AHMAD FALLAHPOUR

1 Hidden Markov Models

1.1 Probability of an observed sequence

$$\begin{aligned}\alpha_1(1) &= P(x_1 = A|z_1 = s_1)P(z_1 = s_1) = 0.4 * 0.7 = 0.28\\ \alpha_1(2) &= P(x_1 = A|z_1 = s_2)P(z_1 = s_2) = 0.2 * 0.3 = 0.06\end{aligned} \tag{1}$$

$$\begin{aligned}\alpha_2(1) &= P(x_2 = G|z_2 = s_1)(a_{11}\alpha_1(1) + a_{21}\alpha_1(2)) = 0.4 * (0.8 * 0.28 + 0.4 * 0.06) = 0.0992\\ \alpha_2(2) &= P(x_2 = G|z_2 = s_2)(a_{12}\alpha_1(1) + a_{22}\alpha_1(2)) = 0.2 * (0.2 * 0.28 + 0.6 * 0.06) = 0.0184\end{aligned} \tag{2}$$

$$\begin{aligned}\alpha_3(1) &= P(x_3 = C|z_3 = s_1)(a_{11}\alpha_2(1) + a_{21}\alpha_2(2)) = 0.1 * (0.8 * 0.0992 + 0.4 * 0.0184) = 0.0087\\ \alpha_3(2) &= P(x_3 = C|z_3 = s_2)(a_{12}\alpha_2(1) + a_{22}\alpha_2(2)) = 0.3 * (0.2 * 0.0992 + 0.6 * 0.0184) = 0.0093\end{aligned} \tag{3}$$

$$\begin{aligned}\alpha_4(1) &= P(x_4 = G|z_4 = s_1)(a_{11}\alpha_3(1) + a_{21}\alpha_3(2)) = 0.4 * (0.8 * 0.0087 + 0.4 * 0.0093) = 0.0043\\ \alpha_4(2) &= P(x_4 = G|z_4 = s_2)(a_{12}\alpha_3(1) + a_{22}\alpha_3(2)) = 0.2 * (0.2 * 0.0087 + 0.6 * 0.0093) = 0.0015\end{aligned} \tag{4}$$

$$\begin{aligned}\alpha_5(1) &= P(x_5 = T|z_5 = s_1)(a_{11}\alpha_4(1) + a_{21}\alpha_4(2)) = 0.1 * (0.8 * 0.0043 + 0.4 * 0.0015) = 0.000404\\ \alpha_5(2) &= P(x_5 = T|z_5 = s_2)(a_{12}\alpha_4(1) + a_{22}\alpha_4(2)) = 0.3 * (0.2 * 0.0043 + 0.6 * 0.0015) = 0.000528\end{aligned} \tag{5}$$

$$\begin{aligned}\alpha_6(1) &= P(x_6 = A|z_6 = s_1)(a_{11}\alpha_5(1) + a_{21}\alpha_5(2)) = 0.4 * (0.8 * 0.000404 + 0.4 * 0.000528) = 0.000214\\ \alpha_6(2) &= P(x_6 = A|z_6 = s_2)(a_{12}\alpha_5(1) + a_{22}\alpha_5(2)) = 0.2 * (0.2 * 0.000404 + 0.6 * 0.000528) = 0.000079\end{aligned} \tag{6}$$

$$P(X_{1:6} = O_{1:6}) = \sum_j \alpha_6(j) = \alpha_6(1) + \alpha_6(2) = 0.000214 + 0.000079 = 0.000293 \tag{7}$$

1.2 Most likely explanation

t=1, initial time:

$$\begin{aligned}\delta_1(1) &= P(x_1 = A|z_1 = s_1)P(z_1 = s_1) = 0.4 * 0.7 = 0.28\\ \delta_1(2) &= P(x_1 = A|z_1 = s_2)P(z_1 = s_2) = 0.2 * 0.3 = 0.06\end{aligned} \tag{8}$$

