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Linear Algebra.

ASSIGNMENT # 02

TOPIC:

DETERMINENTS

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Question #

What is Matrix Determinant?
What are the properties of a determinant?
Explain each property with an example.

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DETERMINENTS OF A MATRIX.

The Determinant is a scalar value.
It is function of the elements of a square matrix.

It is also an element that identifies or determines nature of a matrix.

e.g., A matrix is invertible if its determinant is non-zero.

We can determine a determinant by following method:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc. \Rightarrow \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
—(1)

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - eg)$$
$$= aei - ahf - bdi - bgf + cdh - ceg \quad —(2)$$

For 2x2 matrix —(1)
For 3x3 matrix —(2)

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A general definition on how to determine a matrix's determinant with order $n \times n$ is:

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \dots + (-1)^{1+n} a_{1n}\det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}(\det A_{1j}) \quad -(3)$$

where $(-1)^{1+j} A_{1j}$ is known as Cofactor of an Element.

$\det A_{ij}$ can also be written as A_{ij}.

Cofactor of matrix A

$$C_{ij} = (-1)^{i+j}\det A_{ij} \quad -(4)$$

matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & \cdots & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$

Area of parallelogram.

The determinant of a matrix can also determine the area of a parallelogram.

$$\text{Area of parallelogram} = |\det(A)| \quad (5)$$

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Volume of parallelpiped.

The determinant of a matrix determines the volume of parallelpiped.

$$\text{Volume of parallelpiped} = |\det A| \quad \text{—— (6)}$$

Examples. for calculating Square matrix determinants.

—(1)

let $A = \begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} = 5\times4 - 2\times3 = 20-6 = 14.$

—(2), —(3), —(4)

let $A = \begin{vmatrix} 1 & 3 & 6 \\ 5 & 2 & 9 \\ 4 & 7 & 8 \end{vmatrix} = 1 \begin{vmatrix} 2 & 9 \\ 7 & 8 \end{vmatrix} + (-1)^{1+2}(3) \begin{vmatrix} 5 & 9 \\ 4 & 8 \end{vmatrix}$

$+ (-1)^{1+3}(6) \begin{vmatrix} 5 & 2 \\ 4 & 7 \end{vmatrix}$

$= 1(2\times8 - 7\times9) + (-1)^3(3)(5\times8 - 4\times9) + (-1)^4(6)(5\times7 - 4\times2)$

$= 16 - 63 + (-1)(3)(40-36) + (+1)(6)(35-8)$

$= -47 + (-3)(4) + 6(27)$

$= 162 - 47 - 12 = 103.$

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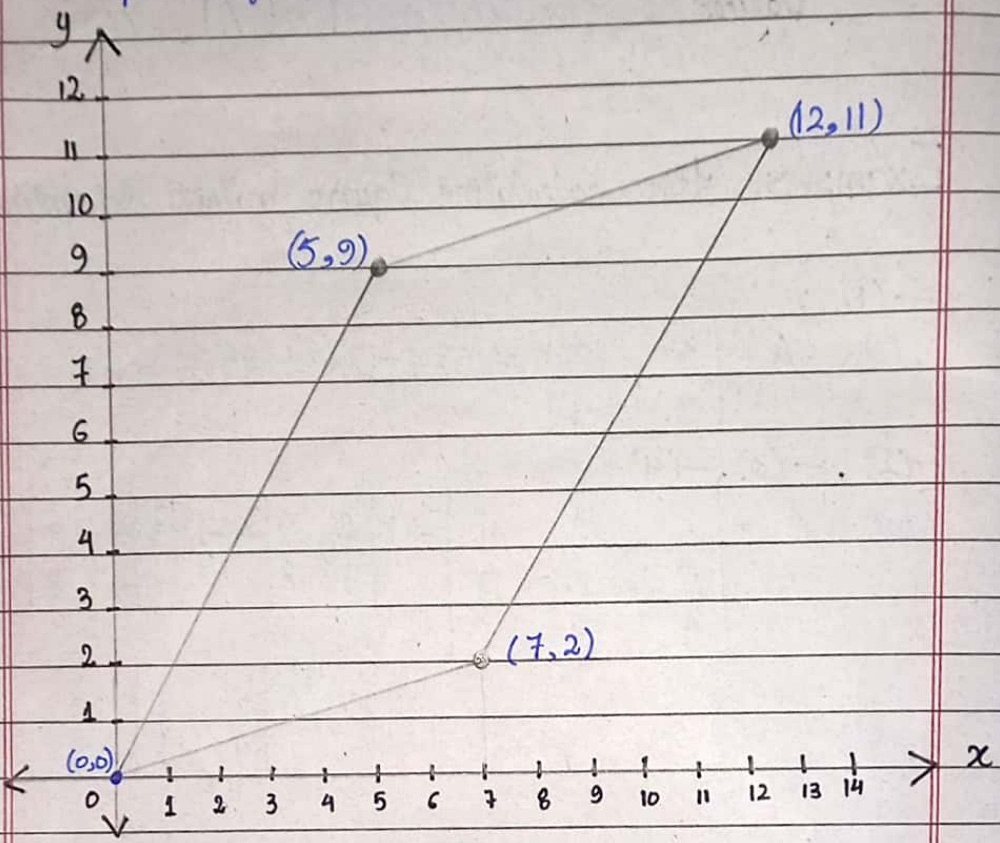
Example for calculating area of parallelogram using determinants.

let. (0,0) (7,2) (5,9) (12,11) be a parallelogram.



