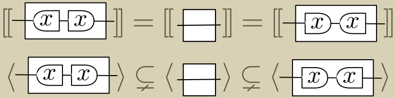


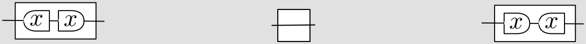

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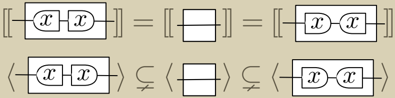

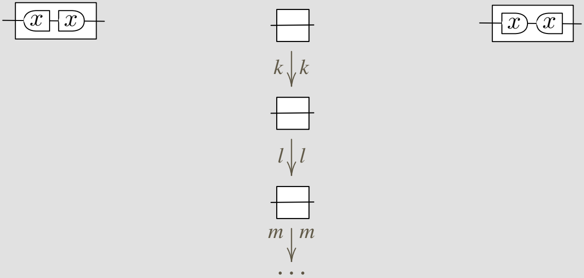

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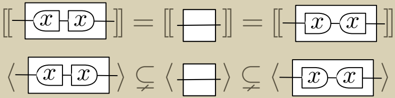


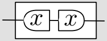


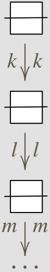


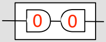

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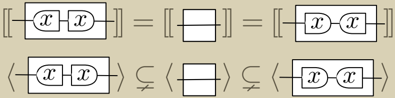


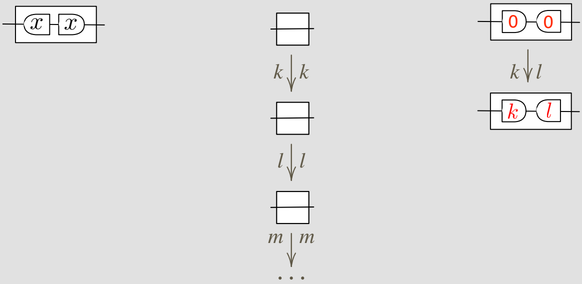

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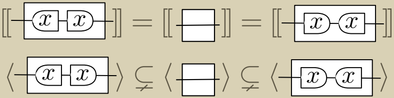


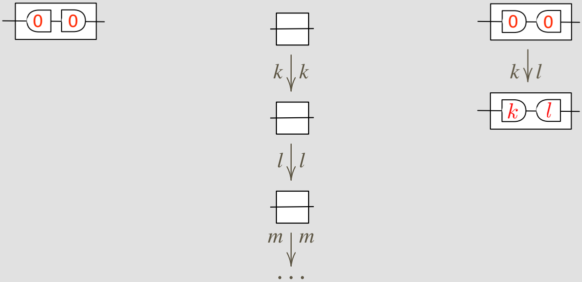

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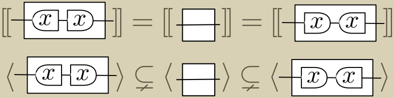


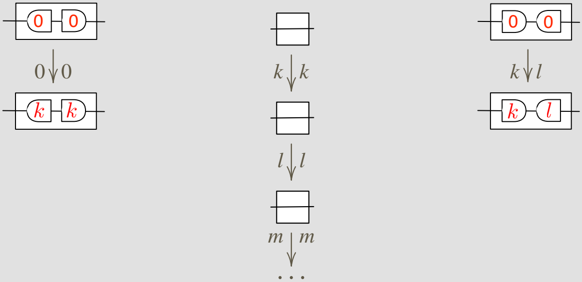

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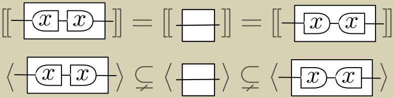


