A bijection between 3-dimensional Catalan objects

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3-dimensional Catalan numbers

- (Bidimensional) Catalan numbers:

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Enumerate the standard Young tableaux of shape (n, n, n)

7	8	11	12
3	4	9	10
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Ex. 2 4 3 1 6 7 5

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$$\sigma = 364512 \;\rightarrow\; 215463 \;\rightarrow\; 562314 = S(\sigma)$$

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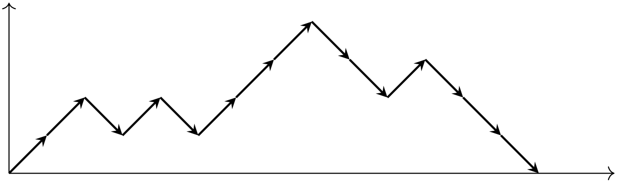
Rem. $1234 \;\to\; 4321 \;\to\; 1234 \;\Rightarrow$ preserves the avoidance

Weighted Dyck paths WD_{2n}

- Dyck path



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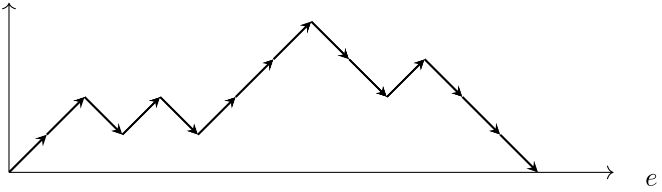

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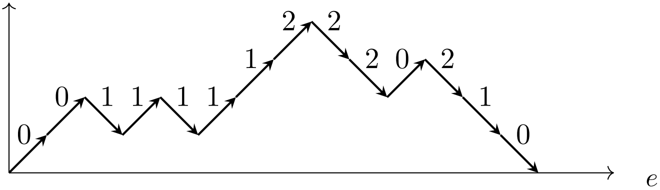

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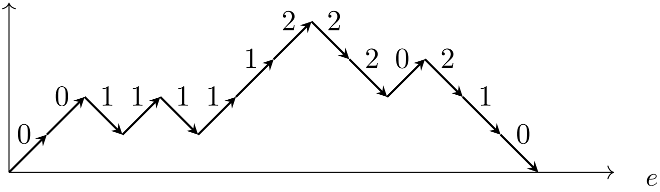

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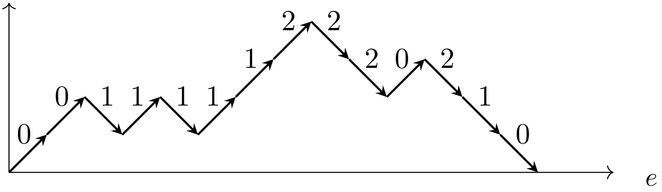

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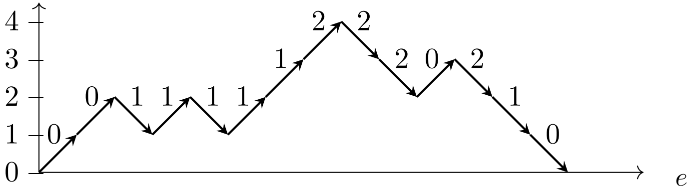

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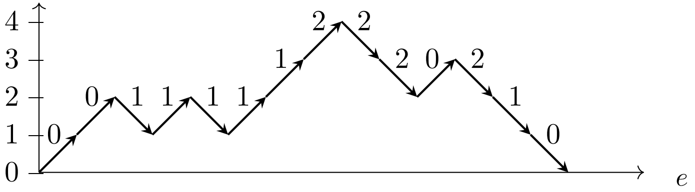

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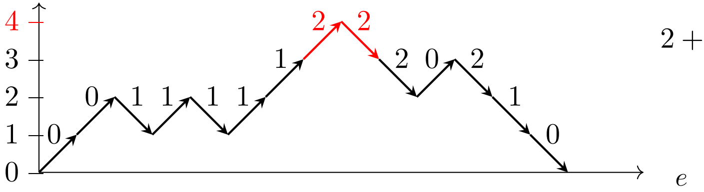

