Single-Crossing Differences on Distributions*

Navin Kartik[†] SangMok Lee[‡] Daniel Rappoport[§]

May 9, 2019

Abstract

In the context of choice among lotteries, we study "interval choice structure", or alternatively, when choices are monotonic (with respect to some order) in an expected-utility agent's preference parameter or type. The requisite property is that the expected-utility difference between any pair of lotteries is single-crossing in the agent's type. We characterize the set of utility functions that have this property. We discuss applications to cheap-talk games, costly signaling games, collective choice problems, and information design. Our analysis provides some new results on monotone comparative statics and aggregating single-crossing functions.

Keywords: monotone comparative statics, choice under uncertainty, interval equilibria

*We thank Nageeb Ali, Federico Echenique, Mira Frick, Ben Golub, Ryota Iijima, Ian Jewitt, Alexey Kushnir, Shuo Liu, Daniele Pennesi, Jacopo Perego, Lones Smith, Bruno Strulovici, and various audiences for helpful comments.

[†]Department of Economics, Columbia University. E-mail: `nkartik@gmail.com`

[‡]Department of Economics, Washington University in St. Louis. E-mail: `sangmoklee@wustl.edu`

[§]Booth School of Business, University of Chicago. E-mail: `Daniel.Rappoport@chicagobooth.edu`

Contents

1. Introduction

1.1. Overview

Motivation. Single-crossing properties and their implications for choices are at the heart of many economic models, as highlighted by Milgrom and Shannon (1994). A limitation of the typical approach is that, either implicitly or explicitly, choices are restricted to deterministic outcomes when it is desirable to accommodate lotteries. For example:

1. In signaling models à la Spence (1973), workers with ability $\theta \in \mathbb{R}$ choose education $e \in \mathbb{R}$ to signal ability and garner a higher wage $w \in \mathbb{R}$. It is assumed—either directly or indirectly—that, in equilibrium, there is a deterministic mapping from education to wage. This assumption underlies the fundamental result that, in any equilibrium, workers with higher ability obtain more education if the workers' utility function $v(w, e, \theta)$ satisfies the Milgrom and Shannon (1994) single-crossing property in wage-education pairs (w, e) and ability θ. More realistically, for any choice of education, workers face a lottery over wages due to aggregate and idiosyncratic uncertainty.

2. In voting models, a voter indexed by θ has preferences over outcomes a given by the utility function $v(a, \theta)$. A single-crossing property of $v(a, \theta)$ in a and θ guarantees a well-behaved majority preference (Gans and Smart, 1996) and the existence of a Condorcet winner. This result is central to various political competition models à la Downs (1957). But the presumption there is that voters or political candidates choose directly among final outcomes. More realistically, the relevant choice is only among some set of policies whose outcomes are uncertain at the time of voting.

In many applications it is not clear what *a priori* restrictions are reasonable on the set of lotteries facing the agent, in particular whether some form of stochastic dominance can order every choice set. For example, in the Spencian setting, higher education choices may well induce wage lotteries that have both higher mean and higher variance; or, in the voting context, the economic and political outcomes are multi-dimensional and so the set of lotteries is unlikely to comply with standard orders. Moreover, the lotteries may be the result of still further interactions—e.g., they may represent continuations in dynamic strategic problems—which would be intractable to structure without severe assumptions.

In this paper, we completely characterize when an agent's expected utility difference between *any* pair of lotteries over outcomes is single crossing in the agent's type or preference parameter. (A real-valued function defined on a partially-ordered set is single crossing if its sign is monotonic.) We term this property *single-crossing expectational differences*, or

SCED. We establish SCED as necessary and sufficient for two valuable comparative statics: (i) choices from any choice set have an interval structure (i.e., the set of types choosing any lottery is an interval); and (ii) there is some order over lotteries that generates choice monotonicity. We demonstrate how SCED is useful in applications such as costly signaling, voting, and cheap talk.

Interval Choice, Single Crossing, and Comparative Statics. Consider a utility function $f : X \times \Theta \to \mathbb{R}$. The set X is an arbitrary—in particular, unordered—set of options while Θ is an ordered preference-parameter or type set in which every pair of types has an upper and lower bound.[1] Our focus, subsequently, will be on X a lottery space and f an expected utility function. A fundamental property of interest is that, no matter the choice set $S \subseteq X$, the set of types choosing any $x \in S$ is an interval. That is:

$$(\forall S \subseteq X)\ (\forall \theta_l < \theta_m < \theta_h) \quad x^* \in \bigcap_{\theta \in \{\theta_l, \theta_h\}} \arg\max_{x \in S} f(x,\theta) \implies x^* \in \arg\max_{x \in S} f(x, \theta_m). \tag{1}$$

Theorem 0 shows that, modulo some details, this property is equivalent to the function f satisfying *single-crossing diffferences*, or SCD:

$$(\forall x, x' \in X) \quad f(x,\cdot) - f(x',\cdot) \text{ is single crossing.} \tag{2}$$

It is intuitive that (1) and (2) are related to monotone comparative statics. We establish a precise connection in Theorem 3: SCD is equivalent to the existence of an order on X such that

$$(\forall S \subseteq X)\ (\forall \theta_l < \theta_h) \quad \arg\max_{x \in S} f(x,\theta_h) \succeq_{SSO} \arg\max_{x \in S} f(x, \theta_l). \tag{3}$$

Here, $\succeq_{SSO}$ is the strong set order generated by the order on X.

It bears emphasis that (2) differs from Milgrom and Shannon's (1994) single-crossing property because we do not take as given an exogenous order on the choice space X. Rather, SCD is necessary and sufficient for the existence of *some* order that generates the choice monotonicity in (3). Necessity owes to requiring choice monotonicity for all subsets $S \subseteq X$; sufficiency builds on Milgrom and Shannon (1994, Theorem 4). Besides the aforementioned comparative statics, the order-independent notion of SCD in (2) has other notable implications; for example, it is the key to guaranteeing that local incentive com-

[1] Throughout, we use "order" to mean "partial order". Subsection 6.1, which includes Theorem 3 mentioned below, assumes that X is minimal with respect to f in a sense defined there.

patibility implies global incentive compatibility (Carroll, 2012, Proposition 4).[2] When one seeks choice monotonicity with respect to a particular order, SCD can be combined with other assumptions on the primitives; Claims 4 and 5 in Subsection 4.3 illustrate in the context of an application.

Single-Crossing Expectational Differences. For the reasons explained at the outset, our focus is choice among lotteries. The bulk of our paper specializes to $X \equiv \Delta A$, where ΔA is the set of (simple) lotteries over an arbitrary outcome space A, and to $f \equiv V$, where V is the expected utility function induced by a von Neumann-Morgenstern utility function $v : A \times \Theta \to \mathbb{R}$. Say that v has single-crossing expectational differences, or SCED, if V has SCD. Theorem 1 establishes that v has SCED if and only if

$$v(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + c(\theta), \tag{4}$$

where f_1 and f_2 are single-crossing functions that satisfy a *ratio-ordering* property we introduce in Subsection 3.1. Roughly speaking, ratio ordering requires that the relative importance placed on $g_1(a)$ versus $g_2(a)$ changes monotonically with type.[3] The idea is transparent when $\Theta \subset \mathbb{R}$ with a minimum $\underline{\theta}$ and a maximum $\overline{\theta}$. v having SCED is then equivalent to the existence of a (type-dependent) representation $\tilde{v}(a, \theta)$ that satisfies

$$\tilde{v}(a, \theta) = \lambda(\theta) \tilde{v}(a, \overline{\theta}) + (1 - \lambda(\theta)) \tilde{v}(a, \underline{\theta}),$$

where $\lambda : \Theta \to [0, 1]$ is increasing (Proposition 1). In other words, SCED is equivalent to each type's preferences being representable by a utility function that is a convex combination of those of the extreme types, with higher types putting more weight on the highest type's utility.

SCED is satisfied by some canonical functional forms: in mechanism design and screening, $v((q, t), \theta) = \theta q - t$ (where $q \in \mathbb{R}$ is quantity, $t \in \mathbb{R}$ is a transfer, and $\theta \in \mathbb{R}$ is the agent's marginal rate of substitution); in optimal delegation without transfers, $v((q, t), \theta) = \theta q + g(q) - t$ (where $q \in \mathbb{R}$ is the allocation, $t \in \mathbb{R}_+$ is money burning, and $\theta \in \mathbb{R}$ is the agent's type; cf. Amador and Bagwell (2013)); in communication and voting, $v(a, \theta) = -(a - \theta)^2 = 2\theta a - a^2 - \theta^2$ (where $a \in \mathbb{R}$ is an outcome and $\theta \in \mathbb{R}$ is the agent's bliss point). On the other hand, our characterization also makes clear that SCED is quite restrictive. For example,

[2] More precisely, Carroll's result implies that local incentive compatibility is sufficient for global incentive compatibility if there is strict single crossing differences as we define it (Definition 2).

[3] More generally, Lemma 1 establishes that ratio ordering is necessary and sufficient for all linear combinations of two single-crossing functions to be single crossing.

within the class of power loss functions, only the quadratic loss function satisfies SCED (Corollary 1).

Section 4 applies SCED to three canonical economic problems. Among other things, the cheap-talk application with uncertain receiver preferences in Subsection 4.1 demonstrates concretely how choices from *all* lotteries emerge naturally. Notwithstanding, there are other contexts in which *a priori* restrictions on feasible lotteries are available, which make the full force of SCED unnecessary. Section 5 provides a relaxation of SCED to a lottery space with linear restrictions; a prominent context is choice among signal structures (e.g., Kamenica and Gentzkow, 2011), a setting in which all lotteries have the same prior expectation. Section 6 discusses some additional general results: monotone comparative statics with respect to some order, and a comparison between SCED and a strengthening to monotonic expectational differences (cf. Kushnir and Liu, 2018).

An Intuition. A key step towards the SCED characterization in Theorem 1 is establishing that every type's utility function over actions is a linear combination of two (type-independent) functions: Equation 4. We can provide a succinct intuition. Suppose $\Theta \subset \mathbb{R}$ with a minimum $\underline{\theta}$ and a maximum $\overline{\theta}$. It suffices to show that there are three actions, a_1, a_2, and a_3, such that any type θ's utility from any action a satisfies

$$v(a,\theta) = \lambda_1(a)v(a_1,\theta) + \lambda_2(a)v(a_2,\theta) + \lambda_3(a)v(a_3,\theta), \tag{5}$$

for some $\lambda(a) \equiv (\lambda_1(a), \lambda_2(a), \lambda_3(a))$ with $\sum_{i=1}^{3} \lambda_i(a) = 1$. (Equation 4 follows by setting, for $i = 1, 2$, $f_i(\theta) = v(a_i,\theta) - v(a_3,\theta)$, $g_i(a) = \lambda_i(a)$, and $c(\theta) = v(a_3,\theta)$.) The desired $\lambda(a)$ is the solution to

$$\begin{bmatrix} v(a,\underline{\theta}) \\ v(a,\overline{\theta}) \\ 1 \end{bmatrix} = \begin{bmatrix} v(a_1,\underline{\theta}) & v(a_2,\underline{\theta}) & v(a_3,\underline{\theta}) \\ v(a_1,\overline{\theta}) & v(a_2,\overline{\theta}) & v(a_3,\overline{\theta}) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1(a) \\ \lambda_2(a) \\ \lambda_3(a) \end{bmatrix},$$

which exists at least when there are three actions for which the 3×3 matrix on the right-hand side is invertible. The interpretation of this matrix equation is that one can find two distinct lotteries over $\{a_1, a_2, a_3, a\}$ such that the lowest and highest types are both indifferent between them.[4] It follows from SCED that all types must be indifferent between these two lotteries, which amounts to Equation 5.

