# A Universal Construction for (Co)Relations

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## Abstract

Calculi of string diagrams are increasingly used to present the syntax and algebraic structure of various families of circuits, including signal flow graphs, electrical circuits and quantum processes. In many such approaches, the semantic interpretation for diagrams is given in terms of relations or corelations (generalised equivalence relations) of some kind. In this paper we show how semantic categories of both relations and corelations can be characterised as colimits of simpler categories. This modular perspective is important as it simplifies the task of giving a complete axiomatisation for semantic equivalence of string diagrams. Moreover, our general result unifies various theorems that are independently found in literature and are relevant for program semantics, quantum computation and control theory.

**1998 ACM Subject Classification** F.3.2 [Semantics of Programming Languages]: Algebraic approaches to semantics.

**Keywords and phrases** corelation, prop, string diagram

**Digital Object Identifier** 10.4230/LIPIcs...

## 1 Introduction

Network-style diagrammatic languages appear in diverse fields as a tool to reason about computational models of various kinds, including signal processing circuits, quantum processes, Bayesian networks and Petri nets, amongst many others. In the last few years, there have been more and more contributions towards a uniform, formal theory of these languages which borrows from the well-established methods of programming language semantics. A significant insight stemming from many such approaches is that a *compositional* analysis of network diagrams, enabling their reduction to elementary components, is more effective when system behaviour is thought as a *relation* instead of a function.

A paradigmatic case is the one of signal flow graphs, a foundational structure in control theory: a series of recent works [3, 1, 5, 6, 12] gives this graphical language a syntax and a semantics where each signal flow diagram is interpreted as a subspace (a.k.a. linear relation) over streams. The highlight of this approach is a sound and complete axiomatisation for semantic equivalence: what is of interest for us is how this result is achieved in [3], namely through a *modular* account of the domain of subspaces. The construction can be studied for any field $\mathsf{k}$: one considers the prop[1] $\mathsf{SV}_{\mathsf{k}}$ whose arrows $n \to m$ are subspaces of $\mathsf{k}^n \times \mathsf{k}^m$, composed as relations. As shown in in [7, 23], $\mathsf{SV}_{\mathsf{k}}$ enjoys a universal characterisation: it is the pushout (in the category of props) of props of *spans* and of *cospans* over $\mathsf{Vect}_{\mathsf{k}}$, the prop

[1] A prop is just a symmetric monoidal category with objects the natural numbers [17]. It is the typical setting for studying both the syntax and the semantics of network diagrams.

with arrows $n \to m$ the linear maps $\mathsf{k}^n \to \mathsf{k}^m$:

$$\begin{tikzcd} \mathsf{Vect}_{\mathsf{k}} + {\mathsf{Vect}_{k}}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{Vect}_{\mathsf{k}}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{Vect}_{\mathsf{k}}) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathsf{SV}_{\mathsf{k}}. \end{tikzcd} \tag{1}$$

In linear algebraic terms, the two factorisation properties expressed by (1) correspond to the representation of a subspace in terms of a basis (span) and the solution set of a system of linear equations (cospan). Most importantly, this picture provides a roadmap towards a complete axiomatisation for $\mathsf{SV}_{\mathsf{k}}$: one starts from the domain $\mathsf{Vect}_{\mathsf{k}}$ of linear maps, which is axiomatised by the equations of Hopf algebras, then combines it with its opposite ${\mathsf{Vect}_{\mathsf{k}}}^{op}$ via two distributive laws of props [16], one yielding an axiomatisation for $\mathsf{Span}(\mathsf{Vect}_{\mathsf{k}})$ and the other one for $\mathsf{Cospan}(\mathsf{Vect}_{\mathsf{k}})$. Finally, merging these two axiomatisations yields a complete axiomatisation for $\mathsf{SV}_{\mathsf{k}}$, called the theory of interacting Hopf algebras [7, 23].

It was soon realised that this modular construction was of independent interest, and perhaps evidence of a more general phenomenon. In [24] it is shown that a similar construction could be used to characterise the prop $\mathsf{ER}$ of equivalence relations, using as ingredients $\mathsf{In}$, the prop of injections, and $\mathsf{F}$, the prop of total functions. The same result is possible by replacing equivalence relations with partial equivalence relations and functions with partial functions, forming a prop $\mathsf{PF}$. In both cases, the universal construction yields a privileged route to a complete axiomatisation, of $\mathsf{ER}$ and of $\mathsf{PER}$ respectively [24].

$$\begin{tikzcd} \mathsf{In} + \mathsf{In}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{In}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{F}) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathsf{ER} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{In} + \mathsf{In}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{In}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{PF}) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathsf{PER}. \end{tikzcd} \tag{2}$$

Even though a pattern emerges, it is certainly non-trivial: for instance, if one naively mimics the linear case (1) in the attempt of characterising the prop of relations, the construction collapses to the terminal prop $\mathbf{1}$.

$$\begin{tikzcd} \mathsf{F} + \mathsf{F}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{F}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{F}) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbf{1} \end{tikzcd} \tag{3}$$

More or less at the same time, diagrammatic languages for various families of circuits, including linear time-invariant dynamical systems [12], were analysed using so-called *corelations*, which are generalised equivalence relations [10, 9, 11, 2]. Even though they were not originally thought of as arising from a universal construction like the examples above, corelations still follow a modular recipe, as they are expressible as a quotient of $\mathsf{Cospan}(\mathcal{C})$, for some prop $\mathcal{C}$. Thus by analogy we can think of them as yielding one half of the diagram

$$\begin{tikzcd} \mathcal{C} + \mathcal{C}^{op} \arrow[d] & \\ \mathsf{Cospan}(\mathcal{C}) \arrow[r] & \mathsf{Corel}(\mathcal{C}). \end{tikzcd} \tag{4}$$

In this paper we clarify the situation by giving a unifying perspective for all these constructions. We prove a general result, which

- implies (1) and (2) as special cases;
- explains the failure of (3);

- extends (4) to a pushout recipe for corelations.

More precisely, our theorem individuates sufficient conditions for characterising the category $\mathsf{Rel}(\mathcal{C})$ of $\mathcal{C}$-relations as a pushout. A dual construction yields the category $\mathsf{Corel}(\mathcal{C})$ of $\mathcal{C}$-corelations as a pushout. For the case of interest when $\mathcal{C}$ is a prop, the two constructions look as follows.

$$\begin{tikzcd} \mathcal{A}+\mathcal{A}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathcal{C}) \arrow[d] \\ \mathsf{Cospan}(\mathcal{A}) \arrow[r] & \mathsf{Rel}(\mathcal{C}). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}+\mathcal{A}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathcal{A}) \arrow[d] \\ \mathsf{Cospan}(\mathcal{C}) \arrow[r] & \mathsf{Corel}(\mathcal{C}). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{5}$$

The variant ingredient $\mathcal{A}$ is a subcategory of $\mathcal{C}$. In order to make the constructions possible, $\mathcal{A}$ has to satisfy certain requirements in relation with the factorisation system $(\mathcal{E},\mathcal{M})$ on $\mathcal{C}$ which defines $\mathcal{C}$-relations (as jointly-in-$\mathcal{M}$ spans) and $\mathcal{C}$-corelations (as jointly-in-$\mathcal{E}$ cospans). For instance, taking $\mathcal{A}$ to be $\mathcal{C}$ itself succeeds in (1) (and in fact, for any abelian $\mathcal{C}$), but fails in (3).

Besides explaining existing constructions, our result opens the lead for new applications. In particular, we observe that under mild conditions the construction lifts to the category $\mathcal{C}^T$ of $T$-algebras for a monad $T\colon \mathcal{C}\to\mathcal{C}$. We leave the exploration of this and other ramifications for future work.

### Synopsis

Section 2 introduces the necessary preliminaries about factorisation systems and (co)relations, and shows the subtleties of mapping spans into corelations in a functorial way. Section 3 states our main result and some of its consequences. We first formulate the construction for categories (Theorem 1), and then for props (Theorem 2), which are our prime object of interest in applications. Section 4 is devoted to show various instances of our construction. We illustrate the case of equivalence relations, of partial equivalence relations, of subspaces, of linear corelations, and finally of relations of algebras. Finally, Section 5 summarises our contribution and looks forward to further work. An appendix contains the proofs of Theorems 1 and 2.

### Conventions

We write $f\,;g$ or $g\circ f$ for composition of $f\colon X\to Y$ and $g\colon Y\to Z$ in a category $\mathcal{C}$. It will be sometimes convenient to indicate an arrow $f\colon X\to Y$ of $\mathcal{C}$ as $X\xrightarrow{f\in\mathcal{C}}Y$ or also $\xrightarrow{\in\mathcal{C}}$, if names are immaterial. Also, we write $X\xleftarrow{f\in\mathcal{C}}Y$ for an arrow $X\xrightarrow{f\in\mathcal{C}^{op}}Y$. We use $\oplus$ for the monoidal product in a monoidal category, with unit object $I$. Monoidal categories and functors will be strict when not stated otherwise.

## 2 (Co)relations

In this section we review the categorical approach to relations, based on the observation that in $\mathsf{Set}$ they are the jointly mono spans. We introduce in parallel the dual notion, called corelations [10]: these are jointly epi cospans and can be seen as an abstraction of the concept of equivalence relation.

