Interacting Bialgebras are Frobenius

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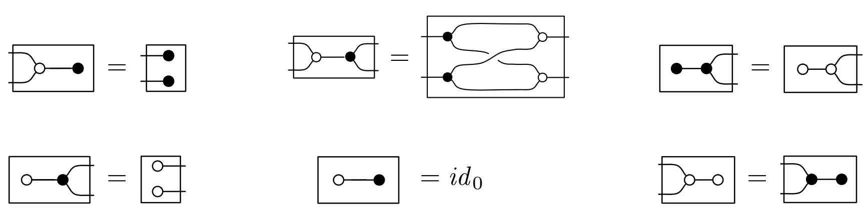
Abstract. Bialgebras and Frobenius algebras are different ways in which monoids and comonoids interact as part of the same theory. Such theories feature in many fields: e.g. quantum computing, compositional semantics of concurrency, network algebra and component-based programming.
In this paper we study an important sub-theory of Coecke and Duncan's ZX-calculus, related to strongly-complementary observables, where two Frobenius algebras interact. We characterize its free model as a category of $\mathbb{Z}_2$-vector subspaces. Moreover, we use the framework of PROPs to exhibit the modular structure of its algebra via a universal construction involving span and cospan categories of $\mathbb{Z}_2$-matrices and distributive laws between PROPs. Our approach demonstrates that the Frobenius structures result from the interaction of bialgebras.

1 Introduction

We report on a surprising meeting point between two separate threads of research. First, Coecke and Duncan [9] introduced the ZX-calculus as a graphical formalism for multi-qubit systems, featuring two interacting separable Frobenius algebras, which we distinguish here graphically via white and black colouring. The following equations capture the interaction for an important fragment of the calculus related to strongly complementary observables [10]:



The aforementioned and related works (see e.g. [11]) emphasise the interaction of two different (here, white and black) Frobenius structures. As we will explain, from an algebraic point of view, it is natural to consider this system as two (anti-separable) *bialgebras* interacting via two distributive laws of PROPs. We will show that the individual Frobenius structures arise as a *result* of these interactions. Consequently, we call the theory above *interacting bialgebras*, and the corresponding (free) PROP $\mathbb{IB}$.

Second, following the work of Katis, Sabadini, Walters and others on the $\mathsf{Span}(\mathbf{Graph})$ algebra [13] of transition systems, the second author introduced the calculus of Petri nets with boundaries [21] and commenced the study of the resulting structures in [22]. That calculus and extensions in [5, 6] are based on the the algebra of stateless connectors [4] of Bruni, Lanese and Montanari, also generated by two monoid-comonoid structures—which again, for sake of uniformity we will refer to as black and white.

Intuitively, in [4] a connector $n \to m$ has n ports on the left boundary and m ports on the right boundary. A black connector forces synchronization on all its ports, while a white one allows only two ports on opposite boundaries to synchronize. The semantics of connectors $n \to m$ are relations $\{0,1\}^n \to \{0,1\}^m$. For example, the black multiplication $2 \to 1$ is the relation $\{(00,0),(11,1)\}$ while the white multiplication is the relation $\{(00,0),(01,1),(10,1)\}$. The black structure (the semantics of comultiplication is the opposite relation) is a Frobenius algebra. The white structure is not Frobenius, but it becomes so if one adds the behaviour $(11,0)$ to the semantics of the white multiplication, making it the graph of addition[3] in $\mathbb{Z}_2$. The resulting theory satisfies the equations of $\mathbb{IB}$.

The meeting point of the two, seemingly disparate, threads is thus the PROP $\mathbb{IB}$. Before accounting for other related work, we outline our contributions.

- We characterise $\mathbb{IB}$ as the PROP $\mathbb{SV}$ of $\mathbb{Z}_2$-sub-vector spaces: the arrows $n \to m$ are sub-vector spaces of $\mathbb{Z}_2^n \times \mathbb{Z}_2^m$, with relational composition.
- We use Lack's framework of distributive laws on PROPs [15] to exhibit the modularity of this theory. The starting point is Lafont's observation [16, Theorem 5] that the theory of anti-separable bialgebras $\mathbb{AB}$ is precisely the PROP $\mathrm{Mat}\,\mathbb{Z}_2$ of $\mathbb{Z}_2$-matrices. $\mathrm{Mat}\,\mathbb{Z}_2$ can be composed with its dual $\mathrm{Mat}\,\mathbb{Z}_2{}^{op}$ via a distributive law given by pullback: the result of this composition is $\mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$, the PROP of spans over $\mathrm{Mat}\,\mathbb{Z}_2$. Dually, $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ arises from the distributive law of $\mathrm{Mat}\,\mathbb{Z}_2{}^{op}$ over $\mathrm{Mat}\,\mathbb{Z}_2$ given by pushout. The theories of $\mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ and $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ are actually the same "up-to exchanging the colours": they are the theory of $\mathbb{IB}$, but *without* the separability equation on precisely one of the white or black structures. We call them, respectively, $\mathbb{IB}^{-w}$ and $\mathbb{IB}^{-b}$. We prove that the top and bottom faces in the cube below are pushout diagrams in the category of PROPs: the isomorphism between $\mathbb{IB}$ and $\mathbb{SV}$ then follows from the universal property.

$$\begin{tikzcd}
 & \mathbb{AB}+\mathbb{AB}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{IB}^{-w} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IB}^{-b} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IB} \arrow[dd, dotted] & \\
 & \mathrm{Mat}\,\mathbb{Z}_2+\mathrm{Mat}\,\mathbb{Z}_2{}^{op} \arrow[rr, dashed] \arrow[dl] & & \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[dl] \\
\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[rr] & & \mathbb{SV} &
\end{tikzcd}$$

The mapping $\mathbb{IB} \to \mathbb{SV}$ gives a *semantics* for $\mathbb{IB}$: it can be presented in inductive form, yielding a simple technique for checking term equality in $\mathbb{IB}$.

[3] This works if one takes the graph of addition in any abelian group, which was pointed out to the second author by RFC Walters.

From a mathematical point of view, the results in this paper are a continuation of the programme initiated by Lack in [15]. In particular, our focus is on systematically extracting from distributive laws (a) complete axiomatisations and (b) factorisation systems for theories. Recent work on capturing algebraic theories using similar techniques includes [12] and [22].

Frobenius algebras [8, 14] have received much attention in topology, physics, algebra and computer science, partly because of the close correspondence with 2D TQFTs. The algebras we consider are the result of the research initiated by Abramsky and Coecke [1] on applying graphical techniques associated with algebras of monoidal categories [20] to model and reason about quantum protocols.

