# Classification of $P$-oligomorphic permutation groups

## Conjectures of Cameron and Macpherson

PhD defense of Justine Falque

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Laboratoire de Recherche en Informatique
Université Paris-Sud (Orsay)

November 29th of 2019

# First notions

- Permutation

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$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

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3 2 7 6 5 1 4

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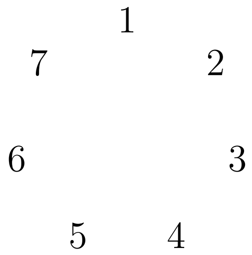

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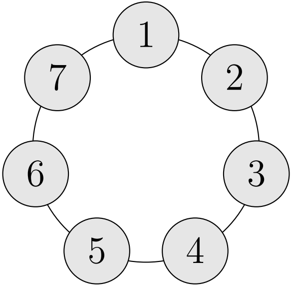

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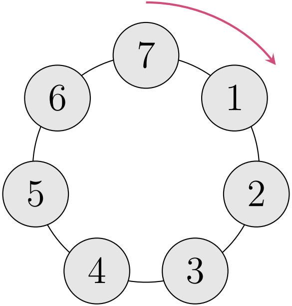

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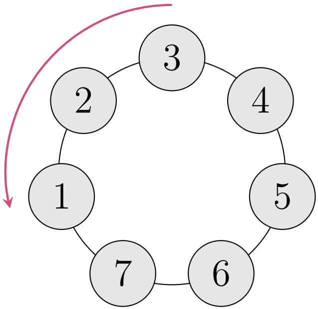

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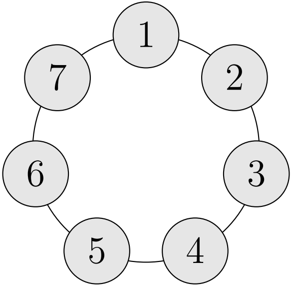

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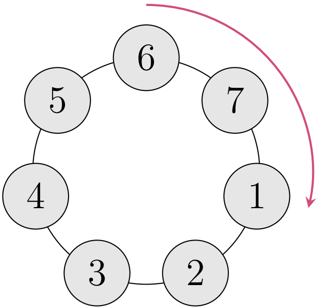

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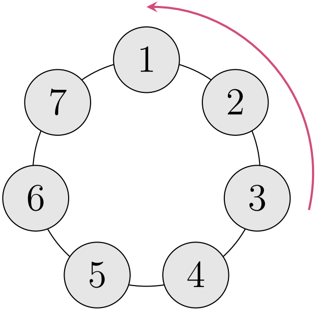

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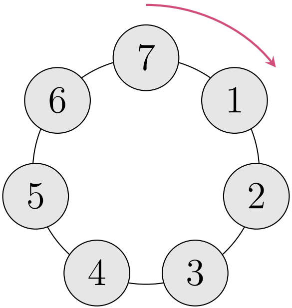

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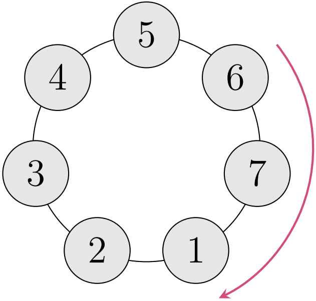

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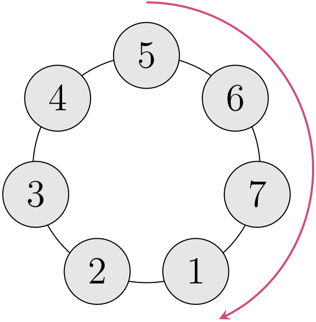

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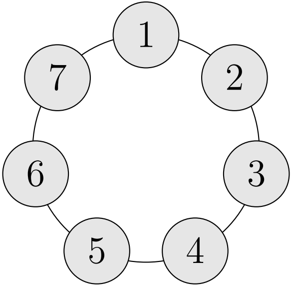

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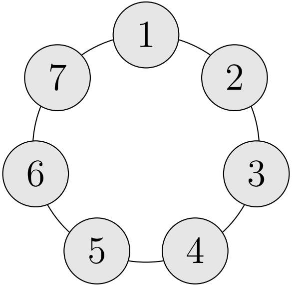

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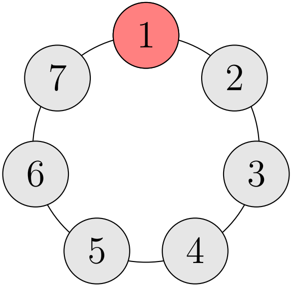

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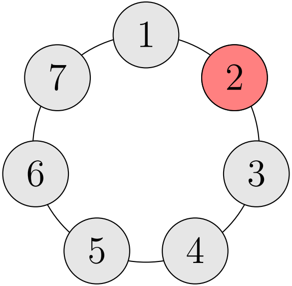

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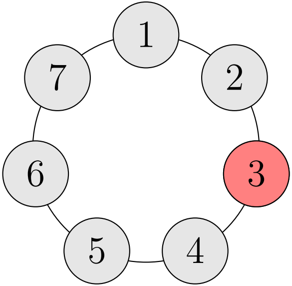

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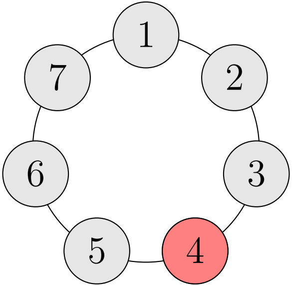

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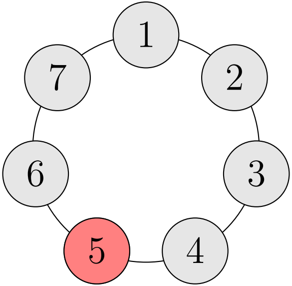

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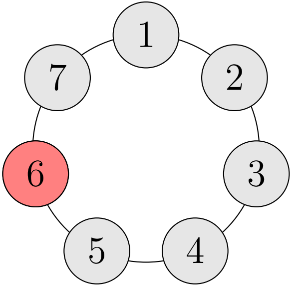

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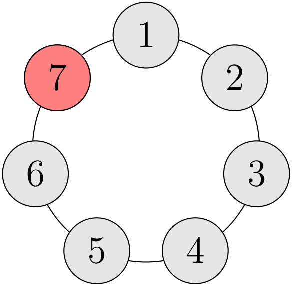

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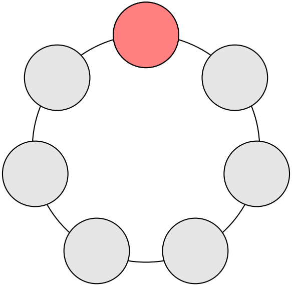

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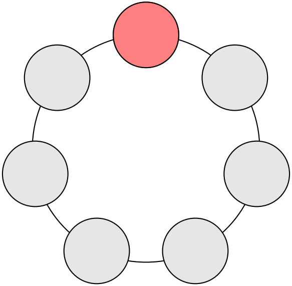

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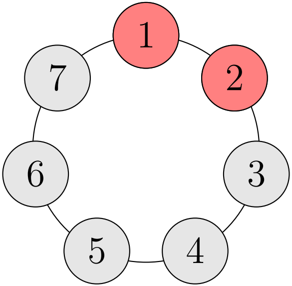

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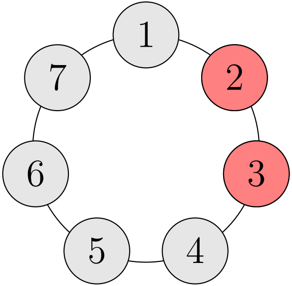

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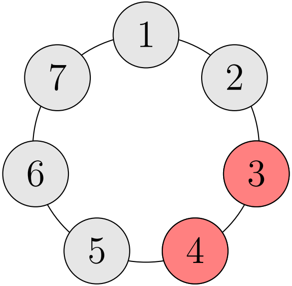

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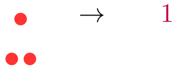

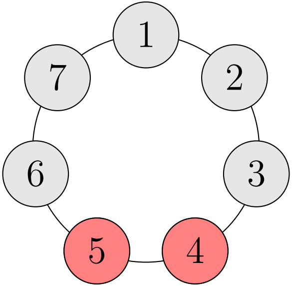

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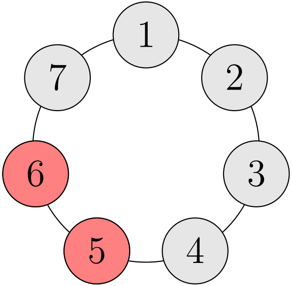


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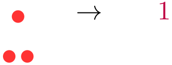

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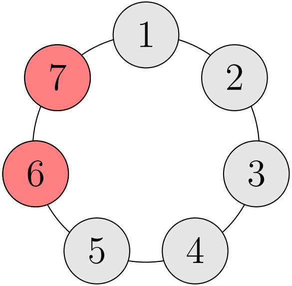


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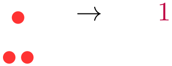


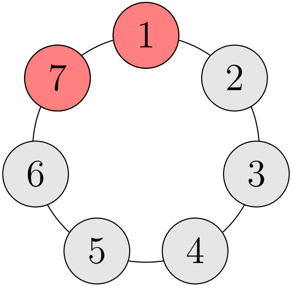

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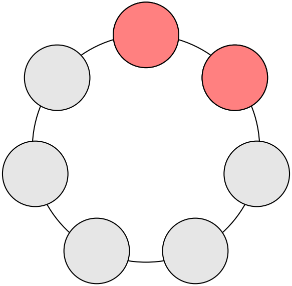

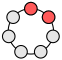

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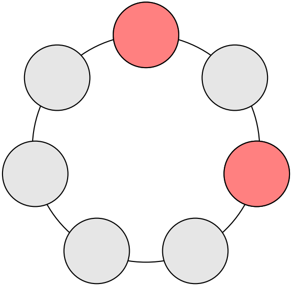

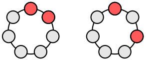

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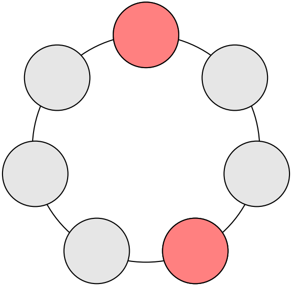

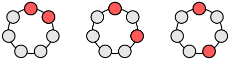

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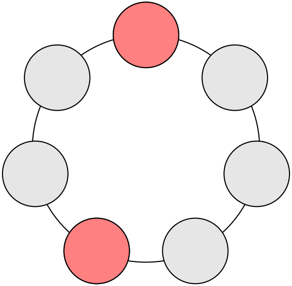

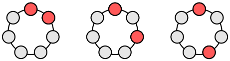

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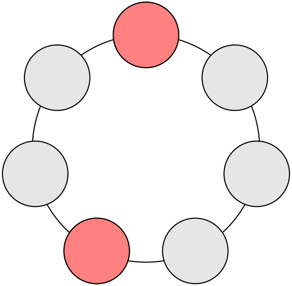
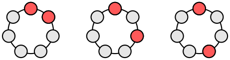
Let's count!

 $\to$ 1

 $\to$ 3





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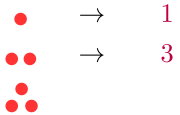


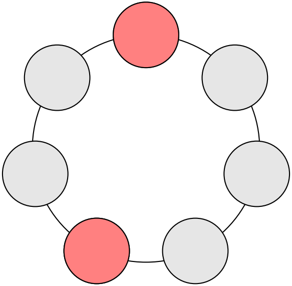

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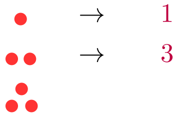

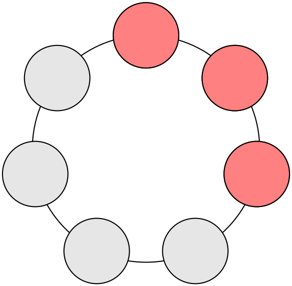

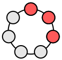

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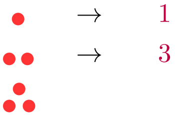


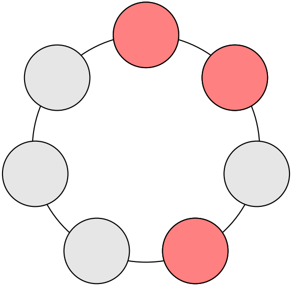

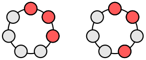

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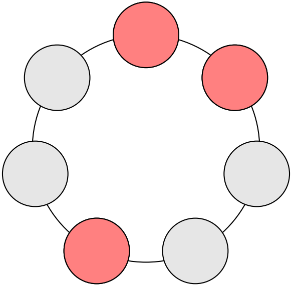
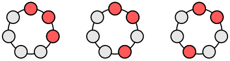
→ 1

→ 3





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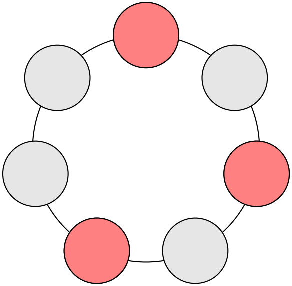

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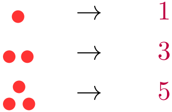


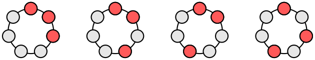

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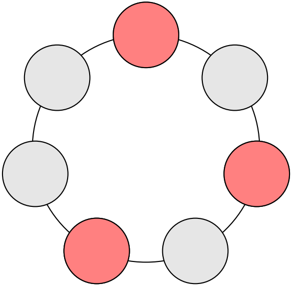

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$\rightarrow$ 1

$\rightarrow$ 3

$\rightarrow$ 5

$\rightarrow$ 5

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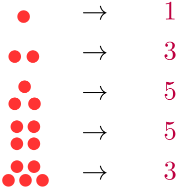


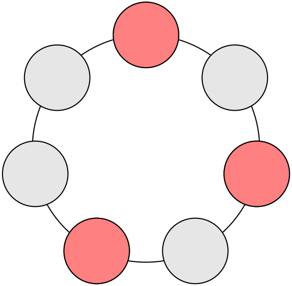

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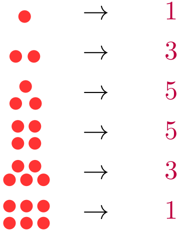


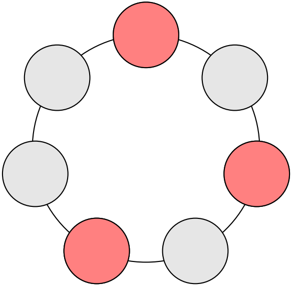

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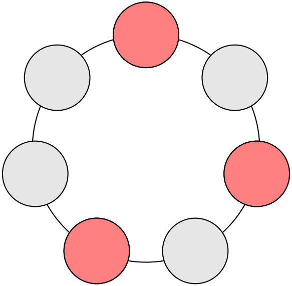

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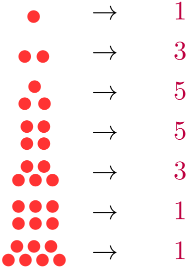

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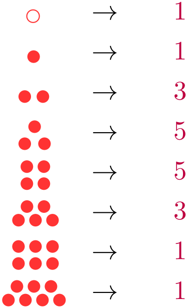


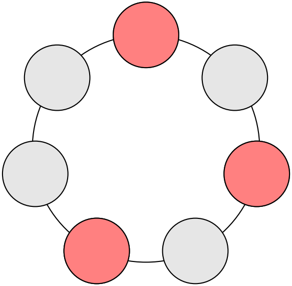

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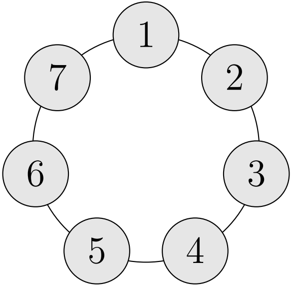
- Orbit of an element

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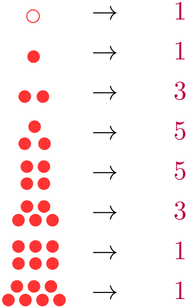

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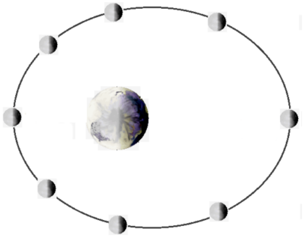

Let's count!