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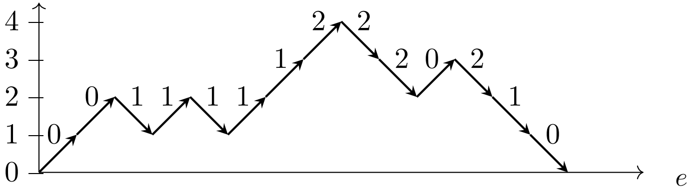

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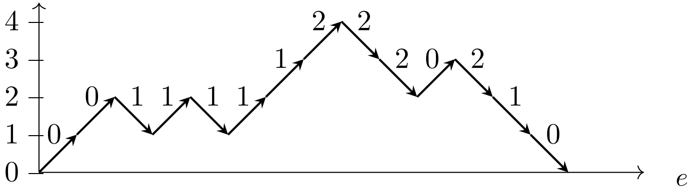

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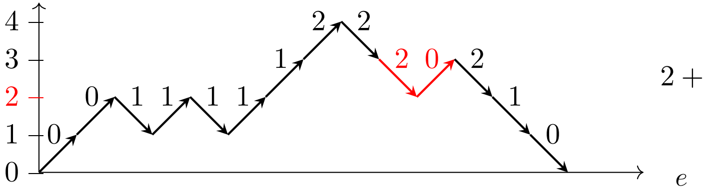

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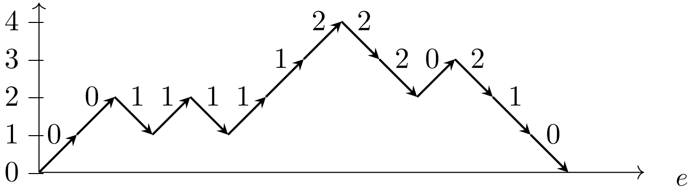

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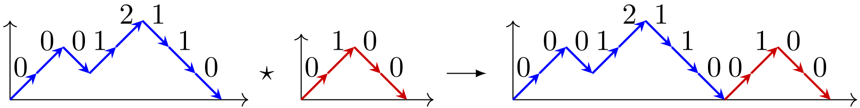

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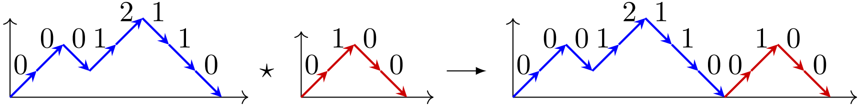

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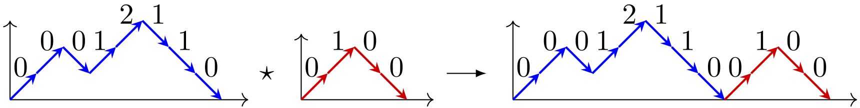

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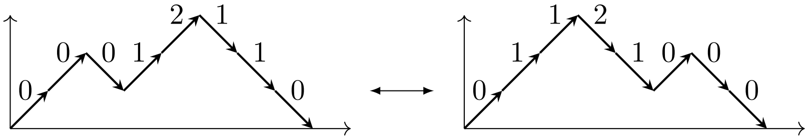

A conjectured statistics-preserving bijection

Conjecture, Borie '17

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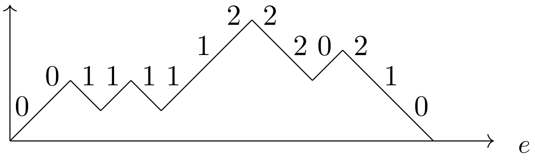
Based on computational exploration (matching numbers of elements according to these statistics).

Principles for our bijection

From a weighted Dyck path wd



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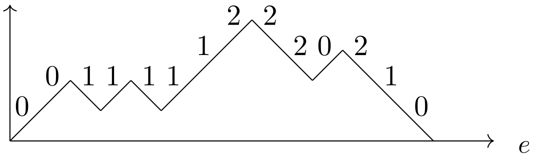

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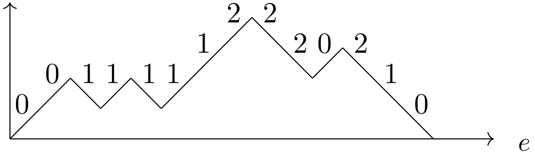

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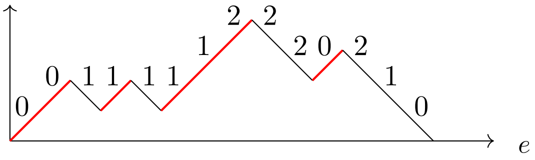