Related monoid-comonoid structures have been studied by computer scientists: amongst several the connectors in network algebra [23] and the wiring operations of REO [2]. Another closely related thread is Lafont's work on the algebraic theory of Boolean circuits [16], following the ideas of Burroni [7].

Structure of the paper. In §2 we recall the background on PROPs. In §3 we introduce the PROP $\mathbb{IB}$ and consider some of its properties. In §4 we recall the theory of anti-separable bialgebras and the characterisation of its free model as $\mathrm{Mat}\,\mathbb{Z}_2$. In §5 we give the details of the two distributive laws that yield $\mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ and $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ and their elementary presentations as the free PROPs $\mathbb{IB}^{-w}$ and $\mathbb{IB}^{-b}$. In §6 we collect our results to construct the cube (⧉).

Notation. Composition of arrows $f\colon a \to b$, $g\colon b \to c$ is denoted by $f \,;\, g\colon a \to c$. $\mathbb{C}[a,b]$ is the set of arrows from a to b in a small category $\mathbb{C}$ and $f^\star \in \mathbb{C}^{op}[b,a]$ is the contravariant counterpart of $f \in \mathbb{C}[a,b]$. Given $\mathcal{F}\colon \mathbb{C}_1 \to \mathbb{C}_2$, we denote with $\mathcal{F}^{op}\colon \mathbb{C}_1^{op} \to \mathbb{C}_2^{op}$ the functor defined by $(a \xrightarrow{f} b) \mapsto (a \xrightarrow{\mathcal{F}(f^\star)^\star} b)$.

2 Background

In this section we recall PROPs and their composition.

2.1 PROPs and Symmetric Monoidal Theories

Let $\mathbb{P}$ be the skeletal symmetric strict monoidal category of finite sets and bijections. It is harmless to take the naturals $\mathbb{N} = \{0, 1, 2, \dots\}$ as the objects, where $n \in \mathbb{N}$ stands for the finite set $\{0, 1, n-1\}$. The tensor product on objects is $n+m$. On arrows, given $f\colon n \to n$ and $g\colon m \to m$, $f \otimes g = f + g\colon n+m \to n+m$ where $+$ is ordinal sum.

Our exposition is founded on symmetric strict monoidal categories called PROPs (**pro**duct and **p**ermutation categories [17, 15]). They have $\mathbb{N}$ as the set of objects and the tensor product on objects is addition. Any PROP $\mathbb{T}$ contains certain arrows called permutations, which yield the symmetric monoidal structure and satisfy the same equations as they do in $\mathbb{P}$—i.e. there is a identity-on-objects symmetric strict (ISS) monoidal functor from $\mathbb{P}$ to $\mathbb{T}$. $\mathbb{P}$ is actually the initial object in **PROP**, the category of PROPs and their *homomorphisms*:

ISS monoidal functors that are homomorphic w.r.t. the permutations. In fact, **PROP** is the slice category $\mathbb{P}/\mathbf{PRO}$ where **PRO** is the category of symmetric strict monoidal categories that have $\mathbb{N}$ as set of objects and ISS functors. The fact that **PROP** is a slice category is vital: e.g. when we calculate the coproduct of two PROPs we must equate the images of the permutations via the injections (coproducts in a slice category are pushouts in the underlying category).

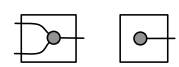
PROPs can encode (one-sorted) symmetric monoidal theories, that are equational theories at the level of abstraction of symmetric monoidal categories. A *symmetric monoidal theory* (SMT) is a pair (Σ, E) where Σ is a signature with elements $o\colon n \to m$. Here o is an operation symbol with *arity* n and *coarity* m. The Σ*-terms* are built by composing operations in Σ, subject to laws of symmetric monoidal categories. The set E consists of equations between Σ-terms.

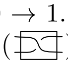
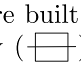
The *free PROP* $\mathbb{T}_{(\Sigma,E)}$ on the theory (Σ, E) is defined by letting $\mathbb{T}_{(\Sigma,E)}[n, m]$ be the set of Σ-terms with arity n and coarity m quotiented by E. When Σ is clear from the context, we will usually refer to terms of a SMT as *circuits*.

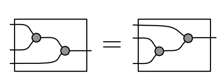
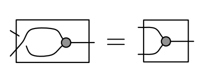
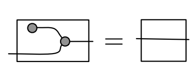
As PROPs describe equational theories, they come equipped with a notion of model: given a PROP $\mathbb{T}$ and a symmetric monoidal category $\mathbb{V}$, a $\mathbb{T}$-*algebra in* $\mathbb{V}$ is any symmetric monoidal functor $\mathcal{A}\colon \mathbb{T} \to \mathbb{V}$. On objects, $\mathcal{A}$ is determined by the assignment $\mathcal{A}(1)$, since $\mathcal{A}(n) \cong \mathcal{A}(1)^{\otimes n}$ for any $n \in \mathbb{N}$. The intuition is that $\mathcal{A}(1)$ is the support carrying the structure specified by $\mathbb{T}$. As expected, if the PROP $\mathbb{T}$ is free on a SMT (Σ, E), then its algebras have a universal characterization in terms of the models of (Σ, E) [18, 12].

Next we recall two important examples of SMTs: commutative monoids, commutative comonoids and the corresponding free PROPs.

The theory (Σ_M, E_M) of commutative monoids has two operation symbols in Σ_M - multiplication and unit - for which we adopt the graphical notation on the right.

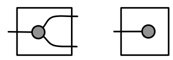


The left diagram represents the multiplication operation $m\colon 2 \to 1$: the two *ports* on the left boundary of the box represent the arity of m, whereas the single port on the right boundary encodes the coarity of m. Similarly, the right diagram depicts the unit operation $u\colon 0 \to 1$. Σ_M-terms are built out of those two components, plus the permutation () and identity () circuits, by sequential (;) and parallel ($\otimes$) composition. The set E_M expresses the following equations, stating associativity (M1), commutativity (M2) and identity (M3).

 (M1)  (M2)  (M3)

The free PROP on (Σ_M, E_M) is isomorphic to the skeletal symmetric strict monoidal category $\mathbb{F}$ of finite sets and functions. Indeed, the graph of a function $f\colon n \to m$ can be represented as a Σ_M-term: the equations (M1)-(M3) guarantee that this is a bijective representation. Consequently, an $\mathbb{F}$-algebra $\mathcal{A}\colon \mathbb{F} \to \mathbb{V}$ is precisely a commutative monoid in $\mathbb{V}$ with carrier $\mathcal{A}(1)$.