▶ **Definition 1.** A **factorisation system** $(\mathcal{E},\mathcal{M})$ in a category $\mathcal{C}$ comprises subcategories $\mathcal{E}$, $\mathcal{M}$ of $\mathcal{C}$ such that

(i) $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms of $\mathcal{C}$.

(ii) every morphism $f \in \mathcal{C}$ admits a factorisation $f = m \circ e$, $e \in \mathcal{E}$, $m \in \mathcal{M}$.
(iii) given $f, f'$, with factorisations $f = m \circ e$, $f' = m' \circ e'$ of the above sort, for every $u$, $v$ such that $v \circ f = f' \circ u$ there exists a unique $s$ making the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[r, "m"] \arrow[d, dashed, "\exists! s"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "e'"'] & \cdot \arrow[r, "m'"'] & \cdot \end{tikzcd}$$

▶ **Definition 2.** Given a category $\mathcal{C}$, we say that a subcategory $\mathcal{A}$ is **stable under pushout** (resp. pullback) if for every pushout square (resp. pullback square)

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

such that $a \in \mathcal{A}$, we also have that $f \in \mathcal{A}$.

A factorisation system $(\mathcal{E}, \mathcal{M})$ is **stable** if $\mathcal{E}$ is stable under pullback, **costable** if $\mathcal{M}$ is stable under pushout, and **bistable** if it is both stable and costable.

Examples of bistable factorisation systems include the trivial factorisation systems $(\mathcal{I}_\mathcal{C}, \mathcal{C})$ and $(\mathcal{C}, \mathcal{I}_\mathcal{C})$ in any category $\mathcal{C}$, where $\mathcal{I}_\mathcal{C}$ is the subcategory containing exactly the isomorphisms in $\mathcal{C}$, the epi-mono factorisation system in any topos, or the epi-mono factorisation system in any abelian category. Stable factorisation systems include the (regular epi, mono) factorisation system in any regular category, such as any category monadic over $\mathsf{Set}$. Dually, costable factorisation systems include the (epi, regular mono) factorisation system in any coregular category, such as the category of topological spaces and continuous maps.

▶ **Definition 3.**

- Given a category $\mathcal{C}$ with pushouts, the category $\mathsf{Cospan}(\mathcal{C})$ has the same objects as $\mathcal{C}$ and arrows $X \to Y$ isomorphism classes of cospans $X \xrightarrow{f}\xleftarrow{g} Y$ in $\mathcal{C}$. The composite of $X \xrightarrow{f}\xleftarrow{g} Y$ and $Y \xrightarrow{h}\xleftarrow{i} Z$ is obtained by taking the pushout of $\xleftarrow{g}\xrightarrow{h}$.
- Given a category $\mathcal{C}$ with pullbacks, the category $\mathsf{Span}(\mathcal{C})$ has the same objects as $\mathcal{C}$ and arrows $X \to Y$ isomorphism classes of spans $X \xleftarrow{f}\xrightarrow{g} Y$ in $\mathcal{C}$. The composite of $X \xleftarrow{f}\xrightarrow{g} Y$ and $Y \xleftarrow{h}\xrightarrow{i} Z$ is obtained by taking the pullback of $\xrightarrow{g}\xleftarrow{h}$.

When $\mathcal{C}$ also has a (co)stable factorisation system, we may define a category of (co)relations with respect to this system.

▶ **Definition 4.**

- Given a category $\mathcal{C}$ with pushouts and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, the category $\mathsf{Corel}(\mathcal{C})$ has the same objects as $\mathcal{C}$. The arrows $X \to Y$ are equivalence classes of cospans $X \xrightarrow{f} N \xleftarrow{g} Y$ under the transitive closure of the following relation: two cospans $X \xrightarrow{f} N \xleftarrow{g} Y$ and $X \xrightarrow{f'} N' \xleftarrow{g'} Y$ are equivalent if there exists $N' \xrightarrow{m} N$ in $\mathcal{M}$ such that

$$\begin{tikzcd} & N' \arrow[d, "m"] & \\ X \arrow[ur, "f"] \arrow[dr, "f'"'] & N & Y \arrow[ul, "g"'] \arrow[dl, "g'"] \end{tikzcd} \tag{6}$$

commutes. This notion of equivalence respects composition of cospans, and so $\mathsf{Corel}(\mathcal{C})$ is indeed a category. We call the morphisms in this category **corelations**.

- Given a category $\mathcal{C}$ with pullbacks and a stable factorisation system, we can dualise the above to define the category $\mathsf{Rel}(\mathcal{C})$ of **relations**.

(Co)stability is needed in order to ensure that composition of (co)relations is associative, *cf.* [10, §3.3]. For proofs it is convenient to give an alternative description of (co)relations.

▶ **Proposition 1.** *When $\mathcal{C}$ has binary coproducts, corelations are in one-to-one correspondence with isomorphism classes of cospans such that the copairing $[p,q]\colon X+Y \to N$ lies in $\mathcal{E}$.*

*When $\mathcal{C}$ has binary products, relations are in one-to-one correspondence with isomorphism classes of spans such that the pairing $\langle f,g\rangle\colon N \to X \times Y$ lies in $\mathcal{M}$.*

We refer to Appendix A.1 for a proof of the proposition.

We call a span $\xleftarrow{f}\xrightarrow{g}$ **jointly-in-$\mathcal{M}$** if the pairing $\langle f,g\rangle$ lies in $\mathcal{M}$, and analogously for $\mathcal{E}$ and for cospans. To each relation there is thus, up to isomorphism, a canonical representation as a jointly-in-$\mathcal{M}$ span, and similarly to each corelation a jointly-in-$\mathcal{E}$ cospan.

▶ **Example 1.** Many examples of relations and corelations are already familiar.

- The category $\mathsf{Set}$ is bicomplete and has a bistable epi-mono factorisation system. Relations with respect to this factorisation system are simply the usual binary relations, while corelations from $X \to Y$ in $\mathsf{Set}$ are surjective functions $X+Y \to N$; thus their isomorphism classes—the arrows of $\mathsf{Corel}(\mathsf{Set})$—are partitions, or equivalence relations on $X+Y$.
- The category of vector spaces over a field $\mathsf{k}$ is abelian, and hence bicomplete with a bistable epi-mono factorisation system. The categories of relations and corelations are isomorphic: a morphism $X \to Y$ in these categories can be thought of as a linear relations, i.e. a subspace of $X \times Y$.
- In any category $\mathcal{C}$ the trivial morphism-isomorphism factorisation system $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ is bistable. Relations with respect to $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ are equivalence classes of isomorphisms $N \xrightarrow{\sim} X \times Y$, and hence there is a unique relation between any two objects. Corelations are just cospans.
- Dually, relations with respect to the isomorphism-morphism factorisation $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ are just spans, and there is a unique corelation between any two objects.

We now study the functorial interpretation of cospans and spans as corelations. This discussion is instrumental in our universal construction for corelations (Theorem 1).

First, given two categories with the same collections of objects, we may speak of **identity-on-objects (ioo)** functors between them, i.e. functors that are the identity map on objects. Four examples of such functors will become relevant in the next section:

$$\begin{array}{lll} \mathcal{C} \to \mathsf{Cospan}(\mathcal{C}) \text{ maps } \xrightarrow{f} \text{ to } \xrightarrow{f}\xleftarrow{id} & \text{and} & \mathcal{C} \to \mathsf{Span}(\mathcal{C}) \text{ maps } \xrightarrow{f} \text{ to } \xleftarrow{id}\xrightarrow{f}. \\ \mathcal{C}^{op} \to \mathsf{Cospan}(\mathcal{C}) \text{ maps } \xleftarrow{g} \text{ to } \xrightarrow{id}\xleftarrow{g} & \text{and} & \mathcal{C}^{op} \to \mathsf{Span}(\mathcal{C}) \text{ maps } \xleftarrow{g} \text{ to } \xleftarrow{g}\xrightarrow{id}. \end{array} \tag{7}$$

We are now ready to discuss the canonical map from cospans to corelations. This is simple: one just interprets a cospan representative as its corelation equivalence class.

▶ **Definition 5.** Let $\mathcal{C}$ be a category equipped with a costable factorisation system $(\mathcal{E}, \mathcal{M})$. We define $\Gamma\colon \mathsf{Cospan}(\mathcal{C}) \to \mathsf{Corel}(\mathcal{C})$ as the ioo functor mapping the isomorphism class of cospans represented by $X \xrightarrow{f} N \xleftarrow{g} Y$ to the corelation represented by this cospan.

It is straightforward to check that this is well-defined. Moreover,

▶ **Proposition 2.** $\Gamma\colon \mathsf{Cospan}(\mathcal{C}) \to \mathsf{Corel}(\mathcal{C})$ *is full.*

**Proof.** Let $a$ be a corelation. Then choosing some representative $X \to N \leftarrow Y$ of $a$ gives a cospan whose $\Gamma$-image is $a$. ◀

Mapping spans to corelations is subtler. Given a span, we may obtain a cospan by taking its pushout. When $\mathcal{C}$ has pushouts and pullbacks, this defines a function on morphisms $\mathsf{Span}(\mathcal{C}) \to \mathsf{Cospan}(\mathcal{C})$. This function is rarely, however, a functor: it may fail to preserve composition. To turn it into a functor, two tweaks are needed: first, we restrict to a subcategory $\mathsf{Span}(\mathcal{A})$ of $\mathsf{Span}(\mathcal{C})$, for some carefully chosen subcategory $\mathcal{A} \subseteq \mathcal{C}$, and second, we take the jointly-in-$\mathcal{E}$ part of the pushout. We call the resulting functor $\Pi$, as it takes the *pushout* and then *projects*.