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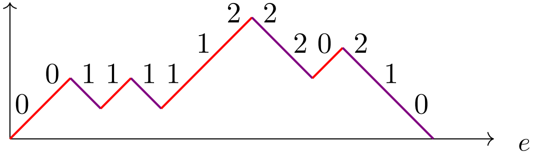
- Second: positions of down-steps, from right to left $\rightarrow Top(\sigma)$



Compatibility with the Schützenberger involution

Assume β is a bijection $WD_{2n} \to A_{2n}(1234)$

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$$\begin{array}{ccc} wd & \xleftrightarrow{\mathrm{S}} & \mathrm{S}(wd) \\ \beta \downarrow & & \downarrow \beta \\ \beta(wd) & \xleftrightarrow[\mathrm{S}]{} & \mathrm{S}(\beta(wd)) \end{array}$$

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Handling of down-steps $\to$ “same” as up-steps

Ideas for the construction of $Bot(\sigma)$

- Observation: $Bot(\sigma)$ must avoid 123

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  $\Rightarrow$ the number of elements to the right of the 1^{st} ascent increases

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  $\Rightarrow$ the number of elements to the right of the 1$^{\text{st}}$ ascent increases
- Idea: the (non-decreasing) weights could measure this number
  $\rightarrow$ if the weight has increased, we insert u at this distance from the right-hand end

$$5 \quad 2 \nearrow \color{red}{6} \quad 1 \quad 4 \quad 3$$

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- Idea: the (non-decreasing) weights could measure this number
  $\rightarrow$ if the weight has increased, we insert u at this distance from the right-hand end; if **not**, at the **left-hand end**.

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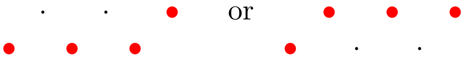
Pattern avoidance (1)

1234 patterns we are able to avoid:



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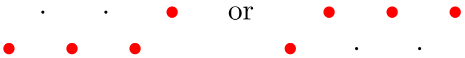

3 other configurations to avoid...

Need of a shift

- This is actually a bijection Borie described between non-decreasing parking functions and 123-avoiding permutations.

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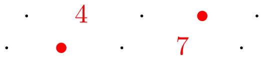

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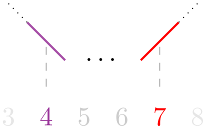

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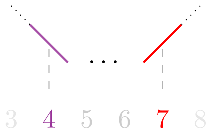

$\rightarrow$ Every down-step d to the left of a given up-step u should be inserted as a top element to the right of u (bottom element).

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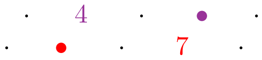
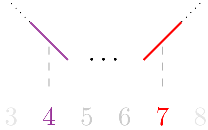


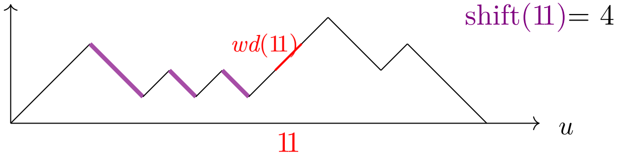


→ Every down-step d to the left of a given up-step u should be inserted as a top element to the right of u (bottom element).

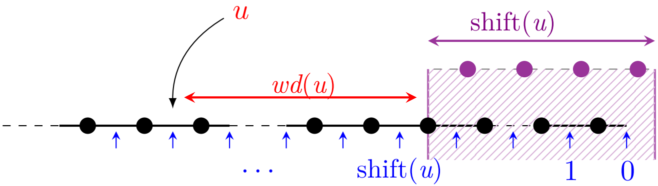
→ shift(u) = # down-steps to the left of the up-step u

We add shift(u) to the weight $wd(u)$ to obtain the insertion position.

Computation of the insertion position



Bottom word



$\longleftarrow$ Insertion position from the right

Jump and valley condition

Recall: Same weight as previous one ⇒ no new leftmost ascent
⇒ **jump**

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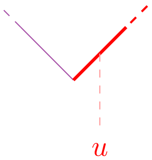
What about valleys ?