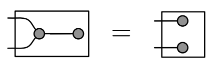
$\mathbb{F}^{op}$ is also a PROP, which is free for the theory (Σ_C, E_C) of commutative comonoids. As $\mathbb{F}^{op}$ is the opposite of $\mathbb{F}$, the operations in Σ_C (called comultiplication and counit, on the right) and the equations in E_C are those of E_M "rotated by 180°".



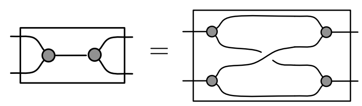
2.2 Composing PROPs

Given SMTs (Σ, E) and (Σ', E'), one can define their *sum* as the theory $(\Sigma \uplus \Sigma', E \uplus E')$. Usually one quotients the sum by new equations, describing the way in which the operations in Σ and Σ' interact. Both our leading examples of this construction are quotients of the sum of the theories of monoid and comonoids:

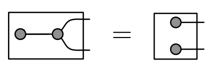
– the theory of (commutative/cocommutative) *bialgebras* is given as $(\Sigma_M \uplus \Sigma_C, E_M \uplus E_C \uplus B)$, where B consists of the following equations.



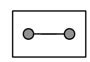
(B1)



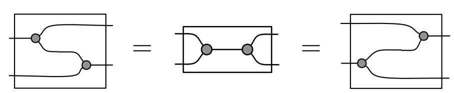
(B3)



(B2)

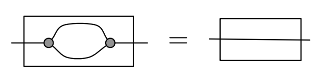
 $= id_0$ (B4)

– The theory of *Frobenius algebras* is given as $(\Sigma_M \uplus \Sigma_C, E_M \uplus E_C \uplus F)$, where F consists of the following two equations.



(Frob)

(Frob) states that circuits are invariant with respect to any topological deformation of their internal structure, provided that the link configuration between the ports is preserved. The theory of *separable* Frobenius algebras (SFAs) is given by adding to F the following equation.



(Sep)

Just as SFAs and bialgebras express different ways of combining a monoid and a comonoid, their free PROPs can be equivalently described as different ways of "composing" the PROPs $\mathbb{F}$ and $\mathbb{F}^{op}$. As we will see, this composition exactly amounts to the sum of the two SMTs quotiented by new equations.

To make this precise, we recall from [15] how PROP composition is defined in terms of distributive laws between monads. As shown in [24], the whole theory of monads can be developed in an arbitrary bicategory. Of particular interest are monads in the bicategory $\mathsf{Span}(\mathbf{Set})$, as they exactly correspond to small categories. A distributive law between two such monads can be seen as a way of forming the composite of the associated categories (with the same objects) [19].

In an analogous way, a PROP can be represented as a monad in a certain bicategory [15] and any two PROPs $\mathbb{T}_1$ and $\mathbb{T}_2$ can be composed via a distributive law $\lambda\colon \mathbb{T}_2 \,;\, \mathbb{T}_1 \to \mathbb{T}_1 \,;\, \mathbb{T}_2$ between the associated monads, provided that λ "respects" the monoidal structure [15].

Remark 1. The monad $\mathbb{T}_1 \,;\, \mathbb{T}_2$ yields a PROP with the following properties [15]:

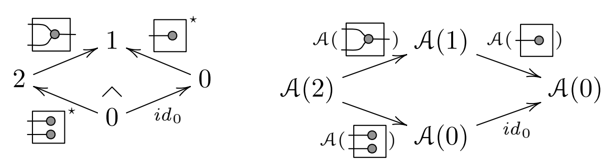
(†) any arrow $f \in \mathbb{T}_1 \,;\, \mathbb{T}_2[n,m]$ can be factorised into $f' \in \mathbb{T}_1[n,z]$ and $f'' \in \mathbb{T}_2[z,m]$, for some $z \in \mathbb{N}$;

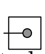
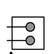
(‡) a $\mathbb{T}_1 \,;\, \mathbb{T}_2$-algebra $\mathcal{A}\colon \mathbb{T}_1 \,;\, \mathbb{T}_2 \to \mathbb{V}$ gives $\mathcal{A}(1)$ the structure of a $\mathbb{T}_1$-algebra and a $\mathbb{T}_2$-algebra, subject to the equations induced by the distributive law.

We provide an example of this construction and refer to [15] for further details.

Example 1. Let us consider what it means to define the composite PROP $\mathbb{F}^{op} \,;\, \mathbb{F}$ via a distributive law $\lambda\colon \mathbb{F} \,;\, \mathbb{F}^{op} \to \mathbb{F}^{op} \,;\, \mathbb{F}$. By its type, λ should map a pair of arrows $f \in \mathbb{F}[n,z]$, $g \in \mathbb{F}^{op}[z,m]$ into a pair $g' \in \mathbb{F}^{op}[n,z]$, $f' \in \mathbb{F}[z,m]$. This amounts to saying that λ maps *cospans* $n \xrightarrow{f} z \xleftarrow{g^\star} m$ into *spans* $n \xleftarrow{g'^\star} z \xrightarrow{f'} m$ in $\mathbb{F}$: a canonical way to define such a mapping is by forming the pullback of the given cospan. This indeed makes λ satisfy the equations of distributive laws [15]. The resulting PROP $\mathbb{F}^{op} \,;\, \mathbb{F}$ is the category of spans on $\mathbb{F}$, obtained by identifying the isomorphic 1-cells of the bicategory $\mathsf{Span}(\mathbb{F})$ and forgetting the 2-cells. With a slight abuse of notation, we call this category $\mathsf{Span}(\mathbb{F})$.

The SMT of $\mathsf{Span}(\mathbb{F})$ is the sum of the theories of the composed categories $\mathbb{F}$ and $\mathbb{F}^{op}$, quotiented by the equations induced by the distributive law. Those equations can be obtained by interpreting the pullbacks defining λ in a generic algebra $\mathcal{A}\colon \mathsf{Span}(\mathbb{F}) \to \mathbb{V}$. In this case, it suffices to consider four pullbacks [15]. One of them is depicted on the left, and its image in $\mathbb{V}$ is depicted on the right.



Since  and  originally belong to the $\mathbb{F}^{op}$-algebra structure, what is interpreted is their contravariant counterpart. Commutativity of the right-hand diagram is implied by $\mathsf{Span}(\mathbb{F})$ being a composite PROP [15] and it yields the equation (B1). The remaining three pullbacks to be considered yield (B2), (B3) and (B4). Therefore imposing the equations induced by λ correspond precisely to quotienting the monoidal and comonoidal structure of $\mathcal{A}(1)$ by the bialgebra equations. It follows that $\mathsf{Span}(\mathbb{F})$ is the free PROP on the theory of bialgebras.

