# Classification of $P$-oligomorphic permutation groups

## Conjectures of Cameron and Macpherson

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joint work with Nicolas M. Thiéry

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Université Paris-Sud (Orsay)

SLC, April 17h of 2019

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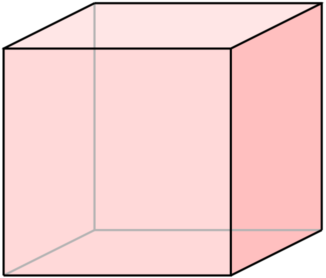

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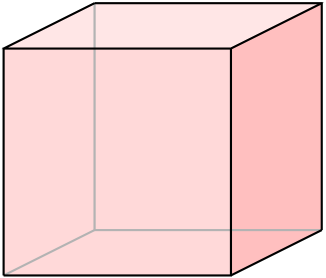

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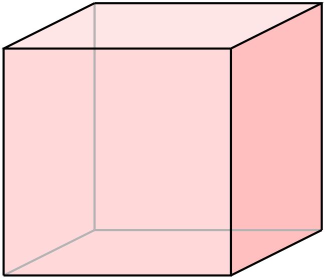

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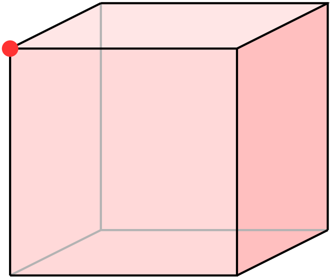

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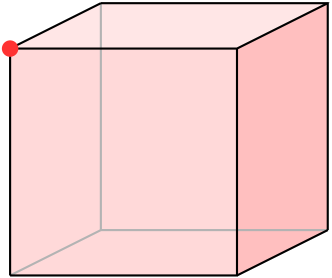

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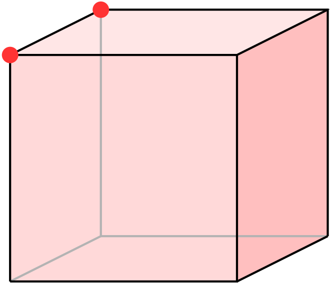

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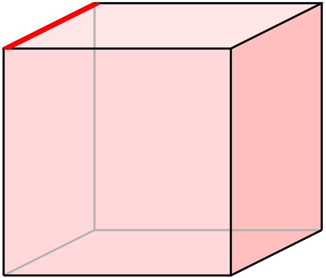

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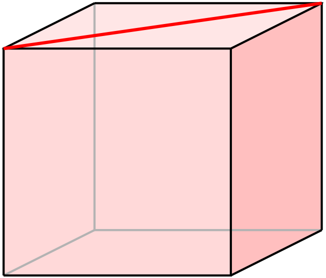

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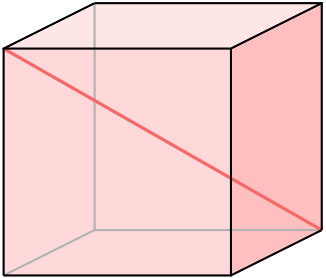

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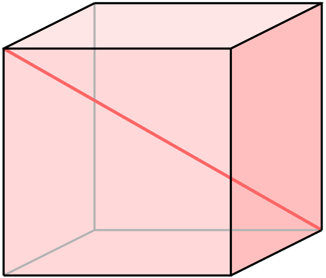

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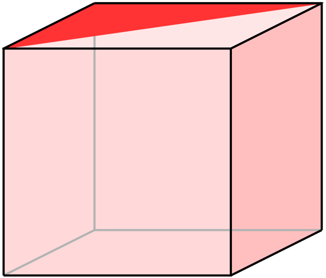

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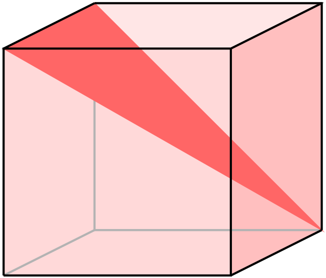

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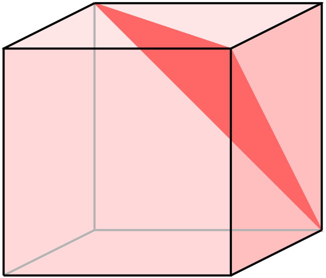

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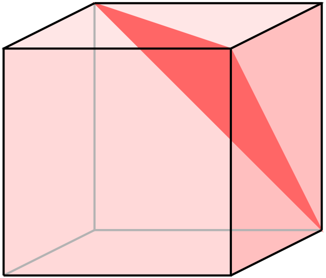

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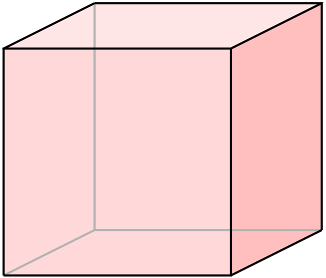

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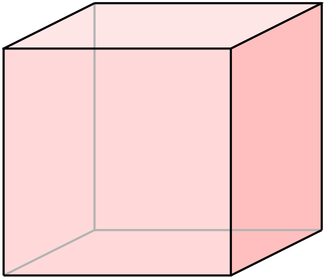

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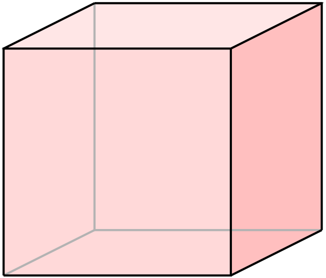

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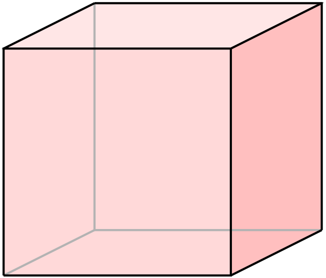

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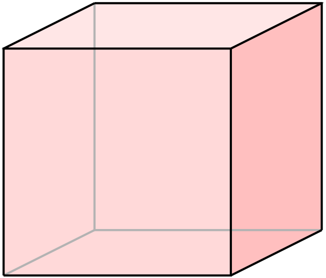

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## Conjecture 1 - Cameron, 70's

$G$ $P$-**oligomorphic** $\;\Rightarrow\; \mathcal{H}_G(z) \; = \; \frac{N(z)}{\prod_i (1-z^{d_i})}$ with $N(z) \in \mathbb{Z}[z]$

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## Conjecture 2 (stronger) - Macpherson, 85

$G$ $P$-oligomorphic $\Rightarrow$ $\mathcal{A}_G$ is finitely generated

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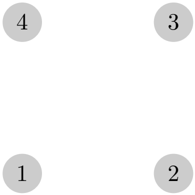

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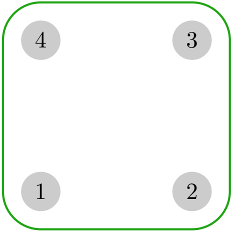

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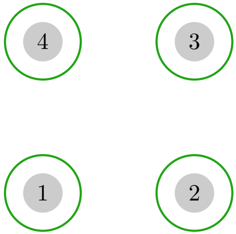

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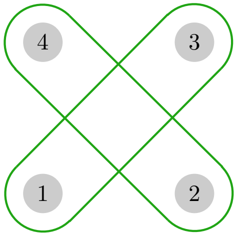

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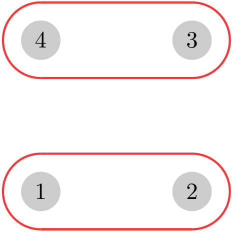
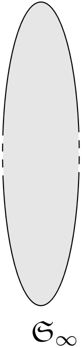
## Example

Block systems of $C_4$

Not a block system $\rightarrow$



# The complete primitive $P$-oligomorphic groups



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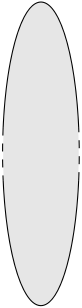
Macpherson:
$G$ $P$-oligomorphic with no non trivial blocks $\Rightarrow \varphi_G(n) = 1$



$\mathfrak{S}_\infty$

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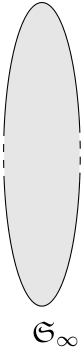


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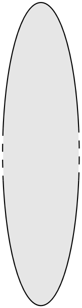

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- $\mathrm{Aut}(\mathbb{Q})$ : automorphisms of the rational chain
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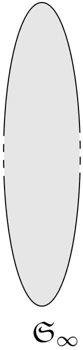


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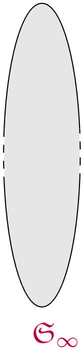
Well known, nice groups (called *highly homogeneous*).
In particular, their orbit algebra is finitely generated.

# An infinite example: $\mathfrak{S}_\infty \wr \mathfrak{S}_3$



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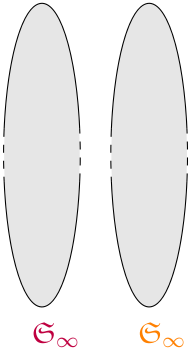

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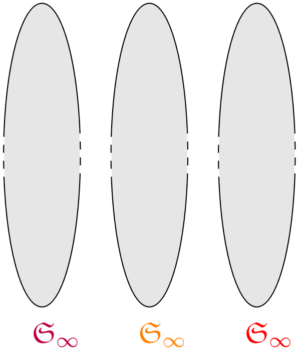

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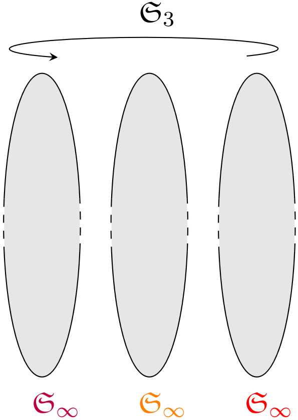

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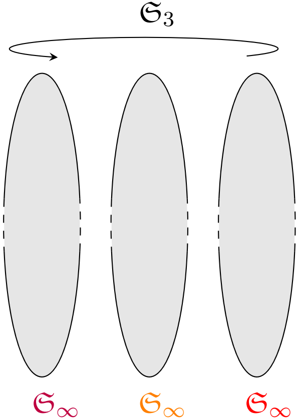


## Wreath product

$\mathfrak{S}_\infty \wr \mathfrak{S}_3$

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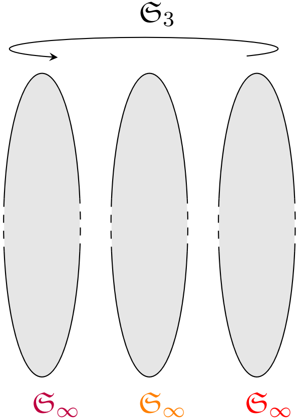


