DISTRIBUTION OF MONOMIAL-PRIME NUMBERS AND MERTENS SUM EVALUATIONS

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Abstract. In this paper, we mainly study the monomial-prime numbers, which are of the form pn^k for primes p and integers $k \geq 2$. First, we show an asymptotic estimate on the number of numbers of a general form $pf(n)$ for arithmetic functions f satisfying certain growth conditions, which generalizes Bhat's recent result on the Square-Prime Numbers. Then, we prove three Mertens-type theorems related to numbers of a more general form, partially extending the recent work of Popa, Tenenbaum, and Qi-Hu on the Mertens sum evaluations. At the end, we evaluate the average and variance of the number of distinct monomial-prime factors of integers by applying our Mertens-type theorems.

1. Introduction and statement of main results

The distribution of integers with certain constraints is a fundamental topic in analytic number theory. Recently, Bhat [3] studied the numbers of the form pn^2 for integers $n > 1$ and primes p, which are named Square-Prime (SP) Numbers. He showed that the number of SP Numbers smaller than x is asymptotic to $(\zeta(2) - 1)x/\log x$, where $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s \, (s > 1)$ is the Riemann zeta function. In this paper, we study the numbers of a general form $pf(n)$ for arithmetic functions $f : \mathbb{N} \to \mathbb{R}_{\geq 1}$. If $f(n) = n^s$ is a monomial for some positive real number $s > 1$, we call pn^s a *monomial-prime number* of power s. Our first result gives an asymptotic estimate for the number of monomial-prime numbers.

Theorem 1.1. *Let S be a set of primes satisfying the following asymptotic estimate*

$$\sum_{p \leq x, p \in S} 1 = \frac{cx^{\gamma}}{\log^{\alpha}(x)} \left(1 + O\left(\frac{1}{\log^{\beta}(x)}\right)\right)$$

for some positive constants c, α, β, and $0 < \gamma \leq 1$. Let $f : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be an increasing positive sequence satisfying $\sum_{f(n) \geq x} 1/f^{\gamma}(n) \ll x^{-\delta}$ for some $\delta > 0$. Then as $x \to \infty$ we have

$$\#\{(p, n) : pf(n) \leq x, p \in S\} \sim \left(\sum_{n=1}^{\infty} \frac{1}{f^{\gamma}(n)}\right) \cdot \frac{cx^{\gamma}}{\log^{\alpha}(x)}. \tag{1}$$

In particular, for any real positive number $s > 1$ we have

$$\#\{(p, n) : pn^s \leq x, p \text{ prime}\} \sim \zeta(s) \cdot \frac{x}{\log x}. \tag{2}$$

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Remark 1.2. For $s = 1$ in (2), in [4, Proposition 5(ii)], Bănescu and Popa proved that

$$\#\{(p, n) : pn \leq x, p \text{ prime}\} \sim x \log\log x.$$

Theorem 1.1 builds a connection between the density of sets of primes and the asymptotic behavior on numbers of the form $pf(n)$. We see that the asymptotic behavior of monomial-prime numbers differs from that of prime numbers by the Riemann zeta function $\zeta(s)$ as a factor. Several common examples of S will be given in section 2.1 including the primes in arithmetic progressions, primes in the Chebatorev density theorem, Beatty sequences, Piatetski-Shapiro squences, and primes with preassigned digits. If S is the set of all primes and $f(n) = n^2$, then Bhat's asymptotic estimate on the number of SP Numbers is recovered from Theorem 1.1 by the prime number theorem. Moreover, analogous to the SP Numbers, we call pn^3 a Cube-Prime (CP) Number[1] for $n > 1$. Taking $f(n) = n^3$, we get the following asymptotic on the number of CP Numbers.

Corollary 1.3. *The number of CP Numbers smaller than x is asymptotic to $(\zeta(3)-1)x/\log x$.*

In [3, Theorem 6.1] Bhat also gave an asymptotic estimate on the number of SP Numbers ending in 1. As another application of Theorem 1.1, following the argument in [3], we get the following asymptotic estimate on the number of CP Numbers ending in 1 as well.

Corollary 1.4. *The number of CP Numbers ending in 1 is asymptotic to*

$$\frac{x}{4000 \log x}\left(\zeta\left(3, \frac{1}{10}\right) + \zeta\left(3, \frac{3}{10}\right) + \zeta\left(3, \frac{7}{10}\right) + \zeta\left(3, \frac{9}{10}\right) - 1000\right), \tag{3}$$

where $\zeta(s, t) = \sum_{n=0}^{\infty} 1/(n + t)^s$ is the Hurwitz zeta function.

Remark 1.5. The asymptotic estimates of CP numbers ending in 3, 7, 9 respectively are the same as that ending in 1. Similarly, one can count the number of CP numbers ending in 2, 4, 5, 6, and 8 in the same way as the case ending in 1, but the asymptotic estimates are slightly different.

Next, we evaluate the sums and products of Mertens type for monomial-prime numbers. Recall that Mertens' theorems are three results related to the reciprocals of primes (e.g., see [17, Chapter I .1]), which are stated as follows:

$$\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1) \qquad \text{(Mertens' first theorem)}, \tag{4}$$

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + M + O\left(\frac{1}{\log x}\right) \qquad \text{(Mertens' second theorem)}, \tag{5}$$

$$\prod_{p\leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right) \qquad \text{(Mertens' third theorem)}, \tag{6}$$

where M is a constant, called Mertens' constant, and γ is Euler's constant.

Generalizations of Mertens' theorems have been widely studied in many literatures. For example, Garcia and Lee [7] recently obtained unconditional and effective number-field analogues of the three Mertens' theorems. In [12], Lichtman proved an asymptotic formula for

[1]Here we require $n > 1$ for CP Numbers to agree with the SP Numbers defined by Bhat. But $n = 1$ is allowed for monomial-prime numbers defined in this paper.

the dissecting sum of reciprocals of numbers with exactly k prime divisors, $k \geq 1$. In this paper, we study the following generalization. Let

$$S_k(x) := \sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k},$$

where p_j denotes a prime number. Then Mertens' second theorem evaluates $S_1(x)$. In [13, 14], Popa proved asymptotic estimates for $S_2(x)$ and $S_3(x)$ respectively. In 2017, Tenenbaum [18, 19] showed the following asymptotic formula for general $S_k(x)$ using the Selberg-Delange method.

Theorem 1.6 (Tenenbaum)**.** *Let $k \geq 1$. We have*

$$S_k(x) = P_k(\log\log x) + O\left(\frac{(\log\log x)^{k-1}}{\log x}\right) \qquad (x \geq 3), \tag{7}$$

where $P_k(X) := \sum_{0 \leq j \leq k} \lambda_{j,k} X^j$, and

$$\lambda_{j,k} := \sum_{0 \leq m \leq k-j} \binom{k}{m, j, k-m-j} (M-\gamma)^{k-m-j} \left(\frac{1}{\Gamma}\right)^{(m)}(1) \qquad (0 \leq j \leq k).$$

Here M is Mertens' constant, γ is Euler's constant as in Mertens' third theorem, $\Gamma(x)$ is the Gamma function, and $(1/\Gamma)^{(m)}$ is the m-th derivative of $1/\Gamma$.