We now focus on the dual situation: one can define the PROP $\mathbb{F} \,;\, \mathbb{F}^{op}$ via a distributive law $\lambda'\colon \mathbb{F}^{op} \,;\, \mathbb{F} \to \mathbb{F} \,;\, \mathbb{F}^{op}$ that forms the pushout of a given span. It follows that $\mathbb{F} \,;\, \mathbb{F}^{op}$ is the category $\mathsf{Cospan}(\mathbb{F})$, obtained from the corresponding bicategory of cospans, analogously to the case of $\mathbb{F}^{op} \,;\, \mathbb{F}$ and $\mathsf{Span}(\mathbb{F})$. One obtains the equations given by λ' by interpreting pushout diagrams, analogously to what we showed for λ. Those correspond to (Frob) and (Sep) [15], meaning that $\mathsf{Cospan}(\mathbb{F})$ is the free PROP on the theory of SFAs.

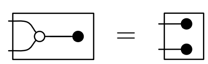
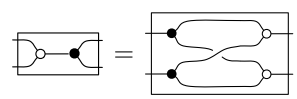
3 Interacting Bialgebras

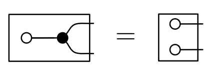
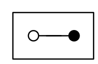
In this section we present a fragment of the ZX-calculus [9] that we call $\mathbb{IB}$. We define it as the free PROP on the SMT of *interacting bialgebras* (below) and we state that it is isomorphic to the PROP $\mathbb{SV}$ of $\mathbb{Z}_2$ vector subspaces. The remainder of the paper is a modular proof of this fact.

Definition 1. *The SMT of* interacting bialgebras $(\Sigma_{\mathbb{IB}}, E_{\mathbb{IB}})$ *consists of a signature* $\Sigma_{\mathbb{IB}}$ *with two copies each of the theory of monoids and of comonoids. In order to distinguish them, we colour one monoid/comonoid white, the other black. We will informally refer to them as the white and the black structures.*

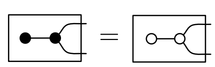
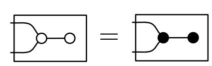
The set $E_{\mathbb{IB}}$ *of equations consists of:*

– *the equations making both the white and the black structures SFAs;*
– *bialgebra equations for the white monoid and the black comonoid;*

 (Q1)  (Q3)

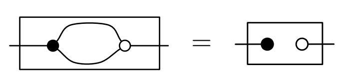
 (Q2)  $= id_0$ (Q4)

– *the following two equations, expressing the equivalence between the white and the black (self-dual) compact closed structure.*

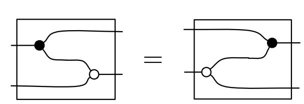
 (Q5)  (Q6)

Remark 2. The given axiomatization enjoys the following properties.

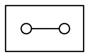
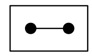
(a) "Rotating any equation by 180°" is sound.
(b) All equations (and thus all derived laws) are completely symmetric up-to swapping of white and black structures.
(c) $\mathbb{IB}$ satisfies the following "anti-separability" law expressing the fact that the white and the black structure cancel each other.

 (ASep)

(d) $\mathbb{IB}$ satisfies the "quasi-Frobenius" law below relating the black and white structures. This, together with the Frobenius black and white structures, amounts to saying that "only the topology matters".

 (QFrob)

(e) $\mathbb{IB}$ has all the "zero laws", expressing that the only circuit with no ports is id_0: they are (Q4), (Q4) "rotated by 180°" — *cf.* (a) — and the following.

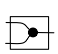
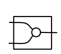
 $= id_0$ (Zero$_w$)  $= id_0$ (Zero$_b$)

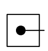
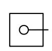
Definition 2. *Let $\mathbb{SV}$ be the following PROP:*

- *arrows $n \to m$ are subspaces of $\mathbb{Z}_2^n \times \mathbb{Z}_2^m$ (considered as a $\mathbb{Z}_2$-vector space).*
- *The composition ; is relational: for subspaces $G = \{(u, v) \mid u \in \mathbb{Z}_2^n, v \in \mathbb{Z}_2^z\}$ and $H = \{(v, w) \mid v \in \mathbb{Z}_2^z, w \in \mathbb{Z}_2^m\}$, their composition is the subspace $\{(u, w) \mid \exists v.(u, v) \in G \wedge (v, w) \in H\}$.*
- *The tensor product $\otimes$ on arrows is given by direct sum of spaces.*
- *The permutations $n \to n$ are induced by bijections of finite sets: to $\rho : n \to n$ we associate the subspace generated by $\{(1_i, 1_{\rho i})\}_{i<n}$ where 1_k stands for the binary n-vector with 1 at the $k{+}1$th coordinate and 0s elsewhere. For instance the twist $2 \to 2$ is the subspace generated by $\{(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}), (\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix})\}$.*

We now introduce a semantics homomorphism $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$ that we will later prove to be an iso. Even if $\mathcal{S}_{\mathbb{IB}}$ is not necessary for proving $\mathbb{IB} \cong \mathbb{SV}$, we present it as a valuable tool to reason about the equivalence of circuits in $\mathbb{IB}$.

Definition 3. *Let $[v_1, \ldots, v_n]$ denote the space generated by the vectors $v_1 \ldots v_n$. The homomorphism $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$ is inductively defined. For the monoids:*

 $\longmapsto [(\begin{pmatrix}1\\1\end{pmatrix}, (1))]$  $\longmapsto [(\begin{pmatrix}0\\1\end{pmatrix}, (1)), (\begin{pmatrix}1\\0\end{pmatrix}, (1))]$

 $\longmapsto [(), (1)]$  $\longmapsto [(), (0)]$

For the comonoids: take the reverse relations of the ones above; for composite circuits: $s \otimes t \mapsto \mathcal{S}_{\mathbb{IB}}(s) \otimes \mathcal{S}_{\mathbb{IB}}(t)$ and $s \mathbin{;} t \mapsto \mathcal{S}_{\mathbb{IB}}(s) \mathbin{;} \mathcal{S}_{\mathbb{IB}}(t)$.

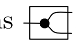
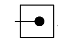
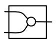
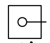
The homomorphism is well-defined since all the equations of $\mathbb{IB}$ are sound w.r.t. $\mathcal{S}_{\mathbb{IB}}$, namely if $s = t$ then $\mathcal{S}_{\mathbb{IB}}(s) = \mathcal{S}_{\mathbb{IB}}(t)$. The following theorem guarantees that the axiomatization is also complete.