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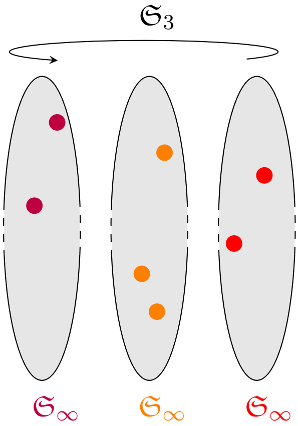


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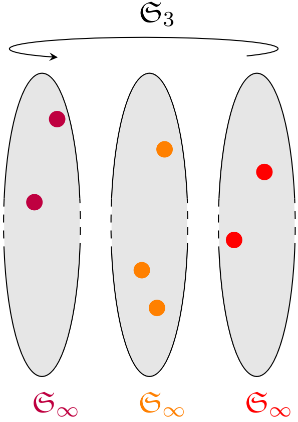


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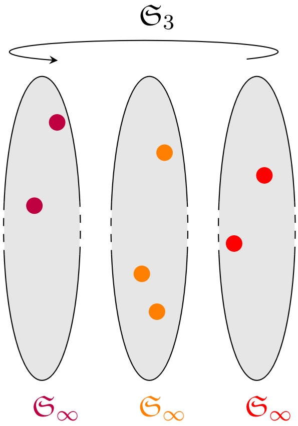


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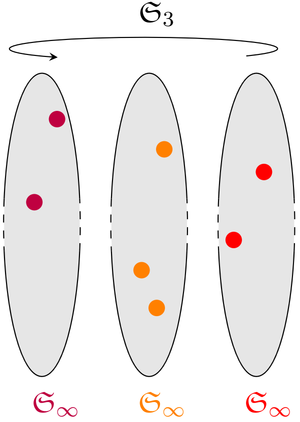


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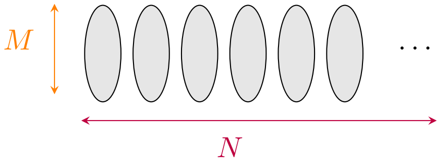
One can obtain functions counting integer partitions, combinations, $P$-partitions (with optional length and/or hight restrictions) as profiles of wreath products...

# Lower bound



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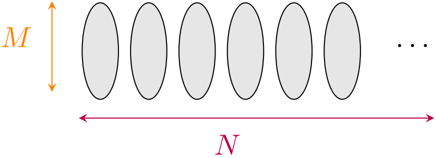


$$\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$$

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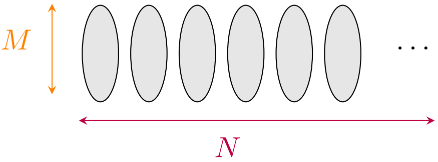


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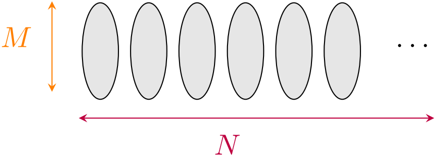


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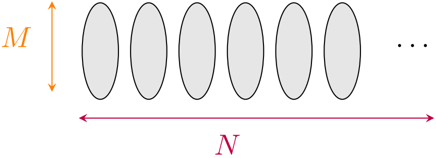


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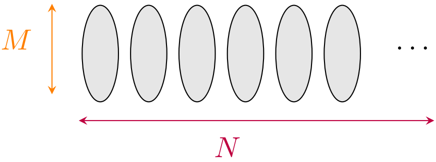


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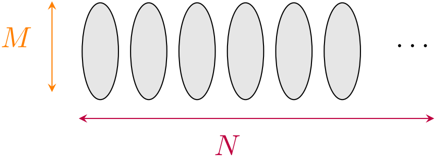


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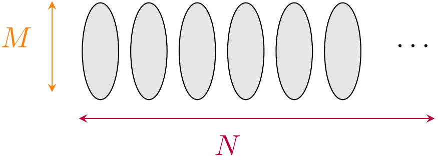


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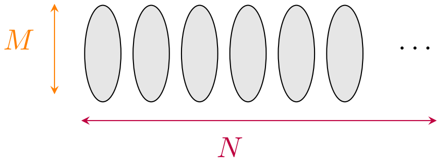


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Better have big finite blocks and/or "small" infinite ones...

# Lattices of block systems

Lattice of partitions $\rightarrow$ structure of *lattice* on block systems

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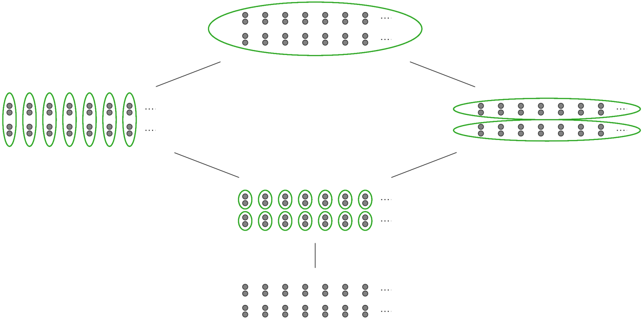

$(\mathcal{C}_2 \times \mathcal{C}_2) \wr \mathfrak{S}_\infty$

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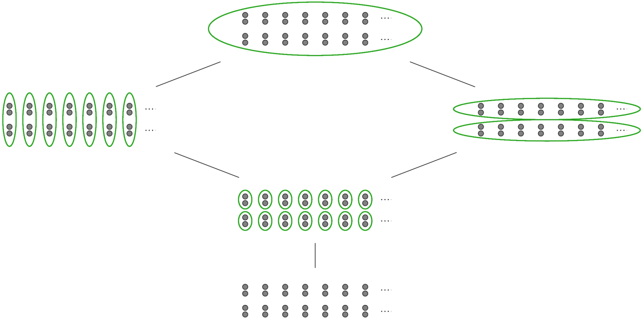

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## Non trivial fact

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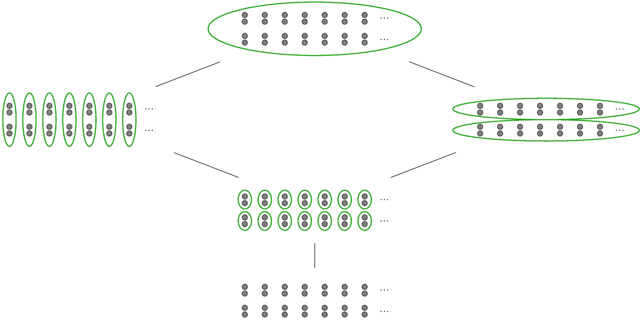

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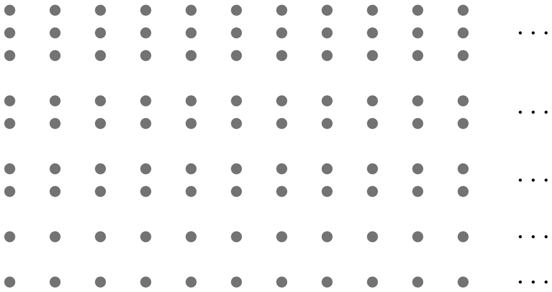
Remark. If $G$ is $P$-oligomorphic, both of them are actually finite!

# The nested block system

## Idea



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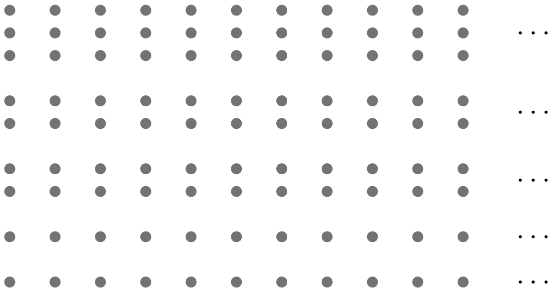
1. Take the *maximal* system of finite blocks



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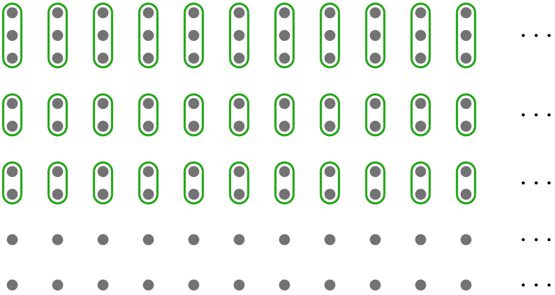

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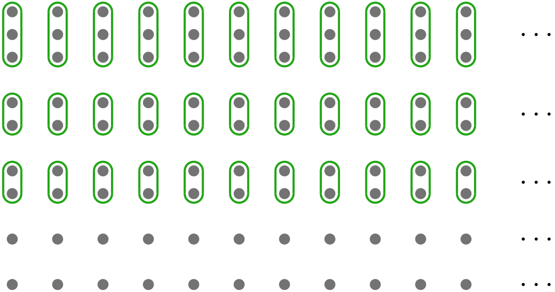

Action on the maximal finite blocks...

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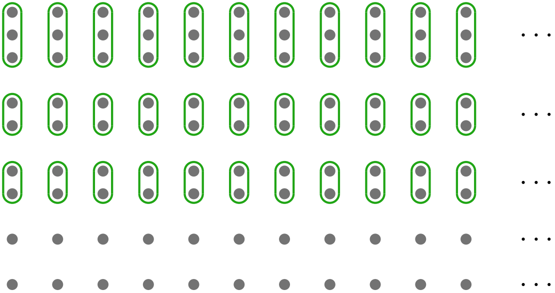

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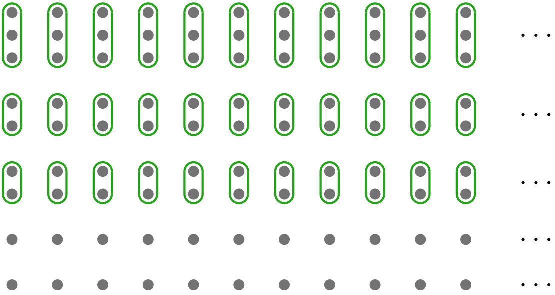


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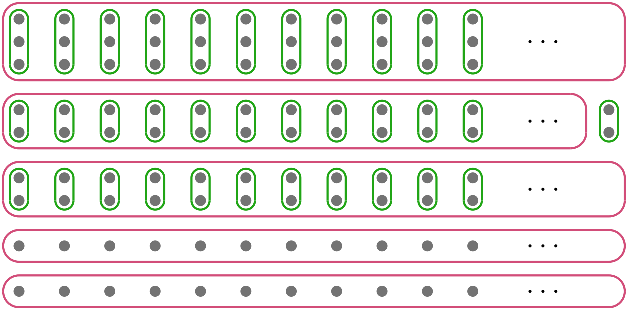

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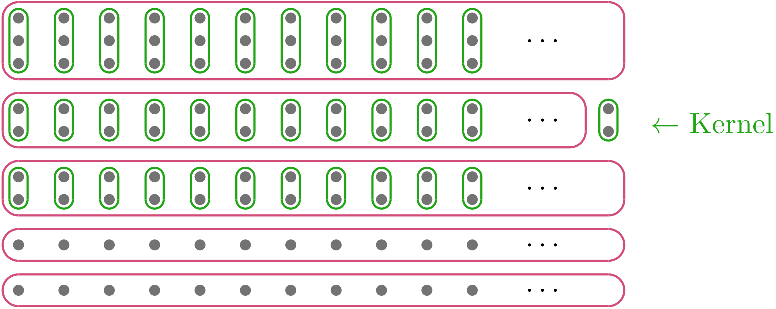
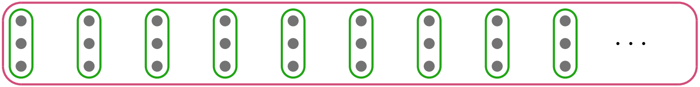



Action on the maximal finite blocks... that has no finite blocks.

