Séminaire Lotharingien de Combinatoire XX (2021)
Article #YY, 13 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

A Bijection Between Weighted Dyck Paths and 1234-Avoiding Alternating Permutations

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Abstract. Three-dimensional Catalan numbers are a variant of the classical (bidimensional) Catalan numbers that count, among other interesting objects, the standard Young tableaux of shape (n, n, n). In this paper, we present a structural, statistics-preserving bijection between two three-dimensional Catalan objects: 1234-avoiding alternating permutations, and a class of weighted Dyck paths.

Keywords: Bijective combinatorics, three-dimensional Catalan numbers, alternating permutations, pattern avoidance, weighted Dyck paths, Young tableaux, prographs

1 Introduction

Among a vast number of combinatorial classes of objects, the famous Catalan numbers enumerate the standard Young tableaux of shape (n, n). Counting the standard tableaux of shape (n, n, n) is a sequence known as the three-dimensional Catalan numbers A005789, whose first entries are $1, 1, 5, 42, 462, 6006, 87516, 1385670, \ldots$. Many other combinatorial objects are enumerated by this sequence, from certain walks in the quarter plane [2], to product-coproduct prographs [1], to 1234-avoiding alternating permutations.

This last case, which this paper dwells on, was proven by Lewis, who provides in [6, 5] two bijections between this class and standard Young tableaux of shape (n, n, n). As observed by Borie in [1] however, these bijections do not highlight any obvious similarities of combinatorial nature.

In this same article, Borie proves that product-coproduct prographs are three-dimensional Catalan objects, by giving a bijection with standard Young tableaux of shape (n, n, n); on the other hand, and with another bijection involving his prographs, he highlights a certain class of weighted Dyck paths as a new three-dimensional Catalan family. About this new family, he makes the (freely rephrased) following conjecture, which is the starting point of this paper.

Conjecture 1.1 (Borie 2017 [1])**.** *There exists a combinatorial bijection between alternating permutations of size* $2n$ *avoiding* 1234 *and a class* WD_{2n} *of weighted Dyck paths.*

Borie additionally presumes that the positions of steps $(1, 1)$ (*i.e.* up-steps) in the paths should correspond to the bottom elements in the permutation, and came up with a partial bijection, in the particular case where these are exactly the elements $1, 2, \dots, n$. He relied on the observation that the product-coproduct prographs, in this case, were essentially pairs of binary trees, and he used a bijection to 123-avoiding permutations on each one in such a way that the two permutations could respectively become the bottom elements and top elements of a 1234-avoiding alternating permutation.

In this extended abstract, we present a general bijection from these weighted Dyck paths to 1234-avoiding alternating permutations. This bijection extends Borie's partial bijection to the whole combinatorial classes and preserves some structural properties and statistics.

2 The combinatorial objects

This section presents, in two separate subsections, both classes of combinatorial objects dealt with in this paper. In each case, we recall the definition, provide examples, and then describe the Schützenberger involution and a natural product on the objects.

2.1 Alternating permutations of $2n$ avoiding 1234

An alternating permutation (or up-down permutation) of $2n$ avoiding 1234 is a permutation of size $2n$ whose descents set is $\{2, 4, 6, \dots\}$, with no increasing subsequence of length 4. We denote by $A_{2n}(1234)$ the set of all these permutations.

For instance, here are the 42 alternating permutations of size 6 avoiding 1234:

$$A_6(1234) = \left\{ \begin{array}{c} 143625, 153624, 154623, 163524, 164523, 241635, 243615, \\ 251436, 251634, 253614, 254613, 261435, 261534, 263514, \\ 264513, 341625, 342615, 351426, 351624, 352416, 352614, \\ 354612, 361425, 361524, 362415, 362514, 364512, 451326, \\ 451623, 452316, 452613, 453612, 461325, 461523, 462315, \\ 462513, 463512, 561324, 561423, 562314, 562413, 563412 \end{array} \right\}. \tag{2.1}$$

We shall use the following convenient notations: for any permutation σ of $A_{2n}(1234)$ seen as a word, we denote by $Bot(\sigma) = \sigma_2\sigma_4 \cdots \sigma_{2n}$ the subword that consists of the letters in even positions — that we may call bottom elements rather than valleys, in order to avoid confusion with the valleys of Dyck paths; and by $Top(\sigma) = \sigma_1\sigma_3 \cdots \sigma_{2n-1}$ the subword of the odd-position letters — that we may call top elements. Of course, σ is determined by $Bot(\sigma)$ and $Top(\sigma)$, for instance:

$$\sigma = 364512 \qquad \begin{array}{cccccc} & 6 & & 5 & & 2 \\ 3 & & 4 & & 1 & \end{array} \qquad \begin{array}{cc} \longleftarrow & Top(\sigma) \\ \longleftarrow & Bot(\sigma)\,. \end{array}$$