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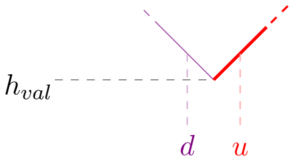

Valley condition to decide the jump: $wd(d) + wd(u) \geqslant h_{val}$

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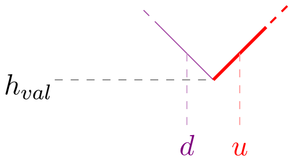

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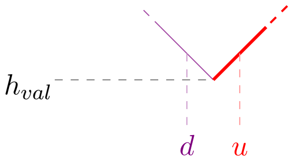

Valley condition to decide the jump: $wd(d) + wd(u) \geqslant h_{val}$

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$\rightarrow$ General jump rule:

compare $wd(u)$ to its minimal possible value to decide the jump

Pattern avoidance (2)

1234 patterns we are able to avoid:

 or  and 

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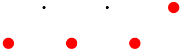
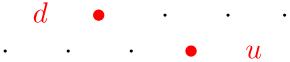
 or  and 

The valley condition will naturally prevent:



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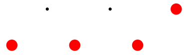
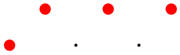
 or  and 

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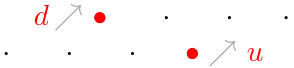

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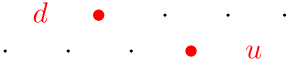
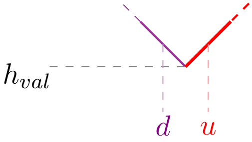


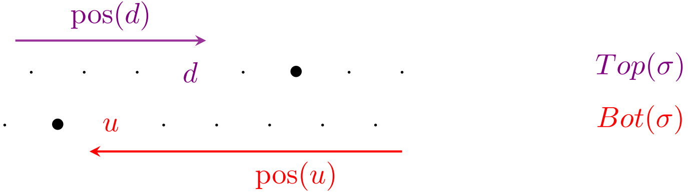
And the peak condition:

Valley condition and insertion position



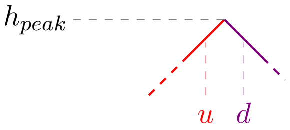
$$wd(d) + wd(u) \geqslant h_{val}$$

$$u > d$$

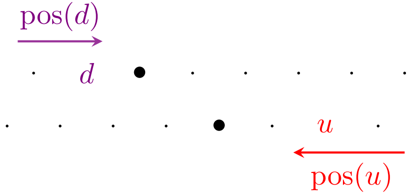


Peak condition and pattern avoidance

Similarly, the peak condition should prevent the last pattern.



$$wd(d) + wd(u) \leqslant h_{peak}$$
$$u < d$$



Something to fix

Heuristical solution for every possible pattern situation...

Is this over?

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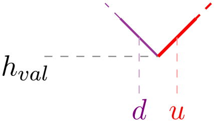

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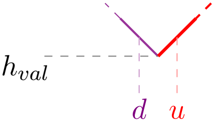

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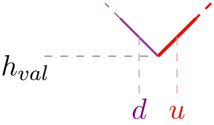

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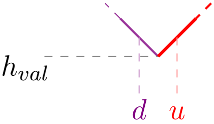

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$\rightarrow$ Not bijective!

Solution

We used to compare every weight to its minimal value

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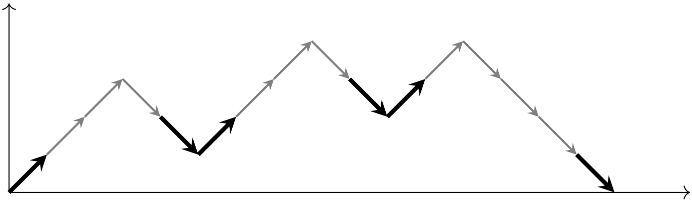
Until now:



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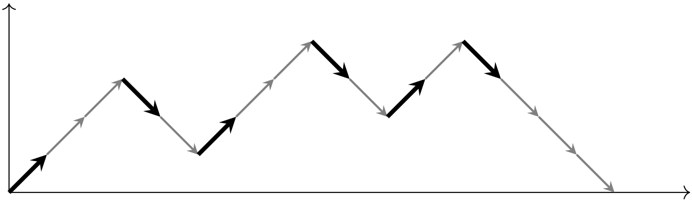
Another idea...



