U.E : Scientific computing.

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Sujet :

Spectral method for 1d Burgers equation.

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1 Introduction

The Burgers' equation is a partial differential equation arising from fluid mechanics. It appears in various fields of applied mathematics, such as modeling gas dynamics, acoustics, or traffic flow. It is named after Johannes Martinus Burgers, who discussed it in 1948. It appears in earlier works by mathematician Andrew Forsyth and Harry Bateman. It is a nonlinear partial differential equation that finds its use in many areas of physics, engineering, and applied mathematics. It was initially formulated to describe turbulent motions of viscous fluids, but it has proven to be a versatile model for various other phenomena.

The one-dimensional Burgers' equation is defined as follows :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

where :
- $u(x,t)$ is the unknown function representing the fluid velocity field as a function of spatial position x and time t.
- $\frac{\partial u}{\partial t}$ is the partial derivative of u with respect to time, representing the rate of change of fluid velocity over time.
- $u\frac{\partial u}{\partial x}$ is the convective term, which describes the transport of quantity u along the spatial direction x.
- $\nu\frac{\partial^2 u}{\partial x^2}$ is the diffusive term, where ν is the kinematic viscosity of the fluid ; this term represents the diffusion of quantity u in space.

Despite its apparent simplicity, the Burgers' equation captures complex physical phenomena such as shock wave formation, turbulence, energy dissipation, and coherent structure formation. Due to this versatility, it is widely used in various research and application domains.

2 Classical Solution Methods

The numerical solution of the Burgers' equation has been approached by various classical methods over the decades. Among these are explicit and implicit approaches, finite difference methods, and others.

Here is a review of the main classical numerical methods used to solve the Burgers' equation :

- Explicit Schemes : These methods, such as "upwind" or "Lax-Wendroff" schemes, are easy to implement and are often used for their simplicity. However, they can be prone to numerical instabilities due to the nonlinear nature of the Burgers' equation.

- Implicit Schemes : These methods, such as the Crank-Nicolson scheme, are more stable than explicit schemes but may require costly iterative calculations to solve the resulting systems of equations.

- Finite Difference Methods discretize the Burgers' equation by replacing derivatives with discrete approximations on a spatial grid. They are widely used due to their simplicity and efficiency, although they may be limited by stability and numerical accuracy issues in some cases.

- Finite Volume Methods discretize the equation by integrating over control volumes around each grid point. They are particularly suitable for mass and momentum conservation and are often used to solve convection-diffusion problems like the Burgers' equation.

- Galerkin Methods, such as the finite element method, discretize the equation by approximating the solution over finite elements in space. They are particularly suited for handling complex geometries and variable boundary conditions but may require intensive computations in terms of solving linear systems.

These classical methods all have their advantages and disadvantages and can be selected based on the specific requirements of the problem to be solved and computational constraints. In general, the selection of an appropriate method will depend on considerations such as required accuracy, geometric complexity of the domain, nature of the desired solution, and available computational resources.

3 Preliminaries

Spectral methods are a powerful approach for solving differential equations by decomposing solutions into series of spectral basis functions, such as Fourier series or orthogonal polynomials. This decomposition allows exploiting the fast convergence of these series to obtain accurate and efficient numerical solutions. Spectral methods play a crucial role in solving differential equations due to their ability to provide precise and efficient solutions for a wide range of problems. Here are some key points to illustrate the importance of these methods :

- High Accuracy and Fast Convergence : Spectral methods exploit the properties of spectral bases, such as trigonometric functions or orthogonal polynomials, which converge rapidly to the exact solutions of differential equations. This enables obtaining numerical solutions with high accuracy, even with a relatively low number of degrees of freedom.

- Adaptability to Various Types of Equations : Spectral methods are versatile and can be applied to different types of differential equations, including nonlinear partial differential equations (PDEs) like the Burgers' equation, the Navier-Stokes equations, the Schrödinger equation, etc. They can also be used to solve ordinary differential equations (ODEs).

