Thèse de Doctorat

Classification of P-oligomorphic groups
Conjectures of Cameron and Macpherson

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Justine Falque

Composition du jury :

M.	Peter CAMERON	Professor Emeritus Queen Mary Univ. of London & Univ. of St Andrews School of Mathematical Sciences	Rapporteur
M.	Pascal WEIL	Directeur de Recherche CNRS, Université de Bordeaux LaBRI	Rapporteur
Mme	Isabelle GUYON	Professeure Université Paris-Sud LRI	Présidente du Jury
M.	Maurice POUZET	Professeur émérite Université Claude Bernard Lyon 1 Institut Camille Jordan	Examinateur
M.	Christophe TOLLU	Maître de Conférences Université Paris-Nord LIPN	Examinateur
Mme	Annick VALIBOUZE	Professeure Sorbonne Universités LIP6	Examinatrice
M.	Nicolas M. THIÉRY	Professeur Université Paris Sud LRI	Directeur

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Drôle d'aventure que la préparation d'une thèse, mais on ne peut certes pas dire qu'elle manque de saveur !

Introduction

Contexte et contribution de la thèse

Avant-propos sur la combinatoire et la combinatoire algébrique

Traditionnellement, la combinatoire est définie comme l'«art de compter», avec des applications allant de l'informatique théorique (compter le nombre d'opérations nécessaires à l'exécution d'un algorithme) à la physique statistique (compter le nombre d'états possibles d'un système), en passant bien entendu par les mathématiques.

Souvent, la quantité à compter est fonction d'un entier n (typiquement la taille des données en entrée pour un algorithme, ou la taille du système en physique). On aura donc une séquence infinie $u_0, u_1, u_2, \cdots$ de nombres, dont on pourra étudier le comportement quand n devient grand (par exemple pour déterminer la complexité de l'algorithme, ou le comportement macroscopique d'un système). Pour manipuler une telle suite, il est commode d'en regrouper tous les termes sous la forme d'une *série formelle* :

$$H(z) = u_0 + u_1 z + u_2 z^2 + \cdots .$$

Très souvent, les propriétés définissant la suite se traduisent sous forme d'équations (algébriques, différentielles, ...) sur la série. Par exemple, u_n vérifie une relation de récurrence linéaire si et seulement si $H(z)$ est une fraction rationnelle :

$$H(z) = \frac{P(z)}{Q(z)} .$$

Ces observations ouvrent la porte à des manipulations algébriques pour étudier le comportement (en particulier asymptotique) de la séquence de départ : on entre dans le domaine de la combinatoire algébrique.

Cette branche de la combinatoire, comme son nom l'indique, utilise des outils algébriques pour étudier des problèmes de combinatoire — et vice-versa. Elle cherche notamment souvent à ajouter de la structure sur les objets étudiés : par exemple une structure d'algèbre en définissant un produit adapté, une relation d'ordre, ou bien d'autres relations plus générales.

La recherche puis l'exploitation de telles structures permettent de voir l'objet sous un jour nouveau, de mettre en évidence son agencement interne, et globalement de mieux le comprendre.

Inversement, faire apparaître un point de vue combinatoire sur des objets algébriques (comme par exemple associer des partitions aux caractères d'un groupe) permet une meilleure visualisation de certains problèmes, et facilite une approche algorithmique.

Profils des structures relationnelles

Revenons à nos fonctions de comptage et aux séries qui les encodent. Une instance classique de cette situation est le comptage du nombre $\varphi(n)$ de *graphes* simples sur n sommets à un isomorphisme près. On obtient

$$H_\varphi(z) = 1 + z + 2z^2 + 4z^3 + 11z^4 + \cdots$$

Ce problème peut être reformulé comme suit. On considère le graphe de Rado R, soit le graphe infini obtenu avec probabilité 1 lorsque, pour chaque paire (v, v') de sommets dans un ensemble infini dénombrable, on tire à pile ou face l'existence d'une arête reliant v à v'. Intuitivement, on définit un sous-graphe induit (du graphe de Rado par exemple) comme un sous-ensemble de sommets doté des même arêtes que dans le graphe d'origine.

Proposition 0.0.1. *Tout graphe simple apparaît, à isomorphisme près, comme sous-graphe induit du graphe de Rado R.*

Il s'ensuit que $\varphi(n)$ peut aussi être défini comme le nombre de sous-graphes induits à n sommets de R, à un isomorphisme près.

Plus généralement, on considère une *structure relationnelle R*, c'est-à-dire un ensemble infini dénombrable muni d'une ou plusieurs relations (binaires ou non). Le *profil φ* de la structure relationnelle R compte, pour chaque entier n, le nombre $\varphi(n)$ de sous-structures de R induites sur des sous-ensembles de taille n, à un isomorphisme près.

Ce cadre très général contient comme cas particuliers les problèmes suivants, qui ont été étudiés indépendamment:

- En *combinatoire des mots* [Lot97] : l'étude de la *complexité d'un mot infini*, c'est-à-dire le nombre de ses facteurs de taille n;

- En *combinatoire des permutations* [KK02; AMB07] : l'énumération de *classes de permutations*, ensembles de permutations définies par évitement de motifs;

- En *théorie des graphes, des ordres partiels, des tournois* [BBM06; Bal+09; BP10] : comptage des sous-graphes finis d'un graphe infini à isomorphisme près;

- En *théorie des groupes* [Cam09; Mac85a; Mac85b] : étude du profil orbital d'un groupe de permutations infini, comptage des orbites d'un groupe.

Pour être précis, il faudrait dans certains cas élargir le cadre aux *classes héréditaires de structures finies*; se reporter à [Kla08; Bol98] pour une vue d'ensemble.

Comportement asymptotique et conjecture de Cameron

L'étude des profils de structures relationnelles constitue un domaine de recherche à part entière depuis les années 1970 [Fra00; Pou06]. Leur comportement asymptotique en particulier a été largement étudié. On sait par exemple ([Pou76][Cam90, Section 3.1]) qu'un profil est toujours croissant au sens large. Les recherches rapportent également, dans chacun des cas mentionnés plus haut, un phénomène de «saut» dans les possibilités de comportement asymptotique du profil : sa croissance semble en effet être soit polynomiale au sens faible : $an^k \leqslant \varphi(n) \leqslant bn^k$ à partir d'un certain rang de n et pour certains a, b et k; soit plus rapide que tout polynôme [Pou78; Pou06; BBM06; Kla08]. Pouzet a démontré que ce phénomène apparaît pour toute structure relationnelle dès lors qu'elle satisfait quelques conditions naturelles.

Des résultats concordant dans les différents contextes [PT13; KK02; Bal+09] suggèrent que l'on pourrait renforcer ce théorème:

Conjecture 0.0.2. *Si un profil est à croissance bornée par un polynôme, alors il est équivalent à un polynôme : $\varphi(n) \sim an^k$; de plus, f est un quasi-polynôme:*

$$\varphi(n) = a_k(n)n^k + \cdots + a_0(n)$$

où les coefficients $a_i(n)$ sont des fonctions périodiques.

Notons au passage que, si un profil se trouve être un quasi-polynôme, alors il est automatiquement équivalent à un polynôme en invoquant le résultat de Pouzet selon lequel le profil est croissant.

Dans le cadre restreint des profils orbitaux, soit quand le profil compte les orbites de sous-ensembles d'un groupe de permutations, cette question a été soulevée par Cameron dans [Cam90, Section 3.6].

Cas des groupes oligomorphes

Compter des objets sous une action de groupe est un thème récurrent en combinatoire, ce qui fait des profils orbitaux un cas particulièrement intéressant. Dans un contexte où nous nous intéressons aux valeurs du profil et à son comportement, il est naturel de nous restreindre aux groupes dont le profil ne prend que des valeurs finies, dits *oligomorphes*.

Il est à noter que ces groupes apparaissent naturellement dans le cadre de la *théorie des modèles*. En effet, comme noté par Cameron dans [Cam90, Section 2.5], une structure relationnelle dénombrable R est $\aleph_0$-catégorique (ce qui signifie que sa théorie est $\aleph_0$-catégorique, ou encore dans ce cas que R est l'unique modèle dénombrable de sa théorie) si et seulement si son groupe d'automorphismes est oligomorphe.

L'une des contributions de la présente thèse est de démontrer la conjecture de Cameron sur le profil des groupes oligomorphes. Elle se présente en réalité comme le corollaire d'une autre conjecture, que nous allons maintenant nous attacher à introduire.

Structure d'algèbre et conjecture de Macpherson

Comme il est bien connu (voir par exemple [Sta97]), la quasi-polynomialité d'une séquence se traduit sur la série génératrice associée par le fait qu'elle soit de la forme

$$\frac{P(z)}{(1 - z^{d_1})(1 - z^{d_2}) \cdots (1 - z^{d_k})} \ , \tag{0.1}$$

avec $1 = d_1 \leqslant \cdots \leqslant d_k$ et $P(z) \in \mathbb{Z}[z]$.

Les séries de cette forme apparaissent naturellement en *algèbre commutative*. Soit en effet A une algèbre commutative graduée connexe. Si A est de type fini, alors sa *série de Hilbert* est nécessairement de la forme (0.1), où les d_i sont les degrés d'un système minimal de générateurs homogènes [CLO97, Chapter 9, §2]; la réciproque n'est pas nécessairement vraie.

Ce type de lien entre algèbre et propriétés du profil a motivé l'introduction par Cameron d'une structure d'algèbre sur les orbites des groupes de permutation, appelée *algèbre des orbites* $\mathbb{Q}[\mathcal{A}_G]$ (plus généralement *algèbre d'âge* $\mathbb{Q}[\mathcal{A}_R]$ pour une structure relationnelle R quelconque) dont la série de Hilbert coïncide avec celle du profil [Cam97].

Une approche possible pour montrer que le profil est un quasi-polynôme dans le cas où il est borné par un polynôme est alors d'étudier l'algèbre des orbites : si elle est de type fini, le profil est un quasi-polynôme équivalent à un polynôme.

La question suivante se pose alors assez naturellement.

Question 0.0.3. Soit R une structure relationnelle à profil borné par un polynôme. Son algèbre d'âge $\mathbb{Q}[\mathcal{A}_R]$ est-elle de type fini?

La réponse à la question 0.0.3 est négative en général (par exemple, les algèbres d'âge de tournois sont de type fini si et seulement si le profil est borné [Pou06, Theorem 27]); dans le cas des groupes, elle a fait l'objet d'une conjecture due à un ancien étudiant de Cameron, Macpherson — lequel, comme souligné par un des rapporteurs de [FT18], ne se sentait pas alors assez confiant pour employer le terme de «conjecture», lui préférant celui de «question».

Conjecture 0.0.4 (Macpherson, 1985 [Mac85a])**.** *Soit G un groupe de permutation dont le profil est borné par un polynôme. Alors, son algèbre des orbites $\mathbb{Q}[\mathcal{A}_G]$ est de type fini.*

Cette conjecture, plus forte donc que celle de Cameron, est l'objectif central de cette thèse. Le cas particulier d'un profil borné avait déjà été résolu par Pouzet, qui avait par ailleurs prouvé que l'algèbre des orbites était un domaine intégral dès lors que le groupe n'avait pas d'orbite (d'éléments) finie [Pou08].

Résultats de la thèse

La présente thèse démontre la conjecture de Macpherson, et avec elle la conjecture de Cameron.

Entre autres notions et outils, l'un des ingrédients essentiels de la démonstration dans le cas général est la notion de *systèmes de blocs*; leur étude dans ce but a

été inspirée par les décompositions monomorphes de structures relationnelles, qu'ils généralisent. En effet, la génération finie de l'algèbre d'une structure relationnelle dotée d'une décomposition monomorphe finie admet une caractérisation combinatoire découverte par Pouzet et Thiéry [PT18]; ce cas était donc très bien connu.

La méthode initialement utilisée reposait ensuite très largement sur la *théorie des invariants* : une adaptation de la preuve du théorème de Hilbert sur l'algèbre des invariants a permis de démontrer que la propriété de génération finie de l'algèbre se relevait à tout surgroupe d'indice fini, ouvrant la voie à une série de réductions successives du problème à des sous-groupes d'indices finis, jusqu'à se ramener à un cas plus simple. Une démarche expérimentale, c'est-à-dire une exploration informatique des cas les plus délicats, a par ailleurs aidé à franchir certains obstacles résistants et s'est révélée cruciale pour compléter la preuve.

Bien que cette première preuve soit intéressante en elle-même, c'est finalement une histoire légèrement différente qui sera narrée dans ce manuscrit. Les résultats suggérés par l'approche informatique y restent cependant déterminants.

De fait, après une résolution positive des conjectures, la question un peu plus fine de la propriété de *Cohen-Macaulay* pouvait être abordée. L'un des intérêts de cette propriété est que le numérateur de la série de Hilbert est dans ce cas à coefficients entiers naturels.

L'idée était légitimée par divers cas particuliers : les algèbres d'invariants de groupes finis, notamment, sont réalisables comme algèbres des orbites, et ont cette propriété bien connue. L'exploration de ce problème, qui demande une compréhension plus exhaustive de la structure de l'algèbre, a conduit à repenser la preuve originale des conjectures, et a mené à une *classification* des groupes oligomorphes à profil borné par un polynôme (que nous appelons *P-oligomorphes*), notre résultat le plus important. Informellement, un groupe *P*-oligomorphe (clos) est décrit de manière unique par un groupe de permutations fini doté d'un système de blocs dont chacun est décoré par une paire de groupes — l'un fini, l'autre infini — obéissant à un système de critères précis. La classification des groupes se traduit naturellement par une classification des algèbres correspondantes, qui sont en fait (à un quotient pour ainsi dire inoffensif près) des algèbres d'invariants à graduation non standard.

Theorem 0.0.5. *Si G est un groupe P-oligomorphe, son algèbre des orbites $\mathbb{Q}[\mathcal{A}_G]$ est isomorphe à l'algèbre d'invariants d'un certain groupe fini $G_{<\infty}$, agissant sur des variables dont les degrés se déduisent de G, quotientée par $x^2 = 0$ pour certaines de ces variables x. En particulier, $\mathbb{Q}[\mathcal{A}_G]$ est de Cohen-Macaulay, et le profil est un quasi-polynôme équivalent à un polynôme.*

Les groupes P-oligomorphes se révèlent donc être une classe de groupes nettement plus rigide que prévu, et leurs algèbres, finalement bien connues, ne constituent pas fondamentalement une nouvelle classe d'algèbres commutatives.

Cette classification est l'angle sous lequel se trouve finalement présenté notre travail. En plus d'admettre les deux conjectures comme corollaires directs, elle amène une compréhension profonde des groupes P-oligomorphes et de leurs algèbres. En particulier, en procurant un encodage fini et explicite de ces groupes, elle permet de les implémenter efficacement et de calculer leur profil en utilisant une généralisation

de l'*énumération de Pólya.* Les groupes P-oligomorphes peuvent donc désormais être construits, manipulés, denombrés.

Contenu et plan de la thèse

Cette thèse est divisée en trois grandes parties, plus une annexe. La première partie est consacrée aux préliminaires : elle expose les prérequis des différents domaines impliqués, et fournit des exemples qui deviendront partie prenante de notre travail et de la résolution des conjectures.

Combinatoire algébrique : comptage, séries, produits; treillis

Nous commençons par introduire quelques notions fondamentales de combinatoire algébrique : ensembles gradués et fonctions de comptage, dont les profils sont une instance, ainsi que leurs séries génératrices. Nous donnons des exemples, et nous basculons résolument du côté «algébrique» en introduisant la notion de produits gradués, par lesquels nous pouvons doter nos ensembles gradués d'une structure d'algèbre pertinente. Nous explorons les interactions entre les algèbres graduées et leurs séries de Hilbert, qui constituent le cas échéant une mine d'informations sur la fonction de comptage.

Dans une section un peu à part, nous abordons les notions d'ordre, de poset et de treillis. Nous donnons quelques propriétés de base de ces objets, qui nous serviront par la suite, ainsi que l'exemple qui nous concerne particulièrement du treillis des partitions d'ensemble.

Théorie des groupes et des invariants

Dans un second chapitre, nous nous penchons sur le cas des groupes de permutations, et plus généralement des actions de groupes, dont nous évoquons les propriétés fondamentales. Nous citons l'exemple, crucial, de l'action induite sur les sous-ensembles.

Nous abordons également les concepts de classes, d'indice et de normalité, dont nous serons appelés à nous servir abondamment.

La section suivante est consacrée à la théorie des invariants : nous introduisons la notion d'algèbre d'invariants d'un groupe de permutations fini, et nous mettons en lumière leurs propriétés structurelles, en particulier celle d'être de Cohen-Macaulay.

Enfin, nous consacrons une section à l'énumeration de Pólya, une méthode extrêmement efficace de dénombrement d'objets sous une action de groupe; forts du résultat de classification que nous aurons obtenu d'ici la fin de ce document, nous pourrons mettre cette méthode à profit pour calculer les profils des groupes P-oligomorphes de manière systématique.

Structures relationnelles et groupes oligomorphes, algèbres des orbites; conjectures

Dans le chapitre suivant, qui ouvre la deuxième grande partie de ce manuscrit, nous plongeons au cœur du sujet qui nous intéresse plus directement, et entreprenons d'exposer les problématiques à l'origine de cette thèse. Nous introduisons tout d'abord les structures relationnelles et leurs profils, ainsi que leurs algèbres d'âge, et nous dressons un état de l'art sommaire du domaine.

Nous glissons ensuite vers le cas particulier des groupes oligomorphes: les profils des structures relationnelles homogènes comptent en fait les orbites d'un groupe; et les profils orbitaux peuvent tous être réalisés comme profils d'une structure homogène. Nous donnons des exemples et présentons ici la conjecture de Cameron.

Avant de doter l'ensemble des orbites de sa structure d'algèbre graduée, nous considérons la notion de clôture d'un groupe pour la topologie de la convergence simple, et concluons que nous pouvons, et devrions, dans le cadre de l'étude des âges et profils, travailler sous l'hypothèse de clôture des groupes. En effet, prendre la clôture d'un groupe ne change pas ses orbites de sous-ensembles.

Nous présentons ensuite une construction de l'algèbre des orbites (un peu différente de celle que nous avons donnée pour les algèbres d'âge, quoique les deux se répondent), ainsi que quelques propriétés pratiques : en particulier, nous décrivons les comportements de l'algèbre vis-à-vis des restrictions, sous-groupes, produits directs... Enfin, nous énonçons la conjecture, ou plutôt question, de Macpherson, ainsi que la résolution du cas particulier d'un profil borné par Pouzet.

Préparation du terrain par des études de cas et quelques notions de théorie de groupes

Nous enchaînons avec un chapitre qui engage la transition vers le travail personnel de l'auteure de la thèse. Les résultats présentés ici ne se prétendent pas originaux et les notions préexistent, mais bon nombre de preuves sont de la main de l'auteure et les raisonnements menant à la réalisation de nos objectifs se trouvent amorcés. En particulier, ce chapitre s'intéresse à la mise en œuvre d'une stratégie largement usitée, à savoir celle qui consiste à tenter de séparer le problème pour l'étudier sur des cas le plus élémentaires possible, avant d'en déduire le cas général.

La première section constitue une étude de cas, naturel car relativement «simple», et qui s'avérera plus essentiel que prévu : celui des *produits en couronne*. Nous en donnons la définition et les abordons sous l'angle des conjectures, qui trouvent chez eux une résolution presque immédiate dès lors qu'ils présentent une certaine forme.

La seconde section introduit *la* notion centrale de notre preuve, dont l'examen a été encouragé par celle de décomposition monomorphe des structures relationnelles,

comme mentionné plus haut : les systèmes de blocs des groupes de permutations.
Nous en citons quelques exemples utiles, à la suite de quoi nous nous intéressons
au cas des groupes *primitifs*, soit ceux qui n'en possèdent pas. Sous l'hypothèse
de P-oligomorphie, ces groupes, dits *hautement homogènes*, ont un profil qui vaut
uniformément 1, et sont classifiés (à clôture près) — ce qui facilitera grandement
notre entreprise.

Un dernier élément nous est nécessaire : le sous-produit direct. Cette notion
de théorie des groupes (en l'occurrence) formalise le concept de synchronisation
entre deux ensembles stables sur lesquels un groupe agit simultanément, comme
par exemple deux orbites d'éléments. Comme il s'avère qu'une synchronisation est
déterminée par un sous-groupe normal, et que les groupes primitifs P-oligomorphes
en sont très pauvres, ces groupes auront un comportement prévisible et agréable
sous ce rapport.

Première étape de la preuve : étude des systèmes de blocs

Toute la suite de la thèse constitue un travail original. Les trois chapitres suivants,
rassemblés dans une troisième grande partie, sont consacrés à la preuve proprement
dite de la classification des groupes P-oligomorphes, et par corollaire des conjectures
présentées plus haut.

Dans ce premier chapitre, nous explorons les informations apportées par la don-
née d'un système de blocs; en particulier, en désignant un surgroupe plus simple
(essentiellement un produit direct de produits en couronne), un tel système fournit
une borne inférieure sur la croissance du profil. Nous amenons aussi doucement
l'idée de considérer des blocs de blocs dans ce même objectif.

Nous exploitons ensuite les structures de treillis sur l'ensemble des systèmes de
blocs (finis, infinis, ou sans restriction) pour tenter de maximiser la borne obtenue,
afin de choisir un système d'étude du groupe le plus pertinent possible.

En dernier lieu, nous en venons à notre construction phare, celle d'un système
de blocs infinis de blocs finis (sobrement nommés *superblocs*) soigneusement sélec-
tionné et défini de manière unique, que nous appelons le *système emboîté*. En plus
d'optimiser la borne, il a l'immense mérite de faire apparaître des groupes haute-
ment homogènes partout où cela est possible, facilitant de ce fait la dissection des
synchronisations au sein du groupe.

Deuxième étape : classification sur la brique de base

Ce chapitre est consacré au cas d'un groupe dont le système emboîté consiste en un
seul superbloc. Nous rappelons quelques exemples naturels, comme le produit en
couronne d'une part, dans lequel les actions au sein de chacun des blocs finis sont
toutes indépendantes; d'autre part l'exact opposé, avec une seule action diagonale,
c'est à dire simultanée, sur le contenu des blocs; enfin, la situation intermédiaire

d'un produit en couronne sur lequel on aurait ajouté une action diagonale supplémentaire. Nous annonçons dans la foulée un résultat de classification dans ce cas particulier, qui affirme très simplement que ces exemples contiennent en fait tous les cas de figure. La section suivante en entreprend la preuve.

La première étape consiste à étudier la manière dont les blocs finis permutent entre eux. Elle établit que nous pouvons, modulo renumérotation, considérer que toutes leurs permutations peuvent s'effectuer sans modifier l'ordre des éléments de chacun des blocs (ce que nous appelons de manière imagée le lemme de l'échelle), donc en particulier sans interaction avec l'action du groupe sur le contenu des blocs.

En vertu de quoi, nous pouvons ensuite et indépendamment nous intéresser au stabilisateur des blocs, seul vecteur de l'action du groupe au sein des blocs. Cette seconde étape définit la notion de tour d'un groupe mono-superbloc, destinée à observer les synchronisations internes de cette action.

Nous classifions les tours de ces groupes, aidés en cela, d'une manière qui est transparente pour le lecteur mais sur laquelle nous reviendrons plus tard, de l'exploration informatique de nombreux exemples. Les tours se révèlent avoir une forme extrêmement rigide et dont toute l'information tient en deux groupes finis : la restriction à un bloc et un sous-groupe normal de cette restriction, qui est en fait la restriction à un bloc en supposant qu'un autre est fixé point par point.

En invoquant le sous-produit direct, nous montrons ensuite que la tour détermine entièrement le stabilisateur, et donc le groupe entier puisque le groupe agissant par permutation des blocs est essentiellement connu — ce qui permet de relever la classification des tours à celle des groupes eux-mêmes.

Troisième étape : généralisation de la classification; résolution des conjectures et autres bénéfices immédiats

Dans le dernier chapitre de cette troisième partie, nous finalisons la preuve de la classification en deux étapes : nous exhibons le sous-groupe normal d'indice fini minimal K à partir du système emboîté, puis nous l'utilisons pour mettre en évidence la structure de produit du groupe P-oligomorphe dont il est issu, et de là définir un encodage fini de ce groupe. Nous montrons ensuite que cet encodage classifie intégralement les groupes P-oligomorphes clos (ou encore les âges des groupes P-oligomorphes).

L'algèbre des orbites de K est de Cohen-Macaulay, et nous en connaissons une décomposition de Hironaka; comme le groupe entier G agit par permutation sur les générateurs, en nombre fini, de cette algèbre, l'algèbre de G se trouve donc (à un quotient naturel près) être une algèbre d'invariants. En particulier, elle est de Cohen-Macaulay, et les conjectures de Macpherson et de Cameron sont démontrées. Nous en profitons pour citer quelques autres conséquences et bénéfices immédiats de la classification : calcul du profil; une structure relationnelle de même âge moins gourmande en relations que la structure traditionnellement associée; dénombrement des groupes P-oligomorphes.

Un aperçu de la démarche expérimentale et de l'implémentation des groupes P-oligomorphes

Ce premier chapitre de l'annexe est consacré au travail de programmation. La première section, sans répertorier tous les tests effectués dans d'autres contextes moins décisifs, donne une idée de la démarche expérimentale qui a conduit à la classification des tours sur un superbloc seul. Nous fournissons et décrivons le code, écrit dans le langage GAP, et nous joignons quelques exemples d'exécutions qui ont contribué à établir des conjectures, que nous avons ensuite pu prouver de manière théorique.

D'autre part, une fois obtenue, la classification a permis d'implémenter les groupes P-oligomorphes, au travers d'une hiérarchie de classes dans le logiciel SageMath. Nous incluons ici quelques courts extraits de ce code, qui serait trop long à joindre intégralement mais dont nous donnons un aperçu d'utilisation type.

La preuve initiale des conjectures

Nous donnons également en annexe l'approche initiale utilisée pour prouver les conjectures de Macpherson et de Cameron, avec des résultats intermédiaires intéressants, bien que cette approche soit globalement plus faible et ne permette pas de classifier les groupes. Elle repose sur de la théorie des invariants, et en particulier sur un résultat de réduction du problème : si un sous-groupe normal d'indice fini de G a son algèbre de type fini, alors G aussi; ceci s'appliquant également à la propriété d'être de Cohen-Macaulay. Il suffit donc de prouver la génération finie sur un tel sous-groupe, et en bonus la propriété de Cohen-Macaulay. Ceci permet une série de réductions pratiques destinées à simplifier le groupe P-oligomorphe et donc le problème.

Nous énumérons les réductions qui nous seront utiles, puis prouvons le théorème qui les autorise en adaptant la preuve du théorème de Hilbert sur l'algèbre des invariants, ainsi qu'une preuve de Stanley.

Enfin, nous appliquons les réductions à un groupe générique et utilisons d'autres résultats de la thèse pour conclure.

Introduction

Context and contribution of the thesis

Foreword on combinatorics and algebraic combinatorics

Traditionally, combinatorics are defined as the "art of counting", with a range a applications from theoretical computer science (counting the necesary operations to run some given algorithm) to statistical physics (counting the possible states of a system), not to mention mathematics of course.

Oftentimes, what needs to be counted depends on an integer n (typically the size of the input data for an algorithm, or the size of the system in physics). We will thus obtain an infinite sequence of numbers $u_0, u_1, u_2, \cdots$, of which we can try to study the behavior when n grows (for instance in order to determine the complexity of the algorithm, or the macroscopic behavior of a system). To manipulate such a sequence, it is convenient to gather its terms into a *formal power series*:

$$H(z) = u_0 + u_1 z + u_2 z^2 + \cdots .$$

Properties of the sequence commonly correspond to (algebraic, differential, ...) equations on the series. For instance, u_n satisfies a relation of linear recurrence if and only if $H(z)$ is a rational fraction:

$$H(z) = \frac{P(z)}{Q(z)} .$$

This paves the way for some algebraic manipulations destined to study the behavior (asymptotic in particular) of the original sequence: we enter the world of algebraic combinatorics.

This branch of combinatorics, as suggested by its name, uses algebraic tools to tackle combinatorics problems — and the other way around. It often implies adding some more structure on the studied objects: for example a structure of algebra by defining an adapted product; an order on the objects; or other relations or operations.

The research and exploitation of such structures allow to see the object in a new light, to highlight its internal arrangement, and more generally to understand it better.

On the other hand, bringing a combinatorial viewpoint to algebraic objects (for instance by associating partitions to the characters of a group) enables to visualize some situations more clearly, and makes it easier to use an algorithmic approach to the problem.

Profiles of relational structures

Let us get back to counting sequences, or equivalently counting functions, and to the series encoding them. A classical instance of this situation is the counting of the number $\varphi(n)$ of graphs on n vertices, up to an isomorphism. One gets

$$H_\varphi(z) = 1 + z + 2z^2 + 4z^3 + 11z^4 + \cdots$$

This problem can be rephrased as follows. Consider the Rado graph R, that is the infinite graph obtained with probability 1 if, for every pair of vertices in a countably infinite set, one flips a coin to decide if they are linked by an edge. Intuitively, we define an induced subgraph (of the Rado graph for example) as a subset of vertices endowed with the same edges as in the original, embedding graph.

Proposition 0.0.6. *Every simple graph appears, up to isomorphism, as an induced subgraph of the Rado graph.*

Consequently, $\varphi(n)$ can also be defined as the number of induced subgraphs on n vertices of R, up to an isomorphism.

More generally, consider a *relational structure R*, that is a countably infinite set endowed with one or more relations, binary or not. The *profile φ* of the relational structure R counts, for each integer n, the number $\varphi(n)$ of induced substructures of R on subsets of size n, up to isomorphism.

This very general case admits the following particular cases, that have been studied independently:

- In combinatorics of words [Lot97]: the study of the complexity of an infinite word, which is the number of its factors of length n;

- In combinatorics of permutations [KK02; AMB07]: the counting of classes of permutations, some permutation sets defined by pattern avoidance;

- In graph theory, posets and tournament theory [BBM06; Bal+09; BP10]: counting of the finite subgraphs of an infinite graph up to isomorphism;

- In group theory [Cam09; Mac85a; Mac85b]: study of the *orbital profile* of an infinite permutation group, counting of the orbits of a group.

To be specific, one would need in some cases to broaden the setting to hereditary classes of finite structures; see [Kla08; Bol98] for an overview.

Behavior and conjecture of Cameron

The study of the profiles of relational structures is a whole research domain since the seventies [Fra00; Pou06]. Their asymptotic behavior in particular has largely been investigated. We know for instance ([Pou76][Cam90, Section 3.1]) that a profile is always non decreasing. Studies also mention, in each of the cases presented above, a phenomenon of "gaps" in the possibilities of asymptotic behavior: the growth of a profile is either weakly polynomial: $an^k \leqslant \varphi(n) \leqslant bn^k$ for n large enough and some a, b and k; or faster than any polynomial [Pou78; Pou06; BBM06; Kla08]. Pouzet proved that this phenomenon occurs for every relational structure under some mild conditions.

Consistent results in the various contexts [PT13; KK02; Bal+09] suggest that this theorem might be strenghened:

Conjecture 0.0.7. *If the profile of a relational structure is bounded above by a polynomial, then it is asymptotically equivalent to a polynomial: $\varphi(n) \sim an^k$; furthermore, f is a quasi-polynomial:*

$$\varphi(n) = a_k(n)n^k + \cdots + a_0(n)$$

where the coefficients $a_i(n)$ are periodic functions.

Note that, if a profile is quasi-polynomial, then it is automatically asymptotically equivalent to a polynomial, knowing that it does not decrease.

In the particular case of orbital profiles, when the profile counts the orbits of a permutation group, this conjecture is due to Cameron in [Cam90, Section 3.6].

The case of oligomorphic groups

Counting objects under a group action is a recurrent endeavor of combinatorics, which makes the case of orbital profiles particularly interesting. In a context where we focus on the values and behaviors of profiles, it is natural to narrow our interest to groups with a profile that only takes finite values, called *oligomorphic*.

Note that these groups spontaneously appear in the domain of model theory. Indeed, as highlighted by Cameron in [Cam90, Section 2.5], a denumerable relational structure R is $\aleph_0$-categorical (which means that its theory is $\aleph_0$-categorical, or in this case that R is the unique denumerable model of its theory) if and only if its automorphism group is oligomorphic.

One of the contributions brought by this thesis is to prove the conjecture of Cameron on the profile of oligomorphic groups. It will actually be a corollary of another conjecture, that we will now proceed to introduce.

Algebra structure and conjecture of Macpherson

It is a commonly used property (see for instance [Sta97]) that the quasi-polynomiality of a sequence corresponds to its series being of shape

$$\frac{P(z)}{(1 - z^{d_1})(1 - z^{d_2}) \cdots (1 - z^{d_k})} \, , \tag{0.2}$$

with $1 = d_1 \leqslant \cdots \leqslant d_k$ and $P(z) \in \mathbb{Z}[z]$.

These series naturally appear in *commutative algebra*. Indeed, let A be a connected graded commutative algebra. If A is finitely generated, then its *Hilbert series* is necesarily of the form (0.2), where the d_i are the degrees of a minimal set of homogeneous generators [CLO97, Chapter 9, §2]; the reciprocal is not true however.

This kind of link between algebra and properties of the profile motivated the introduction by Cameron of a structure of graded algebra on the orbits of a permutation group, called the *orbit algebra* $\mathbb{Q}[\mathcal{A}_G]$ (more generally *age algebra* for a relational structure), of which the Hilbert series coincides with that of the profile [Cam97].

A possible approach to show that the profile is a quasi-polynomial when bounded by a polynomial is then to study the orbit algebra of the group: if it is finitely generated, the profile is a quasi-polynomial.

The following question is then quite natural.

Question 0.0.8. Let R be a relational structure with profile bounded by a polynomial. Is its age algebra finitely generated ?

The answer to question 0.0.8 is negative in general; for instance, the algebras of tournaments are finitely generated if and only if the profile is bounded [Pou06, Theorem 27]). In the particular case of groups however, this property has been conjectured by Macpherson — who, as noticed by one of the reviewers of [FT18], did not feel confident enough about the fact to even call it a conjecture, using the word question instead.

Conjecture 0.0.9 (Macpherson, 1985 [Mac85a])**.** *Let G be a permutation group of profile bounded by a polynomial, then its orbit algebra $\mathbb{Q}[\mathcal{A}_G]$ is finitely generated.*

This conjecture, stronger than Cameron's, is the core aim of this thesis. The particular case of a bounded profile had already been solved by Pouzet, who also proved that the orbit algebra is an integral domain as soon as the group does not have finite orbits of elements [Pou08].

Results of the thesis

This thesis brings a proof to the conjecture of Macpherson, and thus to Cameron's as a consequence.

Among other notions and tools, one of the key ingredients of the proof in the general case is the notion of *block systems*; studying them has been inspired by the monomorphic decompositions of relational structures, that they generalize. Indeed, the finite generation of the age algebra of a relational structure with a finite monomorphic decomposition has a combinatorial characterization uncovered by Pouzet and Thiéry [PT18]; this case was thus very well known.

The initially used method was then widely based on *invariant theory*: an adaptation of Hilbert's proof of his famous theorem enabled to show that the property of finite generation could be lifted to any finite index supergroup, allowing for a series of reductions of the problem (to finite index subgroups), until we find ourselves in a simpler situation. An experimental approach by exploring many examples on a computer proved crucial to complete the solution.

Although this first proof is interesting in itself, we will finally tell another story in this manuscript. (The results suggested by the computational exploration stay decisive nevertheless.)

Indeed, after a positive answer to the conjectures was found, the slightly more subtle question of the Cohen-Macaulay property could be approached. One of the interests of this property is that the numerator of the Hilbert series has then positive integer coefficients.

The idea seemed legitimate, considering various particular cases: the invariant algebras of finite groups, notably, can be realized as orbit algebras and have this well known property. The investigation of this problem, requiring a more exhaustive understanding of the structure of the algebra, led to rethink the original proof of the conjectures, and finally to a *classification* of *P-oligomorphic groups* (up to *closure*), which is how we call the permutation groups with profile bounded by a polynomial. This classification is our most important result. Informally, a *P*-oligomorphic group is uniquely and entirely described by a finite permutation group endowed with a block system, each block of which is decorated by a pair of groups — one finite, the other infinite — satisfying some explicit conditions. The classification of groups also hands a classification of the orbit algebras, that are just (up to a so-to-speak harmless quotient) invariant algebras with an unusual graduation.

Theorem 0.0.10. *Let G be a permutation group whose profile is bounded by a polynomial. Then, $\mathbb{Q}[\mathcal{A}_G]$ is isomorphic to the algebra of invariants of some finite permutation group acting on variables of known degrees, quotiented by the relations $x^2 = 0$ for some of the variables. In particular, $\mathbb{Q}[\mathcal{A}_G]$ is Cohen-Macaulay, and the profile is a quasi-polynomial that is equivalent to a polynomial.*

The class of *P*-oligomorphic groups seems thus to be significantly more rigid than expected, and their algebras, actually well known, cannot be fundamentally considered a new class of commutative algebras.

Our work is finally presented here as a journey to the classification. Not only does it admit the two conjectures as direct corollaries, but it brings a deep understanding of *P*-oligomorphic groups and their algebras. In particular, by providing an explicit finite encoding of the groups, it allows for an efficient implementation of these on a computer and a computation of their profiles using a generalization of *Pólya enumeration*. The *P*-oligomorphic groups can now be constructed, manipulated, enumerated.

Content and plan of the thesis

This thesis is divided into three main parts, plus an appendix. The first part is dedicated to preliminaries: it exposes the prerequisites from the different domains involved, and provides some examples that will become important actors in our work towards the conjectures.

Algebraic combinatorics: counting functions, series, products; lattices

We begin with the introduction of a few fundamentals of algebraic combinatorics: graded sets and counting functions, of which profiles are instances, and their generating series. We give examples and dive into the algebraic aspects by presenting the concept of *graded product*, which we can use to endow our graded sets with a relevant structure of graded algebra. We explore the interactions between the graded algebras and their Hilbert series, that provide substantial information on the counting function.

In a bit of a side section, we evoke the notions of *order*, *poset*, and *lattice*. We cite a few basic properties of these objects, as well as the example of the lattice of set partitions, that will be of particular interest to us.

Group and invariant theories

In a second chapter, we broach the subjet of permutation groups, and more generally group actions, of which we list some fundamental properties. We give the crucial example of the induced action on subsets.

We also bring to mind the notions of *cosets* of a subgroup, *index* and *normality*, that we will be using a lot in the sequel.

The next section is dedicated to invariant theory: we introduce the concept of invariant algebra of a finite permutation group, and we highlight some of their structural properties, in particular their being Cohen-Macaulay.

Finally, we proceed to explain the functioning of Pólya enumeration, an extremely effective method for counting objects under a finite group action; taking advantage of the classification result we will have obtained by the end of this document, we will later use this method to systematically compute the profiles of P-oligomorphic groups.

Relational structures and oligomorphic groups, orbit algebras; conjectures

In the next chapter, that opens the second main part of this manuscript, we finally turn to the domain that this thesis originates from, and expose the problems that we aim at proving. We first introduce relational structures and their profiles, as well as their age algebras, and we briefly outline some important results from this research field.

We then transition to the particular case of oligomorphic groups: the profiles of *homogeneous* relational structures actually count the orbits of a group; and the orbital profiles can always be realized as profiles of a homogeneous relational structure. We provide some examples and present here the conjecture of Cameron.

Before endowing the set of orbits with its graded algebra structure, we dwell on the notion of closure of a group with relation to the simple convergence topology, and conclude that we can and should, while studying ages and profiles, work only with closed groups. Indeed, taking the closure of a group does not impact its orbits of subsets.