[4] Take one lottery to be the uniform distribution $P = (1/4, 1/4, 1/4, 1/4)$ on $\{a_1, a_2, a_3, a\}$ and the other lottery to be $P + (1/M)(\lambda_1(a), \lambda_2(a), \lambda_3(a), -1)$, with any sufficiently large $M > 0$.

1.2. Related Literature

Our paper views an agent's choice as a lottery over outcomes. Quah and Strulovici (2012, QS hereafter), on the other hand, consider choices with uncertainty about preferences. Specifically, they consider an agent with utility $u(x, \theta, t)$, where $x \in \mathbb{R}$ is the choice variable, θ is the type, and t is an unobserved payoff parameter. They characterize when the agent's expected utility has the Milgrom and Shannon (1994) single-crossing property in (x, θ) for every distribution of t that is independent of x and θ.[5] In a nutshell, their sufficient condition is *signed-ratio monotonicity* of certain pairs of functions of θ.

While QS's approach could, in principle, be applied for some problems we are interested in, it would be unwieldy. Consider an agent with utility function $v(a, \theta)$ choosing an action x that leads, perhaps through some strategic interaction, to an outcome function $a^*(x, t)$, where t is an independent random realization. Let $u(x, \theta, t) \equiv v(a^*(x, t), \theta)$. To ensure choice monotonicity, QS's approach would require checking that $u(x, \theta, t) - u(x', \theta, t)$ and $u(x, \theta, t') - u(x', \theta, t')$ have signed-ratio monotonicity for every t, t', $x > x'$, and candidate functions $a^*(\cdot)$. This is daunting, particularly when the set of $a^*(\cdot)$ is large and/or *a priori* difficult to characterize.

Our SCED approach instead views actions x and x' as each inducing arbitrary lotteries over outcomes a, and demands single-crossing differences directly on the lottery space. While this is conceptually more demanding—and leads to our stringent characterization—it provides a condition directly on $v(a, \theta)$ that can be easier to check; moreover, SCED is in fact necessary when the set of $a^*(\cdot)$ functions is sufficiently rich, as in our cheap-talk application in Subsection 4.1. We also note that sufficient conditions for any problem à la QS concerning $u(x, \theta, t)$ can be approached by defining $a \equiv (x, t)$, $v(a, \theta) \equiv u(x, \theta, t)$ and checking for SCED (or imposing assumptions guaranteeing it).[6]

When $\Theta \subseteq \mathbb{R}$, the utility specification $v(a, \theta) = \theta g_1(a) + g_2(a)$ has SCED; indeed, the expected utility difference between any two lotteries is monotonic in θ. The usefulness of this utility specification (or slight variants) to structure choices among arbitrary lotteries has been highlighted by Duggan (2014), Celik (2015), and Kushnir and Liu (2018). We provide in Subsection 6.2 a detailed connection of our work with these authors'. Here we only note

[5] Athey (2002) considers an agent with utility $\hat{u}(x, t)$ who receives a signal θ about the unobservable t. Athey asks when $\mathbb{E}_t[\hat{u}(x, t)|\theta]$ has the single-crossing property in (x, θ). QS's approach is also sufficient for this problem because one can take $u(x, \theta, t) = \hat{u}(x, t) \Pr(t|\theta)$ so that $\mathbb{E}_t[\hat{u}(x, t)|\theta] \propto \int u(x, \theta, t) \mathrm{d}t$.

[6] From a mathematical perspective, both QS' and our results hinge on ensuring that aggregations (i.e., linear combinations) of single-crossing functions remain single crossing. The results differ because of differences in the domains of the functions and the set of aggregations considered. Choi and Smith (2016) observe that QS' results can be related to those of Karlin (1968). We elaborate on these connections after Lemma 1 in Subsection 3.1.

that, modulo some details, Celik (2015) defines the monotonicity property as "extended single-crossing" and mentions the utility specification, while Kushnir and Liu (2018) define the monotonicity property as "increasing differences over distributions" and provide a functional-form characterization. Both those papers' substantive focus is on issues in mechanism design. Duggan (2014), on the other hand, uses the specification in the context of collective choice over lotteries; he observes that what is essential is single-crossing, and discusses why it might be difficult to go beyond positive affine transformations of the specification.

We recently became aware that in the operations research literature, there has been interest in functional forms for "multi-attribute utility functions", i.e., when outcomes are multi-dimensional.[7] When there are two attributes, x and y, Fishburn (1977) suggested the form $u(x, y) = f_1(x)g_1(y) + f_2(x)g_2(y)$. Abbas and Bell (2011) discuss a "one-switch condition" that more or less amounts to requiring that, for any two lotteries over attribute x, the expected utility difference as a function of attribute y must be strictly single crossing. Within their more restrictive environment (e.g., the set of y is linearly ordered), their Theorem 1 claims a result that is related to our monotonic expectational differences characterization (Theorem 5).[8] They do not obtain the (S)SCED characterizations in our Theorem 1 and Theorem 2, and they do not discuss comparative statics characterizations of SCED nor economic applications.

Finally, there is a link between our results and the famous theorem of Harsanyi (1955) concerning utilitarianism. Consider a set of three individuals, $\Theta = \{\underline{\theta}, S, \overline{\theta}\}$ endowed with the order $\underline{\theta} < S < \overline{\theta}$, and utility functions $v(a, \theta)$. Here, SCED is equivalent to S (which stands for society) satisfying a Pareto principle with respect to $\underline{\theta}$'s and $\overline{\theta}$'s preferences over lotteries (specifically, the conjunction of Weak Pareto and Pareto-weak preference, in the terminology of De Meyer and Mongin (1995)). A version of Harsanyi's theorem (De Meyer and Mongin, 1995, Proposition 2) implies that for some $\kappa \in \mathbb{R}_+$ and $\mu \in \mathbb{R}$, $\kappa v(\cdot, S) - \mu$ is a convex combination of $v(\cdot, \underline{\theta})$ and $v(\cdot, \overline{\theta})$; this result is also an implication of our Theorem 1, as seen from Proposition 1. More generally, the monotonicity identified in Proposition 1 owes to SCED's inter-type restrictions, and there are cases—either because of the structure of Θ or the utility function $v(\cdot)$—in which the ratio ordering of Theorem 1 cannot be simplified.[9]

[7] We are grateful to Daniele Pennesi for directing us to this literature after seeing our 2017 working paper.

[8] Equation 12 in Abbas and Bell's (2011) Theorem 1 can be viewed as a type-dependent positive affine transformation of Equation 14 in our Theorem 5.

[9] The latter issue has some connection with the question of signing the aggregation coefficients in Harsanyi's theorem.

2. Single-Crossing Differences and Interval Choice

Our analysis begins by formalizing a comparative statics result that justifies a notion of single crossing differences without reference to an order over the choice space.

Let $(\Theta, \leq)$ be a (partially) ordered set containing upper and lower bounds for all pairs.[10] We often refer to elements of Θ as *types*.

Definition 1. A function $f : \Theta \to \mathbb{R}$ is:

1. **single crossing** (resp., from below or from above) if $\mathrm{sign}[f]$ is monotonic (resp., increasing or decreasing);[11]

2. **strictly single crossing** if it is single crossing and there are no $\theta' < \theta''$ such that $f(\theta') = f(\theta'') = 0$.

Definition 2. Given any set X, a function $f : X \times \Theta \to \mathbb{R}$ has:

1. **single-crossing differences (SCD)** if $\forall x, x' \in X$, the difference $D_{x,x'}(\theta) \equiv f(x, \theta) - f(x', \theta)$ is single crossing in θ;

2. **strict single-crossing differences (SSCD)** if $\forall x, x' \in X$, $D_{x,x'}(\theta)$ is strictly single crossing in θ.

Our definition of (S)SCD is related to but different from Milgrom (2004), who stipulates that $f : X \times \Theta \to \mathbb{R}$ has (strict) single-crossing differences given an order $\succeq$ on X if for all $x' \succ x''$, where $\succ$ is the strict component of $\succeq$, $D_{x',x''}(\theta)$ is (strictly) single crossing from below. We do not presume that X is ordered, but we consider differences for all pairs of elements of X. As established below, our notion characterizes related but distinct comparative statics.

We say that $\Theta_0 \subseteq \Theta$ is *interval closed* if $\theta_l, \theta_h \in \Theta_0$ and $\theta_l < \theta_m < \theta_h$ imply $\theta_m \in \Theta_0$. Let $C : 2^X \times \Theta \rightrightarrows X$ with $C(S, \theta) \subseteq S$ for each $S \subseteq X$ and $\theta \in \Theta$. We say that C *induces an interval structure* over types if $\{\theta : x \in C(S, \theta)\}$ is interval closed for each $S \subseteq X$ and $x \in S$.

[10] A partial order—hereafter, also referred to as just an order—is a binary relation that is reflexive, anti-symmetric, and transitive (but not necessarily complete). An upper (resp., lower) bound of $\Theta_0 \subseteq \Theta$ is $\overline{\theta} \in \Theta$ (resp., $\underline{\theta} \in \Theta$) such that $\theta \leq \overline{\theta}$ (resp., $\underline{\theta} \leq \theta$) for all $\theta \in \Theta_0$. While none of our results require any assumptions on the cardinality of Θ, the results in Subsection 3.1 are trivial when $|\Theta| < 3$. Appendix I discusses how our results extend when $(\Theta, \leq)$ is only a pre-ordered set, i.e., when $\leq$ does not satisfy anti-symmetry.

[11] For $x \in \mathbb{R}$, $\mathrm{sign}[x] = 1$ if $x > 0$, $\mathrm{sign}[x] = 0$ if $x = 0$, and $\mathrm{sign}[x] = -1$ if $x < 0$. A function $h : \Theta \to \mathbb{R}$ is increasing (resp., decreasing) if $\theta_h > \theta_l \implies h(\theta_h) \geq h(\theta_l)$ (resp., $h(\theta_h) \leq h(\theta_l)$); it is monotonic if it is either increasing or decreasing. An equivalent, and perhaps more familiar, definition of f being single crossing from below is $(\forall \theta < \theta')\ f(\theta) \geq (>)0 \implies f(\theta') \geq (>)0$.

That is, interpreting C as a choice correspondence, the set of types choosing any option given any choice set is an interval. Lastly, we say that $f : X \times \Theta \to \mathbb{R}$ *strictly violates SCD* if there are $x, x' \in X$ and $\theta_l < \theta_m < \theta_h$ such that $\min\{D_{x,x'}(\theta_l), D_{x,x'}(\theta_h)\} > 0 > D_{x,x'}(\theta_m)$.