- Efficient Treatment of Boundary Conditions : Spectral bases often allow a natural representation of boundary conditions, simplifying their numerical treatment. Additionally, spectral methods can be adapted to handle a variety of boundary conditions, including periodic conditions, Dirichlet conditions, and Neumann conditions.

- Good Stability Properties : In many cases, spectral methods exhibit good stability properties, meaning they can effectively handle numerical convergence issues and error propagation.

- Applicability to Diverse Domains : Spectral methods are widely used in various scientific and engineering domains, including fluid dynamics, solid mechanics, plasma physics, meteorology, material modeling, quantum mechanics, and many others. Their versatility makes them a valuable tool for researchers and engineers in many fields.

In summary, spectral methods offer a set of powerful tools for the numerical solution of differential equations, providing accurate and efficient solutions for a wide range of problems. Their importance in solving differential equations is undeniable, and their continued use continues to drive research and applications in many scientific and engineering domains.

3.1 Foundations of Spectral Analysis

Spectral methods rely on the fundamental concepts of spectral analysis, which study the properties of functions in the frequency or wavenumber domain. Spectral functions form a complete basis in a Hilbert space, allowing for the representation of any function in this space using a linear combination of these basis functions. Mathematically, a function $f(x)$ can be expressed as a series of spectral functions $\phi_n(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

where c_n are the series coefficients.

Spectral functions $\phi_n(x)$ are functions used to describe the characteristics of a system in a frequency domain. They are often chosen to form an orthogonal basis in an associated Hilbert space. This means the inner product :

$$\langle \phi_m, \phi_n \rangle = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

3.2 Fourier Transform and Fourier Series

- The Fourier transform is a fundamental tool in spectral analysis, allowing for the decomposition of a function into an infinite sum of sine and cosine functions, thus representing the function in the frequency domain. Mathematically, the Fourier transform of a function $f(x)$ is defined as :

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

where $F(k)$ is the Fourier transform of $f(x)$, and k is the spatial frequency. Fourier series are a specific application of the Fourier transform for periodic functions.

- Fourier series are used to decompose a periodic function into an infinite sum of sine and cosine functions, usually in the form :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

Where a_0, a_n, and b_n are the Fourier coefficients that can be calculated from the original periodic function. If we assume $f(x)$ has a period T, we have :

$$a_0 = \frac{1}{T} \int_0^T f(x)\, dx$$

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n x}{T}\right) dx$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n x}{T}\right) dx$$

3.3 Spectral Approximation and Convergence

Spectral methods exploit the rapid convergence of spectral series to obtain accurate approximations of solutions to differential equations. By increasing the number of terms in the spectral series, the approximation becomes increasingly accurate. Mathematically, the approximation of a solution $f(x)$ can be expressed as :

$$f(x) \approx \sum_{n=1}^{N} c_n \phi_n(x)$$

where c_n are then the unknowns to be determined and ϕ_n are the basis functions.

4 Spectral Methods for the 1D Burgers' Equation

The Burgers' equation is a nonlinear partial differential equation that appears in many fields of physics and engineering. It models various phenomena such as wave propagation in fluids, shock formation, and turbulence.

Spectral methods are a family of numerical methods for solving partial differential equations. They are based on approximating the solution by a series of basis functions, typically polynomials or exponentials.

Spectral Method for the 1D Burgers' Equation

The 1D Burgers' equation is given by :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

where
$u(x,t)$ is the velocity field
ν is the viscosity

Step 1 : Define the Basis Functions
The first step is to choose a basis of functions to approximate the solution. Common choices include Legendre polynomials, Chebyshev polynomials, and exponentials. Select basis functions to approximate the solution $u(x,t)$. Chebyshev polynomials, Legendre polynomials, or trigonometric functions (such as Fourier series) are often used depending on the geometry of the problem and boundary conditions.

