A Categorical Semantics of Signal Flow Graphs

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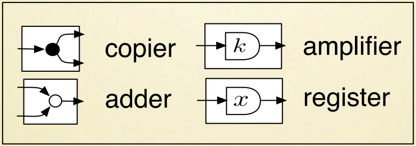
ENS DE LYON

CONCUR 2014

Signal Flow Graphs

- Signal Flow Graphs (SFGs) are **stream processing circuits** widely adopted in Control Theory since at least the 1950s.
- Constructed combining four kinds of gate



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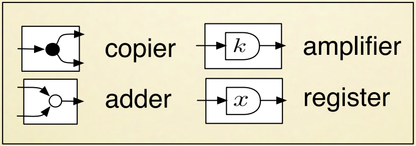

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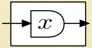

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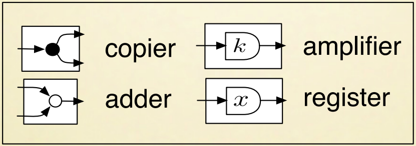

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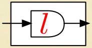

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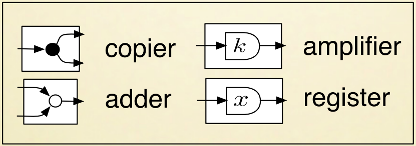

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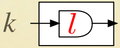

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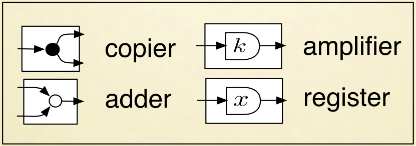

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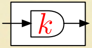

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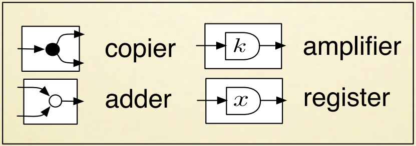

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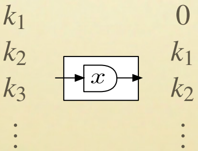

Signal Flow Graphs

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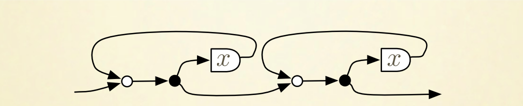

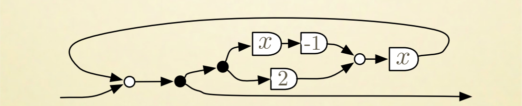

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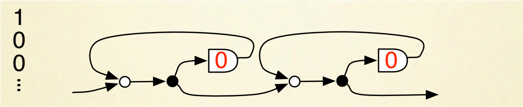

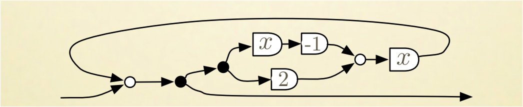

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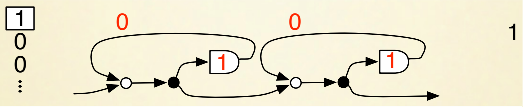

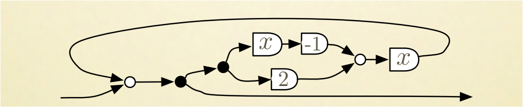

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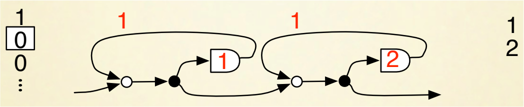

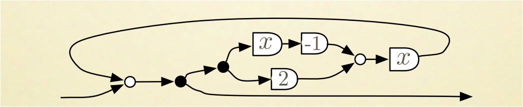

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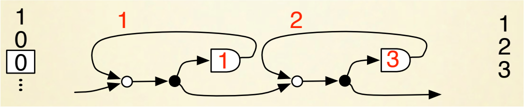

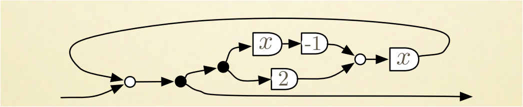

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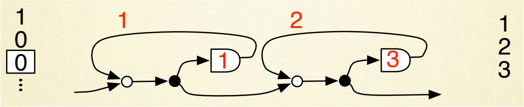