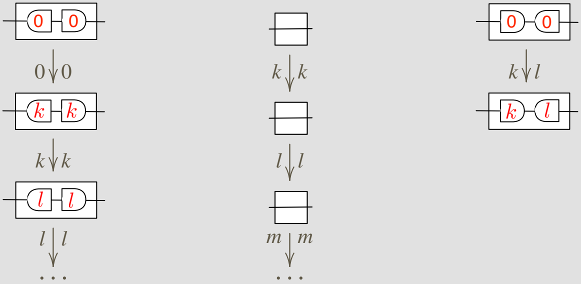

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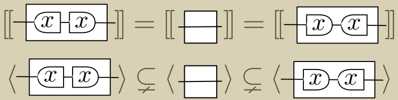
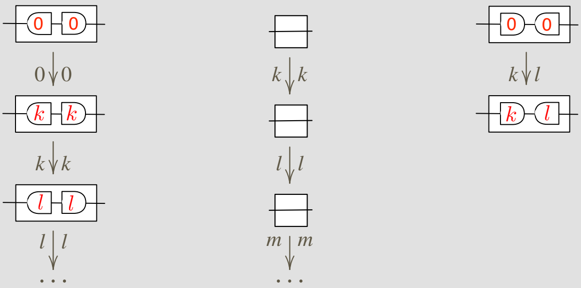
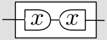
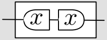





We say that  has *deadlocks* and  needs *initialisation*.

# Full Abstraction

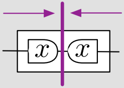
**Theorem**

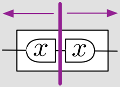
For any $c$ and $d$ in $\mathsf{Circ}$ **deadlock and initialisation free**

$$[\![c]\!] = [\![d]\!] \iff \langle c \rangle = \langle d \rangle$$

# Realisability

In presence of deadlocks or initialisation, we cannot determine directionality of the flow.

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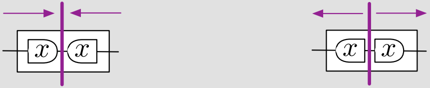


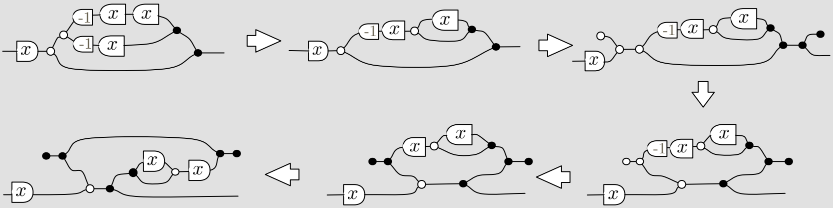

A trace for these circuits cannot be thought as the execution of a state-machine.
However, all the circuit diagrams can be put into an executable form using the equational theory $\stackrel{\mathbb{IH}}{=}$.

**Realisability Theorem**

For any circuit $c$ of $\mathsf{Circ}$ there exists
$d$ deadlock and initialisation free such that $c \stackrel{\mathbb{IH}}{=} d$.

# Realisation via $\mathbb{IH}$-rewriting

Implementing the Fibonacci circuit



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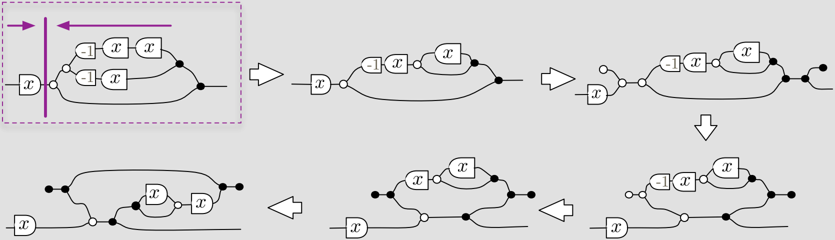

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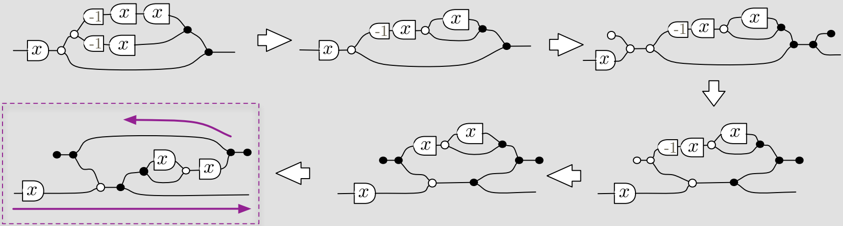
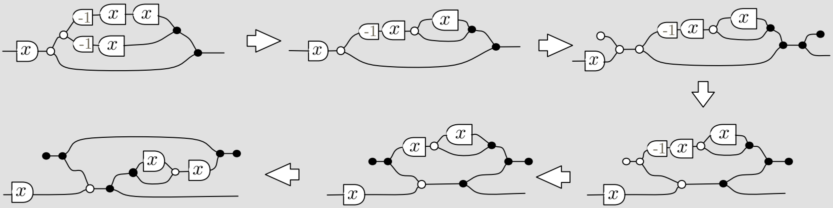