$\rightarrow$ 1
$\rightarrow$ 1
$\rightarrow$ 3
$\rightarrow$ 5
$\rightarrow$ 5
$\rightarrow$ 3
$\rightarrow$ 1
$\rightarrow$ 1

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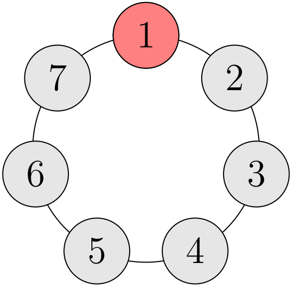
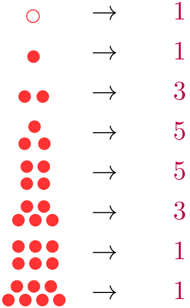
- Permutation group
- Orbit of an element



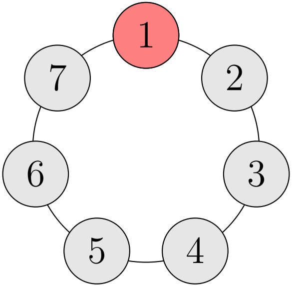

Let's count!




Orbit of 1

# First notions

- Permutation
- Permutation group
- Orbit of an element

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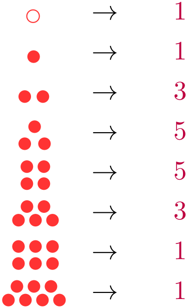
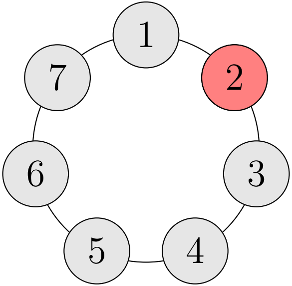



Orbit of 1: 1

# First notions

- Permutation
- Permutation group
- Orbit of an element

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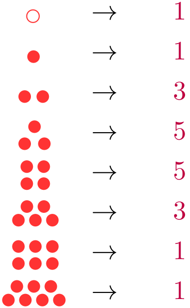
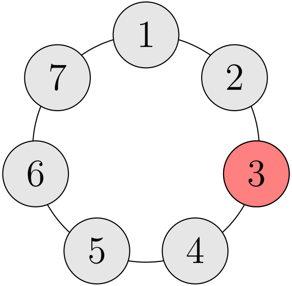


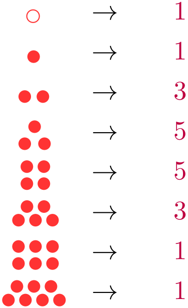

Orbit of 1: 1, 2

# First notions

- Permutation
- Permutation group
- Orbit of an element



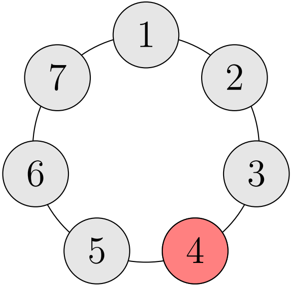

Let's count!



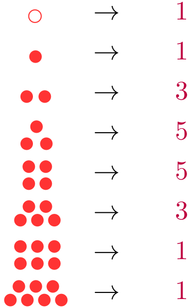

Orbit of 1: 1, 2, 3

# First notions

- Permutation
- Permutation group
- Orbit of an element



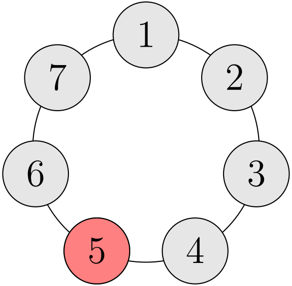

Let's count!



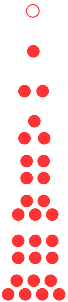

Orbit of 1: 1, 2, 3, 4

# First notions

- Permutation
- Permutation group
- Orbit of an element




Let's count!



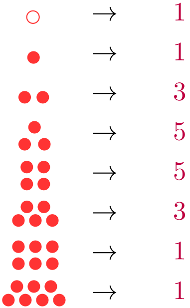
| | |
|---|---|
| $\rightarrow$ | 1 |
| $\rightarrow$ | 1 |
| $\rightarrow$ | 3 |
| $\rightarrow$ | 5 |
| $\rightarrow$ | 5 |
| $\rightarrow$ | 3 |
| $\rightarrow$ | 1 |
| $\rightarrow$ | 1 |

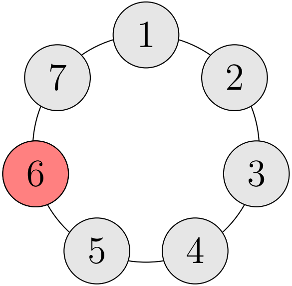
Orbit of 1: 1, 2, 3, 4, 5

# First notions

- Permutation
- Permutation group
- Orbit of an element

Let's count!

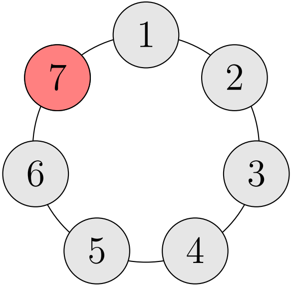



Orbit of 1: 1, 2, 3, 4, 5, 6

# First notions

- Permutation
- Permutation group
- Orbit of an element

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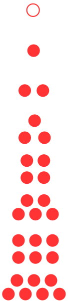
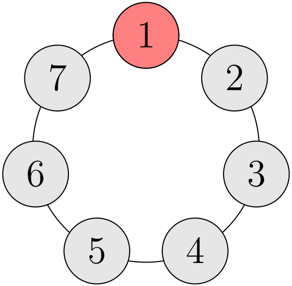


→ 1
→ 1
→ 3
→ 5
→ 5
→ 3
→ 1
→ 1

Orbit of 1: 1, 2, 3, 4, 5, 6, 7

# First notions

- Permutation
- Permutation group
- Orbit of an element

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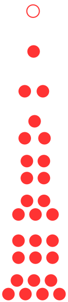
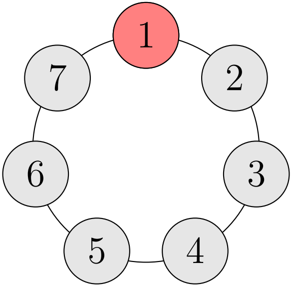


$\rightarrow$ 1
$\rightarrow$ 1
$\rightarrow$ 3
$\rightarrow$ 5
$\rightarrow$ 5
$\rightarrow$ 3
$\rightarrow$ 1
$\rightarrow$ 1

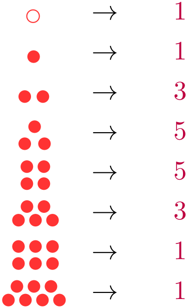
Orbit of 1: 1, 2, 3, 4, 5, 6, 7

# First notions

- Permutation
- Permutation group
- Orbit of an element
- Orbit of a subset (degree)




Let's count!




Orbit of 1: 1, 2, 3, 4, 5, 6, 7

# First notions

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- Permutation group
- Orbit of an element
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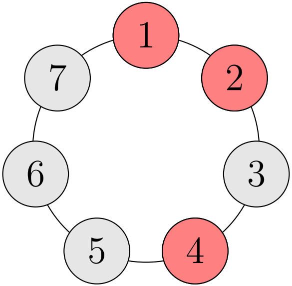
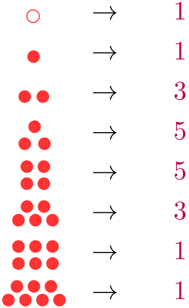



Let's count!




Orbit of $\{1, 2, 4\}$ : $\{1, 2, 4\}$

# First notions

- Permutation
- Permutation group
- Orbit of an element
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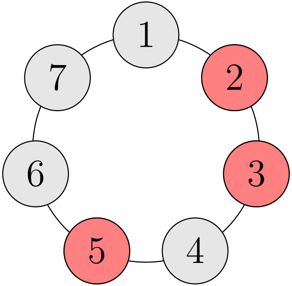


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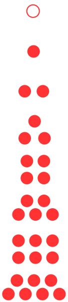

$\rightarrow$ 1
$\rightarrow$ 1
$\rightarrow$ 3
$\rightarrow$ 5
$\rightarrow$ 5
$\rightarrow$ 3
$\rightarrow$ 1
$\rightarrow$ 1

Orbit of $\{1,2,4\}$ : $\{1,2,4\}$, $\{2,3,5\}$

# First notions

- Permutation
- Permutation group
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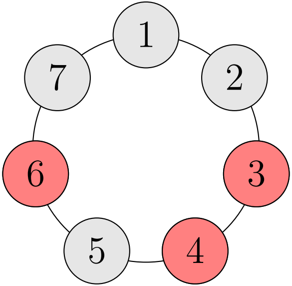
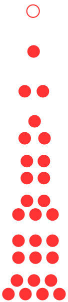



Let's count!

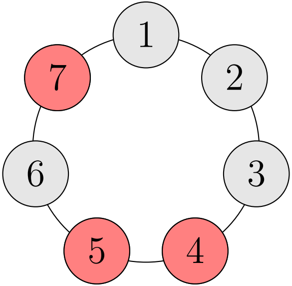


$\rightarrow$ 1
$\rightarrow$ 1
$\rightarrow$ 3
$\rightarrow$ 5
$\rightarrow$ 5
$\rightarrow$ 3
$\rightarrow$ 1
$\rightarrow$ 1

Orbit of $\{1,2,4\}$ : $\{1,2,4\}$, $\{2,3,5\}$, $\{3,4,6\}$

# First notions

- Permutation
- Permutation group
- Orbit of an element
- Orbit of a subset (degree)

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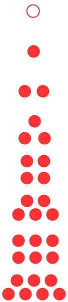

$\rightarrow$ 1

$\rightarrow$ 1

$\rightarrow$ 3

$\rightarrow$ 5

$\rightarrow$ 5

$\rightarrow$ 3

$\rightarrow$ 1

$\rightarrow$ 1

Orbit of $\{1, 2, 4\}$ : $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{3, 4, 6\}$ ...

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- Permutation
- Permutation group
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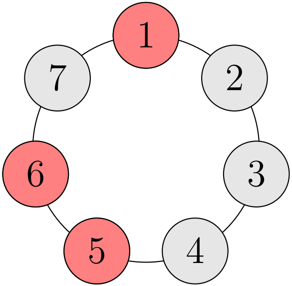
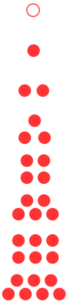



Let's count!



$\rightarrow$ 1
$\rightarrow$ 1
$\rightarrow$ 3
$\rightarrow$ 5
$\rightarrow$ 5
$\rightarrow$ 3
$\rightarrow$ 1
$\rightarrow$ 1

Orbit of $\{1, 2, 4\}$ : $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{3, 4, 6\}$ ...

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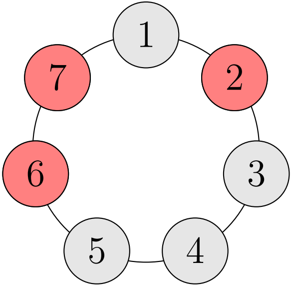


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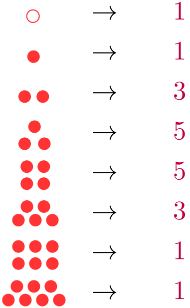


Orbit of $\{1,2,4\}$ : $\{1,2,4\}$, $\{2,3,5\}$, $\{3,4,6\}$ ...

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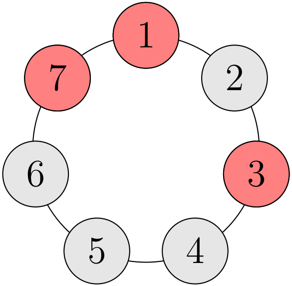


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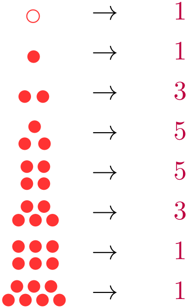


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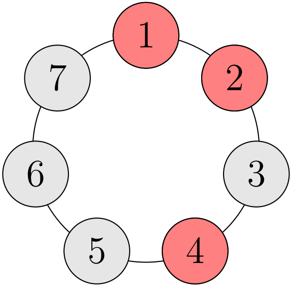


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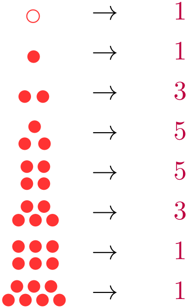


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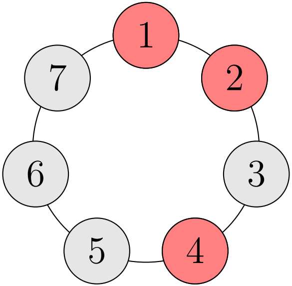


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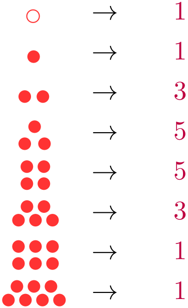


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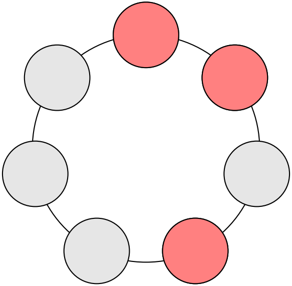

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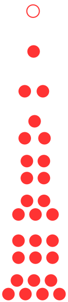

$\rightarrow$ 1

$\rightarrow$ 1

$\rightarrow$ 3

$\rightarrow$ 5

$\rightarrow$ 5

$\rightarrow$ 3

$\rightarrow$ 1

$\rightarrow$ 1

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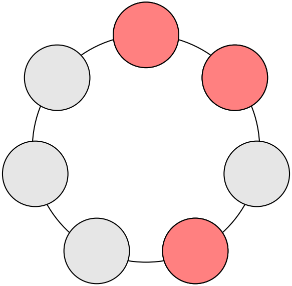