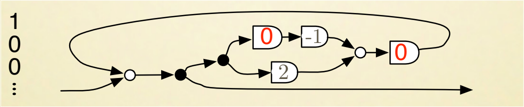

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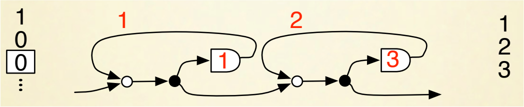

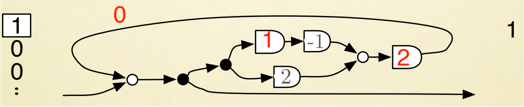

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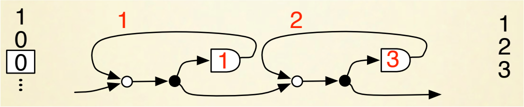

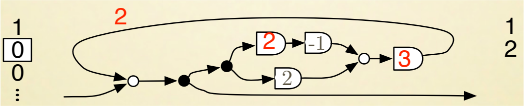

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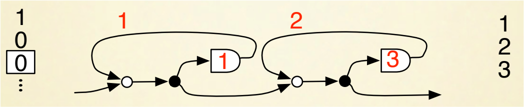

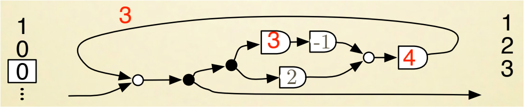

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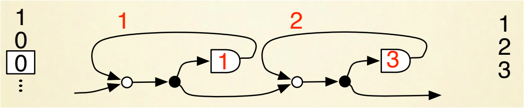

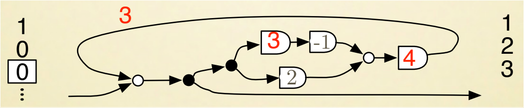

Both circuits implement the generating function

$$\frac{1}{(1-x)^2} = 1x + 2x^2 + 3x^3 + \dots$$

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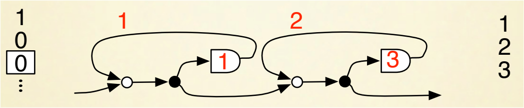

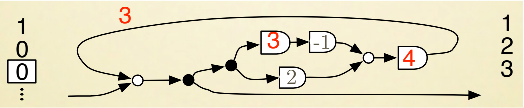

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Can we check this *statically*?

Signal Flow Graphs

- ○ In traditional approaches, SFGs are not treated as interesting mathematical structures per se.
  - $\Rightarrow$ formal analysis typically mean translation into systems of linear equations.
- ○ We study SFGs *directly* as graphical structures.

In this work

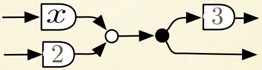
A graphical theory of Signal Flow Graphs

- String diagrammatic syntax for circuits.
- **Compositional** semantics.
- **Sound and complete axiomatisation** for semantic equivalence.
  - $\Rightarrow$ Two circuits implement the same specification if they can be transformed one into the other using the equational theory.

Outline

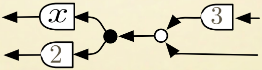
- Functional circuits

  ⇒ the signal flows from left to right



- Reverse functional circuits

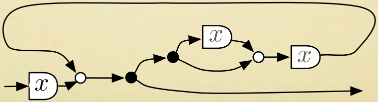
  ⇒ the signal flows from right to left



- Generalised circuits

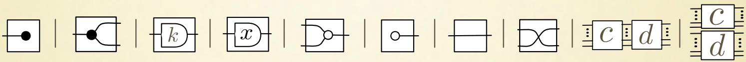
  ⇒ the signal can flow in both directions

  ⇒ environment for modeling signal flow graphs

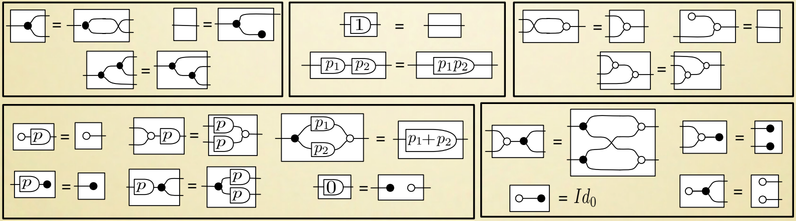


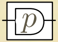
The theory $\mathbb{HA}$ of functional circuits