# One superblock: examples



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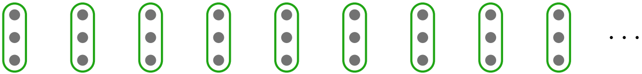

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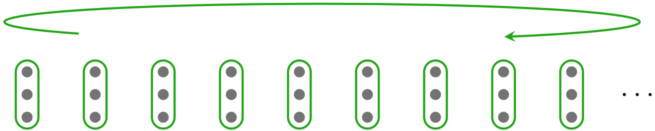

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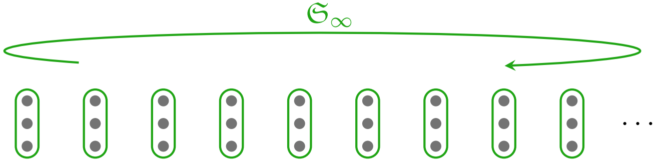

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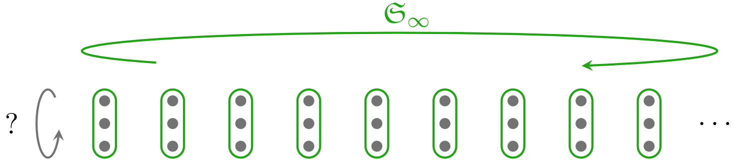

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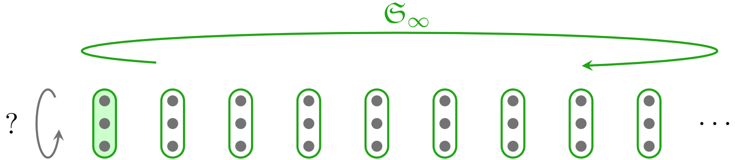

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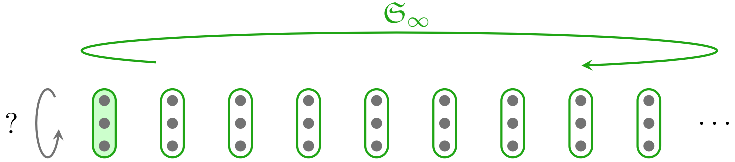


$G_{|B_0} = H_0$

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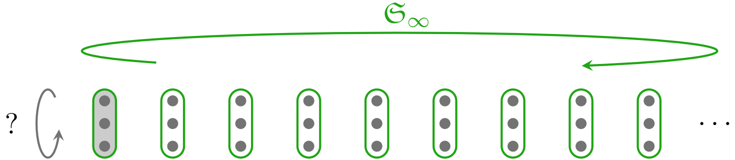


$G_{|B_0} = H_0$ , Fix$(B_0)$

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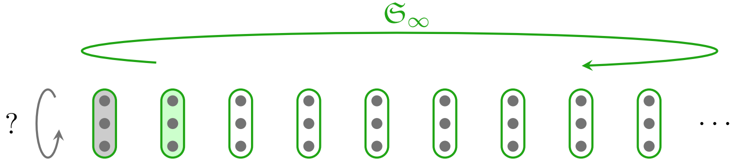


$G_{|B_0} = H_0 \ , \ \mathrm{Fix}(B_0)_{|B_1} = H_1$

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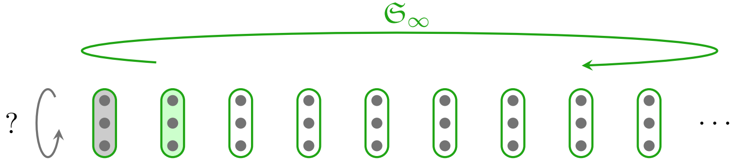


$H_0$ , $H_1$

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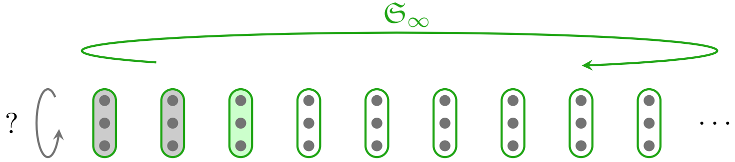


$H_0$ , $H_1$ , $H_2$

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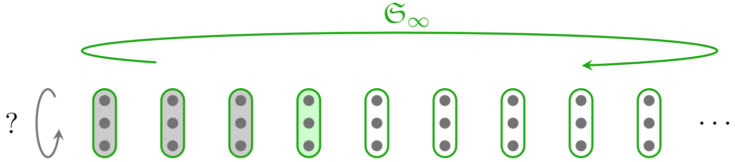


$H_0$ , $H_1$ , $H_2$ , $H_3$

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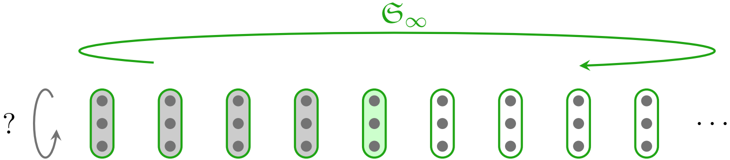


$H_0$ , $H_1$ , $H_2$ , $H_3$ , $H_4$

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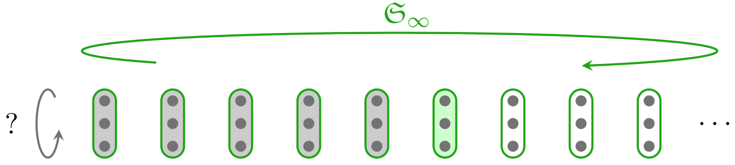


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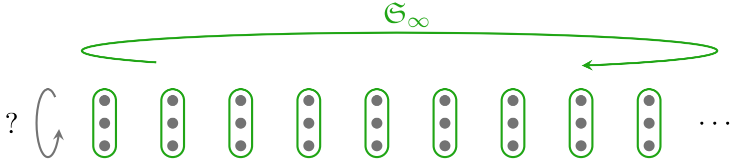


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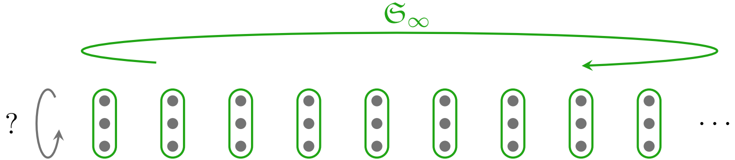


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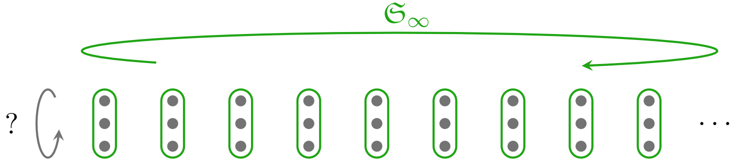


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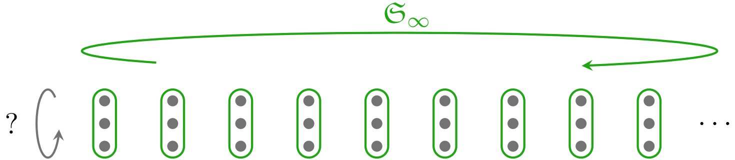


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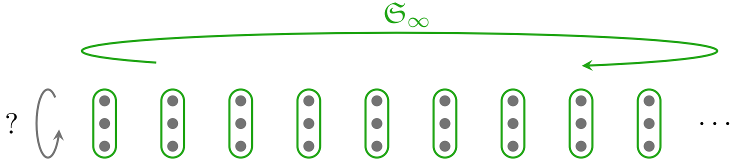


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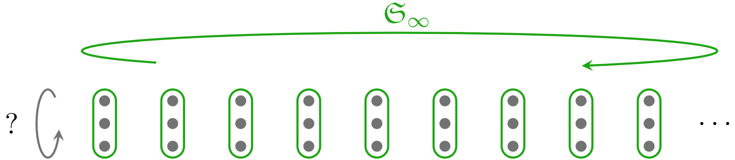


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In particular, both conjectures hold.