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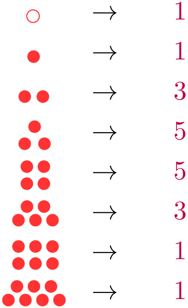
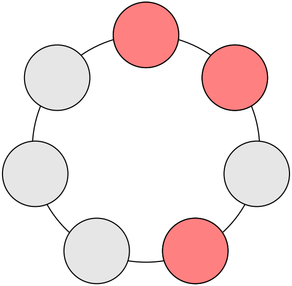



Orbit of $\{1,2,4\}$ : $\{1,2,4\}$, $\{2,3,5\}$, $\{3,4,6\}$ ...

of *degree* 3

# First notions

- Permutation
- Permutation group
- Orbit of an element
- Orbit of a subset (degree)
- Profile of a group



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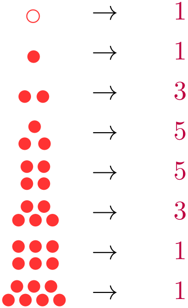


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of *degree* 3

# First notions

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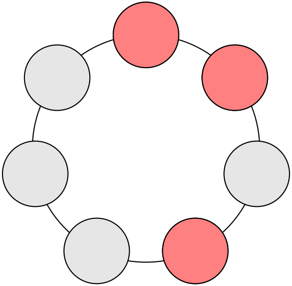

Let's count!

$$\begin{array}{lcl} \varphi(0) & \to & 1 \\ \varphi(1) & \to & 1 \\ \varphi(2) & \to & 3 \\ \varphi(3) & \to & 5 \\ \varphi(4) & \to & 5 \\ \varphi(5) & \to & 3 \\ \varphi(6) & \to & 1 \\ \varphi(7) & \to & 1 \end{array}$$

Orbit of $\{1,2,4\}$ : $\{1,2,4\}$, $\{2,3,5\}$, $\{3,4,6\}$ ...

of *degree* 3

# Series of the profile

$$1 + 1z + 3z^2 + 5z^3 + 5z^4 + 3z^5 + 1z^6 + 1z^7$$

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$$G \textbf{ infinite} \quad \rightarrow \quad \mathcal{H}_G(z) = \sum_n \varphi_G(n) z^n$$

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## Hypothesis

$G$ is $P$**-oligomorphic**: $\varphi_G$ is bounded by a polynomial in $n$

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## Example

$$\mathcal{H}_{\mathfrak{S}_\infty}(z) \;=\; 1 + z + z^2 + \cdots \;=\; \frac{1}{1-z}$$

# Series of the profile

$$1 + 1z + 3z^2 + 5z^3 + 5z^4 + 3z^5 + 1z^6 + 1z^7$$

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## Example

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## Conjecture 1 - Cameron, 70's

$G$ $P$**-oligomorphic** $\Rightarrow \varphi_G(n) \sim a n^k, \quad k \in \mathbb{N}$

# Orbit algebra

## Orbit algebra (Cameron, 80's)

Structure of graded algebra $\mathcal{A}_G = \bigoplus_n \mathcal{A}_n$ on the orbits

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# Orbit algebra

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- combinatorial description of the product

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Hilbert series of the graded algebra

# Conjecture of Macpherson

Example. $$\mathcal{A}_{\mathfrak{S}_\infty} \simeq \mathbb{Q}[X]$$

# Conjecture of Macpherson

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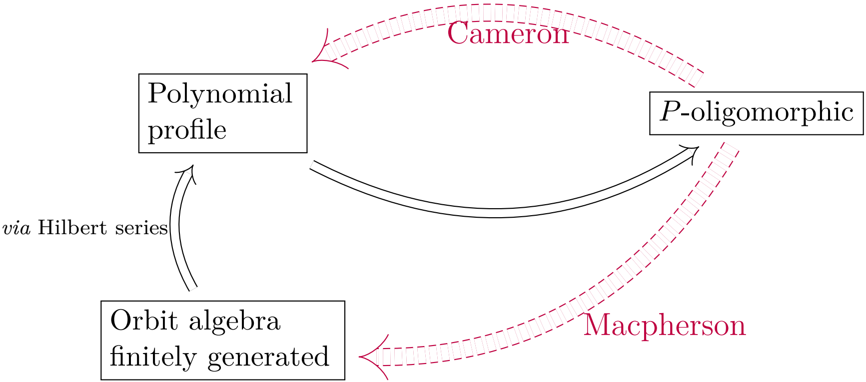
Conjecture 2 (stronger) - Macpherson, 85

$G$ $P$-oligomorphic $\Rightarrow$ $\mathcal{A}_G$ is finitely generated

# Overview


Cameron
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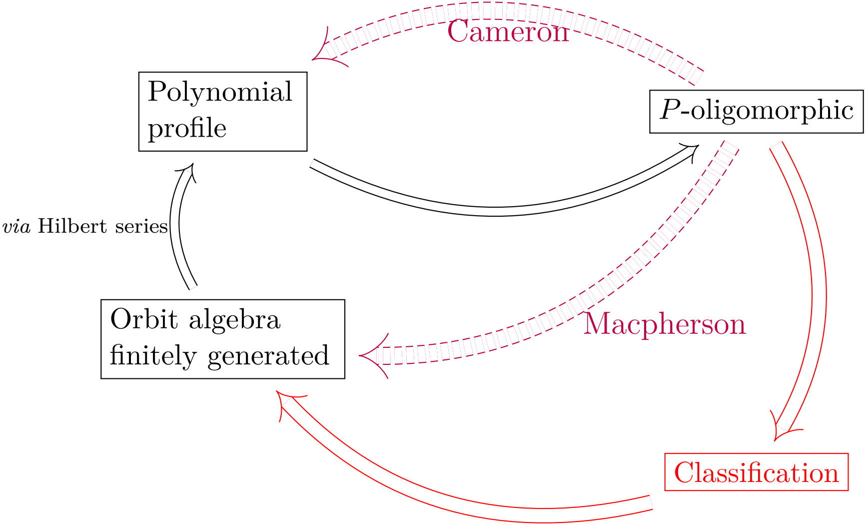
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Classification

# Conjecture of Macpherson

Example. $$\mathcal{A}_{\mathfrak{S}_\infty} \simeq \mathbb{Q}[X]$$

Conjecture 2 (stronger) - Macpherson, 85

$G$ $P$-oligomorphic $\Rightarrow$ $\mathcal{A}_G$ is finitely generated

Theorem (F. 2018)

The orbit algebra of a $P$-oligomorphic group is finitely generated, and Cohen-Macaulay.

In particular, its profile is polynomial in the strong sense.

# Block systems

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- Set partition of the domain into blocks

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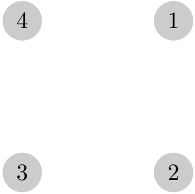
Block systems of $C_4$



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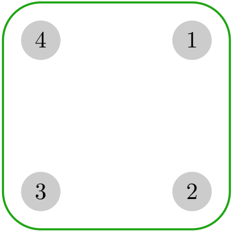

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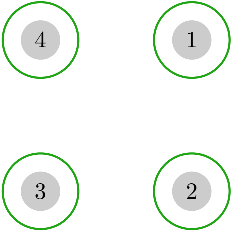

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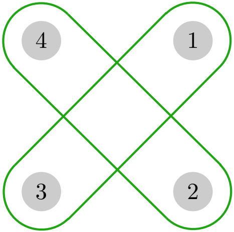

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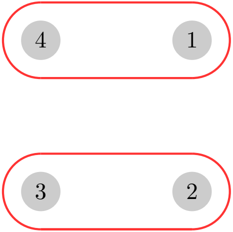
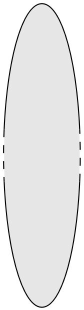
Not a block system $\rightarrow$



# The (closed) primitive $P$-oligomorphic groups



$\mathfrak{S}_\infty$

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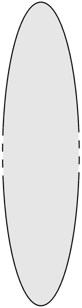
Macpherson:

$G$ $P$-oligomorphic with no (non trivial) blocks $\Rightarrow \varphi_G(n) = 1\ \forall n$



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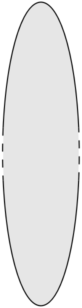

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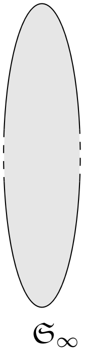


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- $\mathrm{Aut}(\mathbb{Q})$ : automorphisms of the rational chain
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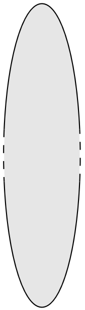


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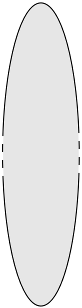
Well known, nice groups (called *highly homogeneous*).
In particular, their orbit algebra is finitely generated.

# An infinite example: $\mathfrak{S}_\infty \wr \mathfrak{S}_3$



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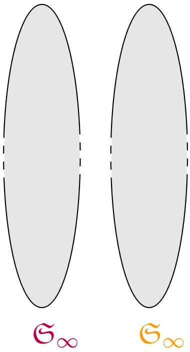

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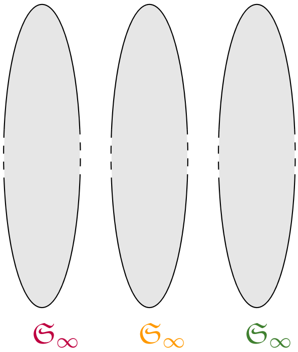

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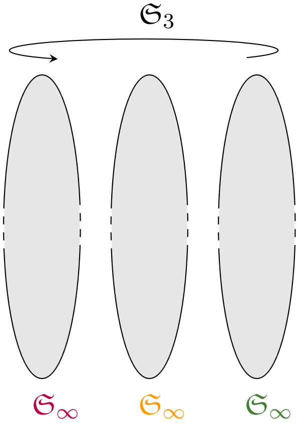

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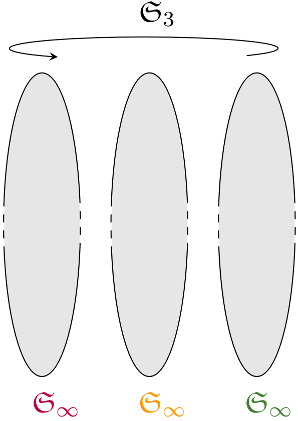


## Wreath product

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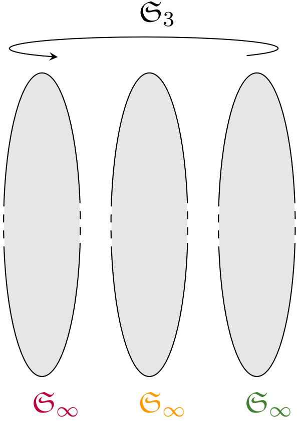


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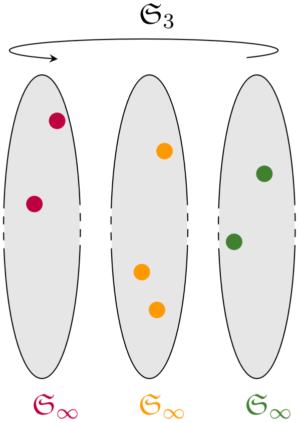


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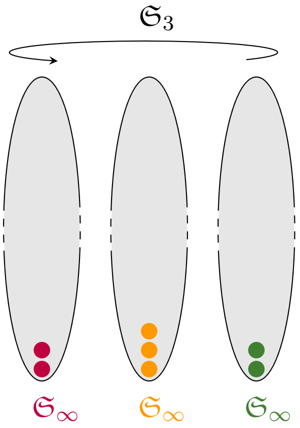


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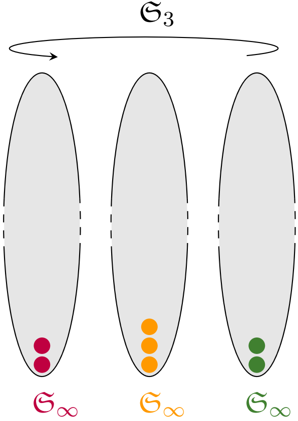


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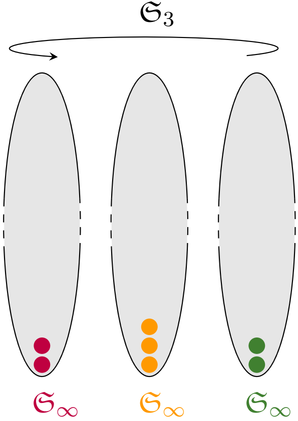


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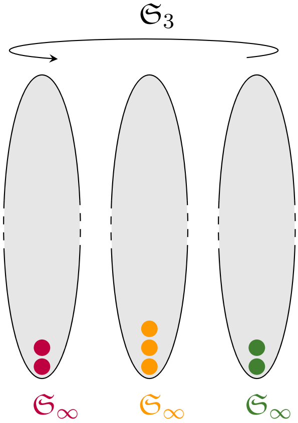


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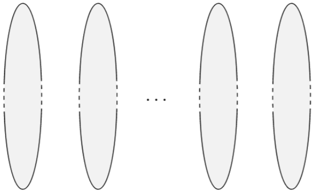
**Examples**
Integer partitions; combinations; $P$-partitions...
(with optional length and/or hight restrictions)

# Further examples

More generally, for $H$ subgroup of $\mathfrak{S}_m$ :

- $G = \mathfrak{S}_\infty \wr H$ :
  $\mathcal{A}_G \simeq \mathbb{Q}[X_1, \ldots, X_m]^H$, the algebra of invariants of $H$



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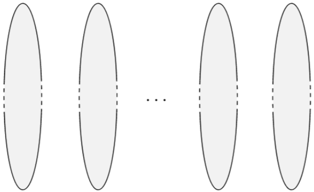

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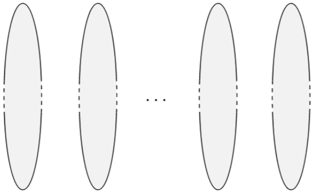

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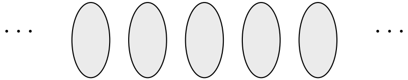

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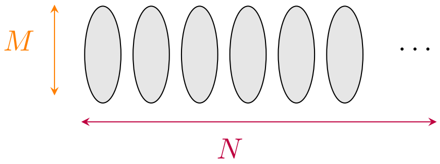

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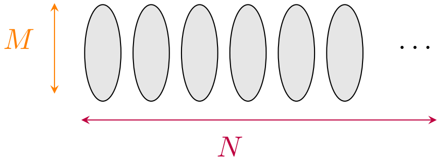


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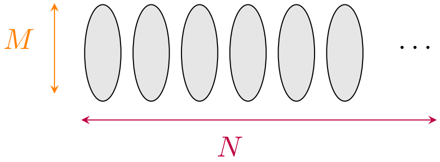


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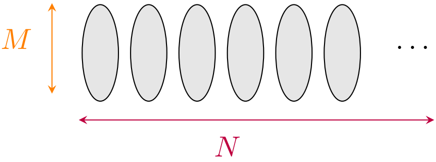


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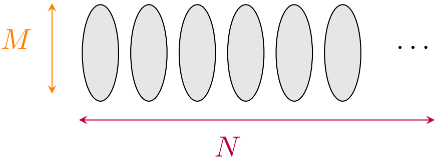


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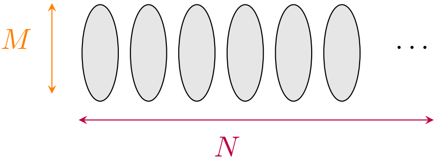


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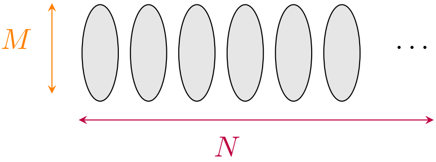


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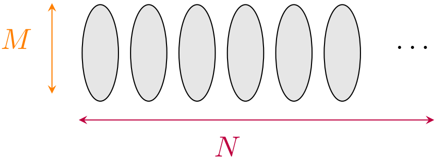


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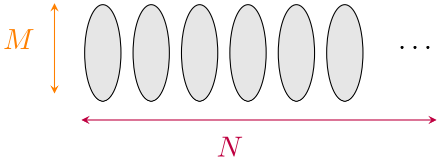


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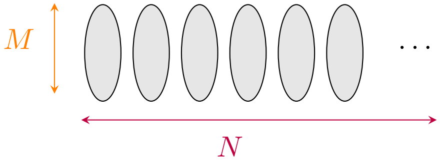


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Better have big finite blocks and/or “small” infinite ones...