Theorem 0. *Let $f : X \times \Theta \to \mathbb{R}$ and $C_f(S, \theta) \equiv \arg\max_{x \in S} f(x, \theta)$ for any $S \subseteq X$ and θ.*

1. *If f has SCD, then the choice correspondence C_f induces an interval structure. If f strictly violates SCD, then C_f does not induce an interval structure.*

2. *Let $|\Theta| \geq 3$. f has SSCD if and only if every selection from C_f induces an interval structure.*

The intuition for the sufficiency of (S)SCD in Theorem 0 is straightforward. Regarding necessity, we note that a violation of SCD—as opposed to a strict violation—is compatible with the choice correspondence inducing an interval structure: e.g., $X = \{x', x''\}$, $\Theta = \{\theta_l, \theta_m, \theta_h\}$ with $\theta_l < \theta_m < \theta_h$, and $\min\{D_{x',x''}(\theta_l), D_{x',x''}(\theta_h)\} > 0 = D_{x',x''}(\theta_m)$.[12] In Part 2 of the Theorem, if $|\Theta| = 2$ then any selection from any choice correspondence trivially induces an interval structure, yet f may not have SSCD: e.g., $D_{x,x'}(\theta) = 0$ for some $x, x' \in X$ and all θ.

Section 4 contains applications demonstrating why an interval structure of choices is desirable. Subsection 6.1 develops a formal connection between (S)SCD and monotone comparative statics with respect to some order over the choice set.

3. Single-Crossing Expectational Differences

We now focus on choice among lotteries for an expected-utility agent and provide a functional-form characterization of the utility functions that have single-crossing differences.

3.1. Main Result

Let A be an arbitrary set and ΔA the set of probability distributions with finite support.[13] Let $v : A \times \Theta \to \mathbb{R}$ be a (type-dependent) utility function. Define the expected utility $V : \Delta A \times \Theta \to \mathbb{R}$ as

$$V(P, \theta) \equiv \int_A v(a, \theta) \mathrm{d}P.$$

[12] On the other hand, a strict violation of SCD is slightly stronger than needed: one could weaken its requirement to $\min\{D_{x',x''}(\theta_l), D_{x',x''}(\theta_h)\} \geq 0 > D_{x',x''}(\theta_m)$. Our formulation with both inequalities being strict amounts to putting aside indifferences, which proves convenient for the applications in Section 4.

[13] We restrict attention to finite-support distributions throughout the paper for ease of exposition, as it guarantees that expected utility is well defined for all utility functions. We could alternatively restrict attention to bounded utility functions. More generally, our results apply so long as the requirement of SCED (Definition 3) is restricted to those distributions for which expected utility is well defined for all types.

For any two probability distributions, also referred to as lotteries, $P \in \Delta A$ and $Q \in \Delta A$,

$$D_{P,Q}(\theta) \equiv V(P,\theta) - V(Q,\theta)$$

is the *expectational difference*.

Definition 3. The utility function $v : A \times \Theta \to \mathbb{R}$ has **single-crossing expectational differences (SCED)** if the expected utility function $V : \Delta A \times \Theta \to \mathbb{R}$ has SCD.

Remark 1. If $A = \{a_1, a_2\}$, then for any two distributions $P, Q \in \Delta A$ with probability mass functions p and q, $D_{P,Q}(\theta) = (p(a_1) - q(a_1))\,(v(a_1,\theta) - v(a_2,\theta))$. It follows that v has SCED if and only if $v(a_1,\theta) - v(a_2,\theta)$ is single crossing, i.e., if and only if v has SCD. $\square$

However, the following example shows that when $|A| > 2$, SCED is not implied by SCD (of v), or even supermodularity.

Example 1. Let $\Theta = [0,2]$ and $A = \{a_0, a_1, a_2\}$ with $a_0 < a_1 < a_2$. Define $v : A \times \Theta \to \mathbb{R}$ by $v(a_0,\theta) = 0$, $v(a_1,\theta) = -(2-\theta)^3 + 9$, and $v(a_2,\theta) = 12\theta + 4$. The function v not only has SCD, but it is strictly supermodular: for any $i > j$, $v(a_i,\theta) - v(a_j,\theta)$ is strictly increasing in θ. Consider the probability distributions $P, Q \in \Delta A$ with respective probability mass functions $p(a_1) = 1$ and $q(a_0) = q(a_2) = 1/2$. See Figure 1, in which the red dot-dashed curve is $\int_A v(a,\theta)\mathrm{d}Q$ while the others depict $v(a_i,\theta)$ for $i = \{0,1,2\}$. $D_{P,Q}(\theta)$ is not single crossing, and so v does not have SCED. $\square$

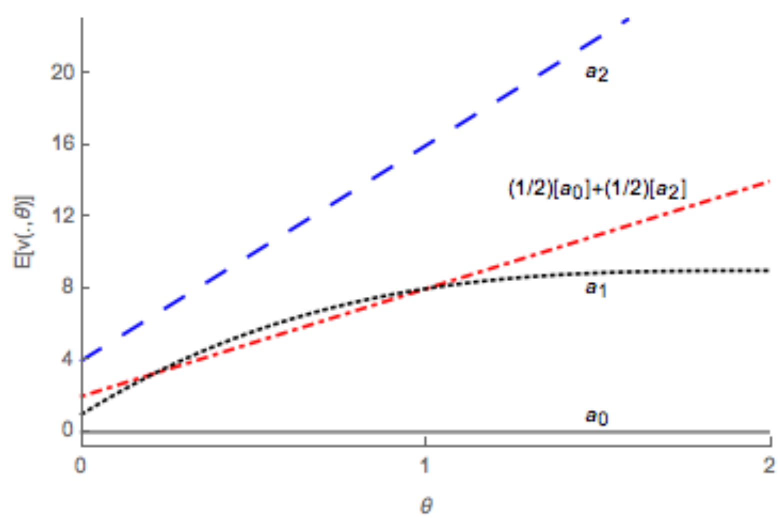


Figure 1: Single-crossing differences does not imply single-crossing expectational differences.

3.1.1. The Characterization

Our characterization of SCED (Theorem 1) uses the following definition.

Definition 4. Let $f_1, f_2 : \Theta \to \mathbb{R}$ each be single crossing.

1. f_1 **ratio dominates** f_2 if

$$(\forall \theta_l < \theta_h) \quad f_1(\theta_l)f_2(\theta_h) \le f_1(\theta_h)f_2(\theta_l), \quad \text{and} \tag{6}$$

$$(\forall \theta_l < \theta_m < \theta_h) \quad f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l) \iff \begin{cases} f_1(\theta_l)f_2(\theta_m) = f_1(\theta_m)f_2(\theta_l), \\ f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m). \end{cases} \tag{7}$$

2. f_1 and f_2 are **ratio ordered** if either f_1 ratio dominates f_2 or f_2 ratio dominates f_1.

Condition (6) contains the essential idea of ratio dominance; Condition (7) only rules out some special cases that we explain later.

Since ratio dominance involves weak inequalities, f_1 can ratio dominate f_2 and vice-versa even when $f_1 \neq f_2$: consider $f_1 = -f_2$. We use the terminology "ratio dominance" because when f_2 is a strictly positive function, (6) is the requirement that the ratio $f_1(\theta)/f_2(\theta)$ must be increasing in θ. Indeed, if both f_1 and f_2 are probability densities of random variables Y_1 and Y_2, then (6) says that Y_1 stochastically dominates Y_2 in the sense of likelihood ratios.[14]

Condition (6) is a natural generalization of the increasing ratio property to functions that may change sign. To get a geometric intuition, suppose f_1 "strictly" ratio dominates f_2 in the sense that (6) holds with strict inequality. For any θ, let $f(\theta) \equiv (f_1(\theta), f_2(\theta))$. For every $\theta_l < \theta_h$, $f_1(\theta_l)f_2(\theta_h) - f_1(\theta_h)f_2(\theta_l) < 0$ implies that the vector $f(\theta_l)$ moves to $f(\theta_h)$ through a rescaling of magnitude and a clockwise—rather than counterclockwise—rotation (throughout our paper, a "rotation" must be no more than 180 degrees); see Figure 2.[15]

Hence, f_1 and f_2 are ratio ordered only if $f(\theta)$ *rotates monotonically* as θ increases, either

[14] From the viewpoint of information economics, think of θ as a signal of a state $s \in \{1,2\}$, drawn from the density $f(\theta|s) \equiv f_s(\theta)$. Condition (6) is Milgrom's (1981) monotone likelihood-ratio property for $f(\theta|s)$.

[15] To confirm this point, recall that from the definition of cross product,

$$\begin{aligned}(f_1(\theta_l), f_2(\theta_l), 0) \times (f_1(\theta_h), f_2(\theta_h), 0) &= \|f(\theta_l)\|\|f(\theta_h)\| \sin(r) e_3 \\ &= (f_1(\theta_l)f_2(\theta_h) - f_1(\theta_h)f_2(\theta_l))\, e_3,\end{aligned}$$

where r is the counterclockwise angle from $f(\theta_l)$ to $f(\theta_h)$, $e_3 \equiv (0,0,1)$, $\times$ is the cross product, and $\|\cdot\|$ is the Euclidean norm. If $\sin(r) < 0$ (resp., $\sin(r) > 0$), then $f(\theta_l)$ moves to $f(\theta_h)$ through a clockwise (resp., counterclockwise) rotation.

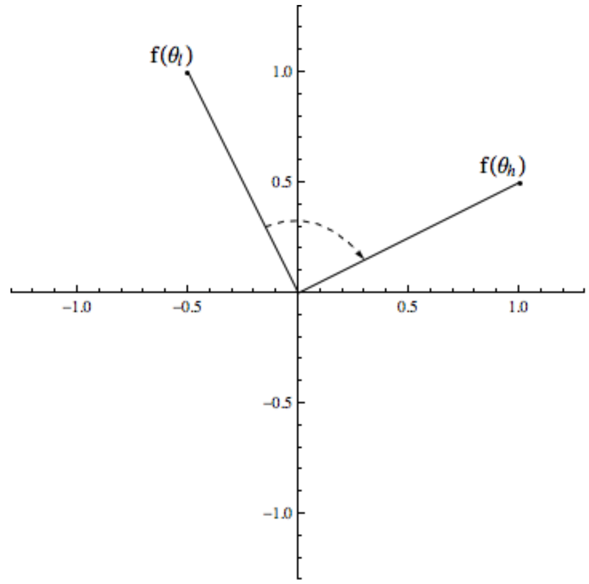


Figure 2: Geometric representation of Condition (6) for two points $\theta_l < \theta_h$.

always clockwise or always counterclockwise.[16] It follows that the set $\{f(\theta) : \theta \in \Theta\}$ must be contained in a closed half-space of $\mathbb{R}^2$ defined by a hyperplane that passes through the origin: otherwise, there will be two pairs of vectors such that an increase in θ corresponds to a clockwise rotation in one pair and a counterclockwise rotation in the other. When f_1 and f_2 are both strictly positive functions, monotonic rotation of $f(\theta)$ and ratio ordering are equivalent to monotonicity of the ratio $f_1(\theta)/f_2(\theta)$.