We remark that recently Qi and Hu [16] proved another formula for (7) and Bayless et al. [2] showed these two formulas ere equivalent to each other.

Next, we consider a generalization of Mertens' first theorem. In [16], Qi and Hu evaluated the following sum

$$\sum_{p_1 \cdots p_k \leq x} \frac{\log^s(p_1 \cdots p_k)}{p_1 \cdots p_k}$$

for positive integers k and s. By their result, for $s = 1$ there is a polynomial $F(X)$ of degree $k-1$ such that

$$\sum_{p_1 \cdots p_k \leq x} \frac{\log(p_1 \cdots p_k)}{p_1 \cdots p_k} = F(\log\log x)\log x + O\left((\log\log x)^k\right). \tag{8}$$

See Theorem 2.1 in Section 2.2 for the explicit description of $F(X)$. In particular, when $k = 2$, one may take $F(X) = 2X + 2M - 2$, see [5, Theorem 3.3] as well.

Our second main result is the following theorem of Mertens type for the numbers of more general form $p_1 \cdots p_k f_1(n_1) \cdots f_r(n_r)$ for some arithmetic functions f_i and integers k, $r \geq 1$, $1 \leq i \leq r$.

Theorem 1.7. *Let k and r be two positive integers. Let $P_k(X)$ and $F(X)$ be the same polynomials as in (7) and (8) respectively. For $1 \leq i \leq r$, let $f_i : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be an increasing positive function satisfying $\sum_{f_i(n) \geq x} 1/f_i(n) \ll x^{-\delta_i}$ for some $\delta_i > 0$. Then we have*

$$\text{(I)}: \qquad \sum_{p_1 \cdots p_k f_1(n_1) \cdots f_r(n_r) \leq x} \frac{\log(p_1 \cdots p_k f_1(n_1) \cdots f_r(n_r))}{p_1 \cdots p_k f_1(n_1) \cdots f_r(n_r)} \tag{9}$$

$$= \left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right) F(\log\log x)\log x + O\left((\log\log x)^k\right);$$

$$\text{(II)}: \quad \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} \frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)} \tag{10}$$

$$= \left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right) P_k(\log\log x) + O\left(\frac{(\log\log x)^k}{\log x}\right);$$

$$\text{(III)}: \quad \prod_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} \left(1 - \frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}\right) \tag{11}$$

$$= e^{-\left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right)P_k(\log\log x)+c(k,f_1,\cdots,f_r)}\left(1+O\left(\frac{(\log\log x)^{k-1}}{\log x}\right)\right).$$

Here the constant $c(k, f_1, \cdots, f_r)$ in (11) depends on k and the functions f_i, $1 \le i \le r$ only.

Taking $k = r$ and $f_i(n) = n^{s_i}$ for any positive real number $s_i > 1$, $1 \le i \le k$, we get the following theorem of Mertens type for the products of k monomial-prime numbers.

Corollary 1.8. *Let $s_1 > 1, \ldots, s_k > 1$ be k positive real numbers. Let $P_k(X)$ and $F(X)$ be as in Theorem 1.7. Then we have*

$$\text{(I)}': \quad \sum_{p_1 n_1^{s_1}\cdots p_k n_k^{s_k}\le x} \frac{\log(p_1 n_1^{s_1}\cdots p_k n_k^{s_k})}{p_1 n_1^{s_1}\cdots p_k n_k^{s_k}} \tag{12}$$

$$= \left(\prod_{i=1}^{k}\zeta(s_i)\right) F(\log\log x)\log x + O\left((\log\log x)^k\right);$$

$$\text{(II)}': \quad \sum_{p_1 n_1^{s_1}\cdots p_k n_k^{s_k}\le x} \frac{1}{p_1 n_1^{s_1}\cdots p_k n_k^{s_k}} = \left(\prod_{i=1}^{k}\zeta(s_i)\right) P_k(\log\log x) + O\left(\frac{(\log\log x)^k}{\log x}\right); \tag{13}$$

$$\text{(III)}': \quad \prod_{p_1 n_1^{s_1}\cdots p_k n_k^{s_k}\le x} \left(1 - \frac{1}{p_1 n_1^{s_1}\cdots p_k n_k^{s_k}}\right) \tag{14}$$

$$= e^{-\left(\prod_{i=1}^{r}\zeta(s_i)\right)P_k(\log\log x)+c(k,s_1,\cdots,s_k)}\left(1+O\left(\frac{(\log\log x)^{k}}{\log x}\right)\right).$$

Here the constant $c(k, s_1, \cdots, s_k)$ in (14) depends on the constants k, s_1, $\cdots$, and s_k.

On Mertens' first theorem, notice that in the case $k = 1$ the error term in (4) is better than that in (8). Due to this observation, the following theorem gives a more precise estimate than (12) for $k = 1$.

Theorem 1.9. *Let $s > 1$ be a positive number. Then*

$$\sum_{pn^s \le x} \frac{\log(pn^s)}{pn^s} = \zeta(s)\log x - s\zeta'(s)\log\log x + O(1). \tag{15}$$

Finally, we apply (13) in Corollary 1.8 to compute the average and variance of the number of distinct monomial-prime factors of integers. Let $k \geq 2$ be an integer. Let $\omega_k(n)$ be the number of distinct monomial-prime factors of n of power k. That is, $\omega_k(n) = \sum_{pm^k|n} 1$.