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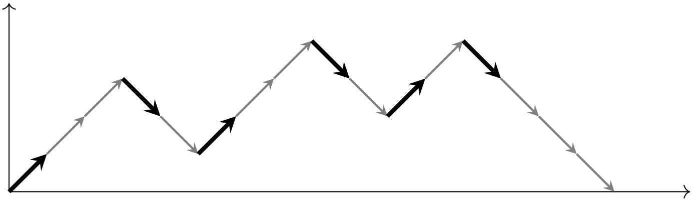
Another idea... Not compatible with S!



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Adjacent slopes (one upward, one downward)

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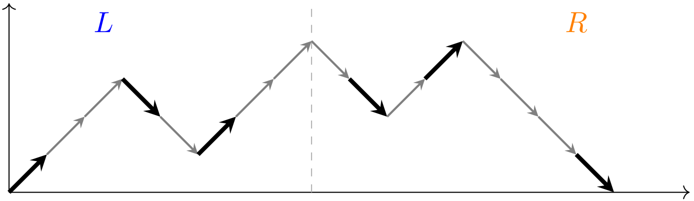
Finally:



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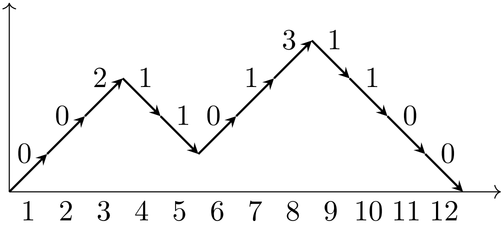
Rem. The image permutation being up-down is checked by a more thorough look at every distance involved.

An example



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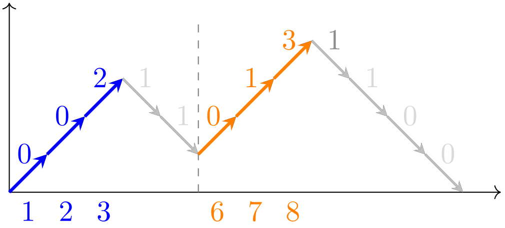

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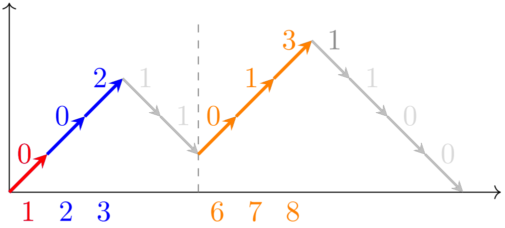
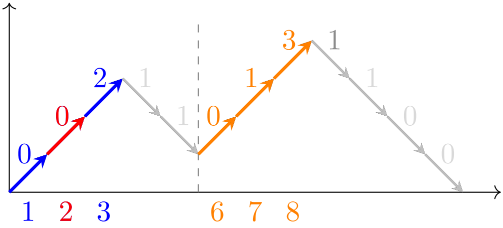


1

An example



1

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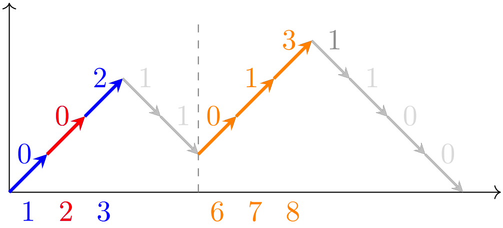
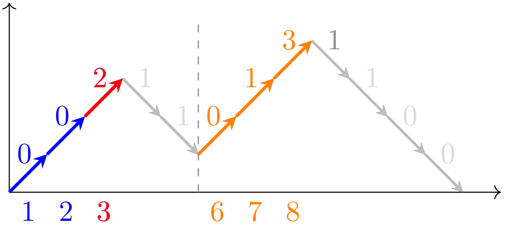


$0 = 0 \quad \rightarrow \quad 2$ jumps

2 1

An example



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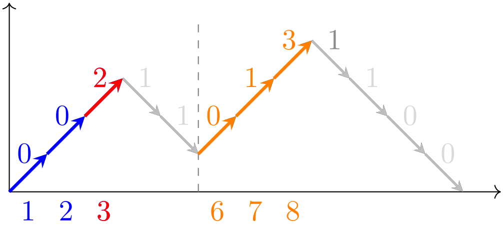

$$2 \neq 0 \quad \rightarrow \quad \mathrm{pos}(3) = 2$$

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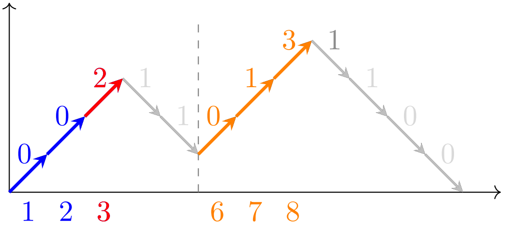