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Partially ordered set (poset) with notions of join $\vee$ and meet $\wedge$ : any subset has a unique supremum (resp. infimum).

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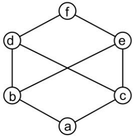

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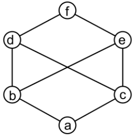
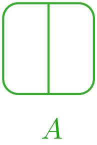



Join and meet in the lattice of set partitions



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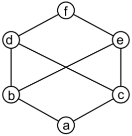


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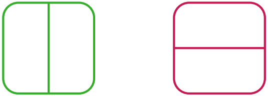

$A$ $B$

# Lattices

## Lattice

Partially ordered set (poset) with notions of join $\vee$ and meet $\wedge$ : any subset has a unique supremum (resp. infimum).

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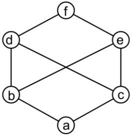


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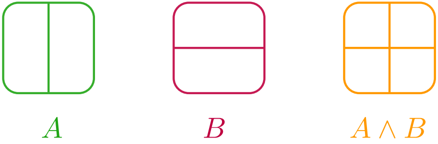

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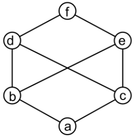


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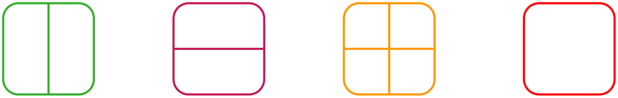

$A$ $B$ $A \wedge B$ $A \vee B$

# Lattices of block systems

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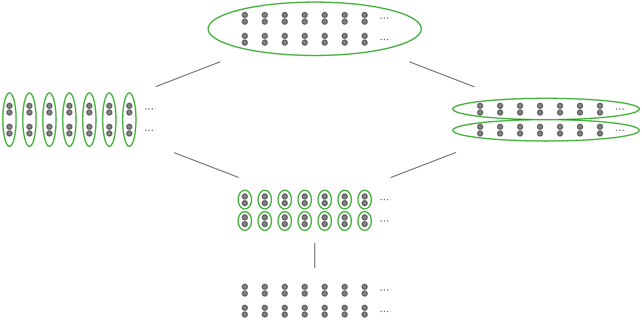

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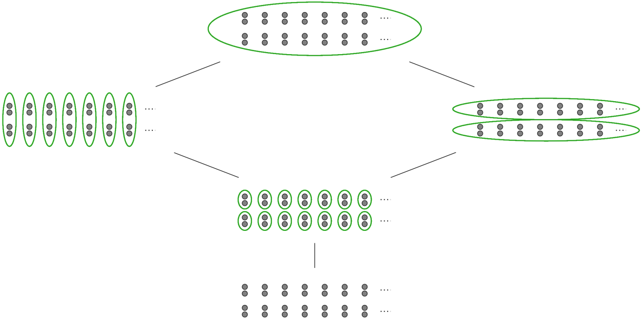

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## Proposition (F.)

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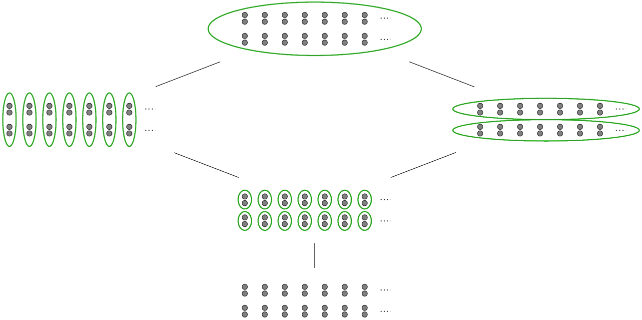

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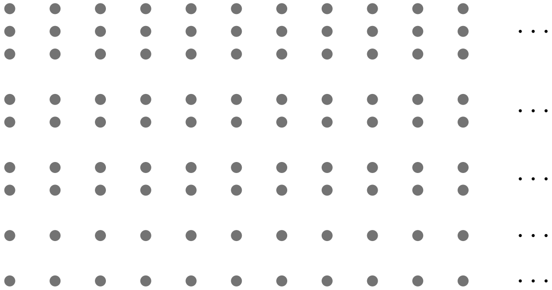
Remark. If $G$ is $P$-oligomorphic, both of them are actually finite !

# The nested block system

## Idea



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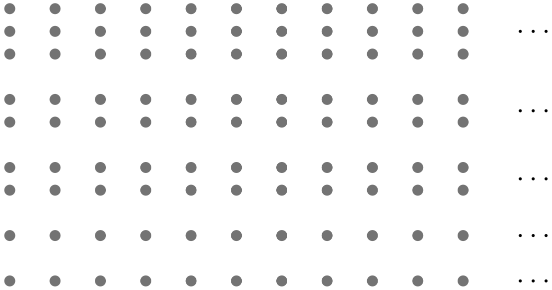
1. Take the *maximal* system of finite blocks



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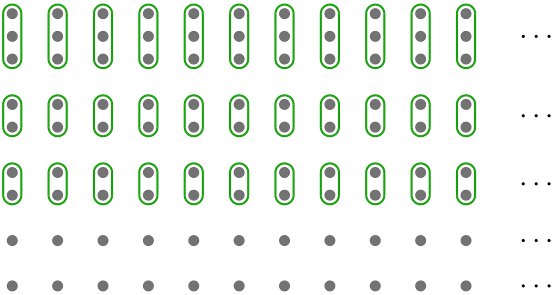

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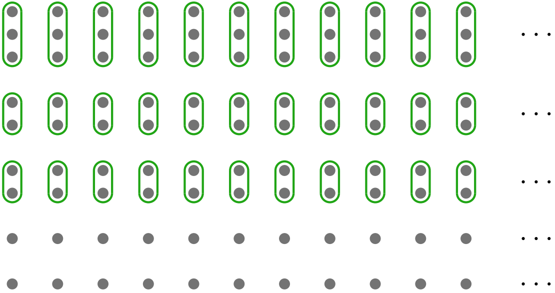

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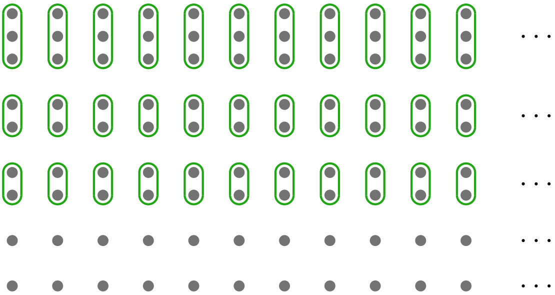

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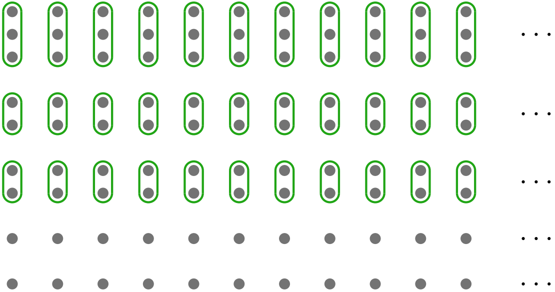

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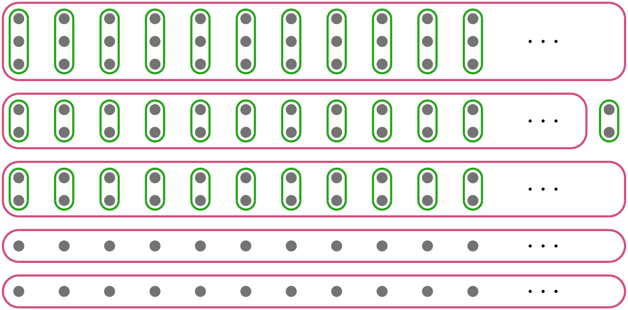

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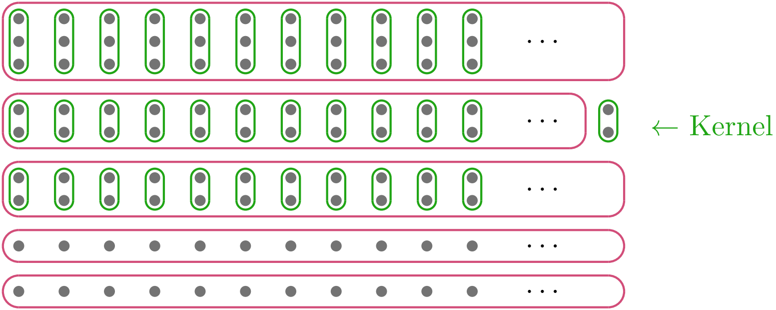


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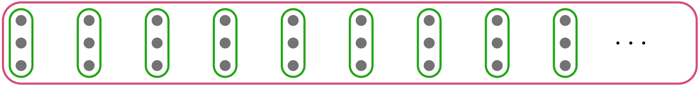

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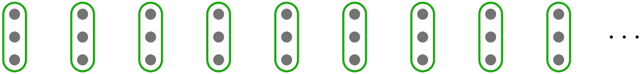

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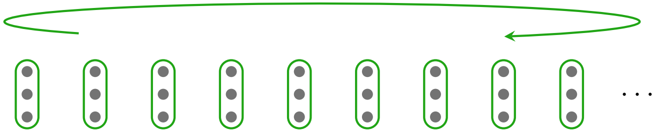

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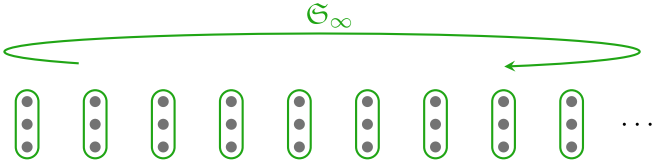

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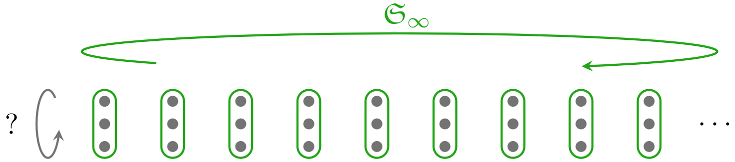

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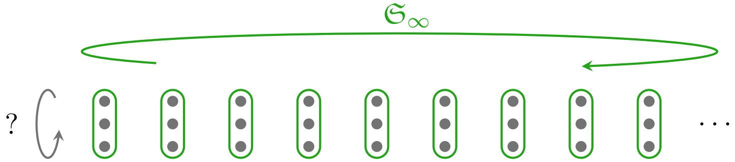


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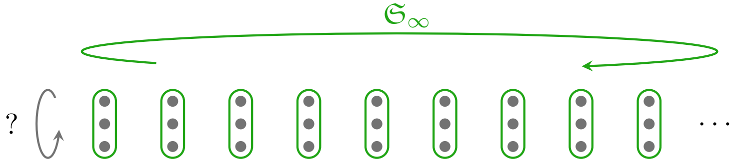


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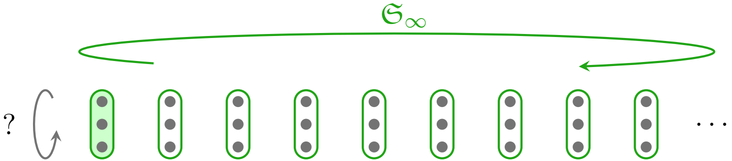


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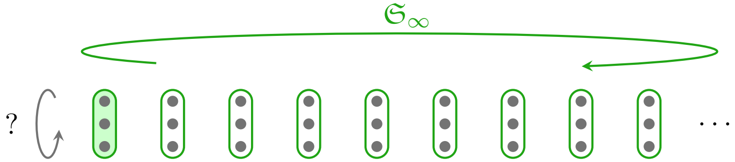


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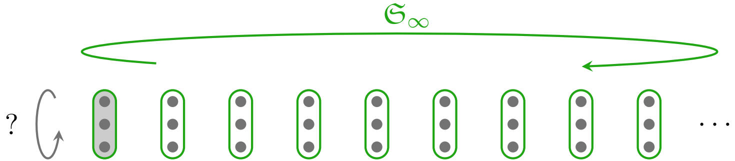


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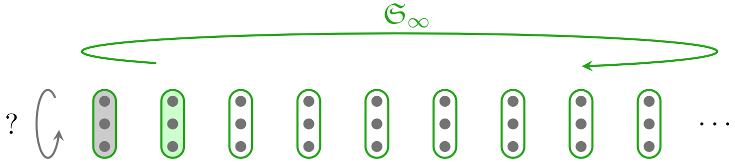


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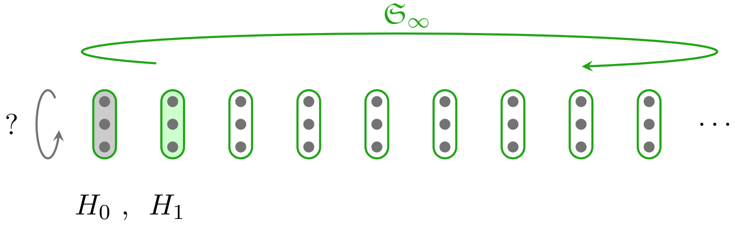


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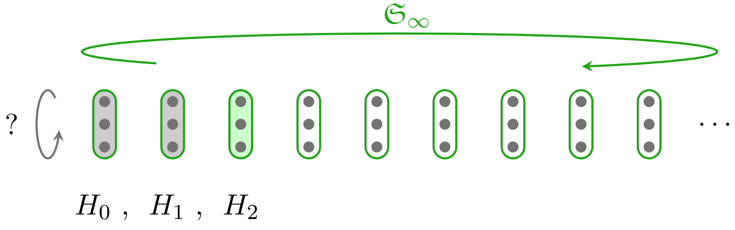