Step 2 : Solution Approximation
The solution $u(x,t)$ is approximated by a series of basis functions, which is a linear combination of the chosen basis functions with unknown coefficients. :

$$u(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x)$$

where :
N is the number of modes
$a_n(t)$ are the series coefficients

$\phi_n(x)$ are the basis functions

Step 3 : Determine the Coefficients
The coefficients $a_n(t)$ are determined by substituting the approximation of the solution into the Burgers' equation and employing the appropriate method.

- Initial and Boundary Conditions : Apply the initial conditions and boundary conditions to the series of basis functions. This yields a system of ordinary differential equations (ODEs) for the unknown coefficients $a_n(t)$.

- Solving the ODEs : Use appropriate numerical methods to solve the resulting ODE system. Methods like Runge-Kutta or explicit Euler method can be used to evolve the coefficients $a_n(t)$ in time.

- Solution Reconstruction : Once the coefficients $a_n(t)$ are calculated, use them to reconstruct the solution $u(x,t)$ using the expression of spectral decomposition.

- Analysis and Interpretation of Results : Analyze the obtained results and ensure they are consistent with physical expectations. You can also perform post-processing analyses to extract relevant information about the system's behavior.

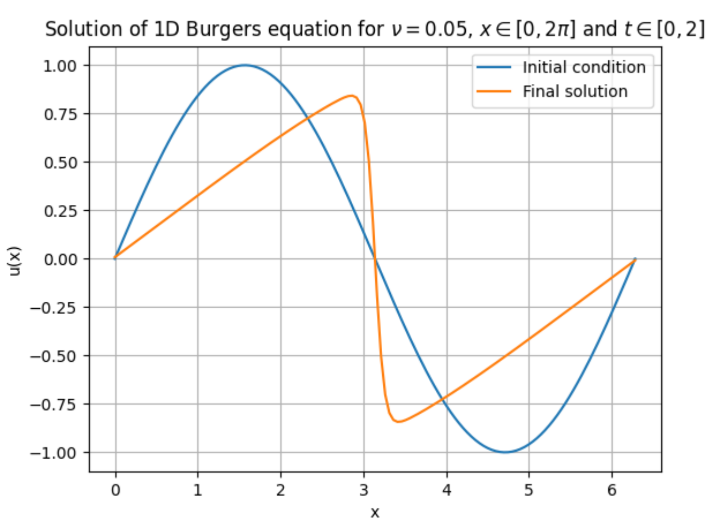
It is worth noting that the spectral method can be highly effective in solving the 1D Burgers' equation, but it may require some expertise to choose appropriate basis functions and to solve the resulting ODE systems stably and efficiently.

Solution of the Burgers' Equation

```

# -*- coding: utf-8 -*-
"""
Created on Thu Mar 14 20:09:16 2024

@author: FALL

"""

import numpy as np
import matplotlib.pyplot as plt
from scipy.fftpack import fft, ifft, fftfreq

# Problem parameters
nx = 128  # Number of grid points in x
nt = 500  # Number of time steps
L = 2 * np.pi  # Length of spatial region
T = 2   # Final time
nu = 0.05  # Viscosity coefficient

dt = T / nt # Time step

# Discretization of the space
x = np.linspace(0, L, nx)
dx = L / nx

# Initial conditions
u_initial = np.sin(2 * np.pi * x / L)

```

```
# Initial conditions
u = np.copy(u_initial)

# Time loop - Temporal integration with explicit Euler method
for n in range(nt):
    # Fourier transform
    u_hat = fft(u)

    # Differential operators in spectral domain
    k = 2 * np.pi * fftfreq(nx, d=dx)
    u_hat = u_hat - dt * 0.5j * k * fft(u * u)
    u_hat = u_hat * np.exp(-nu * dt * k**2)

    # Back to physical domain
    u = np.real(ifft(u_hat))

# Plotting the solution
plt.figure()
plt.plot(x, u_initial, label='Initial condition')
plt.plot(x, u, label='Final solution')
plt.xlabel('x')
plt.ylabel('u(x)')
plt.title('Solution of 1D Burgers equation for $ \\nu = {} $, $ x \\in [0, 2\\pi] $ and $ t \\in [0,{}] $'.format(nu, T))
plt.grid(True)
plt.legend()
plt.show()
```

This code solves the 1D Burgers equation using a fast Fourier transform (FFT)-based approach and temporal integration with the explicit Euler method, then it visualizes the initial condition and the final solution on a graph.