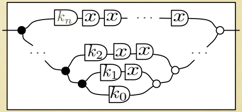
Functional circuits are the string diagrams generated by the grammar

$c, d ::=$ 

subject to the following equations:



where, for a polynomial $p=k_0+k_1x+\cdots+k_nx^n$, 

is 

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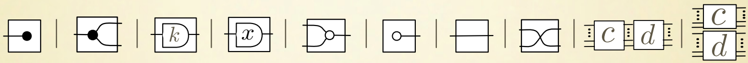
$c, d ::=$ 

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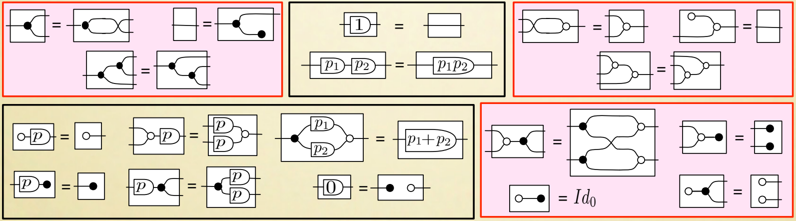
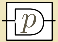
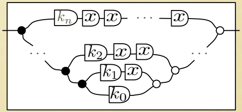


where, for a polynomial $p=k_0+k_1x+\cdots+k_nx^n$, 

is 

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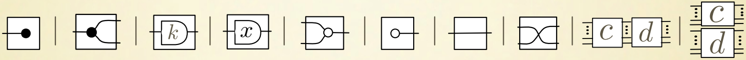
$c, d ::=$ 

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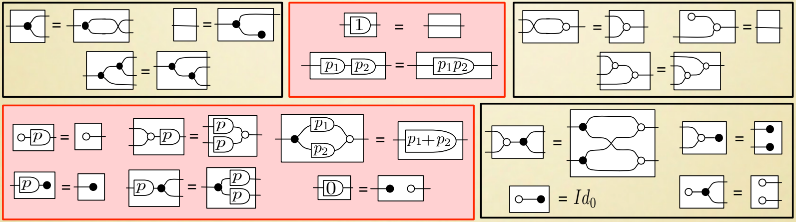
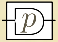
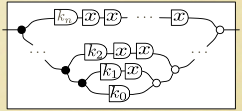


where, for a polynomial $p = k_0 + k_1 x + \cdots + k_n x^n$, 

is 

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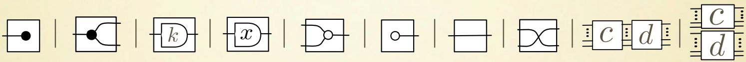
$c, d ::=$ 

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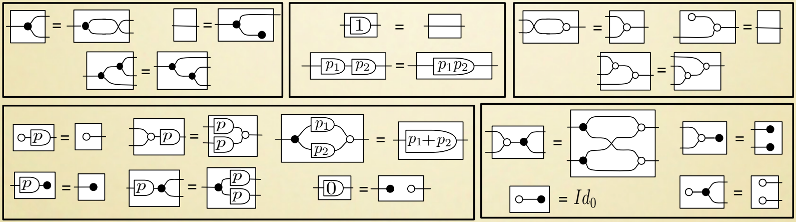
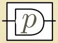
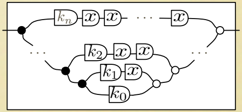


where, for a polynomial $p = k_0 + k_1 x + \cdots + k_n x^n$,  is 

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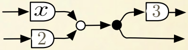
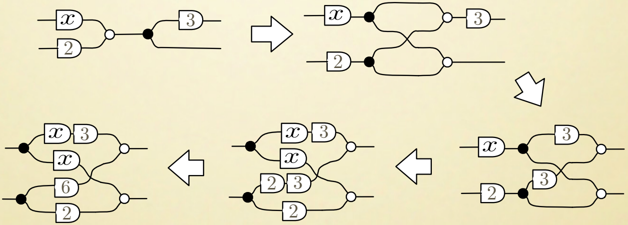
- Example: check the semantics of  using the equational theory $\mathbb{HA}$.