We impose Condition (7) to rule out cases in which, for some $\theta_l < \theta_m < \theta_h$, either (i) $f(\theta_l)$ and $f(\theta_h)$ are collinear in opposite directions while $f(\theta_m)$ is not, or (ii) $f(\theta_l)$ and $f(\theta_h)$ are non-zero vectors while $f(\theta_m)$ is not. See Figure 3, wherein panel (a) depicts case (i) and panel (b) depicts case (ii). Note that Condition (6) is satisfied in both panels.

Our main result is:

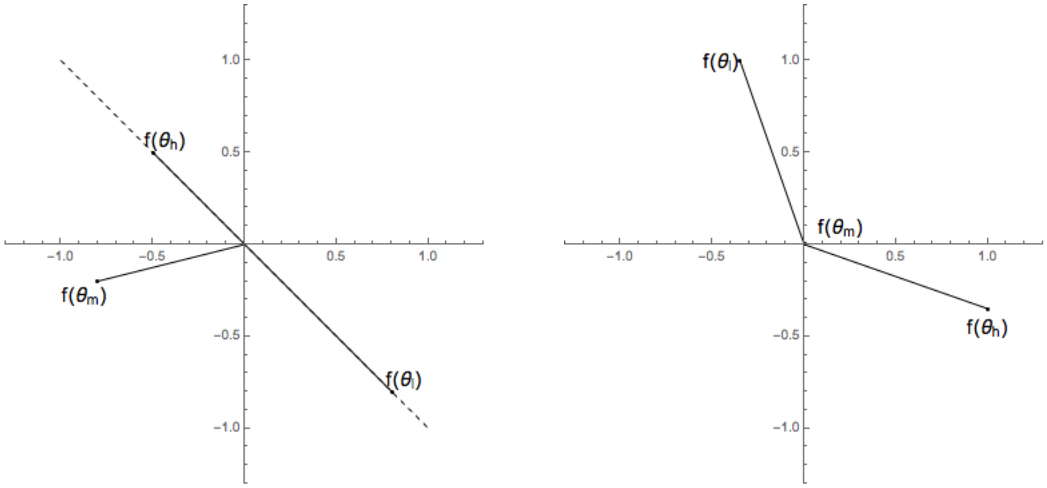
Theorem 1. *The function $v : A \times \Theta \to \mathbb{R}$ has SCED if and only if it takes the form*

$$v(a, \theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta) + c(\theta), \tag{8}$$

with $f_1, f_2 : \Theta \to \mathbb{R}$ each single crossing and ratio ordered, $g_1, g_2 : A \to \mathbb{R}$, and $c : \Theta \to \mathbb{R}$.

A number of observations help interpret Theorem 1. First, the Theorem says that for v

[16] The preceding discussion establishes this point under the presumption that Condition (6) holds strictly; however, because of the hypothesis in Definition 4 that f_1 and f_2 are single crossing and because of Condition (7), the conclusion holds without that presumption. Furthermore, it can be confirmed that a monotonic rotation of $f(\cdot)$ implies ratio ordering if there are no θ' and θ'' such that $f(\theta')$ and $f(\theta'')$ are collinear.



(a) Failure of $\Longrightarrow$.

(b) Failure of $\Longleftarrow$.

Figure 3: f_1 and f_2 are not ratio ordered because Condition (7) fails for $\theta_l < \theta_m < \theta_h$.

to have SCED, it must be possible to write it in the form (8). Notice that given (8), for any $a_0, a \in A$, the function $v(a, \cdot) - v(a_0, \cdot)$ is a linear combination of $f_1(\cdot)$ and $f_2(\cdot)$. Therefore, to rule out the possibility of the form (8), it is sufficient to find $a_0, a_1, a_2, a_3 \in A$ and $\theta_l < \theta_m < \theta_h$ such that the 3×3 matrix $M \equiv [v(a_i, \theta_j) - v(a_0, \theta_j)]_{i \in \{1,2,3\}, j \in \{l,m,h\}}$ is invertible. This procedure is often useful to reject SCED, as illustrated by the following corollary, which identifies quadratic loss as the unique power loss function that satisfies SCED.

Corollary 1. *Let $A = \mathbb{R}$ and $\Theta \subseteq \mathbb{R}$ with $|\Theta| \geq 3$, with the interpretation that a is a decision or policy and θ parameterizes the agent's bliss point. A loss function of the form $v(a, \theta) = -|a - \theta|^z$ with $z > 0$ has SCED if and only if $z = 2$.*

Under quadratic loss, preferences over lotteries are summarized by the lotteries' first and second moments. The sufficiency of two statistics is a general property under SCED; given (8), the relevant statistics for any lottery $P \in \Delta A$ are $\int_A g_1(a) \mathrm{d}P$ and $\int_A g_2(a) \mathrm{d}P$.

The form (8) is not enough for SCED, however: the component f_1 and f_2 functions must be single crossing and ratio ordered. Example 1 illustrates. Take $c(\theta) = 0$, $f_1(\theta) = -(2 - \theta)^3 + 9$, $f_2(\theta) = 12\theta + 4$, and for each $i \in \{1, 2\}$, $g_i(a) = \mathbb{1}_{\{a=i\}}$, where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. The Example then satisfies (8), with f_1 and f_2 each single crossing. But since f_1 and f_2 are both strictly positive while f_1/f_2 is not monotonic, they are not ratio ordered. Indeed, the requirement of ratio ordering underlies the following corollary.

Corollary 2. *Let $A \subseteq \mathbb{R}^2$ with $a \equiv (q, t)$ and $\Theta \subseteq \mathbb{R}$, with the interpretation that q is a quantity or allocation, t is money, and θ parameterizes the agent's marginal rate of substitution.*

1. *$v((q,t),\theta) = g(q)f(\theta) - t$ (as in mechanism design and screening), where g is not constant, has SCED if and only if f is monotonic.*

2. *$v((q,t),\theta) = q\theta + g(q) - t$ (as in delegation with money burning; cf. Amador and Bagwell (2013)) has SCED.*

An asymmetry between a and θ in the functional form (8) bears noting: the function $c : \Theta \to \mathbb{R}$ does not have a counterpart function $A \mapsto \mathbb{R}$. The reason is that whether the expectational difference between a pair of lotteries is single crossing or not could be altered by adding a function of a alone to the utility function $v(a,\theta)$. On the other hand, adding a function of θ alone to $v(a,\theta)$ has no effect on expectational differences. Indeed, SCED is an ordinal property of preferences over lotteries that is invariant to (type-dependent) positive affine transformations of $v(a,\theta)$: if $v(a,\theta)$ has SCED, then so does $b(\theta)v(a,\theta) + c(\theta)$ for any $b : \Theta \to \mathbb{R}_{++}$ and $c : \Theta \to \mathbb{R}$.

If $v(a,\theta)$ has the form (8) with strictly positive functions f_1 and f_2, then up to a positive affine transformation (viz., subtracting $c(\theta)$ and dividing by $f_1(\theta) + f_2(\theta)$), any type's utility becomes a convex combination of two type-independent utility functions over actions, g_1 and g_2. Theorem 1's ratio ordering requirement then simply says that the relative weight on g_1 and g_2 changes monotonically with the agent's type. This idea underlies the following proposition.

Proposition 1. *If Θ has both a minimum and a maximum (i.e., $\exists\, \underline{\theta}, \overline{\theta} \in \Theta$ such that $(\forall\theta)\ \underline{\theta} \le \theta \le \overline{\theta}$), then $v : A \times \Theta \to \mathbb{R}$ has SCED if and only if v has a positive affine transformation $\tilde{v}$ satisfying*

$$\tilde{v}(a,\theta) = \lambda(\theta)\tilde{v}(a,\overline{\theta}) + (1-\lambda(\theta))\tilde{v}(a,\underline{\theta}), \tag{9}$$

with $\lambda : \Theta \to [0,1]$ increasing.

The remainder of this subsection explains the logic behind Theorem 1. As expected utility differences correspond to linear combinations of utilities, the central issue turns out to be when arbitrary such aggregations are single crossing. Lemma 1 below shows that ratio ordering is the characterizing property when aggregating two functions; Proposition 2 then establishes that when aggregating more than two functions, no more than two can be linearly independent, which leads to the form (8).[17]

[17] Real-valued functions $f_1, f_2, \ldots, f_n$ are linearly independent if $(\forall\lambda \in \mathbb{R}^n \backslash \{0\})\ \sum_{i=1}^n \lambda_i f_i$ is not a zero function, i.e., is not everywhere zero.

3.1.2. Aggregating Single-Crossing Functions

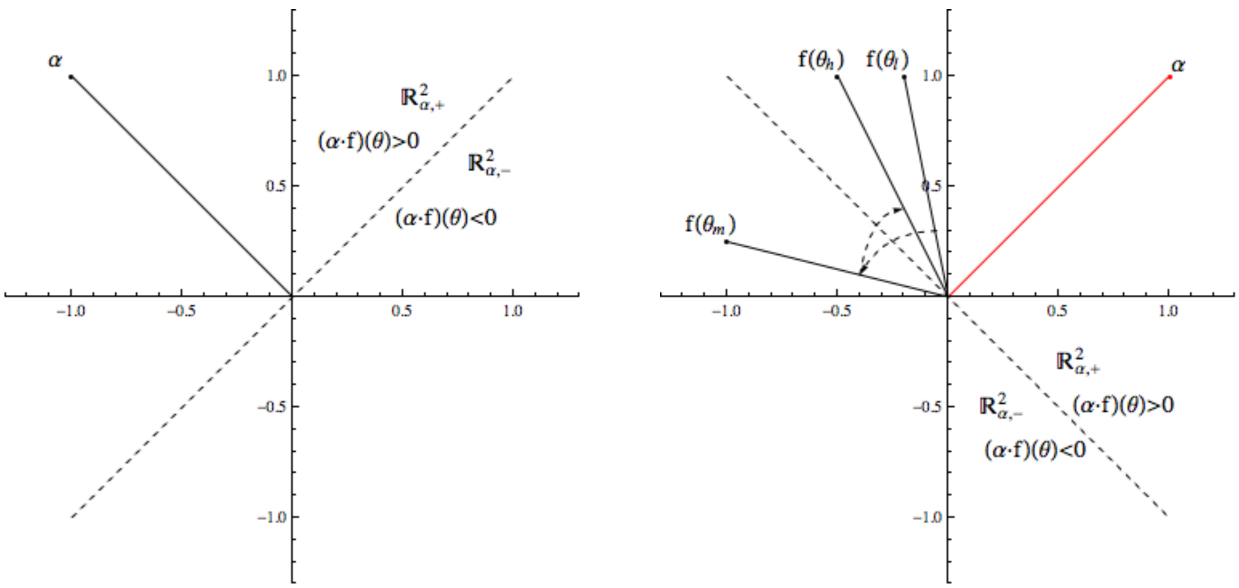
Lemma 1. *Let $f_1, f_2 : \Theta \to \mathbb{R}$. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is single crossing $\forall \alpha \in \mathbb{R}^2$ if and only if f_1 and f_2 are (i) each single crossing and (ii) ratio ordered.*

Lemma 1 is related to Quah and Strulovici (2012, Proposition 1). They establish that for any two functions f_1 and f_2 that are each single crossing from below, $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing from below for all $\alpha \in \mathbb{R}^2_+$ if and only if f_1 and f_2 satisfy a condition they call signed-ratio monotonicity. In general, ratio ordering is not comparable with signed-ratio monotonicity because we consider a different aggregation problem from Quah and Strulovici: (i) the input functions may be single crossing in either direction; (ii) the linear combinations involve coefficients of arbitrary sign; and (iii) the resulting combination can be single crossing in either direction. Example 1 highlights the importance of point (ii): both $f_1(\theta) = -(2-\theta)^3 + 9$ and $f_2(\theta) = 12\theta + 4$ are positive functions (hence, single crossing from below), and so all positive linear combinations are also positive functions, but $2f_1 - f_2$ is not single crossing because f_1 and f_2 are not ratio ordered. If the input functions in Lemma 1 are restricted to be single crossing from below, then ratio ordering implies signed-ratio monotonicity.