Theorem 1.10. *We have*

$$\sum_{n \leq x} \omega_k(n) = \zeta(k)x \log\log x + \zeta(k)Mx + O\left(\frac{x \log\log x}{\log x}\right), \tag{16}$$

where M is Mertens' constant

Theorem 1.11. *We have*

$$\sum_{n \leq x} (\omega_k(n) - \zeta(k)\log\log x)^2 = \frac{\zeta^3(k) - \zeta^2(k)\zeta(2k)}{\zeta(2k)} x(\log\log x)^2 + O(x \log\log x). \tag{17}$$

Remark 1.12. Let $\omega(n)$ be the number of distinct prime factors of n. One may view $\omega_k(n)$ as an analogue of $\omega(n)$ for monomial-prime numbers. Observe that $\omega_k(n)$ is not a multiplicative function, some properties of ω still hold with respect to $\omega_k(n)$ though. For example, let $k = 2$, if there are infinitely many natural numbers n such that $w(n) = w(n+2) = 1$ or $w_2(n) = w_2(n+2) = 1$, then both of them would imply the well-known twin prime conjecture is true. By [8, Theorem 5], we know there are infinitely many n such that $w(n) = w(n+2) = 5$ and $w_2(n) = w_2(n+2) = 9$. Also, one may think of $\omega(n)$ as the limit of $\omega_k(n)$ as $k \to \infty$. That is, $\omega(n) = \lim_{k\to\infty} \omega_k(n)$. Then letting $k \to \infty$ in Theorems 1.10 and 1.11, we recover the following estimates on $\omega(n)$ (e.g., see [6, Theorems 3.1.1 and 3.1.2]):

$$\sum_{n \leq x} \omega(n) = x \log\log x + Mx + O\left(\frac{x \log\log x}{\log x}\right), \tag{18}$$

$$\sum_{n \leq x} (\omega(n) - \log\log x)^2 = O(x \log\log x). \tag{19}$$

This paper is organized as follows. In Section 2 we introduce some examples of interesting sets of prime numbers and some technical theorems and lemmas that will be applied later. In Section 3 we first use the key technical tool, Lemma 2.2, to prove Theorem 1.1, then we apply Theorem 1.1 to prove Corollary 1.4. In Section 4 we prove Theorems 1.7 and 1.9 by Lemma 2.2. In Section 5 we compute the average and variance of $\omega_k(n)$ in Theorems 1.10 and 1.11 respectively, by applying (13) in Corollary 1.8 and Lemma 2.5.

Notation. The letters p, q, $p_1, \ldots,$ and p_k always denote primes. We write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists some constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all x. The implied constant C may depend on some parameters, say k, m, or ε. We write $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. As usual $[a, b]$ is the least common multiple of a and b, (a, b) is the greatest common divisor of a and b, and $\lfloor x \rfloor$ is the floor function.

2. Nuts and bolts

In this section, we list some examples of interesting sets of prime numbers, state a theorem on Mertens sums, and prove some lemmas that will be used in the proofs of main results. In particular, Lemma 2.2 is the key technical tool to be frequently used in the following sections.

2.1. **Examples of primes.** Let $\mathcal{P}$ be the set of all primes. Let S be a set of primes and let $\pi_S(x) = \{p \in S : p \le x\}$ be the number of primes in S up to x. The following list gives several common interesting examples of S in the literature. The asymptotic estimates on $\pi_S(x)$ satisfy the assumptions in Theorem 1.1.

(1) Arithmetic progressions. Let $q \ge 2$ and $1 \le a < q$ be two integers with $(a, q) = 1$. Let $S = \{p \in \mathcal{P} : p \equiv a \pmod q\}$, then by the prime number theorem in arithmetic progressions (e.g., see [17, Theorem II .4.1]), we have
$$\pi_S(x) = \frac{1}{\varphi(q)} \frac{x}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$
(2) Chebatorev density theorem. Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. For any conjugacy class $C \subset G$, let
$$S_C = \left\{p \in \mathcal{P} : p \text{ unramified}, \left[\frac{K/\mathbb{Q}}{p}\right] = C\right\},$$
where $\left[\frac{K/\mathbb{Q}}{p}\right]$ is the conjugacy class of Artin symbols with respect to an unramified prime p. Then by effective versions of the Chebotarev density theorem from Lagarias and Odlyzko [11, Theorems 1.3 and 1.4], we have
$$\pi_{S_C}(x) = \frac{|C|}{|G|} \frac{x}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$
(3) Beatty sequences. Let α be positive and irrational of finite type. Let
$$S_\alpha = \{p \in \mathcal{P} : p = \lfloor \alpha n \rfloor \text{ for some } n \in \mathbb{N}\}.$$
Then by the prime number theorem for Beatty sequences [1, Corollary 5.5], we have
$$\pi_{S_\alpha}(x) = \frac{x}{\alpha \log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$
(4) Piatetski-Shapiro primes. Let $c \ge 1$ be a positive and set
$$S_c := \{p \in \mathcal{P} : p = \lfloor n^c \rfloor \text{ for some } n \in \mathbb{N}\}.$$
Then by Piatetski-Shapiro's work [15], we have
$$\pi_{S_c}(x) = \frac{x^{1/c}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right)$$
for $c \in [1, 12/11)$.

(5) Primes with preassigned digits. Let $q \ge 2$ be an integer and $A_q = \{0, 1, 2, \ldots, q-1\}$. For integers $n \ge 0$ and $j \ge 0$, let $a_j(n) \in A_q$ be defined by $n = \sum_{j=0}^{\infty} a_j(n) q^j$. Let $b \ge 1$ be an integer with q-ary expansion $b = \sum_{j=0}^{r} b_j q^j$ with $b_0, b_1, \ldots, b_r \in A_q$ and $(b_0, q) = 1$. For a sequence of indexes $1 \le l_1 < \cdots < l_r$, we take
$$S_b := \left\{p \in \mathcal{P} : a_0(p) = b_0, a_{l_j}(p) = b_j, \forall\, 1 \le j \le r\right\}.$$
Then by [10, Theorem 1], for $q^N < x < q^{N+1}$, $N \ge 1$, $0 \le r < \sqrt{N}$, $1 \le l_1 < \cdots < l_r \le N$, we have
$$\pi_{S_b}(x) = \frac{1}{\varphi(q) q^r} \frac{x}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

2.2. **Multiple Mertens evaluations.** Suppose $\{a_n\}$ is a sequence related to the Riemann zeta function, that is,

$$a_2 = -\zeta(2), a_3 = 2\zeta(3), a_4 = 3\zeta(2)^2 - 6\zeta(4),$$
$$a_k = \sum_{i=1}^{k-3}(-1)^i C_{k-1}^i i!\zeta(i+1)a_{k-1-i} + (-1)^{k-1}(k-1)!\zeta(k) \quad (k > 4),$$

and $C_k^l = \binom{k}{l}$. Then we set a series of polynomials $\{Q_i(y) : i \geq 0\}$:

$$Q_0(y) = 1, Q_1(y) = y + M,$$
$$Q_k(y) = (y+M)^k + \sum_{m=2}^{k} C_k^m a_m (y+M)^{k-m} \quad (k \geq 2),$$

where M is Mertens' constant.

Now, we state a theorem about multiple Mertens evaluations by Qi and Hu.

Theorem 2.1 ([16, Theorem 1.1])**.** *For any positive integers k and s, the following evaluation holds*

$$\sum_{p_1\cdots p_k \leq x} \frac{\log^s(p_1\cdots p_k)}{p_1\cdots p_k} = \sum_{l=0}^{k-1}(-1)^l \frac{A_k^{l+1}}{s^{l+1}} Q_{k-1-l}(\log\log x)\cdot \log^s(x) + f(2)\log^{s-1}(2) \qquad (20)$$
$$+ O\left(\log^{s-1}(x)\cdot(\log\log x)^k\right),$$

where

$$f(x) = \sum_{l=0}^{k-1}(-1)^l A_k^{l+1} Q_{k-1-l}(\log\log x)\cdot \log x,$$

and the combinatorial number $A_k^l = \binom{k}{l}\cdot l!$.