How do we choose $\mathcal{A}$? Given a cospan $X \to A \leftarrow Y$, we may take its pullback to obtain a span $X \leftarrow P \to Y$, and then pushout this span in $\mathcal{C}$ to obtain a cospan $X \to Q \leftarrow Y$. This gives a diagram

$$\begin{tikzcd}
 & X \arrow[dr] \arrow[drr, bend left] & & \\
P \arrow[ur] \arrow[dr] & & Q \arrow[r, dashed, "f"] & A \\
 & Y \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

where the map $f$ exists and is unique by the universal property of the pushout. We want this map $f$ to lie in $\mathcal{M}$: by Definition 4, this implies that $X \to A \leftarrow Y$ and $X \to Q \leftarrow Y$ represent the same corelation. This condition is reminiscent of that introduced by Meisen in her work on so-called categories of pullback spans [19].

Note that this pullback and pushout take place in $\mathcal{C}$. We nonetheless ask $\mathcal{A}$ to be closed under pullback, so spans $\xleftarrow{f\in\mathcal{A}}\xrightarrow{g\in\mathcal{A}}$ do indeed form a subcategory $\mathsf{Span}(\mathcal{A})$ of $\mathsf{Span}(\mathcal{C})$.[2]

▶ **Proposition 3.** *Let $\mathcal{C}$ be a category equipped with a costable factorisation system $(\mathcal{E}, \mathcal{M})$. Let $\mathcal{A}$ be a subcategory of $\mathcal{C}$ containing all isomorphisms and stable under pullback. Further suppose that the canonical map given by the pushout of the pullback of a cospan in $\mathcal{A}$ lies in $\mathcal{M}$. Then mapping a span in $\mathcal{A}$ to the jointly-in-$\mathcal{E}$ part of its pushout cospan defines an ioo functor $\Pi\colon \mathsf{Span}(\mathcal{A}) \to \mathsf{Corel}(\mathcal{C})$.*

**Proof.** Recall that $\mathsf{Span}(\mathcal{A})$ is generated by morphisms of the form $\xleftarrow{id}\xrightarrow{f\in\mathcal{A}}$ and $\xleftarrow{f\in\mathcal{A}}\xrightarrow{id}$. It is thus enough to show $\Pi$ preserves composition on arrows of these two types. There exist four cases: (i) $\xleftarrow{id}\xrightarrow{f}\xleftarrow{id}\xrightarrow{g}$, (ii) $\xleftarrow{f}\xrightarrow{id}\xleftarrow{g}\xrightarrow{id}$, (iii) $\xleftarrow{f}\xrightarrow{id}\xleftarrow{id}\xrightarrow{g}$, and (iv) $\xleftarrow{id}\xrightarrow{f}\xleftarrow{g}\xrightarrow{id}$. The first three cases are straightforward to prove, and in fact hold when mapping $\mathsf{Span}(\mathcal{C}) \to \mathsf{Cospan}(\mathcal{C})$. It is the case (iv) that needs our restriction to $\mathsf{Span}(\mathcal{A})$. There $\Pi(\xleftarrow{id}\xrightarrow{f})\Pi(\xleftarrow{g}\xrightarrow{id})$ is represented by the cospan $\xrightarrow{f}\xleftarrow{g}$, while $\Pi(\xleftarrow{id}\xrightarrow{f}\xleftarrow{g}\xrightarrow{id})$ is the represented by the pushout $\xrightarrow{p}\xleftarrow{q}$ of the pullback of $\xrightarrow{f}\xleftarrow{g}$. But by hypothesis, there exists a unique $\xrightarrow{m\in\mathcal{M}}$ making the following diagram commute.

$$\begin{tikzcd}
\cdot \arrow[dr, "p"] \arrow[drr, bend left, "f"] & & \\
 & \cdot \arrow[r, "m"] & \cdot \\
\cdot \arrow[ur, "q"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

This implies that $\xrightarrow{p}\xleftarrow{q}$ and $\xrightarrow{f}\xleftarrow{g}$ represent the samse corelation, and so $\Pi$ is functorial. ◀

For example, if the category $\mathcal{M}$ has pullbacks and these coincide with pullbacks in $\mathcal{C}$, then we can take $\mathcal{A} = \mathcal{M}$. If $\mathcal{C}$ is abelian, we can take $\mathcal{A} = \mathcal{C}$.

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[2] Calling this subcategory $\mathsf{Span}(\mathcal{A})$ is a slight abuse of notation: it may be the case that $\mathcal{A}$ itself has pullbacks, and we have not proved that these agree with pullbacks in $\mathcal{C}$. Nonetheless, this conflict does not cause trouble in any of our examples below, and we stick to this convention for notational simplicity.

## 3 Main theorem: a universal property for (co)relations

This section states our main result and some consequences. We first fix our ingredients.

▶ **Assumption 1.** Let $\mathcal{C}$ be a category with

- pushouts and pullbacks;
- a costable factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M}$ a subcategory of the monos in $\mathcal{C}$;
- a subcategory $\mathcal{A}$ of $\mathcal{C}$ containing $\mathcal{M}$, stable under pullback, and such that the canonical map given by the pushout of the pullback of a cospan in $\mathcal{A}$ lies in $\mathcal{M}$.

Building on the results of Section 2, the second requirement above allows us to form a category $\mathsf{Corel}(\mathcal{C})$ of corelations, whereas the third yields a functor $\Pi\colon \mathsf{Span}(\mathcal{A}) \to \mathsf{Corel}(\mathcal{C})$. We shall also use the functor $\Gamma\colon \mathsf{Cospan}(\mathcal{C}) \to \mathsf{Corel}(\mathcal{C})$ (Definition 5) and a category $\mathcal{A}+_{|\mathcal{A}|}\mathcal{A}^{op}$: its objects are those of $\mathcal{A}$ and the morphisms $X \to Y$ are 'zigzags' $X \xrightarrow{f}\xleftarrow{g}\xrightarrow{h} \dots \xleftarrow{k} Y$ in $\mathcal{A}$. There are ioo functors from $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ to $\mathsf{Cospan}(\mathcal{C})$ and to $\mathsf{Span}(\mathcal{C})$, defined on morphisms by taking colimits, respectively limits of zigzags—equivalently, they are defined by pointwise application of the functors in (7).[3] We make all these components interact in our main theorem.

▶ **Theorem 1.** *Let $\mathcal{C}$ and $\mathcal{A}$ be as in Assumption 1. Then the following is a pushout in* $\mathbf{Cat}$*:*

$$\begin{tikzcd}
\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathcal{A}) \arrow[d, "\Pi"] \\
\mathsf{Cospan}(\mathcal{C}) \arrow[r, "\Gamma"'] & \mathsf{Corel}(\mathcal{C}) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{$\star$}$$

We leave a complete proof of this theorem to Appendix A.2. In a nutshell, the key point is that, in light of (6), $\mathsf{Corel}(\mathcal{C})$ differs from $\mathsf{Cospan}(\mathcal{C})$ precisely because it has the extra equations $\xrightarrow{m}\xleftarrow{m} = \xrightarrow{id}\xleftarrow{id}$, with $\xrightarrow{m} \in \mathcal{M}$. But these equations arise by pullback squares in $\mathcal{A}$, and so are contributions of $\mathsf{Span}(\mathcal{A})$. Moreover, the remaining equations induced by $\mathsf{Span}(\mathcal{A})$ are a subset of those holding already in $\mathsf{Cospan}(\mathcal{C})$. Hence we have a pushout square.

We now discuss some observations, consequences and examples.

▶ **Remark 1.** If any such $\mathcal{A}$ exists, then we may always take $\mathcal{A} = \mathcal{M}$ and the theorem holds. We record the above, more general, theorem as it explains preliminary results in this direction already in the literature; see the abelian case and examples for details.

Next, we formulate the dual version of the theorem, which yields a characterisation for relations. It is based on a dual version of Assumption 1.

▶ **Assumption 2.** Let $\mathcal{C}$ be a category with

- pushouts and pullbacks;
- a stable factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E}$ a subcategory of the epis in $\mathcal{C}$;
- a subcategory $\mathcal{A}$ of $\mathcal{C}$ containing $\mathcal{E}$, stable under pushout, and such that the canonical map given by the pullback of the pushout of a span in $\mathcal{A}$ lies in $\mathcal{E}$.

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[3] More abstractly, one can see $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ as the pushout of $\mathcal{A}$ and $\mathcal{A}^{op}$ over the respective inclusions of $|\mathcal{A}|$, the discrete category on the objects of $\mathcal{A}$. The functors $\mathsf{Span}(\mathcal{A}) \leftarrow \mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} \to \mathsf{Cospan}(\mathcal{C})$ are then those given by the universal property with respect to (suitable restrictions of) the functors in (7).