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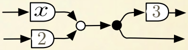
- Example: check the semantics of 

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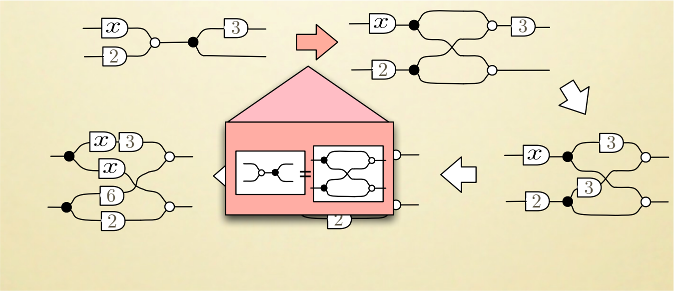

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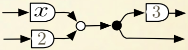
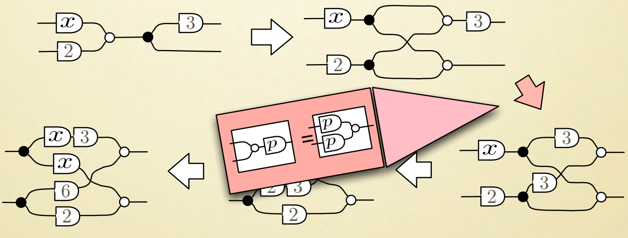
- Example: check the semantics of  using the equational theory $\mathbb{HA}$.



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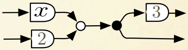
- Example: check the semantics of 

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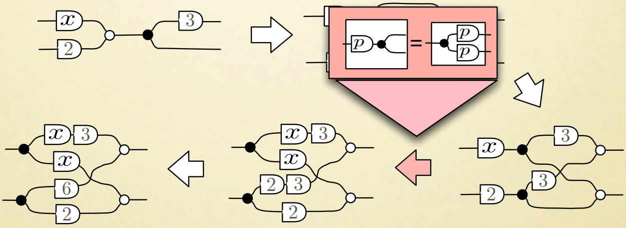

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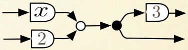
- Example: check the semantics of 

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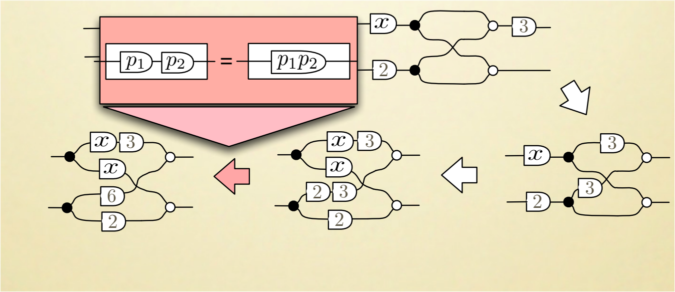

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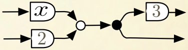
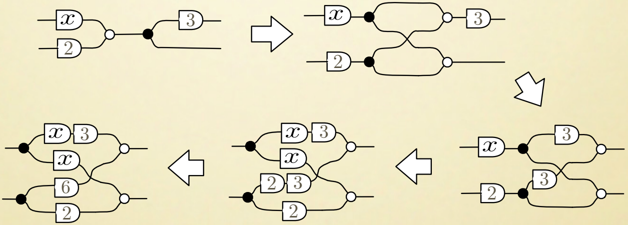
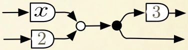
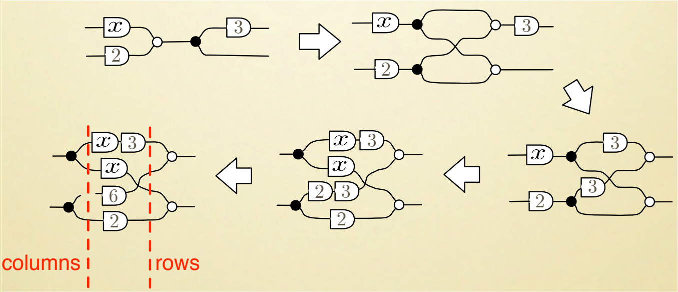
- Example: check the semantics of  using the equational theory $\mathbb{HA}$.