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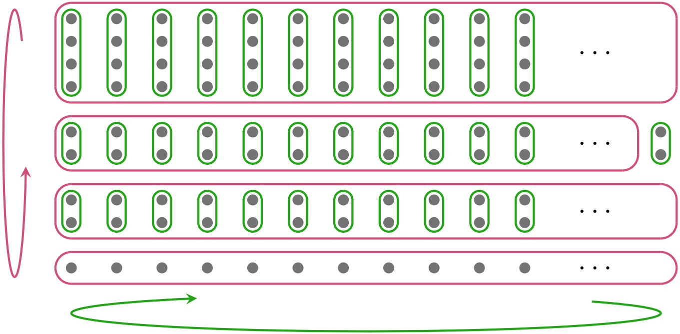

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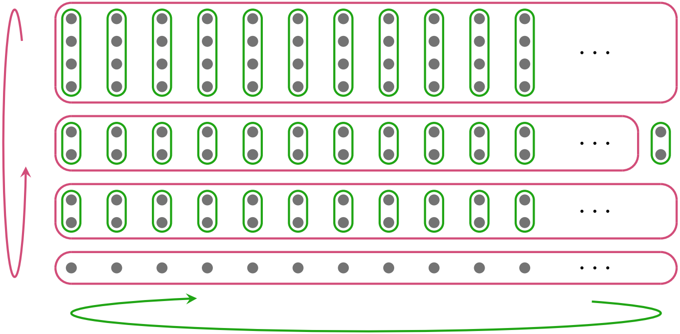

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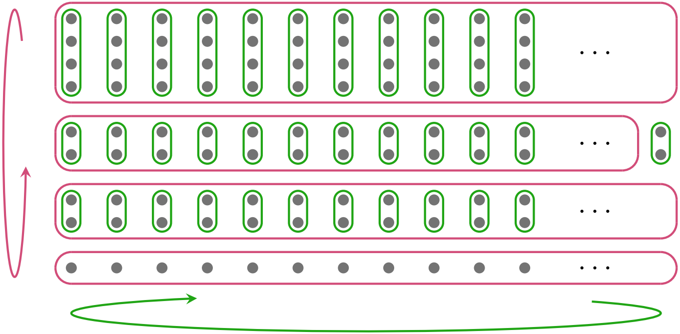

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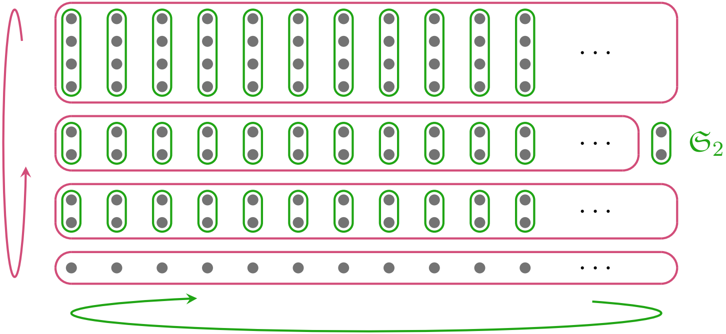

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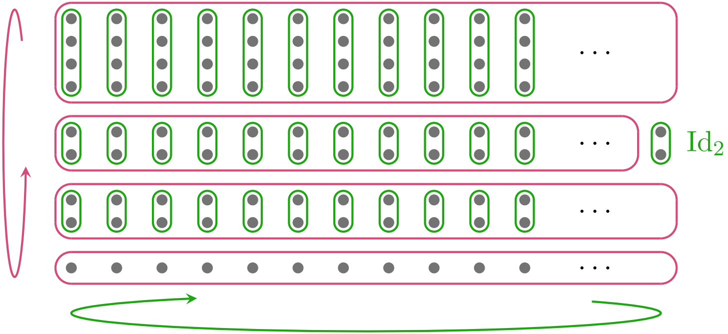

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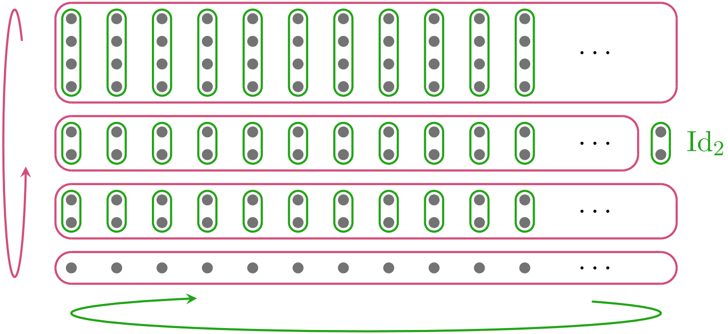

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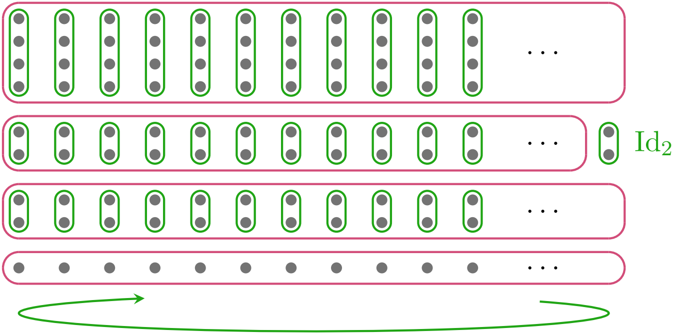

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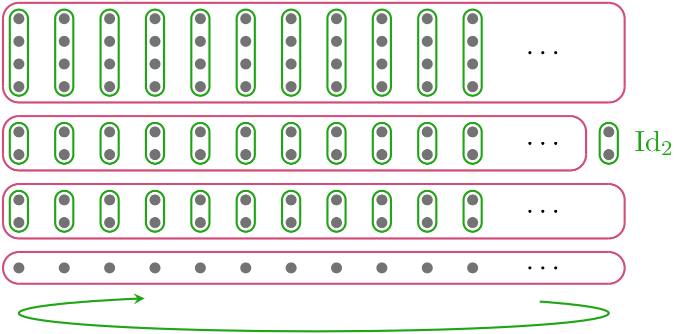

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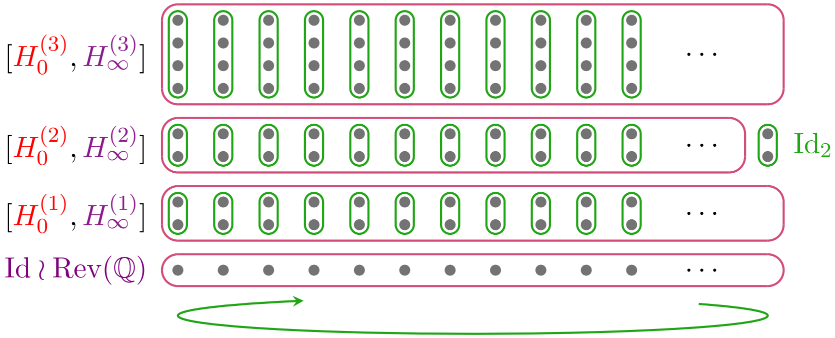

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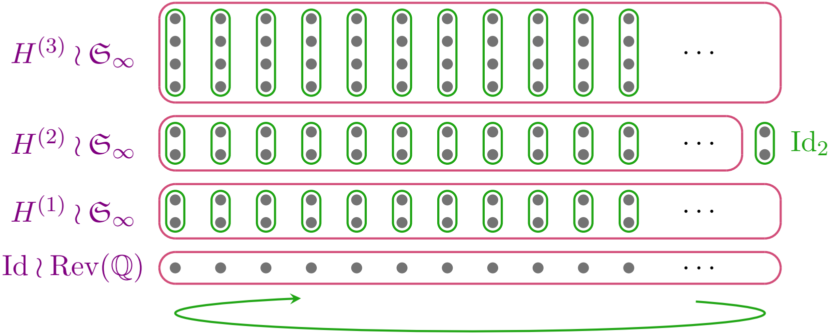

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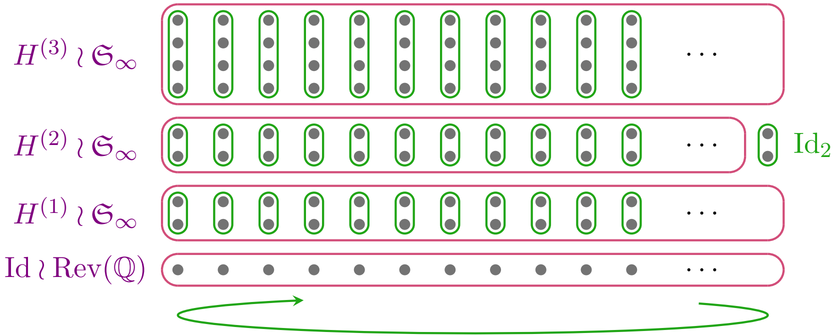

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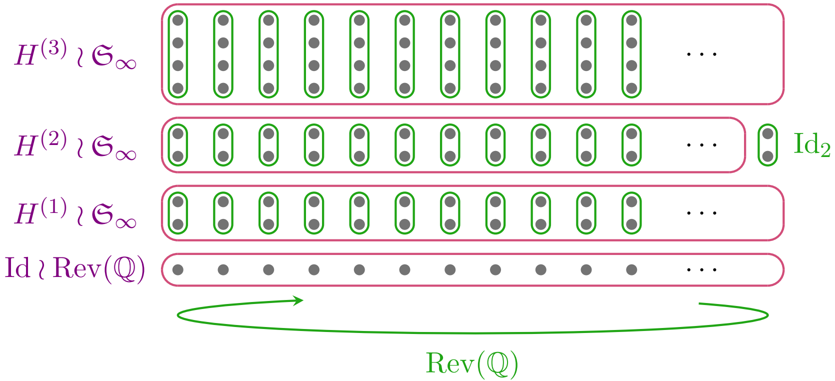

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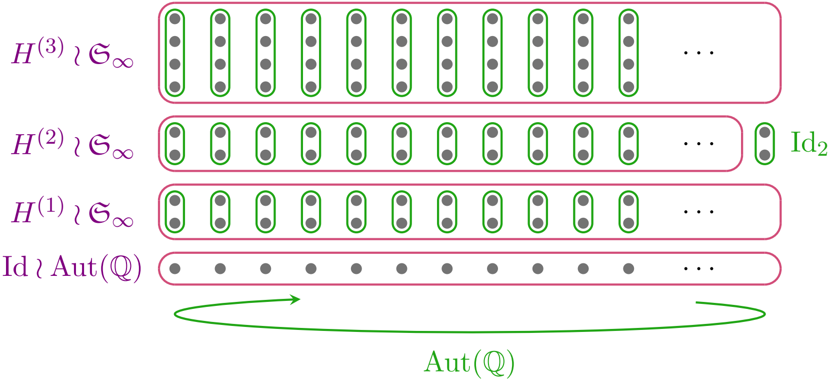

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  Which end the proof of the conjectures !