Next, we expose a construction of the orbit algebra (seemingly a bit different from the one we gave for the age algebras, although they essentially coincide), as well as a few practical properties: in particular, we describe the behavior of the algebra when it comes to restrictions, subgroups, direct products... Finally, we present the conjecture, or rather question, of Macpherson, together with the solution of the case of a bounded profile found by Pouzet.

Preparing the ground by case studies and a few more notions from group theory

We go on with a chapter that transitions to the personal work of the author of the thesis. The results presented here do not claim to be original and the notions pre-exist, but many of the included proves are by the author and the reasoning towards our objectives is initiated. In particular, this chapter intends to lay the groundwork for the use of a quite reknown strategy, that of trying to divide the problem one is facing in order to study it on the most elementary possible cases, before taking on the general case.

The first section is a case study, natural because relatively "simple", that will prove more fundamental than expected: that of *wreath products*. We give the definition and consider them from the angle of the conjectures, that are both easily validated on them as soon as they satisfy some natural conditions.

Then, we introduce *the* central notion of our proof, whose examination, as mentioned earlier, was inspired by that of monomorphic decompositions of relational structures: the block systems of permutation groups. After giving a few examples, we focus on the case of *primitive* groups, those that do not have any non trivial ones. Under the hypothesis of P-oligomorphism, these groups, called *highly homogeneous*, have a constantly equal to 1 profile, and are classified (up to closure) — which will make things easier for us.

We will need a last prerequisite: the subdirect product. This notion from group theory (here) formalizes the concept of synchronization between two stable sets a group acts on simultaneously, such as for instance two orbits of elements. As it turns out that a synchronization is determined by a normal subgroup, and that primitive P-oligomorphic groups have very few, these groups will have a pleasant behavior regarding this matter.

First step of the proof: study of block systems

The sequel of the thesis is an original work. The next three chapters, gathered into a third part, are dedicated to the proof of the classification itself, and consequently of the conjectures presented above.

In this first chapter, we explore the information brought by the knowledge of a block system; in particular, by handing a simpler supergroup of finite index (essentially a direct product of wreath products), such a system provides a lower bound on the growth of the profile. We slowly bring up the idea of considering blocks of blocks with the same goal.

We then exploit the lattice structures on (finite, infinite or unrestricted) block systems to try to maximize the obtained bound, in order to choose a relevant system where to best study the group.

Eventually, we come to our flagship construction, that of a carefully selected and uniquely defined system of infinite blocks of finite blocks — simply named *superblocks*. We call this special system the *nested block system* of the group. Besides optimizing the bound, it has the nice property of revealing highly homogeneous groups wherever possible, making it easier to dissect synchronizations within the group.

Second step: classification on the elementary brick

This chapter is dedicated to the case of groups of which the nested block system consists of only one superblock. We remind a few natural instances, such as wreath products on the one hand, in which the actions within each of the finite blocks are all independent; and the exact opposite on the other hand, with a single diagonal, simultaneous action on the elements of the blocks altogether; finally, the intermediary situation of a wreath product with an additive diagonal action. We assert a result of classification in this case, that simply states that these examples actually incorporate all possibilities. The remainder of the chapter takes on the proof of this theorem.

The first stage consists in the study of the way the finite blocks permute. It shows that we can, up to relabelling, assume that all of their permutations can occur without changing the order of the elements within each block (a fact to which we give the full of imagery name of "ladder lemma"), in particular without interacting with the action of the group within the blocks.

We can then and independently examine the block stabilizer, sole actor of the action within the blocks. This second step defines the tower of a group with a single superblock, destined to study the internal synchronizations of this action.

We classify the towers of these groups, helped by the computer exploration of numerous examples, in a way that is transparent for the reader but that we will dwell on later. Towers turn out to have a very rigid shape, with all the information concentrated into two finite groups: the restriction to a block and a normal subgroup

of this restriction, which is actually the restriction to a block after assuming that another one is fixed.

Using the subdirect product, we then show that the tower entirely determines the block stabilizer, and thus the whole group since the action on the blocks is essentially known — which makes it possible to lift the classification of towers to the groups themselves.

Third step: generalization of the classification; solution of the conjectures and other immediate repercussions

In the last chapter of this third part, we finish the proof of the classification in two steps: we exhibit the minimal normal subgroup of finite index K of G from the nested block system, and then we use it to uncover the product structure of the P-oligomorphic group G and to define from there a finite encoding of this group. We show that this finite encoding classifies the P-oligomorphic groups, up to closure (and thus classifies the ages of all P-oligomorphic groups).

The orbit algebra of K is Cohen-Macaulay, and we do know a *Hironaka decomposition*; as the group G acts on the finitely many generators of this algebra, the algebra of G is therefore (up to a natural quotient) an algebra of invariants. In particular, it is Cohen-Macaulay, and the conjectures of Macpherson and Cameron are positively solved. We take the opportunity to list a few other immediate consequences: computation of the profile; a relational structure of same age that requires less relations than the traditionally associated structure; enumeration of P-oligomorphic groups.

A glimpse at the experimental approach and the implementation of P-oligomorphic groups

This first chapter of the appendix deals with the programming work. The first section, without mentioning all tests that have been performed in less decisive contexts, gives an outline of the experimental approach that led to the classification of towers on a single superblock. We provide the code, written in the GAP language, and describe it. We include a few examples of runnings that contributed to conjecturing some results, before proving them theoretically.

On the other hand, once obtained, the classification allowed to implement P-oligomorphic groups, through a hierarchy of classes in the SageMath software. We include here some short extracts of that code, which is too long to fit in here; we provide an instance of use instead.

The initial proof of the conjectures

We also include in the appendix the original approach used to solve the conjectures of Macpherson and Cameron, with some interesting intermediary results, although this method is weaker as a whole and does not end up classifying the groups. It relies on some invariant theory, particularly on a reduction result: if a normal subgroup of finite index of G has a finitely generated algebra, then so does G; this remains true

for the Cohen-Macaulay property. It is thus sufficient to prove the finite generation on such a subgroup, as well as the Cohen-Macaulay property. This allows for a series of convenient reductions that simplify the group and therefore the problem step by step.

We list the reductions we will need, and then prove the theorem that allows them by adapting the proof the Hilbert theorem on the invariant algebra, as well as a proof of Stanley.

Finally, we apply the reductions to a generic group and use other results of this thesis to conclude.

Part I

Background

A flavor of algebraic combinatorics,
with touches of group theory

Summary

This part exposes the background and necessary material to understand this thesis and the different objects we will be dealing with. No result here is from the author of the thesis, and we try to provide references in which they can be found in a wider perspective.

Chapter 1 will first introduce some basic elements of algebraic combinatorics, starting with graded sets, counting functions, and generating series. We then transition to more algebraic aspects: endowing a graded set with a relevant product allows to turn it into a graded algebra, of which the Hilbert series coincide with the series of the counting function. We recall some general properties of these objects, and provide examples. Eventually, we bring up another kind of structure, which is that of partially ordered sets, often called posets, and we highlight the case of lattices.

Chapter 2 is dedicated to objects and methods from the world of algebra, especially group theory and invariant theory. We first present some generalities about permutation groups — a subclass of which will be our main objects of study — and more generally group actions. Then, we explore a tiny bit of invariant theory, and give some fundamental results. Finally, we describe the enumeration method of Pólya, which can allow to count objets under a finite group action.

Notions of algebraic combinatorics and some features most classical

This chapter is dedicated to some classical objects and tools of algebraic combinatorics.

Section 1.1 first introduces the notion of combinatorial class, or more generally of graded set, and that of counting function. We give classical examples and provide some vocabulary regarding the asymptotic behaviors of counting functions. We introduce the concept of generating series, and illustrate their usefulness with a few properties.

Then, Section 1.2 dwells on graded algebras and their Hilbert series: a first subsection of general commutative algebra recalls some of their properties, and we then give a glimpse of how we can enrich a graded set into a graded algebra.

Last, Section 1.3 defines the notion of relation, and develops the particular cases of orders and lattice structures.

1.1 Combinatorial classes and generating series

1.1.1 Examples of combinatorial classes and counting functions

Informally, a *combinatorial class* C is a countable set of objects endowed with a notion of size, or *degree*. A bit more precisely, it is a *graded set* $C = \sqcup_n C_n$ such that each C_n, the subset of objects of size n, is finite.

A very basic instance of this, the collection of words on a finite alphabet $\mathscr{A}$ form a combinatorial class $\mathscr{A}^*$, with the length of words as degree. If $\mathscr{A}$ is just the 2 letters alphabet $\{a, b\}$, we have

$$\mathscr{A}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, \ldots\} = \{\varepsilon\} \sqcup \{a, b\} \sqcup \{aa, ab, ba, bb\} \sqcup \cdots$$

where ε denotes the empty word. The number $\varphi(n)$ of words of size n is 2^n (more generally m^n if $\mathscr{A}$ contains m letters).

The function $\varphi : \mathbb{N} \to \mathbb{N}$ is an instance of what is called a *counting function*. It can of course equivalently be seen as a sequence (and be called a *counting sequence*) $\varphi = (\varphi_n)_{n \in \mathbb{N}} = (\varphi(n))_{n \in \mathbb{N}}$, but we will usually prefer the function notation, which keeps room for other indices. Counting functions are a core feature in combinatorics, and some of them are quite famous.

Definition 1.1.1. A *partition* (or integer partition) λ is a vector $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that the λ_i's are weakly decreasing integers: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l$. The *length* $\ell(\lambda)$ of λ is l, its *width* is λ_1 and its *size* is the sum n of integers $\lambda_1 + \cdots + \lambda_l$. We also say that λ is a partition of n.

For instance, there are 5 partitions of size 4 (we willingly omit here both commas and parentheses):

$$4$$
$$3\ 1$$
$$2\ 2$$
$$2\ 1\ 1$$
$$1\ 1\ 1\ 1$$

The number of partitions of size n is usually denoted by $p(n)$. One can also count the partitions with a restriction on the length, for instance by considering only those with length less than an integer k. The obtained counting function $p_k(n)$ is actually the same as if we decided to count the partitions with width less than k.

This is easily understandable by considering the reprentation of partitions as *Young diagrams*. For instance, the following Young diagram represents the partition $(6, 3, 2, 2)$, the number of squares of the i-th row being λ_i.
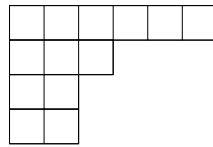
If one chooses to read the diagram by columns rather than rows, one obtains the partition $(4, 4, 2, 1, 1, 1)$, of which the associated Young diagram is the following.
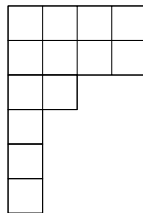
The second one was simply obtained by a reflection according to the first diagonal (the axis $y = x$ using the matrix style coordinates), which obviously exchanges the length and width of the diagram, and thus of the partitions they represent.

As other very classical examples of combinatorial classes and counting functions, one can mention graphs, of size their number of vertices; or binary trees, of size

their number of nodes; or Dyck paths, of size half their length. These last two are actually counted by the same sequence: the most famous Catalan numbers. This kind of observation usually leads to the search for a structural bijection between the two families: in this case, there is a huge variety of Catalan objects and nice bijections between them, bringing a deep understanding of these objects.

1.1.2 Growths

Definition 1.1.2 (Asymptotic behaviour of sequences). Let $f = (f_n)_n$ be a sequence of (real) numbers. We say that f (or with slight abuse f_n) is

1. *bounded (above) by a polynomial*, and we write $f_n = O(n^k)$, if there exist some number α and some integer k such that we have $f_n \leqslant \alpha n^k$ for n large enough.
2. *polynomial in the weak sense* if there exist some numbers α, β and some integer k such that we have $\alpha n^k \leqslant f_n \leqslant \beta n^k$ for n large enough. The integer k is sometimes called the *growth rate* of f. If it is equal to 1 (resp. 2, 3), we say that f has *linear* (resp. *quadratic*, *cubic*) growth.
3. *(asymptotically) equivalent* to another sequence $(g_n)_n$ which takes only finitely many times the value 0, and we write $f_n \sim g_n$, if we have $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$. When g_n is a polynomial in n, we say that f is polynomial in the strong sense, or just polynomial.
4. *exponential* if there exist some numbers α, β, γ such that we have $\alpha \exp(\gamma n) \leqslant f_n \leqslant \beta \exp(\gamma n)$ for n large enough.

Example 1.1.3.

(1) The counting function of words over a finite alphabet is exponential (see previous subsection).

(2) The counting function of *combinations* (basically unsorted partitions) is exponential as well: indeed, there is a bijection between the combinations of size n and the subsets of $\{1, \ldots, n-1\}$ (just identify a combination to the set of its descents), of which there are 2^{n-1}.

(3) The counting function of integer partitions is neither bounded by a polynomial nor exponential or above. We know an equivalent to it, thanks to Hardy and Ramanujan:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

(4) On the other hand, the sequence p_k counting the partitions with a length or width restriction is polynomial, as we have:

$$p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!}.$$

1.1.3 Generating series

When studying a sequence of numbers, it is often a good idea to encode it into a power series, as the coefficients of the series. It is in particular very relevant in the context of the study of profiles, a notion we will introduce in a later chapter and that is at the core of this work.

Although all the results of this subsection remain true for any sequence of numbers, real or complex, we will stick to the vocabulary and notation of our use case, which is that of counting functions.

Definition 1.1.4. The *generating series* (in z) $\mathcal{H}_\varphi(z)$ of a sequence $\varphi = (\varphi(n))_n$ of numbers is the formal power series defined by

$$\mathcal{H}_\varphi(z) = \sum_n \varphi(n) z^n \,.$$

Some properties of the sequence translate into properties of the series, as in the case of the following fundamental result, that can be found in [Sta97] (where it is both stated and proved over $\mathbb{C}$ rather that $\mathbb{Q}$, but the field does not actually matter).

Proposition 1.1.5. *Let $\varphi = (\varphi(n))_n$ be a sequence of rational numbers or integers, and $\alpha_1, \ldots, \alpha_d \in \mathbb{Q}$ be fixed rational numbers with α_d non zero. The following are equivalent:*

(i) *The generating series of φ has a rational shape:*

$$\mathcal{H}_\varphi(z) = \frac{P(z)}{Q(z)}$$

with P some polynomial over $\mathbb{Q}$ of degree less than d and $Q(z) = \alpha_d z^d + \cdots + \alpha_1 z + 1$;

(ii) *For all $n \geqslant 0$, φ satisfies the relation of linear recurrence:*

$$\varphi(n+d) + \alpha_1 \varphi(n+d-1) + \alpha_2 \varphi(n+d-2) + \cdots + \alpha_d \varphi(n) = 0$$

where the α_i are the coefficients of Q.

The proof being rather short, we recall it here.

Proof. Let V_1 and V_2 be the vector spaces over $\mathbb{Q}$ of the sequence that verify (i) and (ii) respectively. Now in (i), we may choose the d coefficients of $P(z)$ arbitrarily, hence V_1 is of dimension d. In (ii) we may choose $\varphi(0), \varphi(1), \ldots, \varphi(d-1)$ and then the other $\varphi(n)$'s are uniquely determined; hence V_2 is of dimension d as well. Last, if φ is in V_1, then equate coefficients of x^n in the identity $Q(x) \sum_{n \geqslant 0} \varphi(n) x^n = P(x)$ to get that φ is in V_2, which ends the proof. $\qquad\square$

Definition 1.1.6. A *quasi-polynomial* (or *pseudo-polynomial*) of degree d is a function $\varphi : \mathbb{N} \mapsto \mathbb{Q}$ of the form:

$$\varphi(n) = \alpha_d(n) n^d + \cdots + \alpha_0(n)$$

where each α_i is a *periodic function* (with integer period) and $\alpha_d(n)$ is not identically zero.

Trivially, the α_i's have a common period N (which is not unique, but one can choose to take N as the least common multiple of the periods); such an integer is called a *quasi-period* of φ. From this observation, one can equivalently define a quasi-polynomial as a function for which there exist $N > 0$ and polynomials $\varphi^{(0)}, \ldots, \varphi^{(N-1)}$ such that we have $\varphi(n) = \varphi^{(i)}(n)$ whenever n verifies $n \equiv i \,(\mathrm{mod}\,N)$.

Proposition 1.1.7. *Let $\varphi : \mathbb{N} \mapsto \mathbb{Q}$ be a function, the following are equivalent.*
(i) *The function φ is a quasi-polynomial of quasi-period N.*
(ii)

$$\sum_n \varphi(n) z^n = \frac{P(z)}{Q(z)}$$

where P and Q are polynomials over $\mathbb{Q}$, every zero z_0 of Q satisfies $z_0^N = 1$ (in particular, Q is essentially a product of cyclotomic polynomials) provided that the fraction is reduced to lowest terms, and the polynomial degrees satisfy $deg(P) < deg(Q)$.

(iii)

$$\sum_n \varphi(n) z^n = \frac{P(z)}{\prod_i (1 - z^{d_i})}$$

not necessarily reduced to lowest terms, with $d_i | N$ for each i, and $deg(P)$ less than that of the denominator.

Furthermore, in case these are satisfied and φ is a counting function (meaning it takes its values in $\mathbb{N}$), P has integer coefficients in (iii).

The first equivalence of the proposition above is a natural consequence of Proposition 1.1.5. The equivalence with the third item is immediate in one direction and only needs numerator and denominator multiplication by some cyclotomic polynomials in the other. (For the last sentence, just notice that if P had some non integer coefficients, that of smallest degree could not be "compensated" by other terms when expanding the product.)

Example 1.1.8. The generating series of p_k is

$$\prod_{i=1}^{k} \frac{1}{1 - z^i}\,.$$

To get convinced of that, one can use the famous identity $\frac{1}{1-Z} = 1 + Z + Z^2 + \cdots$ on each term and then try to develop the product. The sequence is therefore (eventually) a quasi-polynomial. That can be "checked" easily on the case $k = 2$. How many partitions of n into 2 parts at most? It is not difficult to obtain $\lfloor \frac{n}{2} \rfloor + 1$, which is indeed a quasipolynomial since it behaves as $\frac{n}{2} + 1$ on the even integers and as $\frac{n}{2} + \frac{1}{2}$ on the odd integers.

1.2   Combinatorial algebras and Hilbert series

A more advanced technique when studying objects, combinatorial objects in particular, besides counting them and considering the generating series of their counting

function, is to add some *algebraic structure* (over $\mathbb{Q}$) on them. The crucial part is to endow them with a *product* that makes sense, and by that we mean as first requirement that it must be *graded*. We will also require it to be commutative. We will start this section by recalling some definitions and properties about graded algebras as abstract algebraic structures, then we will give a few examples.

1.2.1 Graded algebras and their Hilbert series

In any case that we may be interested in in the sequel, algebras will always be *commutative*, so we state right now that **all algebras are assumed to be commutative** in this thesis and in this section in particular. All algebras will also be considered over $\mathbb{Q}$.

Definition 1.2.1. A *graded algebra* is an algebra A:
  (i) on which there is a notion of *degree*, that is a linear decomposition $A = \bigoplus_n A_n$;
  (ii) of which the product $\cdot$ is *graded*, which means it induces an application from $A_n \times A_m$ to A_{n+m} for each $n, m \in \mathbb{N}$.
We say (as a recursive definition) that the degree $\deg(a)$ of an element a of A is n if a lives in A_n or it is a sum of which the highest degree term is of degree n. A graded product is thus a product $\cdot$ that satisfies:

$$\deg(a \cdot b) = \deg(a) + \deg(b)$$

for all $a, b \in A$.
The subspace A_n is called the *homogeneous component* of degree n of A, and its elements are said to be *homogeneous* (of degree n).
Finally, A is said to be *connected* if A_0 is just the base field (so $\mathbb{Q}$ in our case).

The absolute prototype of a graded (commutative) algebra is of course the algebra of polynomials $\mathbb{Q}[\underline{x}]$, with the classical notion of polynomial degree. It is also connected.

Definition 1.2.2. The *Hilbert function* of a (commutative) graded algebra A is the function $h: n \in \mathbb{N} \mapsto \dim(A_n)$ (where $\dim(A_n)$ is the dimension of A_n as a vector space). The *Hilbert series* $\mathcal{H}_A(z)$ of A is the generating series of its Hilbert function.

Lemma 1.2.3. *Let A, A_1 and A_2 be graded (commutative) algebras. We have then:*

  1. $A = A_1 \oplus A_2 \implies \mathcal{H}_A = \mathcal{H}_{A_1} + \mathcal{H}_{A_2}$

  2. $A = A_1 \otimes A_2 \implies \mathcal{H}_A = \mathcal{H}_{A_1} \mathcal{H}_{A_2}$.

We now introduce a few notions and properties that we will mainly be using in a context of invariant theory, dwelled on a bit later. For this reason, one should expect a scarcity of examples until then (although the polynomial algebras are trivial illustrations, and those familiar with the theory may think of symmetric polynomials).

The algebra A will be assumed to be connected and graded (and still commutative).

Definition 1.2.4. The *Krull dimension* of A is the maximal size of a set of elements of A that are algebraically independent (meaning that they verify no polynomial equation). If A is of Krull dimension r, a *homogeneous system of parameters* (sometimes abreviated into h.s.o.p.) for A is a set $\{\theta_1, \ldots, \theta_r\}$ of homogeneous elements of A that are algebraically independent.

Equivalently, an h.s.o.p. for A can be defined as a set $\{\theta_1, \ldots, \theta_r\}$ of homogeneous elements such that A is a finitely generated module over the subalgebra $\mathbb{Q}[\theta_1, \ldots, \theta_r]$ (then r is necesarily the Krull dimension of A).

Theorem 1.2.5 (Noether's normalization lemma)**.** *If A is finitely generated, an h.s.o.p always exists.*

Refer for instance to [ZS75, Theorem 25] for this theorem.

Corollary 1.2.6. *If a (commutative) graded algebra A is finitely generated, then its Hilbert series is of the form*

$$\mathcal{H}_A(z) = \frac{P(z)}{\prod_{i \in I}(1 - z^{d_i})}$$

with P having integer coefficients and degree less than that of the denominator, and I being a finite set. In particular, with Proposition 1.1.7, the Hilbert function of A is a quasi-polynomial in this case.

Example 1.2.7. The Hilbert series of the multivariate polynomial algebra $\mathbb{Q}[x_1, \ldots, x_n]$ is $\frac{1}{(1-z)^n}$. If x and y are two indeterminates (of degree 1), then the Hilbert series of the algebra of polynomials $\mathbb{Q}[x, y^3]$, for instance, is $\frac{1}{(1-z)(1-z^3)}$.

Of course, the multivariate polynomial algebra $\mathbb{Q}[x_1, \ldots, x_n]$ is of Krull dimension n. One can also notice that a commutative $\mathbb{Q}$-algebra A's being of Krull dimension n means that there exists an injective morphism from $\mathbb{Q}[x_1, \ldots, x_n]$ to A, but not from $\mathbb{Q}[x_1, \ldots, x_{n+1}]$ to A, which can give an idea of the size of A. Recalling the last item of Example 1.1.3, we derive the following lemma.

Lemma 1.2.8. *If a (commutative) graded algebra A has Krull dimension m, then its Hilbert function has a growth rate of at least $m - 1$.*

The classical result that follows can be found for instance in [Sta79b].

Proposition 1.2.9. *Let A be a connected graded commutative algebra. The following sentences are equivalent.*

(i) *A is a* free module *(necessarily finitely generated) over the subalgebra $\mathbb{Q}[\theta_1, \ldots, \theta_r]$:*

$$A = \bigoplus_{j=1}^{s} \eta_j \mathbb{Q}[\theta_1, \ldots, \theta_r]. \tag{1.1}$$

The η_j's may be chosen homogeneous as well.

(ii) *For every h.s.o.p. $\{\psi_1, \ldots, \psi_r\}$, A is a free module over $\mathbb{Q}[\psi_1, \ldots, \psi_r]$.*

In the eventuality of (i) *(and thus* (ii)*), the family of elements* $\{\eta_j\}$ *verify Equation 1.1 if and only if it is a linear basis of the vector space* $A/(\theta_1,\ldots,\theta_r)$ *(the quotient by the* ideal *generated by the* θ_i*'s).*

Definition 1.2.10. A connected graded commutative algebra that verify the sentences of Proposition 1.2.9 is said to be *Cohen-Macaulay* (although the usual definition is a bit wider). Equation 1.1 is called a *Hironaka decomposition* of A.

Proposition 1.2.11. *The Hilbert series of a Cohen-Macaulay algebra A is of the form $\frac{P(z)}{\prod_i(1-z^{d_i})}$ where P has integer coefficients.*

Sketch of proof. This is a direct consequence of the equality $A = \bigoplus_{j=1}^s \eta_j \mathbb{Q}[\theta_1,\ldots,\theta_r]$. The Hilbert series of $\mathbb{Q}[\theta_1,\ldots,\theta_r]$ is $\prod_i(1-z^{d_i})^{-1}$ where d_i is the degree of θ_i, so the series of A is indeed $\frac{P(z)}{\prod_i(1-z^{d_i})}$ with the coefficient of z^k in P being the number of η_j's of degree k. $\square$

1.2.2 Examples of combinatorial vector spaces and algebras

In order to bring more structure to a graded set, the first natural thing to do is to turn it into a vector space, by considering the space of finite formal linear combination of the objects, or put otherwise a vector space of basis indexed by the objects of the graded set. The obtained vector space V is then naturally *graded*: $V = \bigoplus_n V_n$. This is hardly an end in itself but will allow to define interesting graded products on our objects (actually on their vector space).

This will hand graded algebras, from which we might be able to retrieve more information. In particular, a well constructed structure of graded algebra on a family of combinatorial objects will have its Hilbert series equal to the generating series of the counting function of the objects (see previous subsection for some interesting properties).

We give a few simple examples of such algebras.

Example 1.2.12 (Graded products on words). Let $\mathscr{A}$ be an alphabet, and $\mathscr{A}^*$ the set of words over $\mathscr{A}$.

(1) We can endow the vector space formally generated by $\mathscr{A}^*$ with a graded product called the *concatenation* of words:

$$
\begin{array}{ccccc}
\mathscr{A}^* & \times & \mathscr{A}^* & \longrightarrow & \mathscr{A}^* \\
\omega & \cdot & \upsilon & \longmapsto & \omega\upsilon
\end{array}
$$

For instance, over the two letters alphabet $\{a,b\}$, we have $abbab \cdot aba = abbababa$.

(2) The *shuffle product* on words over $\mathscr{A}$ is defined recursively by:

$$
\omega \sqcup\!\sqcup \upsilon = \begin{cases}
\omega & \text{if } \upsilon = \varepsilon, \\
\upsilon & \text{if } y = \varepsilon, \\
\omega_1(\omega' \sqcup\!\sqcup \upsilon) + \upsilon_1(\omega \sqcup\!\sqcup \upsilon') & \text{otherwise,} \\
& \text{with } \omega_1, \upsilon_1 \in \mathscr{A} \text{ such that } \omega = \omega_1\omega', \upsilon = \upsilon_1\upsilon'
\end{cases}
$$

where ε is the empty word. The result of the product is the sum of all possible shufflings that preserve the order of letters of ω and v, respectively. We have for instance:

$$ab \shuffle ba = abba + abba + abab + baba + baab + baab$$
$$= 2abba + 2baab + abab + baba$$

We call the algebra thereby defined a *shuffle algebra* (although this term is traditionally used to designate the associated *Hopf algebra*).

Example 1.2.13 (Set algebra). Let Ω be a countably infinite set, and $\mathscr{E}(\Omega)$ be the set of all of its finite subsets, which is naturally graded by cardinality. We define the following product on the vector space formally generated by $\mathscr{E}(\Omega)$, the *disjoint union product*. It is first defined on subsets by:

$$F_1 \cdot F_2 = \left\{ \begin{array}{ll} F_1 \cup F_2 & \text{if } F_1 \cap F_2 = \emptyset \\ 0 & \text{otherwise} \end{array} \right.$$

where F_1 and F_2 are two finite sets, and then by linearity. This product, which is not integral but still graded, enables one to endow $\mathscr{E}(\Omega)$ with a structure of graded commutative algebra. The unit is the empty set.

In this thesis, we will be using a slightly different version, where one considers the space of *possibly infinite* linear combinations of finite subsets, provided that the cardinality of subsets involved in the combination is bounded. In other words, if we denote by $\mathbb{Q}\mathscr{E}_n(\Omega)$ the vector space of (possibly infinite) linear combinations of subsets of size n, then we will be considering the direct sum $\bigoplus_n \mathbb{Q}\mathscr{E}_n(\Omega)$. The product stays the same.

We call this version the *set algebra* of Ω, and denote it by $\mathbb{Q}[\mathscr{E}_\Omega]$.

1.3 Orders and lattices

1.3.1 Orders, posets

Definition 1.3.1.   • A *relation* ρ on a set Ω is a collection of subsets of Ω of shared cardinality r, or a collection of r-tuples (in this case the relation is *oriented*). If a subset (resp. tuple) is in ρ, we say that E is in relation for ρ.

   • The integer r is called the *arity* of the relation. A relation of arity 1 (resp. 2) is said to be *unary* (resp. *binary*).

Definition 1.3.2. A *poset* $\mathcal{P}$ is a set endowed with a relation of *order*, that is a relation $\leq$ which is
  (i) *reflexive*: $\forall a \in \mathcal{P},\ a \leq a$;
 (ii) *transitive*: $\forall a, b, c \in \mathcal{P}$, if $a \leq b$ and $b \leq c$ then $a \leq c$;
(iii) *antisymmetric*: $\forall a, b \in \mathcal{P}$, if $a \leq b$ and $b \leq a$ then $a = b$.
Furthermore, if for every $a, b \in \mathcal{P}$ we have either $a \leq b$ or $b \leq a$, then the order is said to be *total* (or *linear*). We write $a < b$ when we have $a \leq b$ and $a \neq b$.

A *subposet* $\mathcal{P}'$ of $\mathcal{P}$ is a subset of $\mathcal{P}$ such that the partial order of $\mathcal{P}'$ is that of $\mathcal{P}$ restricted to elements of $\mathcal{P}'$.

Example 1.3.3.

(1) The set of real numbers $\mathbb{R}$ is endowed with the classical linear order on numbers $\leqslant$; $(\mathbb{Q}, \leqslant)$ is a subposet thereof, as well as any subset of $\mathbb{R}$. In $(\mathbb{C}, \leqslant)$, the relation $\leqslant$ is not total any more.

(2) The *lexicographic order* is a total order on words.

(3) The relation of inclusion $\subseteq$ is an order on subsets of $\{1, \ldots, n\}$ (or any set), that is not total: the *boolean order*.

(4) There are many interesting orders on the permutations (of finite degree), such as the *weak order* or the *Bruhat order*, that we mention without detailing.

Definition 1.3.4. For $a, b \in \mathcal{P}$ we say that b *covers* a, and we write $a \lessdot b$, if we have $a < b$ and there is no $c \in \mathcal{P}$ such that we have $a < c < b$. Such relations are called *cover relations*.

Cover relations alone define the whole poset, as the other relations can be deduced by transitivity. Hence, in order to represent a poset, we only need to picture elements and cover relations: the *Hasse diagram* of the poset is used to display them. Usually, the following convention is applied: a is linked by an edge to b and placed below it if and only if $a \lessdot b$ is verified. As a result, smaller elements are located at the bottom of the picture, as pictured in Figure 1.1.

Definition 1.3.5. An element $a \in \mathcal{P}$ is said to be *minimal* (resp. *maximal*) if there is no $b \in \mathcal{P}$ such that we have $b < a$ (resp. $b > a$).

We give a generic abstract example to illustrate all this.
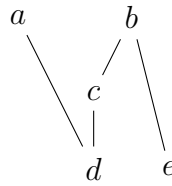
Figure 1.1: Example of a Hasse diagram.

The cover relations of this poset are $d \lessdot c \lessdot b$, $d \lessdot a$ and $e \lessdot b$. The elements d and e are minimal, a and b are maximal. The order is not total since a and b can not be compared.

1.3.2 Joins and meets and lattices

Let E be a subset of elements of a poset $\mathcal{P}$. The *meet* (or greatest lower bound) of E, denoted by $\vee E$, is the unique element l, such that we have

$$a \leq l \iff \forall e \in E, \ a \leq e \tag{1.2}$$

if it exists, and $\varnothing$ otherwise. Symmetrically, the *join* (or least upper bound) of E, denoted by $\wedge E$, is the unique element u, such that we have

$$a \geq u \iff \forall e \in E, \ a \geq e \tag{1.3}$$

if it exists, and $\varnothing$ otherwise. When E is a pair of elements $\{e_1, e_2\}$, we also write $\vee E = e_1 \vee e_2$ and $\wedge E = e_1 \wedge e_2$.

Definition 1.3.6. Let ρ be a relation on a set Ω. Its *transitive closure* is the smallest relation on Ω that contains ρ and is transitive.

For instance if $\mathcal{P}$ is a poset, the transitive closure of its cover relations is the poset itself.

Definition 1.3.7. A chain of a poset $\mathcal{P}$ is a set of elements $\{a_1, a_2, \ldots\}$ such that the natural restriction of the order on this set is total.

Of course, a whole poset may be a chain: such is the case of $(\mathbb{Q}, \leqslant)$, for instance. This specific poset is usually called the *rational chain*.

Definition 1.3.8. A *meet-semilattice* (resp. join semi-lattice) is a poset $\mathcal{P}$ such that $\vee X$ (resp. $\wedge X$) is different from $\varnothing$ for any subset X of $\mathcal{P}$.
A *lattice* is a poset which is a meet-semilattice and a join semilattice.
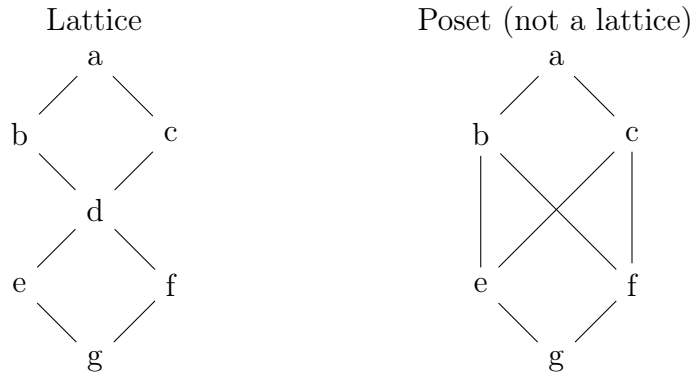
Figure 1.2: Example and counter-example of a lattice.

Lemma 1.3.9. *A meet semilattice (resp. join semillatice) with a unique maximal element (resp. minimal element), then called a* maximum *(resp.* minimum*), is a lattice.*

The Hasse diagram on the right in Figure 1.2 gives an example of a poset which is not a lattice. Indeed we see $b \vee c = \varnothing$ and symmetrically $e \wedge f = \varnothing$. In contrast, the Hasse diagram on the left is this is a lattice.
Another example of a lattice is the already mentioned boolean order on subsets of a finite set, ordered by inclusion. The number of subsets is of course 2^m if m is the size of the full set, and the resulting Hasse diagram is a (hyper)cube of the
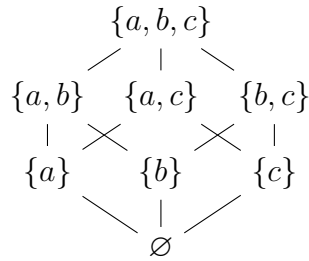
Figure 1.3: Lattice of the boolean order on a set of size 3

matching dimension. Following is the case of a set of size 3 (on which you can get to see the cube shape).

Last, but not least, is the *lattice of set partitions* , ordered by *refinement* . We say that a set partition $\{E_1, E_2, \ldots, E_r\}$ *refines* another one $\{F_1, F_2, \ldots, F_s\}$ if for any E_i there exists F_j such that E_i is a subset of F_j. As an example, $\{\{1,2\}, \{3\}, \{4\}\}$ refines $\{\{1,2\}, \{3,4\}\}$, and the partition with only one part is the maximum of the lattice while the partition into singletons is its minimum. We will have the opportunity to come back to this specific lattice. We include here a nice picture of the lattice of partitions of a set of size 4, made by Tilman Piesk (singletons are not colored).
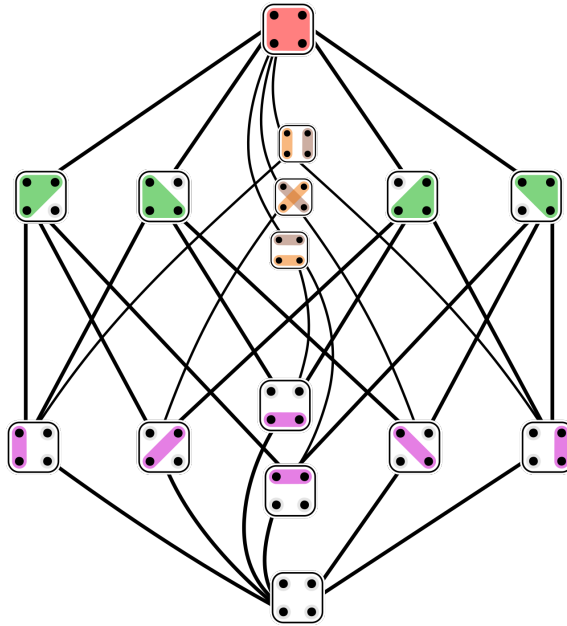
Figure 1.4: Lattice of set partitions on a set of size 4

Group and invariant theories

In this second chapter of preliminaries, we explore the tools we will be needing from group theory and invariant theory.

Section 2.1 is an overview of the basics about permutation groups and group actions in general. We bring to mind notions like orbits, normality, and stabilizers, and we mention examples of group actions.

Section 2.2 then provides a very brief reminder of some fundamental results of invariant theory, and Section 2.3 deals with Pólya enumeration, of which we expose the functioning.

2.1 Elements of group theory

2.1.1 Permutation groups

Definition 2.1.1. Let Ω be a set. The *symmetric group* on Ω, called then the *domain*, is the group of all bijections $\Omega \to \Omega$ with the composition of functions as composition law, denoted by $\mathfrak{S}_\Omega$. Such bijections are called *permutations*.

In this thesis, all domains will be assumed to be countably infinite at most.

When Ω is the finite set $\{1, \dots, n\}$ for $n \in \mathbb{N}$, we usually write $\mathfrak{S}_n$ instead of $\mathfrak{S}_\Omega$.

If Ω is equal to $\mathbb{N}$, or more generally countably infinite, then we denote $\mathfrak{S}_\Omega = \mathfrak{S}_\infty$. We call this group the *infinite symmetric group*.

Definition 2.1.2. A subgroup G of $\mathfrak{S}_n$ for some n possibly infinite is called a *permutation group* . The cardinality n is called the *degree* of G.

Remark 2.1.3. If there exists a bijection α between two sets Ω and Ω', then $\mathfrak{S}_\Omega$ and $\mathfrak{S}_{\Omega'}$ are isomorphic: it is clear indeed that the conjugation by α is an isomorphism between $\mathfrak{S}_\Omega$ and $\mathfrak{S}_{\Omega'}$. For this reason, we will sometimes use the notation $\mathfrak{S}_n$ when the domain is of cardinality n (which might be countably infinite), even if it is not $\{1, \dots, n\}$. In other words, and for the sake of simplicity of exposition, we will sometimes blur the distinction between a permutation group and its class of isomorphism.

Definition 2.1.4. A *fixed point* of a permutation σ is an element a such that we have $\sigma(a) = a$. The *support* of a permutation is the set of elements of Ω that are not fixed points.

The infinite symmetric group is sometimes defined as the union $\cup_n \mathfrak{S}_n$; this is not the case here, since the permutations of $\mathfrak{S}_\infty$ do not have to be finitely supported (*i.e.* of finite support).