Theorem 1. $\mathcal{S}_{\mathbb{IB}} : \mathbb{IB} \to \mathbb{SV}$ *is an isomorphism of PROPs.*

Remark 3. The asymmetry between the black and the white structure in Definition 3 is forced on us because $\mathcal{S}_{\mathbb{IB}}$ will be uniquely determined by the universal property of pushouts in **PROP**. Yet, strikingly, the axioms of $\mathbb{IB}$ describes two algebraic structures—the white and the black—in a completely symmetric way.

In the sequel, we are going to prove Theorem 1 by exploiting PROP composition, as described in Section 2.2. While a more direct proof might be given, our argument reveals the modular structures underlying $\mathbb{IB}$ and $\mathbb{SV}$.

4 Bialgebras and Vector Spaces

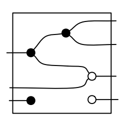
In this section we lay the foundations for our approach, by considering the SMT $\{\Sigma_{\mathbb{AB}}, E_{\mathbb{AB}}\}$ of *anti-separable* bialgebras. The set $\Sigma_{\mathbb{AB}}$ consists of operations , ,  and . The set $E_{\mathbb{AB}}$ contains the equations making the black structure a commutative comonoid, the white structure a commutative monoid, bialgebra equations (Q1)-(Q4) and (ASep). In short, an anti-separable bialgebra is just a bialgebra quotiented by (ASep)[4]. We call its free PROP $\mathbb{AB}$.

By virtue of Remark 2.(b)-(c), $\mathbb{IB}$ contains both a copy of $\mathbb{AB}$ and of $\mathbb{AB}^{op}$. These describe the interaction between the black and white structures of $\mathbb{IB}$.

Remark 4. As the free PROP for bialgebras is the composite $\mathsf{Span}(F) = \mathbb{F}^{op}$; $\mathbb{F}$ (*cf.* Example 1), $\mathbb{AB}$ enjoys the decomposition of Remark 1.(†): any circuit $t \in \mathbb{AB}[n,m]$ can be factorised as s ; $s' \in \mathbb{AB}[n,m]$, where $s \in \mathbb{F}^{op}[n,z]$ is part of the black comonoid and $s' \in \mathbb{F}[z,m]$ is part of the white monoid. Moreover, by (ASep), we can assume that any port on the left boundary has at most one connection with any one on the right boundary.

We say that any circuit s ; s' of the above shape is in *matrix form*: indeed, it has an intuitive representation as a matrix, as shown by the following example.

Example 2. The picture on the left shows a circuit $t \in \mathbb{AB}[3,4]$ in matrix form and on the right its representation as a 4×3 matrix.

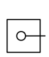
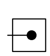
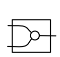
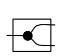


$$M = \begin{pmatrix} 1\,0\,0 \\ 1\,0\,0 \\ 1\,1\,0 \\ 0\,0\,0 \end{pmatrix}$$

The values in M are calculated as follows. For each boundary of t, suppose a top-down enumeration of its ports. Then $M[i,j]$ is 1 if, reading the circuit from the left to the right, one finds a path connecting the j^{th} port on the left boundary to the i^{th} port on the right, and 0 otherwise.

We now make the matrix semantics of $\mathbb{AB}$ formal. Let $\mathrm{Mat}\,\mathbb{Z}_2$ be the PROP with arrows $n \to m$ being $m \times n$ $\mathbb{Z}_2$-matrices, where ; is matrix multiplication and $\otimes$ is defined in the obvious way. The permutations are the rearrangements of the rows of the identity matrix. Clearly, $\mathrm{Mat}\,\mathbb{Z}_2$ is equivalent to the symmetric monoidal category of finite-dimensional $\mathbb{Z}_2$-vector spaces and linear maps.

Definition 4. *The homomorphism* $\mathcal{S}_{\mathbb{AB}} \colon \mathbb{AB} \to \mathrm{Mat}\,\mathbb{Z}_2$ *is defined inductively by*

 $\mapsto\ !$  $\mapsto\ ¡$  $\mapsto (1\ 1)$  $\mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

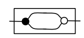
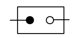
$$s \otimes t \mapsto \mathcal{S}_{\mathbb{AB}}(s) \otimes \mathcal{S}_{\mathbb{AB}}(t) \qquad\qquad s \,;\, t \mapsto \mathcal{S}_{\mathbb{AB}}(s) \,;\, \mathcal{S}_{\mathbb{AB}}(t)$$

where $! \colon 0 \to 1$ *and* $¡ \colon 1 \to 0$ *are the arrows given by initiality and finality of* 0. *It can be checked that* $\mathcal{S}_{\mathbb{AB}}$ *is well defined, as it respects the equations of* $\mathbb{AB}$.

[4] We can consider this as Hopf algebra with a trivial antipode.

Theorem 2 **([16]).** $\mathcal{S}_{\mathbb{AB}} \colon \mathbb{AB} \to \mathrm{Mat}\,\mathbb{Z}_2$ *is an isomorphism of PROPs.*

Proof. Any circuit t is equivalent to one t' in matrix form (*cf.* Remark 4), whose matrix $\mathcal{S}_{\mathbb{AB}}(t')$ can be computed as in Example 2. In fact the encoding of Example 2 is a 1-1 correspondence between matrices and circuits. Then $\mathcal{S}_{\mathbb{AB}}$ is full and faithful and (as $\mathbb{AB}$ and $\mathrm{Mat}\,\mathbb{Z}_2$ have the same objects) thus an isomorphism. $\square$

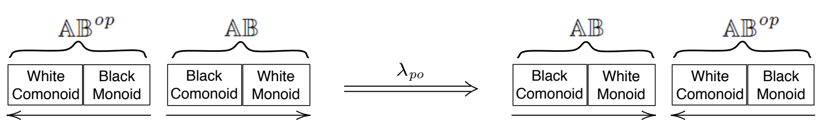
Remark 5. Observe that $\mathcal{S}_{\mathbb{AB}}$ maps the circuits  and  to the same matrix, meaning that equation (ASep) is necessary to establish Theorem 2. On the other hand, the theory of bialgebras with (Sep) in place of (ASep) would yield as free PROP the one of finite sets and relations [3]. The intuition is that in the realm of $\mathbb{Z}_2$-vector spaces a sum $v + v$ of a vector with itself is equal to 0, whereas for matrices representing relations $+$ is idempotent, i.e., $v + v = v$.

5 Composing Bialgebras

The PROPs $\mathbb{AB}$ and $\mathbb{AB}^{op}$ only describe the interaction between the white and black structures in $\mathbb{IB}$. We now study their composition, so that the interaction between the two white and the two black structures may also be observed.