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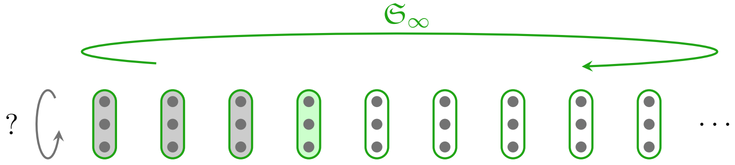


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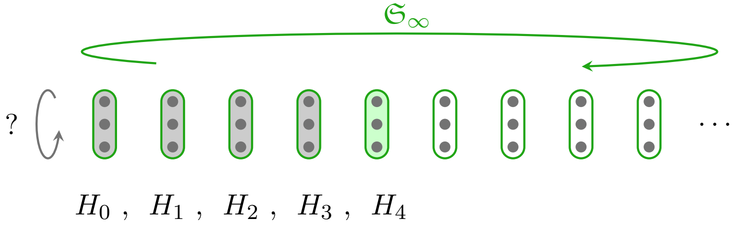


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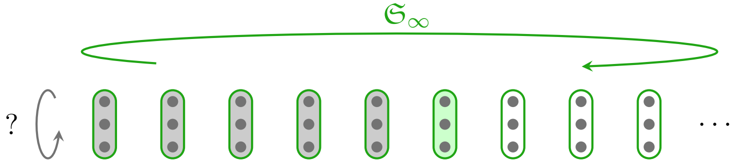


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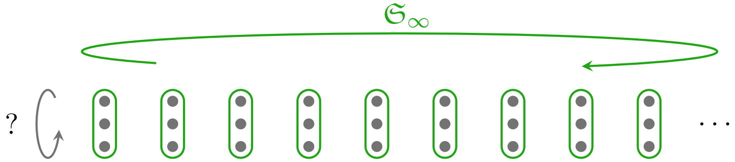


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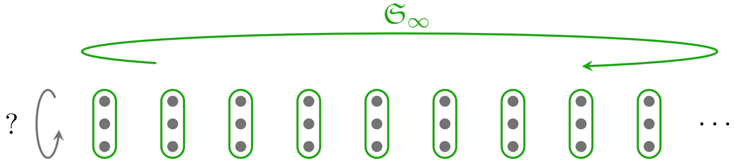


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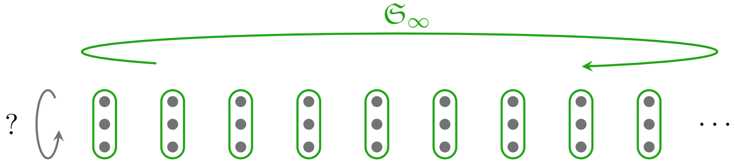


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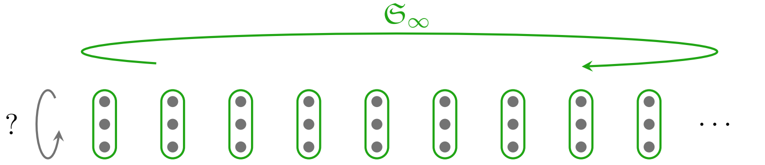


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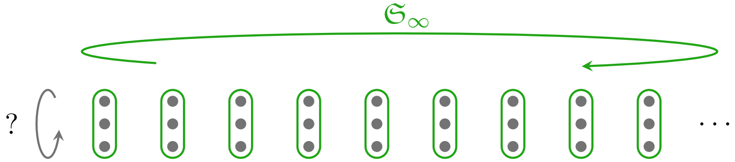


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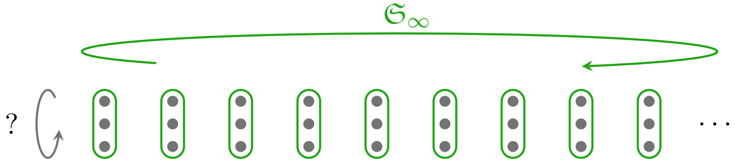


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Notation: $[H_0, H_\infty]$

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How to handle synchronizations between blocks ?

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Remark. The possible synchronizations of a group with another one are linked to its normal subgroups.

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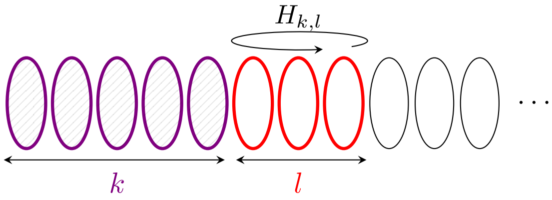

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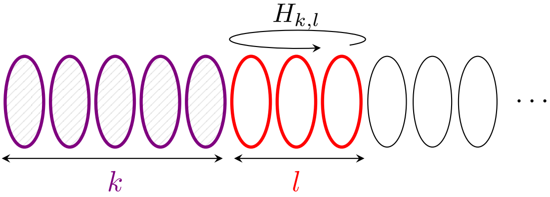


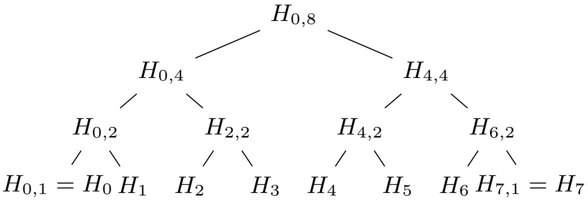

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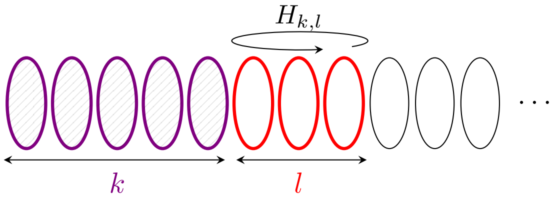


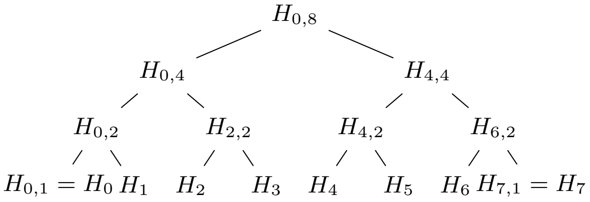


$\leftarrow$ Recursive subdirect product construction

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## Classification

One superblock $\Rightarrow\ G\ =\ [H_0, H_\infty]$

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## Classification

One superblock $\Rightarrow$ $G = [H_0, H_\infty]$

$$\mathbb{Q}[(X_o)_{o \in \mathrm{orb}(H)}]$$

# One superblock: classification

- The tower determines $\mathrm{Stab}_G(\text{blocks})$
- Hence it determines $G$
- Computer exploration on finitely many blocks
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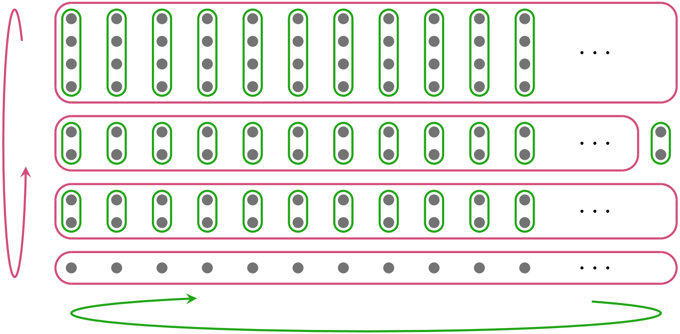

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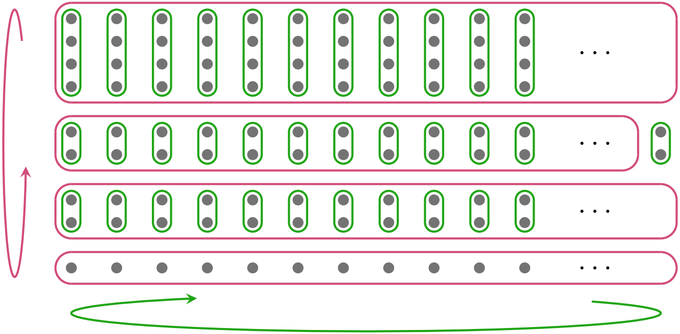

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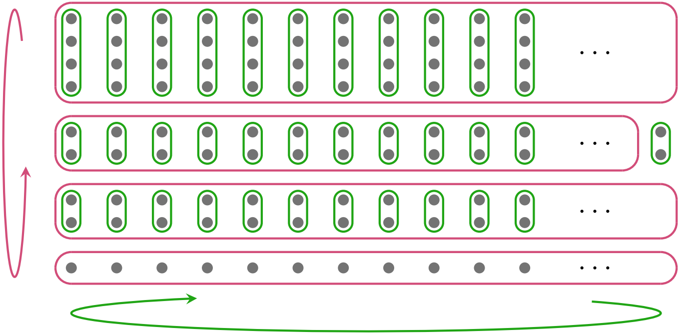

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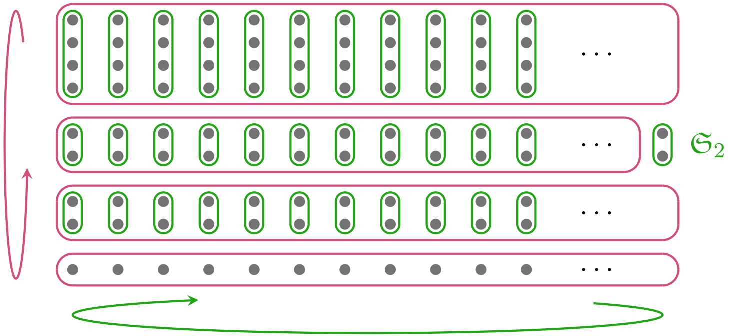

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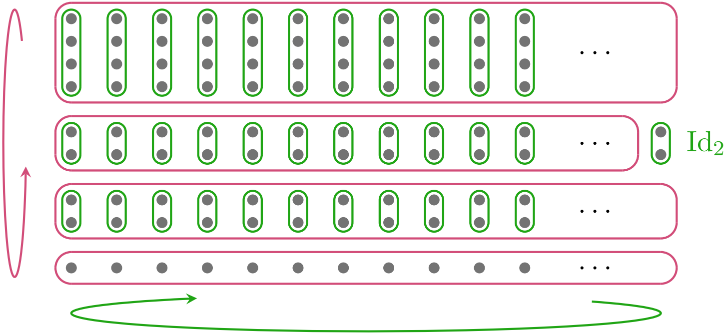

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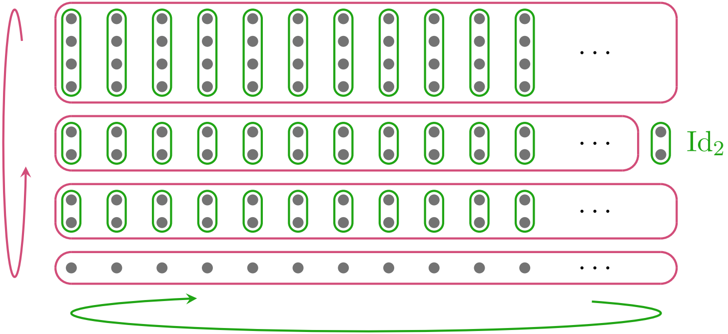

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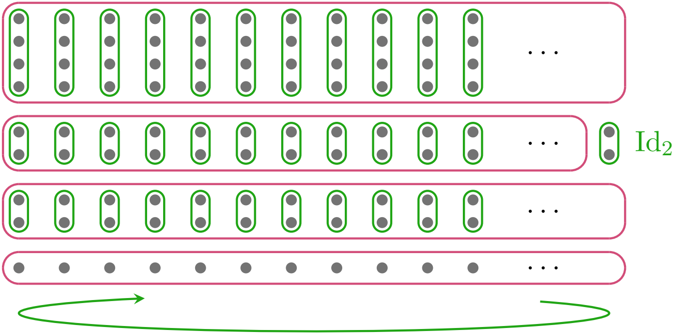

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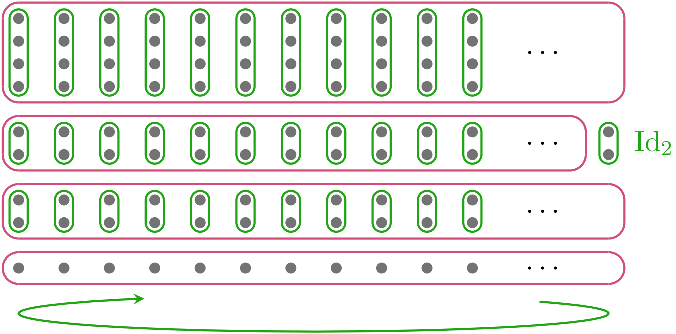

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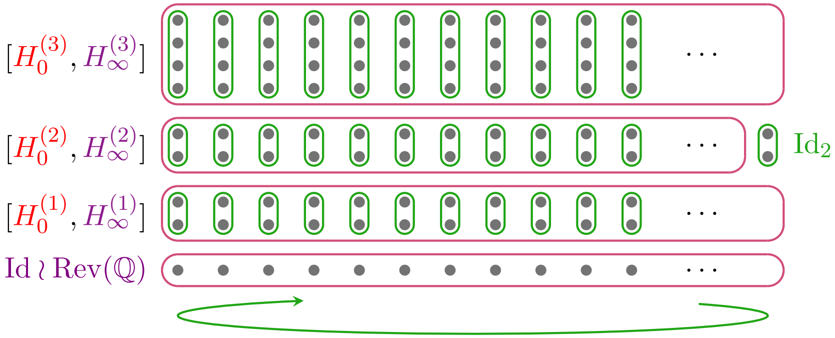

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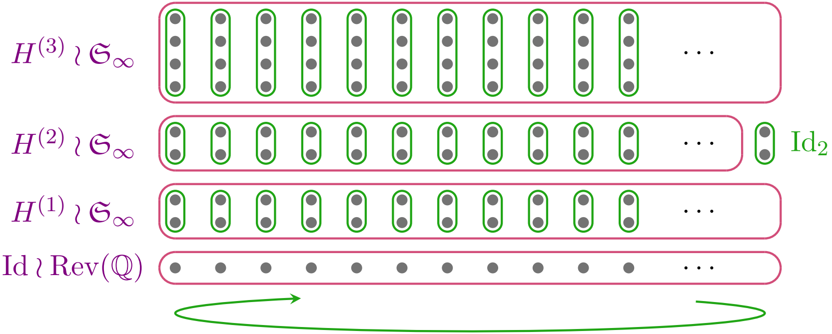

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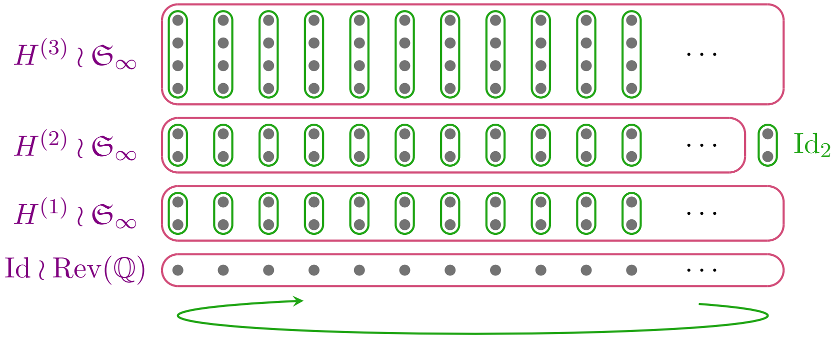