▶ **Corollary 1** (Dual case). *Let $\mathcal{C}$ and $\mathcal{A}$ be as in Assumption 2. Then the following is a pushout square in* **Cat**.

$$\begin{tikzcd}
\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} \arrow[r] \arrow[d] & \mathsf{Cospan}(\mathcal{A}) \arrow[d] \\
\mathsf{Span}(\mathcal{C}) \arrow[r] & \mathsf{Rel}(\mathcal{C}) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{$\circ$}$$

**Proof.** This corollary is obtained by noting that, given a stable factorisation system $(\mathcal{E}, \mathcal{M})$ in $\mathcal{C}$, with $\mathcal{E}$ a subcategory of the epis, we have a costable factorisation system $(\mathcal{M}^{op}, \mathcal{E}^{op})$ in $\mathcal{C}^{op}$, with $\mathcal{E}^{op}$ a subcategory of the monos. Proposition 3 then gives a functor $\mathsf{Cospan}(\mathcal{A}) = \mathsf{Span}(\mathcal{A}^{op}) \longrightarrow \mathsf{Rel}(\mathcal{C}) = \mathsf{Corel}(\mathcal{C}^{op})$. Noting also that $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} = \mathcal{A}^{op} +_{|\mathcal{A}|} (\mathcal{A}^{op})^{op}$ and $\mathsf{Span}(\mathcal{C}) = \mathsf{Cospan}(\mathcal{C}^{op})$, we can hence apply Theorem 1. ◀

As a notable instance of Theorem 1, we can specialise to the case of abelian categories and their epi-mono factorisation system. In this case we can simply pick $\mathcal{A}$ to be $\mathcal{C}$ itself.

▶ **Corollary 2** (Abelian case). *Let $\mathcal{C}$ be an abelian category. Then the following is a pushout square in* **Cat***:*

$$\begin{tikzcd}
\mathcal{C} +_{|\mathcal{C}|} \mathcal{C}^{op} \arrow[r, "{[\alpha, \beta]}"] \arrow[d, "{[\gamma, \delta]}"'] & \mathsf{Span}(\mathcal{C}) \arrow[d, "\Pi"] \\
\mathsf{Cospan}(\mathcal{C}) \arrow[r, "\Gamma"'] & \mathsf{Corel}(\mathcal{C}) \cong \mathsf{Rel}(\mathcal{C}) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{$\triangle$}$$

*where we take (co)relations with respect the epi-mono factorisation system.*

**Proof.** As $\mathcal{C}$ is abelian, it is finitely bicomplete and has a bistable factorisation system given by epis and monos. Furthermore, we need not restrict our spans to some subcategory $\mathcal{A}$: pullbacks are by kernel, pushouts by cokernel, whence the canonical map from the pushout of the pullback of a given cospan to itself is always mono, being the inclusion of the image of the joint map into the apex. Similarly, the map from a span to the pullback of its pushout is simply the joint map with codomain restricted to its image, and hence always epi. Thus $\mathcal{C}$ meets both Assumptions 1 and 2 with $\mathcal{A} = \mathcal{C}$. Then the category of corelations is the pushout of the span $\mathsf{Cospan}(\mathcal{C}) \leftarrow \mathcal{C} +_{|\mathcal{C}|} \mathcal{C}^{op} \rightarrow \mathsf{Span}(\mathcal{C})$. But by the dual theorem (Corollary 1), the pushout of this span is also the category of relations. Thus the two categories are isomorphic. Explicitly, the isomorphism is given by taking a corelation to the jointly mono part of its pullback span, and taking a relation to the jointly epi part of its pushout cospan. ◀

▶ **Remark 2.** In Theorem 1, the diagram $\mathsf{Cospan}(\mathcal{C}) \leftarrow \mathcal{A} +_{|\mathcal{A}|} \mathcal{A} \rightarrow \mathsf{Span}(\mathcal{A})$ 'knows' only about $\mathcal{M}$, not the factorisation system $(\mathcal{E}, \mathcal{M})$. This is enough, however, since if $\mathcal{M}$ is part of a factorisation system, then the factorisation system is unique.

Indeed, suppose we have $\mathcal{E}$, $\mathcal{E}'$ such that both $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M})$ are factorisation systems. Take $e \in \mathcal{E}$. Then the factorisation system $(\mathcal{E}', \mathcal{M})$ gives a factorisation $e = m_1 \circ e'$, while $(\mathcal{E}, \mathcal{M})$ gives a factorisation $e' = m_2 \circ e_2$. By substitution, we have $e = m_1 \circ m_2 \circ e_2$. By uniqueness of factorisation, we can then assume without loss of generality that $e = e_2$ and $m_1 \circ m_2 = id$. Next, using $e = e_2$ and substitution in $e' = m_2 \circ e_2$, we similarly arrive at $m_2 \circ m_1 = id$. Thus $m_1$ is an isomorphism, and hence lies in $\mathcal{E}'$. This implies that $e = e' \circ m_1 \in \mathcal{E}'$, and hence $\mathcal{E} \subseteq \mathcal{E}'$. We may similarly show that $\mathcal{E}' \subseteq \mathcal{E}$, and hence that the two categories are equal.

The next corollary is instrumental in giving categories of (co)relations a presentation by generators and equations.

▶ **Corollary 3.** *Suppose $\mathcal{A}$ and $\mathcal{C}$ are as in Assumption 1. Then* $\mathsf{Corel}(\mathcal{C})$ *is freely generated by the objects of $\mathcal{C}$ and arrows $\xrightarrow{f}$, $\xleftarrow{g}$ of $\mathcal{C}$ quotiented by*

$$\xrightarrow{f\in\mathcal{A}}\xleftarrow{g\in\mathcal{A}} = \xleftarrow{p\in\mathcal{A}}\xrightarrow{q\in\mathcal{A}} \qquad \textit{whenever } \xleftarrow{p}\xrightarrow{q} \textit{ pulls back } \xrightarrow{f}\xleftarrow{g} \tag{8}$$

$$\xleftarrow{f\in\mathcal{C}}\xrightarrow{g\in\mathcal{C}} = \xrightarrow{p\in\mathcal{C}}\xleftarrow{q\in\mathcal{C}} \qquad \textit{whenever } \xrightarrow{p}\xleftarrow{q} \textit{ pushes out } \xleftarrow{f}\xrightarrow{g}. \tag{9}$$

*Equivalently,* $\mathsf{Corel}(\mathcal{C})$ *is the quotient of* $\mathsf{Cospan}(\mathcal{C})$ *by* (8). *A dual statement holds for* $\mathsf{Rel}(\mathcal{C})$.

Note that, in light of Remark 1, one may also replace (8) by the subset of axioms

$$\xrightarrow{f\in\mathcal{M}}\xleftarrow{g\in\mathcal{M}} = \xleftarrow{p\in\mathcal{M}}\xrightarrow{q\in\mathcal{M}} \qquad \text{whenever } \xleftarrow{p}\xrightarrow{q} \text{ pulls back } \xrightarrow{f}\xleftarrow{g}.$$

As $\mathcal{M} \subseteq \mathcal{A}$ by Assumption 1, this may give a smaller presentation.

The importance of the above observation stems from the fact that sets of equations (8) and (9) yield a presentation for categories of spans and cospans over $\mathcal{C}$ respectively. In various interesting cases, they enjoy a *finitary* axiomatisation, which can be elegantly described in terms of distributive laws between categories [20, 16]. Under this light, Corollary 3 provides a recipe for axiomatising categories of (co)relations starting from existing results about spans and cospans. For instance, this is strategy adopted in the literature to axiomatise finite equivalence relations [24, 9], finite partial equivalence relations [24] and finitely-dimensional subspaces [7]. All these are examples of corelations and are treated in Section 4 below.

## The case of props.

As mentioned in the introduction, the motivating examples for our construction are categories providing a semantic interpretation for circuit diagrams. These are typically props (**pro**duct and **p**ermutation categories [17]): it is thus useful to phrase our construction in this setting.

Recall that a prop is a symmetric monoidal category with objects the natural numbers, in which $n \oplus m = n+m$. Props form a category **Prop** with morphisms the ioo strict symmetric monoidal functors. A simplification to Theorem 1 is that the coproduct $\mathcal{C} + \mathcal{C}'$ in **Prop** is computed as $\mathcal{C} +_{|\mathcal{C}|} \mathcal{C}'$ in **Cat**, because the set of objects is fixed for any prop.

For monoidal structure on $\mathcal{C}$ to extend to the categories of (co)spans and (co)relations, it is crucial that the monoidal product respects the ambient structure.

Let $(\mathcal{C}, \oplus)$ be a prop with pushouts, and let $(\mathcal{A}, \oplus)$ be a sub-prop. We say that **the monoidal product preserves pushouts** in $\mathcal{A}$ if, for all spans $N \leftarrow Y \rightarrow M$ and $N' \leftarrow Y' \rightarrow M'$ in $\mathcal{A}$, we have an isomorphism

$$(N \oplus N') +_{Y \oplus Y'} (M \oplus M') \cong (N +_Y M) \oplus (N' +_{Y'} M').$$

Note that this pushout is taken in $\mathcal{C}$. This condition holds, for example, whenever $\mathcal{C}$ is monoidally closed. We say the monoidal product preserves pullbacks if the analogous condition holds for pullbacks.