t=2

$$\begin{aligned}
\delta_2(1) &= max\{P(x_2 = G|z_2 = s_1)P(z_2 = s_1|z_1 = s_1)\delta_1(1), \\
&\quad P(x_2 = G|z_2 = s_1)P(z_2 = s_1|z_1 = s_2)\delta_1(2)\} \\
&= max\{.4 * .8 * .28, .4 * .4 * .06\} = max\{\mathbf{0.0896}, 0.0096\} = 0.0896 \\
\delta_2(2) &= max\{P(x_2 = G|z_2 = s_2)P(z_2 = s_2|z_1 = s_1)\delta_1(1), \\
&\quad P(x_2 = G|z_2 = s_2)P(z_2 = s_2|z_1 = s_2)\delta_1(2)\} \\
&= max\{.2 * .2 * .28, .2 * .6 * .06\} = max\{\mathbf{0.0112}, 0.0072\} = 0.0112
\end{aligned} \tag{9}$$

t=3

$$\begin{aligned}
\delta_3(1) &= max\{P(x_3 = C|z_3 = s_1)P(z_3 = s_1|z_2 = s_1)\delta_2(1), \\
&\quad P(x_3 = C|z_3 = s_1)P(z_3 = s_1|z_2 = s_2)\delta_2(2)\} \\
&= max\{.1 * .8 * .0896, .1 * .4 * .0112\} = max\{\mathbf{0.0072}, 0.0004\} = 0.0072 \\
\delta_3(2) &= max\{P(x_3 = C|z_3 = s_2)P(z_3 = s_2|z_2 = s_1)\delta_2(1), \\
&\quad P(x_3 = C|z_3 = s_2)P(z_3 = s_2|z_2 = s_2)\delta_2(2)\} \\
&= max\{.3 * .2 * .0896, .3 * .6 * .0112\} = max\{\mathbf{0.0054}, 0.0020\} = 0.0054
\end{aligned} \tag{10}$$

t=4

$$\begin{aligned}
\delta_4(1) &= max\{P(x_4 = G|z_4 = s_1)P(z_4 = s_1|z_3 = s_1)\delta_3(1), \\
&\quad P(x_4 = G|z_4 = s_1)P(z_4 = s_1|z_3 = s_2)\delta_3(2)\} \\
&= max\{.4 * .8 * .0072, .4 * .4 * .0054\} = max\{\mathbf{0.0023}, 0.0008\} = 0.0023 \\
\delta_4(2) &= max\{P(x_4 = G|z_4 = s_2)P(z_4 = s_2|z_3 = s_1)\delta_3(1), \\
&\quad P(x_4 = G|z_4 = s_2)P(z_4 = s_2|z_3 = s_2)\delta_3(2)\} \\
&= max\{.2 * .2 * .0072, .2 * .6 * .0054\} = max\{0.0003, \mathbf{0.00065}\} = 0.00065
\end{aligned} \tag{11}$$

t=5

$$\begin{aligned}
\delta_5(1) &= max\{P(x_5 = T|z_5 = s_1)P(z_5 = s_1|z_4 = s_1)\delta_4(1), \\
&\quad P(x_5 = T|z_5 = s_1)P(z_5 = s_1|z_4 = s_2)\delta_4(2)\} \\
&= max\{.1 * .8 * .0023, .1 * .4 * .00065\} = max\{\mathbf{0.000184}, 0.00002\} = 0.000184 \\
\delta_5(2) &= max\{P(x_5 = T|z_5 = s_2)P(z_5 = s_2|z_4 = s_1)\delta_4(1), \\
&\quad P(x_5 = T|z_5 = s_2)P(z_5 = s_2|z_4 = s_2)\delta_4(2)\} \\
&= max\{.3 * .2 * .0023, .3 * .6 * .00065\} = max\{\mathbf{0.000138}, 0.000117\} = 0.000138
\end{aligned} \tag{12}$$

t=6

$$\begin{aligned}
\delta_6(1) &= max\{P(x_6 = A|z_6 = s_1)P(z_6 = s_1|z_5 = s_1)\delta_5(1), \\
&\quad P(x_6 = A|z_6 = s_1)P(z_6 = s_1|z_5 = s_2)\delta_5(2)\} \\
&= max\{.4 * .8 * .000184, .4 * .4 * .000138\} = max\{\mathbf{0.00005888}, 0.000022\} = 0.00005888 \\
\delta_6(2) &= max\{P(x_6 = A|z_6 = s_2)P(z_6 = s_2|z_5 = s_1)\delta_5(1), \\
&\quad P(x_6 = A|z_6 = s_2)P(z_6 = s_2|z_5 = s_2)\delta_5(2)\} \\
&= max\{.2 * .2 * .000184, .2 * .6 * .000138\} = max\{0.000007, \mathbf{0.00001656}\} = 0.00001656
\end{aligned} \tag{13}$$