# Realisation via $\mathbb{IH}$-rewriting

Implementing the Fibonacci circuit



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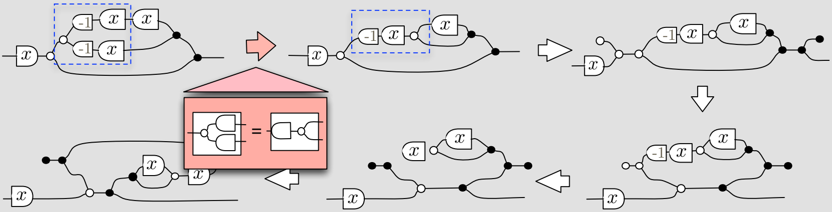

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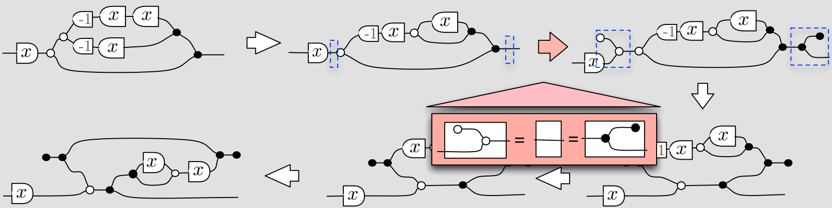

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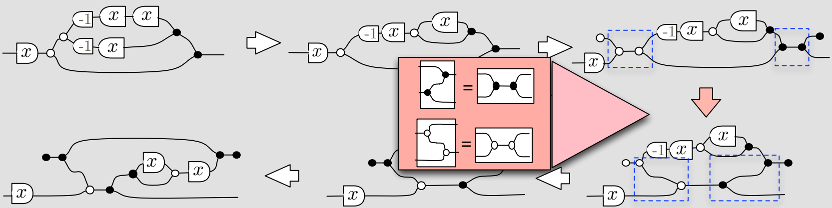

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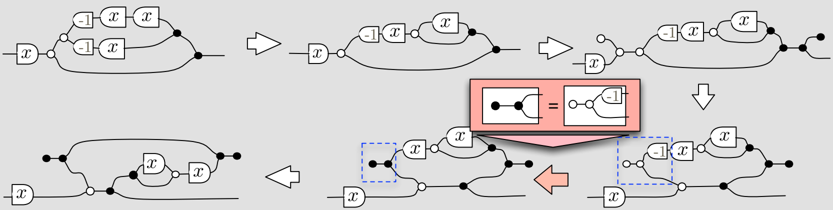

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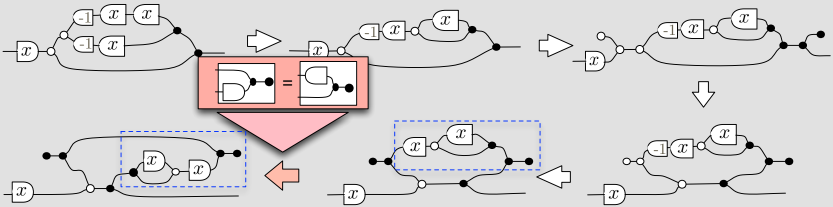

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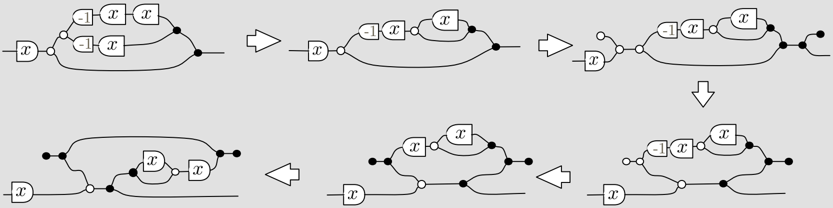

# Conclusions

- The calculus of signal flow diagrams does not rely on flow directionality as a primitive.

  > *The reason why physics has ceased to look for causes is that in fact there are no such things. The law of causality, I believe, like much that passes muster among philosophers, is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm.*
  >
  > *(*Bertrand Russell -1913*)*

- This allows for a more flexible syntax, disclosing a rich and elegant mathematical playground: $\mathbb{IH}$.
- Whenever flow directionality matters, the realisability theorem allows us rewrite any circuit diagram into an executable form.