Semantics of functional circuits

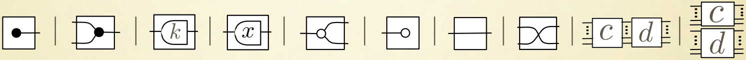
- Functional circuits modulo the equations are in 1-1 correspondence with matrices over the polynomial ring $\mathsf{k}[x]$.
- Example: check the semantics of  using the equational theory $\mathbb{HA}$.



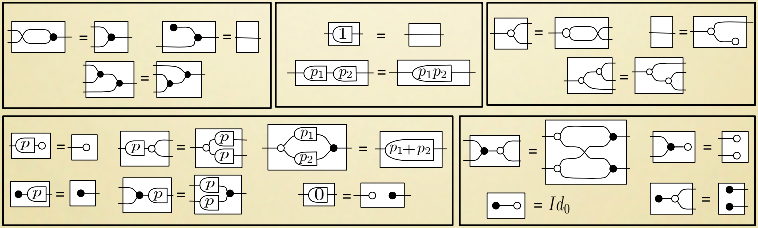
Its semantics is the matrix $\begin{pmatrix} 3x & 6 \\ x & 2 \end{pmatrix}$.

Reverse functional circuits

Reverse functional circuits are functional circuits "reflected about the y-axis". They are the diagrams generated by the grammar

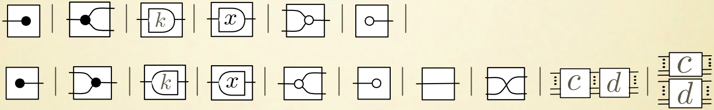
$c, d ::=$ 

subject to equations dual to those of $\mathbb{HA}$:

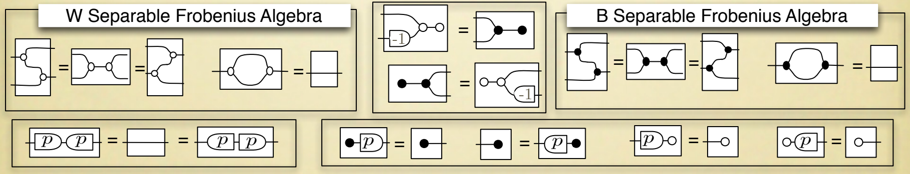


The theory $\mathbb{IH}$ of generalised circuits

Generalised circuits are string diagrams generated by the grammar

$c, d ::=$ 

subject to the equations of the theories of functional and reverse functional circuits, plus the following:

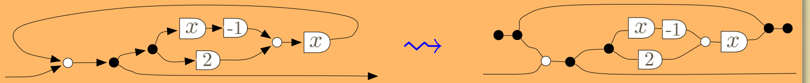


The theory $\mathbb{IH}$ of generalised circuits

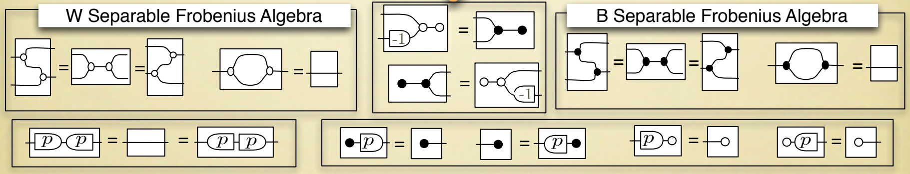
Gene

c, d :

Circuits with feedback loops can be expressed in $\mathbb{IH}$

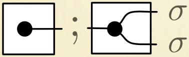


subject to the equations of the the f functional and reverse functional circuits, plus the follow g:



Semantics of Generalised Circuits

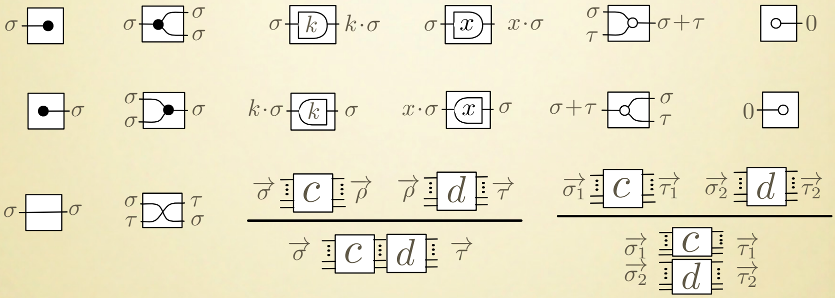
Circuits do not generally have a univocal flow direction — a *relational* model is required.

For instance, 

expresses the diagonal relation.

Semantics of Generalised Circuits

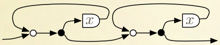
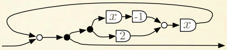
The semantics $[\![\cdot]\!]$ maps a circuit into a linear relation:



The axiomatisation of $\mathbb{IH}$ is sound and complete

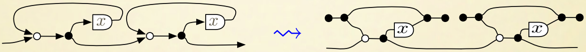
$$[\![c]\!] = [\![d]\!] \quad \Leftrightarrow \quad c \overset{\mathbb{IH}}{=} d$$

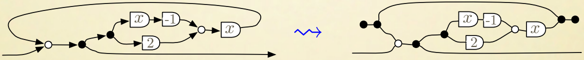
Graphical reasoning in $\mathbb{IH}$

Check:  and  implement $\frac{1}{(1-x)^2}$.

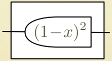
Proof strategy:

- Represent the two SFGs as generalised circuits

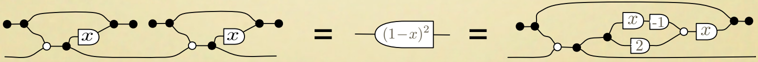




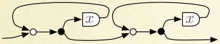
- Represent the specification as a generalised circuit:

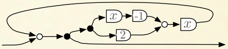
σ  $\sigma \cdot \dfrac{1}{(1-x)^2}$

- Prove the three of them equal using the axioms of $\mathbb{IH}$:

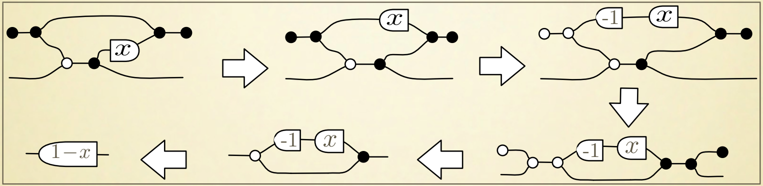


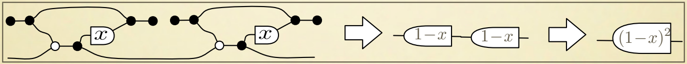
Graphical reasoning in $\mathbb{IH}$

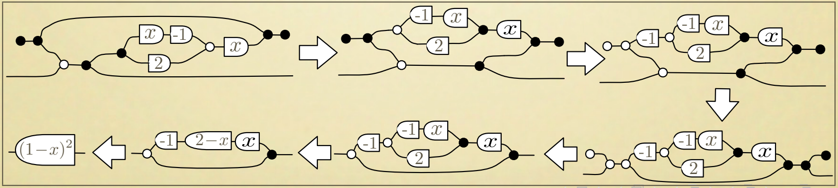
Check: 

and 

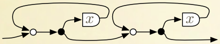
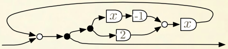
implement $\frac{1}{(1-x)^2}$.