In (8), one may take $F(X) = \sum_{i=0}^{k-1}(-1)^i A_k^{i+1} Q_{k-1-i}(X)$.

2.3. **Lemmas.** Now we establish some lemmas that will be used in the proofs of our main results. In particular, in Lemma 2.2, A and B are two expressions on some variables. For example, we may take $A = p_1 \cdots p_k$, then $p_1, \ldots, p_k$ are variables in this expression. If we take $B = [a^k, b^k]$ or $f_1(n_1)\cdots f_r(n_r)$, then a and b or $n_1, \ldots, n_r$ are variables in B. In our applications, the variables of A and B are easy to see from their explicit expressions related to the summations.

Lemma 2.2. *Let A and B be two expressions. Let $g(A)$ be a nonnegative function on A and $h(B)$ a nonnegative function on B. Suppose that $g(A)$ has the following asymptotic estimate:*

$$\sum_{A\leq x} g(A) = x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right) \qquad (21)$$

for some $\alpha > 0$, $\beta \in \mathbb{R}$, and $\gamma \geq 0$. Here $P(x,y) = \sum_{u,v\in\mathbb{R}} c_{u,v}x^u y^v$ is a finite sum of some monomials of two variables. If $h(B)$ satisfies $\sum_B h(B)/B^\gamma < \infty$ with the following decaying rate

$$\sum_{B\geq x} \frac{h(B)}{B^\gamma} \ll x^{-\delta} \qquad (22)$$

for some $\delta > 0$, then we have

$$\sum_{AB\le x} g(A)h(B) = \left(\sum_{B} \frac{h(B)}{B^\gamma}\right) x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right) + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right). \tag{23}$$

If $P(\log\log x, \log x)$ has no $\log x$ term, then the error term $O\left(\frac{\log\log x}{\log x}\right)$ in (23) can be replaced by $O\left(\frac{1}{\log x}\right)$.

Proof. Let $\ell \ge \max\{\delta^{-1}, \delta^{-1}\alpha\}$. We break the double summations up into two parts:

$$\begin{aligned}\sum_{AB\le x} g(A)h(B) &= \sum_{B\le \log^\ell(x)} h(B) \sum_{A\le x/B} g(A) + \sum_{B>\log^\ell(x)} h(B) \sum_{A\le x/B} g(A) \\ &:= S_1 + S_2.\end{aligned} \tag{24}$$

In S_1, we have $B \le \log^\ell(x)$, which implies $\log(x/B) = \log x\,(1 + O(\log\log x/\log x))$ and $\log\log(x/B) = \log\log x\,(1 + O(1/\log x))$. Since $\log^v(x/B) = \log^v(x)\,(1 + O_{v,\ell}(\log\log x/\log x))$ and $(\log\log(x/B))^u = (\log\log x)^u\,(1 + O_{u,\ell}(1/\log x))$ for any $u, v \in \mathbb{R}$, it follows that

$$P(\log\log(x/B), \log(x/B)) = P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right). \tag{25}$$

By (25) we obtain that

$$\begin{aligned}S_1 &= \sum_{B\le\log^\ell(x)} h(B)(x/B)^\gamma P(\log\log(x/B), \log(x/B))\left(1 + O\left(\frac{(\log\log(x/B))^\beta}{\log^\alpha(x/B)}\right)\right) \\ &= \left(\sum_{B\le\log^\ell(x)} \frac{h(B)}{B^\gamma}\right) x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)\left(1 + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right) \\ &= \left(\sum_{B} \frac{h(B)}{B^\gamma} + O\left(\frac{1}{\log^{\delta\ell}(x)}\right)\right) x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right) + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right) \\ &= \left(\sum_{B} \frac{h(B)}{B^\gamma}\right) x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right) + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right).\end{aligned} \tag{26}$$

For S_2, we have

$$S_2 \le \sum_{B>\log^\ell(x)} h(B) \sum_{A\le x} g(A) \ll \frac{x^\gamma P(\log\log x, \log x)}{\log^{\delta\ell}(x)}. \tag{27}$$

Combining the estimates above for S_1 and S_2 completes the proof. $\square$

Next we prove two lemmas on Mertens-type sums.

Lemma 2.3. *Let $r \ge 1$ be an integer. Let $f : \mathbb{N}^r \to \mathbb{R}_{>0}$ be a function satisfying $\#\{(n_1, \ldots, n_r) \in \mathbb{N}^r : f(n_1, \ldots, n_r) \le a\}$ is finite for any $a \in \mathbb{R}_{>0}$ and*

$$\sum_{f(n_1,\ldots,n_r)\ge x} \frac{1}{f(n_1, \ldots, n_r)} \ll x^{-\delta}$$

for some $\delta > 0$. Then we have

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \frac{\log^s(f(n_1, \ldots, n_r))}{f(n_1, \ldots, n_r)^{1-\eta}} < +\infty \tag{28}$$

for any $0 \le \eta < \delta$ and $s \ge 0$. Furthermore, for any $0 \le \eta < \delta$ and $s \ge 0$, we have

$$\sum_{f(n_1,\ldots,n_r) \ge x} \frac{\log^s(f(n_1, \ldots, n_r))}{f(n_1, \ldots, n_r)^{1-\eta}} \ll x^{\eta-\delta} \log^s(x). \tag{29}$$

Proof. By our assumptions on f, it suffices to prove (29). We write $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. Since $\sum_{f(\mathbf{n}) \ge x} 1/f(\mathbf{n}) = O\left(x^{-\delta}\right)$, we have

$$\sum_{x \le f(\mathbf{n}) < 2x} \frac{1}{f(\mathbf{n})} = O\left(x^{-\delta}\right).$$