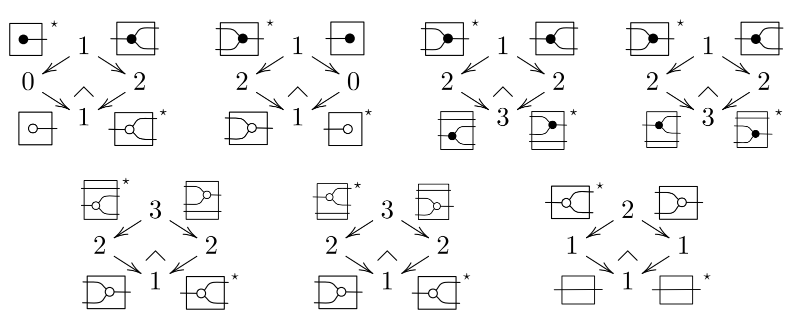
5.1 Cospans

First we obtain the PROP $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ via a distributive law $\lambda_{po} \colon \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \,;\, \mathrm{Mat}\,\mathbb{Z}_2 \to \mathrm{Mat}\,\mathbb{Z}_2 \,;\, \mathrm{Mat}\,\mathbb{Z}_2{}^{op}$ that maps a span in $\mathrm{Mat}\,\mathbb{Z}_2$ into its pushout (*cf.* Example 1). The conclusion of Theorem 2 and the factorisation of circuits in $\mathbb{AB}$ (Remark 4) allow us to understand λ_{po} as transforming circuits:



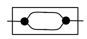
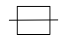
(1)

By Remark 1.(‡), a $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$-algebra will consist of an $\mathbb{AB}$-algebra, an $\mathbb{AB}^{op}$-algebra and equations between them, given by pushouts of spans in $\mathbb{AB}$. A free characterization of $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ can be then given by calculating (in $\mathrm{Mat}\,\mathbb{Z}_2$) those pushouts. Analogously to the case of $\mathsf{Cospan}(\mathbb{F})$, it suffices to consider merely the few cases that we list below.



(2)

The diagrams of the first row yield (Q5) and (Q6) and Frobenius equations for the black structure. The second row implies that the white structure is a SFA.

Therefore, the interaction between $\mathbb{AB}$ and $\mathbb{AB}^{op}$ encoded by λ_{po} has the effect of adding Frobenius structure to $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$. In fact, all equations of the theory of interacting bialgebras are covered, with the notable exception of the equation (Sep) for the black structure, which we denote with (Sep_b). Indeed, the two sides of (Sep_b),  and , denote different cospans in $\mathrm{Mat}\,\mathbb{Z}_2$:

$$1 \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} 2 \xleftarrow{\begin{pmatrix}1\\1\end{pmatrix}^{\star}} 1 \qquad \neq \qquad 1 \xrightarrow{\begin{pmatrix}1\end{pmatrix}} 1 \xleftarrow{\begin{pmatrix}1\end{pmatrix}^{\star}} 1.$$

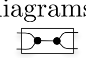
We call $\mathbb{IB}^{-b}$ the free PROP for the theory of interacting bialgebras minus the equation (Sep_b). Of the properties in Remark 2, (a), (c) and (d) also hold for $\mathbb{IB}^{-b}$, whereas (b) does not hold because (Sep_b) is missing. Concerning property (e), (Zero_w) does not hold in $\mathbb{IB}^{-b}$, as its proof requires (Sep_b). Symmetrically, (Zero_b) is derivable, as $\mathbb{IB}^{-b}$ has the white separability equation (Sep_w).

Theorem 3. $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) \cong \mathbb{IB}^{-b}$.

As a result, $\mathbb{IB}^{-b}$ enjoys the properties of composite PROPs. In particular, by Remark 1.(†) we have the following factorisation, where $\tau_1 \colon \mathbb{AB} \to \mathbb{IB}^{-b}$ and $\tau_2 \colon \mathbb{AB}^{op} \to \mathbb{IB}^{-b}$ denote the obvious inclusion maps.

Corollary 1 (Factorisation). *For every circuit $t \in \mathbb{IB}^{-b}[n,m]$, there exist $z \in \mathbb{N}$, $t_1 \in \mathbb{AB}[n,z]$ and $t_2 \in \mathbb{AB}^{op}[z,m]$ such that $t = \tau_1(t_1) \,;\, \tau_2(t_2)$.*

The decomposition of Corollary 1 is the one given in the right-hand side of (1).

Remark 6. The distributive laws for spans and cospans of finite sets [15] determine factorisation systems unique up-to "internal" permutation: i.e. if t factorises as $t_1 \,;\, t_2$ and $t_1' \,;\, t_2'$ then there exists a permutation p such that $t_1 = t_1' \,;\, p$ and $p \,;\, t_2 = t_2'$. The factorisation system of Corollary 1 is strictly weaker, being up-to "internal" isomorphism in $\mathrm{Mat}\,\mathbb{Z}_2$. These are all the invertible $\mathbb{Z}_2$-matrices, not merely the permutations in $\mathrm{Mat}\,\mathbb{Z}_2$. For instance, the two rightmost diagrams in the first row of (2) give different (but isomorphic) decompositions of  .

In order to make the isomorphism between $\mathbb{IB}^{-b}$ and $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ explicit, we define a semantics homomorphism $\mathcal{S}_{\mathbb{IB}^{-b}} \colon \mathbb{IB}^{-b} \to \mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ extending that of $\mathbb{AB}$ on $\mathrm{Mat}\,\mathbb{Z}_2$. It is defined inductively on circuits t in $\mathbb{IB}^{-b}$ as follows[5]:

$$t \mapsto \begin{cases} \kappa_1(\mathcal{S}_{\mathbb{AB}}(t)) & \text{if } t \in \Sigma_{\mathbb{AB}} \\ \kappa_2(\mathcal{S}^{op}_{\mathbb{AB}}(t)) & \text{if } t \in \Sigma_{\mathbb{AB}^{op}} \\ \mathcal{S}_{\mathbb{IB}^{-b}}(t_1) \,;\, \mathcal{S}_{\mathbb{IB}^{-b}}(t_2) & \text{if } t = t_1 \,;\, t_2 \\ \mathcal{S}_{\mathbb{IB}^{-b}}(t_1) \otimes \mathcal{S}_{\mathbb{IB}^{-b}}(t_2) & \text{if } t = t_1 \otimes t_2 \end{cases}$$

[5] For the base cases, recall that the signature $\Sigma_{\mathbb{IB}^{-b}}$ of $\mathbb{IB}^{-b}$ is that of $\Sigma_{\mathbb{AB}} \uplus \Sigma_{\mathbb{AB}^{op}}$.

where $\kappa_1\colon \mathrm{Mat}\,\mathbb{Z}_2 \to \mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ and $\kappa_2\colon \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \to \mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ are the canonical injections mapping $f \in \mathrm{Mat}\,\mathbb{Z}_2[n,m]$ and $g \in \mathrm{Mat}\,\mathbb{Z}_2{}^{op}[n,m]$ in $n \xrightarrow{f} m \xleftarrow{id} m$ and $n \xrightarrow{id} n \xleftarrow{g^\star} m$, respectively. The semantics is well-defined as all the equations of $\mathbb{IB}^{-b}$ are sound w.r.t. $\mathcal{S}_{\mathbb{IB}^{-b}}$.