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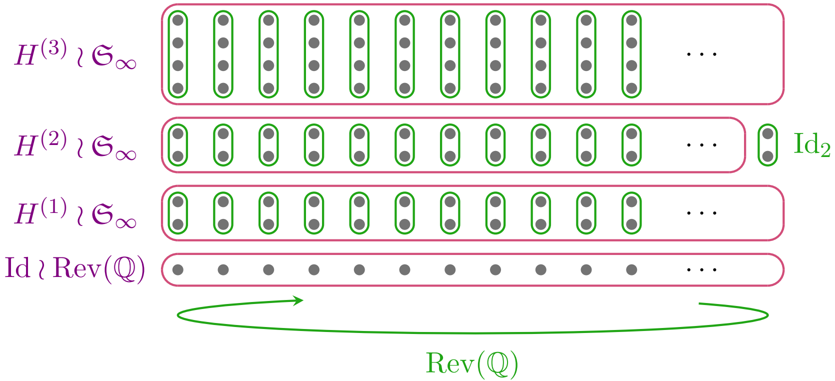

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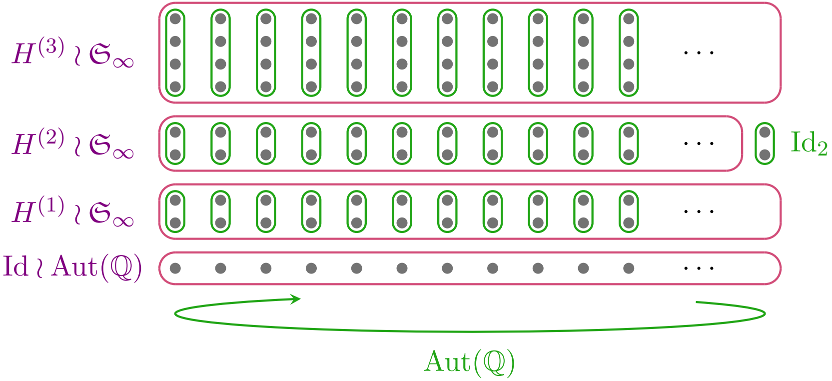

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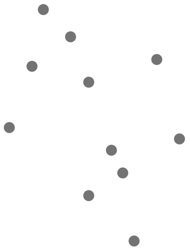
Which ends the proof of the conjectures !

# Classification of $P$-oligomorphic groups (F. 2019)

$G_0$ a finite permutation group



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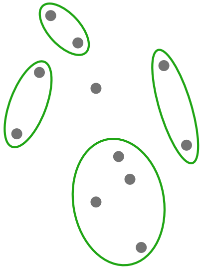

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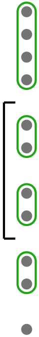

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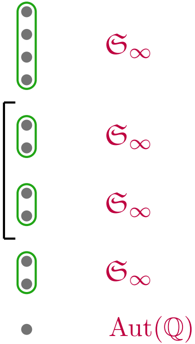

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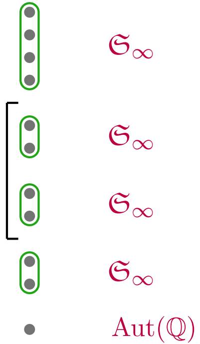

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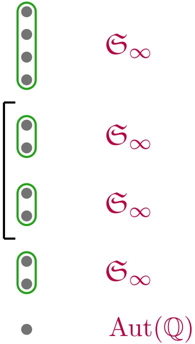

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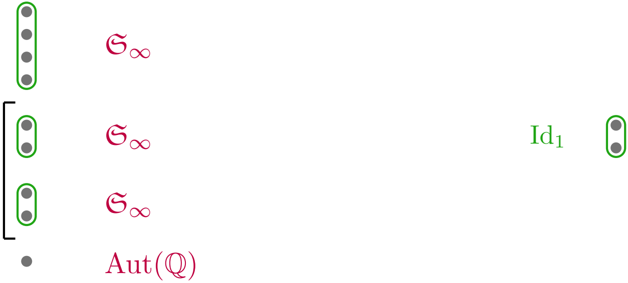

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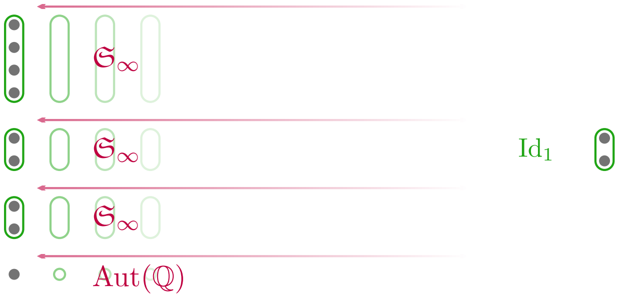

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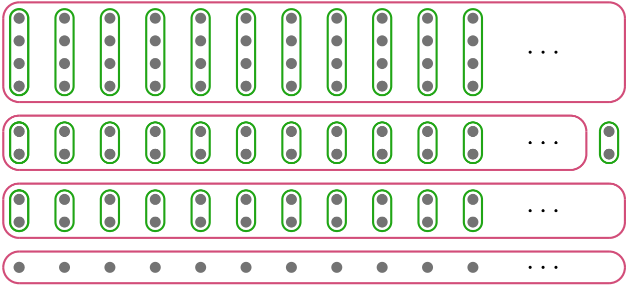

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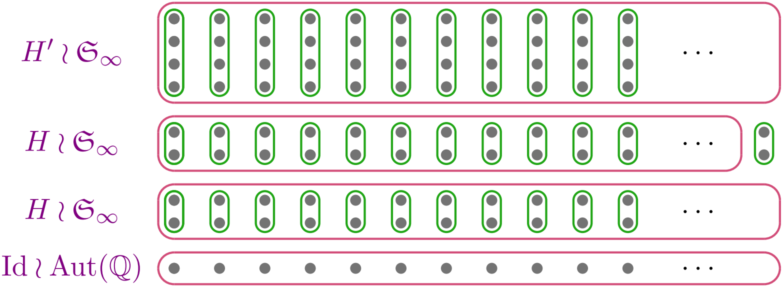

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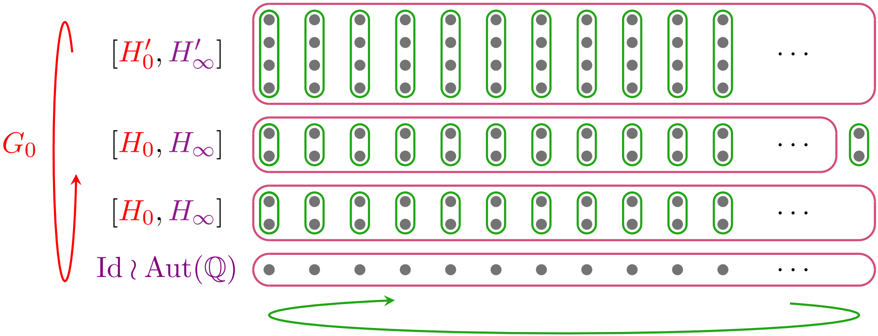

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  - Exponential: wilder primitive groups appear...

# Thank you for your attention !

## Context

- $G$ permutation group of a countably infinite set $E$
- Profile $\varphi_G$: counts the orbits of finite subsets of $E$
- Hypothesis: $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron): $\varphi_G(n) \sim an^k$
- Conjecture (Macpherson): finite generation of the orbit algebra

## Results

- Both conjectures hold !
- Classification of $P$-oligomorphic permutation groups
- The orbit algebra is an algebra of invariants (up to some 2-nilpotent elements)

# Example of a product in the cyclic group $\mathcal{C}_5$

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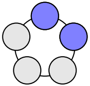
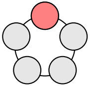


$\times$



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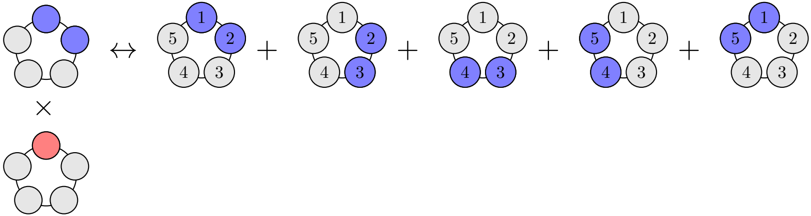

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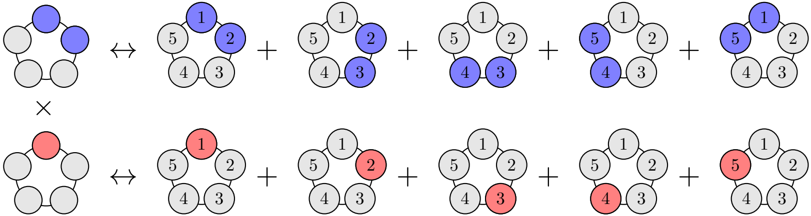

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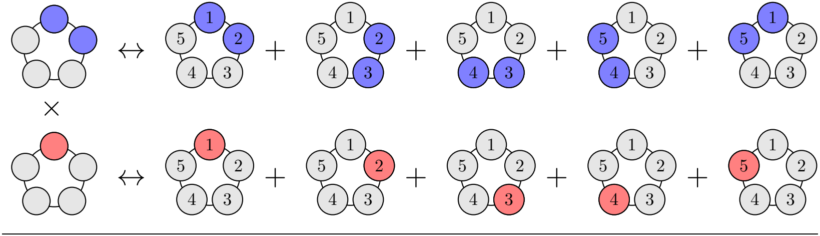

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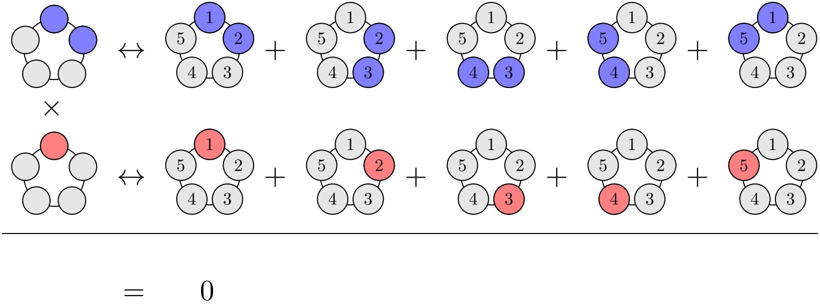

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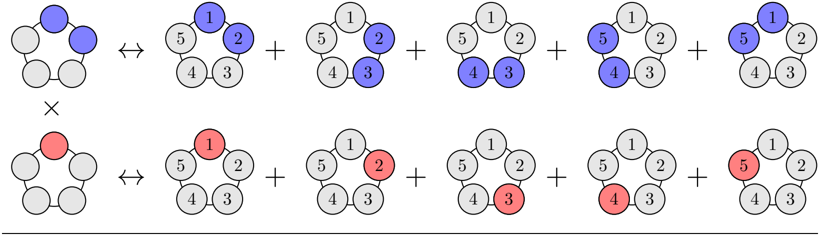

$$= \quad 0 \quad + \quad 0$$

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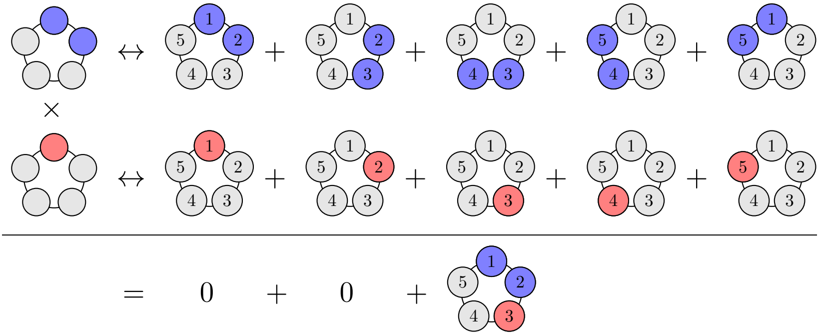

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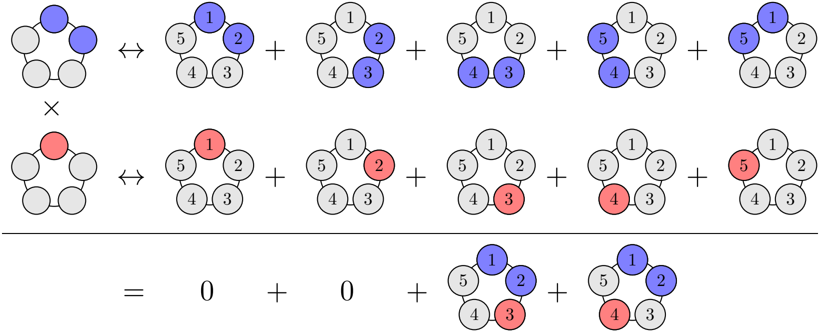

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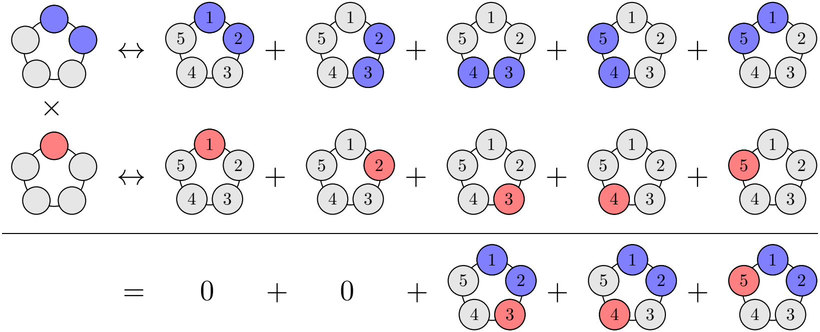

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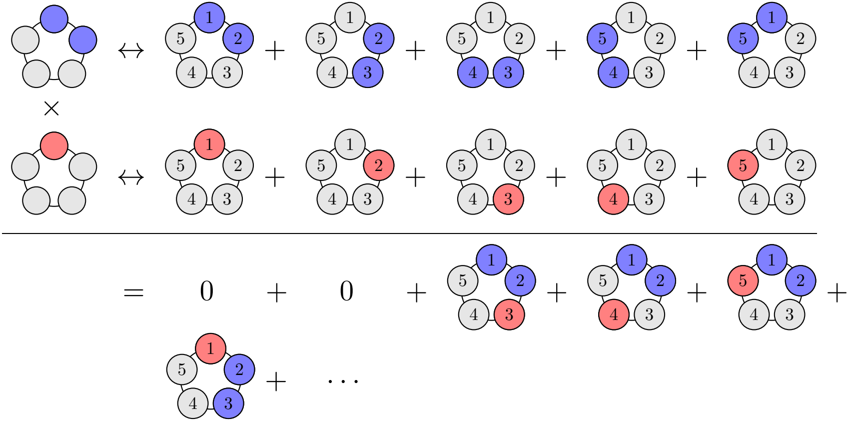