A *cyclic permutation* c, or *cycle* is a permutation for which there exist some elements $a_1, a_2, \ldots, a_k$ such that we have $c(a_1) = a_2$, $c(a_2) = a_3$, $\ldots$, $c(a_k) = a_1$, and the other elements are fixed points. In this case, it is of order k (we may also say it is a k-cycle), and we denote $c = (a_1\ a_2\ \cdots\ a_k)$. For instance, a permutation that just swaps two elements of Ω is a 2-cycle, also called a *transposition*.

It is a folklore result that any permutation can be uniquely written as a composition (in the sense of the composition of functions) of cycles. The tuple of the cycle lengths of the cyclic decomposition of a permutation (by decreasing order) is called its *cycle type*.

Example 2.1.5 (Remarkable permutation groups)**.**

(1) Take $n \in \mathbb{N}$, the cyclic group $\mathcal{C}_n$ is the permutation group of degree n generated by the permutation $(1\ 2\ \cdots\ n)$. It may be seen as the group of rotations of a necklace with n pearls, and is *cyclic* of order n.

(2) A permutation of $\mathfrak{S}_n$ (for $n \in \mathbb{N}$) is said to be *even* when it can be expressed as a product of an even number of transpositions. Altogether, these permutations form a subgroup of $\mathfrak{S}_n$, the *alternating group*. It has an infinite analog in the infinite symmetric group of *finitely supported* permutations, which is defined the same way.

(3) The group of increasing bijections $\mathbb{Q} \to \mathbb{Q}$ is denoted by $\mathrm{Aut}(\mathbb{Q})$. By the above remark, it may be seen as a subgroup of $\mathfrak{S}_\infty$, and as such it is an (infinite) permutation group (which we will not lack opportunities to mention again later on).

2.1.2 Groups actions, orbits and transitivity

Definition 2.1.6. Let Ω be a set and G be a group. A *(right) action* of G on Ω is an application

$$
\begin{array}{ccccc}
G & \times & \Omega & \longrightarrow & \Omega \\
(\ g & , & a\ ) & \longmapsto & g.a
\end{array}
$$

with the additionnal properties $1_G.a = a\ \forall a \in \Omega$, and $h.(g.a) = (gh).a$. We call Ω the *domain* of the group action, and $|\Omega|$ the *degree* of this action.

Note that choosing such a group action is equivalent to choosing a *permutation representation*, that is a morphism $G \to \mathfrak{S}_\Omega$. This double viewpoint between group actions and permutation groups allows the natural use of notions from one of these objects in the context of the other (speaking of, the reader probably noticed the equivalence of the two presented notions of degree).

Examples 2.1.7 (Some remarkable group actions).

(1) The action of a permutation group naturally induced by its elements on its domain is called the *natural action*. We will mostly be using the function notation for this particular action: $g.a = g(a)$.

(2) The (left) *regular action* of a group G is its action on itself by $g.h = gh$ (here the domain is $\Omega = G$).

(3) If G acts on Ω, it induces an action on $\mathscr{E}_n(\Omega)$, the set of subsets of size n (for any $n \in \mathbb{N}$), by $g.\{a_1, a_2, \ldots, a_n\} = \{g.a_1, \ldots, g.a_n\}$. This can be extended to the set $\mathscr{E}(\Omega)$ of all finite subsets of Ω, and is called the *induced action on subsets*.

(4) The *induced action on tuples* is defined the same way as above, with the sets of tuples (of fixed length or not) Ω^n and $\sqcup_n \Omega^n$ (respectively).

These last two actions (especially that on subsets) will be at the core of this thesis.

For the rest of the section, consider a group action of G on Ω.

Definition 2.1.8. The *orbit* of an element $a \in \Omega$ under G is $G.a = \{ g.a : g \in G\}$. The set of orbits of the action is denoted by Ω/G.

Note that the orbits form a partition of the domain.

Definition 2.1.9. A group action of G on Ω is said to be *transitive* whenever $G.a = \Omega$ for some $a \in \Omega$ (hence for every a). It is said to be intransitive otherwise. These notions can be generalized a little: the action is *n-transitive* if the induced action on Ω^n (see last item of Example 2.1.7) is transitive.

Example 2.1.10. The symmetric group is transitive and n-transitive for any n (less than its degree). On the other hand, $\mathrm{Aut}(\mathbb{Q})$ is transitive but not 2-transitive (since it preserves a total order on the elements by definition; we will get back to this later).

2.1.3 Action by conjugation, normality; index

Definition 2.1.11 (Action by conjugation). Any group G can act on itself by *conjugation* : $g.h = h^g = g^{-1}hg$. The orbits for this action are called the *conjugacy classes*, and two elements lying in the same conjugacy class are said to be *conjugate*.

As it is well known, two conjugate permutations share the same cycle type; the reciprocal is true in $\mathfrak{S}_n$, which establishes a bijection between its conjugacy classes and the partitions of n.

Definition 2.1.12 (Normality, simplicity).

- The action by conjugation of G on itself induces an action by conjugation of G on its subgroups: two subgroups H and H' are said to be conjugate if they verify $g^{-1}Hg = H'$. Subgroups that are stable by conjugation, meaning they are alone in their conjugacy class, are called *normal*. Note that if N is normal in G, then the action by conjugation of G on itself may be naturally restricted to N.

- Groups that have no non trivial normal subgroups are called *simple* .

Lemma 2.1.13. *The normal subgroups of a group G are exactly the kernels of the group homomorphisms defined on G.*

The alternating group is a famous example of normal subgroup of $\mathfrak{S}_n$, as the kernel of the *signature* homomorphism, which maps each permutation to 0 if it is even, and 1 otherwise (and is actually the only existing homomorphism from $\mathfrak{S}_n$ onto $\mathbb{Z}/2\mathbb{Z}$). Furthermore, it is well known to be a simple group as soon as n is greater than 5, or equal.

Definition 2.1.14. Let H be a subgroup of G.

- The *left (resp. right) cosets* of H in G are the $gH = \{gh : h \in H\}$ (resp. Hg) for g in G. As it turns out that there are as many right cosets as left cosets for a given subgroup, one can define the *index* of H in G, which is its number of (right or left) cosets. It is usually denoted by $[G : H]$.

Cosets are basically the orbits of the left or right regular action of H on G, and as such, they form a partition of G.

Obviously, for arbitrary elements g and g' in G, the cosets gH and $g'H$ may be equal (which means that g and g' lie in the same coset), so we often appeal to a set of *representatives* of the (here, right) cosets, that is elements $g_1, g_2, \ldots, g_I$ of G such that in $\{Hg_i : 1 \leqslant i \leqslant I\}$, all cosets of H in G appear. Of course, I can be chosen to be the index of H, in which case the set of representatives is said to be minimal: one just needs to take one element g_i in each coset.

Remark 2.1.15. One defines the (right) *coset action* (or action on the cosets) of G on H by $g.Hg_i = Hg_ig = Hg_j$. Note that the (right) regular action is the particular case in which you take $H = \{1_G\}$.

Normal subgroups N are precisely those for which $gN = Ng$ for every $g \in G$, meaning that each left coset coincides with its fellow right coset; furthermore, the set of cosets can then be endowed with a group structure (induced by that of G). The *quotient* of G by N is the resulting group, denoted by G/N. When G is finite, we have: $[G : N] = |G/N| = |G|/|N|$.

For instance, $\mathrm{Aut}(\mathbb{Q})$ is normal of index 2 in $\mathrm{Rev}(\mathbb{Q})$. The coset that elements of $\mathrm{Rev}(\mathbb{Q})$ belong to depends only on their reversing the order of the rational chain, or not.

Note that, if one can exhibit a homomorphism $G \to Q$ of which a given normal subgroup N is the kernel, one can deduce the index of N using the famous first isomorphism theorem: $[G : N] = |Q|$. The alternating group is thereby of index 2 in the symmetric group $\mathfrak{S}_n$.

2.1.4   Stabilizers

Definition 2.1.16 (Stabilizers and fixed points)**.**

- We call *poinwise stabilizer* of a subset $E = \{a_1, a_2, \ldots\}$ of Ω (or sometimes *fixator*) the subgroup $\mathrm{Fix}_G(E) = \{g \in G \mid g.a_i = a_i \ \ \forall i\}$. When $E = \{a\}$ is a singleton, we may also write $\mathrm{Fix}_G(\{a\}) = G_a$.

- The *(setwise) stabilizer* of a subset $E \subseteq \Omega$ is the subgroup $\mathrm{Stab}_G(E) = \{g \in G \mid g.a \in E \ \ \forall a \in \Omega\}$. The *restriction* of an action to a subset $E \subseteq \Omega$ is the group formed by the restrictions of the elements of the stabilizer: $G_{|E} = \mathrm{Stab}_G(E)_{|E}$.

- The set of fixed points $\{a \in \Omega \ : \ g.a = a\}$ of a group element $g \in G$ is denoted by Ω^g.

Lemma 2.1.17 (Properties of stabilizers)**.**
1. $b = g.a \iff G_b = g^{-1} G_a g$
2. $g.a = h.a \iff G_a g = G_a h$

From the second item of Lemma 1, we deduce immediately that if G is transitive, all stabilizers (of singletons) are conjugate.

Theorem 2.1.18 (Orbit stabilizer theorem)**.** *If G is finite, we have:*

$$|G.a| \ = \ [G : G_a] \,.$$

This derives from the bijection $g \mapsto g.a$ that links the set of cosets of G_a in G to the orbit $G.a$, and that does not require the group to be finite.

2.2   A glance at invariant theory

2.2.1   Invariant algebra of a group action, and some fundamentals about symmetric polynomials

The symmetric group $\mathfrak{S}_n$ (resp. any permutation group G of degree n) has a natural action on the algebra of polynomials $\mathbb{Q}[\underline{x}] = \mathbb{Q}[x_1, \ldots, x_n]$, by permutation of the variables:

$$\sigma.P(x_1, x_2, \ldots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

where P is a polynomial and σ a permutation from $\mathfrak{S}_n$ (resp. G).

For instance, if we take $P(\underline{x}) = x_1^3 + x_1 x_2 + x_2 x_3^2 + x_4$ and $\sigma = (1\ 2)$, we will have

$$(1\ 2).P(\underline{x}) = x_2^3 + x_2 x_1 + x_1 x_3^2 + x_4 \,.$$

For the rest of the section, G will be a finite permutation group of degree n, and $\underline{x} = \{x_1, \ldots, x_n\}$ a finite set of variables.

Definition 2.2.1. We say that a polynomial P is *invariant* under the action of G (or G-invariant for short) if it verifies $\sigma.P = P$ for all σ in G. A polynomial in $x_1, \ldots, x_n$ that is $\mathfrak{S}_n$-invariant is said to be *symmetric*.

Example 2.2.2 (Fundamental examples of symmetric polynomials)**.**

(1) The k-th elementary symmetric polynomial in $\underline{x}$ is defined by

$$e_k(\underline{x}) = \sum_{I \subset \{1,\ldots,n\}|I|=k} \left(\prod_{i \in I} x_i \right) .$$

It is the sum of all products of k different variables in $\underline{x}$. For the sake of clarity, we do not specify the variables when there is no need to. For instance, $n = 3$ hands these:

$$e_1 = x_1 + x_2 + x_3$$
$$e_2 = x_1 x_2 + x_2 x_3 + x_3 x_1$$
$$e_3 = x_1 x_2 x_3 .$$

(2) The k-th power sum polynomial in $\underline{x}$ is defined by

$$p_k = \sum_{i=1}^{n} x_i^k .$$

Define by extension, for a tuple of positive integers $k_1, \ldots, k_r$:

$$P_{(k_1,\ldots,k_r)} = \prod_{j=1}^{r} p_{k_j} .$$

Note that each e_k (resp. p_k) is homogeneous of degree k. Furthermore, each of the two families presented is algebraically independent.

Obviously, a sum or product of two invariant polynomials is invariant itself, hence the following definition.

Definition 2.2.3. The *algebra of invariants* of G is the subalgebra of $\mathbb{Q}[\underline{x}]$ that consists of the G-invariant polynomials, denoted by $\mathbb{Q}[\underline{x}]^G$ (sometimes we omit the underline since there is no ambiguity); we denote by $\mathrm{Sym}_n[\underline{x}] = \mathbb{Q}[\underline{x}]^{\mathfrak{S}_n}$ the algebra of symmetric polynomials (when the set of variables is not a relevant piece of information, we may just write Sym_n).

We cite two fundamental theorems about the algebra of symmetric polynomials.

Theorem 2.2.4 (Fundamental theorem of the symmetric functions.)**.** *We have:*

$$\mathrm{Sym}_n \simeq \mathbb{Q}[e_1, \ldots, e_n] .$$

Theorem 2.2.5. *The polynomial algebra $\mathbb{Q}[\underline{x}]$ is a finite dimensional free module (of dimension n!) over the algebra of symmetric polynomials. More explicitely, we have*

$$\mathbb{Q}[\underline{x}] \simeq \bigoplus_{j} \eta_j \mathbb{Q}[e_1, \ldots, e_n] .$$

where the η_j's are polynomials that may be chosen as the "monomials under the stairs": $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that $\alpha_i \leqslant n - i$ for every i.

2.2.2 Fundamental results of invariant theory

For the rest of the section, G will be a finite permutation group of degree n, and $\underline{x} = \{x_1, \ldots, x_n\}$ a finite set of variables.

Theorem 2.2.6 (Hilbert)**.** *The algebra $\mathbb{Q}[\underline{x}]^G$ has Krull dimension n.*

As a consequence, the families of elementary symmetric polynomials and power sums are both homogeneous systems of parameters for $\mathbb{Q}[\underline{x}]^G$; in the context of invariant theory, we also say that they are sets of *primary invariants* for G.

Theorem 2.2.7 (Hilbert)**.** *The algebra $\mathbb{Q}[\underline{x}]^G$ is finitely generated.*

The proof relies on the *Reynolds operator*, which is basically an average of G and is a projection of the polynomial algebra $\mathbb{Q}[\underline{x}]$ onto $\mathbb{Q}[\underline{x}]^G$.

A finer result is found (with a proof) in [Sta79b].

Theorem 2.2.8. *The algebra $\mathbb{Q}[\underline{x}]^G$ is Cohen-Macaulay.*

Obviously, this means that it admits a Hironaka decomposition involving $\{e_k\}_k$ (resp. $\{p_k\}_k$) as primary invariants; the set $\{\eta_j\}_j$ of (homogeneous) polynomials figuring in such a decomposition is then called a set of *secondary invariants* for $\{e_k\}_k$ (resp. $\{p_k\}_k$).

2.3 Pólya enumeration

This much is very clear: combinatorists like to count things. One incredibly efficient tool for this purpose comes from group theory and is known as Pólya's counting theory.

The classical situation is the following: you have a family of combinatorial objects (as in the beginning of the very first section of this thesis) that you want to count modulo some symmetries. As a simple example, we would like to know how many distinct necklaces with a given number N of pearls one can make out of colored pearls, with a fixed pool of colors, say red and blue. On the picture below, N is set to 8. The first two necklaces are the same, but we will consider the last one as different (we could take the turn around into account as well, but we will not in order to keep it simpler).

Let us study the case of $N = 5$, with the choices of color still set to red and blue, as a way to introduce Pólya's theory. The problem comes down to considering the objects under a group action: here, that of the cyclic permutation group $\mathcal{C}_5$ generated by some rotation of the necklace. Colorings that lay in the same orbit for the group action are considered equal, and thus counted only once. Reformulated with a more mathematical vocabulary, the problem is thus to count $\mathcal{C}_5$-orbits of colorings.

It all starts with the famous "Burnside's lemma", although this stage name is misleading (which is the reason why it has occasionally been referred to as "the lemma that is not Burnside's" instead).

For the rest of the section, let G be a finite group, X be a set G acts on (our set of objects), and X/G be the set of orbits of the action of G on X.
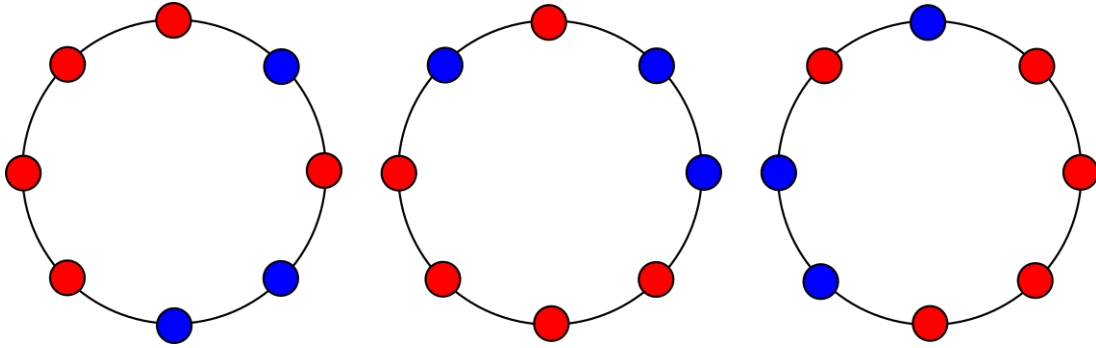
Figure 2.1: Necklaces with red and blue pearls that can be counted by Pólya

Lemma 2.3.1 (Not Burnside's). *We have*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g| \, . \tag{2.1}$$

In other words, the number of orbits for the action of G on X is the average number of fixed points of the group elements. The underlying trend at least seems reasonable: more fixed points means more orbits.

This gives a first, raw answer to our problem (although we will still need to compute the number of fixed points). Indeed, if X is the set of red and blue colorings of $\{1, \ldots, 5\}$ and G is $\mathcal{C}_5$, then X/G is the set of red and blue necklaces up to rotation and has a total number of 8 elements.

This is not bad, but what if we wanted more information, like how many of these 8 have 2 red pearls ? This is what Pólya can do for us. Let us begin with a slightly generalized, weighted version of Burnside's lemma.

Add a notion of weight on X, which is simply a function w on X, chosen to be compatible with the group action: all elements of a given orbit must share the same weight. This allows to extend the notion of weight to the G-orbits: for any h in X, define $w(\bar{h}) = w(h)$, where $\bar{h}$ is the orbit of h.

In practice, the weight will help refining and splitting the information, according to what you are interested in. Equation 2.1 becomes:

$$\sum_{\bar{h} \in X/G} w(\bar{h}) = \frac{1}{|G|} \sum_{g \in G} \sum_{h \in X^g} w(h) \, . \tag{2.2}$$

Now there is only little effort to produce in order to obtain the enumeration theorem of Pólya. In our necklaces example, the elements of X, the colorings, are essentially functions from $\Omega = \{1, \ldots, 5\}$ onto $\chi = \{\text{'blue'}, \text{'red'}\}$, and the action of $\mathcal{C}_5$ on them actually derives from its action on the set of pre-images $\{1, \ldots, 5\}$.

Let thus your X be the set of functions $\Omega \to \chi$, denoted by χ^Ω, where χ and Ω are finite sets such that G acts on Ω, and from here on $X = \chi^\Omega$.

Finally, choose the weight of each function $h \in \chi^\Omega$ to depend only on its set of

images as follows:

$$w(h) = \prod_{a \in \Omega} w'(h(a)) \tag{2.3}$$

for some weight function w' defined on χ. The weight function w' that w derives from should be chosen according to your needs: for instance, if you are interested in counting the number of necklaces per amount of red pearls, you should set $w'(\text{'blue'}) = z^0 = 1$ and $w'(\text{'red'}) = z^1 = z$, where z is an indeterminate. Using z instead of just taking 0 and 1 will help keep the pieces of information properly compartmentalized (whereas choosing weight values in $\mathbb{N}$ or $\mathbb{R}$ would obviously cause everything to merge together in the final sum).

The reader can check that the coefficient of z^n in the sum $\sum_{\bar{h} \in X/G} w(\bar{h})$ is now the number of necklaces with n red pearls, as wished for.

Now, there remains to compute the right-hand member of Equation 2.2

$$\frac{1}{|G|} \sum_{g \in G} \sum_{h \in X^g} \prod_{a \in \Omega} w'(h(a))$$

(in which 2.3 has been injected), and here comes the clever trick.

How do you characterize the functions h that are fixed points for a given g? Remember that G acts on the functions by acting on the pre-images; therefore, a fixed point h for g is a function which is constant on each cycle of g. Once you are convinced of that, choosing a function in X^g comes down to choosing an image for each cycle of g — which totally decorrelates the enumeration from the pre-images Ω (just assimilating G to the permutation group associated with the group action gets complete rid of Ω), and makes it way simpler.

Recall the definition of power sum polynomials from Example 2.2.2, and we are now ready for Pólya's classical theorem.

Theorem 2.3.2 (Pólya enumeration). *We have*

$$\sum_{\bar{h} \in X/G} w(\bar{h}) = \frac{1}{|G|} \sum_{g \in G} p_{\mathrm{CT}(g)}(w'(\chi)) = \frac{1}{|G|} \sum_{g \in G} \prod_{k \in \mathrm{CT}(g)} p_k(w'(\chi)) \tag{2.4}$$

where $\mathrm{CT}(g)$ *denote the cycle type of (the permutation induced by) g and $w'(\chi)$ is the set of images of w' on χ.*

In the case of our necklaces, we obtain $1 + z + 2z^2 + 2z^3 + z^4 + z^5$, which hands the repartition of the 8 necklaces with 5 pearls according their amount of red pearls (or blue, since the situation is symmetric).

Note that each term $p_{\mathrm{CT}(g)}(w'(\chi))$ of the sum only depends on the cycle type of g, not g itself, so one can choose to sum over the conjugacy classes of G, and divide the total by the number of classes instead of the order of the group $|G|$. This can be particularly interesting when dealing with a large group, especially if there are few conjugacy classes.

The applications of this technique are almost infinite (we could have counted cubes with colored vertices, or faces, up to symmetries, or many other things). The

formula can be adapted to trickier cases, for instance by slightly changing the way to define the weight in Equation 2.3. We will eventually come to this as a way to exploit the main result of this thesis.

Part II

Conjectures of Cameron and Macpherson

and some preliminary work towards their resolution

Summary

In this second part, we dive into the domain that this thesis originates from: the study of profiles of relational structures and oligomorphic groups.

Chapter 3 is an overview of the domain and an introduction to the two conjectures that provided this thesis with an aim. It begins with the general case of relational structures, in order to provide both some context and useful notions for when we will be studying oligomorphic groups, as well as an alternative point of view. We then move on to the special case of oligomorphic groups, being careful to expose the precise links between the two viewpoints. Eventually, we present the conjecture we are going to work on for the rest of the thesis, a conjecture of Macpherson according to which the orbit algebra would be finitely generated as soon as the profile of the group is bounded by a polynomial. We will call groups from this class P-oligomorphic.

In Chapter 4, we step towards a proof of the conjecture of Macpherson, by studying examples and uncovering our first hints on the structure of P-oligomorphic groups. In particular, some progress is made towards meeting the hope we may conceive of splitting the problem, and address it on smaller, simpler portions — from which we would hopefully be able to derive the general case. With this goal in mind, and inspired by how enlightening the study of monomorphic decompositions (which we chose not to talk about; see for instance [PT18] to learn more) proved in the case of relational structures, we bring up the notion of block systems and of primitive groups. Then, seeking for appropriate ways to bring pieces (of various natures) back together in our context, we turn to the concept of subdirect product of groups.

Chapter 3

Profiles and orbit algebras
State of the art and conjectures

This chapter deals with relational structures and oligomorphic groups, and intends to present the conjectures we will focus on for the rest of the document.

In Section 3.1, we introduce the profile, the age, and later the age algebra of a structure, in Subsection 3.1.4. We cite some important known facts about them, and also allude to the special case of homogeneous structures, which we will be particularly interested in.

Section 3.2 is then dedicated to oligomorphic groups. We provide some definitions, and the (original) name of "P-oligomorphic" to designate groups with a profile that is bounded by a polynomial. We are then able to expose the conjecture of Cameron, that we aim at proving: the profile of P-oligomorphic groups is asymptotically equivalent to a polynomial. We finish identifying oligomorphic groups to the case of homogeneous structures in Subsection 3.2.2, where we also reduce the study of our problem to closed groups for a certain topology. In Subsection 3.2.3, we present an alternative way of constructing the orbit algebra, the analog of the age algebra in this case, and finally ask, as Macpherson did: is the orbit algebra of a P-oligomorphic group finitely generated? A positive answer would imply Cameron's conjecture.

3.1 Relational structures and their profiles

3.1.1 Relations and relational structures

Recall the notion of *relation* from Definition 1.3.1 of Part I: in a nutshell a collection of subsets (resp. tuples) of a set.

Definition 3.1.1. • A *relational structure* R is a set Ω, called the *domain* or *basis* of the structure, endowed with some relations ρ_i indexed by a finite or countably infinite set I. We use the notation $R = (\Omega, (\rho_i)_{i \in I})$.

• The vector of *arities* of its relations is called the *signature* of R.

- A *substructure* of R is a relational structure of which the domain is a subset E of Ω and the relations are the restrictions to this subset of the ρ_i's: a subset of elements of the substructure is in relation if and only if it is in R (for each ρ_i respectively). The substructure of domain $E \subset \Omega$ may also be referred to as the *restriction* of R to E.

Example 3.1.2.

(1) Graphs are relational structures endowed with a single binary relation, "being linked by an edge", thus of signature 2 (allowing ourselves to forget about the parentheses in this case). One may add some unary relations to obtain a colored graph, or get an oriented graph by giving an orientation to the binary relation, or a hypergraph by adding relations.

(2) Sets endowed with a total order, such as the rational numbers $(\mathbb{Q}, \leqslant)$, are particular cases of relational structures, called *chains*. More generally, posets are relational structures of signature 2.

3.1.2   Local isomorphisms, age and profile

A *morphism of relational structures* f between R and R' is a map between their respective domains $\Omega \mapsto \Omega'$ such that, for any subset (or tuple) $\{a_1, a_2, \ldots, a_n\} \subset \Omega$ and any relation ρ, we have: $\{a_1, a_2, \ldots, a_n\} \in \rho \iff \{f(a_1), f(a_2), \ldots, f(a_n)\} \in \rho$.

Definition 3.1.3. A *local isomorphism* of a relational structure R is a bijective morphism between two restrictions of R to two subsets F and F' of Ω; it is a *local automorphism* if F equals F', and a *global automorphism* (or just *automorphism*) if both are equal to Ω.

The set of local isomorphisms endowed with the classifical composition of functions is not a group, only a *groupoid*: elements cannot always be composed). Nevertheless, one can still speak of its natural action on the elements of (the domain of) the relational structure, and consider the classical notion of orbit for this action.

Just like any action on a set of elements, this one induces an action on the set of (finite) subsets, as we already described in Example 2.1.7 and recall here: if f is a local isomorphism, the image by f of a subset $\{a_1, a_2, \ldots, a_n\}$ of Ω for this action is simply given by

$$f.\{a_1, a_2, \ldots, a_n\} = \{f(a_1), f(a_2), \ldots, f(a_n)\},$$

where the parentheses are used to denote the action on elements.
Of course, this action on subsets may also be seen as an action on substructures. The following notion was introduced by Fraïsse in [Fra00].

Definition 3.1.4. The orbits of substructures of a relational structure R under the action of its local isomorphisms are called the *isomorphism types* of R; we say that substructures of size n lie in isomorphism types of *degree* n. Altogether, these isomorphism types are the *age* $\mathcal{A}_R$ of R, that is partitionned according to the degree: $\mathcal{A}_R = \sqcup_n \mathcal{A}_n(R)$.

Example 3.1.5. Let $\mathscr{A}$ be a finite or countably infinite alphabet, of cardinality γ. Then the set of finite words over $\mathscr{A}$ may be viewed as the age of a colored version of the rational chain $\mathbb{Q}$, such that every color in a set of γ colors (interpreted as a unary relations) appears along the chain between two distinct rational numbers.

Definition 3.1.6. The *profile* of a relational structure R is the function that maps each $n \in \mathbb{N}$ onto the number of substructures of R of size n up to an isomorphism: in other words, it is the counting function of the isomorphism types of the structure per degree.

Example 3.1.7. The graph of vertices indexed by $\mathbb{N}$ and edges $(i, i + 1)$ for all $i \in \mathbb{N}$, like an infinite half-path, has its isomorphism types in natural bijection with the integer partitions.

Figure 3.1: Graph of profile equal to the number of partitions of n

Example 3.1.8. Consider the graph consisting in the juxtaposition (*i.e.* the direct sum of graphs) of two countably infinite complete graphs. Its profile counts the number of partitions of n into at most 2 parts: $\varphi(n) = \lfloor \frac{n}{2} \rfloor + 1$. We find the same profile for infinitely many complete graphs of size 2.
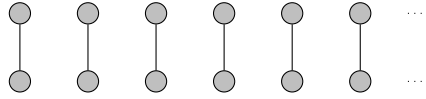
Figure 3.2: Infinitely many complete graphs of size 2.

We can see here that even though two isomorphic structures do yield the same profile, the reciprocal is false: the profile is not a complete invariant of the structure.

It is very natural to wonder about the general behavior of profiles, and in particular about their growths rates (see Subsection 1.1.2 for more details on growth rates).

Under some mild conditions on the choice of the structure R, the pool of potential growth rates for a profile presents "jumps". For instance, no profile grows as $\log(n)$ or $n\log(n)$.

Definition 3.1.9. The *kernel* of a relational structure R is the set $\ker R$ of elements $a \in \Omega$ such that the age of $R_{|\Omega \setminus \{a\}}$ is different from that of R (so smaller).

Example 3.1.10. If, in a relational structure R over Ω, every subset of a given relation ρ contains a fixed element a, then a lies in the kernel of R. In a graph Γ, if there is only one element of a certain arity (meaning here the number of neighbors), it lies in the kernel of Γ. If Γ has only finitely many edges, every vertex involved in an edge is an element of the kernel (since the induced subgraph on these vertices is alone in its isomorphism type).

We state but do not prove the following theorem, due to Pouzet.

Theorem 3.1.11 (Pouzet [Pou78; Pou76]). *Let R be a relational structure on a countably infinite set. Then, provided that either the signature of R is bounded or its kernel is finite, the growth of its profile φ_R is polynomial in the weak sense as soon as φ_R is bounded above by a polynomial.*

(One can find a reminder of the definition of "polynomial in the weak sense" in Definition 1.1.2.)

Considering this, it is natural to ask, for instance:

Question 3.1.12. Let R be a relational structure on a countably infinite set, with finite kernel. Is the profile of R equivalent to a polynomial: $\varphi_R(n) \sim cn^k$ for some $c \geqslant 0$ and $k \in \mathbb{N}$?

This question still does not have an answer in general, although Pouzet and Thiéry proved a particular case in [PT13, Theorem 1.7]; this thesis solves another important particular case, that we will dwell on very soon.

3.1.3   Automorphism group, homogeneous structures

Just like local isomorphisms, the *automorphism group* $\mathrm{Aut}(R)$ of a relational structure R has a natural action on the set $\mathscr{E}(\Omega)$ of finite subsets of Ω (see again the fifth item of Example 2.1.7). As this is the action we are interested in in this thesis and unless stated otherwise, we will always be referring to orbits of subsets when using the word "orbit(s)", rather than orbits of elements.

Let us highlight the following particular case, which will be of special interest to us.

Definition 3.1.13. A relational structure R of which every local isomorphism can be extended into a global automorphism is said to be *homogeneous*.

In this case, the orbits of the groupoid are actually the orbits of a group. Profiles associated to such homogeneous structures are called *orbital profiles* .

Example 3.1.14.

(1) The graph consisting of k countably infinite complete graphs set side by side (direct sum of graphs) is homogeneous, but the half-path of Example 3.1.7 is not, since its only automorphism is the identity.

(2) The rational chain $(\mathbb{Q}, \leqslant)$ is homogeneous.

Remark 3.1.15. The case of homogeneous structures is not that marginal, as suggested by the following example.
The *Rado graph*, defined as the unique countable graph that contains all finite graphs as induced subgraphs, is homogeneous. It turns out that one can alternatively define it as the graph obtained with probability 1 if for any pair of vertices one flips a coin to decide if it is an edge (the reason why this graph is sometimes named the random graph). In other words, a countably infinite graph is homogeneous with probability 1. Of course, graphs are still a very particular case of relational structures.

3.1.4 Age algebras and what they tell us about profiles

In this whole subsection, let $R = (\Omega, (\rho_i)_{i \in I})$ be a relational structure.

As emphasized in Chapter 1 of Part I, when one wants to study a counting function, such as the profile of R, a good idea is to add a structure of graded algebra on the counted objects. This is what was accomplished by Peter Cameron in this case, thanks to a product based on the disjoint union product of the set algebra, that we presented in Example 1.2.13 of Part I.

We chose to present here a quite abstract definition (figuring in [PT13] for instance) of what is called the *age algebra* of a relational structure — as oppposed to the more combinatorial way in which we will later introduce the *orbit algebra* of a permutation group, in order to provide both viewpoints on these closely related objects.

In the set algebra $\mathbb{Q}[\mathscr{E}_\Omega]$ of Ω as we defined it, a way to formalize the notion of infinite linear combination of finite subsets is to see them as maps $\alpha : \mathscr{E}(\Omega) \mapsto \mathbb{Q}$. A finite subset F of Ω, seen as an element of $\mathbb{Q}[\mathscr{E}_\Omega]$, is identified to its indicator function, that maps F to 1 and anything else to 0.

The definition of the disjoint union product can then be rephrased using this new vocabulary. The product of two maps α_1 and α_2 is defined as follows:

$$(\alpha_1 \cdot \alpha_2)(F) = \sum_{F_1, F_2 \in \mathscr{E}(\Omega) \,/\, F_1 \sqcup F_2 = F} \alpha_1(F_1)\alpha_2(F_2)$$

for every F in $\mathscr{E}(\Omega)$.

An R-invariant element of $\mathbb{Q}[\mathscr{E}_\Omega]$ is then a map α that verifies $\alpha(F) = \alpha(F')$ whenever we have $F \simeq F'$ in R (in order words, F and F' have the same isomorphism type).

It is easily seen that the product of two invariant maps is invariant as well, leading to the following definition.

Definition 3.1.16. The *age algebra* of R is the subalgebra of $\mathbb{Q}[\mathscr{E}_\Omega]$ that consists of its R-invariant elements. We denote it by $\mathbb{Q}[\mathcal{A}_R]$.

Viewing an isomorphism type as a collection (presented as a formal sum) of subsets of Ω reveals a correspondence between invariant maps and linear combinations of isomorphism types (finite combinations if R has only finitely many isomorphism types of each degree; refer to the definition of set algebra if this is unclear).

This enables to truly see the age algebra as a structure of commutative algebra on the age of R. One can check that it is graded according to the isomorphism type degree.

Therefore, the Hilbert series of the age algebra matches the generating series of the profile of R:

$$\mathcal{H}_R(z) = \sum_n \varphi_R(n) z^n$$

in which the notation for the Hilbert series of $\mathbb{Q}[\mathcal{A}_R]$ has been simplified for more clarity.

Example 3.1.17. The shuffle algebra introduced in Example 1.2.12 of Part I is isomorphic to the age algebra of the relational structure described in Example 3.1.5. It is also homogeneous [PT13, p. 27].

Note that, as suggested by the name, the age algebra of a relational structure only depends on its age, a fact both noted and proved by Pouzet in his survey [Pou06].

Some properties of the profile of R can be derived from properties of the age algebra, as we will illustrate right away. The following technical result was proved by Cameron in [Cam97].

Proposition 3.1.18. *Let ϵ be the map that maps every singleton to 1 and the rest to 0. If R is infinite, then ϵ is not a zero divisor: $\forall u \in \mathbb{Q}[\mathcal{A}_R],\ u\epsilon = 0 \ \Rightarrow \ u = 0$.*

This has a nice consequence on the profile (see [Pou76] or [Cam90, Section 3.1]).

Theorem 3.1.19. *The profile of an infinite relational structure is non decreasing.*

Indeed, the image of the homogeneous component $\mathbb{Q}\mathcal{A}_n(R)$ by the multiplication by ϵ^m is an independent family of $\mathbb{Q}\mathcal{A}_{n+m}(R)$.

Remark 3.1.20. If the age algebra is finitely generated, Corollary 1.2.6 states that the profile is a quasi-polynomial: $\varphi_R(n) = \alpha_d(n)n^d + \cdots + \alpha_0(n)$ with periodic α_i's. Theorem 3.1.19 implies in this case that the leading coefficient function α_d is actually constant, which answers positively Question 3.1.12 in this particular case.

The following theorem was first conjectured in a particular case by Cameron and then proved in all generality by Pouzet in [Pou08] (see also [Pou06, Theorem 29]).

Theorem 3.1.21. *(Pouzet) If the kernel of R is empty, then its age algebra is an integral domain.*

3.2   Oligomorphic groups and their orbit algebras

3.2.1   Orbital profiles and (P-)oligomorphic groups; conjecture of Cameron

For this whole section, let G be a permutation group on the countably infinite domain Ω.

Definition 3.2.1. For each $n \in \mathbb{N}$, consider the action of G on subsets of size n, as defined in Example 2.1.7, item (5), and set $\varphi_G(n)$ to be the number of orbits for this action. The *profile* of G is the resulting counting function φ_G.
The group is called *oligomorphic* if its profile takes only finite values. The set of all orbits of finite subsets of G can be called (with slight abuse) the *age* of G.

Example 3.2.2.

(1) Take $G = \mathfrak{S}_m$ for some $m \in \mathbb{N}$. The profile is 1 until $n = m$ and 0 beyond. As far as the infinite analog $\mathfrak{S}_\infty$ is concerned, the profile is constantly equal to 1, as well as the one of $\mathrm{Aut}(\mathbb{Q})$.

(2) On the opposite, the trivial permutation group Id_m on m elements (or identity group) has the biggest possible profile for this size of domain, which is the number of subsets of the domain: $\varphi_{\mathrm{Id}_m}(n) = \binom{m}{n}$. The infinite analog of the identity group is not (at all) oligomorphic.

(3) The automorphism group W_k of the direct sum of k infinite complete graphs is oligomorphic: $\varphi_{W_k}(n) = p_k(n)$ the number of integer partitions of n into k parts at most. Indeed, two subsets of which the repartition of the elements in each complete graph define the same integer partition are isomorphic.
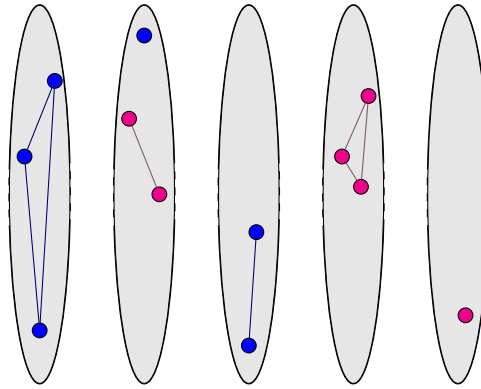
Figure 3.3: Direct sum of infinite complete graphs. The two induced subgraphs, respectively blue and red, both associated to the partition (3,2,1) are isomorphic.

The announced aim of this thesis is to explore some questions regarding a specific class of oligomorphic permutation groups, which we decided after a while to give a proper name to.

Definition 3.2.3. We call a permutation group *P-oligomorphic* if its profile is bounded above by a polynomial.

Not to be pronounced as "polygomorphic" !

Here is now, at last, one of our two star conjectures, that we will be able to solve with this thesis.

Conjecture 3.2.4 (Cameron [Cam90] in 3.6). *The profile of a P-oligomorphic permutation group is asymptotically equivalent to a polynomial.*

3.2.2 A parenthesis on topology and the link with relational structures

This subsection is heavily based on 2.3 and 2.4 of [Cam90].

Let Ω be a countably infinite set, and let us introduce, maybe a bit out of the blue, a natural topology defined on the symmetric group $\mathfrak{S}(\Omega)$, which is that of *pointwise convergence*. Consider any enumeration $a_1, a_2, \ldots$ of the domain Ω. Then,

a sequence $(\sigma_k)_k$ of permutations tends to the limit σ if and only if, for any $i \in \mathbb{N}$, we have $\sigma_k(a_i) = \sigma(a_i)$ for all n large enough. Roughly speaking, the σ_k's coincide with σ on more and more points as k grows.