Lemma 1. $\mathcal{S}_{\mathbb{IB}^{-b}}\colon \mathbb{IB}^{-b} \to \mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$ *is an isomorphism of PROPs.*

Proof. By Corollary 1, any circuit of $\mathbb{IB}^{-b}$ factorises as a cospan $n \xrightarrow{t_1} z \xleftarrow{t_2^\star} m$ in $\mathbb{AB}$. The statement then follows by Theorem 2.

5.2 Spans

Dually, a distributive law $\lambda_{pb}\colon \mathrm{Mat}\,\mathbb{Z}_2 \,;\, \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \to \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \,;\, \mathrm{Mat}\,\mathbb{Z}_2$ given by pullback yields a composite PROP $\mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2) = \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \,;\, \mathrm{Mat}\,\mathbb{Z}_2$. The algebraic characterization of $\mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ follows the same steps as the one of $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$, albeit with the white and black structures swapped.

More formally, let $\mathbb{IB}^{-w}$ be the free PROP on the theory of interacting bialgebras without the *white* separability equation (Sep_w). We define a semantics homomorphism $\mathcal{S}_{\mathbb{IB}^{-w}}\colon \mathbb{IB}^{-w} \to \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ by induction on circuits t of $\mathbb{IB}^{-w}$:

$$t \mapsto \begin{cases} \iota_1(\mathcal{S}_{\mathbb{AB}}(t)) & \text{if } t \in \Sigma_{\mathbb{AB}} \\ \iota_2(\mathcal{S}^{op}_{\mathbb{AB}}(t)) & \text{if } t \in \Sigma_{\mathbb{AB}^{op}} \\ \mathcal{S}_{\mathbb{IB}^{-w}}(t_1) \,;\, \mathcal{S}_{\mathbb{IB}^{-w}}(t_2) & \text{if } t = t_1 \,;\, t_2 \\ \mathcal{S}_{\mathbb{IB}^{-w}}(t_1) \otimes \mathcal{S}_{\mathbb{IB}^{-w}}(t_2) & \text{if } t = t_1 \otimes t_2 \end{cases}$$

where $\iota_1\colon \mathrm{Mat}\,\mathbb{Z}_2 \to \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ and $\iota_2\colon \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \to \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ are the canonical injections mapping $f \in \mathrm{Mat}\,\mathbb{Z}_2[n,m]$ and $g \in \mathrm{Mat}\,\mathbb{Z}_2{}^{op}[n,m]$ in $n \xleftarrow{id} n \xrightarrow{f} m$ and $n \xleftarrow{g^\star} m \xrightarrow{id} m$, respectively.

Lemma 2. $\mathcal{S}_{\mathbb{IB}^{-w}}\colon \mathbb{IB}^{-w} \to \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ *is an isomorphism of PROPs.*

Proof. The proof relies on the transpose homomorphism $\xi\colon \mathrm{Mat}\,\mathbb{Z}_2 \to \mathrm{Mat}\,\mathbb{Z}_2{}^{op}$ mapping matrices to their transposes. This can be equivalently defined for the circuits in $\mathbb{AB}$: taking the transpose of a circuit means to take its *photographic negative*, that is swapping of black and white structures. We call this homomorphism $\nu\colon \mathbb{AB} \to \mathbb{AB}^{op}$. Both ξ and ν are full and faithful and they can be extended to full and faithful homomorphisms $\xi'\colon \mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) \to \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ and $\nu'\colon \mathbb{IB}^{-w} \to \mathbb{IB}^{-b}$. By a simple inductive argument, it holds that $\mathcal{S}_{\mathbb{IB}^{-w}} = \nu' \,;\, \mathcal{S}_{\mathbb{IB}^{-b}} \,;\, \xi'$ and therefore $\mathcal{S}_{\mathbb{IB}^{-w}}$ is full and faithful. Since $\mathbb{IB}^{-w}$ and $\mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ have the same objects, $\mathcal{S}_{\mathbb{IB}^{-w}}$ is thus an isomorphism of PROPs. □

As evident from the above, $\mathbb{IB}^{-w} \cong \mathbb{IB}^{-b}$ and $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) \cong \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ (by self-duality of $\mathrm{Mat}\,\mathbb{Z}_2$). This observation gives a straightforward proof that $\mathbb{IB}^{-w} \cong \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$. However, our explicit characterization via $\mathcal{S}_{\mathbb{IB}^{-w}}$ is instrumental in the construction of the next section.

6 The Cube

We now have all the ingredients in order to construct the diagram (⧉) discussed in the Introduction and to prove Theorem 1.

The backward faces. By definitions of $\mathcal{S}_{\mathbb{IB}^{-w}}$ and $\mathcal{S}_{\mathbb{IB}^{-b}}$, the following diagram commutes, where $\sigma_1 \colon \mathbb{AB} \to \mathbb{IB}^{-w}$ and $\sigma_2 \colon \mathbb{AB}^{op} \to \mathbb{IB}^{-w}$ are inclusions.

$$\begin{tikzcd}
\mathbb{IB}^{-b} \arrow[d, "\mathcal{S}_{\mathbb{IB}^{-b}}"'] & \mathbb{AB} + \mathbb{AB}^{op} \arrow[l, "{[\tau_1,\tau_2]}"'] \arrow[r, "{[\sigma_1,\sigma_2]}"] \arrow[d, "\mathcal{S}_{\mathbb{AB}} + \mathcal{S}_{\mathbb{AB}}^{op}"] & \mathbb{IB}^{-w} \arrow[d, "\mathcal{S}_{\mathbb{IB}^{-w}}"] \\
\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) & \mathrm{Mat}\,\mathbb{Z}_2 + \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \arrow[l, "{[\kappa_1,\kappa_2]}"] \arrow[r, "{[\iota_1,\iota_2]}"'] & \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)
\end{tikzcd} \tag{Back}$$

The bottom face. Given a span $n \xleftarrow{f} z \xrightarrow{g} m$ and a cospan $n \xrightarrow{p} z \xleftarrow{q} m$, we define

$$\varphi(f,g) = \{(u,v) \mid \exists x \in \mathbb{Z}_2^z .\, fx = u,\, gx = v\} \qquad \psi(p,q) = \{(u,v) \mid pu = qv\}.$$

It is easy to show that φ and ψ are homomorphisms and that the diagram

$$\begin{tikzcd}
\mathrm{Mat}\,\mathbb{Z}_2 + \mathrm{Mat}\,\mathbb{Z}_2{}^{op} \arrow[r, "{[\iota_1,\iota_2]}"] \arrow[d, "{[\kappa_1,\kappa_2]}"'] & \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[d, "\varphi"] \\
\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[r, "\psi"'] & \mathbb{SV}
\end{tikzcd} \tag{Bottom}$$

commutes. It is straightforward to verify that it is a pushout in **PROP**.