The set of 1234-avoiding alternating permutations is endowed with a product:

Lemma 2.1 ([1])**.** *The set $\bigcup_{n\in\mathbb{N}} A_{2n}(1234)$ is closed under the shifted concatenation product $\bullet$ on permutations defined by $\sigma \bullet \tau = (\mathrm{shift}_{length(\tau)}(\sigma)) \cdot \tau$.*

For instance, we have $12 \bullet 1423 = 561423$.

Another structure indicator, the classical Schützenberger involution on permutations, consists in reversing the alphabet, then reversing the reading direction.

For example, we have $S(48271635) = 46382715$. As it preserves 1234 patterns, and thus their appearance or avoidance, S stabilizes the set of alternating permutations of $2n$ avoiding 1234.

We shall use a variant on words: for any word ω on the alphabet $\{1, 2, \ldots, 2n\}$, $S_{2n}(\omega)$ is obtained by the same process of reversing the alphabet and the reading direction. The main difference is that not all letters need to appear. For instance, we have $S_8(164) = 538$.

2.2 Weighted Dyck paths

Definition 2.2. We denote by WD_{2n} the set of weighted Dyck paths of length $2n$ whose weights satisfy the following assertions:

1. all weights are non-negative integers smaller than or equal to the lower height;
2. weights are non-decreasing on successive rises;
3. weights are non-increasing on successive descents;
4. on a peak of height h, with d and e the weights of its steps, we have: $e + d \leqslant h$;
5. on a valley of height h, with d and e the weights of its steps, we have: $d + e \geqslant h$.

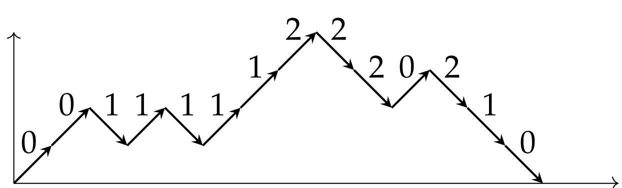


Figure 1: Example of a weighted Dyck path in DW_{14}.

There is a natural concatenation product on these objects, as well as a natural notion of "Schützenberger involution": the reflection according to a vertical axis (see Figure 2).

3 A statistics-preserving bijection

This section presents the results of our work, specifically a solution to the following conjecture.

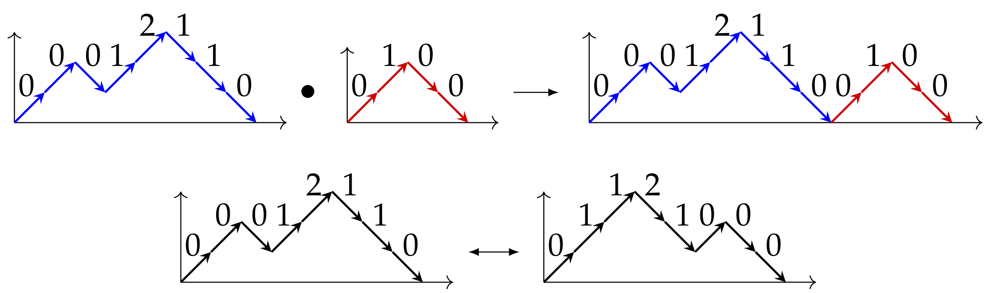


Figure 2: Concatenation product and Schützenberger involution in WD_{2n}.

Conjecture 3.1 (Borie [1])**.** *There exists a bijection between alternating permutations of size* $2n$ *avoiding* 1234 *and weighted Dyck paths of* WD_{2n} *that has the following properties: compatibility with the concatenation product and the Schützenberger involution; correspondence between the positions of steps* $(1,1)$ *in the Dyck path and the bottom elements of the permutation.*

Let wd be an irreducible weighted Dyck path (that is, a weighted Dyck path with no intermediate return to 0) of length $2n$ and let $wd(u)$ denote the weight associated with the step in position $u \in \{1,\ldots,2n\}$. Let us define a map β' from irreducible weighted Dyck paths into the set of permutations. The image permutation is computed by an algorithm that inserts the *positions* of the steps one at a time; moreover, the elements inserted in the bottom word are the positions of steps $(1,1)$ in the Dyck path (and the top word's elements are the positions of steps $(1,-1)$).