The pointwise convergence topology makes multiplication and inversion continuous: we say that $\mathfrak{S}(\Omega)$ is a *topological group*. Explicitly, we have:

$$\begin{cases} \sigma_k \to \sigma \\ \tau_k \to \tau \end{cases} \implies \begin{cases} \sigma_k \tau_k \to \sigma\tau \\ \sigma_k^{-1} \to \sigma^{-1} \end{cases} .$$

This topology may be derived from the following metric:

$$m(\sigma, \tau) = \begin{cases} 0 & \text{if } \sigma = \tau \\ 1/2^i & \text{if } \sigma(a_j) = \tau(a_j) \ \ \forall j < i \ \ \text{but} \ \ \sigma(a_i) \neq \tau(a_i) \end{cases} .$$

Following is a slightly improved version of the metric, in the sense that this one makes $\mathfrak{S}(\Omega)$ into a complete metric space:

$$m'(\sigma, \tau) = \max(m(\sigma, \tau), m(\sigma^{-1}, \tau^{-1})) .$$

It does define the same topology as the inversion is continuous.

In this whole thesis, when we mention a notion of *closure*, we will always be referring to this topology.

Besides, in our study of P-oligomorphic groups, working with closed groups will be both possible and relevant, as justified by the rest of the subsection.

Remark 3.2.5. The infinite symmetric group $\mathfrak{S}_\infty$ as we defined it is the closure of the symmetric group of finitely supported permutations $\cup_n \mathfrak{S}_n$, as any infinitely supported permutation can be approached by a sequence of finitely supported ones.

The closure of the infinite analog of the alternating group is the whole $\mathfrak{S}_\infty$: indeed, every permutation of $\mathfrak{S}_\infty$ can be obtained as a limit of finitely supported even permutations (just take a sequence of permutations that does the job and "correct" the uneven ones with a transposition involving "far enough" elements). In a nutshell, when considering only closed permutation groups, the notion of alternating group is not relevant any more.

Remark 3.2.6. The property of closure is reasonably robust, as it is stable under taking restrictions or setwise or pointwise stabilizers for instance.

We highlighted in the previous section that, sometimes, the isomorphism types of a relational structure R are the orbits (of finite subsets, as always) of a group, its automorphism group $\mathrm{Aut}(R)$: R is homogeneous. On the other hand:

Proposition 3.2.7. *Given any permutation group G on a countably infinite set Ω one can associate to G a relational structure R of domain Ω such that:*

  (i) *G is a subgroup of $\mathrm{Aut}(R)$*
  (ii) *G and $\mathrm{Aut}(R)$ have the same age.*

Proof. This can be done by brute force. Indeed, consider the age of G: $\{O_1, O_2, \ldots\}$, and create for every orbit O_i a relation ρ_i on Ω, of arity the degree of the orbit, such that a subset is in ρ_i if and only if it is in O_i. This ensures (i), and the fact that $\mathrm{Aut}(R)$-orbits are unions of G-orbits. Furthermore, take two subsets E_1 and E_2 in the same $\mathrm{Aut}(R)$; then they are in the same relations by definition of $\mathrm{Aut}(R)$, which here means one and only one relation since these ones are a partition of the set of all finite subsets of Ω. Hence, they lie in the same G-orbit. $\qquad\square$

Note that this structure can be made *homogeneous* by considering tuples (those with distinct elements are enough) instead of subsets, and oriented relations. This way, the only local isomomorphisms still allowed are restrictions of elements of the group.

The homogeneous structure we just described is called the *canonical relational structure* associated with G. Although this structure and the exposed construction do the job for any permutation group, they are not quite economical in relations...

Remark 3.2.8. What Proposition 3.2.7 demonstrates is that profiles of groups can be regarded as a particular case of profiles of relational structure, the homogeneous case. Put otherwise, the definitions of profiles of permutation groups and orbital profiles are equivalent.

The link with the topology of pointwise convergence is displayed below.

Proposition 3.2.9. *A permutation group on Ω is closed if and only if it is the automorphism group of a relational structure R on Ω.*

Proof. Assume $\sigma_n \to \sigma$, with $\sigma_n \in \mathrm{Aut}(R)$. For any tuple t of Ω, we have $\sigma_n.t = \sigma.t$ for n sufficiently high, so $\sigma.t$ satisfies a relation of R if and only if t does (since σ_n preserves the relations of the structure by definition). Hence σ is in $\mathrm{Aut}(R)$, which is closed. Conversely, if G is closed and R is its canonical relational structure, then for any σ in $\mathrm{Aut}(M)$ and tuple t of Ω, according to condition (ii) of 3.2.7, there exists a $\sigma' \in G$ such that we have $\sigma.t = \sigma'.t$. Let $a_1, a_2, \ldots$ be an enumeration of Ω and σ_n be the element σ' obtained when t is $(a_1, a_2, \ldots, a_n)$, then we have $\sigma_n \to \sigma$, which implies that σ is in G since G is closed. $\qquad\square$

What we actually proved is the slightly more specific result:

Proposition 3.2.10. *The closure of G in $\mathfrak{S}(\Omega)$ is the automorphism group of its canonical structure. In particular, a permutation group and its closure have the same age (and thus the same profile).*

Let us get back to the *kernel* of the canonical structure associated to G, which we can call the kernel of G for short. Observe that if the G-orbit of an element e of Ω is finite, then this orbit, seen as a subset, is a fixed point for (the action on subsets of) G. In other words, it is alone in its own orbit of subset(s), and thus e is necessarily an element of the kernel. Conversely, if the orbit of e is infinite, then each orbit of subsets of G will have a representative that does not contain e. Indeed, consider the orbit of an arbitrary subset

$$G.\{a_1, \ldots, a_k\} = \{\{g(a_1), \ldots, g(a_k)\} \; : \; g \in G\}.$$

If e is in the same orbit of elements as some of the a_i's, there are still infinitely many choices of g, for each concerned i, such that $g(a_i)$ is different from e; and there are only finitely many a_i's. This enables us to give the following, equivalent definition of the kernel of a group, which will be easier to manipulate in the sequel of the thesis.

Definition 3.2.11. The *kernel* of a permutation group G is the union of its finite orbits of elements.

As a direct consequence of the definition, one has the following lemma.

Lemma 3.2.12. *The kernel of an oligomorphic permutation group is finite.*

This is just saying that a union of $\varphi_G(1) < \infty$ finite subsets is still finite. It allows to refine a bit our definition (or knowledge) of P-oligomorphic groups. Indeed, using Theorem 3.1.11, one can derive:

Proposition 3.2.13. *The profile of a P-oligomorphic group is polynomial in the weak sense.*

3.2.3   Orbit algebra of an oligomorphic permutation group

Let G be an oligomorphic permutation group, typically of countably infinite domain Ω. Then again, one can define a structure of graded algebra on the orbits (of finite subsets, always) of G, such that the Hilbert series of the algebra and the generating series of φ_G coincide.

Let us consider the vector space $\mathbb{Q}\mathcal{A}_G$ formally generated by the age of G, as described at the beginning of Subsection 1.2.2 of Part I; we need to endow it with a graded product. We will define such a product on the orbits, before extending the definition by linearity.

This vector space embeds naturally into (the underlying vector space of) the set algebra $\mathbb{Q}[\mathscr{E}_\Omega]$ of Ω, by identifying each orbit to the formal sum of the subsets it contains:

$$\iota \; : \; O = \{e_1, e_2, \ldots\} \; \longmapsto \; \iota(O) = e_1 + e_2 + \cdots .$$

For instance, an orbit of degree 3 will be mapped to a formal sum of subsets of size 3 of Ω. Now, to define the product of two orbits O and O', identified to a sum of subsets, we are going to use the disjoint union product of the set algebra.

This will be clearer on an example, that we chose finite for the sake of both simplicity and saving trees.

Consider the cyclic permutation group $\mathcal{C}_5$. Figure 3.4 shows how to (naively) perform the multiplication between the orbits $O_1 = \mathcal{C}_5.\{1\}$ (represented in pink) and $O_2 = \mathcal{C}_5.\{1, 2\}$ (in blue): first, proceed to the identification; then, develop and use the product on subsets to obtain a linear combination of subsets of size $1+2 = 3$ (this is automatic since the disjoint union product is naturally graded).

The subsets are represented with labels on the elements, the orbits without labels. Colors are just meant to make visualization easier. Last line shows that some subsets appear several times in the sum, and can be put together. In the end, subsets with contiguous elements appear twice, once for each possible position of
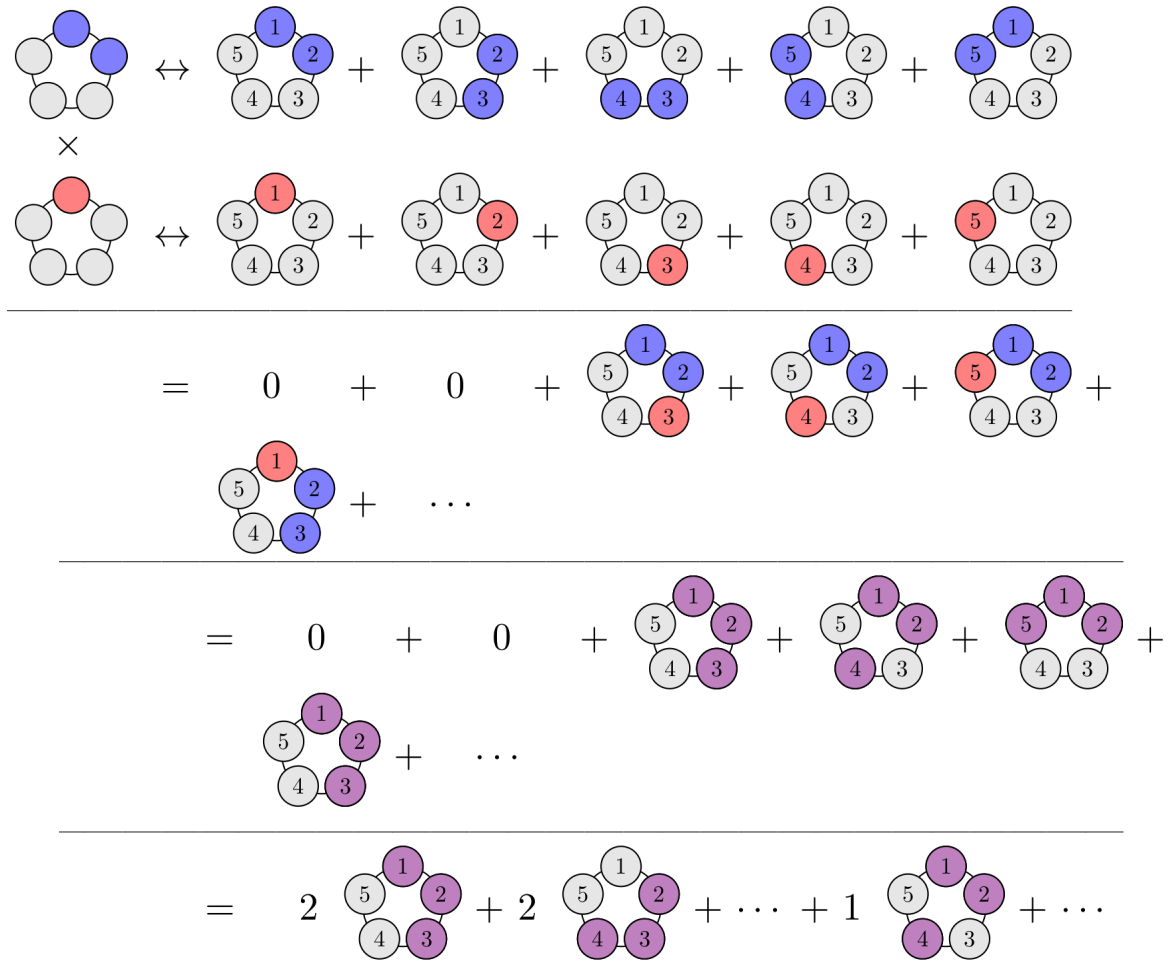
Figure 3.4: Exemple of an orbital product on the finite case of $\mathcal{C}_5$

the pink element; and those with one element apart from the two others appear once, since there is only one way to obtain them as a disjoint union of blue and pink subsets. What we obtain is thus actually a linear combination of orbits (by reversing the identification ι), hence the final result:

This is actually a general fact: $\mathbb{Q}\mathcal{A}_G$, seen as a subspace of the set algebra, is stable by the product of $\mathbb{Q}[\mathscr{E}_\Omega]$, which one can begin to get convinced of by observing this example. In the infinite case, one might fear that the linear combination be infinite, or that the coefficients be so.

However, to begin with, the result's being homogeneous of a certain degree d (the sum of degrees of the two orbits multiplied) implies that the terms of the linear combination are necesarily finitely many for an oligomorphic group: indeed, G admits only $\varphi_G(d) < \infty$ orbits of degree d, by definition. Furthermore, the

coefficients of the result are obtained following a certain rule, that one can observe on the example. Let O be an orbit, and F be any subset in that orbit; then the coefficient of O in the expansion of $O_1 \cdot O_2$ is

$$C_O^{O_1,O_2} = \#\{(F_1, F_2) \,:\, F_1 \sqcup F_2 = F, \; F_1 \in O_1, \; F_2 \in O_2\}$$

a definition that does not depend on the choice of F in O.

In the appropriate vocabulary, this means that the $C_O^{O_1,O_2}$'s are the *structure constants* of the algebra we are creating, for the basis of G-orbits. And since they count ways to split a finite subset (with additional constraints), they are obviously finite.

If one needs more formalism on all this, one can refer to the construction of the age algebra of a relational structure given in Subsection 3.1.4, which can be immediately adapted to the case of permutation groups.

Since it stabilizes $\mathbb{Q}\mathcal{A}_G$ in $\mathbb{Q}[\mathscr{E}_\Omega]$, the disjoint union product can be lifted to define a product on $\mathbb{Q}\mathcal{A}_G$:

$$O_1 \cdot O_2 = \iota^{-1}\left(\,\iota(O_1) \cdot \iota(O_2)\,\right).$$

We obtain a connected graded commutative algebra on the orbits, called the *orbit algebra* of G and denoted by $\mathbb{Q}[\mathcal{A}_G]$.

Example 3.2.14. Seemingly slightly off the subject, let us mention the finite case: if G is a finite permutation group, then its orbit algebra is finitely generated (likely with redundancy) by its age, of which all elements are nilpotent of finite order (bounded by the degree of the finite group G). It is thus of Krull dimension 0.

Before diving into some more details on the orbit algebra, recall Proposition 3.2.7. It implies that the orbit algebra of G is essentially the age algebra of its canonical relational structure — so properties of age algebras can be transposed to orbit algebras, which are a particular case of these.

3.2.4 A few properties of orbit algebras, and the conjecture of Macpherson

We recall here a few technical basics about orbit algebras and orbital profiles, in particular dealing with subgroups or restrictions, that one expects indeed to be able to manipulate in a natural way.

Finally, we come to the case of P-oligomorphic groups and expose the conjecture of Macpherson.

Lemma 3.2.15 (Relations between orbit algebras)**.**

1. *Let G be a permutation group acting on E, and F be a stable subset of E. Then, $\mathbb{Q}[\mathcal{A}_{G_{|F}}]$ is both a subalgebra and a quotient of $\mathbb{Q}[\mathcal{A}_G]$.*

2. *Let G be a permutation group acting on E, and H be a subgroup, both of which being oligomorphic. Then, $\mathbb{Q}[\mathcal{A}_G]$ is a subalgebra of $\mathbb{Q}[\mathcal{A}_H]$.*

Proof. We exhibit the natural morphisms for each one of these cases.

1. We have the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{Q}[\mathscr{E}_F] & \overset{\subseteq}{\longhookrightarrow} & \mathbb{Q}[\mathscr{E}_E] \\
{\scriptstyle \iota_{G_{|F}}}\big\uparrow & & \big\uparrow{\scriptstyle \iota_G} \\
\mathbb{Q}[\mathcal{A}_{G_{|F}}] & \underset{\psi}{\overset{\phi}{\rightleftarrows}} & \mathbb{Q}[\mathcal{A}_G] \\
{\scriptstyle \iota_{G_{|F}}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_G} \\
\mathbb{Q}[\mathscr{E}_F] & \overset{\pi}{\longleftarrow} & \mathbb{Q}[\mathscr{E}_E]
\end{array}
$$

   where π is the linear morphism mapping a subset of E to itself if it is a subset of F and to 0 otherwise. The injective morphisms ι_G and $\iota_{G_{|F}}$ are reversible where needed, which allows to define the respectively injective and surjective morphisms ϕ and ψ by composition.

2. In the following diagram, ι_G and ι_H are the canonical embeddings of the orbit algebras into their set algebras.

$$
\begin{array}{ccc}
 & & \mathbb{Q}[\mathscr{E}_E] \\
 & {\scriptstyle \iota_G}\nearrow & \big\uparrow{\scriptstyle \iota_H} \\
\mathbb{Q}[\mathcal{A}_G] & \overset{\phi}{\longrightarrow} & \mathbb{Q}[\mathcal{A}_H]
\end{array}
$$

   The orbits of G are unions of orbits (of same degree) of H, and since the groups are oligomorphic these unions are finite. The image of ι_G is thus a subset of the image of ι_H, and therefore the diagram is commutative. $\qquad\square$

Lemma 3.2.16 (Direct product). *Let G and H be permutation groups acting on E and F respectively. Take $G \times H$ endowed with its natural action on the disjoint union $E \sqcup F$. Then, $\mathcal{A}_{G \times H} \simeq \mathcal{A}_G \times \mathcal{A}_H$, and $\mathbb{Q}[\mathcal{A}_{G \times H}] \simeq \mathbb{Q}[\mathcal{A}_G] \otimes \mathbb{Q}[\mathcal{A}_H]$; it follows that $\mathcal{H}_{G \times H} = \mathcal{H}_G \mathcal{H}_H$.*

Lemma 3.2.17. *Let G be a permutation group and K be a normal subgroup of finite index. Then,*

$$
\varphi_G(n) \ \leq \ \varphi_K(n) \ \leq \ [G:K]\varphi_G(n) \, .
$$

In particular, K and G share the same profile growth.

Proof. Let O be a G-orbit of elements. Since K is a normal subgroup, O splits into K-orbits on which G – and actually G/K – acts transitively by permutation; there are thus finitely many such K-orbits, all of the same size. In particular, infinite G-orbits split into infinite K-orbits, and similarly for finite ones. $\qquad\square$

We finally come to the conjecture that was the origin of this thesis subject, which was originally enunciated as a "question".

Conjecture 3.2.18 (Macpherson, 1985 [Mac85a]). *The orbit algebra of a P-oligomorphic group is finitely generated.*

In the broader context of relational structures, the analog of this conjecture was already known to be false in general: in [PT18], Pouzet and Thiéry gave a combinatorial characterization of the conditions of satisfaction of this property in a particular case (of the broader context — stay with me).

On the other hand, the property was known to be satisfied in the particular case of a bounded profile, and even a refined version (see [Pou06, Theorem 26] and [PT13, Theorem 1.5]):

Theorem 3.2.19 (Pouzet). *Let R be an infinite relational structure. Then, the following properties are equivalent:*

(i) *The profile of R is bounded.*

(ii) *The Hilbert series is of the form $\mathcal{H}_R(z) = \frac{P(z)}{1-z}$ with $P \in \mathbb{N}[z]$ and $P(1) \neq 1$.*

(iii) *The age algebra $\mathbb{Q}[\mathcal{A}_R]$ is a finite dimensional free module over the free algebra $\mathbb{Q}[\epsilon]$; in particular, it is finitely generated and Cohen-Macaulay.*

Macpherson's conjecture is stronger than Cameron's, as one can establish by combining Corollary 1.2.6 and Remark 3.1.20. In consequence, the rest of this document will be dedicated to proving this conjecture (and actually, eventually, a stronger result).

Laying the groundwork: First hints on the structure

This chapter intends to begin the work on the conjectures, by trying to get some insight into the structure of a P-oligomorphic group, and how we could manage their study.

We start in Section 4.1 with the examination of a natural particular case, which is that of infinite wreath products. When involving the infinite symmetric group and a finite group, they have some pleasant properties: namely, they are P-oligomorphic and verify the conjectures. They form a class of groups that will turn out to be more essential than one might think.

Section 4.2 introduces the key notion of block system, which is easily illustrated on wreath products. We mention the existing classification of the P-oligomorphic groups that do not have any non trivial system; their profile is actually constantly equal to 1.

Eventually, we present in Section 4.3 the concept of subdirect product, that we will need in order to handle synchronizations between different actions induced by a group — for instance actions on different orbits (of elements), or blocks.

4.1 A fundamental case study: wreath products of permutation groups

Let G and H be permutation groups acting on E and F respectively. Intuitively, the *wreath product* $G \wr H$ acts on $|F|$ copies $(E_f)_{f \in F}$ of E, by permuting each copy of E independently according to G and permuting the copies according to H. For our convenience in this thesis, we see this action directly as a permutation group, isomorphic as a group to the semidirect product $(\prod_{f \in F} G) \rtimes H$.

For instance, $\mathcal{C}_3 \wr \mathfrak{S}_2$ is the permutation group generated by the three permutations $(1\ 2\ 3)$, $(4\ 5\ 6)$ and $(1\ 4)(2\ 5)(3\ 6)$.

Examples 4.1.1 (Algebras of wreath products)**.**

1. Let G be the wreath product $\mathfrak{S}_\infty \wr \mathfrak{S}_k$. The profile counts integer partitions with at most k parts. The orbit algebra is the algebra of symmetric polynomials over
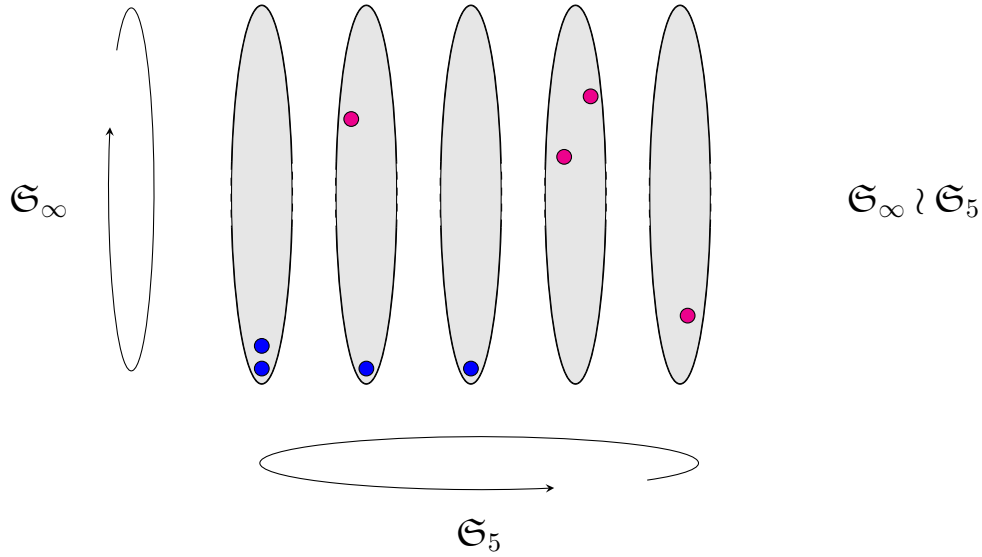
Figure 4.1: An example of a wreath product: $\mathfrak{S}_\infty \wr \mathfrak{S}_5$; the two highlighted subsets, in red and blue respectively, are in the same orbit.

k variables, that is the free commutative algebra with generators of degrees $1, \ldots, k$. The generating series of the profile is given by $\mathcal{H}_G = \frac{1}{\prod_{d=1,\ldots,k}(1-z^d)}$.

See also Figure 4.1, on which the red and blue subsets are in the same orbit. The associated integer partition $(3, 1)$ can be read on the blue subset.

2. Let G' be a finite permutation group. Then, the orbit algebra of $G = \mathfrak{S}_\infty \wr G'$ is isomorphic to the *invariant algebra* $\mathbb{Q}[x]^{G'}$ which consists of the polynomials in $\mathbb{Q}[\underline{x}] = \mathbb{Q}[x_1, \ldots, x_k]$ that are invariant under the action of G'.

3. Let G' be a finite permutation group. Then, the orbit algebra of $G = G' \wr \mathfrak{S}_\infty$ is the free commutative algebra generated by the set $\mathcal{A}_{G'}^+$ of the G'-orbits of non trivial subsets. The generating series of the profile is given by $\mathcal{H}_G = \frac{1}{\prod_d(1-z^d)}$, where d runs through the degrees of $\mathcal{A}_{G'}^+$, taken with multiplicity.

Sketch of proof. The first item is a special case of the second one, that we examine now. Two subsets having the same number of elements in each infinite block are in the same orbit, so if one canonically embeds the orbit algebra of G into the set algebra of Example 1.2.13, it is a subspace of that generated by the infinite sums

$$S_\alpha = \sum_{\text{card}(e \cap B_i) = \alpha_i} e \ , \qquad \alpha = (\alpha_1, \ldots, \alpha_k)$$

for each multi-index α of length the number of infinite blocks (that is the degree of G'), B_i being the i-th block and e the subsets of Ω. Now use the morphism $S_\alpha \mapsto X^\alpha = \prod_i X_i^{\alpha_i}$ to embed $\mathbb{Q}[\mathcal{A}_G]$ into the algebra of polynomials. The action of G' on the blocks acts the variables the same way, so the image of $\mathbb{Q}[\mathcal{A}_G]$ is the

algebra of invariants of G'.

Set $G = G' \wr \mathfrak{S}_\infty$ in order to prove the third item. A canonical one-to-one correspondence can easily be established between G-orbits and multisets of $\mathcal{A}_{G'}^+$: a finite subset of Ω consists of a disjoint union of subsets that are included in the blocks, and thus each G-orbit is determined by the non trivial G'-orbits of these subsets, while the order does not matter. Since, the Hilbert series only depends on the structure of graded vector space, it is then $\frac{1}{\prod_d (1-z^d)}$, where the d's are the orbital degrees of $\mathcal{A}_{G'}^+$, the set of generators.

Define now an alternative notion of degree δ, the number of blocks involved in (the representatives of) an orbit. In the orbit algebra $\mathbb{Q}[\mathcal{A}_G]$, the product of two orbits O_1 and O_2 has one and only one dominant term for δ, followed by lower degree terms (we say that δ is a *filtration*, but we will not go into the details about this notion). It is easy to see that the dominant term is the orbit that corresponds to the multiset $\{O_1, O_2\}$. Therefore, every G-orbit can be obtained as the dominant term of such a product of G'-orbits in $\mathbb{Q}[\mathcal{A}_G]$, and since δ decreases on the other terms, it can actually be realized as a linear combination of products of G'-orbits (this can be argued by induction on δ). Hence $\mathcal{A}_{G'}^+$ generates all of $\mathbb{Q}[\mathcal{A}_G]$.

On the other hand, and by homogeneity, the shape of the Hilbert series imply that it is also a free family of elements: one can consider the canonical morphism with the corresponding polynomial algebra (with indeterminates of the degrees of $\mathcal{A}_{G'}^+$), and deduce by dimension that it is an isomorphism. $\qquad\square$
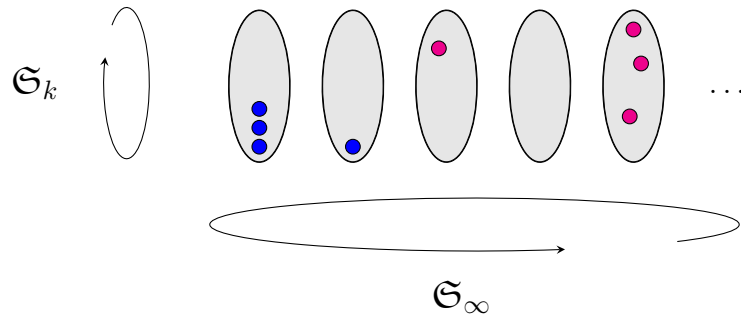
Figure 4.2: Example of a wreath product: $\mathfrak{S}_k \wr \mathfrak{S}_\infty$, with two subsets in the same orbit

We enunciate in the sequel some properties of wreath products that will prove helpful later on.

Proposition 4.1.2. *Let F_1, F_2, P_1 and P_2 be permutation groups such that F_1 (resp. P_1) is a normal subgroup of F_2 (resp. P_2). Then, $F_1 \wr P_1$ is a normal subgroup of $F_2 \wr P_2$, and we have $(F_2 \wr P_2)/(F_1 \wr P_1) \simeq (F_2/F_1) \wr (P_2/P_1)$.*

Sketch of proof. The property of normality is trivial. Denote by $((g_i)_i, p)$ the elements of $F_2 \wr P_2$, with i running through the domain of P_2, g_i in F_2 for each i and p in P_2. There is a natural correspondence between the cosets $g.(F_1 \wr P_1)$ of $(F_2 \wr P_2)/(F_1 \wr P_1)$ and the cosets $((g_i.F_1)_i, p.P_1)$ of $(F_2/F_1) \wr (P_2/P_1)$. $\qquad\square$

Corollary 4.1.3. *If P_1 and P_2 have the same finite degree m, we have*

$$[F_1 \wr P_1 : F_2 \wr P_2] = [F_1 : F_2]^m [P_1 : P_2] \,.$$

If the degree of P_1 and P_2 is infinite, $F_1 \wr P_1$ is of infinite index in $F_2 \wr P_2$ as soon as F_1 is a proper subgroup of F_2.

Sketch of proof. This is a direct consequence of Proposition 4.1.2 and the fact that a wreath product $F \wr P$ is of order $|F|^m |P|$ where m is the degree of the permutation group P. □

4.2   Blocks of imprimitivity, block systems

A key notion when studying permutation groups is that of block systems; they are the discrete analogues of quotient modules in representation theory. This notion is not specific to P-oligomorphic groups but it is the first brick in our solution for the conjecture of Macpherson, that we introduce in the first subsection. As they play in a sense a role of elementary groups, the P-oligomorphic groups with no non trivial block system will be examined (and classified up to closure) in the second subsection, based on the work of Macpherson and Cameron.

4.2.1   Blocks and block systems

Definition 4.2.1. A *set partition* $\{E_1, E_2, \ldots\}$ of the domain Ω is said to be *G-invariant* if for any $g \in G$ and any index i we have $g.E_i = E_j$ for some j (with the case $j = i$ allowed).

Definition 4.2.2. A *block system* for a permutation group G (or *system of imprimitivity* if G is transitive) is a G-invariant partition of Ω; its parts are called *blocks of imprimitivity*, or just *blocks*. Another way of phrasing it is that a block system is an equivalence relation on the domain that is preserved by the action of G.

One can give an equivalent and independent definition of the notion of block: it is a subset B of Ω such that for every $g \in G$ we have either $g.B = B$ or $g.B \cap B = \varnothing$. Indeed, starting from one such B, one can build a block system by taking the orbit of B under the action of G, and then iterate with a block B' included in the complement of B in Ω (the complement itself may be chosen as B'); and the parts of a G-invariant partition obviously satisfy the property of the second definition.

Example 4.2.3.

(1) The partitions $\{\Omega\}$ and $\{\{e\} \mid e \in \Omega\}$ are always block systems and are therefore called the *trivial block systems*.

(2) Following is the list of all block systems of the cyclic permutation group $\mathcal{C}_4$: $\{\{1,2,3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, $\{\{1\},\{2\},\{3\},\{4\}\}$.
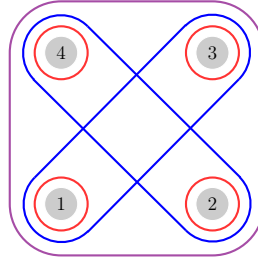
Figure 4.3: Block systems of $\mathcal{C}_4$

(3) Recall the definition of a wreath product $G \wr H$ from the beginning of Section 4.1 and use the same notations. Then, by construction, the partition $(E_f)_{f \in F}$ forms a block system, and $G \wr H$ is not primitive (unless G or H is and F or E, respectively, is of size 1).

(4) If K is a normal subgroup of G, the K-orbits (of elements) form a block system for G. This generalizes to any group action induced by G.

4.2.2 Primitive groups and classification of the (closed) P-oligomorphic ones

Definition 4.2.4. A permutation group is *primitive* if it admits no non trivial block system. By extension, an orbit of elements is primitive if the restriction of the group to this orbit is primitive.

The following two theorems will be fundamental in this thesis.

Theorem 4.2.5 (Macpherson [Mac85b] Theorem 1.1; see also [Cam90] (3.21))**.** *The profile of an oligomorphic primitive permutation group is either 1 or bounded below by an exponential.*

In the context of P-oligomorphic groups, primitive groups have thus always profile 1. These groups are classified (up to *closure*; see comment below).

Theorem 4.2.6 (Cameron [Cam90] (Section 3.4))**.** *There are only five closed permutation groups of profile 1, also called* highly homogeneous*:*
  1. The automorphism group $\mathrm{Aut}(\mathbb{Q})$ *of the rational chain (order-preserving bijections on* $\mathbb{Q}$*);*
  2. $\mathrm{Rev}(\mathbb{Q})$*, generated by* $\mathrm{Aut}(\mathbb{Q})$ *and a reflection;*
  3. $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$*, preserving the cyclic order (see* $\mathbb{Q}/\mathbb{Z}$ *as a circle);*
  4. $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$*, generated by* $\mathrm{Cyc}(\mathbb{Q}/\mathbb{Z})$ *and a reflection;*
  5. $\mathfrak{S}_\infty$*.*

In the vocabulary of model theory, these are the groups preserving, respectively, the dense linear order, the betweenness order, the circular order, the separation relation, and a pure set.

The notion of closure refers here to the topology of simple convergence, that was described in Subsection 3.2.2. Thanks to Proposition 3.2.10, it plays only a minor role for our purposes.

We make the following remark, that will prove crucial later on.

Remark 4.2.7. The collection of the five (closed) highly homogeneous groups is stable under taking finite index normal subgroups. We will sometimes refer to the three of them having no proper finite index normal subgroup $\mathrm{Aut}(\mathbb{Q})$, $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ and $\mathfrak{S}_\infty$ by "the three minimal highly homogeneous groups".

Note further that none of the proper normal subgroups of these groups (thus of infinite index) is closed. For instance and for the record, $\mathrm{Aut}(\mathbb{Q})$ has three: the subgroups of elements with supports that are bounded above, below, or below and above, respectively.

Lemma 4.2.8. *Let G be a (closed) P-oligomorphic permutation group, endowed with a block system. If an infinite G-orbit of blocks is primitive and the blocks of the orbit are not singletons, then the action on these blocks is isomorphic to $\mathfrak{S}_\infty$.*

Proof. Using Theorem 4.2.6, the action on the set of blocks is given by one of the five closed highly homogeneous groups. Assume first it is $\mathrm{Aut}(\mathbb{Q})$. Since there are at least 2 elements in each block, that makes at least two subsets which are not in the same G-orbit: simply consider one singleton and one pair of elements. Then every word using only the letters 1 and 2 can be related to at least one orbit of subsets: 1 means you take only one element in a given block and 2 that you take a pair. Since $\mathrm{Aut}(\mathbb{Q})$ permutes the blocks, the exact block you take the subsets from does not matter, only the ordering of the blocks, which is the ordering of the letters. Several orbits of subsets might be a match for a given word, but picking just one will be good enough to us. Since there are 2^m words of length m, we have highlighted this way 2^m orbits of degrees between m and $2m$, each of them contributing by 1 to a value of the profile between m and $2m$. Therefore, we may bound the partial sum of the profile below: $\sum_{n=0}^{m} \varphi_G(n) \geq \sum_{n=0}^{\lfloor m/2 \rfloor} 2^n$. The second term is clearly exponential, which cannot be with a P-oligomorphic group.

Now, if the action on the blocks was described by another one of the four non symmetric highly homogeneous groups, the words we used would just have to be considered up to a reflection or a cyclic permutation of the letters, which does not change the exponential growth. $\qquad\square$

4.3   Subdirect products and synchronization

The actions of a permutation group on two of its orbits are not independent in general; intuitively, there may be partial or full synchronization, which has consequences on the profile and the orbit algebra. A classical tool to handle this phenomenon is that of subdirect products.

4.3.1   Definitions

Definition 4.3.1. Let G_1 and G_2 be groups. A *subdirect product* of G_1 and G_2 is a subgroup of $G_1 \times G_2$ such that the canonical projections $\pi_1 : (g_1, g_2) \in G \mapsto g_1 \in G_1$

and $\pi_2 : (g_1, g_2) \mapsto g_2$ are surjective.

For instance, suppose G is a permutation group that has exactly two orbits of elements E_1 and E_2. If G_i is the group induced on E_i by G, G is a subdirect product of G_1 and G_2.

Denote by $N_1 = \mathrm{Fix}_G(E_2)$ and $N_2 = \mathrm{Fix}_G(E_1)$ the pointwise stabilizers of E_2 and E_1 respectively. Then, N_1 and N_2 are normal subgroups of G; and $N_1 \cap N_2 = \{1\}$, so we have $\langle N_1, N_2 \rangle \simeq N_1 \times N_2$, the direct product of N_1 and N_2.

Definition 4.3.2. We call *synchronization* between G_1 and G_2 the following iso-morphic quotients (up to restrictions where needed):

$$\frac{G_1}{N_1} \simeq \frac{G}{\langle N_1, N_2 \rangle} \simeq \frac{G_2}{N_2}.$$

Heuristically, these quotients describe the parts of each group that are bound together, whereas the N_i are the independent components.

Proposition 4.3.3. *Let G be a subdirect product of G_1 and G_2. With the above notations, we have*

$$G \simeq \{(g_1, g_2) \in G_1 \times G_2 \mid g_1 N_1 = g_2 N_2\}.$$

This in particular means that a permutation group arising as a subdirect product is uniquely defined by the associated G_1, G_2, N_1 and N_2.

4.3.2   Interest and first practical application

As we just saw, possible synchronizations between two groups are directly linked to their normal subgroups, which is a useful thing to know.

Recall indeed the classification of highly homogeneous groups presented in The-orem 4.2.6. From Remark 3.2.6 and Remark 4.2.7, we derive the following remark about the N_i's introduced in the previous subsection (the pointwise stabilizers asso-ciated to a subdirect product).

Remark 4.3.4. If G is a subdirect product of $G_1 \times G_2$ and G_1 (or G_2, resp.) is a closed highly homogeneous group, then N_1 (resp. N_2) is a closed normal subgroup of G_1 (resp. G_2), and as such it is either trivial or a (closed) highly homogeneous group itself.

We derive a very convenient result.

Corollary 4.3.5. *Let G be a closed P-oligomorphic permutation group. Possibilities of synchronizations between primitive orbits (of points or finite blocks) of G are limited to the following:*
  (i) *no synchronization,*
 (ii) *total synchronization,*
(iii) *synchronized reflection in the cases of $\mathrm{Rev}(\mathbb{Q})$ and $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$ (synchronization of order 2).*
On non trivial finite blocks, only the first two may occur by Lemma 4.2.8.

Part III

P-oligomorphic permutation groups
Resolution of the conjectures
and classification

Summary

This third part of the thesis is mainly dedicated to the proof of our main result: a classification of P-oligomorphic groups.

First, Chapter 5 studies how block systems can provide useful information on the structure of the group and on the algebra, from which we derive a lower bound on the growth of the profile. In search for a block system that would be most fitted for the study of a generic P-oligomorphic group, we examine the structure of lattice of the block systems of such a group, and observe that we can take advantage of this structure to try to maximize the lower bound. We approach the idea of considering block of blocks for a better efficiency, and finally construct a special "block system", called the nested block system, that we will be exploiting to prove the conjectures. It consists of finitely many infinite blocks of finite blocks, plus maybe one finite block, with some additional properties that will help us separate the problem into simpler cases.