The top face. Take $Sep_w \colon \mathbb{IB}^{-w} \to \mathbb{IB}$ and $Sep_b \colon \mathbb{IB}^{-b} \to \mathbb{IB}$ to be the homomorphisms quotienting the arrows in $\mathbb{IB}^{-w}$ and $\mathbb{IB}^{-b}$ w.r.t. the equations (Sep_w) and (Sep_b), respectively. It is immediate to see that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{AB} + \mathbb{AB}^{op} \arrow[r, "{[\sigma_1,\sigma_2]}"] \arrow[d, "{[\tau_1,\tau_2]}"'] & \mathbb{IB}^{-w} \arrow[d, "Sep_w"] \\
\mathbb{IB}^{-b} \arrow[r, "Sep_b"'] & \mathbb{IB}
\end{tikzcd} \tag{Top}$$

To see that (Top) is a pushout, take any $\alpha \colon \mathbb{IB}^{-w} \to \mathbb{C} \leftarrow \mathbb{IB}^{-b} \colon \beta$ such that $[\sigma_1, \sigma_2]\ ;\ \alpha = [\tau_1, \tau_2]\ ;\ \beta$. The mediating homomorphism $\chi \colon \mathbb{IB} \to \mathbb{C}$ is defined inductively on circuits t in $\mathbb{IB}$ as follows:

$$t \mapsto \begin{cases} \alpha(\sigma_1(t)) = \beta(\tau_1(t)) & \text{if } t \in \Sigma_{\mathbb{AB}} \\ \alpha(\sigma_2(t)) = \beta(\tau_2(t)) & \text{if } t \in \Sigma_{\mathbb{AB}^{op}} \\ \chi(t_1)\ ;\ \chi(t_2) & \text{if } t = t_1\ ;\ t_2 \\ \chi(t_1) \otimes \chi(t_2) & \text{if } t = t_1 \otimes t_2 \end{cases} \tag{3}$$

This is well-defined as all equations of $\mathbb{IB}$ hold in either $\mathbb{IB}^{-w}$ or in $\mathbb{IB}^{-b}$.

The front faces. By commutativity of (Back) and (Bottom), the universal property of (Top) induces an homomorphism making the following diagram commute.

$$\begin{tikzcd}
\mathbb{IB}^{-b} \arrow[r, "Sep_b"] \arrow[d, "\mathcal{S}_{\mathbb{IB}^{-b}}"'] & \mathbb{IB} \arrow[d, dotted] & \mathbb{IB}^{-w} \arrow[l, "Sep_w"'] \arrow[d, "\mathcal{S}_{\mathbb{IB}^{-w}}"] \\
\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[r, "\psi"'] & \mathbb{SV} & \mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2) \arrow[l, "\varphi"]
\end{tikzcd} \tag{Front}$$

This homomorphism is defined as in (3). By induction, one can show that this is exactly $\mathcal{S}_{\mathbb{IB}}$ in Definition 3. Fullness and faithfulness follow from fullness and faithfulness of the other semantics homomorphisms and from the fact that (Top) and (Bottom) are pushouts.

7 Conclusions

We have studied the theory of interacting bialgebras $\mathbb{IB}$ which is relevant for both categorical quantum computing [9–11] and compositional models of concurrent systems [4, 21, 5]. We have shown that the PROP $\mathbb{SV}$ of $\mathbb{Z}_2$ sub-vector spaces freely characterizes $\mathbb{IB}$ and provided an inductive semantics which is useful for reasoning about equality of circuits in $\mathbb{IB}$.

Most importantly, we have exhibited the modular structure of $\mathbb{IB}$. The theory of antiseparable bialgebras $\mathbb{AB}$—freely characterized by $\mathrm{Mat}\,\mathbb{Z}_2$, the PROP of $\mathbb{Z}_2$-matrices—can be composed with its dual $\mathbb{AB}^{op}$ in two different ways, resulting in two different, albeit isomorphic theories: $\mathbb{IB}^{-w}$ and $\mathbb{IB}^{-b}$. These have the same equations as $\mathbb{IB}$ but without the white and the black separability axioms, respectively. The former is freely characterized by $\mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ and the latter by $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$. Finally, by gluing $\mathbb{IB}^{-b}$ and $\mathbb{IB}^{-w}$ we obtain $\mathbb{IB}$ and, by gluing $\mathsf{Span}(\mathrm{Mat}\,\mathbb{Z}_2)$ and $\mathsf{Cospan}(\mathrm{Mat}\,\mathbb{Z}_2)$, we arrive at $\mathbb{SV}$.

In fact, a similar story can be told in the simpler setting of the theory of monoids and $\mathbb{F}$ (the PROP of functions) in place of $\mathbb{AB}$ and $\mathrm{Mat}\,\mathbb{Z}_2$. Following essentially the same script, one obtains in place of $\mathbb{IB}^{-w}$ and $\mathbb{IB}^{-b}$ the theory of bialgebras and the theory of SFAs, as shown in [15]. Instead of $\mathbb{SV}$, one gets the PROP of equivalence relations over finite sets and, in place of $\mathbb{IB}$, the gluing of the theories of bialgebras and SFAs which, as shown in [3], can be presented by the equations (Frob), (Sep) and (B4).

It is thus natural to ask whether this general pattern reoccurs in other settings. For example, we are interested in *sets and relations with contention* which, as shown in [22], are structures underlying the compositional semantics of C/E Petri nets. We are confident that, following the work of Lafont [16], our results can be generalized to vector spaces over arbitrary fields. Following in this direction, one could take aim at the ZX-calculus in its entirety.

Acknowledgment. The first and third author acknowledge support by project ANR 12IS02001 PACE.

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