Lemma 1 implies a characterization of likelihood-ratio ordering for random variables with single-crossing densities, e.g., those with strictly positive densities. While this likelihood-ratio ordering characterization is not well-known among economists (to our knowledge), it is a special case of Karlin's (1968) results on the variation diminishing property of totally positive functions. More generally, however, we believe the full force of Lemma 1 cannot be derived from the variation diminishing property. See Appendix H for further discussion of Karlin (1968) and Quah and Strulovici (2012).

Here is Lemma 1's intuition. For sufficiency, consider any linear combination $\alpha_1 f_1 + \alpha_2 f_2$. Assume $\alpha \in \mathbb{R}^2 \backslash \{0\}$, as otherwise the linear combination is trivially single crossing. The vector α defines two open half spaces $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$ and $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, where $\cdot$ is the dot product; see Figure 4(a). As explained earlier, ratio ordering of f_1 and f_2 implies that the vector $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ rotates monotonically as θ increases. If the rotation is from $\mathbb{R}^2_{\alpha,-}$ to $\mathbb{R}^2_{\alpha,+}$ (resp., from $\mathbb{R}^2_{\alpha,+}$ to $\mathbb{R}^2_{\alpha,-}$), then $\alpha \cdot f \equiv \alpha_1 f_1 + \alpha_2 f_2$ is single crossing only from below (resp., only from above). If $\bigcup_{\theta \in \Theta} f(\theta) \subseteq \mathbb{R}^2_{\alpha,-}$ or $\bigcup_{\theta \in \Theta} f(\theta) \subseteq \mathbb{R}^2_{\alpha,+}$, then $\alpha \cdot f$ is single crossing both from below and above. Other cases are similar.

To see why Condition (6) of ratio ordering is necessary, suppose the vector $f(\theta)$ does not rotate monotonically. Figure 4(b) illustrates a case in which, for $\theta_l < \theta_m < \theta_h$, $f(\theta_l)$ rotates counterclockwise to $f(\theta_m)$, but $f(\theta_m)$ rotates clockwise to $f(\theta_h)$. As shown in the Figure, one can find $\alpha \in \mathbb{R}^2$ such that $f(\theta_m) \in \mathbb{R}^2_{\alpha,-}$ while both $f(\theta_l), f(\theta_h) \in \mathbb{R}^2_{\alpha,+}$, which implies



(a) Sufficiency of ratio ordering.

(b) Necessity of ratio ordering, with $\theta_l < \theta_m < \theta_h$.

Figure 4: Ratio ordering and single crossing of all linear combinations.

that $\alpha \cdot f$ is not single crossing. The necessity of Condition (7) can be seen by returning to Figure 3. In panel (a), $(f_1 + f_2)(\theta_l) = (f_1 + f_2)(\theta_h) = 0$ while $(f_1 + f_2)(\theta_m) < 0$; in panel (b), $(f_1 + f_2)(\theta_l) > 0$ and $(f_1 + f_2)(\theta_h) > 0$ while $(f_1 + f_2)(\theta_m) = 0$.

Theorem 1 requires an extension of Lemma 1 to more than two functions. Consider any set X and $f : X \times \Theta \to \mathbb{R}$. We say that f is **linear combinations SC-preserving** if $\sum_{x \in X} f(x, \theta)\mu(x)$ is single-crossing in θ for every function $\mu : X \to \mathbb{R}$ with finite support.

Proposition 2. *Let $f : X \times \Theta \to \mathbb{R}$ for some set X. The function f is linear combinations SC-preserving if and only if there exist $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 : X \to \mathbb{R}$ such that*

1. *$f(x_1, \cdot) : \Theta \to \mathbb{R}$ and $f(x_2, \cdot) : \Theta \to \mathbb{R}$ are (i) each single crossing and (ii) ratio ordered, and*

2. *$(\forall x)\ f(x, \cdot) = \lambda_1(x) f(x_1, \cdot) + \lambda_2(x) f(x_2, \cdot)$.*

Proposition 2 says that a family of single-crossing functions $\{f(x, \cdot)\}_{x \in X}$ preserves single crossing of all finite linear combinations if and only if the family is "linearly generated" by two single-crossing functions that are ratio ordered. In particular, given any three single-crossing functions, f_1, f_2, and f_3, all their linear combinations will be single crossing if and only if there is a linear dependence in the triple, say $\lambda_1 f_1 + \lambda_2 f_2 = f_3$ for some $\lambda \in \mathbb{R}^2$, and f_1 and f_2 are ratio ordered.

The sufficiency direction of Proposition 2 follows from Lemma 1, as does necessity of the "generating functions" being ratio ordered. The intuition for the necessity of linear dependence is as follows. Assume Θ is completely ordered. For any θ, let $f(\theta) \equiv (f_1(\theta), f_2(\theta), f_3(\theta))$. If $\{f_1, f_2, f_3\}$ is linearly independent, then there exist $\theta_l < \theta_m < \theta_h$ such that $\{f(\theta_l), f(\theta_m), f(\theta_h)\}$ spans $\mathbb{R}^3$. Take any $\alpha \in \mathbb{R}^3 \setminus \{0\}$ that is orthogonal to the plane S_{θ_l,θ_h} that is spanned by $f(\theta_l)$ and $f(\theta_h)$, as illustrated in Figure 5. The linear combination $\alpha \cdot f$ is not single crossing because $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$ while $(\alpha \cdot f)(\theta_m) \neq 0$.

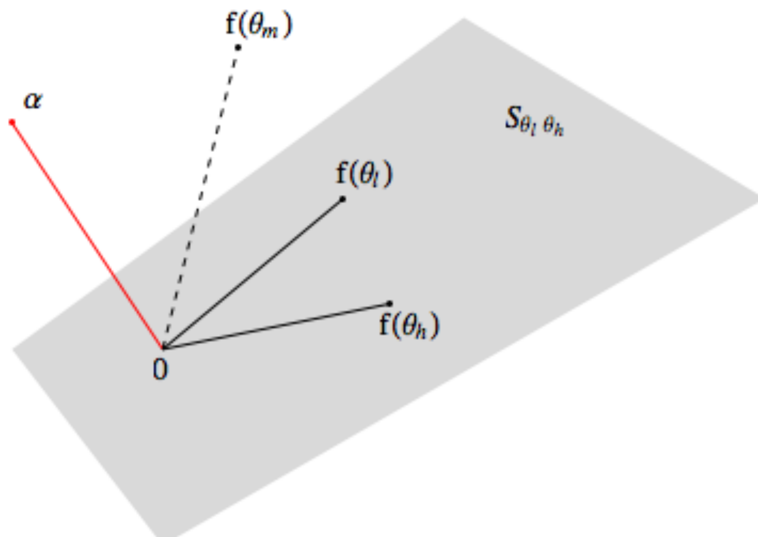


Figure 5: The necessity of linear dependence in Proposition 2.

While the necessity portion of Proposition 2 only asserts ratio ordering of the "generating functions", Lemma 1 implies that if $f : X \times \Theta \to \mathbb{R}$ is linear combinations SC-preserving, then for all $x, x' \in X$, the pair $f(x, \cdot) : \Theta \to \mathbb{R}$ and $f(x', \cdot) : \Theta \to \mathbb{R}$ must be ratio ordered.

3.1.3. Proof Sketch of Theorem 1

We can now sketch the argument for Theorem 1. That its characterization is sufficient for SCED is straightforward from Lemma 1. For necessity, suppose, as a simplification, $A = \{a_0, \dots, a_n\}$ and $v : A \times \Theta \to \mathbb{R}$ is such that $(\forall\theta)\ v(a_0, \theta) = 0$.[18] For any $(\lambda_0, \dots, \lambda_n)$, we build on the Hahn-Jordan decomposition of $(\lambda_1, \dots, \lambda_n)$ to write the linear combination $\sum_{i=0}^n \lambda_i v(a_i, \theta)$ as $M \sum_{i=0}^n (p(a_i) - q(a_i)) v(a_i, \theta)$, where p and q are probability mass functions on A, and M is a scalar.[19] (Unless $\sum_{i=1}^n \lambda_i = 0$, we have $\sum_{i=1}^n p(a_i) \neq \sum_{i=1}^n q(a_i)$;

[18] The latter is a normalization, since $v(a, \theta)$ has SCED if and only if $\tilde{v}(a, \theta) \equiv v(a, \theta) - v(a_0, \theta)$ has SCED.

[19] Let $L \equiv \sum_{i=1}^n \lambda_i$. For $i > 0$, set $p'(a_i) \equiv \max\{\lambda_i, 0\}$ and $q'(a_i) \equiv -\min\{\lambda_i, 0\}$. If $L \geq 0$, set $p'(a_0) = 0$ and $q'(a_0) \equiv L$; if $L < 0$, set $p'(a_0) \equiv -L$ and $q'(a_0) \equiv 0$. Let $M \equiv \sum_{i=0}^n p'(a_i) = \sum_{i=0}^n q'(a_i)$. Finally, for all $a \in A$,

the assumption that $v(a_0, \cdot) = 0$ permits us to assign all the "excess difference" to a_0, as detailed in fn. 19.) Since v has SCED, every such linear combination is single crossing, and so Proposition 2 guarantees a' and a'' such that for all a, $v(a, \cdot) = g_1(a)v(a', \cdot) + g_2(a)v(a'', \cdot)$, with $v(a', \cdot)$ and $v(a'', \cdot)$ each single crossing and ratio ordered.

3.2. Strict Single-Crossing Expectational Differences

This subsection provides a "strict variant" of Theorem 1. We now assume the existence of a strictly increasing real-valued function on $(\Theta, \leq)$.[20] This requirement is satisfied, for example, when Θ is finite, or $\Theta \subseteq \mathbb{R}^n$ is endowed with the usual order.

In general, an expected utility function V will have utility-indistinguishable lotteries (i.e., $P, Q \in \Delta A$ such that $D_{P,Q}(\theta) = 0$ for all θ) and thus violate SSCD; for example, multiple lotteries may have the same expectation and v may be linear in a. Accordingly, we consider utility-indistinguishable lotteries as equivalence classes and denote the resulting partition of ΔA by $\widetilde{\Delta} A$. For readability, we abuse notation and treat elements of $\widetilde{\Delta} A$ as lotteries rather than equivalence classes.

Definition 5. The utility function $v : A \times \Theta \to \mathbb{R}$ has **strict single-crossing expectational differences (SSCED)** if the expected utility function $V : \widetilde{\Delta} A \times \Theta \to \mathbb{R}$ has SSCD.