In Chapter 6, we classify the groups of which the nested block system consists of one block only. We first study the action by permutation of the finite blocks, and find out that it can be decorrelated from the action inside each block. Then we study this latter action, for which we need to study synchronization between the blocks. The possibilities turn out to be quite limited, and we classify them — a step that was greatly facilitated by a computer exploration. We deduce the classification of all (closed) P-oligomorphic groups of this form.

Next, Chapter 7 puts the pieces back together. It uncovers a finite index normal subgroup, actually the minimal one, with a convenient shape. We then use this subgroup to describe a classification of all P-oligomorphic groups (up to closure) Finally, we obtain the shape of the orbit algebra from that of the group, and derive the conjectures of Macpherson and Cameron as corollaries, along with some other nice consequences.

5

Lattices of block systems and the nested system

In this whole chapter, G will be a (closed) P-oligomorphic permutation group.

In Section 5.1, we go back to the notion of block system and take a closer look at how we can exploit it for our purposes. First, we show that each block system provides a lower bound on the growth of the profile, and we highlight the fact that studying blocks of blocks can provide an improved bound.

Seeking to maximize this lower bound, we establish the nature of finite lattices of the posets of block systems in Section 5.2, and use it to derive, in Section 5.3 a construction of a special "block system" (for an extended version of the notion) satisfying appropriate properties. The later chapters will show that this so-called *nested block system* minimizes synchronization and provides a tight lower bound.

5.1   Lattice structures and lower bound

5.1.1   How block systems provide a lower bound on the profile

We first consider the case where the block system is *transitive*, that is G acts transitively on its blocks. In this case, all the blocks are conjugated and thus share the same cardinality.

Lemma 5.1.1. *Let G be a P-oligomorphic permutation group, endowed with a transitive block system $\mathcal{B}$. Then,*
1. *Case 1: $\mathcal{B}$ has finitely many infinite blocks, as in Example 4.1.1 (1) and (2). Then G is a subgroup of $\mathfrak{S}_\infty \wr \mathfrak{S}_k$ (where k is the number of blocks), and $\mathbb{Q}[\mathcal{A}_G]$ contains Sym_k which is a free algebra with generators of degrees $(1, \ldots, k)$.*
2. *Case 2: $\mathcal{B}$ has infinitely many finite blocks, as in Example 4.1.1 (3). Then, G is a subgroup of $G_{|B} \wr \mathfrak{S}_\infty$, and $\mathbb{Q}[\mathcal{A}_G]$ contains the free algebra with generators of degrees given by that of the non trivial orbits of $G_{|B}$.*

Note that the first case can be refined by stating that the orbit algebra contains the algebra of invariants of the finite group H acting on the blocks (which may be smaller than the full symmetric group $\mathfrak{S}_k$); but this algebra is typically not free.
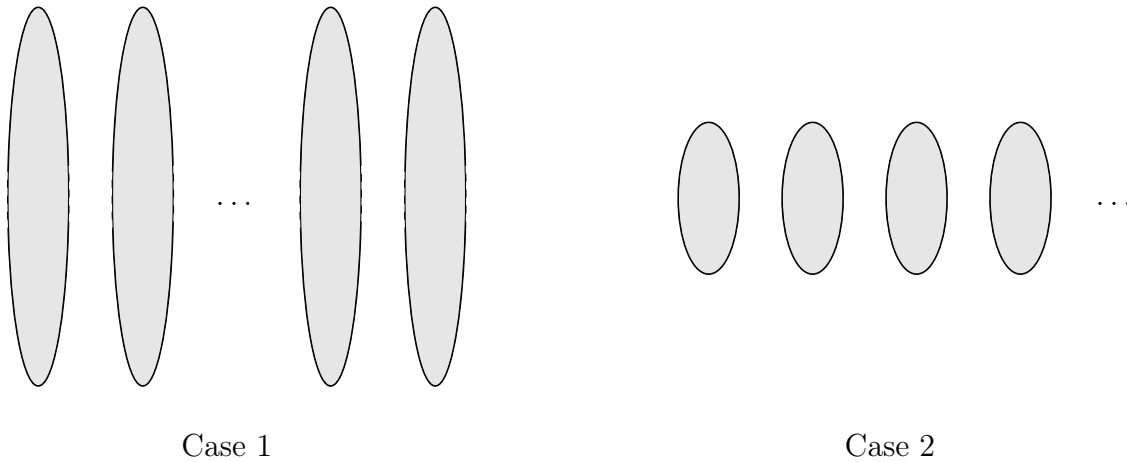
Figure 5.1: Essential cases for a *transitive* block system of a *P*-oligomorphic group

Sketch of proof. The blocks share the same size by transitivity, and if their size (resp. number) is infinite, then their number (resp. size) has to be finite in order to keep the group *P*-oligomorphic (indeed, G is otherwise a subgroup of $\mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ and its profile is bounded below by the number of integer partitions). Use Lemma 3.2.15 and Examples 4.1.1 in each case. $\square$

Assume that G is endowed with a block system. Then, the proof of the above lemma applies in the same fashion to the restrictions of G to (the support of) its orbits of blocks, leading the whole $\mathbb{Q}[\mathcal{A}_G]$ (with just one more use of Lemma 3.2.15) to also contain the mentionned subalgebras (in the case with finitely many finite blocks, refer instead to Example 3.2.14). Recall also that in Case 2, we have the convenient property of Lemma 4.2.8.

Remark 5.1.2. Let G be an oligomorphic permutation group, and $E_1, \ldots, E_k$ be a partition of E such that each E_i is stable under G. In our use case, we have a block system $\mathcal{B}$, and each E_i is the support of one of the orbits of blocks in $\mathcal{B}$.

Then, G is a subgroup of $G_{|E_1} \times \cdots \times G_{|E_k}$ (precisely, it is a *subdirect product* of this direct product). Therefore, by Lemma 3.2.15, $\mathbb{Q}[\mathcal{A}_G]$ contains $\mathbb{Q}[\mathcal{A}_{G_{|E_1}}] \otimes \cdots \otimes \mathbb{Q}[\mathcal{A}_{G_{|E_k}}]$ as a subalgebra. In particular, the algebraic dimension of $\mathbb{Q}[\mathcal{A}_G]$ is bounded below by the sum of the algebraic dimensions of the $\mathbb{Q}[\mathcal{A}_{G_{|E_i}}]$.

When in addition the actions of G on each E_i are completely independent, the containments above are equalities; then, $\mathbb{Q}[\mathcal{A}_G]$ is finitely generated if and only if each $\mathbb{Q}[\mathcal{A}_{G_{|E_i}}]$ is.

Remark 5.1.3. Combining Lemma 5.1.1 and Remark 5.1.2, each block system of G provides a lower bound on the algebraic dimension of $\mathbb{Q}[\mathcal{A}_G]$ — and therefore on the growth rate of the profile by Lemma 1.2.8.

The following example illustrates that the lower bound on the profile highly depends on the chosen block system.
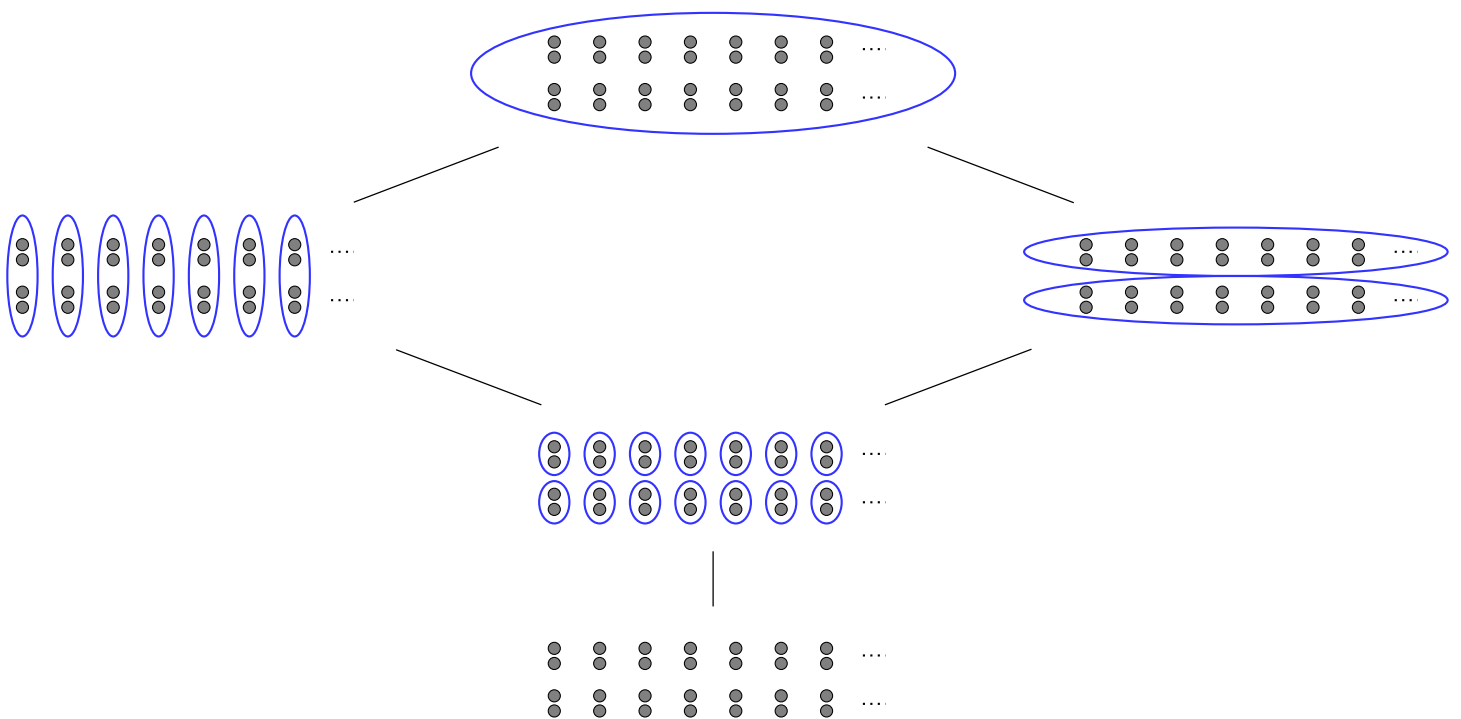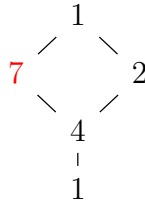
Figure 5.2: Example of a lattice of block systems, on $G = (\mathfrak{S}_2 \times \mathfrak{S}_2) \wr \mathfrak{S}_\infty$

Example 5.1.4. Let $G = (\mathfrak{S}_2 \times \mathfrak{S}_2) \wr \mathfrak{S}_\infty$. Following is the poset of all its block systems, ordered by refinement:

The picture below displays the lower bounds on the algebraic dimension that can be deduced respectively from each of these block systems, using Lemma 5.1.1 and Remark 5.1.2:

$$
\begin{array}{ccc}
 & 1 & \\
\diagup & & \diagdown \\
7 & & 2 \\
\diagdown & & \diagup \\
 & 4 & \\
 & | & \\
 & 1 &
\end{array}
$$

For instance, for the block system with two orbits of blocks of size 2, the lower bound on the algebraic dimension is $4 = 2 + 2$ since we have $G_{|B} = \mathfrak{S}_2$ in each orbit; for the block system with finite blocks of size 4, the lower bound is 7 because $G_{|B} = \mathfrak{S}_2 \times \mathfrak{S}_2$ has this many orbits of non empty subsets. This latter lower bound is obviously tight since the inclusion $G_{|B} \wr \mathfrak{S}_\infty \subset G$ is an equality: the algebraic dimension of $\mathbb{Q}[\mathcal{A}_G]$ is 7 and the growth rate of the profile of G is 6.

This example suggests that better lower bounds are obtained when maximizing the size of the finite blocks (and then maximizing the number of infinite blocks; consider also the example $\mathfrak{S}_\infty \wr \mathrm{Id}_n$ for that).

Nevertheless, the bound provided by this heuristic alone can be improved at rather low cost, as advertised by the following example.

5.1.2 Towards blocks of blocks

We study here a generic example, and bring up the idea of considering not only blocks, but also blocks of blocks. We give some hints on how we could or should do it, and describe how this may oftentimes provide a refined lower bound.

Example 5.1.5. Consider the permutation group $G = \mathcal{C}_4 \wr (\mathfrak{S}_\infty \wr \mathcal{C}_3)$; we use here the parentheses regardless of the associativity to emphasize the action of G on its natural system of infinitely many (maximal) blocks of size 4.

By Remark 5.1.3, this block system provides a lower bound of 4 on the algebraic dimension. As we will see, it is very crude; a lot of information was lost when embedding the action on the blocks $\mathfrak{S}_\infty \wr \mathcal{C}_3$ into $\mathfrak{S}_\infty$. This action was not even primitive to begin with: one can form 3 infinite blocks (of 4-blocks). Let us exploit that information. Consider the stabilizer S of the three infinite blocks of finite blocks. This is a normal subgroup of finite index of G, and therefore it has the same algebraic dimension using Lemma 3.2.17. But now that these infinite blocks of blocks are stable parts of the domain, their contributions to the algebraic dimension can be treated separately; which hands a bound of $3 * 4 = 12$ for S, and thus for G.

Let us step toward a generalization and a formalization of the phenomenon observed in the above example.
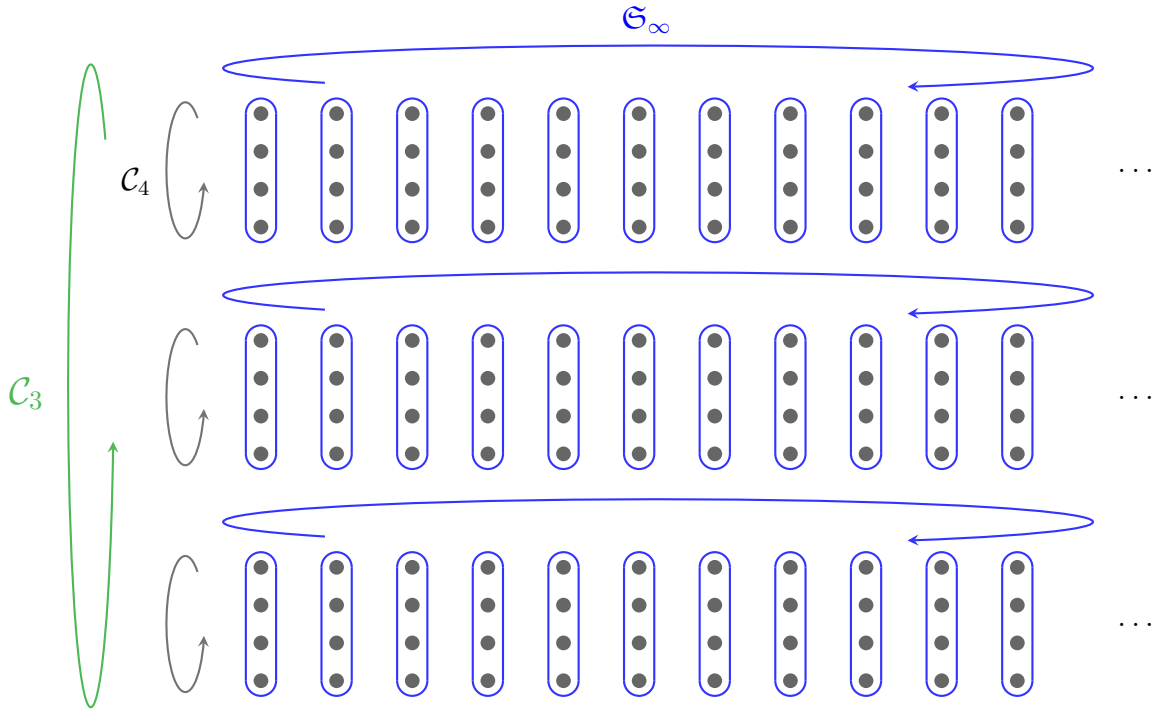
Figure 5.3: Example of $\mathcal{C}_4 \wr (\mathfrak{S}_\infty \wr \mathcal{C}_3)$

Let G be a P-oligomorphic permutation group. Take a block system $\mathcal{B}^{<\infty}$ of finite blocks only. Assume further that these finite blocks are maximal: G does not have any strictly coarser system of finite blocks. This choice is motivated by the earlier observations (see Example 5.1.4); we will see in Section 5.2 that $\mathcal{B}^{<\infty}$ always exists and is unique.

If G has some finite orbits of elements, their union forms a stable finite block which contains all other stable finite blocks (a stable block is a union of orbits, that are obviously finite if the block is).

By definition of a block system, G acts on the set of blocks of $\mathcal{B}^{<\infty}$. Furthermore, this induced action does not admit any non trivial finite block, for else the blocks of $\mathcal{B}^{<\infty}$ would not be maximal. It has no special reason to be primitive though: its block systems will just have infinite blocks only (plus possibly one singleton, if $\mathcal{B}^{<\infty}$ has a stable block) – and finitely many of them, for the same reasons as in Lemma 5.1.1. By choosing one such system, we end up with two nested block systems: an inner one with finite blocks, and an outer one with (finitely many) infinite blocks; in other words, *a finite system of infinite blocks of finite blocks*.

Remark 5.1.6. A lower bound can be obtained from such a double, nested block system by first stabilizing the infinite blocks of finite blocks (which does not change the growth of the group, as stated by Lemma 3.2.17), and then applying the same method as in Remark 5.1.3. For the same choice of (maximal) finite blocks, the lower bound L provided by this method is better than the one deduced from the matching simple block system *via* Remark 5.1.3.

This is pretty much obvious if you consider the typical case highlighted in Example 5.1.5.

As for the choice of the infinite blocks of blocks, the general intuition remains the same as with classical, simple block systems: we feel that the more the better, as far as the lower bound is concerned.

The next section formalizes these intuitions to construct a canonical block system (in fact a system of blocks of blocks) that will hopefully maximize the lower bound.

5.2 Optimizing the lower bound through lattice structures

As suggested by the previous subsection, maximizing the lower bound will involve maximizing or minimizing block systems with certain properties. To this end, we will exploit the lattice structure of the poset of block systems, which we recall now.

Proposition 5.2.1. *Let G be a permutation group G acting on a set E, finite or infinite. The poset $\mathcal{L}(G)$ of all its block systems, endowed with the refinement order, is a sublattice of the lattice of set partitions of E. Its maximum and minimum are respectively the trivial block systems $\top = \{E\}$ and $\bot = \{\{e\} \mid e \in E\}$.*

Proof. Take two block systems $\mathcal{B}$ and $\mathcal{B}'$, and consider their meet in the lattice of set partitions, namely the set partition:

$$\mathcal{B} \wedge \mathcal{B}' = \{B \cap B' \mid B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}.$$

It is straightforward to check that this is still a block system for the group. Hence this is the meet of $\mathcal{B}$ and $\mathcal{B}'$ in $\mathcal{L}(G)$.

Similarly, consider the join $\mathcal{B} \vee \mathcal{B}'$ in the lattice of set partitions. It is obtained by taking the equivalence classes of the closure of the relation "being in the same block in $\mathcal{B}'$ or in $\mathcal{B}$". There remains to check that $\mathcal{B} \vee \mathcal{B}'$ is a block system: if x and y are in the same part and σ is an element of G, then $\sigma(x)$ and $\sigma(y)$ are in the same part as well. To this end, consider a sequence $x_0, \ldots, x_k$ such that we have $x_0 = x$, $x_k = y$ and any two consecutive elements in the same block for either $\mathcal{B}$ or $\mathcal{B}'$; then the same holds for the sequence $\sigma(x_0), \ldots, \sigma(x_k)$.

In conclusion, $\mathcal{L}(G)$ is stable under both join and meet operations, and therefore a sublattice of the lattice of set partitions of E. □

In the sequel, we will consider block systems with only finite blocks (resp. only infinite blocks, up to kernel); the following propositions state that those block systems form finite sublattices. This will provide us with a canonical maximal (resp. minimal) block system from which we will derive bounds.

Proposition 5.2.2. *Let G be an oligomorphic permutation group, and $\mathcal{L}^{<\infty}(G)$ be the subposet of block systems consisting of finite blocks only. Then, $\mathcal{L}^{<\infty}(G)$ is a sublattice of $\mathcal{L}(G)$, with the trivial block system as minimum. If in addition G is P-oligomorphic, then $\mathcal{L}^{<\infty}(G)$ is finite, with a maximum $\mathcal{B}^{<\infty}$.*

Proposition 5.2.3. *Let G be a P-oligomorphic permutation group, and $\mathcal{L}^\infty(G)$ be the subposet of block systems consisting of infinite blocks only; if the kernel of G is non trivial, then finite blocks contained in the kernel are allowed as well. Then, $\mathcal{L}^\infty(G)$ is a finite sublattice of $\mathcal{L}(G)$, with a minimum and the trivial block system as maximum.*

Proving Propositions 5.2.2 and 5.2.3 will require a couple of lemmas.

Lemma 5.2.4. *Let G be a P-oligomorphic group, and L be the function that maps a block system onto the associated lower bound described in the previous subsection (Remark 5.1.3). Let $\mathcal{B} < \mathcal{B}'$ be a cover (not involving the kernel) in the lattice $\mathcal{L}^{<\infty}(G)$, then we have $L(\mathcal{B}) < L(\mathcal{B}')$.*
If instead $\mathcal{B} < \mathcal{B}'$ is a cover (not involving the kernel) in the lattice $\mathcal{L}^\infty(G)$, we have $L(\mathcal{B}) > L(\mathcal{B}')$.

Proof. Assume that $\mathcal{B} < \mathcal{B}'$ is a cover in $\mathcal{L}^{<\infty}(G)$. Pick one of the finite blocks B in $\mathcal{B}'$ that splits into two new blocks B_1 and B_2 in $\mathcal{B}$; by conjugation, the same can be said about all the other blocks in the orbit of B. There are two cases: either B_1 and B_2 may swap or they may not.

If not, then the support O_B of the orbit of B is the union of the supports O_1 and O_2 of the orbits of B_1 and B_2 (resp.), and the age of $G_{|O_B}$ contains the (disjoint) ages of the restrictions G_1 and G_2 to O_1 and O_2 (resp.). It also contains the additionnal orbits of subsets that have non empty intersections with both B_1 and B_2, so the inclusion is strict. Using Lemma 5.1.1, the provided bound on the profile is strictly better with the coarser system (since we took a cover, the situation in $\mathcal{B}$ and $\mathcal{B}'$ is the same everywhere else).

If B_1 and B_2 do swap, then we get a single orbit of (small) blocks in $\mathcal{B}$, just as in $\mathcal{B}'$; except that if one denotes by H the restriction of G to one of the small blocks in $\mathcal{B}$, the restriction to one block of $\mathcal{B}'$ is $H \wr \mathfrak{S}_2$, which has a strictly larger age. Hence, $\mathcal{B}'$ provides a better bound.

As for $\mathcal{L}^\infty(G)$, the result is rather obvious from Lemma 5.1.1. $\square$

Lemma 5.2.5. *Let G be an oligomorphic permutation group. The poset $\mathcal{L}^{<\infty}(G)$ is closed under taking joins (as defined in the lattice of set partitions).*

The following simple example illustrates that this statement may fail without the hypothesis of oligomorphism.

Example 5.2.6. Recall that the (non oligomorphic) permutation group $\mathrm{Aut}(\mathbb{Z})$ is generated by the translation $x \mapsto x + 1$, and take $G = \mathrm{Aut}(\mathbb{Z}) \times \mathrm{Aut}(\mathbb{Z})$, acting on two copies of $\mathbb{Z}$: $E = \{1, 2\} \times \mathbb{Z}$. This group admits an infinite family of block systems $(\mathcal{B}_j)_{j \in \mathbb{Z}}$ with non trivial finite blocks of size 2:

$$\mathcal{B}_j := \{\{(1, i), (2, i + j)\} \mid i \in \mathbb{Z}\}.$$

The following picture illustrates the block systems $\mathcal{B}_0$ and $\mathcal{B}_1$; their join is the trivial block system with a single infinite block.

$\mathcal{B}_0$ and $\mathcal{B}_1$
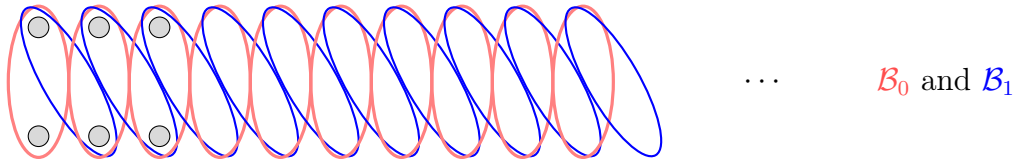
In general, the join of two of block systems $\mathcal{B}_i$ and $\mathcal{B}_j$ with $i \neq j$ is composed of infinite blocks.

Proof of Lemma 5.2.5. Assume that the join of two systems of finite blocks $\mathcal{B}$ and $\mathcal{B}'$ from $\mathcal{L}^{<\infty}(G)$ contains at least one infinite block. This block is thus a union of infinitely many blocks from both $\mathcal{B}$ and $\mathcal{B}'$, in which every block from one system intersects at least one block from the other one. If all of the blocks of $\mathcal{B}$ involved were singletons, each of them would be included in one block from $\mathcal{B}'$ and so the join would not have an infinite block; hence at least one of them, call it B_0, is not.

Consider the stabilizer S_0 of this B_0 (in red in the center of Figure 5.4). In this subgroup, the union of the blocks from $\mathcal{B}'$ having a non empty intersection with B_0 (in blue) is also stable, so as well is their set difference with B_0. One can iterate the argument with the union of blocks from $\mathcal{B}$ intersecting this stable domain (the outer crown of red blocks), and so on.
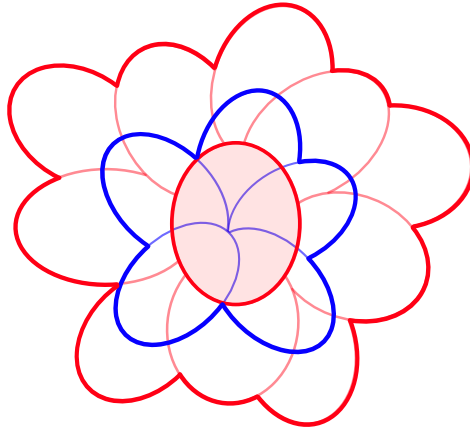
Figure 5.4: Nested stable areas arising when stabilizing one block of $\mathcal{B}$

This reveals an infinite sequence of finite disjoint (by taking the set difference every time) domains that are stable under the action of S_0, and of which the first item is B_0. Take now two distinct subsets A_1 and A_2 of E, each of them consisting of two elements in B_0 and just one in any of the other S_0-stable domains. An element of G mapping A_1 to A_2, if there is any, necessarily belongs to S_0, since the pair included in B_0 has no other choice but to be mapped onto the corresponding pair of A_2. Therefore, changing the S_0-stable domain in which we take the singleton for A_2 (or A_1) exhibit infinitely many non isomorphic subsets of size 3 for G, which is to say infinitely many orbits of degree 3, and makes G a non oligomorphic group. $\square$

Proof of Proposition 5.2.2. Thanks to Lemma 5.2.5, we already know that $\mathcal{L}^{<\infty}(G)$ is table under taking the join. We will successively prove that $\mathcal{L}^{<\infty}(G)$ is stable

under meets, locally finite, and that it admits no infinitely increasing chain. We will then conclude that it is bounded and finite.

Take two block systems $\mathcal{B}$ and $\mathcal{B}'$ in $\mathcal{L}^{<\infty}(G)$. Consider their meet in the lattice of block systems:

$$\mathcal{B} \wedge \mathcal{B}' = \{B \cap B' \neq \varnothing \mid B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}.$$

By construction, it has again finite blocks, which proves that $\mathcal{L}^{<\infty}(G)$ is stable under taking either joins or meets. In addition to this, the trivial block system $\bot = \{\{e\} \mid e \in E\}$ is obviously its minimal element.

Let $\mathcal{B}$ be an element of $\mathcal{L}^{<\infty}(G)$. Consider the interval $[\bot, \mathcal{B}]$, and take a block system $\mathcal{B}'$ in that interval. The way a block B in $\mathcal{B}$ splits into blocks in $\mathcal{B}'$ forces the way the blocks in the same orbit split in $\mathcal{B}'$ themselves. Since the blocks of B are finite, and there are finitely many orbits thereof for G is oligomorphic, there are finitely many ways of splitting these blocks. Therefore the interval $[\bot, \mathcal{B}]$ is finite, and the same holds for any interval: $\mathcal{L}^{<\infty}(G)$ is locally finite.

Take a strict chain C in $\mathcal{L}^{<\infty}(G)$. Using the local finiteness, embed this chain in a strict chain C' where each step is a cover. Thanks to Lemma 5.2.4, L is strictly increasing along that chain. Since G is P-oligomorphic, L is also bounded, and it follows successively that C' and C are finite.

This ensures the existence of a maximum $\mathcal{B}^{<\infty}$, for else we could construct an infinite chain by starting with an element and then recursively take the join with an incomparable element. We conclude by remarking that $\mathcal{L}^{<\infty}(G) = [\bot, \mathcal{B}^{<\infty}]$ is finite. $\qquad \square$

Proof of Proposition 5.2.3. The poset $\mathcal{L}^{\infty}(G)$ obviously has $\top = \{E\}$ as maximal element, and it is stable under joins: take indeed two block systems $\mathcal{B}$ and $\mathcal{B}'$ in $\mathcal{L}^{<\infty}(G)$: their blocks are infinite or included in the kernel. It is straightforward to check that the blocks of their join satisfy the same property.

Let us prove that $\mathcal{L}^{\infty}(G)$ is stable under meet. Consider the meet of two block systems in $\mathcal{L}^{\infty}(G)$:

$$\mathcal{B} \wedge \mathcal{B}' = \{B \cap B' \mid B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}.$$

It has finitely many blocks. The union of all the finite ones is finite and stable by G; it is therefore included in the kernel of G. It follows that the blocks of $\mathcal{B} \wedge \mathcal{B}'$ are either infinite or included in the kernel of G, as desired.

Consider an interval $[\mathcal{B}, \top]$ in $\mathcal{L}^{\infty}(G)$. Every block system $\mathcal{B}'$ from the interval is obtained by merging together some of the finitely many blocks of $\mathcal{B}$. Hence this interval is finite, and $\mathcal{L}^{\infty}(G)$ is locally finite.

We conclude as in the proof of Proposition 5.2.2: there are no infinite chains in $\mathcal{L}^{\infty}(G)$ (a bit of care needs to be taken since $L(\mathcal{B})$ may not be strictly increasing at steps where two finite blocks are merged; but there can be only finitely many such steps). This in turn ensures the existence of a minimal element $\mathcal{B}^{\infty}$ and the finiteness of $\mathcal{L}^{\infty}(G)$. $\qquad \square$

5.3   The nested block system

We may now use the structure of finite lattice on the block systems of a P-oligomorphic group to select a block system of a special kind (actually a system of blocks of blocks), which will maximize the associated lower bound by construction, and thus hopefully provide a best fitted set-up for the study of the group.

Definition 5.3.1. Let G be a P-oligomorphic permutation group. Take:
1. the maximal (coarsest) element $\mathcal{B}^{<\infty}$ of $\mathcal{L}^{<\infty}(G)$
2. the minimal (finest) element $\mathcal{B}^{\infty}$ of the lattice of block systems for the induced action of G on $\mathcal{B}^{<\infty}$.

We call the pair formed by the nested two partitions of E defined this way the *nested block system* $\mathcal{B}_{\mathcal{B}}(G)$ of G.
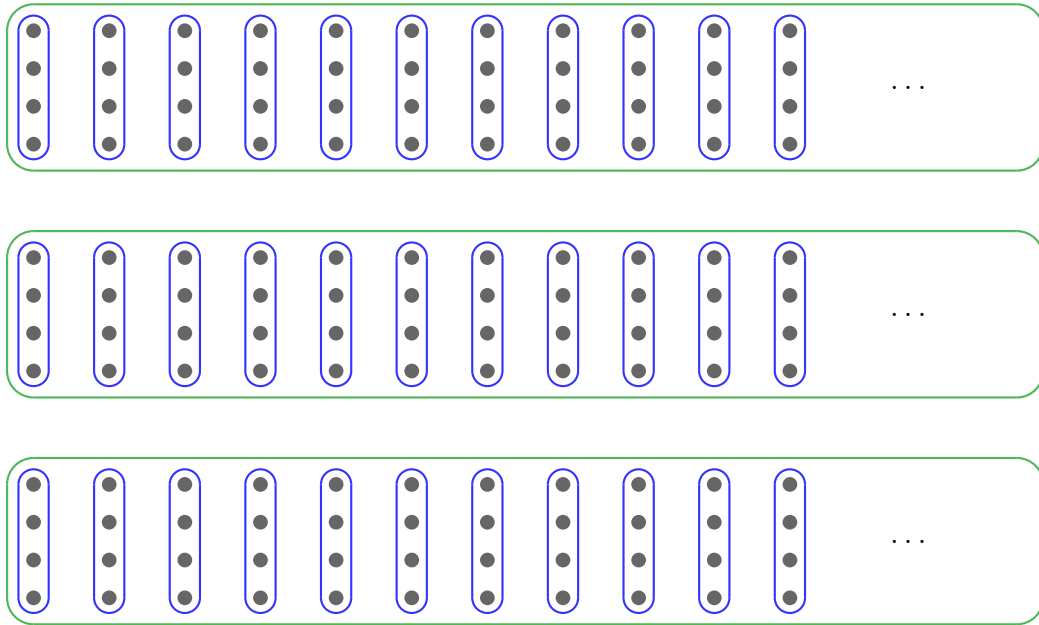
Figure 5.5: Nested block system of $\mathcal{C}_4 \wr (\mathfrak{S}_\infty \wr \mathcal{C}_3)$

Definition 5.3.2. We call an infinite primitive block of maximal finite blocks a *superblock*.

Note that, under the preliminary assumption of maximality, the primitivity requirement is equivalent to asking that the infinite block be minimal, so the above results on lattice structures imply that there is no "choice" for such superblocks–as opposed to the many choices of blocks, for the classical notion. From this point of view, the following (straightforward from the construction process of the nested block system) proposition offers an alternative definition of the superblocks , as the infinite blocks of blocks in $\mathcal{B}_{\mathcal{B}}(G)$.

Proposition 5.3.3 (Structure of the nested system)**.** *The nested block system consists of finitely many superblocks, and maybe one stable finite block.*

Given $\mathcal{B}_{\mathcal{B}}(G)$, the deduced lower bound on the growth rate of φ_G is $\sum_i m_i - 1$, where m_i is the size of finite blocks of the i-th superblock in $\mathcal{B}_{\mathcal{B}}(G)$ and the sum is over all superblocks.

Besides providing a competitive lower bound on the profile growth, the nested block system can pride itself on some pleasant properties of manageability.

Lemma 5.3.4. *Take two stable superblocks; the actions induced by G on their sets of maximal finite blocks are independent (up to taking a normal subgroup of finite index).*

Proof. By definition, the actions on the finite blocks of each superblock are isomorphic to one of the five highly homogeneous groups. Recall then Remark 4.3.5, and if need be take the finite index subgroup in which the actions of type $\mathrm{Rev}(\mathbb{Q})$ and $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$ are replaced by $\mathrm{Aut}(\mathbb{Q})$ and $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ to avoid synchronizations of order 2. Now the maximality of the finite blocks allows to eliminate the case of total synchronizations, which leaves none possible. $\qquad\square$

Put otherwise, superblocks in $\mathcal{B}_{\mathcal{B}}(G)$ are not that far from independence, which would allow to use Remark 5.1.2. This thesis will eventually clarify what "not that far" actually means.

Lemma 5.3.5. *Let G be a P-oligomorphic permutation group, and K be a finite index normal subgroup of G. Then we have $\mathcal{B}_{\mathcal{B}}(K) = \mathcal{B}_{\mathcal{B}}(G)$.*

Proof. We aim to prove that K has the same superblocks as G. Observe first that blocks of imprimitivity of any permutation group are still blocks for any subgroup, as a direct consequence of the definition. Let BB be a superblock, and $\mathfrak{M}$ be the action of G (implicitely after stabilization and restriction to the support of BB) on the set of finite blocks of BB. Then, $\mathfrak{M}$ is one of the five highly homogeneous groups; as the action of K on the same set of finite blocks is necessarily a finite index subgroup of $\mathfrak{M}$, it is highly homogeneous as well. We now just need to justify that the maximal finite blocks of G are still maximal for K. Assume some of them are not, then there exists $m \geq 2$ superblocks $(BB^{(j)})_{1 \leqslant j \leqslant m}$ in $\mathcal{B}_{\mathcal{B}}(G)$ and an ordering of their respective finite blocks $(B_i^{(j)})_i$ such that the unions $\cup_j B_i^{(j)}$ form new blocks for K (up to taking the join in $\mathcal{L}^{<\infty}(K)$). This can only happen if some of the actions of K on distinct $(B_i^{(j)})_i$ fully synchronize for $1 \leqslant j \leqslant m$. Since they are infinite (highly homogeneous) actions, this is in contradiction with K's being of finite index. (Indeed, the action of K on the blocks would be of infinite index in that of G, which is not possible.) $\qquad\square$

The reader has probably already wondered at this point why to stop here. We already have blocks of blocks, why not blocks of blocks of blocks... etc.? The blocks of blocks of the nested block system allow a good description of wreath products of type $F_1 \wr P \wr F_2$, where F_1 and F_2 are two finite groups that may be trivial and P is an infinite permutation group (recall Example 5.1.5). But what if we add a layer of wreath product: $F_1 \wr P \wr F_2 \wr G$? Well, it simply turns out that if G is not finite then the group is not P-oligomorphic anymore; and if it is, we are actually back to the same configuration as earlier (by associativity). Of course, if we had not made any hypothesis on the growth of the profile, it would be relevant to consider any number of layers of blocks.
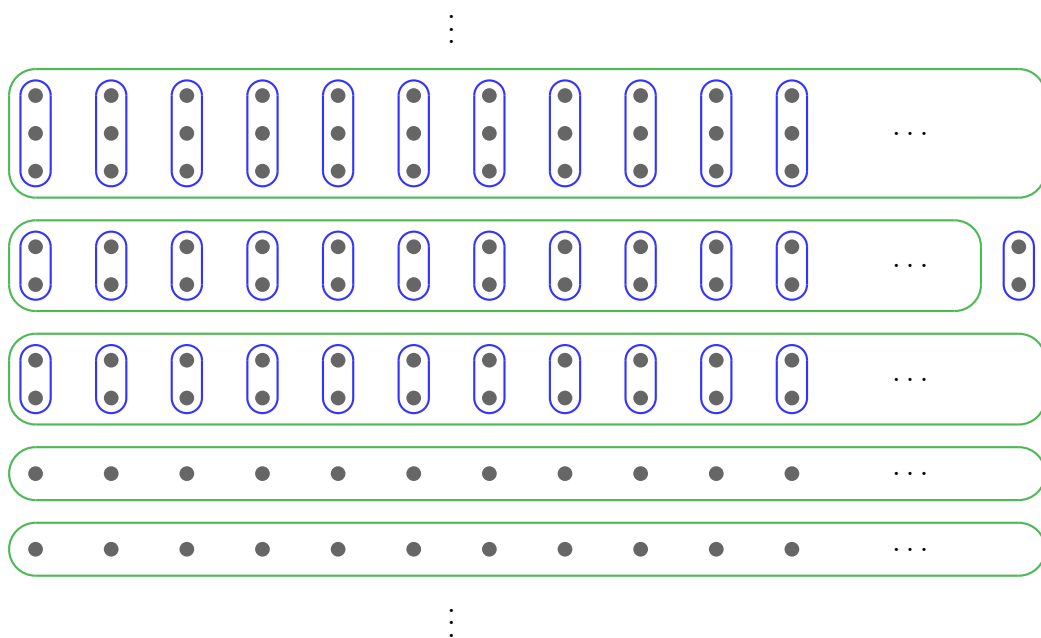
Figure 5.6: Generic look of the nested block system of a P-oligomorphic group

Classification in the case of a single superblock

In this chapter, we consider the class of closed P-oligomorphic permutation groups G with a single superblock, of which we denote by $B_1, B_2, \ldots$ the maximal finite blocks. This class includes wreath products $G = H \wr \mathfrak{S}_\infty$ where H is finite. In Section 6.1.1 we construct other examples by direct products; then, by combining wreath products and direct products, we introduce a family of permutation groups that subsumes all these examples. We show that their orbit algebras are invariant rings of permutation groups, hence finitely generated and Cohen-Macaulay.
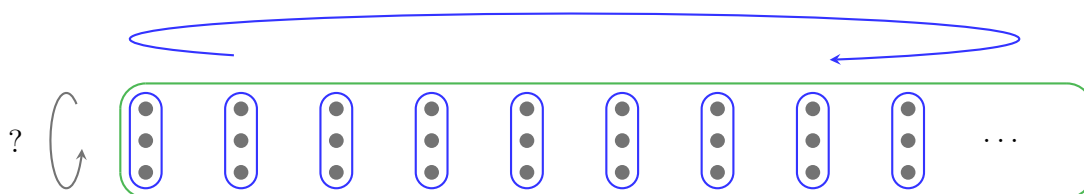
Figure 6.1: Case of a single superblock

In Section 6.1.2 we announce a classification theorem: any instance of this class is isomorphic to exactly one permutation group in the family. This answers positively Macpherson's question for this class of permutation groups.