Definition 3.2. Let $\beta'(wd)$ be the permutation σ whose bottom word is obtained by the insertion algorithm *ins* defined below; and the top word is obtained by applying the Schüztenberger involution to wd, then the algorithm *ins*, and then the (shifted) Schützenberger involution again :

$$Bot(\sigma) = ins(wd)$$
$$Top(\sigma) = S_{2n}(ins(S(wd)))\,.$$

The function *ins* is computed using the following algorithm.

1. Split wd into the first, left-hand half of slopes L and the second, right-hand half of slopes R (see Figures 4 & 5 for examples).

2. Start with $\tau = \varepsilon$ the empty permutation. The weighted Dyck path wd is scanned from the left, and elements are inserted in τ as we go.

3. For every upward slope U in wd, starting from the left
   (a) Let $shift$ be the total number of steps $(1,-1)$ that are to the left of the slope in wd.

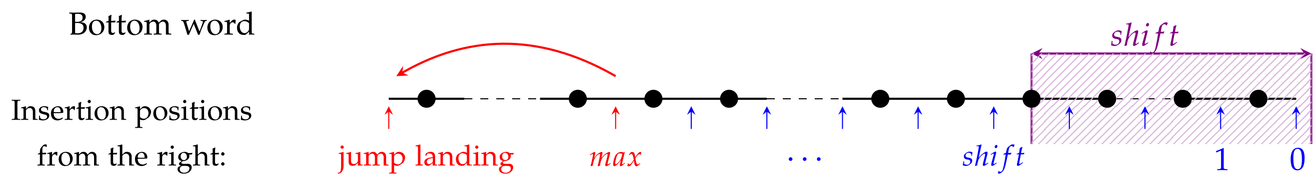


Figure 3: Insertion positions in the building bottom word (dots are word's elements).

(b) For every step $(1,1)$ of the current upward slope, starting from the left, denote by u its position and proceed to insert u in τ as follows:

- In case the slope U is in L: if $wd(u)$ is *minimal*, insert u to the left of τ (we say that u *jumps*, or that $wd(u)$ is a jumping weight); otherwise, insert u in τ at distance $wd(u) + shift - 1$ from the right-hand end of τ.
- In case the slope U is in R: consider instead the *maximality* of $wd(u)$ to decide if u jumps, and insert at distance $wd(u) + shift$ from the right-hand end otherwise.

The minimality (resp. maximality) of the value is decided by considering either the increment condition on the slope or the valley (resp. peak) condition of Definition 2.2: if the corresponding bound is achieved, then the tested value is indeed extremal.

Remark 3.3. Since a bottom (resp. top) element in an alternating permutation is automatically followed (resp. preceded) by a larger top (resp. smaller bottom) element, a top element and a bottom element, in that order, can never form an increasing subsequence (a 12 pattern) in $A_{2n}(1234)$. This is the reason for the $shift$ value: as $shift$ elements, among the top elements of the permutation, need to be to the right of the element u that is being inserted as a bottom element, the insertion position of u from the right needs to be at least $shift$ if we aim at obtaining (as we really do) a 1234-avoiding permutation.

Remark 3.4. Keeping all the way through the path the comparison to the minimum (resp. maximum) as the unique criterion for jumps may seem tempting, but it would make the map β non-injective. Indeed, a single valley (resp. peak) condition would decide if both of its adjacent steps wield jumping weights — this being generally the case with several different pairs of weights. On the other hand, a choice like comparing to the minimum all weights of steps $(1,1)$ and to the maximum all those of steps $(1,-1)$ would make this map incompatible with the Schützenberger involution (refer to Figure 4 for visual support).

Remark 3.5. Why jump at all? Without diving into the detail, it allows to avoid completing a 1234 pattern (recall that it is an increasing subsequence of size 4). The *ins*

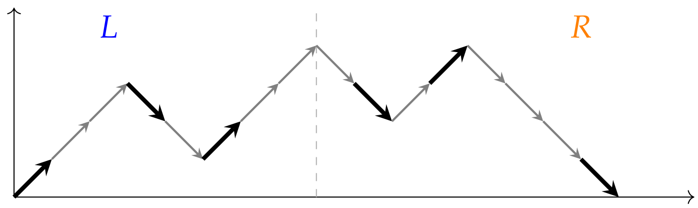


Figure 4: Locations where the jump is decided using the peak or valley condition.