Furthermore, we say that a subcategory $\mathcal{A}$ is **closed under** $\oplus$ if, given morphisms $f, g$ in $\mathcal{A}$, the morphism $f \oplus g$ is also in $\mathcal{A}$.

▶ **Theorem 2.** *Let $\mathcal{C}$ and $\mathcal{A}$ be props satisfying Assumption 1. Suppose that the monoidal product of $\mathcal{C}$ preserves pushouts in $\mathcal{C}$ and pullbacks in $\mathcal{A}$, and that $\mathcal{M}$ is closed under the* *monoidal product. Then we have a pushout square in* **Prop**

$$\begin{tikzcd} \mathcal{A} + \mathcal{A}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathcal{A}) \arrow[d, "\Pi"] \\ \mathsf{Cospan}(\mathcal{C}) \arrow[r, "\Gamma"'] & \mathsf{Corel}(\mathcal{C}). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{10}$$

We also state the prop version of the abelian case. An abelian prop is just a prop which is also an abelian category and where the monoidal product is the biproduct.

▶ **Corollary 4.** *Suppose that $\mathcal{C}$ is an abelian prop. The following is a pushout in* **Prop**.

$$\begin{tikzcd} \mathcal{C} + \mathcal{C}^{op} \arrow[r, "{[\alpha,\beta]}"] \arrow[d, "{[\gamma,\delta]}"'] & \mathsf{Span}(\mathcal{C}) \arrow[d, "\Pi"] \\ \mathsf{Cospan}(\mathcal{C}) \arrow[r, "\Gamma"'] & \mathsf{Corel}(\mathcal{C}) \cong \mathsf{Rel}(\mathcal{C}) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{$\triangle$}$$

We leave the proofs of these results to Appendix A.3.

# 4 Examples

## 4.1 From Injections to Equivalence Relations

Our first example concerns the construction of equivalence relations starting from injective functions. For $n \in \mathbb{N}$, write $\overline{n}$ for the set $\{0, 1, \ldots, n\}$, and $\uplus$ for the disjoint union of sets. We fix a prop $\mathsf{ER}$ whose arrows $n \to m$ are the equivalence relations on $\overline{n} \uplus \overline{m}$. For composition $e_1 \,;\, e_2 \colon n \to m$ of equivalence relations $e_1 \colon n \to z$ and $e_2 \colon z \to m$, one first defines an equivalence relation on $\overline{n} \uplus \overline{z} \uplus \overline{m}$ by gluing together equivalence classes of $e_1$ and $e_2$ along common witnesses in $\overline{z}$, then obtains $e_1 \,;\, e_2$ by restricting to elements of $\overline{n} \uplus \overline{m}$.

Equivalence relations are equivalently described as corelations of functions. For this, let $\mathsf{F}$ be the prop whose arrows $n \to m$ are functions from $\overline{n}$ to $\overline{m}$. $\mathsf{F}$ has the usual factorisation system $(\mathsf{Su}, \mathsf{In})$ given by epi-mono factorisation, where $\mathsf{Su}$ and $\mathsf{In}$ are the sub-props of surjective and of injective functions respectively. Given these data, one can check that $\mathsf{ER}$ is isomorphic to $\mathsf{Corel}(\mathsf{F})$, the prop of corelations on $\mathsf{F}$.

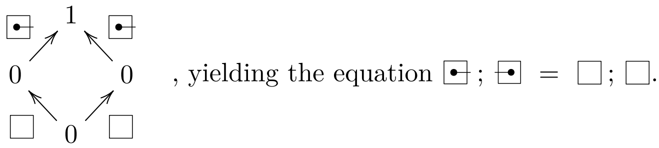
We are now in position to apply our construction of Theorem 2. First, we verify Assumption 1 with $\mathcal{C}$ instantiated as $\mathsf{F}$ and $\mathcal{A}$ as $\mathsf{In}$. The only point requiring some work is the third, which goes as follows: given a cospan of monos $X \to P \leftarrow Y$, consider $X$, $Y$ as subsets of $P$. Then the pullback-pushout diagram looks like

$$\begin{tikzcd} & X \arrow[dr] \arrow[drr, bend left] & & \\ X \cap Y \arrow[ur] \arrow[dr] & & X \cup Y \arrow[r, dashed] & P \\ & Y \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

and $X \cup Y \to P$ is the inclusion map, hence a mono in $\mathsf{In}$. Therefore, we can construct the pushout diagram (10) as follows:

$$\begin{tikzcd} \mathsf{In} + \mathsf{In}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{In}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{F}) \arrow[r] & \mathsf{ER} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{11}$$

This modular reconstruction easily yields a presentation by generators and relations for (the arrows of) $\mathsf{ER}$. Following the recipe of Corollary 3, $\mathsf{ER}$ is the quotient of $\mathsf{Cospan}(\mathsf{F})$ by all the equations generated by pullbacks in $\mathsf{In}$, as in (8). Now, recall that $\mathsf{In}$ is presented (in string diagram notation [22]) by the generator  $: 0 \to 1$, and no equations. Thus, in order to present all the equations of shape (8) it suffices to consider a single pullback square in $\mathsf{In}$:


(12)

On the other hand, we know $\mathsf{Cospan}(\mathsf{F})$ is presented by the theory of special commutative Frobenius monoids (also termed separable Frobenius algebras), see [16]. Therefore $\mathsf{ER}$ is presented by the generators and equations of special commutative Frobenius monoids, with the addition of (12). This is known as the theory of extraspecial commutative Frobenius monoids [9]. This result also appears in [24], in both cases without the realisation that it stems from a more general construction.

## 4.2 From Functions to the Terminal Prop

It is instructive to see a non-example, to show that the assumptions on $\mathcal{A}$ are not redundant. One may want consider an obvious variation of (11), where instead of $\mathsf{In}$ one takes the whole $\mathsf{F}$ as $\mathcal{A}$. However, with this tweak the construction collapses: the pushout is the terminal prop $\mathbf{1}$ with exactly one arrow between any two objects.

$$\begin{tikzcd} \mathsf{F} + \mathsf{F}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{F}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{F}) \arrow[r] & \mathbf{1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{13}$$

This phenomenon was noted before ([7], see also [14, Th. 5.6]), however without an understanding of its relationship with other (non-collapsing) instances of the same construction. Theorem 2 explains why this case fails where others succeed: the problem lies in the choice of $\mathsf{F}$ as the subcategory $\mathcal{A}$. Indeed, the canonical map given by the pullback of the pushout of any span in $\mathcal{A} = \mathsf{F}$ does not necessarily lie in $\mathsf{In}$, i.e. it may be not injective. An example is given by the cospan $0 \to 1 \leftarrow 2$, with the canonical map from the pushout of the pullback cospan the non-injective map $2 \to 1$:

$$\begin{tikzcd} & 2 \arrow[dr] \arrow[drr, bend left] & & \\ 0 \arrow[ur] \arrow[dr] & & 2 \arrow[r, dashed] & 1. \\ & 0 \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

## 4.3 From Injections to Partial Equivalence Relations

Partial equivalence relations (PERs) are common structures in program semantics, which date back to the seminal work of Scott [21] and recently revamped in the study of quantum computations (e.g., [15, 13]). Our approach yields a characterisation for the prop $\mathsf{PER}$ whose arrows $n \to m$ are PERs on $\overline{n} \uplus \overline{m}$, with composition as in $\mathsf{ER}$. The ingredients of the construction generalise Example 4.1 from total to partial maps. Instead of $\mathsf{F}$ one starts with $\mathsf{PF}$, the prop of partial functions, which has a factorisation system involving the sub-prop of partial surjections and the sub-prop of injections. The resulting prop of PF-corelations is isomorphic to PER. Theorem 2 yields the following pushout

$$\begin{tikzcd} \mathsf{In} + \mathsf{In}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{In}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{PF}) \arrow[r] & \mathsf{PER}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{14}$$

As in Example 4.1, following Corollary 3, (14) reduces the task of axiomatising PER to the one of axiomatising Cospan(PF) and adding the single equation (12) from Span(In). Cospan(PF) is presented by "partial" special commutative Frobenius monoids, studied in [24].

## 4.4 From Linear Maps to Subspaces

We now consider an example for the abelian case: the prop $\mathsf{SV}_{\mathsf{k}}$ whose arrows $n \to m$ are k-linear subspaces of $\mathsf{k}^n \times \mathsf{k}^m$, for a field k. Composition in $\mathsf{SV}_{\mathsf{k}}$ is relational: $V \,;\, W = \{(v,w) \mid \exists u.(v,u) \in V, (u,w) \in W\}$. Interest in $\mathsf{SV}_{\mathsf{k}}$ is motivated by various recent applications. We mention the case where k is the field of Laurent series, in which $\mathsf{SV}_{\mathsf{k}}$ constitutes a denotational semantics for signal flow graphs [3, 1, 5, 6], and the case $\mathsf{k} = \mathsf{Z}_2$, in which $\mathsf{SV}_{\mathsf{k}}$ is isomorphic to the phase-free ZX-calculus, an algebra for quantum observables [8, 4].