The next sections undertake the proof of the classification theorem, which splits into two main poles: understanding the way the blocks permute, and understanding the structure of the stabilizer of blocks. Subsection 6.2.1 handles the action on the set of blocks: the result we obtain here may seem a bit technical but it will simplify our work of both visualization and manipulation.

Section 6.2.2 introduces the tower of G in order to deal with the action within the blocks, an object that will be the key tool in the rest of the proof, and that turns out to be classified.

Finally, Section 6.2.3 shows that this classification can be lifted to the groups themselves.

6.1   Examples and classification results

6.1.1   A family of examples beyond wreath products

Definition 6.1.1. We call **direct product on blocks** of two permutation groups H and S and denote by $H \square S$ the permutation group defined by the action of $H \times S$ on $\deg(S)$ blocks of size $\deg(H)$ by

$$b_{r,i}.(\tau, \sigma) = b_{\tau(r), \sigma(i)} \,,$$

where $b_{1,i}, \ldots b_{m,i}$ is an arbitrary ordering of the elements of each block B_i. It is isomorphic to the natural action of $H \times S$ on the cartesian product of the supports.

This can be thought of as the action of permuting (by H and S respectively) the rows and columns of a potentially infinite matrix.

As opposed to the wreath product, for which every action of H within a block is independent from what happens within the others, here there is actually only one diagonal action of H, on all finite blocks at once. These two cases are in this regard the two oppposite ends of the spectrum of all possible synchronizations between blocks.

It is then natural to think of a class of groups that would complete the spectrum. We introduce such groups, as hybrids of wreath products and direct products.

Definition 6.1.2. Let $H \lhd H_0$ and $\mathfrak{M}$ be three permutation groups, with H and H_0 finite. Denote by $[H_0, H^\infty, \mathfrak{M}]$ the permutation group generated by the elements of $H \wr \mathfrak{M}$ and $H_0 \square \mathfrak{M}$. For short, denote by $[H_0, H^\infty] = [H_0, H^\infty, \mathfrak{S}_\infty]$.

Remark 6.1.3. The group $[H_0, H^\infty, \mathfrak{M}]$ is P-oligomorphic if and only if we have $\mathfrak{M} = \mathfrak{S}_\infty$ or $H_0 = H = \mathrm{Id}_1$: indeed $[H_0, H^\infty, \mathfrak{S}_\infty]$ contains $H \wr \mathfrak{S}_\infty$ as a subgroup; it is therefore P-oligomorphic; the other implications are trivial using Lemma 4.2.8.

Lemma 6.1.4. *The permutation group $G = [H_0, H^\infty, \mathfrak{M}]$ contains $H \wr \mathfrak{M}$ as a normal subgroup of finite index.*

Proof. First note that G can be defined equivalently as the group generated by $H \wr \mathrm{Id}_\infty = \langle H^\infty, \mathrm{Id} \wr \mathfrak{M} \rangle$, and the finite group $H_0 \square \mathrm{Id}_\infty$. For the sake of notation, and when there is no ambiguity, we identify an element h_0 of H_0 with the element $(h_0, h_0, \ldots)$ of $H_0 \square \mathrm{Id}_\infty$, and identify H_0 with $H_0 \square \mathrm{Id}_\infty$.

Note that h_0 commutes with the elements of $\mathrm{Id} \wr \mathfrak{M}$ and, by normality of H in H_0, skew-commutes with those of $H \wr \mathrm{Id}_\infty$, meaning $H^\infty h_0 = h_0 H^\infty$. It follows that we have

$$G = \bigcup_{h_0 \in H_0} h_0.(H \wr \mathfrak{M}) \,.$$

This union becomes a decomposition into cosets if the range is restricted to some collection of representatives of the cosets of H in H_0. Therefore $H \wr \mathfrak{M}$ is normal and of finite index $[H : H_0]$ in G, as desired. $\square$

We now describe the orbit algebra of $[H_0, H^\infty, \mathfrak{M}]$ as an invariant ring of a permutation group. Recall that the orbit algebra $\mathbb{Q}[\mathcal{A}_{H \wr \mathfrak{M}}]$ of $H \wr \mathfrak{M}$ is the free

commutative algebra $\mathbb{Q}[X]$, with $X = (X_{\overline{A}})_{\overline{A}}$ where $\overline{A}$ ranges through the non-trivial H-orbits, and $X_{\overline{A}}$ denotes the $H \wr \mathfrak{M}$-orbit of A, seen as an element of the orbit algebra. Finally, lift the action of H_0 on the H-orbits $\overline{A}$ to an action on the variables $X_{\overline{A}}$.

Proposition 6.1.5. *With the above notations, the orbit algebra $\mathbb{Q}[\mathcal{A}_G]$ of $G = [H_0, H^\infty, \mathfrak{M}]$ is isomorphic to the invariant ring $\mathbb{Q}[X]^{H_0}$.*

Sketch of proof. Take an element $h_0 \in H_0$ and E a subset of the support of H. Check that the $H \wr \mathfrak{M}$-orbit of E is mapped onto another such $H \wr \mathfrak{M}$-orbit, as prescribed by the announced action of H_0 on the variables $X_{\overline{E}}$. $\qquad\square$

Remark 6.1.6. The variables of invariant rings are commonly taken of degree 1; this is not the case here: the degree of the variable $X_{\overline{E}}$ is given by $|E|$. This must be taken into account when computing the Hilbert series using Molien's formula or Pólya enumeration.

6.1.2 Classification and application to Macpherson's conjecture

We may now state the main theorem of this section which classifies the trivial case when $\mathfrak{M}$ is highly homogeneous (case $H_0 = H = \mathrm{Id}_1$ below). Nevertheless, the core of this section is about the case of non trivial finite blocks, in which $\mathfrak{M}$ is necessarily $\mathfrak{S}_\infty$.

Theorem 6.1.7 (Classification on one superblock)**.** *Let G be a closed P-oligomorphic permutation group such that $\mathcal{B}_\mathcal{B}(G)$ consists of a single superblock. Then G is isomorphic as a permutation group to $[H_0, H^\infty, \mathfrak{M}]$, where $H \lhd H_0$ are two finite permutation groups and $\mathfrak{M}$ is one of the five P-oligomorphic groups with profile 1. In addition, H, H_0, and $\mathfrak{M}$ are unique, and satisfy the condition of Remark 6.1.3.*

Proof. The statement is obvious if the finite blocks are singletons. Otherwise, by Lemma 4.2.8, G acts on the set of finite blocks as $\mathfrak{M} = \mathfrak{S}_\infty$. Use the upcoming Proposition 6.2.6 to classify the action of G on its blocks (the *tower of G*) and the upcoming Proposition 6.2.9 to lift this classification to G itself. $\qquad\square$

A positive answer to Macpherson's question follows immediately thanks to the description of the orbit algebras of the groups $[H_0, H^\infty, \mathfrak{M}]$ from Proposition 6.1.5.

Corollary 6.1.8 (Macpherson on one superblock)**.** *Let G be a closed P-oligomorphic permutation group such that $\mathcal{B}_\mathcal{B}(G)$ consists of a single superblock. Then, $\mathbb{Q}[\mathcal{A}_G]$ is the invariant ring of a finite permutation group, hence finitely generated, Cohen-Macaulay, and its algebraic dimension is given by the number of H-orbits (of non trivial subsets), where H is as defined by the classification.*

Remark 6.1.9. Until now, the lower bound provided by the nested block system evoked in Remark 5.1.6 was calculated using Example 4.1.1 when it came to stable superblocks in $\mathcal{B}_\mathcal{B}(G)$. With the notations of this section, it was based on the (possibly infinite index) supergroup $H_0 \wr \mathfrak{M}$: namely, the provided lower bound for

the algebraic dimension was the cardinality of the age of H_0. Corollary 6.1.8 hands a refinement of this bound, that is based on the subgroup $H \wr \mathfrak{M}$ and tight on the relevant restriction of the group.

In Chapter 7 the strategy to tackle a group G with several superblocks will be to consider the restrictions of G on each of its superblocks, and patch together their properties. This will use the following technical corollary.

Corollary 6.1.10. *Let G be a closed P-oligomorphic permutation group such that $\mathcal{B}_{\mathcal{B}}(G)$ consists of a single superblock; write it as $G = [H_0, H^\infty, \mathfrak{M}]$ using the classification of Theorem 6.1.7, and let M be the minimal finite index normal subgroup of $\mathfrak{M}$.*

Then, any finite index normal subgroup $\tilde{G}$ of G is of the form $[\tilde{H}_0, H^\infty, \tilde{M}]$, with $H \le \tilde{H}_0 \le H_0$ and $M \le \tilde{M} \le \mathfrak{M}$. In particular, $H \wr M$ is the minimal finite index normal subgroup of G.

Proof. Since $\tilde{G}$ is of finite index, its nested block system is still equal to $\mathcal{B}_{\mathcal{B}}(G)$ by Lemma 5.3.5, and its action on the maximal finite blocks is a normal subgroup of finite index of $\mathfrak{M}$. Using the classification of Theorem 6.1.7, $\tilde{G}$ is of the form $[\tilde{H}_0, \tilde{H}^\infty, \tilde{M}]$, with the expected group inclusions: $\tilde{H}_0 \lhd H_0$, $\tilde{H} \lhd H$, and $\tilde{M} \lhd \mathfrak{M}$. Lemma 6.1.4 also states that it contains $\tilde{H} \wr \tilde{M}$ as a finite index normal subgroup, while G contains $H \wr \mathfrak{M}$ and thus $H \wr \tilde{M}$ as finite index normal subgroups. Considering Lemma 4.1.3, we need to have $\tilde{H} = H$ for $\tilde{G}$ to have finite index in G. $\square$

6.2 Proof of the classification

6.2.1 Action on the set of blocks

The sequel of Section 6 is devoted to the statement and proof of the two propositions used in the proof of Theorem 6.1.7.

From now on, **we assume that G acts on the set of finite blocks as $\mathfrak{M} = \mathfrak{S}_\infty$**. The following two technical lemmas strengthen this assumption by showing that, for an appropriate enumeration of the elements within in each block, G can permute the blocks while preserving that enumeration.

Lemma 6.2.1. *Take any finite collection $(B_{i_1}, \dots, B_{i_k})$ of blocks; then $\mathrm{Fix}_G(B_{i_1}, \dots, B_{i_k})$ acts on the remaining blocks as $\mathfrak{S}_\infty$.*

Proof. Take k in $\mathbb{N}$. As $\mathrm{Fix}_G(B_1, \dots, B_k)$ is a normal subgroup of finite index of $\mathrm{Stab}_G(B_1, \dots, B_k)$, it acts on the remaining blocks as a subgroup of finite index of $\mathfrak{S}_\infty$, which may only be $\mathfrak{S}_\infty$ itself. By conjugation of the blocks, the same holds for any collection $(B_{i_1}, \dots, B_{i_k})$ of blocks. $\square$

Lemma 6.2.2 ("Ladder lemma")**.** *There exists an ordering $b_{1,i}, \dots b_{m,i}$ of the elements within each block B_i such that (the closure of) G contains $\mathrm{Id}_m \square \mathfrak{S}_\infty = \mathrm{Id}_m \wr \mathfrak{S}_\infty$ as a permutation subgroup.*

Proof. Since G acts by $\mathfrak{S}_\infty$ on the blocks, there exists for each $i > 1$ a permutation $\tau_{1,i}^{(0)} \in G$ that swaps B_1 and B_i and stabilizes all the other blocks. Take now $k \geq 0$; using Lemma 6.2.1 there exists a permutation $\tau_{1,i}^{(k)}$ that not only swaps B_1 and B_i, but also fixes all the (other) blocks in $B_2, \ldots, B_k$.

Take an infinite sequence $\tau_{1,i}^{(0)}, \ldots, \tau_{1,i}^{(k)}, \ldots$. Noting that there are only finitely many possibilities for the restriction of $\tau_{1,i}^{(k)}$ to $B_1 \cup B_i$, we can extract a subsequence with always the same restriction. Thus, using the closure, there exists in G a permutation $\tau_{1,i}$ which swaps B_1 and B_i and fixes all the other blocks. This permutation need not be of order 2 though.

Say that $\tau_{1,i}$ and $\tau_{1,j}$ are equivalent if their restrictions to $B_1 \cup B_i$ and $B_1 \cup B_j$ coincide up to renaming the elements of B_i (or B_j).

Now consider the map $i \mapsto \tau_{1,i}$. It takes finitely many values, and therefore there exists i and j such that $\tau_{1,i}$ and $\tau_{1,j}$ are equivalent. Define $\tau'_{i,j} = \tau_{1,i}\tau_{1,j}^{-1}\tau_{1,i}$.
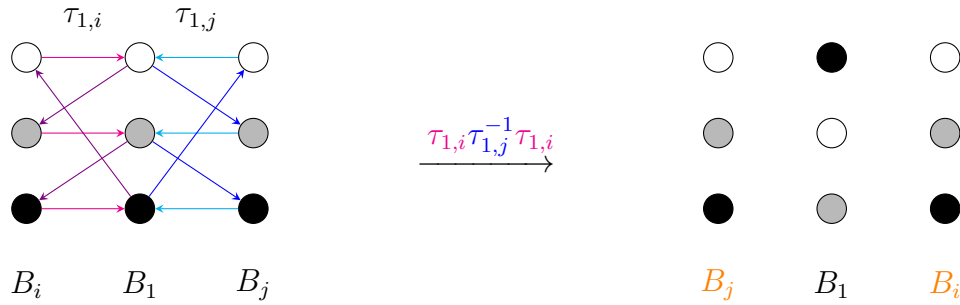

Figure 6.2: Example: Straight swap of B_i and B_j (also permuting B_1)

Now check that

- $\tau'_{i,j}$ swaps B_i and B_j "straightforwardly": that is its restriction to $B_i \cup B_j$ is of order 2 (see Figure 6.2);

- $\tau'_{i,j}$ stabilizes B_1;

- $\tau'_{i,j}$ fixes all the other blocks (pointwise).

We may then conjugate $\tau'_{i,j}$ to stabilize some block B_k instead of B_1, with k as large as desired, and still swap straightforwardly B_i and B_j while fixing the remaining blocks.

Therefore there exists in G, which we recall is assumed to be closed, a permutation $\tau_{i,j}$ of order 2 that swaps B_i and B_j and fixes all the other blocks. By conjugation, we can find for each n a similar permutation τ_n swapping B_n and B_{n+1}.

Choose an arbitrary ordering $b_{1,1}, \ldots, b_{m,1}$ of B_1. Define the ordering $b_{1,2}, \ldots, b_{m,2}$ of B_2 so that τ_1 is the trivial swap, meaning that it swaps $b_{1,r}$ and $b_{2,r}$ for each r. Proceed similarly to order the elements of B_3 so that τ_2 is the trivial swap, and so on (Figure 6.3 shows the stage $k-1$).

Conclusion: the τ_n's generate $\mathrm{Id}_m \,\square\, \mathfrak{S}_\infty$ as a permutation subgroup of G, as desired. $\square$
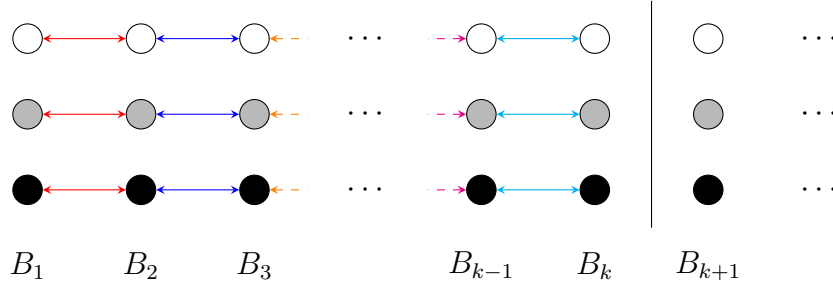
Figure 6.3: Straight swaps between the first k blocks

6.2.2   Towers and their classification

While the previous subsection dealt with the way the finite blocks could permute, this subsection is going to focus on what can happen within the blocks when they do not permute (the results obtained above state that the actions on and within the blocks can be decorrelated anyway).

Definition 6.2.3. Let $S_{\mathcal{B}} = S_{\mathcal{B}}^G = \mathrm{Stab}_G(\mathcal{B})$ be the kernel of the morphism that maps G onto its induced action on the set of blocks, and, for $i \geq 0$, set $H_i = H_i^G = \mathrm{Fix}_{S_{\mathcal{B}}}(B_1, \ldots, B_i)_{|B_{i+1}}$. We call the sequence $H_0, H_1, H_2 \cdots$ the *tower* of G with respect to the block system $\mathcal{B}$. The groups H_i are considered up to a permutation group isomorphism.

Remark 6.2.4.   • By conjugation, using Lemma 4.2.8, the tower does not depend on the ordering $B_1, B_2, \ldots$ of the blocks. In others words, H_i can be obtained by fixing (pointwise) any i blocks and taking the restriction to any other block.

• Each H_{i+1} can be naturally considered a subgroup of H_i. Indeed, choose a block "far enough", or take an arbitrary block out of the numbering by relabelling the blocks, call it say B_0, and consider all H_i's on this block: $H_i = \mathrm{Fix}_{S_{\mathcal{B}}}(B_1, \ldots, B_i)_{|B_0}$. This realizes the H_i's as a chain of decreasing subgroups (that can be renormalized, if not already, to the domain $\{1, 2, \ldots, m\}$, where m is the cardinality of the blocks). Furthermore, each H_i is normal in H_{i-1}.

The above definition and remark also apply to a permutation group of a finite set, as long as it acts on the (finitely many) blocks as the full symmetric group.

Example 6.2.5 (Fundamental examples)**.** Let H be a finite permutation group. The tower of $H \wr \mathfrak{S}_\infty$ (resp. $H \, \Box \, \mathfrak{S}_\infty$) for its natural block system is $H, H, H \cdots$ (resp. $H, \mathrm{Id}, \mathrm{Id} \cdots$). The tower of $[H_0, H^\infty]$ is $H_0, H, H, H \cdots$.

We aim to prove that these are the only possibilities for a tower (so there is actually only one prototype of a tower, since the first two examples are a specialization of the third one).

Proposition 6.2.6. *Let G be a closed P-oligomorphic permutation group with $\mathcal{B}_{\mathcal{B}}(G)$ consisting of a single superblock. Then, the tower of G has the form $H_0, H, H, H \cdots$, where H_0 is a finite permutation group and H is a normal subgroup of H_0.*

$$
\begin{array}{ccl}
H_0 \wr \mathfrak{S}_\infty & \longleftrightarrow & H_0,\ H_0,\ H_0,\ H_0 \ \cdots \\
\ \vdots & & \\
[H_0, H^\infty] & \longleftrightarrow & H_0,\ H\ ,\ H\ ,\ H\ \cdots \\
\ \Big|{\scriptstyle <\infty} & & \\
H \wr \mathfrak{S}_\infty & \longleftrightarrow & H\ ,\ H\ ,\ H\ ,\ H\ \cdots \\
\ \vdots & & \\
H \square\, \mathfrak{S}_\infty & \longleftrightarrow & H,\ \mathrm{Id},\ \mathrm{Id},\ \mathrm{Id}\ \cdots
\end{array}
$$

Figure 6.4: Towers of subgroups

Proof. Consider, for any $i \in \mathbb{N}$, the restriction G_i of $\mathrm{Fix}_G(\cup_{j<i} B_j)$ to the four next blocks. The tower of this permutation group (for a natural extension of the notion to finite groups of the adequate shape) is $H_i, H_{i+1}, H_{i+2}, H_{i+3}$. We aim to show that $H_{i+1} = H_{i+2}$, which will conclude the proof.

An element s of the blockwise stabilizer S_i of G_i is determined by its action on each block, which we write as a quadruple. Let g be an element of H_{i+1}. Then S_i has an element x that may be written $(1, g, h, h')$, with h and h' also in H_{i+1}. Let σ be an element of G_i that permutes "straightforwardly" the first two blocks and fixes the other two (Lemma 6.2.2 states that such an element actually exists). By conjugating x with σ in G_i, we get an element y in S_i that we may write $(g, 1, h, h')$, so that $x^{-1} y = (g, g^{-1}, 1, 1)$. Hence, using Remark 6.2.4, g is actually in H_{i+2}. $\qquad\square$

6.2.3 Lifting of the classification from towers to groups

The two results to follow will show that G is uniquely defined by its tower, by first recovering the blockwise stabilizer of the group from the tower, and then using the result of "straightforward" permutation we proved in Lemma 6.2.2.

Lemma 6.2.7. *The tower of G w.r.t. $\mathcal{B}$ uniquely determines its blockwise stabilizer $S_{\mathcal{B}}$.*

Proof. Let $(H_i)_i$ be the tower of G w.r.t. $\mathcal{B}$ and $S_{\mathcal{B}}$ be the blockwise stabilizer of $\mathcal{B}$.

Using that S is closed, it is sufficient to prove that, for any $l \geq 0$, the restriction S_r of S to the first r blocks is determined by the tower; or equivalently to any r blocks (recall that the order of the blocks is irrelevant). To this end, we will show that S_r admits an expression that involves only explicit subdirect products and the H_i's (which will do the job thanks to Proposition 4.3.3).

In order to proceed by induction on r, we consider the larger family $(H_{k,r})_{k \geq 0, r > 0}$, where $H_{k,r}$ is the restriction on r blocks of the fixator of k other blocks in S. Of course, $H_{0,r} = S_r$.
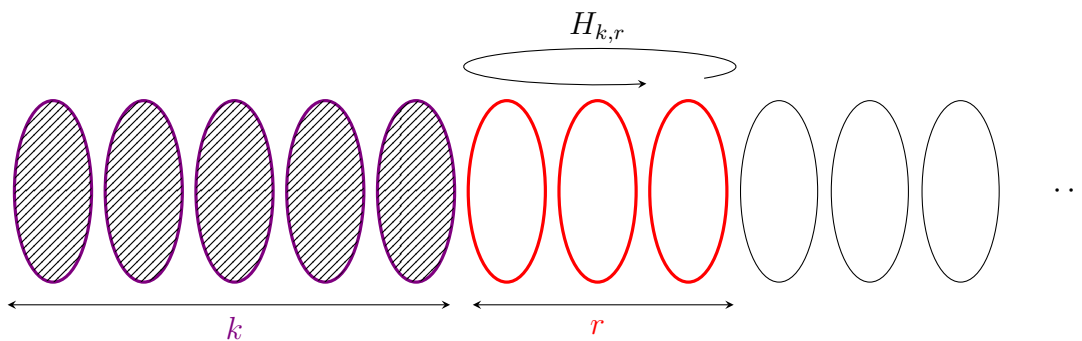
Figure 6.5: Fixing the first k blocks and restricting the action to r others (in proof of Lemma 6.2.7

First, note that we have $H_{k,1} = H_k$ for all k. This gives the base case for the induction. We now take $r > 1$, and express $H_{k,r}$ as a subdirect product involving only $H_{k',r'}$ with $r' < r$ (and incidentally also $k' + r' \leqslant k + r$).

Write $r = r_1 + r_2$ with $r_1 > 0$ and $r_2 > 0$, partition the r blocks into r_1 and r_2 blocks, and let E_1 and E_2 be their respective union. Considering the action of $H_{k,r}$ on E_1 and E_2 provides the desired expression:

$$H_{k,r} = \mathrm{Subdirect}((G_1, G_2), (N_1, N_2)),$$

where:

$$
\begin{aligned}
G_1 &= H_{k,r|E_1} &&= H_{k,r_1} \\
G_2 &= H_{k,r|E_2} &&= H_{k,r_2} \\
N_1 &= \mathrm{Fix}_{H_{k,r}}(E_2)_{|E_1} &&= H_{k+r_1,r_2} \\
N_2 &= \mathrm{Fix}_{H_{k,r}}(E_1)_{|E_2} &&= H_{k+r_2,r_1} \quad \square
\end{aligned}
$$

Remark 6.2.8. If desired, more explicit formulae can be obtained, by imposing the partition. For instance, even splittings have the pleasant property that $G_1 = G_2$ and $N_1 = N_2$, which allows to illustrate the process by a binary tree: following is for example a recursion tree to express $H_{0,8}$ as a subdirect product of H_i's, assuming that the left (resp. right) hand child of a group is the G_i (resp. N_i) of the subdirect product making this group (so a group is determined by its two children). This recursion tree generalizes immediately to any $H_{0,2^n}$, which is sufficient to retrieve S by closure.
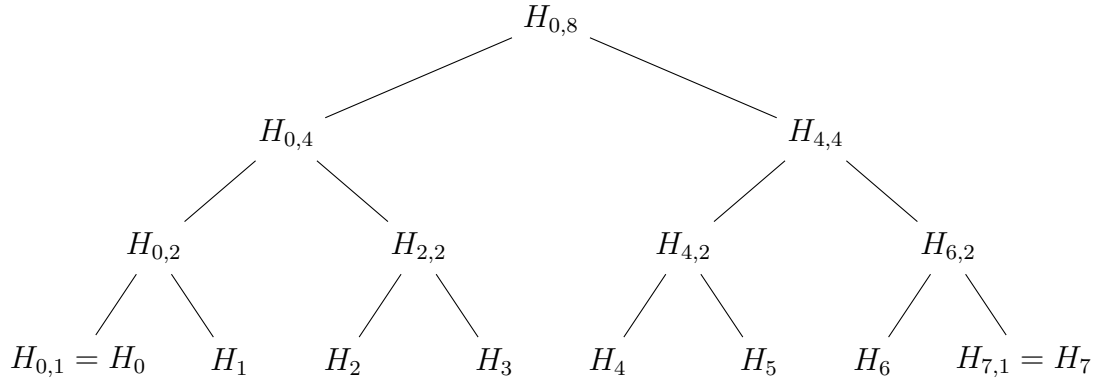
Proposition 6.2.9. *The permutation group G is the natural semi-direct product of its blockwise stabilizer $S_{\mathcal{B}}$ and $L = \mathrm{Id}_m \,\square\, \mathfrak{S}_\infty$. In particular, it is uniquely defined by its tower w.r.t. $\mathcal{B}$.*

Proof. Use first Lemma 6.2.2 to state that G contains $L = \mathrm{Id}_m \,\square\, \mathfrak{S}_\infty$ (for "ladder") as a permutation subgroup.

Take $k > 0$, the stabilizer $\mathrm{Stab}_{\tilde{G}}(B_1, \ldots, B_k)$ of the first k blocks is isomorphic to $\mathrm{Stab}_G(B_1, \ldots, B_k)$ by Lemma 6.2.7.

Now, the group generated by $L_{|B_1 \cup \cdots \cup B_k}$ and $\mathrm{Stab}_G(B_1, \ldots, B_k)_{|B_1 \cup \cdots \cup B_k}$ is a subgroup of the restriction of G (actually of $\mathrm{Stab}_G(B_1 \cup \cdots \cup B_k)$) to the same domain. Moreover, the latter is of size $|\mathfrak{S}_k|.|\mathrm{Stab}_G(B_1, \ldots, B_k)|$ (consider the morphism that projects onto the action on the blocks); therefore the two groups are equal.

Finally, since G acts on the blocks as the symmetric group, which is the closure of the group of all finitely supported permutations, each of its elements is also the simple limit of a sequence of finitely supported permutations. Since we just showed that the restrictions to any finite number of blocks are uniquely defined by the tower, so is the whole (closed) group G. $\qquad\square$

This hands the final element to the proof of the theorem of classification in the particular case of this section, and we are now free to move on to the general case.

Classification of P-oligomorphic groups and its corollaries

Let G be a closed P-oligomorphic group. In Section 7.1 we exploit the results from Section 5.1 on the blocks systems of G and the classification of closed P-oligomorphic groups with one single superblock of Section 6 to give a constructive description of the minimal finite index subgroup K of G. This subgroup is the first piece of the classification of G. Its orbit algebra is (a simple quotient of) a graded polynomial algebra.

In Section 7.3, we use it to define a classification system for P-oligomorphic groups, up to some slight simplification first, and we explain afterwards how to handle the general case.

We deduce in Section 7.4 that the orbit algebra of G is (a simple quotient of) the invariant algebra of a finite permutation group F, and that Macpherson's conjecture thus holds: $\mathbb{Q}[\mathcal{A}_G]$ is finitely generated, and even Cohen-Macaulay. In addition, K prescribes the algebraic dimension of the orbit algebra of G and provides a natural system of parameters, and thus (a choice of) the degrees appearing in the denominator of the Hilbert series. We also show how the classification allows to apply a variant of Pólya enumeration to systematically compute the profile of a P-oligomorphic group; we evoke the natural relational structure suggested by this classification to encode a given group; and we outline how it can be used to enumerate the P-oligomorphic groups.

7.1 The minimal finite index subgroup

Following Definition 5.3.1, let $\mathcal{B}_{\mathcal{B}}(G)$ be the nested block system of G; recall that it consists in a partition of the set of maximal finite blocks (but the kernel of G) into finitely many superblocks $(BB^{(j)})_j$. Let $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}}(G))$ be the stabilizer of the superblocks; it is a finite index normal subgroup of G and therefore, by Lemma 5.3.5, has the same superblocks.

We now consider the restriction $G^{(j)} = \mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}})_{|E^{(j)}}$ of $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}}(G))$ on the support $E^{(j)}$ of each superblock $BB^{(j)}$. It admits a single superblock, so that we can use the classification result of Section 6. Consider the minimal finite index normal

subgroup $K^{(j)} = H^{(j)} \wr M^{(j)}$ of $G^{(j)}$ of Corollary 6.1.10.

To be concrete, use Lemma 6.2.2 to choose a *coherent enumeration* of the elements of each block $B_i^{(j)}$ of $B^{(j)}$: for each i, i' there exists $g \in G$ that maps $B_i^{(j)}$ to $B_{i'}^{(j)}$ while preserving the enumeration. From now on, we use this chosen enumeration to implicitly identify permutations of $B_i^{(j)}$ and of $B_{i'}^{(j)}$ when meaningful. Recall that M is obtained by considering the homomorphic image $\mathfrak{M}$ of $G^{(j)}$ acting on the blocs in $BB^{(j)}$ and, if needed, taking its minimal finite index normal subgroup; $H^{(j)}$ can be obtained by picking arbitrarily two blocks $B_0^{(j)}$ and $B_1^{(j)}$ in $BB^{(j)}$, and taking the restriction to $B_1^{(j)}$ of the subgroup of G that fixes $B_0^{(j)}$ and stabilizes $B_1^{(j)}$. Recall also that $M^{(j)} = \mathfrak{S}_\infty$ whenever the blocks are non trivial; otherwise, $H^{(j)}$ is the trivial permutation group on one element, and $K^{(j)} = M^{(j)}$ is one of $\mathfrak{S}_\infty$, $\mathrm{Aut}(\mathbb{Q})$ or $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$.

In addition, set by convention $B_0^{(0)} = \ker G$, $BB^{(0)} = \{\ker G\}$, and $K^{(0)} = \mathrm{Id}_{\ker G}$.

Remark 7.1.1. The groups $K^{(j)}$ and $K^{(j')}$ are conjugate whenever the superblocks $BB^{(j)}$ and $BB^{(j')}$ are in the same G-orbit.

Proposition 7.1.2. *Let G be a closed P-oligomorphic group. Then $K = \prod_j K^{(j)}$ is isomorphic to the minimal finite index normal subgroup of G.*

Let us start by proving the following result.

Lemma 7.1.3. *A permutation group of shape $\prod_j H^{(j)} \wr M^{(j)}$ (where $H^{(j)}$ is finite and $M^{(j)}$ is a minimal highly homogeneous group) admits no proper finite index normal subgroup.*

Proof. Let K be such a permutation group. If a subgroup $\tilde{K}$ of K is normal and of finite index, Lemma 5.3.5 states that K's superblocks are still superblocks for $\tilde{K}$; and since they are stable by K, they are for any subgroup. Using the classification of Corollary 6.1.10, the restrictions of K to any superblocks have no finite index normal subgroup, so $\tilde{K}$ has the same restrictions and is thus a subdirect product of these. There remains to show that there is no synchronization between these parts. Again, the $K^{(j)}$ have no finite index normal subgroups, so no finite synchronization (that would be linked to a proper normal subgroup of finite index) is to consider; and the case of infinite synchronizations is excluded by Lemma 5.3.4. $\qquad\qquad\square$

Proof of the proposition. We are going to reduce G down to K by applying two successive reductions to a normal subgroup of finite index, which will conclude the proof using Lemma 7.1.3.

Recall that the intersection of two finite index normal subgroups is again a finite index normal subgroup. Hence the finite index normal subgroups of G form a lattice. It is not guaranteed *a priori* to have a minimal element though.

We consider the nested block system $\mathcal{B}_\mathcal{B}(G)$ introduced in Section 5.1. By Lemma 5.3.5, at each step, the nested block system of the finite index normal subgroup will still be $\mathcal{B}_\mathcal{B} = \mathcal{B}_\mathcal{B}(G)$. In particular, the kernel will not grow bigger. Denote as earlier by $(BB^{(j)})_j$ the superblocks of $\mathcal{B}_\mathcal{B}(G)$ and by $(E^{(j)})_j$ their respective supports.

Let $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}})$ be the finite index normal subgroup of G that stabilizes the superblocks in $\mathcal{B}_{\mathcal{B}}(G)$, which is the first reduction.

Assume first $j \neq 0$. We may apply the classification result of Lemma 6.1.4 : $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}})_{|E^{(j)}}$ contains as a finite index normal subgroup some wreath product $H^{(j)} \wr M^{(j)}$, where $M^{(j)}$ is given by the action of $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}})$ on the set of maximal finite blocks of $E^{(j)}$ (possibly up to taking an index 2 normal subgroup), while $H^{(j)}$, which acts within the blocks, is given by H in the tower $H_0, H, H, H, \ldots$ of $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}})_{|E^{(j)}}$. This subgroup is isomorphic to $K^{(j)}$, which also implies that it contains no proper normal subgroup of finite index; therefore, thanks to the aforementionned lattice structure, it is the minimal finite index normal subgroup of $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}})_{|E^{(j)}}$.

The same conclusions can be reached trivially for $j = 0$ (recall that $E^{(0)}$ is the kernel of G and $K^{(0)}$ is the trivial group thereupon).

Now is the time for the second reduction. Consider the finitely many cosets of $K^{(j)}$ in $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}})_{|E^{(j)}}$; the latter (and therefore $\mathrm{Stab}_G(\mathcal{B}_{\mathcal{B}})$) acts by permutation on these cosets. Now denote by $\tilde{K}$ the kernel of its simultaneous action on the whole set of cosets for all j. At this point, for each given j, the restriction of $\tilde{K}$ to $E^{(j)}$ is a subgroup of $K^{(j)}$; it could be a proper subgroup at first glance, due to the constraints inheritated from the action on the other sets of cosets (the cosets from other $E^{(j)}$, for different j). However, thanks to the minimality of $K^{(j)}$, we do have $\tilde{K}_{|E^{(j)}} = K^{(j)}$, for every j : $\tilde{K}$ is a subdirect product of the direct product K. Conclude the proof by replaying that of Lemma 7.1.3 to show that $\tilde{K}$ is actually the whole direct product, namely K itself, and use Lemma 7.1.3 and the lattice structure to state its status of minimum.

□

Remark 7.1.4. From Remark 5.1.2, $\mathbb{Q}[\mathcal{A}_K]$ is a free algebra, possibly tensored with some finite dimensional diagonal algebra, which is finitely generated. Explicitely, we may write:

$$\mathbb{Q}[\mathcal{A}_K] = \bigotimes_i \mathbb{Q}[\mathcal{A}_{K_i}] \otimes \mathbb{Q}[\mathcal{A}_{\ker K}] = \mathcal{A}_f \otimes \mathbb{Q}[\mathcal{A}_{\ker K}] = \bigoplus_k e_k \, \mathcal{A}_f$$

the e_k being the subsets of the kernel of K; in other words, $\mathbb{Q}[\mathcal{A}_K]$ is a Cohen-Macaulay algebra over the free subalgebra $\mathcal{A}_f$.

Corollary 7.1.5. *The lower bound provided by the nested block system according to Remark 6.1.9 is tight.*

Proof. The algebraic dimension of the algebra of K is the sum of the dimensions of the $\mathcal{A}_f$, which coincide with the lower bound handed by the nested system; and the algebraic dimension of $\mathbb{Q}[\mathcal{A}_G]$ is the same by finiteness of the index of K. □

7.2 Semi-direct product structure and diagonal action

We aim at proving that G is a product FK, where F is a finite group acting diagonally on $\mathcal{B}_{\mathcal{B}}(G)$.

We first provide an informal description the concept of diagonal action on $\mathcal{B}_{\mathcal{B}}(G)$, that we will properly define a bit later in the subsection. It generalizes the diagonal action we already came across in Section 6. Informally, such an action will induce the same action on each block of a given superblock, and possibly permute the superblocks but not the blocks inside each superblock. Considering Figure 5.6, one may intuitively picture it as a purely vertical action, that permutes the rows formed by the elements of Ω on the figure — and preserves the superblock structure of course.

Define $\mathcal{U}$ as the collection of all $U = \sqcup U^{(j)}$ where each $U^{(j)}$ is some block $B_i^{(j)}$ in $BB^{(j)}$. Note that U always contains the kernel $B^{(0)}$.

Choose arbitrarily U_0 in $\mathcal{U}$ and denote its blocks by $(B^{(j)})_j$.

For each U, U', let $k_{U,U'}$ be the permutation in K that exchanges U and U', preserves the coherent enumeration, and fixes all the blocks that are neither in U nor U'.

For $U \in \mathcal{U}$, define the map

$$\Phi_U : \begin{cases} G \longrightarrow \mathfrak{S}_{U_0} \\ g \longmapsto k_{U_0,g(U)} \, g \, k_{U,U_0|U_0} \end{cases}.$$

Define $K_{<\infty} = \prod_j H^{(j)}$, and observe that we have $\Phi_U(K) = K_{<\infty}$.

Define then $\overline{\Phi}_U$ by $\overline{\Phi}_U(g) = \pi(\Phi_U(g)_{|U_0})$, where π is the canonical projection of $\mathfrak{S}_{U_0}$ onto the right $K_{<\infty}$ cosets.