construction has two assets. First, as we explain later, elements that do not jump are those forming a 12 pattern with their fellow bottom elements; actually, at the time of insertion, all elements to the left are the 1 (the smallest element in the increasing subsequence of length 2) of a 12 pattern, the element not jumping being the 2. Second (as a consequence), the insertion position of a no-jump element is the number of elements strictly following the (current) first ascent in the word; transposed to $Top(\sigma)$, this means the weights of steps $(1, -1)$ enable to locate the rightmost ascent. For bottom elements, jumping allows to escape the threat of this ascent as a potential 34 in the 1234 pattern. Since the leftmost insertion position that could be computed, if it was not for jumps, is the problematic one, one could have thought of making only the weights achieving this bound jumping weights... but then, not all elements would be able to jump — which brings back to Remark 3.4.

Remark 3.6. The minus 1 in the computation of the insertion position as of the left-hand half L compensates the fact that elements jump if their weight is minimal, so the non-jumping weights have values between 1 and ℓ (the current lower height), whereas the valid insertion positions are $0 + shift$ to $(\ell - 1) + shift$.

In the sequel, whilst we will be applying the algorithm to weighted Dyck paths wd in order to realize $\sigma = \beta'(wd)$, we will be refering to the transitional word that will become $Bot(\sigma)$ as the *bottom word*, and to the word that will become $Top(\sigma)$ as the *top word*.

Example 3.7. Take again the example in Figure 1, of an element of WD_{14}. Here, the set L consists of the slopes to the left of the valley in position 5 (the steps whose positions are to be inserted are in dotted blue on Figure 5), and R consists of the slopes to the right (steps to be inserted in dashed orange). The big dots on the figure mark the places where one must use the peak/valley condition in order to decide if the element jumps.

1. First to be handled is the leftmost upward slope. Both elements jump to the left: 2 1 .

2. We overrun the first slope down, which is one step long, so we have $shift = 1$ as we handle the 4. Since this upward slope still belongs to the first half L, we compare the weight 1 to its minimum value. In the absence of a previous weight on the slope to compare it to, it is determined by the valley condition: here, $1 + 1$

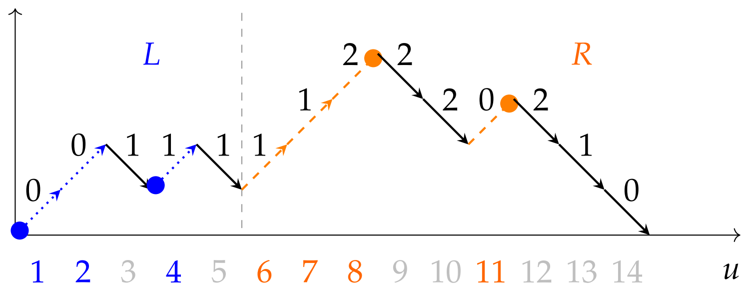


Figure 5: Left-hand and right-hand halves of the upward slopes (L and R, respectively).

is strictly greater than the valley height 1, so the weight is not minimal and 4 does not jump. Its insertion position from the right is $1 + shift - 1 = 1$, so we obtain 2 4 1 .

3. We overrun the second slope down and add its length to $shift$, which is now 2. We leave the left-hand half L and enter the right-hand half R, so we need to check if the next weight is maximal to decide if the elements jump. The value 1 is indeed maximal, both because of its height and because the next element is 1 too, so 6 jumps. However, 7 does not, since 1 is less than both its lower height and the following weight 3. In R, we do not subtract 1 to the insertion position anymore; we get $1 + shift = 3$ for 7, and: 6 7 2 4 1 . Finally, 8 does jump because 3 is maximal with respect to the peak condition: $2 + 2 = 4 = h_{peak}$. We have now: 8 6 7 2 4 1 .

4. We add the length of the next slope down to $shift$, which brings it to 4, and test if 0 is maximal. It is obviously smaller than its height, and since it is the last element of the slope we need to test the peak condition: $0 + 2$ is less than 3, so 11 does not jump and is inserted at position $0 + shift = 4$ from the right, which hands:
$$Bot(\sigma) = 8\ 6\ 11\ 7\ 2\ 4\ 1\,.$$

5. The subsequence $Top(\sigma)$ is obtained by applying the Schützenberger involution to the path, basically reversing left and right, then executing the algorithm, and finally applying the (shifted) Schützenberger involution so as to be back with the right element values. Note that it boils down to using the same algorithm on the slopes down as on the upward slopes, except they are scanned from right to left (with the rules related to L and R swapped), the insertion position computed is from the left, and elements that jump go all the way to the right. In the end we get $Top(\sigma) = 13\ 12\ 14\ 10\ 9\ 5\ 3$ and $\sigma = 8\ 13\ 6\ 12\ 11\ 14\ 7\ 10\ 2\ 9\ 4\ 5\ 1\ 3$.