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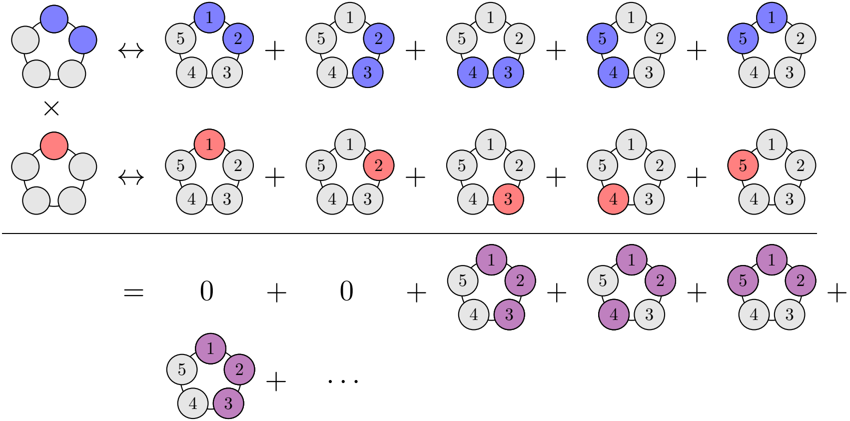

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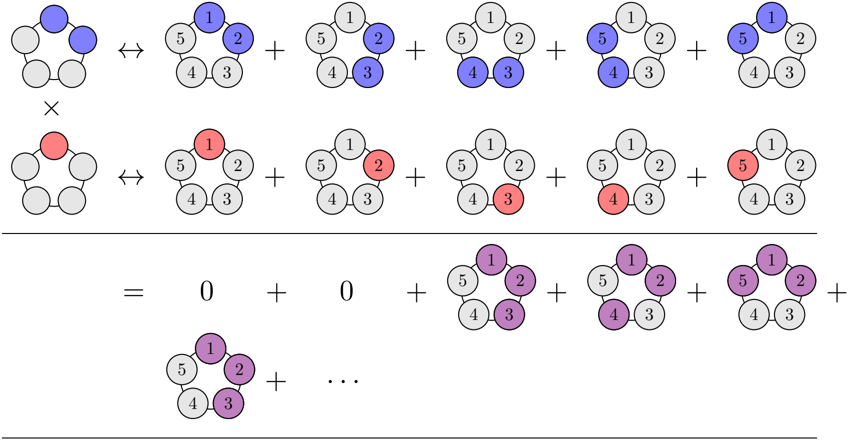

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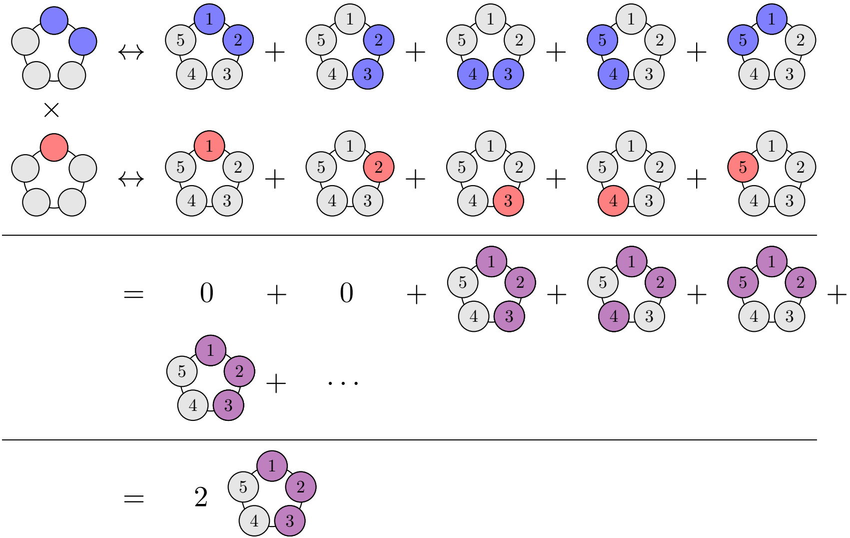

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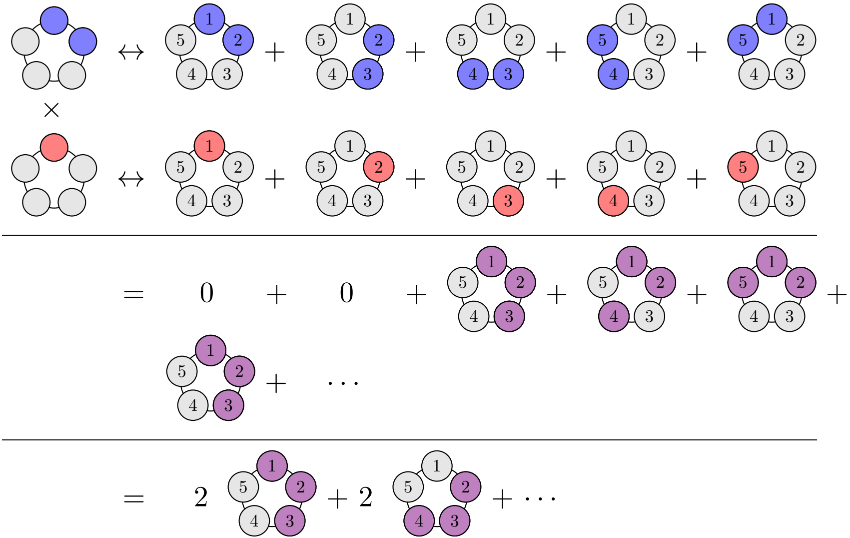

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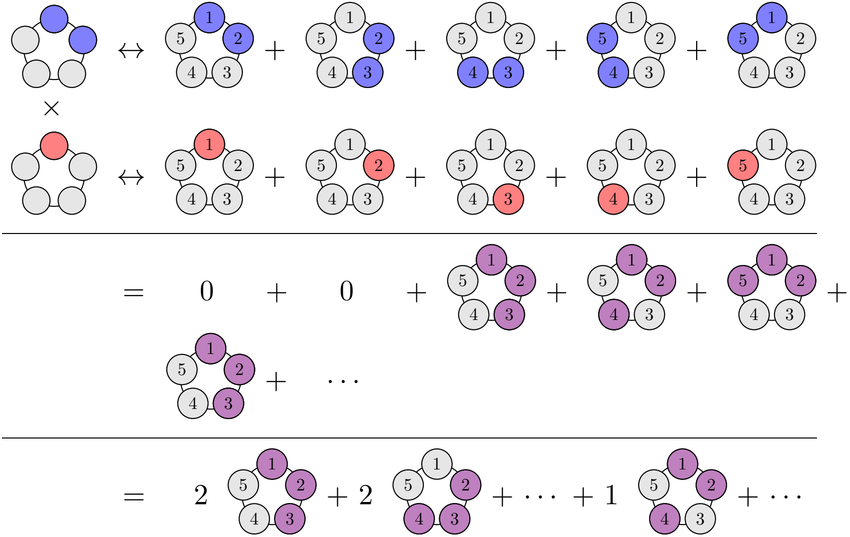
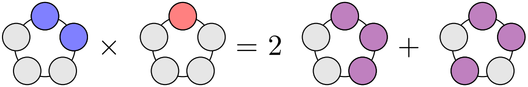


# Conjecture of Macpherson

In the end:



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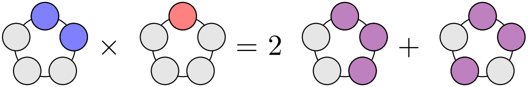
In the end: 

## Non trivial fact

Product well defined (and graded) on the space of orbits.

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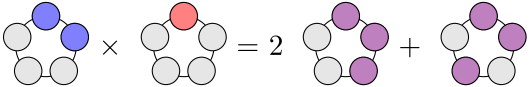


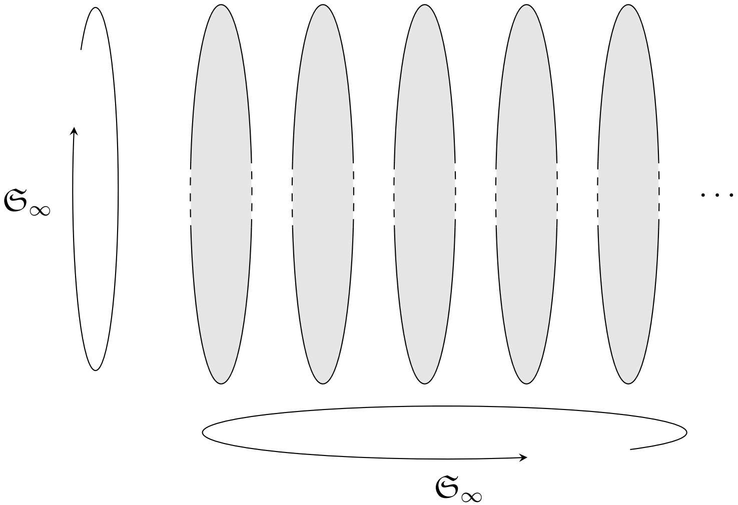

## Non trivial fact

Product well defined (and graded) on the space of orbits.

$\longrightarrow$ **Orbit algebra of a permutation group**

# Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) = \ ?$



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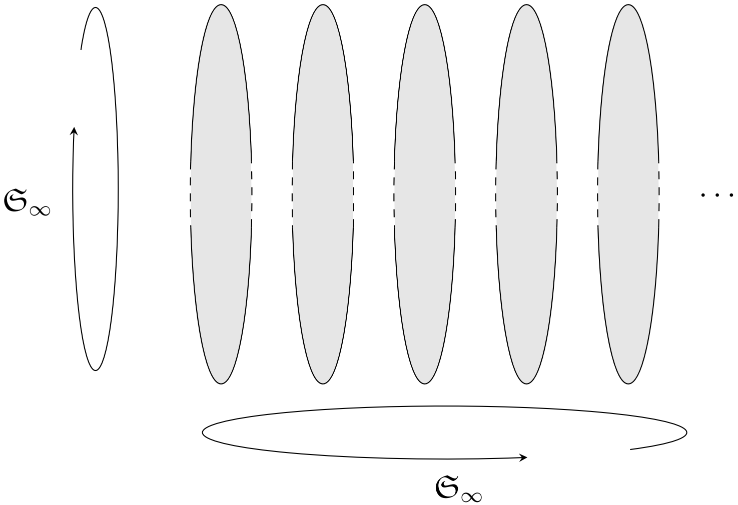
An orbit of degree $n \longleftrightarrow$ a partition of $n$



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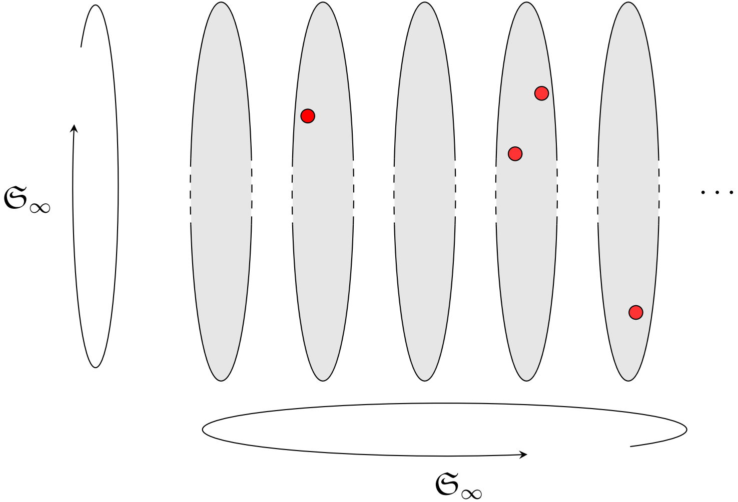

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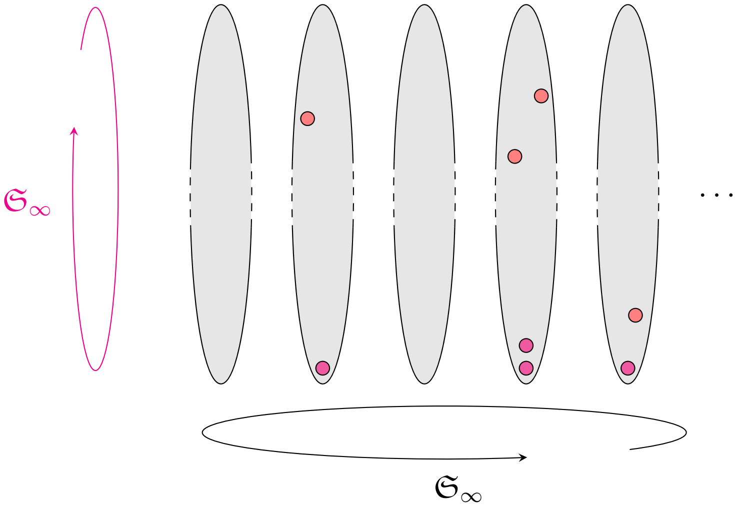

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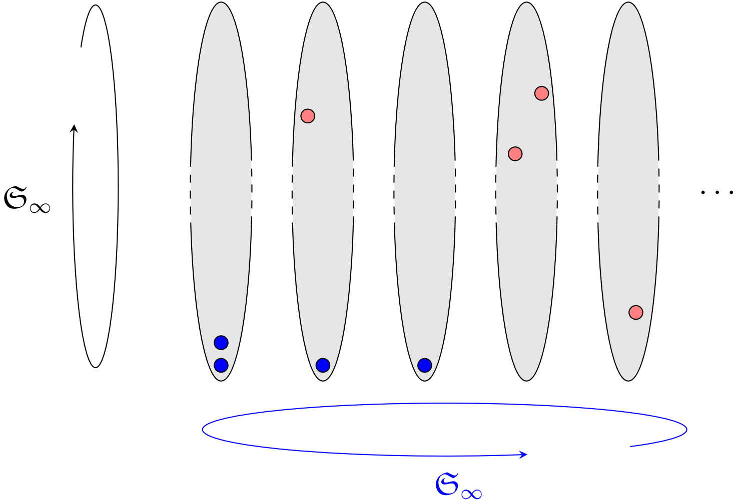

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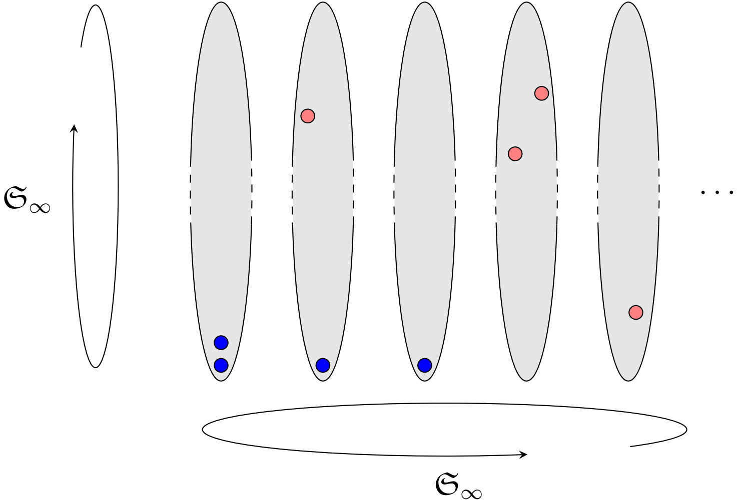

# The tower determines the group (1): "straight $\mathfrak{S}_\infty$"

$G$ contains a set of "straight" swaps of blocks

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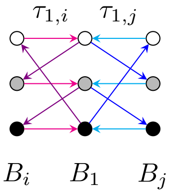
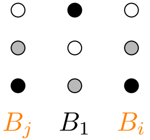



$$\xrightarrow{\tau_{1,i}\tau_{1,j}^{-1}\tau_{1,i}}$$



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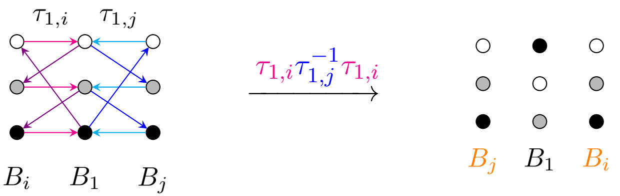
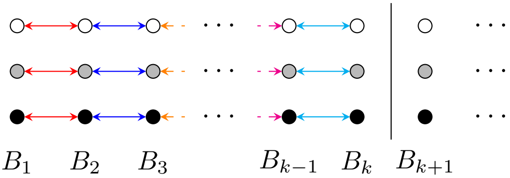



Hence the actions on and within the blocks are independent.