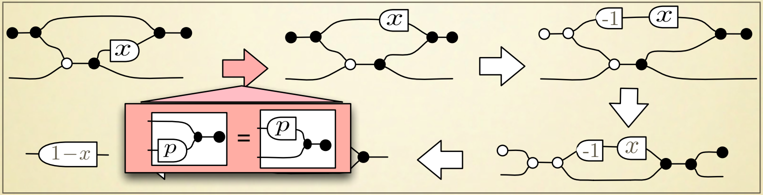


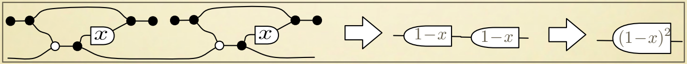


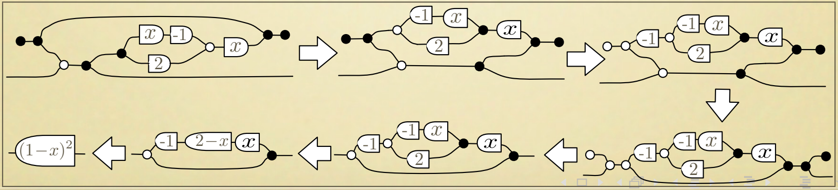


Graphical reasoning in $\mathbb{IH}$

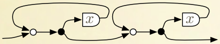
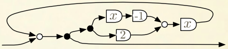
Check:  and  implement $\frac{1}{(1-x)^2}$.

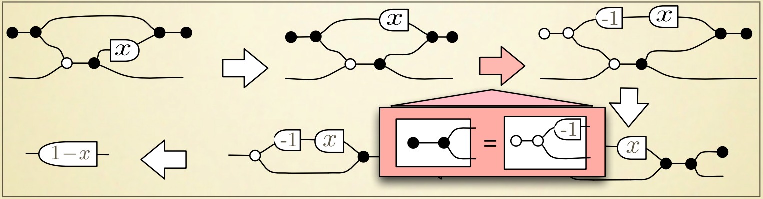


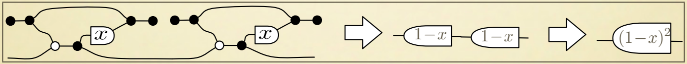


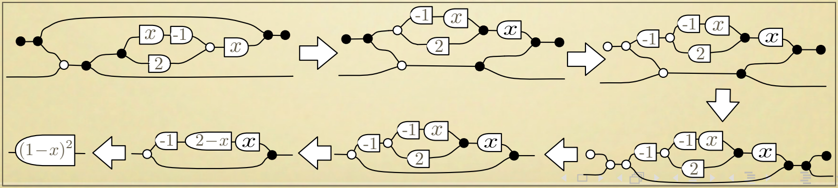


Graphical reasoning in $\mathbb{IH}$

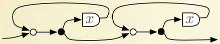
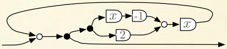
Check:  and  implement $\frac{1}{(1-x)^2}$.

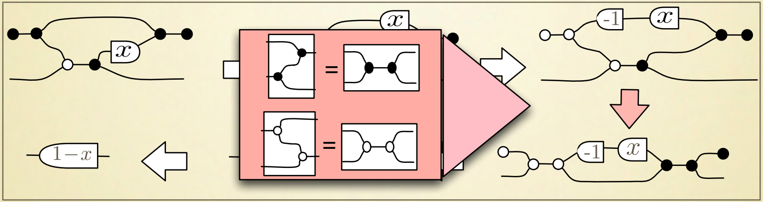


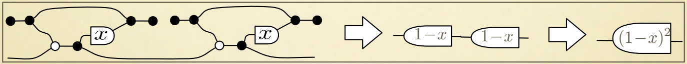


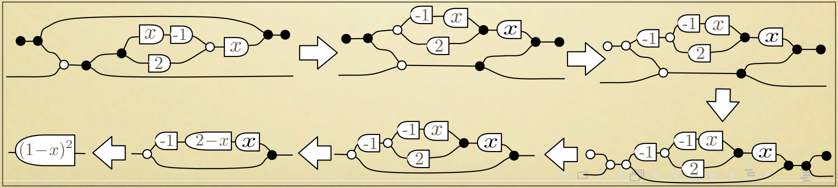


Graphical reasoning in $\mathbb{IH}$

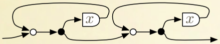
Check:  and  implement $\frac{1}{(1-x)^2}$.