Definition 3.8. Let wd be a weighted Dyck path in WD_{2n}. We define $\beta(wd)$ as the map obtained by applying β' to the irreducible factors of wd and then taking the shifted concatenation product of the image permutations.

Remark 3.9. By construction, the map β is compatible with the concatenation products as well as the Schützenberger involutions of both WD_{2n} and $A_{2n}(1234)$.

Theorem 3.10. *The map β is a bijection between WD_{2n} and $A_{2n}(1234)$.*

4 Proof of the bijection

4.1 Proof of injectivity

We have the following lemma.

Lemma 4.1. *Let wd be an irreducible element of WD_{2n}. Suppose we are applying β' to wd using the algorithm of Definition 3.2, and call u the next element to be inserted in the bottom word. Assume further that u does not jump. Then the possible values of the weight $wd(u)$ produce distinct, valid insertion positions.*

Proof. We prove it for the bottom word, and under the hypothesis that u belongs to an upward slope that is in R, as the proof is essentially the same for the other cases. The current length of the word is the number ℓ of steps $(1,1)$ already handled, that is, the number of elements already inserted. As u does not jump, it will be inserted at position $wd(u) + shift$, where $shift$ is the number of down-steps that have been overrun at this stage of the algorithm. The range that should be granted to $wd(u)$ is thus between 0 and the length of the allowed insertion zone, that is $\ell - shift$ (or actually $\ell - shift - 1$ since we exclude jump cases); this is exactly the lower height of the step corresponding to u, so this condition is ensured by the definition of WD_{2n} (and the fact that u does not jump). □

Proposition 4.2. *The map β is injective.*

Sketch of proof. By definition, two weighted Dyck paths with different underlying Dyck paths have different images, since the steps $(1,1)$ correspond exactly to the elements in even positions in the image permutation.

Let us consider wd and wd' two weighted Dyck paths from WD_{2n} with the same underlying Dyck path and the same image, and assume that they are different. Consider the position of the first difference a in the weight values, starting from the left: we have $wd(a) \neq wd'(a)$ and $wd(e) = wd'(e)$ for all $e < a$.

Using Lemma 4.1, for the element a to be inserted in the same position in the image permutations, regardless of the difference of values, it needs to jump in both cases.

Since $wd(a-1)$ is equal to $wd'(a-1)$, the slope a needs to be one whose jumps are decided by considering the next value to the right (so we are in R, and we will stay so all the way until the right-hand end).

By checking every configuration, one can show that every weight from there on to the right has only one possible value, determined by the weight of a and that differs in both paths. This is necessary all the way to the last element, which can only assume the weight 0 by definition; so this is actually impossible. □

4.2 The image is a 1234-avoiding alternating permutation

In this subsection, we rely on the following criteria.

Proposition 4.3 ([1]). *For n a non-negative integer and σ an alternating permutation of size $2n$, σ avoids 1234 if and only if the following four conditions are satisfied:*

1. *the sequence $Top(\sigma)$ avoids 123;*
2. *the sequence $Bot(\sigma)$ avoids 123;*
3. *each value of $Top(\sigma)$ smaller than a bottom element k appears to the right of k in σ;*
4. *if a bottom element k has a smaller bottom element to its left, all peak values greater than k to its right must be ordered in σ decreasingly.*

Proposition 4.4. *Let wd be a weighted Dyck path and σ be its image under the map β. Then, σ is an alternating permutation avoiding 1234.*

Sketch of proof. We shall use the above criteria to prove the proposition.

First and foremost, let us prove that $Bot(\sigma)$ avoids 123 (the first item is proven essentially the same way).

Note that the insertion process on the first upward slope (which hands a permutation of size the length of the slope) is actually a bijection between non-decreasing parking functions (here starting from 0) and 123-avoiding permutations which is described in [1].