From Eq.(13) we have $\delta_6(1) > \delta_6(2)$, then final state would be s_1. By tracking back we will have following as the most likely path:

$$s_1, s_1, s_1, s_1, s_1, s_1 \tag{14}$$

1.3 Prediction

$$P(X_7 = x|X_{1:6} = O_{1:6}) = \frac{1}{P(X)} \sum_j P(x_7 = x|z_7 = s_j) \sum_i P(z_7 = s_j|z_6 = s_i)\alpha_6(i) \tag{15}$$

$$P(X) = \sum_{i=1}^{2} \alpha_6(i) \tag{16}$$

$$\begin{aligned} P(X_7 = x|X_{1:6} = O_{1:6}) &= \frac{1}{P(X)} \sum_j P(x_7 = x|z_7 = s_j) \sum_i P(z_7 = s_j|z_6 = s_i)\alpha_6(i) \\ &= \frac{1}{P(X)} \sum_j P(x_7 = x|z_7 = s_j)[P(z_7 = s_j|z_6 = s_1)\alpha_6(1) + P(z_7 = s_j|z_6 = s_2)\alpha_6(2)] \end{aligned} \tag{17}$$

$$\begin{aligned} P(X_7 = x|X_{1:6} = O_{1:6}) = \frac{1}{P(X)}\Big(&P(x_7 = x|z_7 = s_1)P(z_7 = s_1|z_6 = s_1)\alpha_6(1) + \\ &P(x_7 = x|z_7 = s_1)P(z_7 = s_1|z_6 = s_2)\alpha_6(2) + \\ &P(x_7 = x|z_7 = s_2)P(z_7 = s_2|z_6 = s_1)\alpha_6(1) + \\ &P(x_7 = x|z_7 = s_2)P(z_7 = s_2|z_6 = s_2)\alpha_6(2)\Big) \end{aligned} \tag{18}$$

$$\begin{aligned} P(X_7 = x|X_{1:6} = O_{1:6}) = \frac{1}{P(X)}\Big(&P(x_7 = x|z_7 = s_1) * 0.8 * 0.000214 + \\ &P(x_7 = x|z_7 = s_1) * 0.4 * 0.000079 + \\ &P(x_7 = x|z_7 = s_2) * 0.2 * 0.000214 + \\ &P(x_7 = x|z_7 = s_2) * 0.6 * 0.000079\Big) \end{aligned} \tag{19}$$

$$\begin{aligned} P(X_7 = x|X_{1:6} = O_{1:6}) = \frac{1}{P(X)}\Big(&0.0002028 * P(x_7 = x|z_7 = s_1) + \\ &0.0000902 * P(x_7 = x|z_7 = s_2)\Big) \end{aligned} \tag{20}$$

$$P(X_7 = x|X_{1:6} = O_{1:6}) = \frac{0.0001}{P(X)}\Big(2.028 * P(x_7 = x|z_7 = s_1) + 0.902 * P(x_7 = x|z_7 = s_2)\Big) \tag{21}$$

$$argmax_x\Big(2.028 * P(x_7 = x|z_7 = s_1) + 0.902 * P(x_7 = x|z_7 = s_2)\Big) \tag{22}$$

$$\begin{aligned} x = A &\Rightarrow 2.028 * P(x_7 = A|z_7 = s_1) + 0.902 * P(x_7 = A|z_7 = s_2) = 0.9916 \\ x = C &\Rightarrow 2.028 * P(x_7 = C|z_7 = s_1) + 0.902 * P(x_7 = C|z_7 = s_2) = 0.4734 \\ x = G &\Rightarrow 2.028 * P(x_7 = G|z_7 = s_1) + 0.902 * P(x_7 = G|z_7 = s_2) = 0.9916 \\ x = T &\Rightarrow 2.028 * P(x_7 = T|z_7 = s_1) + 0.902 * P(x_7 = T|z_7 = s_2) = 0.4734 \end{aligned} \tag{23}$$

So, the prediction for the $x_7 = A$ or G!