Now, in order to apply our construction, note that $\mathsf{SV}_{\mathsf{k}}$ is isomorphic to $\mathsf{Rel}(\mathsf{Vect}_k)$, where $\mathsf{Vect}_k$ is the abelian prop whose arrows $n \to m$ are the linear maps of type $\mathsf{k}^n \to \mathsf{k}^m$ (the monoidal product is by direct sum). This follows from the observation that subspaces of $\mathsf{k}^n \times \mathsf{k}^m$ of dimension $z$ correspond to mono linear maps from $\mathsf{k}^z$ to $\mathsf{k}^n \times \mathsf{k}^m$, whence to jointly mono spans $n \leftarrow z \to m$ in $\mathsf{Vect}_k$.

We are then in position to use Corollary 4, which yields the following pushout characterisation for $\mathsf{SV}_{\mathsf{k}}$.

$$\begin{tikzcd} \mathsf{Vect}_k + {\mathsf{Vect}_k}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{Vect}_k) \arrow[d] \\ \mathsf{Cospan}(\mathsf{Vect}_k) \arrow[r] & \mathsf{SV}_{\mathsf{k}}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{15}$$

This very same pushout has been studied in [4] for the $\mathsf{k} = \mathsf{Z}_2$ case. As before, the modular reconstruction suggests a presentation by generators and relations for $\mathsf{SV}_{\mathsf{k}}$, in terms of the theories for spans and cospans in $\mathsf{Vect}_k$. The axiomatisation of $\mathsf{SV}_{\mathsf{k}}$ is called the theory of interacting Hopf algebras [7, 23] , as it features two Hopf algebras structures and axioms expressing their combination.

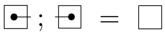
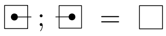
On the top of existing results on $\mathsf{SV}_{\mathsf{k}}$, our Corollary 4 suggests a novel perspective, namely that $\mathsf{SV}_{\mathsf{k}}$ can be also thought as the prop of *corelations* over $\mathsf{Vect}_{\mathsf{k}}$. This representation can be understood by recalling the 1-1 correspondence between subspaces of $\mathsf{k}^n \times \mathsf{k}^m$ and (solution sets of) homogeneous systems of equations $Mv = 0$, where $M$ is a $z \times (n+m)$ matrix. Writing the block decomposition $M = (M_1 \mid -M_2)$, where $M_1$ is a $z \times n$ matrix and $M_2$ a $z \times m$ matrix, this is the same as solutions to $M_1 v_1 = M_2 v_2$. These systems then yield jointly epi cospans $n \xrightarrow{M_1} z \xleftarrow{M_2} m$ in $\mathsf{Vect}_{\mathsf{k}}$, that is, corelations.

## 4.5 From Free Module Homomorphisms to Linear Corelations

We now consider the generalisation of the linear case from fields to principal ideal domains (PIDs). In order to form a prop, we need to restrict our attention to finitely-dimensional *free* modules over a PID R. The symmetric monoidal category of such modules and module homomorphisms, with monoidal product by direct sum, is equivalent to the prop $\mathsf{FMod}_{\mathsf{R}}$ whose arrows $n \to m$ are $\mathsf{R}$-module homomorphisms $\mathsf{R}^n \to \mathsf{R}^m$ or, equivalently, $m \times n$-matrices in $\mathsf{R}$. Because of the restriction to free modules, $\mathsf{FMod_R}$ is not abelian. However, it is still finitely bicomplete and has a costable (epi, split mono)-factorisation system.[4] Note that the fact that the ring $\mathsf{R}$ is a PID matters for the existence of pullbacks, as it is necessary for submodules of free $\mathsf{R}$-modules to be free—pushouts exist by self-duality of $\mathsf{FMod_R}$.

Write $\mathsf{MFMod_R}$ for the prop of split monos in $\mathsf{FMod_R}$. It is a classical, although nontrivial, theorem in control theory that this category obeys the required condition on pushouts of pullbacks [12]. Hence Theorem 2 yields the pushout square

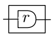
$$\begin{tikzcd} \mathsf{FMod_R} + \mathsf{FMod_R}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{MFMod_R}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{FMod_R}) \arrow[r] & \mathsf{Corel}(\mathsf{FMod_R}) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{16}$$

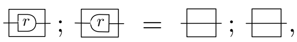
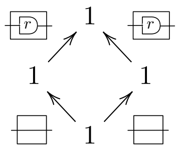
in **Prop**. This modular account of $\mathsf{Corel}(\mathsf{FMod_R})$ is relevant for the semantics of dynamical systems. When $\mathsf{R} = \mathbb{R}[s, s^{-1}]$, the ring of Laurent polynomials in some formal symbol $s$ with coefficients in the reals, the prop $\mathsf{Corel}(\mathsf{FMod}_{\mathbb{R}[s,s^{-1}]})$ models complete linear time-invariant discrete-time dynamical systems in $\mathbb{R}$; more details can be found in [12]. In that paper, it is also proven that $\mathsf{Corel}(\mathsf{FMod_R})$ is axiomatised by the presentation of $\mathsf{Cospan}(\mathsf{FMod_R})$ with the addition of the law . By Corollary 3, it follows that  originates by a pullback in $\mathsf{Span}(\mathsf{MFMod_R})$ and in this case it is the only contribution of spans to the presentation of corelations.

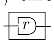
It is worth noticing that, even though $\mathsf{FMod_R}$ is not abelian, the pushout of spans and cospans over $\mathsf{FMod_R}$ does not have a trivial outcome as for the prop $\mathsf{F}$ of functions (Example 4.2). Instead, in [7, 23] it is proven that we have the pushout square

$$\begin{tikzcd} \mathsf{FMod_R} \oplus \mathsf{FMod_R}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathsf{FMod_R}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{FMod_R}) \arrow[r] & \mathsf{SV_k} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{17}$$

in **Prop**, where $\mathsf{k}$ is the *field of fractions* of $\mathsf{R}$.

The pushout (17) is relevant for the categorical semantics for signal flow graphs pursued in [3, 5, 6]. Even though it is not an instance of Theorem 2 or Corollary 4, our developments shed light on (17) through the comparison with (16). First, note that any element $r \in \mathsf{R}$ yields a module homomorphism $x \mapsto rx$ in $\mathsf{FMod_R}$ of type $1 \to 1$, represented as a string diagram . The key observation is that, in (17), $\mathsf{Span}(\mathsf{FMod_R})$ is contributing to the axiomatisation of $\mathsf{SV_k}$ (*cf.* Corollary 3) by adding, for each $r$, an equation

, corresponding to a pullback 

in $\mathsf{FMod_R}$. Back to (16), the only equations of this kind that $\mathsf{Span}(\mathsf{MFMod_R})$ is contributing with are those in which  is a *split mono*, that means, when $r$ is *invertible* in $\mathsf{R}$. Therefore,

[4] The factorisation given by (epi, mono) morphisms is not unique up to isomorphism, whence the restriction to split monos—see [12].

the difference between (16) and (17) is that in the latter one is adding formal inverses $\boxed{r}$ also for elements $\boxed{r}$ which are not originally invertible in $\mathsf{R}$. This explains the need of the field of fractions $\mathsf{k}$ of $\mathsf{R}$ in expanding the pushout object from $\mathsf{Corel}(\mathsf{FMod}_{\mathsf{R}})$ (in (16)) to $\mathsf{Corel}(\mathsf{Vect}_{\mathsf{k}}) \cong \mathsf{SV}_{\mathsf{k}}$ (in (17)).

### 4.6 From Maps of Algebras to Relations of Algebras

Let $\mathcal{C}$ be a regular category in which all regular epimorphisms split, and let $T$ be a monad on $\mathcal{C}$. Then the Eilenberg–Moore category $\mathcal{C}^T$ is regular. As for any regular category, the (regular epi,mono)-factorisation system is stable, so we can construct the category $\mathsf{Rel}(\mathcal{C}^T)$ of relations in $\mathcal{C}^T$. With a few further conditions on $T$, we can apply Corollary 1 to realise $\mathsf{Rel}(\mathcal{C}^T)$ as a pushout of categories.

To see this, note that the regular epis in $\mathcal{C}^T$ are simply the algebra maps with underlying map in $\mathcal{C}$ a regular epi. Indeed, since by assumption regular epimorphisms in $\mathcal{C}$ split, coequalizers in $\mathcal{C}^T$ can be computed using coequalizers in $\mathcal{C}$. Moreover, as the forgetful functor $U \colon \mathcal{C}^T \to \mathcal{C}$ is monadic, it creates finite limits, and the canonical map from a span to the pullback of its pushout can also be computed in $\mathcal{C}$. Thus $\mathcal{C}^T$ satisfies Assumption 2 whenever it is finitely cocomplete and $\mathcal{C}$ satisfies Assumption 2. (Given the finite completeness of $\mathcal{C}$, it is in fact enough for $\mathcal{C}^T$ to have reflexive coequalizers: this implies finite cocompleteness.) This allows us to apply the construction of Corollary 1.

These conditions are met, for example, for any monad $T$ over $\mathsf{Vect}_{\mathsf{k}}$. Hence, for example, we can apply Corollary 1 to the construction of the category of relations between algebras over a field (that is, vector spaces equipped with a bilinear product).