To obtain the subsequence $Bot(\sigma)$, one just needs to consider the upward slopes, as well as whether their valley/peak steps need to jump (which is usually determined by looking at the value of the adjacent step $(1,-1)$). Following this viewpoint, we define a transformation of the upward slopes, whose image is a single upward slope; this slope will be such that applying the function *ins* to it hands a permutation that is the standardized version of $Bot(\sigma)$. To obtain the weights of the new upward slope from the initial (upward slopes of the) weighted Dyck path, consider each step $(1,1)$, from left to right and determine if it jumps in the initial Dyck path. If so, give it the same image weight as the previously obtained weight; if not, the image weight will be the pre-image weight to which one adds a *correction*, which is the number $shift$ of steps $(1,-1)$ that are on its left, plus 1 if the pre-image weight belongs to the right-hand half of the upward slopes (this in order to compensate the minus one that is applied to left-hand slopes when determining the position of insertion according to *ins*).

The new weights are weakly increasing. Indeed, it is obvious when the weight corresponds to a jump (it is equal to the previous one) or for weights formerly belonging to

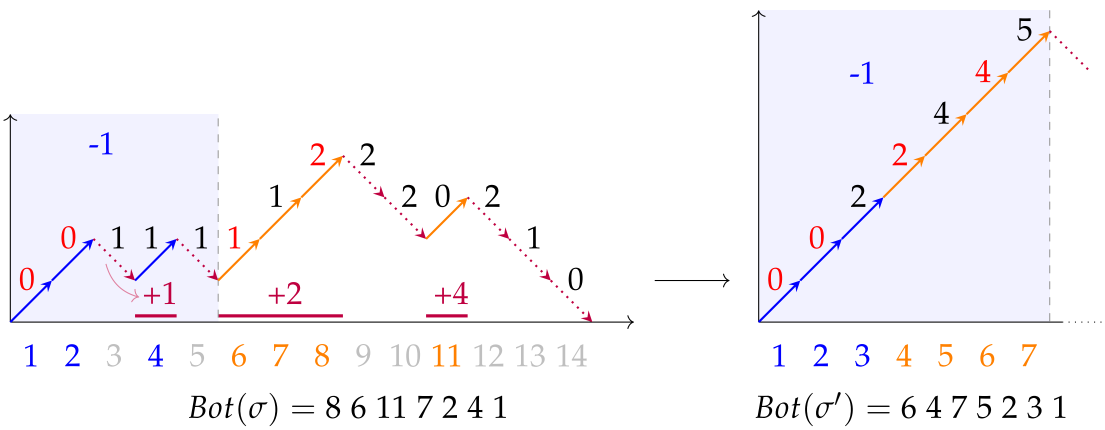
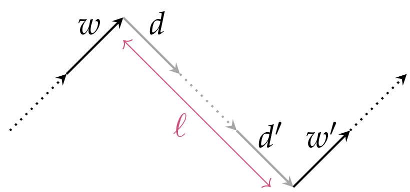


Figure 6: Transformation to a single slope. Weights are red when there is a jump.

the same slope (since one adds a constant value to a slope); consider now a formerly last element of a slope w and the formerly first element of the following slope w'. The peak condition demands that u be less than or equal to $h_{peak} - d$, where h_{peak} is the height of the peak and d the adjacent weight. On the other hand, we have $w' \geqslant h_{val} - d'$, where h_{val} is the height of the valley of w' and d' the adjacent weight on this valley. But $h_{val} - d'$ is greater than $h_{val} - d = h_{peak} - \ell - d$, which is in turn greater than $w - \ell$, where ℓ is the length of the slope down between the two considered upward slopes, and also exactly the difference between the *corrections* added by the transformation to w and w' (except for maybe an additional 1 that works in our favour).



$$w' \geqslant h_{val} - d' \geqslant h_{val} - d = h_{peak} - \ell - d \geqslant w - \ell$$

In addition, the new weights are still bounded by the height. It is straightforward for the left-hand half of upward slopes (in blue on the figures), since the weight and height of the step are increased by the same value. As for the right-hand half (in orange), to which an additionnal 1 is added, the only values that could bring trouble are maximal values with respect to the height and, as part of the right-hand half, that means they jump: they will thus take the value of the previous weight in the image, so they necessarily stay below the bound.

The new weights are therefore a non-decreasing parking function (starting from 0), on which the function *ins* hands a 123-avoiding permutation, which ends the proof.

We now move on to the third criterion. Refering to the notations of Figure 7, we need to show that the distance between d and u in the image permutation (in grey) is non-negative. We compute it as the difference between the distances from the right of u and d (in green and violet, respectively). We assume that neither d nor u jumps, which is the worst-case scenario for the distance. More specifically, we may assume without loss of generality that at least one element from each one of the two slopes does not jump; the lowest height element of the slope that does not jump is then the one we should examine since it is the closest, in the permutation, to its counterparts from the other slope: replace d (resp. u) by this element.