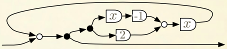






Graphical reasoning in $\mathbb{IH}$

Check: 

and 

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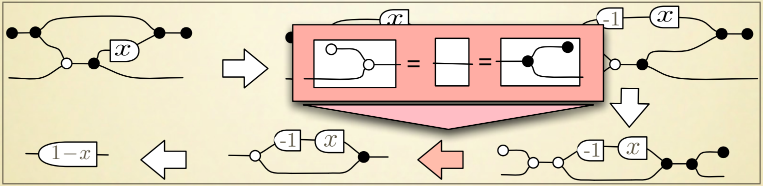

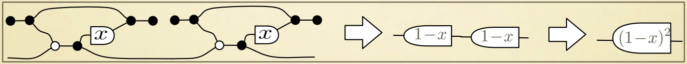

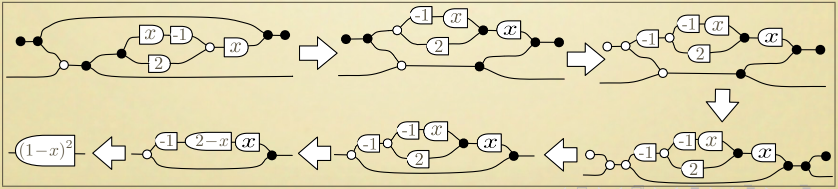

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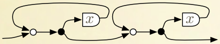
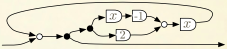
Check: 

and 

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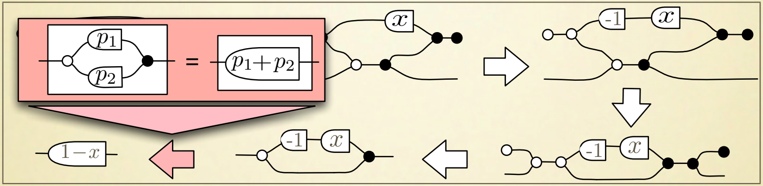

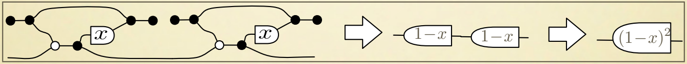

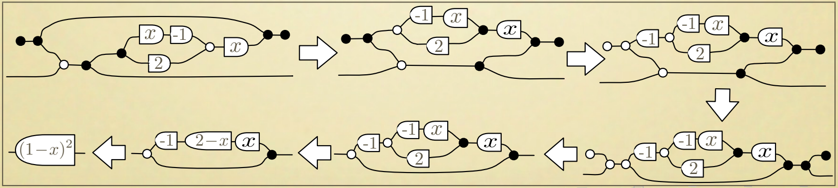

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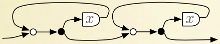
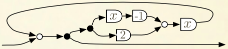
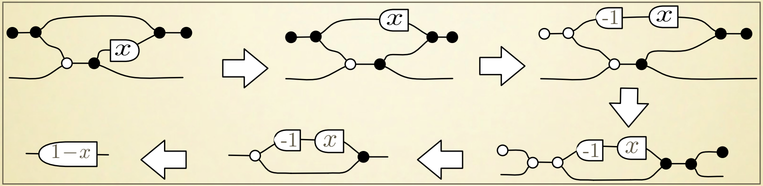
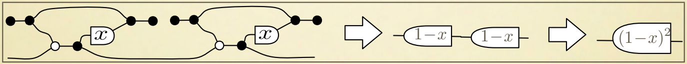
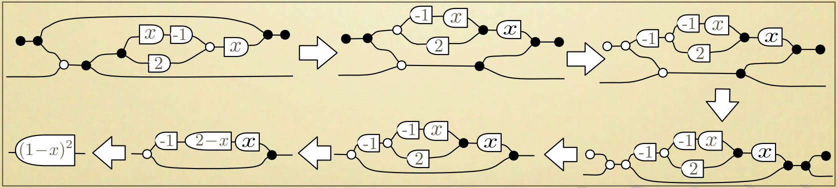
Check:  and  implement $\frac{1}{(1-x)^2}$.







Conclusions

We proposed a categorical environment for signal flow graphs

- compositional semantics in terms of linear relations
- sound and complete axiomatisation
    - graphical proof system

      implementation = implementation

      specification $\Rightarrow$ implementation

    - rich mathematical playground