This implies that $\#\{\mathbf{n} : x \le f(\mathbf{n}) < 2x\} = O(x^{1-\delta})$ and $\delta \le 1$. Then

$$\begin{aligned}
\sum_{f(n_1,\ldots,n_r) \ge x} \frac{\log^s(f(n_1, \ldots, n_r))}{f(n_1, \ldots, n_r)^{1-\eta}} &= \sum_{k=0}^{\infty} \sum_{2^k x \le f(\mathbf{n}) < 2^{k+1} x} \frac{\log^s(f(\mathbf{n}))}{f(\mathbf{n})^{1-\eta}} \\
&\le \sum_{k=0}^{\infty} \frac{\log^s(2^{k+1}x)}{(2^k x)^{1-\eta}} \sum_{2^k x \le f(\mathbf{n}) < 2^{k+1} x} 1 \\
&\ll \sum_{k=0}^{\infty} \frac{\log^s(2^{k+1}x)}{(2^k x)^{1-\eta}} \cdot (2^k x)^{1-\delta} \\
&= x^{\eta-\delta} \sum_{k=0}^{\infty} \frac{\log^s(2^{k+1}x)}{2^{k(\delta-\eta)}} \\
&\ll x^{\eta-\delta} \sum_{k=0}^{\infty} \frac{\left(\log(2^{k+1}) \log x\right)^s}{2^{k(\delta-\eta)}} \\
&\ll x^{\eta-\delta} \log^s(x),
\end{aligned}$$

which completes the proof. $\square$

Lemma 2.4. *Let $f_i : \mathbb{N} \to \mathbb{R}_{\ge 1}$ be an increasing positive sequence satisfying $\sum_{f_i(n) \ge x} 1/f_i(n) \ll x^{-\delta_i}$ for some $\delta_i > 0$, $1 \le i \le r$. Then we have*

$$\sum_{f_1(n_1) \cdots f_r(n_r) \ge x} \frac{1}{f_1(n_1) \cdots f_r(n_r)} \ll x^{-\delta} \tag{30}$$

for some $\delta > 0$.

Proof. Indeed, notice that $\sum_{n=1}^{\infty} 1/f_i(n) < \infty$ for each $1 \le i \le r$. We have

$$\sum_{f_1(n_1) \cdots f_r(n_r) \ge x^r} \frac{1}{f_1(n_1) \cdots f_r(n_r)}$$

$$
\begin{aligned}
&= \sum_{\substack{f_1(n_1)\cdots f_r(n_r)\geq x^r\\ f_1(n_1)\cdots f_{r-1}(n_{r-1})\geq x^{r-1}}} \frac{1}{f_1(n_1)\cdots f_r(n_r)} + \sum_{\substack{f_1(n_1)\cdots f_r(n_r)\geq x^r\\ f_1(n_1)\cdots f_{r-1}(n_{r-1})< x^{r-1}}} \frac{1}{f_1(n_1)\cdots f_r(n_r)}\\
&\leq \sum_{f_1(n_1)\cdots f_{r-1}(n_{r-1})\geq x^{r-1}} \frac{1}{f_1(n_1)\cdots f_r(n_r)} + \sum_{f_r(n_r)\geq x} \frac{1}{f_1(n_1)\cdots f_r(n_r)}\\
&= \left(\sum_{f_1(n_1)\cdots f_{r-1}(n_{r-1})\geq x^{r-1}} \frac{1}{f_1(n_1)\cdots f_{r-1}(n_{r-1})}\right)\left(\sum_{n_r}\frac{1}{f_r(n_r)}\right)\\
&\quad + \left(\sum_{n_1\ldots,n_{r-1}} \frac{1}{f_1(n_1)\cdots f_{r-1}(n_{r-1})}\right)\left(\sum_{f_r(n_r)\geq x}\frac{1}{f_r(n_r)}\right)\\
&\ll \sum_{f_1(n_1)\cdots f_{r-1}(n_{r-1})\geq x^{r-1}} \frac{1}{f_1(n_1)\cdots f_{r-1}(n_{r-1})} + x^{-\delta_r}\\
&\ll x^{-\delta_1}+\cdots+x^{-\delta_r},
\end{aligned}
$$

where the last line above holds by induction on r. Thus, we may take $\delta = r^{-1}\min\{\delta_1,\ldots,\delta_r\}$ for (30). This completes the proof. □

The following two lemmas will be applied in the study of the average and variance of $\omega_k(x)$.

Lemma 2.5. *Let $k \geq 2$ be an integer. Then we have*

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{[a^k,b^k]} = \frac{\zeta^3(k)}{\zeta(2k)}. \tag{31}$$

Proof. Set $d=(a,b)$, $a=da'$, $b=db'$, then we get $[a^k,b^k]=d^k a'^k b'^k$. It follows that

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{[a^k,b^k]} = \sum_{d=1}^{\infty}\sum_{\substack{a'=1\\(a',b')=1}}^{\infty}\sum_{b'=1}^{\infty}\frac{1}{d^k a'^k b'^k} = \zeta(k)\sum_{\substack{a'=1\\(a',b')=1}}^{\infty}\sum_{b'=1}^{\infty}\frac{1}{a'^k b'^k}. \tag{32}$$

Now, we compute the square of $\zeta(k)$ as follows:

$$\zeta^2(k) = \sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{a^k b^k} = \sum_{d=1}^{\infty}\sum_{\substack{a'=1\\(a',b')=1}}^{\infty}\sum_{b'=1}^{\infty}\frac{1}{d^{2k} a'^k b'^k} = \zeta(2k)\sum_{\substack{a'=1\\(a',b')=1}}^{\infty}\sum_{b'=1}^{\infty}\frac{1}{a'^k b'^k}.$$

This implies that

$$\sum_{\substack{a'=1\\(a',b')=1}}^{\infty}\sum_{b'=1}^{\infty}\frac{1}{a'^k b'^k} = \frac{\zeta^2(k)}{\zeta(2k)}. \tag{33}$$

Then (31) following by plugging (33) into (32). □

For the tail of the double series in (31) we have the following bound.

Lemma 2.6. *For any $\varepsilon > 0$, we have*

$$\sum_{[a^k,b^k]\geq x} \frac{1}{[a^k,b^k]} = O\left(x^{-1+\frac{1}{k}+\varepsilon}\right). \tag{34}$$

Proof. By [9, (1.81)], we have $\tau_3(n) \ll n^{\varepsilon}$. Then we have

$$\sum_{[a^k,b^k]\geq x} \frac{1}{[a^k,b^k]} = \sum_{l^k\geq x, [a^k,b^k]=l^k} \frac{1}{l^k} \ll \sum_{l^k\geq x} \frac{\tau_3(l)}{l^k} \ll \sum_{l\geq x^{1/k}} \frac{l^{\varepsilon}}{l^k} \ll x^{-1+\frac{1}{k}+\varepsilon}.$$

□

3. Proofs of Theorem 1.1 and Corollary 1.4

In this section we prove Theorem 1.1 and Corollary 1.4. Indeed, if we take $A = p$, $B = f(n)$, $g(A) = 1_S(p)$, and $h(B) = 1$ in Lemma 2.2, where 1_S is the indicator function on S, then Theorem 1.1 follows immediately. Therefore, it suffices to prove Corollary 1.4, in which we apply Theorem 1.1.