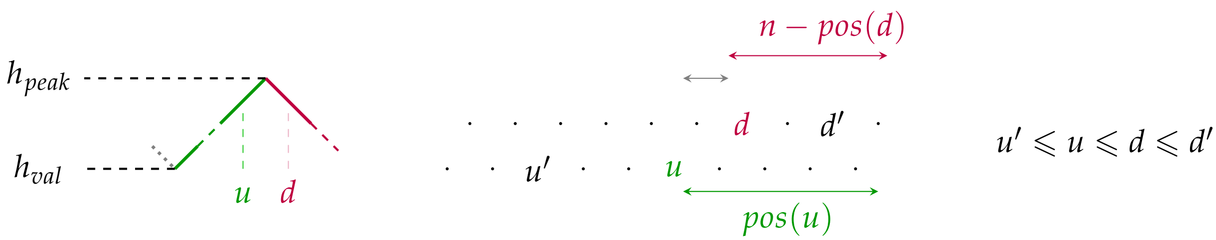
In the algorithm of insertion defined by β, u is inserted at a distance from the right equal to $pos(u) = wd(u) + shift(u)\ (-1)$, where $wd(u)$ is the weight of u in the Dyck path, $shift(u)$ is defined in the algorithm (see also the legend of Figure 7), and 1 may or may not be subtracted depending on the position of the slope in the path. What the transformation described in the previous item of the proof shows is that the steps $(1,1)$ that are to the right of u correspond to elements that will be inserted to its left in the permutation (this is only true if u does not jump). Therefore, this position of insertion is also the final distance from the right of u. For the same reason (except that the insertion position is computed as a distance from the left), the final distance from the right of d is $n - pos(d) = n - wd(d) + shift(d)\ (+1)$. Note also, by exhaustion of cases, that at most one of the two insertion positions requires a minus 1.

Now the difference of positions between d and the element following u in the permutation is:

$$\begin{aligned} dist(u,d) &= pos(u) + pos(d) - n \ \ (-1) \\ &= wd(u) + wd(d) + shift(u) + shift(d) - n \ \ (-1) \\ &= wd(u) + wd(d) + shift(u) - lenBot(u) \ \ (-1) \\ &= wd(u) + wd(d) - h_{val} \ \ (-1) \end{aligned}$$

where $lenBot(u)$ is the length of the bottom word right before the insertion of the slope of u. The condition on the valley is $wd(u) + wd(d) \geqslant h_{val}$, but since neither u nor d jumps, this inequality is strict, so that the distance is non-negative, and d is before u in the permutation, as needed.

Finally, let us prove the last criterion is satisfied, that is to say that the configuration of 1234 pattern on the following figure cannot occur.



Observe first, by considering once again the transformation at the beginning of this proof, that the elements in $Bot(\sigma)$ (resp. $Top(\sigma)$) that have a smaller bottom (resp. larger top) element to their left (resp. right) in the permutation are exactly those who do not jump. Consider thus a bottom element u and a larger top element d, both of