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$$2 \quad 3 \quad 1$$

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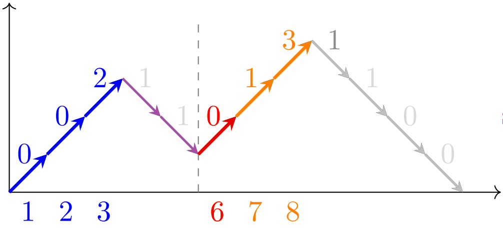

shift(6) = 2

$2 \neq 0 \quad \rightarrow \quad \mathrm{pos}(3) = 2$ - 1

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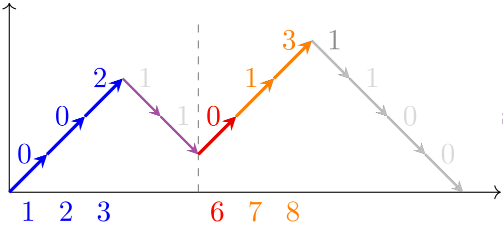
An example

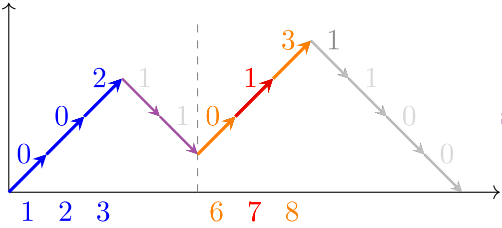


$\text{shift}(6) = 2$

$$0 \neq 1 \quad \rightarrow \quad \text{pos}(6) = 0 + 2$$

2 6 3 1

An example

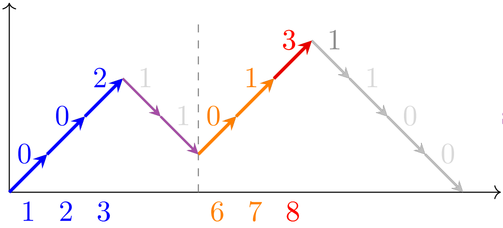


$\mathrm{shift}(6) = 2$

$$1 \neq 3 \quad \rightarrow \quad \mathrm{pos}(7) = 1 + 2$$

2 7 6 3 1

An example

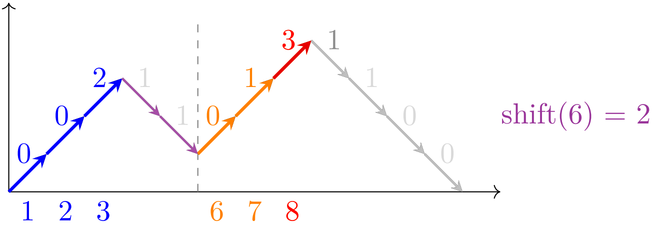


shift(6) = 2

3+1 = 4

2 7 6 3 1

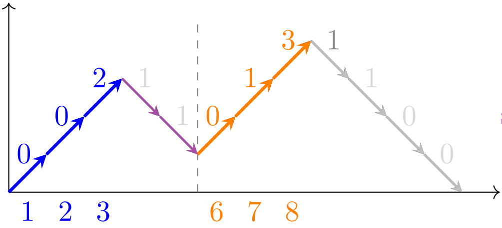
An example



$$3{+}1 = 4 = h_{peak} \quad \to \quad 8 \text{ jumps!}$$

8 2 7 6 3 1

An example



shift(6) = 2

$Top(\sigma)$ is obtained by scanning the down-steps from right to left

10	12	11	9	5	4
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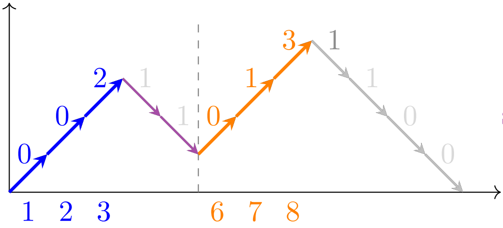
An example



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Thank you for your attention!