# Classification of $P$-oligomorphic groups

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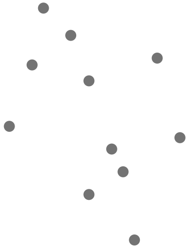

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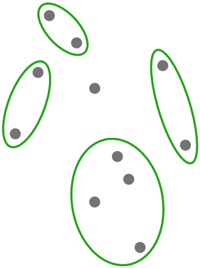

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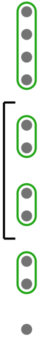

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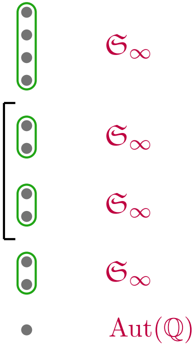

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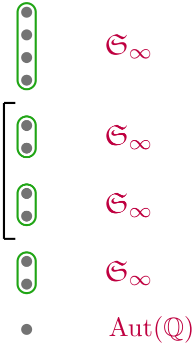

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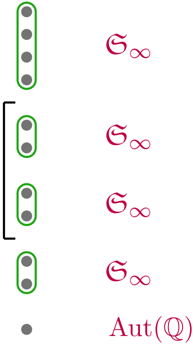

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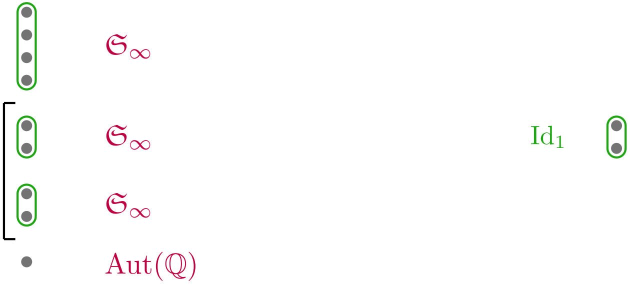

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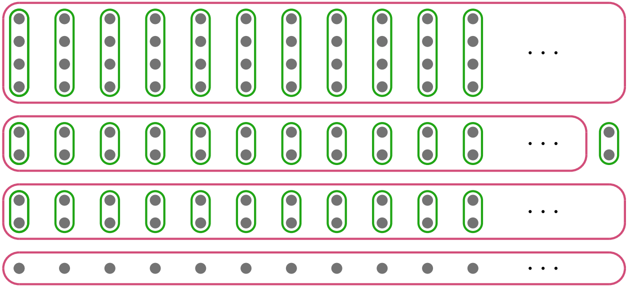

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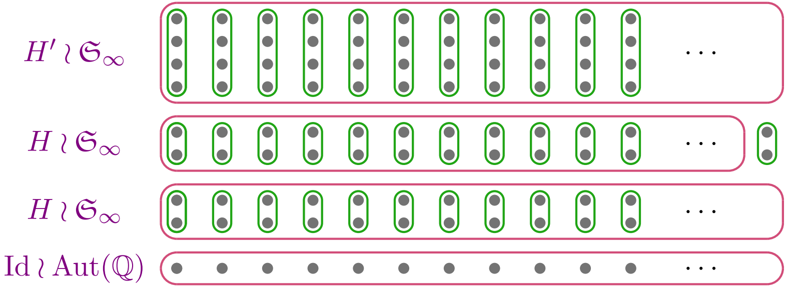

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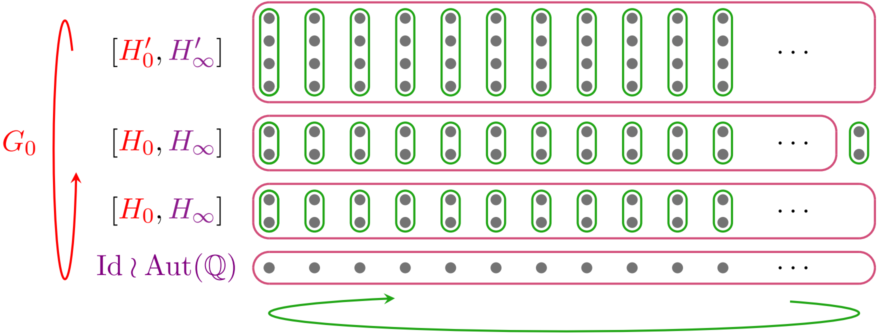


# Thank you for your attention !

## Context

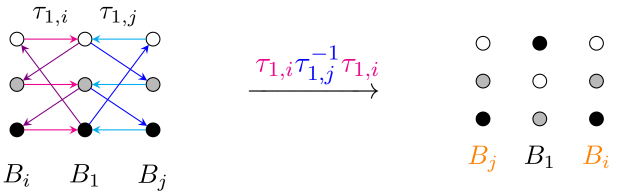
- $G$ permutation group of a countably infinite set $E$
- Profile $\varphi_G$: counts the orbits of finite subsets of $E$
- Hypothesis: $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron): rational form of the generating series
- Conjecture (Macpherson): finite generation of the orbit algebra

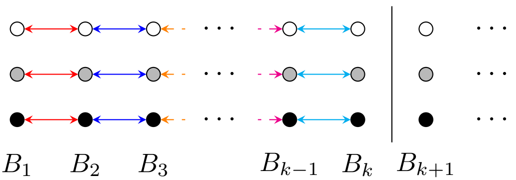
## Results

- Both conjectures hold !
- Classification of $P$-oligomorphic permutation groups
- The orbit algebra is an algebra of invariants (up to some idempotents)

# The tower determines the group (1): "straight $\mathfrak{S}_\infty$"

$G$ contains a set of "straight" swaps of blocks



# Subdirect product

## Subdirect product of $G_1$ and $G_2$

- Formalizes the *synchronization* between $G_1$ and $G_2$
- Subgroup of $G_1 \times G_2$ (with canonical projections $G_1$ and $G_2$)
- $E = E_1 \sqcup E_2$ stable $\Rightarrow G$ subdirect product of $G_{|E_1}$ and $G_{|E_2}$

## Synchronization in a subdirect product

Let $N_1 = \mathrm{Fix}_G(E_2)$ and $N_2 = \mathrm{Fix}_G(E_1)$.

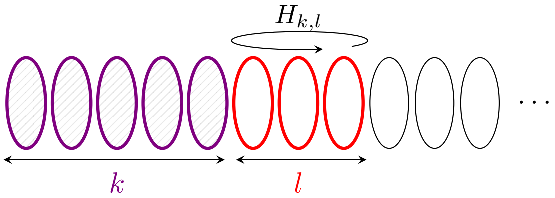
$$\frac{G_1}{N_1} \simeq \frac{G}{N_1 \times N_2} \simeq \frac{G_2}{N_2}$$

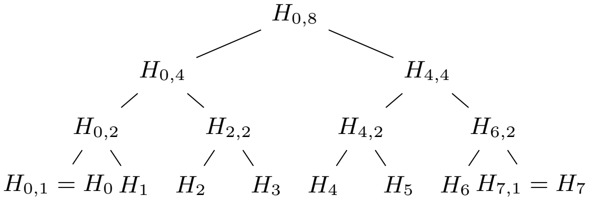
A subdirect product with explicit $N_i$'s is explicit.

Remark. $N_1$ and $N_2$ are *normal* in $G_1$ and $G_2$, so the possibilities of synchronization of a group is linked to its normal subgroups.

# The tower determines the group (2): $\mathrm{Stab}_G(\text{blocks})$

$\mathrm{Stab}_G(\text{blocks})$ = explicit subdirect product of the $H_i$




← The tower determines $\mathrm{Stab}_G(\text{blocks})$

$G \simeq \mathrm{Stab}_G(\text{blocks}) \rtimes$ "straight $\mathfrak{S}_\infty$" $\rightarrow$ Ok

# Example of a product in the cyclic group $\mathcal{C}_5$

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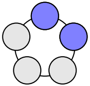
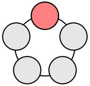


$\times$



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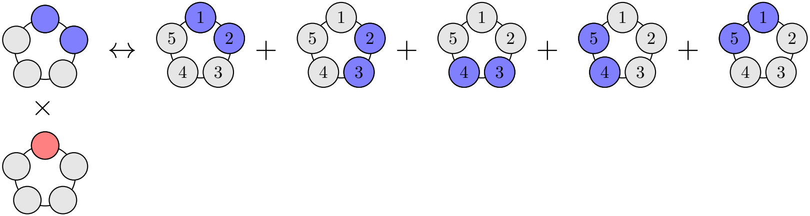

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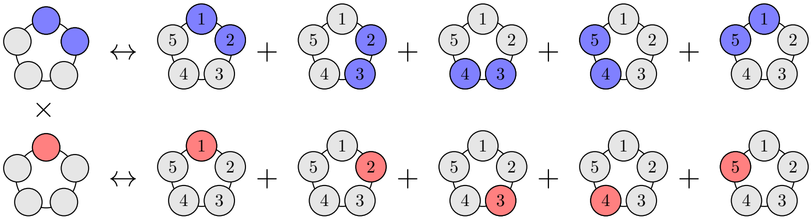

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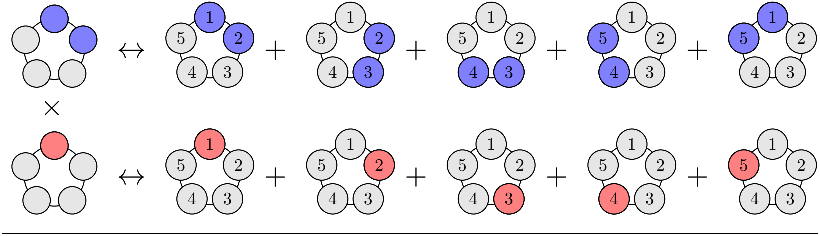

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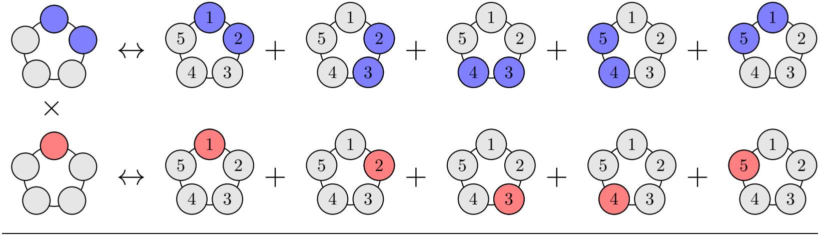

$$= \quad 0$$

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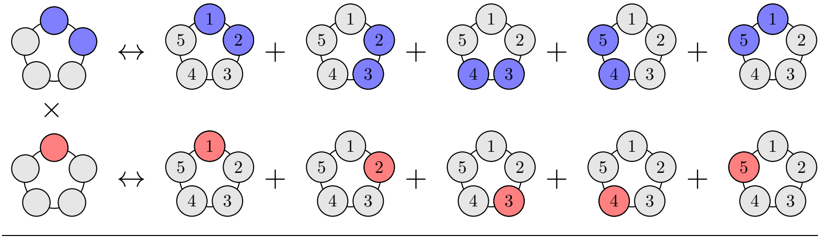

$$= \quad 0 \quad + \quad 0$$

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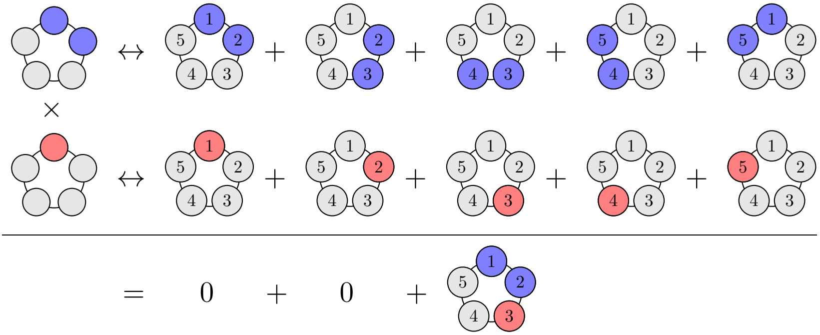