Proof of Corollary 1.4. Using the fact that primes are asymptotically equi-distributed in the reduced residues mod 10, we calculate the asymptotic estimates for CP numbers ending in 1 as follows. We have the following four cases: when $p \equiv 1 \pmod{10}$, we require a^3 to end in 1, thus a is congruent to 1 modulo 10; when $p \equiv 3 \pmod{10}$, we require a^3 to end in 7, thus a is congruent to 3 modulo 10; when $p \equiv 7 \pmod{10}$, we require a^3 to end in 3, thus a is congruent to 7 modulo 10; and finally when $p \equiv 9 \pmod{10}$, we require a^3 to end in 9, thus a is congruent to 9 modulo 10. From the definition of CP numbers, we know that $a > 1$. Therefore, the number of pairs (p, a) such that $pa^3 \equiv 1 \pmod{10}$ with $pa^3 \leq x$ is

$$\begin{aligned}
&\#\left\{(p,k) : p \equiv 1 \pmod{10},\ p(10k+1)^3 \leq x,\ k \geq 1\right\}\\
&+ \#\left\{(p,k) : p \equiv 3 \pmod{10},\ p(10k+3)^3 \leq x,\ k \geq 0\right\}\\
&+ \#\left\{(p,k) : p \equiv 7 \pmod{10},\ p(10k+7)^3 \leq x,\ k \geq 0\right\}\\
&+ \#\left\{(p,k) : p \equiv 9 \pmod{10},\ p(10k+9)^3 \leq x,\ k \geq 0\right\}.
\end{aligned}$$

Then from Theorem 1.1, we have

$$\begin{aligned}
&\#\left\{(p,k) : p \equiv 1 \pmod{10},\ p(10k+1)^3 \leq x,\ k \geq 1\right\}\\
&\sim \frac{x}{4\log x} \sum_{k=1}^{\infty} \frac{1}{(10k+1)^3} = \frac{x}{4000 \log x}\left(\zeta\left(3, \frac{1}{10}\right) - 1000\right).
\end{aligned}$$

Similarly, we obtain

$$\#\left\{(p,k) : p \equiv 3 \pmod{10},\ p(10k+3)^3 \leq x,\ k \geq 0\right\} \sim \frac{x}{4000\log x}\zeta\left(3, \frac{3}{10}\right),$$

$$\#\left\{(p,k) : p \equiv 7 \pmod{10},\ p(10k+7)^3 \leq x,\ k \geq 0\right\} \sim \frac{x}{4000\log x}\zeta\left(3, \frac{7}{10}\right),$$

and

$$\#\left\{(p,k) : p \equiv 9 \pmod{10},\ p(10k+9)^3 \leq x,\ k \geq 0\right\} \sim \frac{x}{4000\log x}\zeta\left(3, \frac{9}{10}\right).$$

Hence the asymptotic formula (3) holds by adding these four asymptotic estimates up. □

4. Proofs of Theorems 1.7 and 1.9

In this section we prove Theorems 1.7 and 1.9, in which we frequently apply Lemma 2.2.

4.1. **Proof of Theorem 1.7.** First, we prove (9). We write the partial sum in (9) as two parts:

$$\begin{aligned}
&\sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} \frac{\log(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)} \\
&= \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} \frac{\log(p_1\cdots p_k)}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)} \\
&\quad + \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} \frac{\log(f_1(n_1)\cdots f_r(n_r))}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)} \\
&:= S_3 + S_4.
\end{aligned}$$

For S_3, we take $A = p_1p_2\cdots p_k$, $B = f_1(n_1)f_2(n_2)\cdots f_r(n_r)$, $g(A) = (\log A)/A$, and $h(B) = 1/B$ in Lemma 2.2. Combining with (8) and Lemma 2.4, we have

$$S_3 = \left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right)\cdot F(\log\log x)\cdot\log x + O\left((\log\log x)^k\right). \tag{35}$$

For S_4, we take $A = p_1p_2\cdots p_k$, $B = f_1(n_1)f_2(n_2)\cdots f_r(n_r)$, $g(A) = 1/A$, and $h(B) = (\log B)/B$ in Lemma 2.2. Combining with Theorem 1.6, Lemma 2.3, and Lemma 2.4, we get

$$\begin{aligned}
S_4 &= \sum_{i=1}^{r}\sum_{n_i=1}^{\infty}\frac{\log(f_1(n_1)\cdots f_r(n_r))}{f_1(n_1)f_2(n_2)\cdots f_r(n_r)}\cdot P_k(\log\log x)\left(1+O\left(\frac{1}{\log\log x\cdot\log x}\right)\right) \\
&= O\left((\log\log x)^k\right).
\end{aligned}\tag{36}$$

Combining (35) and (36) together gives the desired formula (9).

Similarly, if we take $A = p_1\cdots p_k$, $B = f_1(n_1)\cdots f_r(n_r)$, $g(A) = 1/A$, and $h(B) = 1/B$ in Lemma 2.2, then by Lemma 2.4 we immediately obtain (10).

Finally, we use (10) to show (11). Let

$$V_{k,r}(x) := \prod_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\left(1-\frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}\right).$$

After taking logarithm, we have

$$\begin{aligned}
-\log V_{k,r}(x) &= \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} -\log\left(1-\frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\right) \\
&= \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\sum_{t\ge1}\frac{1}{t(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))^t} \\
&= \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}
\end{aligned}$$

$$+\sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\sum_{t\ge 2}\frac{1}{t(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))^t}.$$

By our assumptions on the functions f_i, $1\le i\le r$, the tail in the second sum satisfies

$$\begin{aligned}
&\sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)> x}\sum_{t\ge 2}\frac{1}{t(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))^t}\\
&\ll \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)> x}\frac{1}{(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))^2}\\
&\ll \sum_{p_1\ge x^{\frac12}}\frac{1}{p_1^2}+\sum_{p_2\cdots p_k f_1(n_1)\cdots f_r(n_r)^2\ge x^{\frac12}}\frac{1}{(p_1\cdots p_k f_1(n)\cdots f_r(n_r))^2}\\
&\ll \cdots\cdots\\
&\ll \sum_{i=1}^{k}\sum_{p_i\ge x^{\frac{1}{2^i}}}\frac{1}{p_i^2}+\sum_{j=1}^{r}\sum_{f_j(n_j)\ge x^{\frac{1}{2^{k+j}}}}\frac{1}{(f_j(n_j))^2}\\
&\ll x^{-\varepsilon}
\end{aligned}$$

for some constant $\varepsilon>0$. Therefore, by (10) we obtain

$$-\log V_{k,r}(x)=\left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right)P_k(\log\log x)-c(k,f_1,\cdots,f_r)+O\left(\frac{(\log\log x)^k}{\log x}\right)$$

for some constant $c(k,f_1,\cdots,f_r)$ that depends on k and the functions f_i, $1\le i\le r$. Therefore, we know

$$V_{k,r}(x)=e^{-\left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right)P_k(\log\log x)+c(k,f_1,\cdots,f_r)}\left(1+O\left(\frac{(\log\log x)^k}{\log x}\right)\right).$$

We have finished the proof of Theorem 1.7.