Lemma 7.2.1. *For g in G, $\overline{\Phi}_U(g)$ does not depend on U.*

Proof. Consider

$$h = \Phi_U(g) \, \Phi_{U_0}(g)^{-1} = k_{U_0,g(U)} \, g \, k_{U,U_0} \, g^{-1} \, k_{U_0,g(U_0)}.$$

Take a block $U_0^{(j)}$. Use that g permutes the superblocks $BB^{(j)}$ to argue that h stabilizes $U_0^{(j)}$. We now prove that the restriction $h^{(j)}$ of h to $U_0^{(j)}$ is in $H^{(j)}$.

If $j = 0$, then we have $U_0^{(j)} = U^{(j)}$ and $h^{(j)}$ is the identity. Otherwise, take some block $B_1^{(j)}$ which is not in either of U_0, U, $g(U_0)$, or $g(U)$, and observe that h is the identity on $B_1^{(j)}$. It follows from the tower of G on the superblock $BB_1^{(j)}$ that $h^{(j)}$ is in $H^{(j)}$.

Conclusion: $\Phi_U(g)$ and $\Phi_{U_0}(g)$ are in the same right $K_{<\infty}$-coset, as desired. $\square$

Since $\overline{\Phi}_U$ does not depend on U, we may define $\overline{\Phi} = \overline{\Phi}_U$.

Lemma 7.2.2. $\overline{\Phi}$ *is a group morphism.*

Proof. Write $H = K_{<\infty}$ and identify it, as needed, with the pointwise stabilizer, in K, of all the blocks that are not in U_0. Then we have (using the normality of K and the defining property of H in K to commute it past $k_{U_0,g'(g(U))} g' k_{g(U),U_0}$):

$$\overline{\Phi}(g')\overline{\Phi}(g) = \overline{\Phi}_{g(U)}(g') \, \overline{\Phi}_{U_0}(g) = H(k_{U_0,g'(g(U))} \, g' \, k_{g(U),U_0})_{|U_0} H(k_{U_0,g(U)} \, g)_{|U_0}$$

$$= (H \, k_{U_0,g'(g(U))}) \, g' \, k_{g(U),U_0} \, H \, k_{U_0,g(U)} \, g)_{|U_0}$$

$$= (H\, k_{U_0,g'(g(U))}\, g'\, k_{g(U),U_0}\, k_{U_0,g(U)}\, g)_{|U_0}$$
$$= (H\, k_{U_0,g'(g(U))}\, g'\, g)_{|U_0}$$
$$= H(k_{U_0,g'(g(U))}\, g'\, g)_{|U_0}$$
$$= \overline{\Phi}(g'g)\,.$$

$\square$

Define $G_{<\infty} = \Phi_{U_0}(G)$.

Proposition 7.2.3. *The morphism* $\overline{\Phi} : G \to G_{<\infty}/K_{<\infty}$ *realizes the canonical quotient map from G to G/K. In other words, G is a semi direct product of $G_{<\infty}$ and K.*

Proof. Check that K is the kernel of $\overline{\Phi}$. $\square$

A permutation $g \in G$ acts diagonally on $\mathcal{B}_{\mathcal{B}}(G)$ if:
(i) $\Phi_U(g)$ does not depend on the choice of U in $\mathcal{U}$;
(ii) g maps each block $B_i^{(j)}$ to some block $B_i^{(j')}$.
Note: this definition is relative to the choice of the indexing of the blocks within each superblock. It assumes that, within an orbit of superblocks, the index set is the same for each superblock, with the same action of $M^{(j)}$ on that index set.

Proposition 7.2.4. *G is the product $G_{<\infty}K$, where $G_{<\infty}$ is embedded in G by acting diagonally on $\mathcal{B}_{\mathcal{B}}(G)$.*

Proof. It is sufficient to prove that, for any $g_0 \in G_{<\infty}$, some permutation of G acts diagonally on $\mathcal{B}_{\mathcal{B}}(G)$ according to g_0.

Take any g in G such that $\Phi_0(g) = g_0$.

We may assume without loss of generality that $\Phi_U(g) = g_0$ for all U. Otherwise: for each block $B_j^{(j)}$ and any U containing it, write $\Phi_U(g) = g_0 h$, and define $h_{i,j} = h_{|U_0}$; observe that $h_{i,j}$ does not depend on U; define $k \in K$ by having it fix all blocks, and act on each $B_j^{(j)}$ by $h_{i,j}$; replace g by gk^{-1}.

Take each superblock $BB^{(j)}$ in turn, and let $BB^{(j')} = g(BB^{(j)})$. For each block $g(B_i^{(j)})$ let $\sigma(i)$ be such that $g(B_i^{(j)}) = B_{\sigma(i)}^{(j')}$. We may assume without loss of generality that $\sigma(i) = i$. Otherwise: observe that σ is in $M^{(j)}$; take $k \in K$ that permutes the blocks of $BB^{(j)}$ according to σ leaving everything else untouched; replace g by gk^{-1}.

Hence g acts diagonally on $\mathcal{B}_{\mathcal{B}}(G)$, as desired. $\square$

7.3 Classification

We now have all the ingredients to classify P-oligomorphic groups. We first use the previous sections to extract from a P-oligomorphic group G a finite piece of information $\mathrm{Data}(G)$. It consists of a finite permutation group, endowed with a block system where each block is decorated with a permutation group of its elements and one of the five permutation groups with profile 1 (or the trivial group). Conversely,

we show that, starting from such a permutation group with decorated blocks Δ, one can construct an oligomorphic permutation group $\mathrm{Group}(\Delta)$.

We check that, up to a natural isomorphism, G can be reconstructed from $\mathrm{Data}(G)$ by Group. More generally, we check that, up to an isomorphism, Data and Group give a one-to-one correspondence between finite permutation groups with decorated blocks and P-oligomorphic permutation groups. This concludes the classification.

7.3.1   Classification of Rev-free P-oligomorphic groups

For the sake of simplicity of exposition, we first tackle the subclass of Rev-*free groups*; that is groups that do not act by $\mathrm{Rev}(\mathbb{Q})$ or $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$ on any of their superblocks. This is actually sufficient to classify ages of P-oligomorphic groups. In the following section, we detail how $\mathrm{Data}(G)$ can be extended to also preserve this piece of information.

Let G be a closed P-oligomorphic group. Take again the notations introduced at the beginning of the previous subsection: $\mathcal{B}_{\mathcal{B}}(G) = \{BB^{(j)}\}_j$, $K = \prod_j K^{(j)}$, with $K^{(j)} = H^{(j)} \wr M^{(j)}$, and finally the finite blocks $(B_0^{(j)})_j$ arbitrarily picked in each superblock.

Consider the subgroup of G that stabilizes $\sqcup_j B_0^{(j)}$, and take its restriction to this union of finite blocks; call it $G_{<\infty}$. Similarly, consider the same restriction but calculated from K, and call it $K_{<\infty}$. Observe that $K_{<\infty}$ is a subgroup of $G_{<\infty}$ that is isomorphic to $\prod_j H^{(j)}$.

Definition 7.3.1. Define $\mathrm{Data}(G) = (G_{<\infty}, (B_0^{(j)})_j, (H^{(j)})_j, (M^{(j)})_j)$.

Definition 7.3.2. A **permutation group with decorated blocks** $(F, B, (H^{(j)})_j, (M^{(j)})_j)$ consists of a finite permutation group F endowed with a block system $B = \{B^{(j)}\}_j$ together with the choice, for each block $B^{(j)}$, of

- a normal subgroup $H^{(j)}$ of the restriction $\mathrm{Fix}_F(\cup_{i \neq j} B^{(i)})_{|B^{(j)}}$ of the pointwise stabilizer of the other blocks,

- $M^{(j)}$, one of the three minimal groups of profile 1, or the trivial group,

satisfying the following constraints:

(i) the choices must be the same for $i \neq j$ whenever $B^{(i)}$ and $B^{(j)}$ are in the same F-orbit,

(ii) $M^{(j)}$ must be $\mathfrak{S}_\infty$ whenever $B^{(j)}$ is not a singleton,

(iii) at most one of the $M^{(j)}$ is trivial, and when it is $K^{(j)} = H^{(j)}$ is trivial too.

Remark 7.3.3. Let G be a P-oligomorphic permutation group. Based on the results of Section 7.1, $\mathrm{Data}(G)$ is a permutation group with decorated blocks.

Definition 7.3.4. Let $\Delta = (F, B, (H^{(j)})_j, (M^{(j)})_j)$ be a permutation group with decorated blocks, and Ω the disjoint union $\sqcup_j \Omega^{(j)}$, where $\Omega^{(j)}$ is the cartesian product of $B^{(j)}$ and the domain of $M^{(j)}$. For each j, take the wreath product $K^{(j)} = H^{(j)} \wr M^{(j)}$ acting naturally on $\Omega^{(j)}$. Finally, let K be the direct product $\prod_j K^{(j)}$, acting naturally on Ω.

We define $\mathrm{Group}(\Delta)$ as the smallest permutation group on Ω containing both K and F acting diagonally on $\sqcup_j \Omega^{(j)}$. Denote additionally $H = \prod_j H^{(j)} = K_{<\infty}$.

Proposition 7.3.5. *Let Δ be a permutation group with decorated blocks. Define $G = \mathrm{Group}(\Delta)$, and use the notations above. Then, G is a P-oligomorphic permutation group.*

Proof. The subgroup K of G is the direct product of the wreath products $H^{(j)} \wr M^{(j)}$, and therefore P-oligomorphic. This implies the result by Lemma 3.2.17. $\square$

We proceed by defining the notion of isomorphism for groups with decorated blocks, and checking that it matches the classical notion of isomorphism for P-oligomorphic groups.

Definition 7.3.6. Let Δ and Δ' be two permutation groups with decorated blocks. Then, Δ and Δ' are isomorphic if there exists an isomorphism between the underlying groups F and F' that transports the block system B and the groups $H^{(j)}$ and $M^{(j)}$ to their equivalents in Δ'.

Lemma 7.3.7. *Let Δ be a permutation group with decorated blocks. Then $\mathrm{Group}(\Delta)$ is the natural semidirect product $K \rtimes F/H$ with the notations of Definition 7.3.4, and $\Delta' = \mathrm{Data}(\mathrm{Group}(\Delta))$ is isomorphic to Δ.*

Proof. Denote $G = \mathrm{Group}(\Delta)$. The groups K, F and H obtained in Section 7.2 from G correspond to the groups of same name from Definition 7.3.4, so the results from that subsection apply and the first part is immediate. In particular, K is a normal subgroup of G of finite index, which in addition, by Proposition 7.1.2, admits no finite index normal subgroup and is therefore the unique minimal finite index normal subgroup of G. This in turn implies the uniqueness of all the other pieces of $\mathrm{Data}(G)$: the nested block system of G is given by $\mathcal{B}_{\mathcal{B}}(G) = (BB^{(j)})_j$, with $BB^{(j)} = (B_i^{(j)})_i$ and, for i in the support of $M^{(j)}$, $B_i^{(j)} = B^{(j)} \times \{i\}$; the permutation subgroups $G_{<\infty}$ and $K_{<\infty}$ induced respectively by G and K on $\sqcup_j B_0^{(j)}$ are respectively trivially isomorphic to F and H. $\square$

Lemma 7.3.8. *Let G be a Rev-free (closed) P-oligomorphic group. Then, $G' = \mathrm{Group}(\mathrm{Data}(G))$ is isomorphic to G.*

Proof. We use the coherent enumeration to identify the elements of each $B_i^{(j)}$ with that of $B^{(j)} \times \{i\}$. Through this identification and by construction, we have $K' = K$; since the finite groups acting diagonally are the same as well and using Proposition 7.2.4, we have indeed $G' = G$. $\square$

Theorem 7.3.9. Rev-*free P-oligomorphic permutation groups are classified by finite permutation groups with decorated blocks through the* Data *and* Group *reciprocal correspondences.*

Proof. Lemma 7.3.7 together with Lemma 7.3.8 asserts that Data and Group are reciprocal correspondences, as desired. ☐

7.3.2   Extending the classification to all P-oligomorphic groups

We start with an example illustrating that the straightforward extension of Data to all P-oligomorphic groups does not give a proper correspondence.

Example 7.3.10. Consider the P-oligomorphic group $G = \operatorname{Rev}\mathbb{Q} \times \operatorname{Rev}\mathbb{Q}$, and the index 2 subgroup generated by $G' = \operatorname{Aut}\mathbb{Q} \times \operatorname{Aut}\mathbb{Q}$ on the one hand and the reversal acting simultaneously on the two copies of $\mathbb{Q}$ on the other hand.

Let us try to define Data on G and G' as before; in both cases, we get the same data:
$$(\operatorname{id}(\{1,2\}), (\{i\})_{i=1,2}, (\operatorname{id}(\{i\}))_{i=1,2}, (\operatorname{Rev}(\mathbb{Q}))_{i=1,2}) .$$
The information about the synchronization of the reversal on the two superblock is lost.

We now tweak the definition of Data to keep track of reversals in the finite group $G_{<\infty}$. To achieve this, each copy of $R = \mathbb{Q}$ (or of $R = \mathbb{Q}/\mathbb{Z}$) where a reversal can occur will be compressed into a block of two points instead of a single one.

Let $BB^{(j)}$ be a superblock; if its blocks are of size 1 and G acts on them by $M^{(j)} = \operatorname{Rev}(R)$, then define $\overline{B}_0^{(j)}$ by choosing any two points of $\Omega^{(j)}$; note that $\overline{B}_0^{(j)}$ is not a block anymore, but this is fine. Otherwise, define $\overline{B}_0^{(j)}$ as $B_0^{(j)}$.

Define $\overline{G}_{<\infty}$ as before, but using $\sqcup_j \overline{B}_0^{(j)}$ instead of $\sqcup_j B_0^{(j)}$.

Example 7.3.11. With G and G' as in the previous example, $\overline{G}_{<\infty}$ and $\overline{G}'_{<\infty}$ both act on $\{1,2\} \sqcup \{1,2\}$. However $\overline{G}_{<\infty}$ is of size 4, permuting independently the two blocks, whereas $\overline{G}'_{<\infty}$ is of size 2, permuting simultaneously the two blocks.

Definition 7.3.12. Define $\overline{\operatorname{Data}}G = (\overline{G}_{<\infty}, (\overline{B}_0^{(j)})_j, (H^{(j)})_j, (M^{(j)})_j)$.

The definition of permutation group with decorated blocks must be extended accordingly: each $M^{(j)}$ can now be any one of the five closed highly homogeneous groups; however $\overline{B}^{(j)}$ must be of size 2 whenever $M^{(j)}$ is of the form $\operatorname{Rev}(R)$ and of size 1 whenever $M^{(j)}$ is of the form $\operatorname{Aut}(R)$.

The definition of Group must be adjusted as well: if $M^{(j)}$ is of the form $\operatorname{Rev}(R)$ and therefore $B^{(j)}$ is of size 2, then $\Omega^{(j)}$ consists of a single copy of the support of $M^{(j)}$. Also, the diagonal action of an element f of F on Ω must be adjusted: assumes that $M^{(j)}$ is of the form $\operatorname{Rev}(R)$ and therefore $B^{(j)}$ is of size 2; let j' be such that f maps $B^{(j)}$ to $B^{(j')}$. Then, f maps the elements of $\Omega^{(j)}$ onto those of $\Omega^{(j')}$, with a reversal whenever the elements of $B^{(j)}$ are swapped by f in $B^{(j')}$.

Theorem 7.3.13. *Closed P-oligomorphic permutation groups are classified by finite permutation groups with (extended) decorated blocks through the* $\overline{\operatorname{Data}}$ *and* $\overline{\operatorname{Group}}$ *reciprocal correspondences.*

Proof. Replay each step of the proof described in Subsection 7.3.1. ☐

7.4 Corollaries

7.4.1 Resolution of the conjectures and the property of Cohen-Macaulay

Let G be a (closed) P-oligomorphic group, $K = \prod_j K^{(j)}$ its minimal subgroup of finite index, and use again the notations of the previous section. Let D_G be the set of degrees of the non zero degree elements of the ages $\mathcal{A}_{H^{(j)}}$ of the $H^{(j)}$'s.

Theorem 7.4.1. *Let G be a permutation group whose profile is bounded by a polynomial. Then, $\mathbb{Q}[\mathcal{A}_G]$ is isomorphic to the algebra of invariants of some finite permutation group acting on variables of degrees D_G, quotiented by the relations $x^2 = 0$ for some of the variables. In particular, $\mathbb{Q}[\mathcal{A}_G]$ is Cohen-Macaulay.*

Proof. For each superblock $BB^{(j)}$, let S_j be the collection of all the non-trivial subsets of all blocks of $BB^{(j)}$. Let $S = \sqcup_j S_j$. By the definition of block systems, K acts on each S_j and on S. Denote by $(\theta_{i,j})_i$ the K-orbits in S_j and observe that they are in bijection with the positive degree part $\mathcal{A}_G^+(H^{(j)})$ of the age of $H^{(j)}$.

As in Example 4.1.1, the orbit algebra $\mathbb{Q}[\mathcal{A}_{K^{(j)}}]$ of $K^{(j)}$, for $j \neq 0$, is the free algebra $\mathbb{Q}[(\theta_{i,j})_i]$; for $j = 0$, the orbit algebra is the finite dimensional algebra $\mathbb{Q}[(\theta_{i,0})_i]/(\theta_{i,0}^2 = 0\ \forall i)$ instead. The orbit algebra of K itself is the tensor product $\bigotimes_j \mathbb{Q}[\mathcal{A}_{K^{(j)}}]$, generated by $(\theta_{i,j})_{i,j}$.

The group G itself also acts on S; since K is normal in G, this lifts to an action on the finitely many K-orbits $(\theta_{i,j})_{i,j}$ in S. Let G_0 be the finite permutation group induced by this action, and let $\mathbb{Q}[(\theta_{i,j})_{i,j}]^{G_0}$ be its invariant ring. Then, $\mathbb{Q}[\mathcal{A}_G]$ is the following quotient thereof:

$$\mathbb{Q}[\mathcal{A}_G] = \mathbb{Q}[(\theta_{i,j})_{i,j}]^{G_0}/(\theta_{i,0}^2 = 0\ \forall i)\,. \qquad \square$$

Recall that invariant rings of permutations groups are finitely generated and Cohen-Macaulay. Then, the following corollary, our goal since the beginning, is immediate.

Corollary 7.4.2. *The orbit algebra of a P-oligomorphic group is finitely generated, and even Cohen-Macaulay.*

In particular, the conjectures of Macpherson and Cameron both hold.

7.4.2 Computation of the profile using Pólya enumeration

Recall the basics about Pólya enumeration from Section 2.3. We slightly modify the notation system: a group F acts on a set X, which is a set of functions $\Omega \to \chi$.

We want to count the orbits of subsets of a P-oligomorphic group G (precisely, we want the corresponding generating series, in say z). This boils down to counting the G-orbits of the K-orbits of subsets, where K is the minimal normal subgroup of finite index described in Section 7.1; which in turn, thanks to the classification of Section 7.3 boils down to counting the F-orbits of K-orbits, where F is endowed with its diagonal action on the nested block system.

We thus want X to emulate the K-orbits of subsets. Consider the union $\mathcal{A} = \sqcup_j \mathcal{A}_{H^{(j)}}$ of the finite ages of all $H^{(j)}$ (taken as in Section 7.1); then a K-orbit of subsets essentially corresponds to a function $\mathcal{A} \to \mathbb{N}$. Indeed, each superblock is stable under the action of K, and an orbit of subsets of the restriction of K to a superblock $BB^{(j)}$ can be described by a function $\mathcal{A}_{H^{(j)}} \to \mathbb{N}$. We therefore take $\Omega = \mathcal{A}$, $\chi = \mathbb{N}$ and $X = \mathbb{N}^{\mathcal{A}}$.

We now only need an adapted weight function on X for Pólya to operate. Remember that it usually derives from a weight function on the image set χ. Here, it is natural to choose $w' : n \mapsto z^n$, for the more times a subset will appear in a representative of a K-orbit, the heavier this K-orbit needs to be. But of course, the final weight of the K-orbit also depends on the cardinality of the subset (or on the orbital degree of the $H^{(j)}$-orbit of that subset). This is why we need to define a notion of weight on the set of pre-images $\mathcal{A}$ as well: actually simply the orbital degree d. The total weight of an element k of X is then $w(k) = \prod_{a \in \mathcal{A}} w'(k(a))^{d(a)}$.

This is all nice, except for one detail: the reason why Pólya works this well is because we can forget about the set of pre-images — which we cannot here, since the weight of k directly depends on the pre-images a *via* the weight d. This issue can be addressed by partitioning the computation of the contribution of each $g \in F$, in Equation 2.4 of Theorem 2.3.2. Indeed, if $\mathcal{A}_i$ denotes the homogeneous component of degree i in $\mathcal{A}$, then $\mathcal{A}_i$ is of course stable under the action of F (and more generally of G). Let us denote by g_i the restriction of $g \in F$ to $\mathcal{A}_i$ and m the maximal size of a finite block in the nested block system (hence the maximal orbital degree encountered in $\mathcal{A}$), we then obtain:

$$\sum_{\bar{h} \in X/G} w(\bar{h}) = \frac{1}{|G|} \sum_{g \in G} P_g = \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{m} \prod_{l \in CT(g_i)} \mathrm{p}_{li}(w'(\mathbb{N})). \qquad (7.1)$$

Note that since we chose $\chi = \mathbb{N}$, the set of weights $w'(\mathbb{N})$ is infinite this time, so $\mathrm{p}_{li}(w'(\mathbb{N}))$ is a series and not a polynomial. Explicitely, we have:

$$\mathrm{p}_{li}(w'(\mathbb{N})) = 1 + z^{li} + z^{2li} + \cdots = \frac{1}{1 - z^{li}}.$$

The thorough reader will have noticed that the method described here overlooks the kernel. Minor arrangements are needed to take it into account — an implementation of the complete method can be found at the end of Section A.2 of the Appendix.

7.4.3   Reasoned encoding relational structure

Having access to a complete and explicit classification simplifies a lot of things. Recall for instance the canonical relational structure that one can associate to a permutation group, and of which we give a construction in the proof of Proposition 3.2.7. At that time, we noted that this canonical structure, although it did verify the properties announced in the statement of the proposition, was not a very subtle one to do so. Thanks to the classification, we can build a more economical and reasoned structure. One needs:

(i) The relevant order relations to describe the highly homogeneous actions (see Theorem 4.2.6 and the remark that follows immediately). For instance, that is nothing on a superblock with non trivial blocks, but the relation of linear order on a superblock of singletons permuted by $\mathrm{Aut}(\mathbb{Q})$.

(ii) A set of relations describing F, of which one copy will be added to each member of the G-orbit of $\sqcup_j B_0^{(j)}$ (namely the $\sqcup_j B_{i(j)}^{(j)}$ for each choice of $i(j)$ in the set of possible indices). This will account for the synchronizations between different superblocks (but not for synchronizations internal to one superblock and not involving any other).

(iii) For each orbit of superblocks, a set of relations of which there will be a copy on each pair of blocks both taken from the same superblock; these will basically describe the gap between $[H_0, H^\infty]$ and the underlying wreath product $H \wr M$ (where the (j) indices have been omitted) within each superblock; and they only depend on the orbit of the superblock.

7.4.4 Enumeration of P-oligomorphic groups

Another nice outcome of the classification is an algorithmic way to browse all closed P-oligomorphic permutation groups with a computer, to enumerate them. Let us assume for now that we narrow our research to transitive groups.

Fix a finite cardinality m. You first need to browse all finite transitive permutation groups, for which many computer algebra softwares have nowadays a complete library until some decent degree: 32 for GAP, which represents a very big amount of data. The second step is to scan the related block systems, and then the normal subgroups of the restrictions to the blocks — two tasks that, again, GAP is pretty good at. Deciding eventually what highly homogeneous groups act on each superblock, and which ones among the non minimal are involved in a synchronization of order two is then essentially a matter of basic combinatorics, and will not have any effect on the age. The only real issue we need to be careful with is the risk of counting the same group several times, or rather counting several isomorphic groups. A notion of equivalence relation should be defined, as well as canonical representatives, in order to address this.

Once a group has been entirely chosen (or even up to the last step, as mentioned), one can determine the growth of its profile, by calculating the finite profiles of the normal subgroups $H^{(j)}$ (actually the cardinalities of the finite ages would be enough), which is just a simple application of the classical Pólya method.

We extensively discussed how the nested block system provides a tight lower bound on the growth rate of the profile. If m_j is the size of blocks of $BB^{(j)}$ in $\mathcal{B}_\mathcal{B}(G)$, recall that this lower bound is simply $m = \sum_j m_j - 1$. And since, for a given m, we had only finitely many choices to make all the way through the algorithmics described above, one can conclude that there are only finitely many transitive P-oligomorphic groups of a given growth — and by extension of growth bounded by any given $r \in \mathbb{N}$. It could be interesting to compute the first terms of this (new?)

counting sequence.

Removing the hypothesis of transitivity is tricky but not unreasonable regarding some aspects: the orbits will be stable parts of the domain of F, and thus F can be obtained by subdirect product of the restrictions thereupon. An exhaustive browsing per size m of the finite domain of F should thus be possible.

Nevertheless, there is one new source of trouble in this case: the possibility of a kernel in the P-oligomorphic group. Since it can be of any finite size, and does not raise the growth rate of the profile, the pleasant remark we made about the transitive P-oligomorphic groups is not true anymore if one allows a kernel — it actually fails immediately: there is of course infinitely many finite permutation groups, of which the profile is ultimately 0. Similarly, one can add, by direct product for instance, an arbitrarily large kernel to any P-oligomorphic group without changing the growth rate of the profile. There are thus clearly infinitely many groups for each growth rate in general; but there should be finitely many if and only if one only considers the kernel-free groups.

Part IV

Appendix

Algorithmic aspects and another (weaker) way to the conjectures

Before and after the classification: experimental approach and resulting programs

As a crucial part of my research process, I sometimes rely on computer experiments in order to forge an intuition and guess some fundamental results – which then "only" need proving. We have nowadays a huge asset that our predecessors did not, and that is ours to use and benefit from. Besides this renowned technique of systematic "testing - conjecture - proof", thinking about how to implement and manipulate theoretical objects, and the more natural or efficient way to do so, is an excellent means to get a deep understanding of the objects and to give a concrete and rigorous shape to an informal intuition, leading to clean and compact mathematical descriptions. Of course, it seems very important to share the code we wrote, for the scientific community to benefit from it. This is how we end up having access to powerful, lively and more and more complete softwares and facilities, such as SageMath and GAP-system, which are the two main languages I have been using and have begun contributing to. In addition and from a purely pragmatic point of view, sharing results as usable code is a good way to guarantee their diffusion and permanence.

In this chapter, we will dwell on these computational aspects. We will first have a look at a situation where the computer exploration saved the day, and at the GAP code that was written and used.

In a second section, we will give a flavour of how the classification found in this thesis made an implementation of P-oligomorphic groups possible, along with methods allowing, for instance, an efficient computation of their profile — all of that using SageMath this time. Unlike in the first section, the code advertised here is more a result of our work than a tool we used; but it can itself (and that is the whole point) now be used as a tool, for people to play with or to forge an intuition, just like I did with my GAP code, using other people's programs!

A.1 Towers of finite groups with GAP-system

In order to make some progress in the proof of Macpherson's conjecture, I was some-
times confronted to a lack of vision, and of a starting point. In this situation, I could
turn to a fantastic tool: computer assisted mathematics. In order words, program-
ming. I naturally followed a protocol that has been successfully experimented for
decades now, although it was not always as popular, and most of all as easy as
today: the protocol of test - conjecture - proof. You explore plenty of examples,
giving them an appropriate shape for reading the information you are interested in,
and guess some results from that material — results that are then yours to prove.

For my own computer experimentation, I started with the SageMath software
(that we will present in next section), but it turned out that I was mostly interested
in the features that actually called to another calculator, GAP-system. Since the
interface between the two still presented some bugs at the time, some of which got
me into quite some trouble, I finally switched to GAP itself.

GAP-system (or just GAP for short) is a free system that allows for mathematical
computations in the domain of discrete algebra. In particular, it provides a vast
range of tools and efficient algorithms to manipulate groups and permutation groups.
It comes with its own programming language (the GAP language), an interactive
environment and a full documentation, of which the reference manual can be found
online.

GAP possesses a large and active community of users and developers, with some
hot spots located in Germany and Scotland among others. It can be installed by
following the instructions on the dedicated webpage.

In the sequel of the section, we will review some of the GAP programs I used
to solve the conjectures (and actually come up with the classification). We will
focus on one of the problems I encountered, which was getting some insight into
the kind of phenomena that could occur on a single superblock. As described in
Subsection 6.2.2, a specific notion was introduced to study this case, the notion of
tower. Only, I was still unsure of how wild such towers could get, and I needed
to forge an intuition. Computational exploration was just the right answer to this
issue, and finally brought way more precise and decisive results than expected, since
it was eventually the key to Theorem 6.1.7, the first brick of the classification.

The goal was to have a look at many towers, and observe what they looked
like; question was: how to proceed? First, I would have to restrain myself to finite
groups, since infinite ones are currently not dealt with by mathematical calculators
in all generality. That being said, I was not interested in all of them. For instance,
considering a block system, I wanted the group to be transitive on the blocks, just
like a *P*-oligomorphic group is on one of its superblock. It seemed reasonable at
this point to just begin with transitive groups, especially since GAP had all the
infrastructure one could possibly need to explore them; the slightly more general
cases could be studied later — even more so since, in first approximation, the aim

was more to restrain the scope of possibilities than to open it up. Not to mention that the classical notion of block system, widened a bit in the context of this thesis, only truly exists for transitive groups, and that the pre-existing related GAP functions would assume (or ask for) the group to be so.

This was not quite enough yet: Lemma 4.2.8 further suggested that research needed to focus on groups that acted on the blocks as $\mathfrak{S}_\infty$ (as long as the blocks were not trivial, but towers of primitive groups were obviously of no interest to us anyway). Even more precise was the condition handed by Lemma 6.2.1, which I was to include. Note that a block system satisfying even just the first condition would necesarily be maximal, which would nicely narrow the field of research.

For convenience in the (hopefully) limited time of the exploration, I would say that a group is "nice" (or acts "nicely") on a given block system if the induced action on the blocks satisfied the condition, and it was likely that I would need to implement a function destined to test this. Since the property obviously depends on the choice of the block system, it was likely I would need to browse through some transitive groups (probably of some given degree), and then through the maximal block systems for each of them, and select the pair (G, B) if G was nice on B to have its tower computed and printed...

Now, for the sake of comparing the obtained towers, I had in mind a *model of use* where I would just type something like

```
f(card, Nb)
```

where `Nb` would be the number of blocks desired and `card` the cardinality of each block, or of the whole domain, and get *some* towers of this shape, that would come from relevant, "nice" groups. Of course, I would begin with very small values and increase them step by step, in order to avoid waiting forever or making my computer explode.

At this point, the plan was clear enough. The first step was to find, if possible, for a given permutation group `gp`, an "adapted block system", that is a block system such that the group would be "nice" on it. Having my model of use (above) in mind, I wanted to be able to provide one additionnal argument to the function, say `Nb`, so that it would look for block systems with this many blocks. Later, I decided to add yet another argument `card` that would specify the degree of the group: even though the piece of information is technically redundant, it would prevent the programs from having to recompute it at every step, for every tested group, which could gain a tiny little bit of time.

(Note that the method just described is obviously not optimal and today, I would do things differently. For a start, instead of testing for every group a quite rigid condition on the shape of the block system and throw away the majority of them, which will then require other rounds of extensive and costly tests to be handled as well, I would probably build a database as groups pass by, and explore it afterwards.)

The following three functions, each one using the next, handle this first part of the job. Some explanations will be given below.

Lines behind a # symbol are comments and are ignored when one runs the pro-
gram. Although they are supposed to help make the code user-friendly, they can be
ignored by the reader as well :) .

For now the programs appear (almost) as I used them in the context of my per-
sonal research, and they do not follow GAP's conventions of proper documentation
or syntax.

```
adapted_block_system := function(gp, card, Nb)
  # gp is a permutation group, card its degree, Nb the desired number of blocks
  # computes, if possible, a block system on which gp is "nice"
  # returns the block system as a list
  # used in tower; uses is_nice_on_B
  local repres, B;
  B := MaximalBlocks(gp, [1..card]);
  # only tries one such system, but it is likely to be enough for our purposes
  if Size(B[1]) = card/Nb and is_nice_on_B(gp, B) then
    return B;
  else
    return [];
  # if not found
  fi;
end;


is_nice_on_B := function(gp, B)
  # gp is a permutation group and B a block system for gp
  # checks if gp is a semi direct product
  # of the actions on and within the blocks of B
  # (concretely checks that however many blocks are fixed pointwise,
  # the action on the remaining blocks is symmetric)
  # returns a boolean
  # used in adapted_block_system; uses Gonblocks_is_sym
  local Nb, Gonblocks, stab, moving_blocks, indep_on_blocks, n, hom, Fix;
  Fix := gp;
  # initialization
  moving_blocks := List(B);
  Nb := Size(B);
  indep_on_blocks := Gonblocks_is_sym(Fix, moving_blocks);
  n := 1;
  while n <= Nb-2 and indep_on_blocks do
    Fix := Stabilizer(Fix, B[n], OnSets);
    # fixes the first n-1 blocks, stabilizes the next one
    hom := ActionHomomorphism(Fix, B[n]);
    Fix := Kernel(hom);
    # fixes the first n, the others are free
    Remove(moving_blocks, 1);
    # one less moving block
    indep_on_blocks := Gonblocks_is_sym(Fix, moving_blocks);
    n := n+1;
  od;
```

```
  return indep_on_blocks;
end;

Gonblocks_is_sym := function(gp, B)
  # gp is a permutation group and B a block system for gp
  # checks if G acts as the symmetric group on the blocks of the block system
B
  # returns a boolean
  # used in is_nice_on_B
  local Gonblocks;
  Gonblocks := Action(gp, B, OnSets);
  return Gonblocks = SymmetricGroup(Size(B));
end;
```

Most GAP functions used here do what one would expect them to do: `MaximalBlocks` computes a maximal non trivial block system; `ActionHomomorphism` computes the permutation representation of the group associated with the specified action (see Subsection 2.1.2), *i.e.* the associated injective morphism that embeds the group in the relevant symmetric group (the keyword `OnSets` is used to precise the type of action); `Action` directly computes the image of this morphism, thus returning a permutation group; `Remove` removes the element of specified position in the list provided as first argument; `Size`, `List`, `Stabilizer`, `Kernel`, `SymmetricGroup` do (resp. construct) what you expect them to do (resp. construct). The function `is_nice_on_B` proceeds by pointwise fixing more and more blocks of B, starting with 0 and ending with all but 2, and checking at every step that the action on the blocks that we did not ask (yet) to be fixed is still symmetric.

The following function, `stab_blocks`, computes the block stabilizer of the group `gp` on a block system B. It will be used to compute the tower of `gp` on B.

```
stab_blocks := function(gp, B)
  # gp is a permutation group and B a block system for gp
  # returns the block stabilizer of the group gp on the block system B
  # used in tower
  local hom;
  hom := ActionHomomorphism(gp, B, OnSets);
  return Kernel(hom);
end;
```

The following two functions are meant to compute and print in a convenient way the tower of the group provided as input, searched for of specified shape. Note that, the requirements being quite rare among all permutation groups of degree `card`, even transitive, the use of these functions on random groups would probably be fastidious. For instance, only 59 of the 25000 transitive groups of degree 24 will be selected and provide some actual data. Since we will use the programs for exploration of a rather large amount of instances, called by a function that will do all the work for us, this will not be an issue in this context. The optional argument `print_group_option` (that GAP passes along packed into a list, since it allows for

several optional arguments — but we will provide at most one here) enables one to
ask for the group to be printed, before the tower is. This may occasionally be useful
when printing several towers using a loop, although it did not turned out decisive
for us.

```
tower := function(gp, card, Nb)
  # gp is a permutation group, card its degree, Nb the desired number of blocks
  # returns the tower of gp on a block system of the specified shape, as a list
  # returns [] instead if gp cannot act "nicely" on such a block system
  # or if gp admits no block system of the specified shape
  # used in print_tower; uses adapted_block_system and stab_blocks
  local B, hom, Gonblocks, stab, H0, tower, n, Fix;
  B := adapted_block_system(gp, card, Nb);
  # B = [] if such a system does not exist
  tower := [];
  if B <> [] then
    Fix := stab_blocks(gp, B);
    # stabilizer of all blocks, will then fix more and more
    n := 1;
    while n < Nb+1 do
      hom := ActionHomomorphism(Fix, B[n]);
      H0 := Image(hom);
      Add(tower, H0);
      # save of result
      Fix := Kernel(hom);
      # one more fixed block, for next H
      n := n+1;
    od;
  fi;
  return tower;
end;

print_tower := function(gp, card, Nb, print_group_option...)
  # gp is a permutation group, card its degree, Nb the desired number of blocks
  # argument print_group_option is optional and should be a boolean
  # prints the tower if the group has the required minimal blocks
  # used in print_all_towers; uses tower
  local tentative_tower, print_group, H, to_print;
  tentative_tower := tower(gp, card, Nb);
  # computation of the tower
  if Length(print_group_option) <> 0 then
    # handling of the option
    print_group := print_group_option[1];
  else
    print_group := false;
    # default value
  fi;
  if print_group and tentative_tower <> [] then
    # if we asked to see the group and it has the required shape
```

```
      Print("Group : ", StructureDescription(gp),"\n");
      Print(" Tower : ");
    fi;
    for H in tentative_tower do
      # prints the terms H of the tower
      to_print := StructureDescription(H);
      while Size(to_print) < 8 do
        # for the terms of towers to appear aligned (for small blocks)
        to_print := Concatenation(to_print, " ");
      od;
      Print(to_print, " ");
    od;
    if tentative_tower <> [] then
      Print("\n");
    fi;
end;
```

Finally, `print_all_towers` function can be used to print the towers of a certain shape (`Nb` blocks) for transitive groups of degree `card` under the conditions previously described. Option `print_groups` can still be used in case there is a need for it, but it makes the comparison of towers less convenient.

```
print_all_towers := function(card, Nb, print_groups...)
  # card is the size of domain, Nb the desired number of blocks
  # argument print_groups is optional and should be a boolean
  # prints all towers that meet the conditions
  # uses print_tower
  local print_option, index, gp;
  if Length(print_groups) <> 0 then
    print_option := print_groups[1];
  else
    print_option := false;
  fi;
  for index in [1..NrTransitiveGroups(card)] do
    # runs through all transitive groups of that degree
    gp := TransitiveGroup(card, index);
    print_tower(gp, card, Nb, print_option);
  od;
end;
```

Examples of outputs we can get are exposed below; the third one takes a few seconds to compute. The first one, for instance, is looking for towers of transitive groups acting "nicely" on 6 blocks of 3 elements. Each tower is printed on a different row, and the H_i's of distinct towers are aligned on the i-th column. The printed groups are named using GAP's library of group names (for instance, `D10` represents the dihedral group on 5 elements, and the symbols : and x are used for the semi-direct and direct products, respectively).

```
gap> print_all_towers(18, 6);
C3      1        1        1        1        1
```

```
C3         C3         C3         C3         C3         1
S3         C3         C3         C3         C3         1
C3         C3         C3         C3         C3         C3
S3         C3         C3         C3         C3         C3
S3         S3         S3         S3         S3         C3
S3         S3         S3         S3         S3         C3
S3         S3         S3         S3         S3         S3
gap>
gap> print_all_towers(25, 5);
C5         1          1          1          1
D10        1          1          1          1
C5 : C4    1          1          1          1
A5         1          1          1          1
S5         1          1          1          1
C5         C5         C5         C5         1
D10        C5         C5         C5         1
C5 : C4    C5         C5         C5         1
C5         C5         C5         C5         C5
D10        C5         C5         C5         C5
C5 : C4    C5         C5         C5         C5
D10        D10        D10        D10        C5
D10        D10        D10        D10        C5
D10        D10        D10        D10        D10
C5 : C4    D10        D10        D10        D10
C5 : C4    C5 : C4    C5 : C4    C5 : C4    C5
C5 : C4    C5 : C4    C5 : C4    C5 : C4    C5
C5 : C4    C5 : C4    C5 : C4    C5 : C4    D10
C5 : C4    C5 : C4    C5 : C4    C5 : C4    D10
C5 : C4    C5 : C4    C5 : C4    C5 : C4    C5 : C4
A5         A5         A5         A5         A5
S5         A5         A5         A5         A5
S5         S5         S5         S5         A5
S5         S5         S5         S5         A5
S5         S5         S5         S5         S5
gap>
gap> print_all_towers(24, 8);
C3         1          1          1          1          1          1          1
C3         C3         C3         C3         C3         C3         C3         1
S3         C3         C3         C3         C3         C3         C3         1
C3         C3         C3         C3         C3         C3         C3         C3
S3         C3         C3         C3         C3         C3         C3         C3
S3         S3         S3         S3         S3         S3         S3         C3
S3         S3         S3         S3         S3         S3         S3         C3
S3         S3         S3         S3         S3         S3         S3         S3
```

What to retain from all this? One seemingly odd observation: in between the first and last elements of the tower, all groups are equal; there can only be a strict decrease after the first group, and before the last one. The fact seemed so surprising to me at first that I did not give it proper consideration right away. When I showed

the results to my advisor and he asked "Did you notice that these towers only decrease at the beginning or the very end?", my answer was "Yes of course, I don't know why, maybe my programs are wrong or my conditions too restrictive". Fortunately, he was more confident than me and said "OR... you could try to prove it's always true"... which it was indeed. This very crucial point led to the infinite analog, that the tower of a P-oligomorphic group has shape H_0, H, H, H $\cdots$; and from there to the classification on a single superblock, which I believe to be the hardest part of the work presented in this thesis.