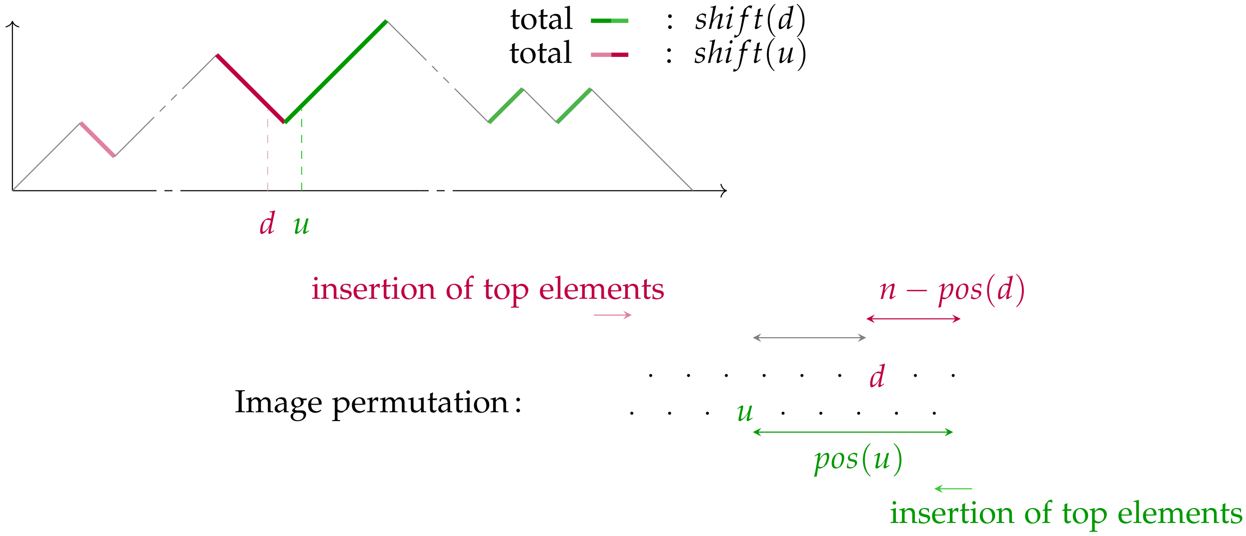


Figure 7: No smaller top element on the left of a given bottom element.

which do not jump. Since all top elements larger than d are inserted to its right, we may assume without loss of generality that d is the position of a step from the slope down following that of u. We need to show that u is put to the right of d in the image permutation, so we compute the same difference as before, but this time we need it to be negative (not just non-positive). Note that, unlike in the previous case, at least one of the insertion positions will receive a minus 1 when computed from the weight. Hence, with $lenSlope(u)$ the length of the slope of u and h_{val} the height of this slope's valley in the Dyck path:

$$\begin{aligned}
dist(u,d) &= pos(u) + pos(d) - n - 1 \ \ (-1) \\
&= wd(u) + shift(u) + wd(d) + shift(d) - n - 1 \ \ (-1) \\
&= wd(u) + wd(d) + shift(u) + n - (lenSlope(u) + L(u)) - n - 1 \ \ (-1) \\
&= wd(d) + wd(u) - h_{val} - lenSlope(u) - 1 \ \ (-1) \\
&\leqslant h_{peak} - h_val - lenSlope(u) - 1 \ \ (-1) = -1 \ \ (-1)\ .
\end{aligned}$$

□

Corollary 4.5. *Let n be a positive integer. The map β_n is a bijection from WD_{2n} to $A_{2n}(1234)$ and is compatible with the concatenation product and the Schützenberger involution. Furthermore, the set of positions of the steps $(1,1)$ of the Dyck path is the set of bottom elements (valleys) in its image alternating permutation.*

As mentionned earlier, Borie gives in [1] a bijection between weighted Dyck paths and product-coproduct prographs on the one hand, and another between these prographs and standard Young tableaux of shape (n,n,n). They can be combined into a rather natural bijection between weighted Dyck paths and this family of tableaux (or alternatively the excursions in the quarter plane with steps $(1,0)$, $(-1,1)$ and $(0,-1)$). Using the bijection presented in this article, this provides a direct bijection between the tableaux, or excursions, and the 1234-avoiding alternating permutations. More generally speaking, the new bijection unveils some bridges between 3-dimensional Catalan objects and deepens our understanding thereof.

Since it looks possible to endow the product-coproduct prographs with a lattice structure, this bijection will also allow to study such a structure (different from those already known on permutations) in the world of 1234-avoiding alternating permutations, and then maybe even generalize it to some larger class of permutations.

Acknowledgements

This research relied to the open-source software `SageMath` [4] and the combinatorics features developed by the `Sage-Combinat` community [3].

References

[1] Nicolas Borie. "Three-dimensional Catalan numbers and product-coproduct prographs." English. In: *Sémin. Lothar. Comb.* 78B (2017), 78b.39, 12. ISSN: 1286-4889/e.

[2] Mireille Bousquet-Mélou and Marni Mishna. "Walks with small steps in the quarter plane". In: *Contemp. Math.* 520 (Oct. 2008). DOI: 10.1090/conm/520/10252.

[3] The Sage-Combinat community. *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics*. `http://combinat.sagemath.org`. 2008.

[4] The Sage Developers. *Sage Mathematics Software (Version 6.4)*. `http://www.sagemath.org`. 2014.

[5] Joel Brewster Lewis. "Generating trees and pattern avoidance in alternating permutations". In: *Electron. J. Combin.* 19.1 (2012), Paper 21, 21. ISSN: 1077-8926.

[6] Joel Brewster Lewis. "Pattern avoidance in alternating permutations and tableaux (extended abstract)". In: *22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010)*. Discrete Math. Theor. Comput. Sci. Proc., AN. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010, pp. 391–402.