Definition 6. A function $f_1 : \Theta \to \mathbb{R}$ **strictly ratio dominates** $f_2 : \Theta \to \mathbb{R}$ if Condition (6) holds with strict inequality; f_1 and f_2 are **strictly ratio ordered** if either f_1 strictly ratio dominates f_2 or vice-versa.

The definition of strict ratio dominance does not make reference to Condition (7) because that condition is vacuous when Condition (6) holds with strict inequality.

Theorem 2. *The function $v : A \times \Theta \to \mathbb{R}$ has SSCED if and only if it takes the form (8), with $f_1, f_2 : \Theta \to \mathbb{R}$ strictly ratio ordered, $g_1, g_2 : A \to \mathbb{R}$, and $c : \Theta \to \mathbb{R}$.*

4. Applications

This section illustrates the usefulness of our results in three applications.

---

set $p(a) \equiv p'(a)/M$ and $q(a) \equiv q'(a)/M$.

[20] That is, we assume $\exists h : \Theta \to \mathbb{R}$ such that $\underline{\theta} < \overline{\theta} \implies h(\underline{\theta}) < h(\overline{\theta})$. This requirement is related to utility representations for possibly incomplete preferences (Ok, 2007, Chapter B.4.3). A sufficient condition is that Θ has a countable order dense subset, i.e., if there exists a countable set $\Theta_0 \subseteq \Theta$ such that $(\forall \underline{\theta}, \overline{\theta} \in \Theta \setminus \Theta_0) \quad \underline{\theta} < \overline{\theta} \implies \exists \theta_0 \in \Theta_0$ s.t. $\underline{\theta} < \theta_0 < \overline{\theta}$ (Jaffray, 1975, Corollary 1).

4.1. Cheap Talk with Uncertain Receiver Preferences

There are two players, a sender (S) and a receiver (R). The sender's type is $\theta \in \Theta$, where Θ is ordered by $\leq$. After learning his type, S chooses a payoff-irrelevant message $m \in M$, where $|M| > 1$. After observing m but not θ, R takes an action $a \in A$. The sender's von Neumann-Morgenstern utility function is $v(a, \theta)$; the receiver's is $u(a, \theta, \psi)$, where $\psi \in \Psi$ is a preference parameter that is unknown to S when choosing m, and known to R when choosing a. Note that ψ does not affect the sender's preferences. The variables θ and ψ are independently drawn from commonly-known probability distributions.

An example is $\Theta = [0, 1]$, $A = \mathbb{R}$, $\psi \in \Psi \subseteq \mathbb{R}^2$, $v(a, \theta) = -(a - \theta)^2$ and $u(a, \theta, \psi) = -(a - \psi_1 - \psi_2\theta)^2$. Here the variable ψ_1 captures the receiver's "type-independent bias" and ψ_2 captures the relative "sensitivity" to the sender's type. If ψ were commonly known and θ uniformly distributed, this would be the model of Melumad and Shibano (1991), which itself generalizes the canonical example from Crawford and Sobel (1982) that obtains when $\psi_1 \neq 0$ and $\psi_2 = 1$.

We focus on (weak Perfect Bayesian) equilibria in which S uses a pure strategy, $\mu : \Theta \to M$, and R plays a possibly-mixed strategy, $\alpha : M \times \Psi \to \Delta A$.[21] Given any α, every message m induces some lottery over actions from the sender's viewpoint, $P_\alpha(m)$. An equilibrium (μ, α) is: (i) an *interval equilibrium* if every message is used by an interval of sender types, i.e., if $(\forall m)\ \{\theta : \mu(\theta) = m\}$ is interval closed, and (ii) *sender minimal* if for all on-the-equilibrium-path $m \neq m'$, there is some θ such that $V(P_\alpha(m), \theta) \neq V(P_\alpha(m'), \theta)$. In other words, sender minimality rules out all sender types being indifferent between two distinct on-path messages.[22]

Claim 1. *If v has the form stated in Theorem 2, and hence has SSCED, then every sender-minimal equilibrium is an interval equilibrium.*

Proof. From the sender' viewpoint, the lottery over the receiver's actions that is induced by any message (given any receiver strategy) is independent of θ because ψ and θ are independent. The result follows from Theorem 0, as sender-minimality implies that one can restrict attention to the sender choosing among action lotteries that are utility-distinguishable (i.e., if P' and P'' are equilibrium action lotteries, then $D_{P,P''}$ is not a zero function). *Q.E.D.*

[21] Our notion of equilibrium requires optimal play for every (not just almost every) type of sender. The restriction to pure strategies for the sender is for expositional simplicity.

[22] In Crawford and Sobel (1982) and Melumad and Shibano (1991), all equilibria are outcome equivalent to sender-minimal equilibria. More generally, all equilibria are sender minimal when there is a complete order over messages under which higher messages are infinitesimally more costly for all sender types.

Claim 1 relates to Seidmann (1990), who first considered an extension of Crawford and Sobel (1982) to sender uncertainty about the receiver's preferences. His goal was to illustrate how such uncertainty could facilitate informative communication even when the sender always strictly prefers higher actions. Example 2 in Seidmann (1990) constructs a non-interval and sender-minimal equilibrium that is informative. Claim 1 clarifies that the key is a failure of (S)SCED.

The strict single crossing property in standard cheap-talk models (e.g., Crawford and Sobel (1982) and Melumad and Shibano (1991)) not only yields interval equilibria, but it also implies that local incentive compatibility is sufficient for global incentive compatibility. This additional tractability also holds under SSCED. Let $\Theta = \mathbb{N}$ for convenience, and $P : \Theta \to \Delta A$ be a candidate equilibrium allocation (i.e., the distribution of receiver actions that each sender type induces in equilibrium). Under SSCED, it is sufficient for sender incentive compatibility that $(\forall i \in \mathbb{N})\ V(P(\theta_i), \theta_i) \geq \max\{V(P(\theta_{i-1}), \theta_i), V(P(\theta_{i+1}), \theta_i)\}$.

Besides being sufficient, (S)SCED is also necessary to guarantee interval cheap-talk equilibria so long as the environment—specifically, the receiver utility function u and the distribution of his preference parameter ψ—is rich enough to generate appropriate pairs of lotteries. Say that v *strictly violates* SCED if V strictly violates SCD: i.e., if there are $P, Q \in \Delta A$ and $\theta_l < \theta_m < \theta_h$ such that $\min\{D_{P,Q}(\theta_l), D_{P,Q}(\theta_h)\} > 0 > D_{P,Q}(\theta_m)$.

Claim 2. *Let $\Theta \subseteq \mathbb{R}$, $A = \mathbb{R}$, $\Psi \subseteq \mathbb{R}^2$, and $u(a, \theta, \psi) \equiv -(a - \psi_1 - \psi_2\theta)^2$. If v strictly violates SCED, then for some pair of distributions of θ and ψ there is a non-interval equilibrium in which each sender type plays its unique best response.*

Proof. Assume v strictly violates SCED and let P and Q be the distributions and $\theta_l < \theta_m < \theta_h$ the types in that definition. Let $M \equiv \{m', m''\}$ and consider the sender's strategy

$$\mu(\theta) = \begin{cases} m' & \text{if } \theta \in \{\theta_l, \theta_h\} \\ m'' & \text{if } \theta = \theta_m. \end{cases}$$

Let F_θ be any distribution with support $\{\theta_l, \theta_m, \theta_h\}$ and $\theta' \equiv \mathbb{E}_{F_\theta}[\theta | \theta \in \{\theta_l, \theta_h\}] \neq \theta_m$. Then, the unique best response against μ for a receiver of type $\psi = (\psi_1, \psi_2)$ is

$$\alpha(m', \psi) = \psi_1 + \psi_2\theta' \quad \text{and} \quad \alpha(m'', \psi) = \psi_1 + \psi_2\theta_m.$$

It can be verified that there is a distribution F_ψ such that, from the sender's viewpoint, the message m' leads to the distribution P and the message m'' leads to the distribution Q, and so μ is the sender's unique best response.[23] *Q.E.D.*

The particular specification of u in Claim 2 is not critical; what is important, as suggested earlier, is that there be enough flexibility to generate appropriate distributions of actions from the sender's viewpoint using best responses for the receiver.[24] For example, the result would also hold—more straightforwardly, but less interestingly—if the receiver were totally indifferent over all actions for some preference realization. On the other hand, if $\psi \in \mathbb{R}$ and $u(a, \theta, \psi) \equiv -(a - \theta - \psi)^2$, then SCED is not necessary, because any pair of lotteries that the sender may face are ranked by first order stochastic dominance. Strict supermodularity of $v(a, \theta)$ then guarantees that all sender-minimal equilibria are interval equilibria; however, strict supermodularity does not imply SCED, as noted in Example 1.

In our cheap-talk application it is uncertainty about the receiver's preferences that leads to the sender effectively choosing among lotteries over the receiver's action. Similar results could also be obtained when the receiver's preferences are known but communication is noisy, à la Blume, Board, and Kawamura (2007).

4.2. Collective Choice

Collective choice over lotteries arises naturally in many contexts: elections entail uncertainty about what policies politicians will implement if elected; and a board of directors may view each candidate for CEO as a probability distribution over firm earnings. Zeckhauser (1969) first pointed out that pairwise-majority comparisons in these settings can be cyclical, even when comparisons over deterministic outcomes are not. Our results shed light on when such difficulties do not arise.

Consider a finite group of individuals indexed by $i \in \{1, 2, \ldots, N\}$. The group must choose from a set of lotteries, $\mathcal{A} \subseteq \Delta A$, where A is the space of outcomes (political policies,

[23] Let x and y be random variables with distributions P and Q, respectively. Let F_ψ be the distribution of a random vector $\psi = (\psi_1, \psi_2)$ defined by

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \equiv \begin{bmatrix}1 & \theta'\\ 1 & \theta_m\end{bmatrix}^{-1} \begin{bmatrix}x\\ y\end{bmatrix}.$$

As $\psi \sim F_\psi$ (i.e., ψ has distribution F_ψ),

$$\begin{bmatrix}1 & \theta'\\ 1 & \theta_m\end{bmatrix} \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = \begin{bmatrix}\psi_1 + \psi_2\theta'\\ \psi_1 + \psi_2\theta_m\end{bmatrix}$$

is stochastically equivalent to (x, y). Thus, $\alpha(m', \psi) = \psi_1 + \psi_2\theta' \sim P$ and $\alpha(m'', \psi) = \psi_1 + \psi_2\theta_m \sim Q$.

[24] In particular, the result in Claim 2 holds under the following more general assumptions: $\Theta, A \subseteq \mathbb{R}$, the receiver's preferences are represented by $u(a, \theta, \psi) = g_1(\psi, a)\theta + g_2(\psi, a)$, and for every $\theta_l < \theta_h$ and $a', a'' \in A$, there exists $\psi \in \Psi$ such that $a' \in \arg\max_a u(a, \psi, \theta_l)$ and $a'' \in \arg\max_a u(a, \psi, \theta_h)$. The proof is very similar to that of Claim 2.

earnings, etc.) with generic element a. Each individual i has von Neumann-Morgenstern utility function $v(a, \theta_i)$, where $\theta_i \in \Theta$ is a preference parameter or i's type. We assume Θ is completely ordered; without further loss of generality, let $\Theta \subset \mathbb{R}$ and $\theta_1 \leq \cdots \leq \theta_N$. The expected utility for an individual of type θ from lottery $P \in \mathcal{A}$ is $V(P, \theta) \equiv \int_A v(a, \theta) \mathrm{d}P$. We denote individual i's preference relation over lotteries by $\succeq_i$, with strict component $\succ_i$.