The area can be determined by vectors

$v_1 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 5 & 7 \\ 9 & 2 \end{bmatrix}$ $\Rightarrow$ Area of parallelogram $= |\det A|$

$\Rightarrow$ Area of parallelogram $= \left| \begin{vmatrix} 5 & 7 \\ 9 & 2 \end{vmatrix} \right| = |5 \times 2 - 9 \times 7|$

$= |10 - 63| = |-53| = 53.$

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Examples of calculating volume of parallelpiped using determinents.

let (0,0,0), (2,2,-1) (1,3,0) (-1,1,4) makes a parallelpiped

Then Volume of parallelpiped will be absolute value of determinent of matrix

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{bmatrix}$$

$$\text{Volume of parallelpiped} = \left| \begin{vmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{vmatrix} \right|$$

$$= \left| 2 \begin{vmatrix} 3 & 0 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} \right|$$

$$= \left| 2(3\times4-0) - 2(1\times4-0) - 1(1\times1-(-1)(3)) \right|$$

$$= \left| 2(12) - 2(4) - 1(4) \right| = \left| 24 - 8 - 4 \right| = |12| = 12.$$

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PROPERTIES OF DETERMINENTS.

(1) The matrix is only invertible when determinant is non-zero.
i.e. matrix A is only invertible i.e. A^{-1} when $\det A \neq 0$.
because if it is 0 then

$$A^{-1} = \frac{Adj\,A}{\det A}$$ - will be undefined.

$$A^{-1} = \frac{Adj\,A}{\det A}, \quad \det A \neq 0.$$

(2) The matrix's determinant has same value as the determinant of its transpose.
i.e., $\det A = \det A^t$.
⇒ if A is a matrix then the determinant of its transpose will be same as its determinant.

Proof

e.g., let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$\det A = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{21}a_{12}$$

$$= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$$

Example

if $A = \begin{bmatrix} 6 & 8 \\ 7 & 11 \end{bmatrix}$ ⇒ $\det A = 11\times6 - 7\times8 = 66 - 56$
$= 10.$

then $A^t = \begin{bmatrix} 6 & 7 \\ 8 & 11 \end{bmatrix}$ ⇒ $\det A^t = 6\times11 - 8\times7 = 66 - 56$
$= 10.$

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(3) When rows or columns of a matrix are interchanged the determinant is negative

i.e. if we interchange rows or columns of matrix A then its determinant is
$-\det A$

Proof

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det A = a_{11}a_{22} - a_{12}a_{21}$

if $A = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$ → Rows interchanged.

$\det A = a_{21}a_{12} - a_{22}a_{11} = a_{12}a_{21} - a_{11}a_{22}$
$= -(a_{11}a_{22} - a_{12}a_{21}) = -\det A.$

Example

$A = \begin{bmatrix} 11 & 10 \\ 2 & 3 \end{bmatrix} \Rightarrow \det A = 33 - 20 = 13.$

$A_1 = \begin{bmatrix} 10 & 11 \\ 3 & 2 \end{bmatrix} \Rightarrow \det A = 20 - 33 = -13$

$\Rightarrow -\det A.$

(4) The determinant of a matrix is 0.
$\det A = 0$ when
(i) A matrix A has two identical rows or two identical columns.
(ii) A matrix A has all elements zero.
i.e. when all the entries of a matrix are zero.

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(i)

$$A = \begin{bmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \end{bmatrix} \Rightarrow \det A = a_{11}a_{21} - a_{11}a_{21} = 0.$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \det A = 2 - 2 = 0.$$

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 8 \\ 3 & 3 & 9 \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} 1 & 1 & 4 \\ 1 & 1 & 8 \\ 3 & 3 & 9 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 8 \\ 3 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & 8 \\ 3 & 9 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}$$

$$= 1(9-24) - 1(9-24) + 0. = -15 + 15 = 0.$$

(ii)

It is obvious that if all the elements of a matrix are 0 then determinant will be zero.
which concludes that.

A null matrix A of order $n \times n$ has
$\det A = 0$ (always).

(5) Determinant of the Identity matrix is always 1.

i.e. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(I) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$= 1 \times 1 - 0 \times 0 = 1.$

Same goes for order $n \times n$.

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(6) If each entry of a row or a column consists of two terms as a sum then its determinant can be expressed as.

Example:

$$A = \begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

(7) When to each entry of a column or a row of a matrix A is added a non-zero multiple of the corresponding entries of another row or column then determinant of the matrix remains the same.

Example.

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$\Rightarrow$ if $|B| = \begin{vmatrix} a_{11} & a_{12}+ka_{11} \\ a_{21} & a_{22}+ka_{21} \end{vmatrix}$

then $|B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & ka_{11} \\ a_{21} & ka_{21} \end{vmatrix}$

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$$\Rightarrow |B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + k \begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \end{vmatrix}$$

$$\Rightarrow |B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + 0$$

$$\Rightarrow |B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |A|.$$

(8). If a matrix is triangular then product of the entries in its diagnol gives the determinant of that matrix.

Example.

e.g. let $A = \begin{bmatrix} 21 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$\Rightarrow |A| = \begin{vmatrix} 21 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 21 \times 6 \times 2 = 252.$$

let $B = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 7 & 1 \\ 0 & 0 & 6 \end{bmatrix}$

$$|B| = \begin{vmatrix} 3 & 2 & 6 \\ 0 & 7 & 1 \\ 0 & 0 & 6 \end{vmatrix} = 3 \times 7 \times 6 = 126.$$

Other way round.

$$|B| = 3 \begin{vmatrix} 7 & 1 \\ 0 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ 0 & 6 \end{vmatrix} + 6 \begin{vmatrix} 0 & 7 \\ 0 & 0 \end{vmatrix}$$

$$= 3(7 \times 6 - 0) - 2(0) + 6(0)$$

$$= 3(42) = 126.$$ Hence proved

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