A.2   An implementation of P-oligomorphic groups with SageMath

Born in 2005, SageMath, or just Sage for short, is a software of computer algebra based on the Python language, under the GNU General Public License. It aims to offer a free alternative to traditional softwares such as Maple or Magma, with a vast range of possibilities brought by functional interfaces with other more specialized systems (among which GAP-system, Pari, Singular...), along with its own original tools. Sage is based on free collaborative development, just like GAP, and has grown more and more complete and more and more used over the years, especially in the fields of combinatorics and algebra. Users can submit tickets with modifications or additions they would like to see into Sage, in order to share them with the community. If the ticket is reviewed and validated by other developers, the change will be added into Sage in its next release version (and, in the meantime, it will be patched on the development version). A whole documentation is available online, but [Cas+13] is a very good way to get familiar with the various uses of the software (English and German translations are avaible and all pdf files are free to download).

This chapter exposes some Sage programs I implemented, not so much as a tool to help me prove the conjecture of Macpherson, unlike the GAP code of the previous section, but rather as a result of my research. I first present a piece of code that was successfully submitted to be integrated into Sage; it is based on Pólya enumeration and did not need new theory to be implemented.

Then, I will dwell on the code I wrote in order to be able to manipulate infinite generic P-oligomorphic groups. This was made possible by the classification result of Chapter 7, so it is entirely new technology (as noted in the previous section, computer algebra softwares still lack effective algorithmics to handle infinite groups in general). Since the code is a bit massive as a whole to be included here, I will just give some example of use of the features.

First, two methods where added to the category of finite permutation groups, that allow to compute the finite profile of a finite permutation group, as well as its generating series (a polynomial in that case, hence the alias `profile_polynomial`). These gave birth to a (positively reviewed) Sage ticket, in order to be integrated in the next release version. I present here the submitted version, with full standardized documentation. The parts delimited by `r""" ... """` are the documentation

strings of the methods; they are destined to the user and ignored by the programs.

```python
class FinitePermutationGroups(CategoryWithAxiom):
    [...]

    class ParentMethods:
        [...]

        @cached_method
        def profile_series(self, variable='z'):
            r"""
            Return the (finite) generating series of the (finite) profile
            of the group.

            The profile of a permutation group G is the counting function that
            maps each nonnegative integer n onto the number of orbits of the
            action induced by G on the n-subsets of its domain.
            If f is the profile of G, f(n) is thus the number of orbits of
            n-subsets of G.

            INPUT:

            - ``variable`` -- a variable, or variable name as a string
              (default: ``'z'``)

            OUTPUT:

            - A polynomial in ``variable`` with nonnegative integer coefficients.
              By default, a polynomial in z over ZZ.

            .. SEEALSO::

                - :meth:`profile`

            EXAMPLES::

                sage: C8 = CyclicPermutationGroup(8)
                sage: C8.profile_series()
                z^8 + z^7 + 4*z^6 + 7*z^5 + 10*z^4 + 7*z^3 + 4*z^2 + z + 1
                sage: D8 = DihedralGroup(8)
                sage: poly_D8 = D8.profile_series()
                sage: poly_D8
                z^8 + z^7 + 4*z^6 + 5*z^5 + 8*z^4 + 5*z^3 + 4*z^2 + z + 1
                sage: poly_D8.parent()
                Univariate Polynomial Ring in z over Rational Field
                sage: D8.profile_series(variable='y')
                y^8 + y^7 + 4*y^6 + 5*y^5 + 8*y^4 + 5*y^3 + 4*y^2 + y + 1
                sage: u = var('u')
                sage: D8.profile_series(u).parent()
                Symbolic Ring
```

```python
        """
        from sage.rings.integer_ring import ZZ

        if isinstance(variable, str):
            variable = ZZ[variable].gen()
        cycle_poly = self.cycle_index()
        return cycle_poly.expand(2).subs(x0 = 1, x1 = variable)

    profile_polynomial = profile_series


    def profile(self, n, using_polya=True):
        r"""
        Return the value in ``n`` of the profile of the group ``self``.

        Optional argument ``using_polya`` allows to change the
        default method.

        INPUT:

        - ``n`` -- a nonnegative integer

        - ``using_polya`` (optional) -- a boolean: if ``True`` (default),
          the computation uses Polya enumeration (and all values of the
          profile are cached, so this should be the method used in case
          several of them are needed);
          if ``False``, uses the GAP interface to compute the orbit.

        OUTPUT:

        - A nonnegative integer that is the number of orbits of
          ``n``-subsets under the action induced by ``self`` on
          the subsets of its domain (i.e. the value of the profile
          of ``self`` in ``n``)

        .. SEEALSO::

            - :meth:`profile_series`

        EXAMPLES::

            sage: C6 = CyclicPermutationGroup(6)
            sage: C6.profile(2)
            3
            sage: C6.profile(3)
            4
            sage: D8 = DihedralGroup(8)
            sage: D8.profile(4, using_polya=False)
```

```
            8

        """
        if using_polya:
            return self.profile_polynomial()[n]
        else:
            from sage.libs.gap.libgap import libgap
            subs_n = libgap.Combinations(list(self.domain()), n)
            return len(libgap.Orbits(self, subs_n, libgap.OnSets))
```

Besides this (rather short) addition to Sage, I have implemented classes allowing to build, manipulate, and compute the profile of infinite P-oligomorphic. The infrastructure involves the following classes.

```
class PoligomorphicGroup(Parent)
    class HighlyHomogeneousGroup(PoligomorphicGroup)
        class AutQQ(HighlyHomogeneousGroup)
        class RevQQ(HighlyHomogeneousGroup)
        class AutQQCircle(HighlyHomogeneousGroup)
        class RevQQCircle(HighlyHomogeneousGroup)
        class SymInfinity(HighlyHomogeneousGroup)
    class DirectProductOfPoligomorphicGroups(PoligomorphicGroup)
    class PermutingSuperblocks(PoligomorphicGroup)
        class WreathProductOnInfiniteBlocks(PermutingSuperblocks)
        class SingleSuperblock(PermutingSuperblocks)
            class WreathProductOnFiniteBlocks(SingleSuperblock)
    class PoligomorphicGroup_generic(PoligomorphicGroup)
```

The code is hosted on a public Git repository. It is not perfect yet and is thus likely to change over time (the outputs presented here, or the names used, may therefore slightly vary as well).

Following are some examples of standard use.

```
sage: RevQQCircle()
The group Rev(QQ/ZZ) generated by the automorphisms of the rational circle
and one reflection
sage: G1 = WreathProductInfiniteBlocks(SymmetricGroup(5), RevQQCircle())
sage: G1.number_of_superblocks()
5
sage: G1.as_PoligomorphicGroup().parent()
<class '__main__.PoligomorphicGroup_generic'>
sage: factor(G1.profile_series())
(-1) * (z - 1)^-5 * (z + 1)^-2 * (z^2 + 1)^-1 * (z^2 + z + 1)^-1 *
(z^4 + z^3 + z^2 + z + 1)^-1
sage: G1.profile(10)
30
sage: G1.profile_first_values(15)
[1, 1, 2, 3, 5, 7, 10, 13, 18, 23, 30, 37, 47, 57, 70, 84]
```

```
sage: var('u')
sage: hilbs5 = 1/((1-u)*(1-u**2)*(1-u**3)*(1-u**4)*(1-u**5))
sage: hilbs5.series(u, 16).coefficients(u, sparse=False)
[1, 1, 2, 3, 5, 7, 10, 13, 18, 23, 30, 37, 47, 57, 70, 84]
```

We are now going to build a more generic *P*-oligomorphic group `G2`.

```
sage: AutQQ()
The group of automorphisms of the rational chain Aut(QQ)
sage: C3 = CyclicPermutationGroup(3)
sage: F = PermutationGroup([(1,2,3),(1,2),(4,5)])
sage: G2 = PoligomorphicGroup_generic(F, block_system = [[1,2,3],[4],[5]],
....: wreath_bases = [[[1,2,3], C3]], hhomogeneous_groups = [[[4], AutQQ()],
....: [[5], AutQQ()]])
```

The group `F` that will act diagonally is isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_2$. A non trivial block system is given, which determines the number of superblocks (3 here). The other argument allow to precise how the blocks of the finite group will be extended into superblocks of the resulting *P*-oligomorphic group.

All arguments except the first one (a finite permutation group) are optional; nevertheless, if specified, they need to obey to the system of criteria required by the classification (see Section 7.3). Default values are the simplest cases (nearly): a block system that consists of the orbits if `F` is not transitive, the default block system handed by GAP if it is (a seed can also be given in this case, see the GAP documentation about block systems to learn more); and the wreath bases and highly homogeneous groups set such that each restriction of `G2` to a superblock is a wreath product with $\mathfrak{S}_\infty$ (but `G2` is not necesarily a direct product of those: for instance, with the block system that we gave here, two of the superblocks permute).

Remark that, in our example, the second list in `hhomogeneous_groups` is redundant and does not have to be specified: the highly homogeneous group of the superblock associated to `[5]` would be deduced from the information given for the other superblocks in the same orbit (as long as there is no contradiction). On the other hand, it would raise an error if wrongly specified (if the group was not identical to the one given for `[4]`).

Since we specified a proper normal subgroup $\mathcal{C}_3$ of $\mathfrak{S}_3$, the first superblock will not support a pure wreath product. Note that, if the subgroup given was not normal, it would automatically be corrected to its normal closure.

```
sage: G2.diagonal_action() == F
True
sage: G2.cardinality()
+Infinity
sage: BB = G2.list_of_superblocks()  # superblocks along with the action thereupon
sage: len(BB)  # number of superblocks
3
sage: BB0 = BB[0]
sage: BB0.parent()
```

```
<class '__main__.SingleSuperblock'>
sage: BB0.size_of_blocks()
3
sage: BB0.action_on_blocks()
The closed infinite symmetric group S_infinity
sage: BB0.restriction_to_a_block()  # or BB0._H0()
Symmetric group of order 3! as a permutation group
sage: BB0.finite_wreath_base()  # or BB0._H()
Cyclic group of order 3 as a permutation group
sage: # a normal subgroup of H0
sage: BB0.print_tower()
H0  H1  H1  H1  H1  H1   ...
with H0 = Symmetric group of order 3! as a permutation group
     H1 = Cyclic group of order 3 as a permutation group
sage:
sage: BB0.underlying_wreath_product()
Cyclic group of order 3 as a permutation group  wreath
                           The closed infinite symmetric group S_infinity
sage: BB1 = BB[1]
sage: BB1.is_highly_homogeneous()
True
sage: BB1.action_on_blocks()
The group of automorphisms of the rational chain Aut(QQ)
```

For the record, we include here the core of the method that computes the series of the profile using Pólya in the general case (this is extracted from the class `PoligomorphicGroup_generic`).

```
max_deg = self._max_size_of_finite_block()          # upper bound of computation
finite_orbits = self._union_finite_ages_per_degree() # finite orbits except in kernel
homom =     [None]
dom =       [None]
homom_ker = [None]
dom_ker =   [None]
for d in range(1, max_deg+1): # will enable to consider the actions on orbits of degr
    # actions on orbits outside the kernel
    homom.append(libgap.ActionHomomorphism(self.diagonal_action(), finite_orbits[d], 
    dom.append(list(range(1, len(finite_orbits[d])+1))) # to compute cycle lengths be
    # same with the kernel, if relevant
    if self.has_kernel() :
        subsets_d = list(Combinations(self.kernel(), d))
        homom_ker.append(libgap.ActionHomomorphism(self.diagonal_action(), subsets_d,
        # in the kernel, each set is alone in its K-orbit,
        # so here (above) sets are identified to their orbits,
        # hence OnSets rather then OnSetsSets
        dom_ker.append(list(range(1, len(subsets_d)+1)))
CCl = libgap.ConjugacyClasses(self.finite_group)
polya_sum = 0
for gbar in CCl:
    g = libgap.Representative(gbar)
```

```
            # computation of the contribution of g
            W_g = 1
            for d in range(1, max_deg+1):
            # run through the orbital degrees of the orbits of the H1's
                g_d = libgap.Image(homom[d], g)   # g acting on the orbits of degree d
                CT_d = libgap.CycleLengths(g_d, dom[d])
                for k in CT_d:
                    W_g /= (1 - z**(d*Integer(k)))
                # same with the kernel, only each subset may only be selected once
                if self.has_kernel() and d <= len(self.kernel()):
                    g_ker_d = libgap.Image(homom_ker[d], g)
                    CT_d = libgap.CycleLengths(g_ker_d, dom_ker[d])
                    for k in CT_d:
                        W_g *= 1 + z**(d*Integer(k))
        polya_sum += W_g*len(libgap.List(gbar))
    result = polya_sum / self.finite_group.order()
```

Appendix B

The original way to the conjectures: an alternative (weaker) proof

This appendix chapter describes what was our first strategy to prove the conjecture of Macpherson and the Cohen-Macaulay property. Although it does not end up to the result of the orbit algebra of a P-oligomorphic group being an algebra of invariants, this approach makes a heavy use of invariant theory techniques.

First, we study how the orbit algebra of an oligomorphic permutation group G relates to the orbit algebra of a subgroup K of G of finite index, and derive three important reductions for Macpherson's question. We then give a proof of the lifting theorem in Section B.4. Finally, we apply all these reductions in Section B.5 to obtain a proof of Macpherson's conjecture, and we get a different proof of the Cohen-Macaulay property by using some more invariant theory.

B.1 Lifting theorem and a first reduction

Let us look back to Lemma 3.2.17. This simple result suggests that some properties of the the group (such as its profile group) may be deduced from the study of some finite index (normal) subgroup. The following, more subtle theorem gives an interesting preservation property at the level of the orbit algebras.

Theorem B.1.1. *Let G be an oligomorphic permutation group and K be a subgroup of finite index. If the orbit algebra $\mathbb{Q}[\mathcal{A}_K]$ of K is finitely generated, then so is its subalgebra $\mathbb{Q}[\mathcal{A}_G]$. If in addition $\mathbb{Q}[\mathcal{A}_K]$ is Cohen-Macaulay, then $\mathbb{Q}[\mathcal{A}_G]$ is Cohen-Macaulay too.*

Note that the subgroup does not have to be normal, although it will be in our use case.

This is a close variant of Hilbert's theorem stating that the ring of invariants of a finite group is finitely generated; the orbit algebra $\mathbb{Q}[\mathcal{A}_K]$ plays the role of the polynomial ring $\mathbb{Q}[X]$, while the orbit algebra $\mathbb{Q}[\mathcal{A}_G]$ plays the role of the invariant ring $\mathbb{Q}[X]^G$.

The key ingredient in Hilbert's proof is the *Reynolds operator*, a finite averaging operator over the group. In the setting of orbit algebras, G is not finite; however, we

will compensate by using the *relative Reynolds operator* with respect to K, which is a finite averaging operator over the coset representatives. Then we will just proceed as in Hilbert's proof. The same approach can be used to prove that $\mathbb{Q}[\mathcal{A}_G]$ is Cohen-Macaulay as soon as $\mathbb{Q}[\mathcal{A}_K]$ is.

Before diving into the proof of Theorem B.1.1, we can appreciate its relevance by immediately deriving a series of practical applications, each of which helping simplify the problem. The first one is a reduction of Macpherson's conjecture to groups whose orbits of elements admit no finite non trivial block systems.

Corollary B.1.2 (Reduction 1). *Let G be an oligomorphic group that admits a non trivial finite transitive block system. Let K be the subgroup of the elements of G that stabilize each block. Then, if the orbit algebra of K is finitely generated, so is the orbit algebra of G.*

Proof. By construction, K is the kernel of the canonical projection of G onto its action by permutation on the blocks and thus a normal subgroup of finite index, and we may apply Theorem B.1.1.   □

B.2   Second reduction: ignoring some synchronizations of order 2

Lemma B.2.1 (Reduction 2). *Let G be a (closed) P-oligomorphic permutation group. Take the normal subgroup of G that is G in which the restrictions to trivial superblocks (those with singletons as finite blocks) that are of type $\mathrm{Rev}(\mathbb{Q})$ (resp. $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$) have been replaced by $\mathrm{Aut}(\mathbb{Q})$ (resp. $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$). If the orbit algebra of this subgroup is finitely generated, then so is the orbit algebra of G.*

This reduction is a straightforward consequence of Theorem B.1.1: just take the kernel of the morphism that maps an element g of G to the tuple $(r_i)_{i\in\{1,\dots,m\}}$, m being the number of trivial superblocks of type $\mathrm{Rev}(\mathbb{Q})$ or $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$ and r_i being 1 if the action of g on the i-th such superblock is in $\mathrm{Aut}(\mathbb{Q})$ or $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ and -1 if it is not (if it induces a "flip" on this orbit).

This technically generalizes to superblocks in general, but Lemma 4.2.8 and the results of Section 4.3 already imply that synchronizations of order 2 may not exist in a P-oligomorphic group if the finite blocks are non trivial.

Note that in order to simply ignore synchronizations of order 2 (synchronized "flips") between restrictions of type $\mathrm{Rev}(\mathbb{Q})$ or $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$, it would be sufficient to reduce only those that do synchronize in the original group, and keep the ones that do not.

We derive the following convenient remark from the reduction above and the construction of the nested block system (one may recall intuitively that we chose the finite blocks as big as possible and then took the smallest possible infinite blocks of the induced action on the finite blocks; see proof of Theorem 5.3.1).

Remark B.2.2. As the actions of G on the blocks of superblocks are primitive by construction, and thus highly homogeneous in our setting, they are all independent.

B.3 Third reduction: assuming there are no finite orbits of elements

The third application is a reduction of Macpherson's conjecture to groups with empty kernel.

Theorem B.3.1 (Reduction 3)**.** *Let G be a P-oligomorphic permutation group. Assume that the orbit algebra of $K_{|E-\ker K}$, where K is the subgroup of G acting trivially on the kernel of G, is finitely generated. Then the orbit algebra of G is finitely generated.*

   This reduction allows, when trying to prove the conjecture of Macpherson, to make the assumption that our P-oligomorphic group has no finite orbits of elements.
   We will need the following two simple results to prove this theorem. It is a simple corollary of Lemma 3.2.17.

Lemma B.3.2. *Let G be an oligomorphic permutation group and K be a normal subgroup of finite index. Then $\ker(K) = \ker(G)$.*

Proof of Reduction 3. Let Ψ be the canonical projection from G to its finite restriction $G_{|\ker G}$, and K be its kernel, that is the subgroup of the elements of G acting trivially on $\ker G$ (care that there are two entirely distinct notions of "kernel" in this situation). By construction, K is a normal subgroup of G of finite index. In particular $\ker K = \ker G$. Using Lemma 3.2.16,

$$\mathbb{Q}[\mathcal{A}_K] \simeq \mathbb{Q}[\mathcal{A}_{K_{|E-\ker K}}] \otimes \mathbb{Q}[\mathcal{A}_{\mathbb{K}_{\ker G}}], \tag{B.1}$$

where the right hand side of the tensor product is just the set algebra of $\ker G$. Since it is finitely generated by its singletons, this concludes the proof. □

Remark B.3.3. At the level of Hilbert series, Equation B.1 becomes

$$\mathcal{H}(\mathbb{Q}[\mathcal{A}_K], z) = \mathcal{H}(\mathbb{Q}[\mathcal{A}_{K_{|E-\ker K}}], z)(1 + z)^{|\ker G|}.$$

Hence $K_{|E-\ker K}$ has the same growth rate as K and therefore, by Lemma 3.2.17, as G.

B.4 Proof of the lifting theorem

Let us now turn to the *relative Reynolds operator* R_K^G. It is defined by choosing some representatives $(g_i)_i$ of the left cosets of G w.r.t. K:

$$R_K^G = \frac{1}{[G : K]} \sum_i g_i.$$

Lemma B.4.1. *Let G be an oligomorphic permutation group, and K be a subgroup of finite index. Then, the relative Reynolds operator R_K^G defines a projection from $\mathbb{Q}[\mathcal{A}_K]$ onto $\mathbb{Q}[\mathcal{A}_G]$ which does not depend on the choice of the g_i's, and it is a $\mathbb{Q}[\mathcal{A}_G]$-module morphism.*

Proof. Take $p \in \mathbb{Q}[\mathcal{A}_K]$. Note that, for any $h \in K$, $h.p = p$. Therefore, if g_i and g_i' are representatives of the same cosets (i.e. $g_i' = g_i h_i$), we have:

$$\frac{1}{[G:K]} \sum_i g_i'.p = \frac{1}{[G:K]} \sum_i g_i h_i.p = \frac{1}{[G:K]} \sum_i g_i.p \,.$$

Hence the map defined by the Reynolds operator on $\mathbb{Q}[\mathcal{A}_K]$ does not depend on the choice of representatives, as desired.

Let us now compute:

$$g.(R_K^G.p) = g.\frac{1}{|G:K|} \sum_i g_i.p = \frac{1}{[G:K]} \sum_i gg_i.p = \frac{1}{[G:K]} \sum_i g_{\sigma(i)} h_i.p$$

$$= \frac{1}{[G:K]} \sum_i g_{\sigma(i)}.p = \frac{1}{[G:K]} \sum_i g_i.p = R_K^G.p$$

where we used that g acts by a permutation σ on the cosets and thus $gg_i = g_{\sigma(i)} h_i$ for some h_i's. This proves that R_K^G maps $\mathbb{Q}[\mathcal{A}_K]$ onto $\mathbb{Q}[\mathcal{A}_G]$.

Finally, take $p \in \mathbb{Q}[\mathcal{A}_G]$, and $q \in \mathbb{Q}[\mathcal{A}_K]$. Then,

$$R_K^G.(pq) = \frac{1}{[G:K]} \sum_i (g_i.p)(g_i.q) = \frac{1}{[G:K]} \sum_i p(g_i.q) = p\frac{1}{[G:K]} \sum_i g_i.q = pR_K^G.q \,.$$

Therefore, R_K^G is a $\mathbb{Q}[\mathcal{A}_G]$-module morphism; furthermore taking $q = 1$, we get that R_K^G projects $\mathbb{Q}[\mathcal{A}_K]$ onto $\mathbb{Q}[\mathcal{A}_G]$. $\qquad\square$

We can now finally prove the main result of this section.

Proof of Theorem B.1.1. Consider $I := \langle \mathbb{Q}[\mathcal{A}_G]^+ \rangle$ the graded ideal of all the non zero degree elements of $\mathbb{Q}[\mathcal{A}_G]$ (in other words the ideal generated by the non zero degree orbits) in $\mathbb{Q}[\mathcal{A}_K]$. Since $\mathbb{Q}[\mathcal{A}_K]$ is finitely generated, it satisfies the ascending condition on chains of ideals. Therefore, I is finitely generated as an ideal. Denote by $(p_i)_i$ a generating set for I. Since I is finite dimensional for each degree, we may assume without loss of generality that the p_i's belong to $\mathbb{Q}[\mathcal{A}_G]^+$.

Claim: The p_i's generate $\mathbb{Q}[\mathcal{A}_G]$ as an algebra, which is therefore finitely generated.

To prove the claim, we proceed by induction on the degree d: assume that any element of $\mathbb{Q}[\mathcal{A}_G]$ of degree at most $d - 1$ lives in the algebra generated by the p_i's. This is obviously the case for $d = 1$.

Take p a homogeneous element of degree d of $\mathbb{Q}[\mathcal{A}_G]$, and express it in the ideal generated by the p_i's as $p = \sum p_i q_i$, where the q_i are homogeneous elements of $\mathbb{Q}[\mathcal{A}_K]$ of degree at most $d - 1$. Then,

$$p = R_K^G.p = \sum_i R_K^G.(p_i q_i) = \sum_i p_i R_K^G.q_i \,.$$

Since each $R_K^G.q_i$ belongs to $\mathbb{Q}[\mathcal{A}_G]$, we can apply induction and deduce that it lives in the algebra generated by the p_i's, and therefore the same holds for p.

Now we prove the second part of the theorem : the lifting of the Cohen-Macaulay property.

The proof is adapted from [Sta79b].
We first claim that $\mathbb{Q}[\mathcal{A}_K]$ is a finitely generated $\mathbb{Q}[\mathcal{A}_G]$-module. This is equivalent to saying that $\mathbb{Q}[\mathcal{A}_K]$ is integral over $\mathbb{Q}[\mathcal{A}_G]$ (see [Sta79b] for the proof of equivalence), which means that every element of $\mathbb{Q}[\mathcal{A}_K]$ satisfies a polynomial relation with coefficients in $\mathbb{Q}[\mathcal{A}_G]$ and the one of greatest power being 1. The integrality may be proved by considering, for $f \in \mathbb{Q}[\mathcal{A}_K]$, the polynomial $P_f(X) = \prod_{g_i}(X - g_i(f))$, where the g_i's are representatives of the left cosets of K in G as previously. Each coefficient is a symmetric polynomial in the images of f, so it is an element of $\mathbb{Q}[\mathcal{A}_G]$. The polynomial $P_f(X)$ is also monic as required, and we do have $P_f(f) = 0$ since $X - f$ is one of the factors.
Let now $\theta_1, \ldots, \theta_s$ be an h.s.o.p (homogeneous system of parameters) for $\mathbb{Q}[\mathcal{A}_G]$ (we know such elements exist thanks to the Noether normalization lemma). We aim to prove that $\mathbb{Q}[\mathcal{A}_G]$ is a finitely generated free module over $\mathbb{Q}[\theta_1, \ldots, \theta_s]$. By definition, $\mathbb{Q}[\mathcal{A}_G]$ is finite over $\mathbb{Q}[\theta_1, \ldots, \theta_s]$, and we just showed that $\mathbb{Q}[\mathcal{A}_K]$ is finite over $\mathbb{Q}[\mathcal{A}_G]$ so it is also finite over $\mathbb{Q}[\theta_1, \ldots, \theta_s]$, which means that $\theta_1, \ldots, \theta_s$ is an h.s.o.p for $\mathbb{Q}[\mathcal{A}_K]$.
The orbit algebra of K is Cohen-Macaulay by hypothesis, which implies in particular that it is also a finitely generated free module on $\mathbb{Q}[\theta_1, \ldots, \theta_s]$ (as with any other h.s.o.p). We will use a certain free module basis to deduce (the existence of) a free module basis of $\mathbb{Q}[\mathcal{A}_G]$ on $\mathbb{Q}[\theta_1, \ldots, \theta_s]$.
We now show that we can write $\mathbb{Q}[\mathcal{A}_K] = \mathbb{Q}[\mathcal{A}_G] \oplus U$, where U is a $\mathbb{Q}[\mathcal{A}_G]$-module. Since the Reynolds operator $R = R_K^G$ is a morphism of $\mathbb{Q}[\mathcal{A}_G]$-modules and a projection from $\mathbb{Q}[\mathcal{A}_K]$ onto $\mathbb{Q}[\mathcal{A}_G]$, we may take $U = \{f \in \mathbb{Q}[\mathcal{A}_K]/R(f) = 0\} = \{f - R(f) : f \in \mathbb{Q}[\mathcal{A}_K]\}$.

From the obtained decomposition, we get a decomposition of the following quotient:

$$\mathbb{Q}[\mathcal{A}_K]/(\theta_1, \ldots, \theta_s) = \mathbb{Q}[\mathcal{A}_G]/(\theta_1, \ldots, \theta_s) \oplus U/(\theta_1 U + \cdots + \theta_s U) \ .$$

We now use the well know property of Cohen-Macaulay algebras (Proposition 3.1 of [Sta79b]) by which, for any h.s.o.p $\theta_1, \ldots, \theta_s$ of such a $\mathbb{Q}$-algebra $\mathcal{A}$, a given set of elements forms a free $\mathbb{Q}[\theta_1, \ldots, \theta_s]$-module basis if and only if they are a vector space basis of $\mathcal{A}/(\theta_1, \ldots, \theta_s)$. We take a homogeneous $\mathbb{Q}$-basis for the quotient that is the concatenation of a basis for the left and right-hand term of the decomposition respectively, and then lift the elements of this basis to homogeneous elements of $\mathbb{Q}[\mathcal{A}_G]$ or U (depending on the part of the basis they are taken from). These homogeneous elements form a free module basis of $\mathbb{Q}[\mathcal{A}_K]$ over $\mathbb{Q}[\theta_1, \ldots, \theta_s]$, which leads the ones lifted to $\mathbb{Q}[\mathcal{A}_G]$, thanks to decomposition of $\mathbb{Q}[\mathcal{A}_K]$, to be a free $\mathbb{Q}[\theta_1, \ldots, \theta_s]$-module basis for it. $\qquad\square$

B.5   Results: Theorem of Macpherson and the Cohen-Macaulay property

Theorem B.5.1. *The orbit algebra of a P-oligomorphic permutation group is finitely generated.*

Sketch of proof. Consider the nested block system $\mathcal{B}_{\mathcal{B}}(G)$ introduced in Section 5.3. Recall that it consists *a priori* of superblocks and possibly one finite stable block; use the Reduction 3 to assume that there is no such finite isolated block (meaning G has no kernel).

We aim to prove the existence of a normal subgroup K of finite index of G with a simple form, ensuring that its orbit algebra is a finitely generated (almost free) algebra.

Start with $K = G$. Replace K by the kernel of its action on the set of superblocks of $\mathcal{B}_{\mathcal{B}}(G)$. This ensures that K stabilizes each of them. Use Lemma 5.3.5 to argue that superblocks are still superblocks. Now take one of them. Using Corollary 6.1.8, and replacing K if needed, we may assume that the restriction of K to the support of the superblock is some $H \wr \mathfrak{S}_\infty$.

[Note from the author: when this proof was first written, the classification result on one superblock was not as sharp and only stated that the restriction to a superblock had the same age as a supergroup of finite index of such a wreath product. So the rest of the proof was formulated using "has the same age as" and not "is", which still led to the right conclusion about the conjecture of Macpherson.]

Repeat for the other superblocks.

Now use Corollary 6.1.10 and Lemma 5.3.5 to argue that there is no synchronization (neither finite nor infinite) left between superblocks.

Then, K is some direct product of groups of the form $\mathfrak{S}_\infty$, $\mathrm{Aut}(\mathbb{Q})$, $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, and $G' \wr \mathfrak{S}_\infty$. From Remark 5.1.2, $\mathbb{Q}[\mathcal{A}_K]$ is a free algebra which is finitely generated. Using Theorem B.1.1, it follows that $\mathbb{Q}[\mathcal{A}_G]$ is finitely generated. $\qquad\square$

Theorem B.5.2. *The orbit algebra of a P-oligomorphic permutation group is Cohen-Macaulay.*

To get this theorem, one just needs to apply Theorem B.1.1 to a suitable subgroup K; so start as in the above proof, but instead of just ignoring the kernel of the group at the beginning and cutting it off, replace K by the kernel of the homomorphism (careful, not the same notions!) projecting G onto its action on the stable block.

Then $\mathbb{Q}[\mathcal{A}_K]$ is the tensor product of a finite dimensional algebra (due to the kernel; see Example 3.2.14) and a finitely generated free algebra $\mathscr{A}$. If we call $\mu_1, \ldots, \mu_r$ a linear basis for the finite dimensional part, and $\theta'_1, \ldots, \theta'_m$ an h.s.o.p for $\mathscr{A}$ (so it actually generates it all), this means we have $\mathbb{Q}[\mathcal{A}_K] = \bigoplus_i \mu_i \mathbb{Q}[\theta'_1, \ldots, \theta'_m]$; in other words, $\mathbb{Q}[\mathcal{A}_K]$ is Cohen-Macaulay.

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Index

Titre : Classification des groupes P-oligomorphes, conjectures de Cameron et Macpherson

Mots-clés : Informatique fondamentale; Combinatoire algébrique; Profils; Groupes de permutations oligomorphes; Théorie des invariants; Systèmes de blocs d'imprimitivité.

Résumé : Les travaux présentés dans cette thèse de doctorat relèvent de la combinatoire algébrique et de la théorie des groupes. Précisément, ils apportent une contribution au domaine de recherche qui étudie le comportement des profils des groupes oligomorphes.

La première partie de ce manuscrit introduit la plupart des outils qui nous seront nécessaires, à commencer par des éléments de combinatoire et combinatoire algébrique. Nous présentons les fonctions de comptage à travers quelques exemples classiques, et nous motivons l'addition d'une structure d'algèbre graduée sur les objets énumérés dans le but d'étudier ces fonctions. Nous évoquons aussi les notions d'ordre et de treillis. Dans un second temps, nous donnons un aperçu des définitions et propriétés de base associées aux groupes de permutations, ainsi que quelques résultats de théorie des invariants. Nous terminons cette partie par une description de la méthode d'énumération de Pólya, qui permet de compter des objets sous une action de groupe.

La deuxième partie est consacrée à l'introduction du domaine dans lequel s'inscrit cette thèse, celui de l'étude des profils de structures relationnelles, et en particulier des profils orbitaux. Si G est un groupe de permutations infini, son profil est la fonction de comptage qui envoie chaque $n \in \mathbb{N}$ sur le nombre d'orbites de n-sous-ensembles, pour l'action induite de G sur les sous-ensembles finis d'éléments. Cameron a conjecturé que le profil de G est équivalent à un polynôme dès lors qu'il est borné par un polynôme. Une autre conjecture, plus forte, a été plus tard émise par Macpherson : elle implique une certaine structure d'algèbre graduée sur les orbites de sous-ensembles, créée par Cameron et baptisée algèbre des orbites, soutenant que si le profil est borné par un polynôme, alors l'algèbre des orbites est de type fini. Comme amorce de notre étude de ce problème, nous développons quelques exemples et faisons nos premiers pas vers une résolution en examinant les systèmes de blocs des groupes de profil borné par un polynôme — que nous appelons P-oligomorphes —, ainsi que la notion de sous-produit direct.

La troisième partie démontre une classification des groupes P-oligomorphes, notre résultat le plus important et dont la conjecture de Macpherson se révèle un corollaire. Tout d'abord, nous étudions la combinatoire du treillis des systèmes de blocs, qui conduit à l'identification d'un système généralisé particulier, constitué de blocs de blocs ayant de bonnes propriétés. Nous abordons ensuite le cas particulier où il se limite à un seul bloc de blocs, pour lequel nous établissons une classification. La preuve emprunte à la notion de sous-produit direct pour gérer les synchronisations internes au groupe, et a requis une part d'exploration informatique afin d'être d'abord conjecturée.

Dans le cas général, nous nous appuyons sur les résultats précédents et mettons en évidence la structure de G comme produit semi-direct impliquant son sous-groupe normal d'indice fini minimal et un groupe fini. Ceci permet de formaliser une classification complète des groupes P-oligomorphes, et d'en déduire la forme de l'algèbre des orbites : (à peu de choses près) une algèbre d'invariants explicite d'un groupe fini. Les conjectures de Macpherson et de Cameron en découlent, et plus généralement une compréhension exhaustive de ces groupes.

L'annexe contient des extraits du code utilisé pour mener la preuve à bien, ainsi qu'un aperçu de celui qui a été produit en s'appuyant sur la nouvelle classification, qui permet de manipuler les groupes P-oligomorphes en usant d'une algorithmique adaptée. Enfin, nous joignons ici notre première preuve, plus faible, des deux conjectures.

Title: Classification of P-oligomorphic groups, conjectures of Cameron and Macpherson

Keywords: Theoretical computer science; Algebraic combinatorics; Profiles; Oligomorphic permutation groups; Invariant theory; Block systems.

Abstract: This PhD thesis falls under the fields of algebraic combinatorics and group theory. Precisely, it brings a contribution to the domain that studies profiles of oligomorphic permutation groups and their behaviors.

The first part of this manuscript introduces most of the tools that will be needed later on, starting with elements of combinatorics and algebraic combinatorics. We define counting functions through classical examples; with a view of studying them, we argue the relevance of adding a graded algebra structure on the counted objects. We also bring up the notions of order and lattice. Then, we provide an overview of the basic definitions and properties related to permutation groups and to invariant theory. We end this part with a description of the Pólya enumeration method, which allows to count objects under a group action.

The second part is dedicated to introducing the domain this thesis comes within the scope of. It dwells on profiles of relational structures, and more specifically orbital profiles. If G is an infinite permutation group, its profile is the counting function which maps any $n \in \mathbb{N}$ to the number of orbits of n-subsets, for the induced action of G on the finite subsets of elements. Cameron conjectured that the profile of G is asymptotically equivalent to a polynomial whenever it is bounded by a polynomial. Another, stronger conjecture was later made by Macpherson: it involves a certain structure of graded algebra on the orbits of subsets created by Cameron, the orbit algebra, and states that if the profile of G is bounded by a polynomial, then its orbit algebra is finitely generated. As a start in our study of this problem, we develop some examples and get our first hints towards a resolution by examining the block systems of groups with profile bounded by a polynomial — that we call P-oligomorphic —, as well as the notion of subdirect product.

The third part is the proof of a classification of P-oligomorphic groups, with Macpherson's conjecture as a corollary. First, we study the combinatorics of the lattice of block systems, which leads to identifying one special, generalized such system, that consists of blocks of blocks with good properties. We then tackle the elementary case when there is only one such block of blocks, for which we establish a classification. The proof borrows to the subdirect product concept to handle synchronizations within the group, and relied on an experimental approach on computer to first conjecture the classification.

In the general case, we evidence the structure of a semi-direct product involving the minimal normal subgroup of finite index and some finite group. This allows to formalize a classification of all P-oligomorphic groups, the main result of this thesis, and to deduce the form of the orbit algebra: (little more than) an explicit algebra of invariants of a finite group. This implies the conjectures of Macpherson and Cameron, and a deep understanding of these groups.

The appendix provides parts of the code that was used, and a glimpse at that resulting from the classification afterwards, that allows to manipulate P-oligomorphic groups by apropriate algorithmics. Last, we include our earlier (weaker) proof of the conjectures.