## 5 Concluding remarks

In summary, we have shown that categories of (co)relations may, under certain general conditions, be constructed as pushouts of categories of spans and cospans. In particular, especially since categories of spans and cospans can frequently be axiomatised using distributive laws, this offers a method of constructing axiomatisations of categories of (co)relations. Our results extend to the setting of props, and more generally symmetric monoidal categories. Moreover, these results are readily illustrated, unifying a diverse series examples drawn from algebraic theories, program semantics, quantum computation, and control theory.

Looking forward, note that in the monoidal case the resulting (co)relation category is a so-named hypergraph category: each object is equipped with a special commutative Frobenius structure. Hypergraph categories are of increasing interest for modelling network-style diagrammatic languages, and recent work, such as that of decorated corelations [10] or the generalized relations of Marsden and Genovese [18], gives precise methods for tailoring constructions of these categories towards chosen applications. Our example on relations in categories of algebras for a monad (Subsection 4.6) hints at general methods for showing the present universal construction applies to these novel examples. We leave this as an avenue for future work.

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# A Omitted Proofs

## A.1 Proof of Proposition 1

**Proof of Proposition 1.** We focus on relations, the proof for corelations being dual. It suffices to show that two spans $\xleftarrow{f}\xrightarrow{g}$ and $\xleftarrow{f'}\xrightarrow{g'}$ represent the same relation if and only if the $\mathcal{M}$ parts of the factorisations of $\langle f, g\rangle$ and $\langle f', g'\rangle$ are equal.

For the backward direction, write factorisations $\langle f, g\rangle = e;m$ and $\langle f', g'\rangle = e';m$, and note that $m = \langle m;p_1, m;p_2\rangle$, where the $p_i$ are the canonical projections. Thus the following diagrams commute

$$
\begin{tikzcd}
 & & {} \arrow[dll, "f"'] \arrow[drr, "g"] \arrow[dd, "e"] & & \\
{} & {} \arrow[l, "p_1"] & & {} \arrow[r, "p_2"'] & {} \\
 & & {} \arrow[ul, "m"] \arrow[ur, "m"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & {} \arrow[dll, "f'"'] \arrow[drr, "g'"] \arrow[dd, "e'"] & & \\
{} & {} \arrow[l, "p_1"] & & {} \arrow[r, "p_2"'] & {} \\
 & & {} \arrow[ul, "m"] \arrow[ur, "m"'] & &
\end{tikzcd}
$$

Therefore $\xrightarrow{e\in\mathcal{E}}$ and $\xrightarrow{e'\in\mathcal{E}}$ witness that both $\xleftarrow{f}\xrightarrow{g}$ and $\xleftarrow{f'}\xrightarrow{g'}$ are in the equivalence class of $\xleftarrow{p_1}\xleftarrow{m}\xrightarrow{m}\xrightarrow{p_2}$ and so represent the same relation.

For the forward direction, note that if $\xleftarrow{f}\xrightarrow{g}$ and $\xleftarrow{f'}\xrightarrow{g'}$ represent the same relation, then there exists a sequence of spans $\xleftarrow{f_i}\xrightarrow{g_i}$ in $\mathcal{C}$ together with morphisms $\xrightarrow{e_i\in\mathcal{E}}$, $i = 0, \ldots n$, such that $f_1 = f$, $g_1 = g$, $f_n = f'$, $g_n = g'$, and for all $i = 1, \ldots, n$ either (i) $e_i f_i = f_{i-1}$, $e_i g_i = g_{i-1}$ or (ii) $f_i = e_i f_{i-1}$, $g_i = e_i g_{i-1}$. This implies either (i) $e_i\langle f_i, g_i\rangle = \langle f_{i-1}, g_{i-1}\rangle$ or (ii) $\langle f_i, g_i\rangle = e_i\langle f_{i-1}, g_{i-1}\rangle$. In either case, by the uniqueness of factorisations, we see that the $\mathcal{M}$ parts of $\langle f_i, g_i\rangle$ are the same for all $i$. ◀

## A.2 Proof of Theorem 1

We devote this section to give a step-by-step argument for Theorem 1.

▶ **Proposition 4.** *The square* $(\star)$ *commutes.*

**Proof.** As $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ is a pushout, it is enough to show $(\star)$ commutes on the two injections of $\mathcal{A}$, $\mathcal{A}^{op}$ into $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$. This means that we have to show, for any $f\colon a \to b$ in $\mathcal{A}$, that

$$\Pi(\xleftarrow{id}\xrightarrow{f}) = \Gamma(\xrightarrow{f}\xleftarrow{id}) \quad \text{and} \quad \Pi(\xleftarrow{f}\xrightarrow{id}) = \Gamma(\xrightarrow{id}\xleftarrow{f}).$$

These are symmetric, so it suffices to check one. This follows immediately from the fact that the pushout of $\xleftarrow{id}\xrightarrow{f}$ is $\xrightarrow{f}\xleftarrow{id}$. ◀

Suppose we have a cocone over $\mathsf{Cospan}(\mathcal{C}) \longleftarrow \mathcal{A} +_{|\mathcal{A}|} \mathcal{A} \longrightarrow \mathsf{Span}(\mathcal{A})$. That is, suppose we have the commutative square:

$$
\begin{tikzcd}
\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathcal{A}) \arrow[d, "\Psi"] \\
\mathsf{Cospan}(\mathcal{C}) \arrow[r, "\Phi"'] & \mathcal{X}.
\end{tikzcd}
\tag{†}
$$

We prove two lemmas, from which the main theorem follows easily.

▶ **Lemma 1.** *If* $\xrightarrow{p}\xleftarrow{q}$ *is a cospan in* $\mathcal{A}$ *with pullback* $\xleftarrow{f}\xrightarrow{g}$ *(in* $\mathcal{C}$*), then* $\Psi(\xleftarrow{f}\xrightarrow{g}) = \Phi(\xrightarrow{p}\xleftarrow{q})$. *Similarly, if* $\xleftarrow{f}\xrightarrow{g}$ *is a span in* $\mathcal{A}$ *with pushout* $\xrightarrow{p}\xleftarrow{q}$ *(in* $\mathcal{C}$*), then* $\Psi(\xleftarrow{f}\xrightarrow{g}) = \Phi(\xrightarrow{p}\xleftarrow{q})$.

**Proof.** Consider $\xrightarrow{p}\xleftarrow{q} \in \mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$. Its image in $\mathcal{X}$ via the lower left corner of the commutative square (†) is $\Phi(\xrightarrow{p}\xleftarrow{q})$ while, recalling that $\xrightarrow{p}\xleftarrow{q} \in \mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ is mapped to $\xleftarrow{f}\xrightarrow{g}$ in $\mathsf{Span}(\mathcal{A})$, its image via the upper right corner is $\Psi(\xleftarrow{f}\xrightarrow{g})$. Thus $\Phi(\xrightarrow{p}\xleftarrow{q}) = \Psi(\xleftarrow{f}\xrightarrow{g})$.

The second claim is analogous, beginning instead with the span $\xleftarrow{f}\xrightarrow{g}$. ◀

▶ **Lemma 2.** *If $\xrightarrow{p_1}\xleftarrow{q_1}$ and $\xrightarrow{p_1}\xleftarrow{q_1}$ are cospans in $\mathcal{C}$ such that $\Gamma(\xrightarrow{p_1}\xleftarrow{q_1}) = \Gamma(\xrightarrow{p_2}\xleftarrow{q_2})$, then $\Phi(\xrightarrow{p_1}\xleftarrow{q_1}) = \Phi(\xrightarrow{p_2}\xleftarrow{q_2})$.*

**Proof.** Suppose $\Gamma(\xrightarrow{p_1}\xleftarrow{q_1}) = \Gamma(\xrightarrow{p_2}\xleftarrow{q_2})$ as per hypothesis. Then by Proposition 1 there exists $\xrightarrow{m_1}, \xrightarrow{m_2} \in \mathcal{M}$ and $\xrightarrow{p}\xleftarrow{q} \in \mathsf{Cospan}(\mathcal{C})$ such that

$$\xrightarrow{p_1}\xleftarrow{q_1} \; = \; \xrightarrow{p}\xrightarrow{m_1}\xleftarrow{m_1}\xleftarrow{q} \quad \text{and} \quad \xrightarrow{p_2}\xleftarrow{q_2} \; = \; \xrightarrow{p}\xrightarrow{m_2}\xleftarrow{m_2}\xleftarrow{q}\ .$$

Then

$$\begin{aligned}
\Phi(\xrightarrow{p_1}\xleftarrow{q_1}) \quad &= \quad \Phi(\xrightarrow{p}\xrightarrow{m_1}\xleftarrow{m_1}\xleftarrow{q}) \\
&= \quad \Phi(\xrightarrow{p}\xleftarrow{id})\,;\,\Phi(\xrightarrow{m_1}\xleftarrow{m_1})\,;\,\Phi(\xrightarrow{id}\xleftarrow{q}) \\
&\overset{(\clubsuit)}{=} \quad \Phi(\xrightarrow{p}\xleftarrow{id})\,;\,\Psi(\xleftarrow{id}\xrightarrow{id})\,;\,\Phi(\xrightarrow{id}\xleftarrow{q}) \\
&= \quad \Phi(\xrightarrow{p}\xleftarrow{id})\,;\,\Phi(\xrightarrow{id}\xleftarrow{q}) \\
&= \quad \Phi(\xrightarrow{p}\xleftarrow{q}),
\end{aligned}$$

and similarly for $\xrightarrow{p_2}\xleftarrow{q_2}$. The equality (♣) holds because, by Assumption 1, $\mathcal{M} \subseteq \mathcal{A}$ and $\xrightarrow{m_1 \in \mathcal{M}}$ is mono, thus the pullback of $\xrightarrow{m_1}\xleftarrow{m_1}$ is $\xleftarrow{id}\xrightarrow{id}$ and via Lemma 1 $\Phi(\xrightarrow{m_1}\xleftarrow{m_1}) = \Psi(\xleftarrow{id}\xrightarrow{id})$. ◀

**Proof of Theorem 1.** Suppose we have a commutative diagram (†). It suffices to show that there exists a functor $\theta\colon \mathsf{Corel}(\mathcal{C}) \to \mathcal{X}$ with $\theta\Gamma = \Phi$ and $\theta\Pi = \Psi$. Uniqueness is automatic by fullness (Proposition 2) and bijectivity on objects of $\Gamma$.