4.2. **Proof of Theorem 1.9.** The proof of Theorem 1.9 is similar to that of (9). We write

$$\sum_{pn^s\le x}\frac{\log(pn^s)}{pn^s}=\sum_{pn^s\le x}\frac{\log p}{pn^s}+\sum_{pn^s\le x}\frac{\log(n^s)}{pn^s}:=S_5+S_6.$$

For S_5, we take $A=p$, $B=n^s$, $g(A)=(\log A)/A$, and $h(B)=1/B$ in Lemma 2.2. Combining with Mertens' first theorem, we obtain

$$S_5=\zeta(s)\log x+O(1). \tag{37}$$

For S_6, we take $A=p$, $B=n^s$, $g(A)=1/A$, and $h(B)=(\log B)/B$. Notice that

$$\sum_{n=1}^{\infty}\frac{\log(n^s)}{n^s}=-s\zeta'(s).$$

From Mertens' second theorem, similar to the argument of Lemma 2.2, we obtain

$$S_6=-s\zeta'(s)\log\log x+O(1). \tag{38}$$

Combining (37) and (38), we obtain

$$\sum_{pn^s \le x} \frac{\log (pn^s)}{pn^s} = \zeta(s) \log x - s\zeta'(s) \log\log x + O(1),$$

which completes the proof of Theorem 1.9.

5. Proofs of Theorems 1.10 and 1.11

In this section we prove Theorems 1.10 and 1.11 by applying (13) in Corollary 1.8.

5.1. Proof of Theorem 1.10. By definition, we have $\omega_k(n) = \sum_{pm^k|n} 1$. Then

$$\begin{aligned}\sum_{n\le x} \omega_k(n) &= \sum_{n\le x} \sum_{pm^k|n} 1 \\ &= \sum_{pm^k\le x} \sum_{n\le x/pm^k} 1 \\ &= \sum_{pm^k \le x} \left\lfloor \frac{x}{pm^k} \right\rfloor \\ &= x \sum_{pm^k\le x} \frac{1}{pm^k} + O\left(\sum_{pm^k \le x} 1 \right) \end{aligned} \tag{39}$$

On the one hand, from (13) we know

$$\sum_{pm^k\le x} \frac{1}{pm^k} = \zeta(k) \left(\log\log x + M\right) + O\left(\frac{\log\log x}{\log x}\right). \tag{40}$$

On the other hand, by Theorem 1.1 we have

$$\sum_{pm^k\le x} 1 \sim \zeta(k) \frac{x}{\log x}. \tag{41}$$

Combining (39), (40), and (41), we get

$$\sum_{n\le x} \omega_k(n) = \zeta(k) \log\log x + \zeta(k) M x + O\left(\frac{x \log\log x}{\log x}\right),$$

which completes the proof of Theorem 1.10.

5.2. Proof of Theorem 1.11. We first compute the second moment of $\omega_k(n)$. It is easy to see

$$\begin{aligned}\sum_{n\le x} \omega_k^2(n) &= \sum_{n\le x} \sum_{pa^k|n} \sum_{qb^k|n} 1 \\ &= \sum_{pa^k\le x} \sum_{qb^k\le x} \sum_{\substack{n\le x \\ pa^k|n, qb^k|n}} 1 \\ &= \sum_{pa^k\le x} \sum_{qb^k \le x} \left\lfloor \frac{x}{[pa^k, qb^k]} \right\rfloor \end{aligned}$$

$$
\begin{aligned}
&= x \sum_{[pa^k, qb^k] \le x} \frac{1}{[pa^k, qb^k]} + O\left(\sum_{[pa^k, qb^k] \le x} 1 \right) \\
&:= x \cdot S_7 + S_8. \qquad (42)
\end{aligned}
$$

For the sum S_7, we will prove

$$
S_7 = \frac{\zeta^3(k)}{\zeta(2k)} \left(\log \log x\right)^2 + O\left(\log \log x\right). \qquad (43)
$$

Lemma 2.5 and Mertens' second theorem imply

$$
\begin{aligned}
\sum_{p=q, [pa^k, qb^k] \le x} \frac{1}{[pa^k, qb^k]} &= \sum_{p[a^k, b^k] \le x} \frac{1}{p[a^k, b^k]} \\
&\le \sum_{[a^k, b^k] \le x} \frac{1}{[a^k, b^k]} \sum_{p \le x} \frac{1}{p} \\
&\ll \sum_{p \le x} \frac{1}{p} \\
&\ll \log \log x.
\end{aligned}
$$

So we only need to consider the terms with $p \neq q$. In the sum

$$
S_7' = \sum_{p \neq q, [pa^k, qb^k] \le x} \frac{1}{[pa^k, qb^k]},
$$

we let $a = p^{e_1} q^{e_2} a_1$ and $b = p^{f_1} q^{f_2} b_1$, where $(a_1, pq) = (b_1, pq) = 1$. Then $[pa^k, qb^k] = p^g q^h [a_1^k, b_1^k]$, where $g = \max\{ke_1 + 1, kf_1\} \ge 1$ and $h = \max\{ke_2, kf_2 + 1\} \ge 1$. Next we discuss the contributions of the terms according to their e_1, e_2, f_1, and f_2 values. If $e_1 \ge 1$, then $[pa^k, qb^k] \ge p^3 q [a_1^k, b_1^k]$, and thus

$$
\sum_{p \neq q, [pa^k, qb^k] \le x, e_1 \ge 1} \frac{1}{[pa^k, qb^k]} \ll \sum_{q \le x} \frac{1}{q} \sum_{p=1}^{\infty} \frac{1}{p^3} \sum_{a_1=1}^{\infty} \sum_{b_1=1}^{\infty} \frac{1}{[a_1^k, b_1^k]} \ll \log \log x. \qquad (44)
$$

If $f_1 \ge 1$, then $[pa^k, qb^k] \ge p^2 q [a_1^k, b_1^k]$, which implies

$$
\sum_{p \neq q, [pa^k, qb^k] \le x, f_1 \ge 1} \frac{1}{[pa^k, qb^k]} \ll \sum_{q \le x} \frac{1}{q} \sum_{p=1}^{\infty} \frac{1}{p^2} \sum_{a_1=1}^{\infty} \sum_{b_1=1}^{\infty} \frac{1}{[a_1^k, b_1^k]} \ll \log \log x. \qquad (45)
$$

Similarly, for the terms with $e_2 \ge 1$ or $f_2 \ge 1$, we also have

$$
\sum_{\substack{[pa^k, qb^k] \le x \\ p \neq q, e_2 \ge 1 \text{ or } f_2 \ge 1}} \frac{1}{[pa^k, qb^k]} = O\left(\log \log x\right) \qquad (46)
$$