Define the group's preference relation, $\succeq_{maj}$, over lotteries $P, Q \in \mathcal{A}$ by majority rule:

$$P \succeq_{maj} Q \text{ if } |\{i : P \succeq_i Q\}| \geq N/2.$$

The relation $\succeq_{maj}$ is said to be *quasi-transitive* if the corresponding strict relation, $\succ_{maj}$, is transitive. Quasi-transitivity of $\succeq_{maj}$ is the key requirement for collective choice to be "rational": it ensures—given completeness, which $\succeq_{maj}$ obviously satisfies—that a preference-maximizing choice (equivalently, a Condorcet Winner) exists for the group under standard conditions on the choice set, e.g., if $\mathcal{A}$ is finite. We say there is a *unique median* if either (i) N is odd, in which case we define $M \equiv (N+1)/2$ or (ii) N is even and $\theta_{N/2} = \theta_{N/2+1}$, in which case $M \equiv N/2$.

Claim 3. *If v has the form stated in Theorem 1, and hence has SCED, then the group's preference relation is quasi-transitive. If in addition there is a unique median, then the group's preference relation is transitive and is represented by $V(\cdot, \theta_M)$.*

Our proof is related to the monotone comparative statics arguments of Gans and Smart (1996), but highlights the interval choice structure implication of SC(E)D.

Proof. Let $\overline{M} \equiv \{(N+1)/2\}$ if N is odd, and $\overline{M} \equiv \{N/2, N/2+1\}$ if N is even. The interval choice structure under SCED (Part 1 of Theorem 0) implies that: (i) if $P \succeq_i Q$ for any $i \in \overline{M}$, then $P \succeq_{maj} Q$; and (ii) if $P \succ_i Q$ for all $i \in \overline{M}$, then $P \succ_{maj} Q$. Consequently, if $P \succ_{maj} Q$ and $Q \succ_{maj} R$, then $P \succ_i Q \succ_i R$ for all $i \in \overline{M}$ (by point (i)), and hence $P \succ_{maj} R$ (by point (ii)). This proves quasi-transitivity of $\succeq_{maj}$. Furthermore, points (i) and (ii) also imply that if there is a unique median, M, then $P \succeq_{maj} Q \iff P \succeq_M Q$, and therefore $\succeq_{maj}$ is transitive. *Q.E.D.*

Claim 3 can be applied to a well-known problem in political economy (Shepsle, 1972). The policy space is a finite set $A \subset \mathbb{R}$ (for simplicity) and there are an odd N number of voters ordered by their ideal points in $\mathbb{R}$, $\theta_1 \leq \cdots \leq \theta_N$ (i.e., for each voter i, $\{\theta_i\} = \arg\max_{a \in \mathbb{R}} v(a, \theta_i)$). Let $M \equiv (N+1)/2$. There are two office-motivated candidates, L and R; each $j \in \{L, R\}$ can commit to any lottery from some given set $\mathcal{A}_j \subseteq \Delta A$. A restricted set $\mathcal{A}_j$ may capture various kinds of constraints; for example, Shepsle (1972) assumed the incumbent candidate could only choose degenerate lotteries. In our setting, what ensures the existence of an equilibrium, and which policy lotteries are offered in an equilibrium?[25]

Claim 3 implies that if voters' utility functions v have SSCED, and if voter M is indifferent between her most-preferred lottery in $\mathcal{A}_L$ and in $\mathcal{A}_R$ (e.g., if $\mathcal{A}_L = \mathcal{A}_R$ and they are compact sets, or if both sets contain the degenerate lottery on θ_M, denoted δ_{θ_M} hereafter), then there is a unique equilibrium: each candidate offers the best lottery for voter M; in particular, both candidates converge to δ_{θ_M} if that is feasible for both. A special case is when $v(a,\theta) = -(a-\theta)^2$ and $\delta_{\theta_M} \in \mathcal{A}_L \cap \mathcal{A}_R$. It bears emphasis, however, that there will be policy convergence at the median ideal point (so long as $\delta_{\theta_M} \in \mathcal{A}_L \cap \mathcal{A}_R$) given SSCED not because all voters need be globally "risk averse"; rather, it is because SSCED ensures the existence of a decisive voter whose most-preferred lottery is degenerate.[26]

There is a sense in which (S)SCED is necessary to guarantee that each candidate j will offer the median ideal-point voter's most-preferred lottery from the feasible set $\mathcal{A}_j$. Suppose $v(a,\theta)$ strictly violates SCED, i.e., there are $P, Q \in \Delta A$ and $\theta_l < \theta_m < \theta_h$ such that $\min\{D_{P,Q}(\theta_l), D_{P,Q}(\theta_h)\} > 0 > D_{P,Q}(\theta_m)$. Then, if the population of voters is just $\{l, m, h\}$ and $\mathcal{A}_L = \mathcal{A}_R = \{P, Q\}$, the unique equilibrium is for both candidates to offer lottery P, which is voter m's less preferred lottery.

4.3. Costly Signaling

Consider a version of Spence's (1973) signaling model. A worker is privately informed of his type θ that is drawn from some distribution with support $\Theta \subseteq \mathbb{R}$ and then chooses education $e \in \mathbb{R}_+$. There is a reduced-form market that observes e (but not θ) and allocates wage, or some other statistic of job characteristics, $w \in \mathbb{R}$ to the worker. The worker's von Neumann-Morgenstern payoff is given by $v(w, e, \theta)$. It is convenient to let $a \equiv (w, e)$, so that we can also write $v(a, \theta)$.

In the standard model, (i) w is an exogenously-given strictly increasing function of the market's expectation $\mathbb{E}[\theta|e]$, (ii) $v(w, e, \theta)$ is strictly increasing in w and strictly decreasing in e, and (iii) $v(w, e, \theta)$ has a strict single-crossing property in $((w, e), \theta)$.[27]

[25] More precisely: the two candidates simultaneously choose their lotteries, and each voter then votes for his preferred candidate (assuming, for concreteness, that a voter randomizes between the candidates with equal probability if indifferent). A candidate wins if he receives a majority of the votes. Candidates maximize the probability of winning. We seek a Nash equilibrium of the game between the two candidates.

[26] An example may be helpful. Let $A = [-1, 1]$, $\Theta = \{-1, 0, 1\}$, and $v(a, \theta) = a\theta + 1/(|a| + 1) + 1$. The corresponding functions $f_1(\theta) = \theta$ and $f_2(\theta) = 1$ are each strictly single crossing from below and strictly ratio ordered. For all θ, $v(\cdot, \theta) : A \to \mathbb{R}$ is maximized at $a = \theta$ but convex on some sub-interval of A.

[27] Given point (ii), point (iii) is implied by the Spence-Mirlees single crossing condition: v_w/v_e is increasing in θ, where a subscript on v denotes a partial derivative (assuming differentiability).

Our results allow us to generalize some central conclusions about education signaling to settings in which there is uncertainty about what wage the worker will receive, even conditional on the market belief about his type. Such uncertainty is, of course, plausible for many reasons, e.g., economic fluctuations during the course of one's education. Accordingly, in our specification, we allow for $w \sim F_{\mu}$, i.e., w is drawn from an exogenously-given cumulative distribution F that depends on μ, the market belief about (i.e., probability distribution over) θ. Let $V(F, e, \theta) \equiv \int_w v(w, e, \theta)\mathrm{d}F$ and $\mathcal{F} \equiv \{F_{\mu}\}_{\mu}$ be the family of feasible wage distributions. We assume that for any two beliefs μ and μ', if μ (strictly) *support-dominates* μ' (i.e., $\inf \operatorname{Supp}[\mu] \geq (>) \sup \operatorname{Supp}[\mu']$), then $(\forall e, \theta)\ V(F_{\mu}, e, \theta) \geq (>) V(F_{\mu'}, e, \theta)$. This is a weak sense in which the worker wants to convince the market that his type is higher; it is assured if $v(w, e, \theta)$ is strictly increasing in w and (strictly) support-dominating beliefs imply (strictly) first-order stochastically dominating wage distributions, but in general two wage distributions F_{μ} and $F_{\mu'}$ need not be ordered in any standard sense. We assume that $v(w, e, \theta)$ is strictly decreasing in e.

A (weak Perfect Bayesian) equilibrium is described by a pair of functions $\sigma^*(\cdot)$ and $\mu^*(\cdot)$, where for each θ, $\sigma^*(\theta)$ is type θ's probability distribution over education, and for each education level e, $\mu^*(e)$ is the market belief about the worker's type when that education is observed. For notational and technical simplicity, we will restrict attention to equilibria in which for all θ, $\sigma^*(\theta)$ has countable support. When the equilibrium is pure we write $e^*(\theta)$ instead of $\sigma^*(\theta)$. A pure-strategy equilibrium exists: all types pool on $e = 0$ and off-path beliefs are the same as the prior.

A fundamental conclusion of the standard model is that in any equilibrium higher types acquire more education. Our results deliver this conclusion in our specification; see Liu and Pei (2017) for related work. We say that a strategy σ is *increasing* if for all $\underline{\theta} < \overline{\theta}$, $\sigma(\overline{\theta})$ support-dominates $\sigma(\underline{\theta})$. In other words, a strategy is increasing if a higher type never acquires (with positive probability) strictly less education than a lower type.

Claim 4. *Assume $v(a, \theta) \equiv v(w, e, \theta)$ has the form stated in Theorem 2, and hence has SSCED. If*

$$F, G \in \mathcal{F}, \overline{e} \neq \underline{e} \implies (\exists \theta)\ V(F, \overline{e}, \theta) \neq V(G, \underline{e}, \theta), \tag{10}$$

then in any equilibrium $\sigma^*(\theta)$ is increasing.