The Burgers equation is a nonlinear partial differential equation that combines terms of nonlinear convection and diffusion. It can generate complex phenomena such as shock formation and dissipative structures.

In this code, we used a numerical method to solve the one-dimensional Burgers equation, specifically with the explicit Euler method in the spectral domain.

Initial condition : The initial condition is defined as a sine wave. This means that the profile of the solution at the beginning is a sinusoidal wave with a certain amplitude and frequency.

Temporal integration : Using the explicit Euler method, we iterate through time to update the solution at each time step. The Fourier transform is used to transform the solution into the spectral domain, where differential operations are simpler, and then we return to the physical space to obtain the updated solution.

Results obtained : The final result is the solution of the Burgers equation at a given final time. The solution can exhibit various behaviors depending on the chosen parameters, such as shock formation, dissipative structures, or solitary waves, due to the combination of nonlinear and diffusive terms in the equation.

In general, the evolution of the solution can be influenced by parameters such as viscosity ν, spatial step dx, and time step dt. By observing the solution, we can study how these parameters affect the behavior of the solution evolution, especially regarding the formation and propagation of shocks, as well as the dissipation of energy.

Stability :

To analyze the stability of the spectral method for solving the 1D Burgers equation, we will use the Courant-Friedrichs-Lewy (CFL) condition. This condition ensures that the numerical solutions remain stable based on the time and grid steps used.

The CFL condition for the Burgers equation is given by :

$$\text{CFL} = \frac{\nu \Delta t}{\Delta x^2}$$

where : - ν is the viscosity,
- Δt is the time step,
- Δx is the grid step.

For the Burgers equation, a common stability criterion is given by the CFL condition, which is similar to that for advection equations :

$$\text{CFL} \leq \frac{1}{2}$$

If this condition is not satisfied, numerical instabilities may appear in the solution. To satisfy this condition, we must choose Δt small enough compared to Δx and ν.

Convergence :
To assess the convergence of the numerical method used to solve the 1D Burgers equation, we can perform a convergence analysis using reference solutions or numerical convergence studies.
A common approach to study convergence is to fix a grid resolution level and gradually decrease the time step (Δt). Then, we observe how the numerical solutions converge towards a reference solution as the time step size decreases.

Conservation of Physical Properties :
When numerically solving differential equations such as the Burgers equation, it is important to ensure the conservation of physical properties, such as mass or energy conservation. This ensures that the obtained numerical solution faithfully reflects the physical behavior of the system.
In the case of the 1D Burgers equation, there are no globally conserved quantities such as mass or energy. However, it is still important to check if certain local quantities are properly preserved. For example, one can verify if the sum of the absolute values of the solution remains constant over time.

- - Advantages of Spectral Methods

Spectral methods are mathematical and numerical approaches used to solve a variety of problems in various scientific fields, such as physics, engineering, and applied mathematics. Here are some of the advantages of spectral methods :

High Precision : Spectral methods are known for their ability to provide highly accurate solutions for many types of problems, especially for problems involving eigenvalues and initial values.

Fast Convergence : They can often converge to a precise solution with a relatively small number of degrees of freedom, making them very efficient for solving complex problems.

Treatment of Nonlinearities : Spectral methods are also effective for solving nonlinear problems, as they can often be formulated in a way that simplifies the treatment of nonlinearities.

Adaptability to Boundary Conditions : They can effectively handle a wide variety of boundary conditions, including periodic, Dirichlet, and Neumann conditions.

Effective Handling of High Dimensions : Although they can be memory and computationally expensive for very large problems, spectral methods can often be adapted to effectively handle high-dimensional problems.