Noting that $e_1 = e_2 = f_1 = f_2 = 0$ implies $[pa^k, qb^k] = pq[a_1^k, b_1^k]$, we can combine the estimates (44), (45), and (46) to obtain

$$
S_7' = \sum_{\substack{pq[a^k, b^k] \le x \\ p \neq q, (a, pq) = (b, pq) = 1}} \frac{1}{pq[a^k, b^k]} + O\left(\log \log x\right).
$$

In fact, we can remove the constrains $p \neq q$, $(a, pq) = (b, pq) = 1$ in S_7' since

$$\begin{aligned}\sum_{pq[a^k,b^k]\leq x, p=q} \frac{1}{pq[a^k, b^k]} &= \sum_{p^2[a^k,b^k]\leq x} \frac{1}{p^2[a^k, b^k]} \\ &\leq \sum_{p\leq x} \frac{1}{p^2} \sum_{[a^k,b^k]\leq x} \frac{1}{[a^k, b^k]} \\ &< +\infty,\end{aligned}$$

$$\begin{aligned}\sum_{\substack{pq[a^k,b^k]\leq x, p\neq q \\ p|a, p|b}} \frac{1}{pq[a^k, b^k]} &= \sum_{p^{k+1}q[a^k/p^k,b^k/p^k]\leq x} \frac{1}{p^{k+1}q[a^k/p^k, b^k/p^k]} \\ &\leq \sum_{p\leq x} \frac{1}{p^{k+1}} \sum_{[a^k,b^k]\leq x} \frac{1}{[a^k, b^k]} \sum_{q\leq x} \frac{1}{q} \\ &\ll \log\log x,\end{aligned}$$

and

$$\sum_{\substack{pq[a^k,b^k]\leq x, p\neq q \\ p|a, p\nmid b}} \frac{1}{pq[a^k, b^k]} = \sum_{\substack{p^{k+1}q[a^k/p^k,b^k]\leq x, p\neq q \\ p|a, p\nmid b}} \frac{1}{p^{k+1}q[a^k/p^k, b^k]} \ll \log\log x.$$

Therefore, we have proved

$$S_7 = \sum_{pq[a^k,b^k]\leq x} \frac{1}{pq[a^k, b^k]} + O\left(\log\log x\right).$$

Now equation (43) holds when we apply Lemma 2.2 with $A = pq$, $B = [a^k, b^k]$, $g(A) = 1/A$, and $h(B) = 1/B$, Lemma 2.5, and Lemma 2.6.

By similar arguments we can prove

$$S_8 = O\left(x\log\log x\right). \tag{47}$$

From (43) and (47) we know

$$\sum_{n\leq x} \omega_k^2(n) = \frac{\zeta^3(k)}{\zeta(2k)} x\left(\log\log x\right)^2 + O\left(x\log\log x\right). \tag{48}$$

Combining (48) and Theorem 1.10, we obtain

$$\begin{aligned}&\sum_{n\leq x} \left(\omega_k(n) - \zeta(k)\log\log x\right)^2 \\ &= \sum_{n\leq x} \omega_k^2(n) - 2\zeta(k)\log\log x \cdot \sum_{n\leq x} \omega_k(n) + \zeta^2(k)x\left(\log\log x\right)^2 \\ &= \frac{\zeta^3(k)}{\zeta(2k)} x\left(\log\log x\right)^2 - 2\zeta^2(k)x\left(\log\log x\right)^2 + \zeta^2(k)x\left(\log\log x\right)^2 + O\left(x\log\log x\right) \\ &= \frac{\zeta^3(k) - \zeta^2(k)\zeta(2k)}{\zeta(2k)} x\left(\log\log x\right)^2 + O\left(x\log\log x\right),\end{aligned}$$

which completes the proof of Theorem 1.11.

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References

[1] William D. Banks and Igor E. Shparlinski. Prime numbers with Beatty sequences. *Colloq. Math.*, 115(2):147–157, 2009.

[2] Jonathan Bayless, Paul Kinlaw, and Jared Duker Lichtman. Higher Mertens constants for almost primes. *J. Number Theory*, 234:448–475, 2022.

[3] Raghavendra N. Bhat. Distribution of square-prime numbers. *Missouri J. Math. Sci.*, 34(1):121–126, 2022.

[4] Magdalena Bănescu and Dumitru Popa. Asymptotic evaluations for some double sums in number theory. *Int. J. Number Theory*, 11(7):2073–2085, 2015.

[5] Magdalena Bănescu and Dumitru Popa. A multiple Abel summation formula and asymptotic evaluations for multiple sums. *Int. J. Number Theory*, 14(4):1197–1210, 2018.

[6] Alina Carmen Cojocaru and M. Ram Murty. *An introduction to sieve methods and their applications*, volume 66 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006.

[7] Stephan Ramon Garcia and Ethan Simpson Lee. Unconditional explicit Mertens' theorems for number fields and Dedekind zeta residue bounds. *Ramanujan J.*, 57(3):1169–1191, 2022.

[8] Daniel A. Goldston, Sidney W. Graham, Apoorva Panidapu, Janos Pintz, Jordan Schettler, and Cem Y. Yıldı rım. Small gaps between almost primes, the parity problem, and some conjectures of Erdős on consecutive integers II. *J. Number Theory*, 221:222–231, 2021.

[9] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[10] I. Kátai. Distribution of digits of primes in q-ary canonical form. *Acta Math. Hungar.*, 47(3-4):341–359, 1986.

[11] J. C. Lagarias and A. M. Odlyzko. Effective versions of the Chebotarev density theorem. In *Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975)*, pages 409–464. Academic Press, London, 1977.

[12] Jared Duker Lichtman. Mertens' prime product formula, dissected. *Integers*, 21A(Ron Graham Memorial Volume):Paper No. A17, 15, 2021.

[13] Dumitru Popa. A double Mertens type evaluation. *J. Math. Anal. Appl.*, 409(2):1159–1163, 2014.

[14] Dumitru Popa. A triple Mertens evaluation. *J. Math. Anal. Appl.*, 444(1):464–474, 2016.

[15] I. I. Pyateckiĭ-Šapiro. On the distribution of prime numbers in sequences of the form $[f(n)]$. *Mat. Sbornik N.S.*, 33(75):559–566, 1953.

[16] Tianfang Qi and Su Hu. Multiple Mertens evaluations. *arXiv e-prints*, page arXiv:1909.10930, September 2019.

[17] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015.

[18] Gérald Tenenbaum. Generalized Mertens sums. In *Analytic number theory, modular forms and q-hypergeometric series*, volume 221 of *Springer Proc. Math. Stat.*, pages 733–736. Springer, Cham, 2017.

[19] Gérald Tenenbaum. Generalized Mertens sums. *arXiv e-prints*, page arXiv:1910.02781, September 2019.

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