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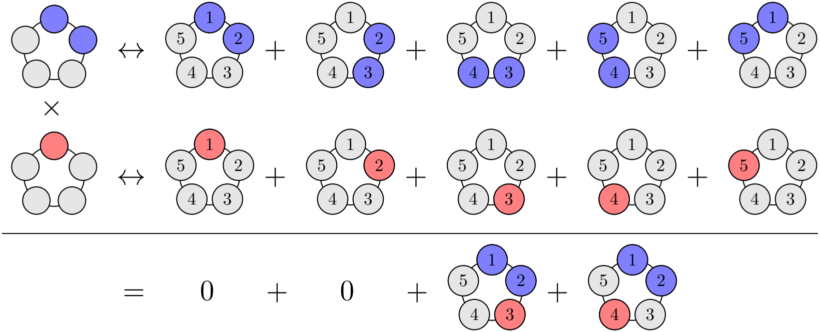

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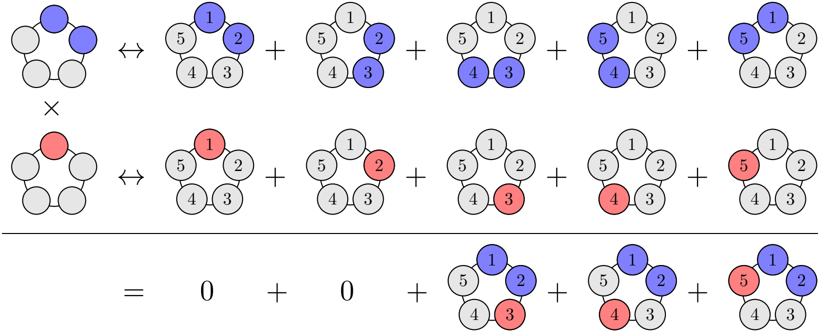

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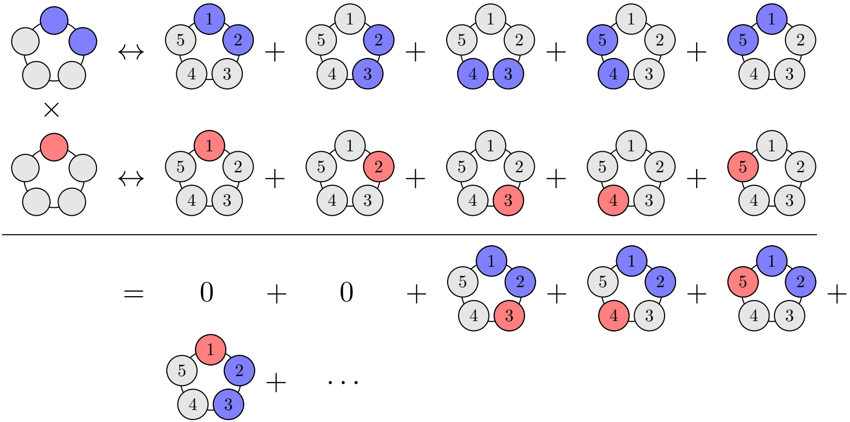

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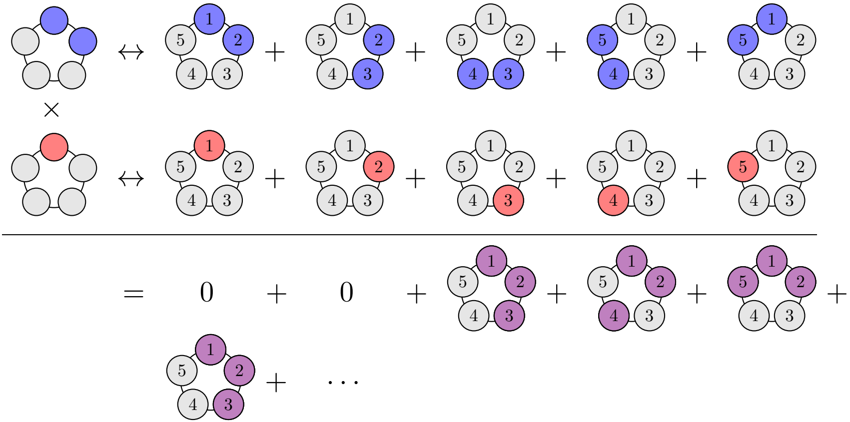

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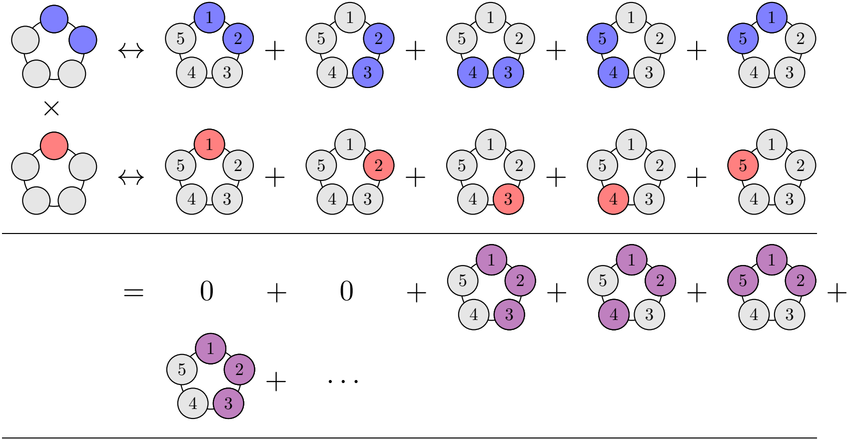

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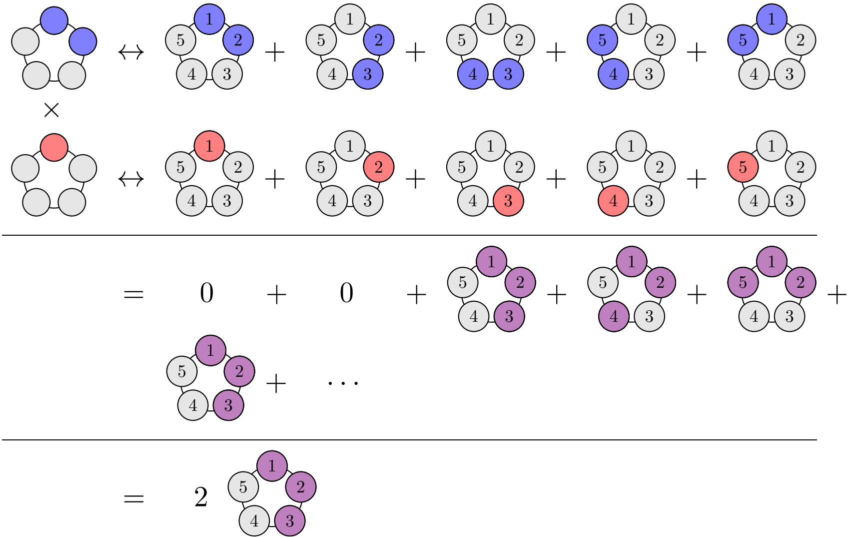

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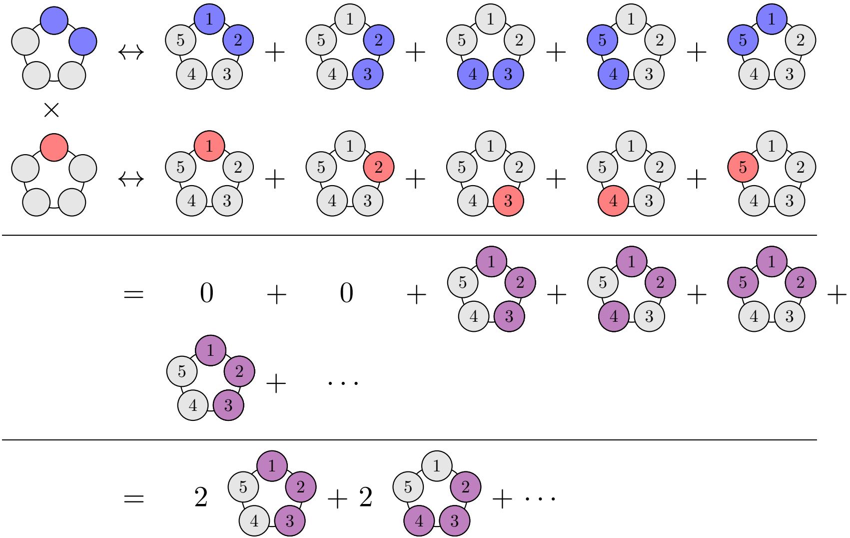