Applicability to Complex Equations : They can be applied to a wide range of differential equations, including partial differential equations, integral equations, ordinary differential equations, etc.

Ease of Implementation : Spectral methods can be relatively straightforwardly implemented using standard techniques such as Fourier series expansions, Chebyshev series expansions, Legendre series expansions, etc.

Numerical Stability : They can offer high numerical stability in many cases, although this may depend on the specific formulation of the problem and the method used.

In summary, spectral methods are often preferred for their combination of precision, fast convergence, and ability to handle a wide variety of problems, especially in the context of differential equations and boundary value problems.

- - Disadvantages of spectral methods

Despite their numerous advantages, spectral methods also have some disadvantages. :

High computational cost : While spectral methods may converge rapidly, the computations involved can be costly in terms of computational time, especially for high-dimensional problems or problems involving significant nonlinearities.

Implementation difficulties : Spectral methods can be challenging to implement.

Implementation complexity : Although the mathematical foundations of spectral methods are often straightforward, their practical implementation can sometimes be complex, especially for three-dimensional problems or non-standard formulations.

Sensitivity to boundary conditions : Spectral methods can be sensitive to boundary conditions, especially if these conditions are poorly defined or difficult to implement.

Dependency on geometry : Spectral methods may be less flexible when it comes to handling domains with complex or varying geometries.

In summary, while spectral methods offer many advantages in terms of accuracy and fast convergence, they may not always be the best option for all types of problems, and their implementation can pose challenges in certain situations.

Applications

Spectral methods have been employed to solve a wide range of fluid flow, wave propagation, and heat transfer problems.

- Fluid dynamics : The Burgers' equation is often used as a simplified model to study more complex phenomena in fluids, due to its nonlinearity and its ability to capture the formation of shock waves and coherent structures.

- Meteorology and oceanography : In the field of meteorological and oceanographic modeling, the Burgers' equation can be used to study phenomena such as the propagation of weather fronts and the formation of vortices.

- Fluid engineering : In engineering, the Burgers' equation is used to model transport phenomena in viscous fluids, such as heat convection, fluid mixing, and the propagation of pressure waves.

- Mathematical physics : The Burgers' equation is also studied for its interesting mathematical properties, such as the formation of singularities and the stability of solutions.

Conclusion

The spectral method for the 1D Burgers' equation provides a powerful and accurate approach to solve this nonlinear problem. By employing series expansions in terms of spectral basis functions, such as Chebyshev polynomials or Fourier functions, this method enables an efficient representation of Burgers' equation solutions in terms of modal coefficients.

By harnessing the properties of fast convergence and intrinsic accuracy of spectral methods, it's possible to attain precise solutions for a broad range of initial and boundary conditions. Moreover, this approach can also be extended to handle more complex cases, such as multidimensional systems or non-homogeneous conditions.

However, despite its many advantages, the spectral method for the 1D Burgers equation has some limitations. It may encounter difficulties with solutions containing shocks or discontinuities, and its implementation can be complex for problems with varying geometry or complex boundary conditions.

In conclusion, the spectral method for the 1D Burgers equation represents a robust and accurate approach for solving this nonlinear problem, providing a balance between precision, efficiency, and complexity. Its judicious use can yield reliable and accurate solutions for a variety of scenarios, but it is important to consider its limitations and choose appropriate strategies to overcome them.

References

- http ://lyoncalcul.univ-lyon1.fr/ed/DOCS_2019/methodes-spectrales.pdf
- J. P. Boyd, Chebyshev and Fourier Spectral Methods.
- Numerical Analysis of Spectral Methods : Theory and Applications
- C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, Spectral Methods : Fundamentals in Single Domains
- https ://fr.wikipedia.org/wiki/Analyse_spectrale
- http ://dspace.univ-tlemcen.dz/bitstream/112/9290/1/Methodes-spectrales-et-application-a-des-Problemes-hyperboliques-et-paraboliques.PDF
- https ://ipag.osug.fr/l̃esurg/theselesurPart2.pdf