Given a corelation $a$, fullness yields a cospan $\xrightarrow{f}\xleftarrow{g}$ such that $\Gamma(\xrightarrow{f}\xleftarrow{g}) = a$. We then define $\theta(a) = \Phi(\xrightarrow{f}\xleftarrow{g})$. This is well-defined by Lemma 2.

For commutativity, clearly $\theta\Gamma = \Phi$. Moreover, $\theta\Pi = \Psi$: given a span $\xleftarrow{f}\xrightarrow{g}$ in $\mathcal{M}$, let $\xrightarrow{p}\xleftarrow{q}$ be its pushout span in $\mathcal{C}$. Thus by Lemma 1,

$$\Psi(\xleftarrow{f}\xrightarrow{g}) = \Phi(\xrightarrow{p}\xleftarrow{q}) = \theta\Gamma(\xrightarrow{p}\xleftarrow{q}) = \theta\Pi(\xleftarrow{f}\xrightarrow{g}).$$ ◀

## A.3 Proof of Theorem 2

We first discuss how to put monoidal structures on $\mathsf{Cospan}(\mathcal{C})$, $\mathsf{Corel}(\mathcal{C})$, and $\mathsf{Span}(\mathcal{A})$, and show that $\Gamma$ and $\Pi$ are prop morphisms in this case.

▶ **Proposition 5.** *Let $(\mathcal{C}, \oplus)$ be a prop with pullbacks, and let $\mathcal{A}$ be a sub-prop of $\mathcal{C}$ stable under pullback. If $\oplus$ preserves pullbacks in $\mathcal{A}$, then $(\mathsf{Span}(\mathcal{A}), \oplus)$ is a prop.*

**Proof.** We need to show the map

$$\oplus\colon \mathsf{Span}(\mathcal{A}) \times \mathsf{Span}(\mathcal{A}) \longrightarrow \mathsf{Span}(\mathcal{A})$$

is functorial. That is, given two pairs $(X \leftarrow N \rightarrow Y,\ X' \leftarrow N' \rightarrow Y')$ and $(Y \leftarrow M \rightarrow Z,\ Y' \leftarrow M' \rightarrow Z')$ of spans in $\mathcal{A}$, we need to show that the composite of their images under $\oplus$:

$$X \oplus X' \longleftarrow (N \oplus N') \times_{Y \oplus Y'} (M \oplus M') \longrightarrow Z \oplus Z'$$

is isomorphic to the image under $\oplus$ of their composite:

$$X \oplus X' \longleftarrow (N \times_Y M) \oplus (N' \times_{Y'} M') \longrightarrow Z \oplus Z'.$$

This is precisely the hypothesis that pullbacks commute with $\oplus$ in $\mathcal{A}$. ◀

Note that dualising the above argument with $\mathcal{A} = \mathcal{C}$ yields the fact that $(\mathsf{Cospan}(\mathcal{C}), \oplus)$ is a prop whenever $\oplus$ preseves pushouts. Also note that the inclusions $\mathcal{A} \to \mathsf{Span}(\mathcal{A})$ and $\mathcal{A}^{op} \to \mathsf{Span}(\mathcal{A})$ are prop functors.

▶ **Proposition 6.** *If $\mathcal{C}$ is a prop with a costable factorisation system, and $\mathcal{M}$ is closed under $\oplus$, then $\mathsf{Corel}(\mathcal{C})$ is a prop. Moreover, the quotient functor*

$$\Gamma \colon \mathsf{Cospan}(\mathcal{C}) \to \mathsf{Corel}(\mathcal{C})$$

*is a prop morphism.*

**Proof.** The first task is to show that $\mathsf{Corel}(\mathcal{C})$ is indeed a prop. We show that $\oplus$ induces a monoidal product, which we shall also write $\oplus$, on $\mathsf{Corel}(\mathcal{C})$. Given two corelations $a$ and $b$, with representatives $\xrightarrow{f}\xleftarrow{g}$ and $\xrightarrow{h}\xleftarrow{k}$ we define their monoidal product $a \oplus b$ to be the corelation represented by the cospan $\xrightarrow{f\oplus h}\xleftarrow{g\oplus k}$. This is well defined: given $\xrightarrow{f'}\xleftarrow{g'}$, $\xrightarrow{h'}\xleftarrow{k'}$ and $m_1, m_2$ in $\mathcal{M}$ such that $f' = f;m_1$, $g' = g;m_1$, $h' = h;m_2$, $k' = k;m_2$, the monoidality of $\oplus$ in $\mathcal{C}$ implies $f' \oplus g' = (f \oplus g);(m_1 \oplus m_2)$ and $h' \oplus k' = (h \oplus k);(m_1 \oplus m_2)$. Since $\mathcal{M}$ is closed under $\oplus$, $m_1 \oplus m_2$ again lies in $\oplus$, and the product corelation is independent of choice of representatives.

As prop morphisms are strict monoidal functors, to show that $\Gamma$ is a prop morphism we just need to check $\Gamma(a \oplus b) = \Gamma a \oplus \Gamma b$, where $a$ and $b$ are cospans. This follows immediately from the definition: the monoidal product of the corelations that two cospans represent is by definition the corelation represented by the monoidal product of the two cospans. ◀

▶ **Proposition 7.** $\Pi \colon \mathsf{Span}(\mathcal{A}) \to \mathsf{Corel}(\mathcal{C})$ *is a prop morphism.*

**Proof.** Again, we just need to check that $\Pi(a \oplus b) = \Pi a \oplus \Pi b$. For this we need that the monoidal product preserves pushouts. Indeed, given spans $a = X \xleftarrow{f} N \xrightarrow{g} Y$ and $b = X' \xleftarrow{f'} N' \xrightarrow{g'} Y'$, we have $\Pi(a \oplus b)$ represented by the cospan

$$X \oplus X' \longrightarrow (X \oplus X') +_{(N \oplus N')} (Y \oplus Y') \longleftarrow Y \oplus Y',$$

and $\Pi a \oplus \Pi b$ represented by the cospan

$$X \oplus X' \longrightarrow (X +_N Y) \oplus (X' +_{N'} Y') \longleftarrow Y \oplus Y'.$$

These cospans are isomorphic by the fact $\oplus$ preserves pushouts, and hence represent the same corelation. ◀

Now having described how to interpret $(\star)$ in the category of props, it remains to show that it is a pushout.

**Proof of Theorem 2.** From Theorem 1, we know that $(\star)$ commutes, and that given some other cocone

$$\begin{tikzcd} \mathcal{A} +_{|\mathcal{C}|} \mathcal{A}^{op} \arrow[r] \arrow[d] & \mathsf{Span}(\mathcal{A}) \arrow[d, "\Psi"] \\ \mathsf{Cospan}(\mathcal{C}) \arrow[r, "\Phi"'] & \mathcal{X}, \end{tikzcd}$$

there exists a unique functor $\theta$ from $\mathsf{Corel}(\mathcal{C})$ to $\mathcal{X}$. All we need do here is check that $\theta$ is a prop functor.

Suppose we have corelations $a$ and $b$, and write $\tilde{a}$ and $\tilde{b}$ for cospans that represent them. Recall that by definition $\theta a = \Phi\tilde{a}$. Then the strict monoidality of $\Phi$ gives

$$\theta(a \oplus b) = \Phi(\tilde{a} \oplus \tilde{b}) = \Phi\tilde{a} \oplus \Phi\tilde{b} = \theta a \oplus \theta b.$$

This proves the theorem. ◀

**Proof of Corollary 4.** Note that in an abelian prop the biproduct, being both a product and a coproduct, preserves both pushouts and pullbacks, and that the monos are closed under the biproduct. Thus we can apply Theorem 2. ◀

Finally, note that the above arguments, with the routine care paid to coherence maps, extend easily to the more general case of symmetric monoidal categories. In this case the pushout square is a pushout in both the category of symmetric monoidal categories and lax symmetric monoidal functors, and as well as the category of symmetric monoidal categories and strict monoidal functors.