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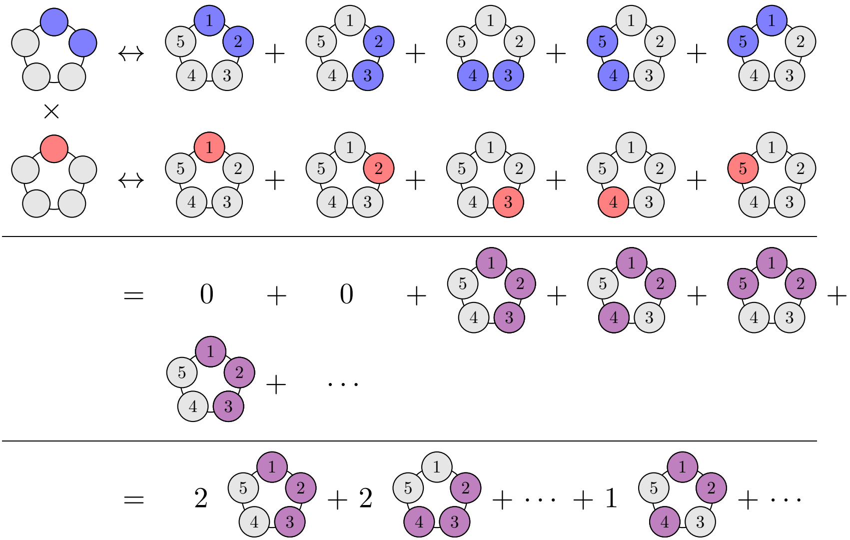
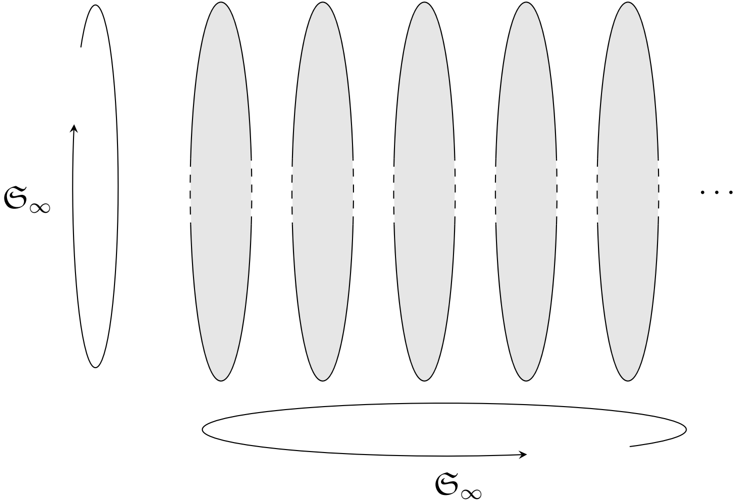


# Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) = \ ?$



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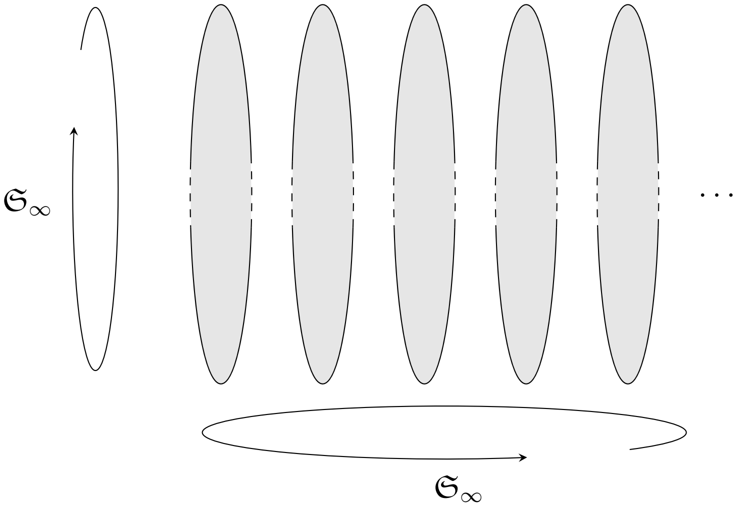
An orbit of degree $n \longleftrightarrow$ a partition of $n$



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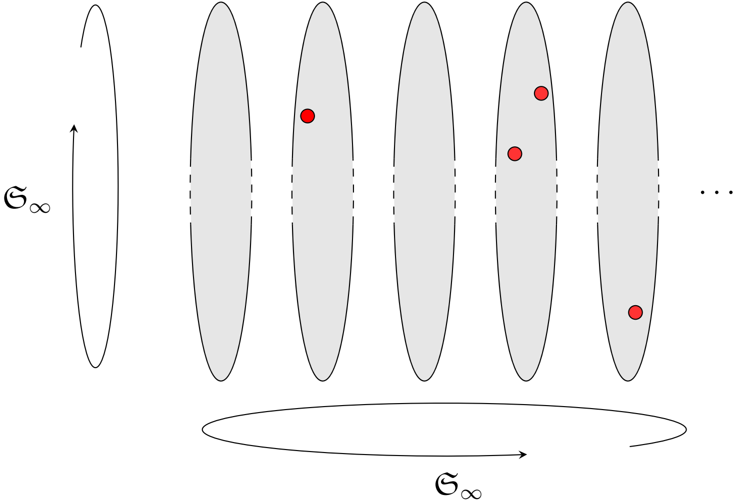

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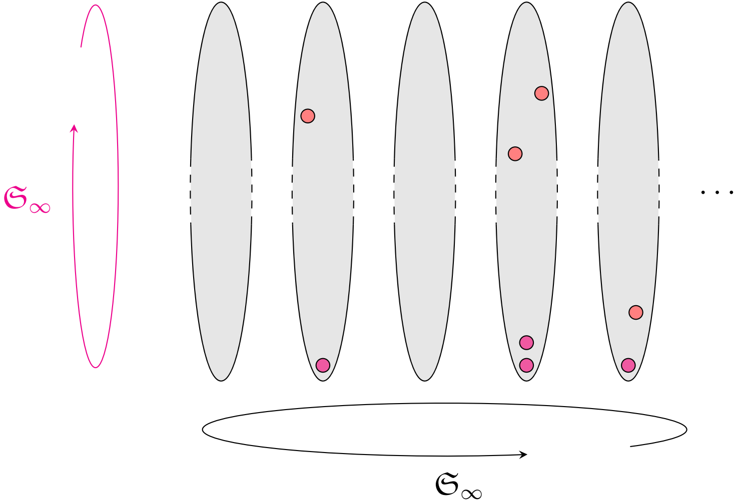

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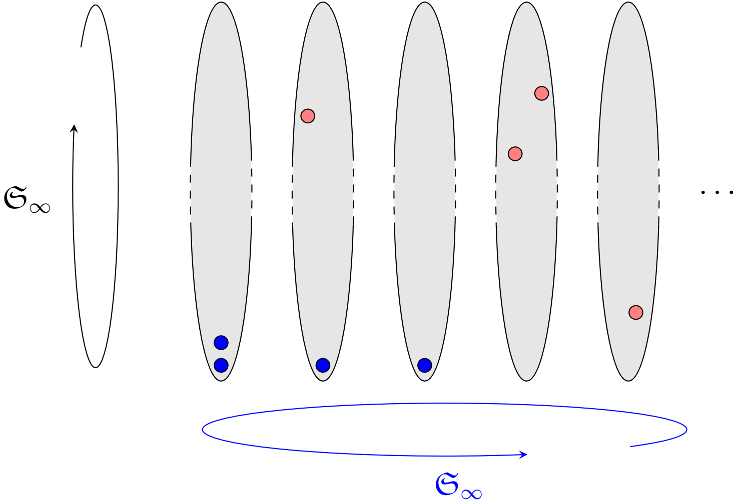

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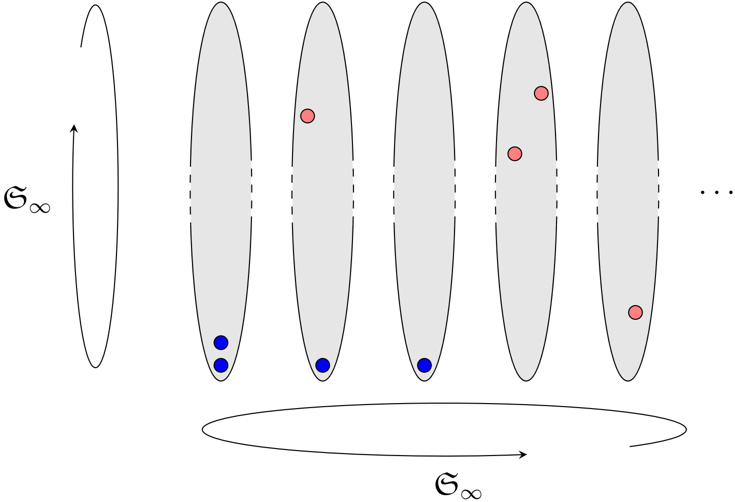

# Examples of orbit algebras (1)

## Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all $n$, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

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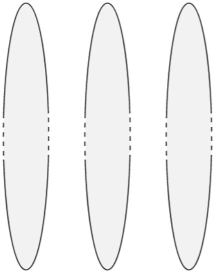
$G = \mathfrak{S}_\infty \wr \mathfrak{S}_3$, recall that $\varphi_G(n) = p_3(n)$.

$A_n$ = homogeneous symmetric polynomials of degree $n$ in $x_1, x_2, x_3$



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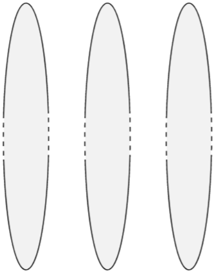


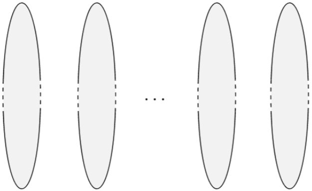
$\rightarrow \mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr \mathfrak{S}_3) = \mathbb{K}[x_1, x_2, x_3]^{\mathfrak{S}_3}$

# Examples of orbit algebras (2)

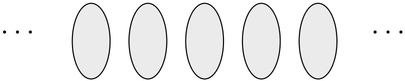
More generally, for $H$ subgroup of $\mathfrak{S}_m$ :

- $G = \mathfrak{S}_\infty \wr H$ :
  $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of $H$

  $\mathbb{Q}\mathcal{A}(G)$ is finitely generated by Hilbert's theorem.

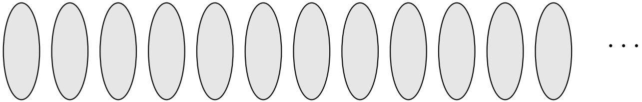


- $G = H \wr \mathfrak{S}_\infty$ :
  $\mathbb{Q}\mathcal{A}(G) =$ the free algebra generated by the age of $H$



# Direct product in the case of finite blocks

"Speak, friend..."

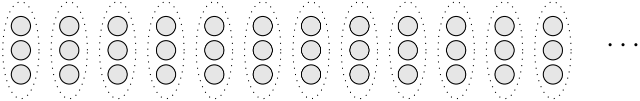


# Direct product in the case of finite blocks

"Speak, friend..."

## Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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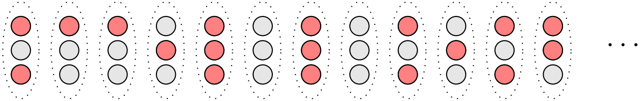

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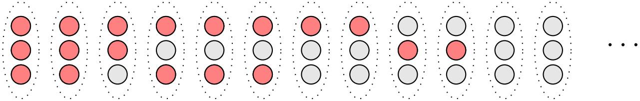

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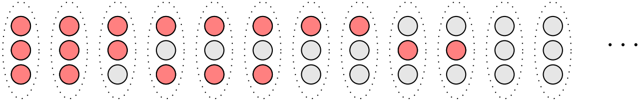

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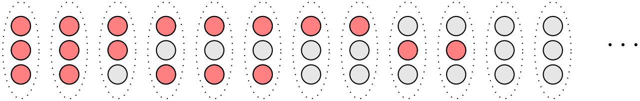

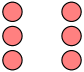

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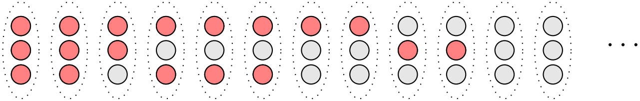

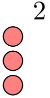

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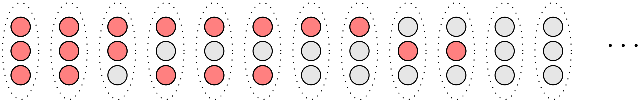

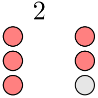

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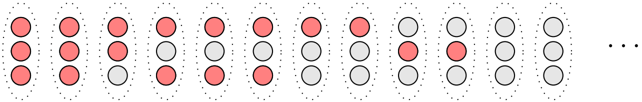

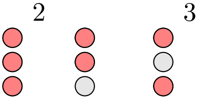

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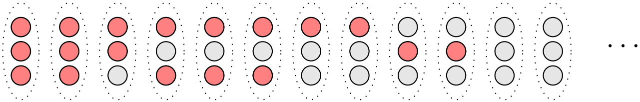

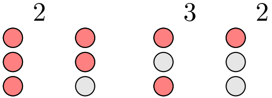

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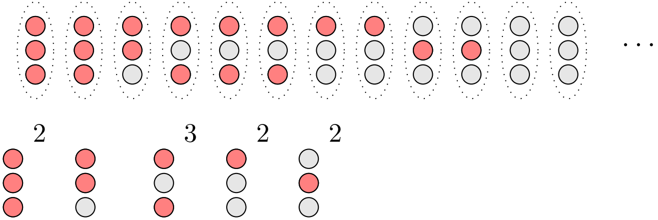

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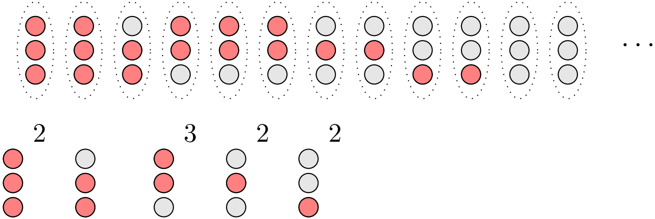

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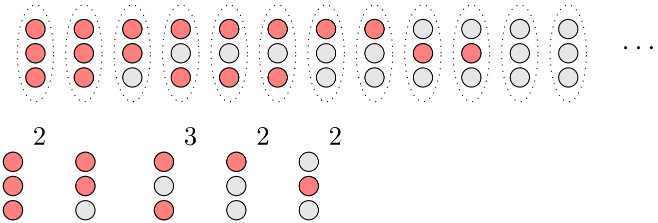

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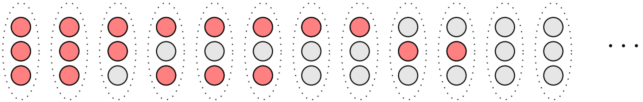

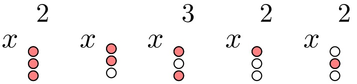

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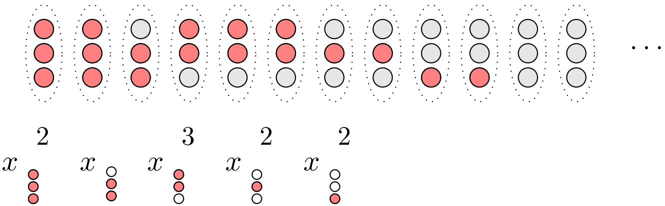

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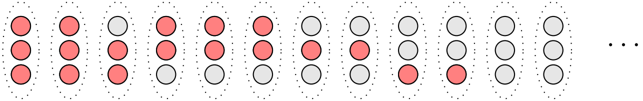

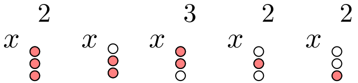

$\rightarrow$ $C_3$ acts on monomials

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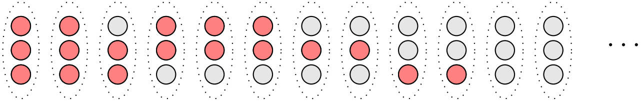

$G' = C_3$ *acting on (non empty) subsets*

$$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad \text{Orbit algebra of } C_3 \times \mathfrak{S}_\infty \ ?$$

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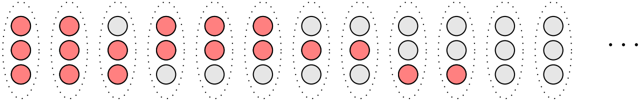
## Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$ ?

$x$ 

$x$ 

# Direct product in the case of finite blocks

"Speak, friend..."

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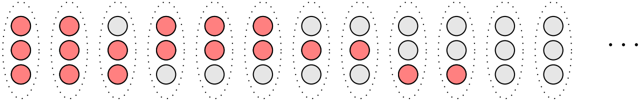

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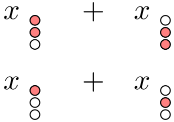

# Direct product in the case of finite blocks



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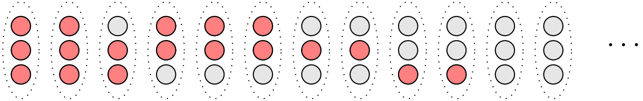

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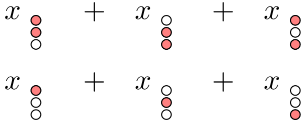

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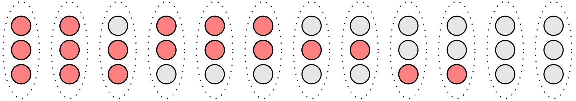 $\cdots$

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O( $x$  )

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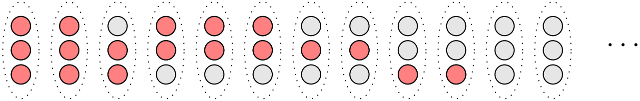

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$\mathrm{O}(\ x\ ).\mathrm{O}(\ x\ )$

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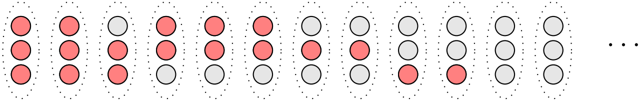

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$$\mathrm{O}(\ x\ ).\mathrm{O}(\ x\ ) = \ \mathrm{O}(\ x\ x\ )$$

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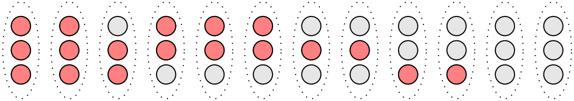

…

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$\mathrm{O}(\ x_{\bullet\bullet\circ}).\mathrm{O}(\ x_{\bullet\circ\circ}) = \ \mathrm{O}(\ x_{\bullet\bullet\circ} x_{\bullet\circ\circ}) + \mathrm{O}(\ x_{\bullet\bullet\circ} x_{\circ\bullet\circ})$

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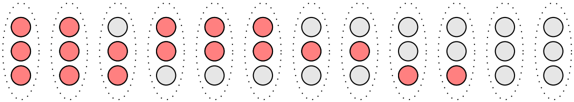

$\dots$

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# Direct product in the case of finite blocks

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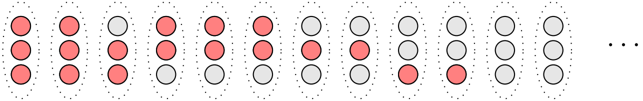

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$\mathrm{O}(\quad).\mathrm{O}(\quad)$

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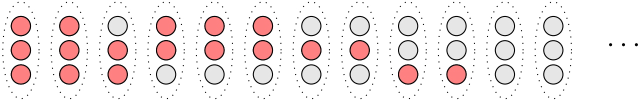

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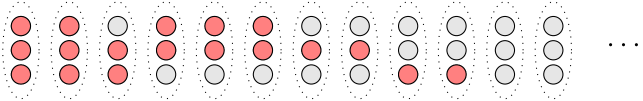

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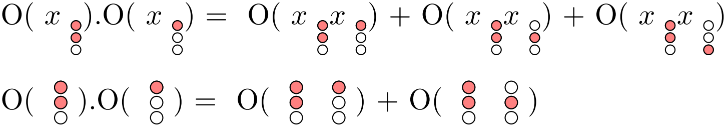

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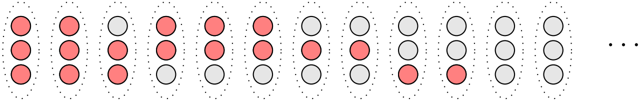

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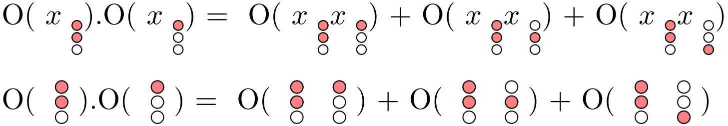

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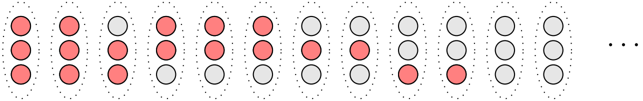

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$\mathrm{O}(\ ).\mathrm{O}(\ ) = \ \mathrm{O}(\ ) + \mathrm{O}(\ ) + \mathrm{O}(\ ) + 3\ \mathrm{O}(\ )$

# The tower has shape $H_0,\ H,\ H,\ H\ \cdots$

## Lemma to prove

$G$ has tower $H_0\ H_1\ H_2\ H_3 \Rightarrow H_1 = H_2$

## Proof.

An element $s \in G$ stabilizing the blocks $\leftrightarrow$ a quadruple
$g \in H_1 \quad \rightarrow \quad \exists\ (1, g, h, k),\ \ h, k \in H_1$.
Let $\sigma$ be an element of $G$ that permutes "straightforwardly" the first two blocks and fixes the other two.
Conjugation of $x$ by $\sigma$ in $G \quad \rightarrow \quad y = (g, 1, h, k)$
Then: $x^{-1}y = (g, g^{-1}, 1, 1)$
By arguing that the tower does not depend on the ordering of the blocks, $g^{-1}$ and therefore $g$ are in $H_2$.

In the infinite case, apply to each restriction to four consecutive blocks of the fixator of the previous ones in $G$.