# The tower has shape $H_0, \ H, \ H, \ H \ \cdots$

Lemma to prove

$G$ has tower $H_0 \ H_1 \ H_2 \ H_3 \Rightarrow H_1 = H_2$

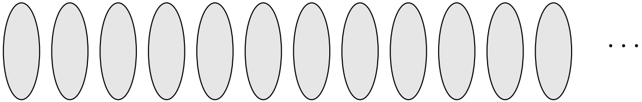
Proof.

An element $s \in G$ stabilizing the blocks $\leftrightarrow$ a quadruple
$g \in H_1 \quad \rightarrow \quad \exists \ (1, g, h, k), \ \ h, k \in H_1$.
Let $\sigma$ be an element of $G$ that permutes "straightforwardly" the first two blocks and fixes the other two.
Conjugation of $x$ by $\sigma$ in $G \quad \rightarrow \quad y = (g, 1, h, k)$
Then: $x^{-1}y = (g, g^{-1}, 1, 1)$
By arguing that the tower does not depend on the ordering of the blocks, $g^{-1}$ and therefore $g$ are in $H_2$.

In the infinite case, apply to each restriction to four consecutive blocks of the fixator of the previous ones in $G$.

# Direct product in the case of finite blocks

"Speak, friend..."

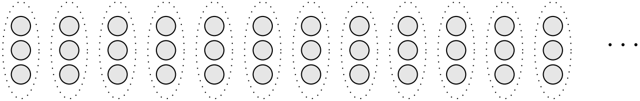


# Direct product in the case of finite blocks

"Speak, friend..."

## Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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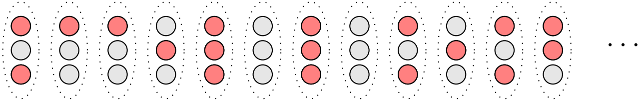

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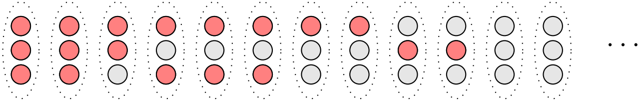

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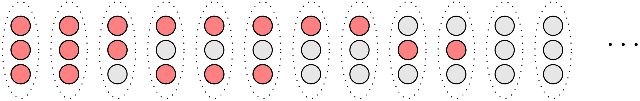

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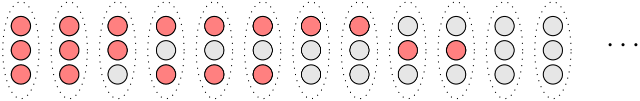

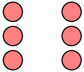

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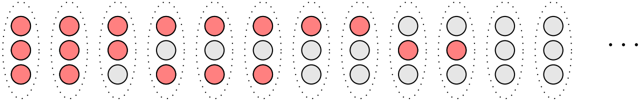


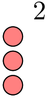

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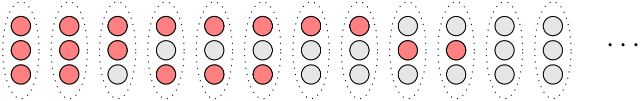

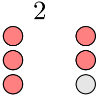

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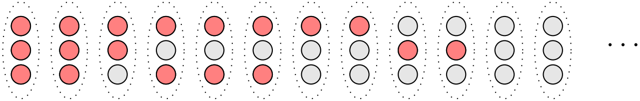

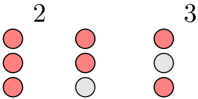

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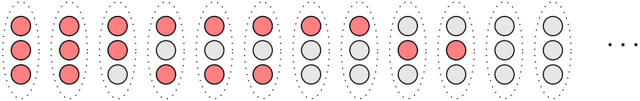

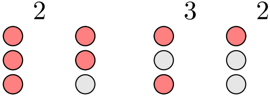

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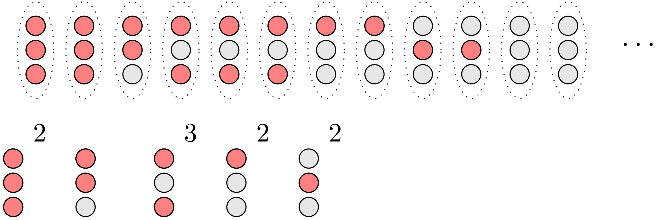

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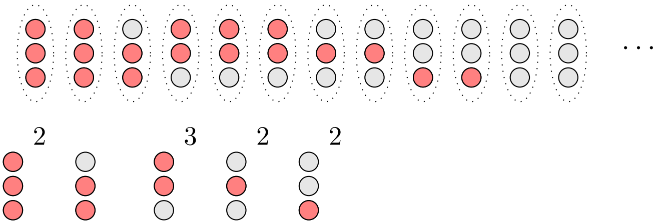

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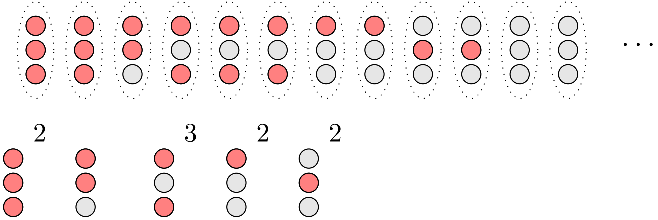

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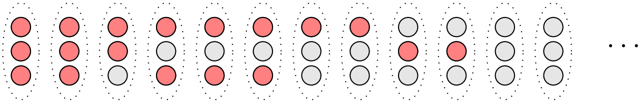

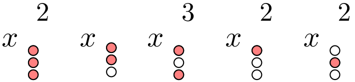

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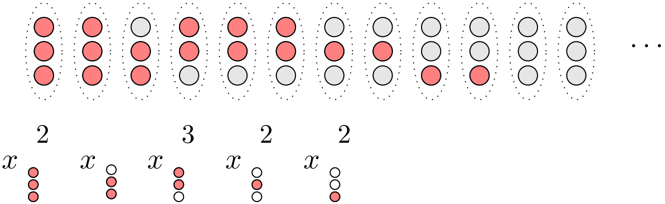

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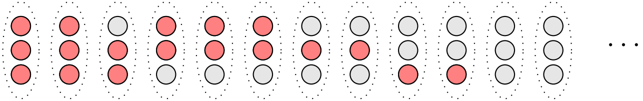

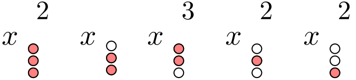

$\rightarrow$ $C_3$ acts on monomials

# Direct product in the case of finite blocks

"Speak, friend..."

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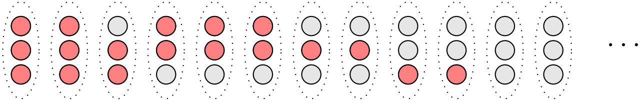

$G' = C_3$ *acting on (non empty) subsets*

$$\mathbb{Q}[\ x\ ]^{G'} \quad \longleftrightarrow \quad \text{Orbit algebra of } C_3 \times \mathfrak{S}_\infty \ ?$$

# Direct product in the case of finite blocks

"Speak, friend..."

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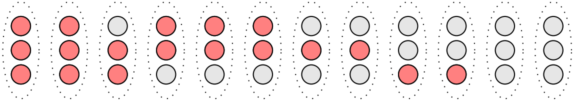 …

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$x$ 

$x$ 

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"Speak, friend..."

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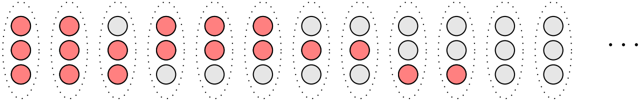

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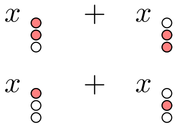

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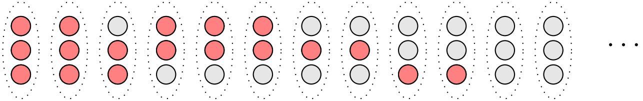

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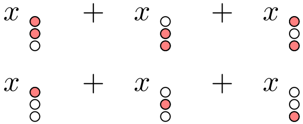

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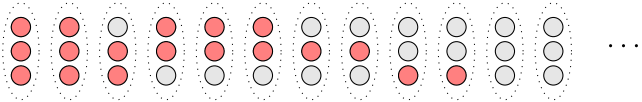

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O( $x$  )

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# Direct product in the case of finite blocks



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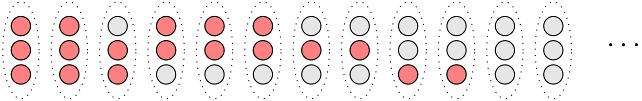

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$\mathrm{O}(\ x\ ).\mathrm{O}(\ x\ )$

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"Speak, friend..."

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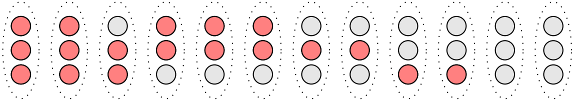

$\ldots$

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$$\mathrm{O}(\ x\ ).\mathrm{O}(\ x\ ) = \ \mathrm{O}(\ x\ x\ )$$

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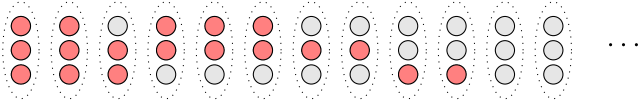
$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$$\mathbb{Q}[\ x\ ]^{G'} \quad \longleftrightarrow \quad \text{Orbit algebra of } C_3 \times \mathfrak{S}_\infty \ ?$$

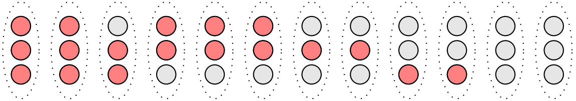
$$\mathrm{O}(x_{\bullet\bullet\circ}).\mathrm{O}(x_{\bullet\circ\circ}) = \mathrm{O}(x_{\bullet\bullet\circ}x_{\bullet\circ\circ}) + \mathrm{O}(x_{\bullet\bullet\circ}x_{\circ\bullet\circ})$$

# Direct product in the case of finite blocks

"Speak, friend..."

## Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$\dots$

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$$\mathrm{O}(\ x\ ).\mathrm{O}(\ x\ ) = \ \mathrm{O}(\ x\ x\ ) + \mathrm{O}(\ x\ x\ ) + \mathrm{O}(\ x\ x\ )$$

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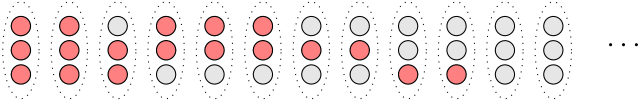

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$\mathrm{O}(\quad).\mathrm{O}(\quad)$

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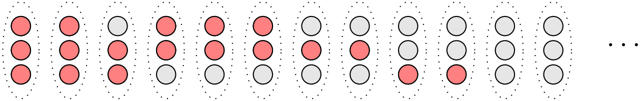
$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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$$\mathrm{O}(\ x_{\substack{\bullet\\\bullet\\\circ}}).\mathrm{O}(\ x_{\substack{\bullet\\\circ\\\circ}}) = \mathrm{O}(\ x_{\substack{\bullet\\\bullet\\\circ}} x_{\substack{\bullet\\\circ\\\circ}}) + \mathrm{O}(\ x_{\substack{\bullet\\\bullet\\\circ}} x_{\substack{\circ\\\bullet\\\circ}}) + \mathrm{O}(\ x_{\substack{\bullet\\\bullet\\\circ}} x_{\substack{\circ\\\circ\\\bullet}})$$

$$\mathrm{O}(\ \substack{\bullet\\\bullet\\\circ}\ ).\mathrm{O}(\ \substack{\bullet\\\circ\\\circ}\ ) = \mathrm{O}(\ \substack{\bullet\\\bullet\\\circ}\ \ \substack{\bullet\\\circ\\\circ}\ )$$

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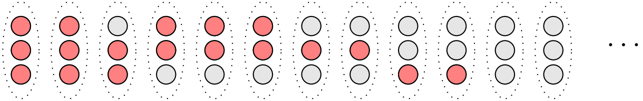

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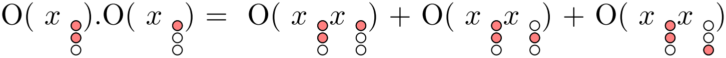
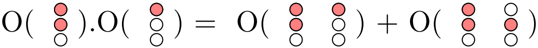


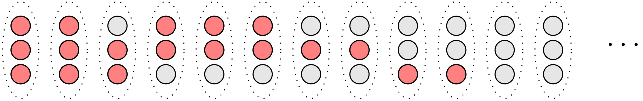



# Direct product in the case of finite blocks



## Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3

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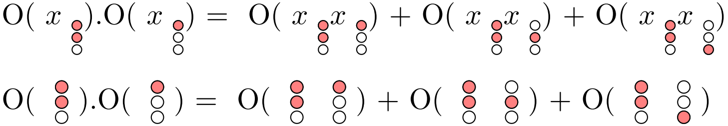
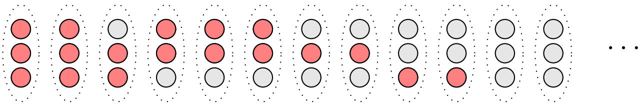


# Direct product in the case of finite blocks

"Speak, friend..."

## Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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