The role of SSCED in the claim is to ensure that every equilibrium is monotonic (either increasing or decreasing). The fact that equilibria must then be increasing essentially stems from our assumptions that more education is more costly and higher beliefs are preferred. Condition (10) is a richness condition that plays an analogous role to the sender-minimality requirement in Claim 1; given SSCED, it is automatically satisfied if the worker's utility is separable in wage and education.[28]

Proof of Claim 4. Suppose, to contradiction, that $\sigma^*(\theta)$ is not increasing. Then there exist $\underline{\theta} < \overline{\theta}$ and $\underline{e} < \overline{e}$ such that $\min\{\sigma^*(\overline{e}|\underline{\theta}), \sigma^*(\underline{e}|\overline{\theta})\} > 0$. Let $\underline{F}$ and $\overline{F}$ be the wage distributions resulting from $\underline{e}$ and $\overline{e}$ respectively and let $S \equiv \{(\underline{e}, \underline{F}), (\overline{e}, \overline{F})\}$. Condition (10) implies that $(\underline{e}, \underline{F})$ and $(\overline{e}, \overline{F})$ are utility-distinguishable; moreover, $(\underline{e}, \underline{F}) \in C(\overline{\theta}, S)$ and $(\overline{e}, \overline{F}) \in C(\underline{\theta}, S)$. As v has SSCED, Part 2 of Theorem 0 implies that $\inf\{\theta : (\underline{e}, \underline{F}) \in C(\theta, S)\} \geq \sup\{\theta : (\overline{e}, \overline{F}) \in C(\theta, S)\}$; otherwise, there would be a non-interval selection. It follows that $\mu^*(\underline{e})$ support-dominates $\mu^*(\overline{e})$. Given the support-dominance, type $\underline{\theta}$ can profitably deviate from switching mass from $\overline{e}$ to $\underline{e}$, because the reduction in education is strictly preferred and the change in market belief is weakly preferred, a contradiction. *Q.E.D.*

We can also study when there is a separating equilibrium. Let F_θ denote the wage distribution when the market puts probability one on θ.

Claim 5. *Let $\Theta \equiv \{\theta_1, \theta_2, \ldots\}$, either finite or infinite. If $v(a, \theta) \equiv v((w, e), \theta)$ has the form stated in Theorem 2 and hence has SSCED, and in addition:*

1. *$v(w, e, \theta)$ is continuous in e,*
2. *$\lim_{e\to\infty} V(F_{\theta_n}, e, \theta_{n-1}) = -\infty$ for all $n > 1$, and*
3. *for all $n > 1$ and $\overline{e} > \underline{e}$:*

$$V(F_{\theta_n}, \overline{e}, \theta_{n-1}) = V(F_{\theta_{n-1}}, \underline{e}, \theta_{n-1}) \implies V(F_{\theta_n}, \overline{e}, \theta_n) \geq V(F_{\theta_{n-1}}, \underline{e}, \theta_n), \tag{11}$$

then there is a pure-strategy equilibrium in which $e^*(\theta)$ is strictly increasing.

The conditions in the result above are related to those in Cho and Sobel (1990). Part 2 of the Claim merely requires that no type θ_n $(n > 1)$ would acquire arbitrarily high education to shift the market's belief from probability one on θ_{n-1} to probability one on θ_n. Despite a resemblance, Condition (11) is not by itself a single-crossing condition; rather, it augments SSCED to provide the requisite direction of single crossing in the proof below.

[28] Suppose v has SSCED and is separable in w and e, so that it has the form $v(w, e, \theta) = g_1(w)f_1(\theta) + g_2(e)f_2(\theta) + c(\theta)$. Fix any $F, G \in \mathcal{F}$ and $\overline{e} \neq \underline{e}$. We compute the expectational difference $V(F, \overline{e}, \theta) - V(G, \underline{e}, \theta) = \left[\int g_1(w)\mathrm{d}F - \int g_1(w)\mathrm{d}G\right] f_1(\theta) + [g_2(\overline{e}) - g_2(\underline{e})]\, f_2(\theta)$. The maintained assumption that v is strictly decreasing in e implies $g_2(\overline{e}) - g_2(\underline{e}) \neq 0$. As strict ratio ordering of f_1 and f_2 implies they are linearly independent, it follows that the expectational difference is non-zero for some θ.

Proof of Claim 5. Set $e_1 = 0$. For $n > 1$, inductively construct e_n as the solution to

$$V(F_{\theta_n}, e_n, \theta_{n-1}) = V(F_{\theta_{n-1}}, e_{n-1}, \theta_{n-1}). \tag{12}$$

Our assumptions ensure there is a unique solution and that $e_n > e_{n-1}$ for all $n > 1$.

We claim $(\forall n)\ e^*(\theta_n) = e_n$ can be supported as an equilibrium. To see this, take any $n' > n$. Conditions (11) and (12) imply $V(F_{\theta_{n'}}, e_{n'}, \theta_{n'}) \geq V(F_{\theta_{n'-1}}, e_{n'-1}, \theta_{n'})$ and $V(F_{\theta_{n'}}, e_{n'}, \theta_{n'-1}) = V(F_{\theta_{n'-1}}, e_{n'-1}, \theta_{n'-1})$, and hence SSCED implies $V(F_{\theta_{n'-1}}, e_{n'-1}, \theta_n) \geq V(F_{\theta_{n'}}, e_{n'}, \theta_n)$. A symmetric argument establishes that for $n' < n$, $V(F_{\theta_{n'+1}}, e_{n'+1}, \theta_n) \geq V(F_{\theta_{n'}}, e_{n'}, \theta_n)$. Therefore, no type has a profitable deviation to any on-path effort. Off-path deviations can be deterred by simply setting off-path beliefs to put probability one on θ_1. *Q.E.D.*

5. Information Design: SCED with Linear Restrictions

In this subsection, we derive an extension of SCED when there is a restricted set of distributions over the set A.[29] The restriction we study is motivated by the recent literature on Bayesian persuasion (Kamenica and Gentzkow, 2011) or, more broadly, information design (Bergemann and Morris, 2017). Take A to be the set of beliefs about some random variable ω. When beliefs or posteriors about ω are generated by Bayesian updating, it is well known that every feasible distribution of posteriors has the same expectation: the prior distribution of ω. Hence, choices among experiments—distribution of posteriors—only involve those subsets of ΔA whose elements have the same expectation. Plainly, given any function $v(a,\theta)$ that satisfies SCED, the function $v(a,\theta) + f(a,\theta)$ with $f(\cdot,\theta)$ linear for each θ will satisfy a weaker form of SCED that only considers the aforementioned subsets of ΔA. We establish below in Proposition 3 that this is essentially the only change to Theorem 1. At the end of the subsection, we mention applications of the result.

Formally, let Ω be an arbitrary set. We refer to $\Delta\Omega$ as the set of beliefs or posteriors with finite support and $\Delta\Delta\Omega$ as the set of experiments with finite support. Let p denote a generic posterior and Q denote a generic experiment. We write δ_ω and δ_p for a degenerate belief on ω and a degenerate experiment on p as usual. The *average* of an experiment $Q \in \Delta\Delta\Omega$ is the posterior $\overline{Q} \in \Delta\Omega$ defined by $\overline{Q} \equiv \int_{\Delta\Omega} p \mathrm{d}Q$.

Let $v : \Delta\Omega \times \Theta \to \mathbb{R}$ denote a utility function, where for each θ, $v(\cdot,\theta)$ captures type θ's preferences over posteriors in a reduced form. Note that $v(\cdot,\theta)$ is convex when induced by a single-person decision problem, while it can have arbitrary shape when induced by

[29] Smith (2011) provides sufficient conditions for a single-crossing property of the expectational difference between lotteries and their certainty equivalents.

strategic settings such as those in Kamenica and Gentzkow (2011). Let $V : \Delta\Delta\Omega \times \Theta \to \mathbb{R}$ be the expected utility function corresponding to v.

Definition 7. The utility function $v : \Delta\Omega \times \Theta \to \mathbb{R}$ has **single-crossing expectational differences over experiments** (SCED-X) if for every $Q, R \in \Delta\Delta\Omega$ with $\overline{Q} = \overline{R}$, the expected utility difference $D_{Q,R}(\theta) \equiv V(Q,\theta) - V(R,\theta)$ is single crossing in θ.

The above definition of SCED-X is simply the analog of SCED specialized to the current setting of experiments, with the twist that one only considers expected utility differences between experiments with the same average.

Proposition 3. *The function $v : \Delta\Omega \times \Theta \to \mathbb{R}$ has SCED-X if and only if*

$$v(p,\theta) = g_1(p)f_1(\theta) + g_2(p)f_2(\theta) + \sum_{\omega\in\Omega} v(\delta_\omega,\theta)p(\omega), \tag{13}$$

for some $f_1, f_2 : \Theta \to \mathbb{R}$ each single crossing and ratio ordered, and $g_1, g_2 : \Delta\Omega \to \mathbb{R}$.

Proposition 3 parallels Theorem 1. The $c(\theta)$ term from Equation 8 is subsumed into the summation term in Equation 13 because each p is a posterior. This summation term, being linear in the posterior, does not affect the expected utility difference between any two experiments with the same average. This means that if v has SCED-X, then insofar as comparisons between pairs of experiments with the same average go, there is a representation of preferences that has SCED. The heart of Proposition 3, then, is in establishing that the restriction to comparing experiments with the same average introduces no further flexibility into the form of v.

As mentioned earlier, when there is a prior on Ω, say p^*, Bayesian updating restricts the agent's choice among experiments to (subsets of) those with average p^*. To capture this point, one would require the difference $D_{Q,R}(\theta)$ to be single crossing only when $\overline{Q} = \overline{R} = p^*$; call this requirement *SCED-X with prior p^**. It is clear that SCED-X implies SCED-X with any prior. If $\mathrm{Supp}[p^*] = \Omega$, then the converse also holds: the characterization in Proposition 3 is necessary for SCED-X with prior p^*. For an intuition, take any experiments Q and R with average $p \neq p^*$. Since p^* has full support, $p^* = \alpha p + (1-\alpha)q$ for some $q \in \Delta\Omega$ and $\alpha \in (0,1)$. The mixture experiments $Q^* = \alpha Q + (1-\alpha)\delta_q$ and $R^* = \alpha R + (1-\alpha)\delta_q$ have average p^*, and hence the difference $D_{Q,R}(\theta) = D_{Q^*,R^*}(\theta)/\alpha$ is single crossing.

The logic behind Proposition 3 generalizes to other kinds of linear restrictions on pairs of distributions. Let $v : A \times \Theta \to \mathbb{R}$ and $h : A \to \mathbb{R}^k$. Consider the requirement that $D_{P,Q}(\theta)$ be single crossing for all pairs $P, Q \in \Delta A$ such that $\int_A h(a)\mathrm{d}P = \int_A h(a)\mathrm{d}Q$. (For instance,

$A \subseteq \mathbb{R}$ and $h(a) = a^2$ impose the equality of second moments.) It can be shown that this requirement holds if and only if

$$v(a, \theta) = u(a, \theta) + h(a) \cdot \tilde{f}(\theta),$$

with u satisfying SCED, $\tilde{f} : \Theta \to \mathbb{R}^k$, and $\cdot$ being the dot product. Moreover, this characterization also holds when the requirement is weakened to only consider pairs $P, Q \in \Delta A$ such that $\int_A h(a)\mathrm{d}P = \int_A h(a)\mathrm{d}Q = \varphi$, for any fixed $\varphi \in \mathbb{R}^k$ that is in the relative interior of the convex hull of $\bigcup_{a \in A} h(a)$. When $A = \Delta\Omega$ and $h(a) = a$ these requirements reduce to SCED-X and SCED-X with a fixed full-support prior, respectively.

To illustrate the use of Proposition 3, consider Kamenica and Gentzkow's (2011) leading prosecutor-judge example. Let $\theta \in [0, 1]$ denote the judge's threshold belief for conviction (or more pertinently, the prosecutor's view about this threshold). The prosecutor's von Neumann-Morgenstern utility is $v(p, \theta) = \mathbb{1}_{\{p \geq \theta\}}$, where $p \in [0, 1]$ is the probability that the defendant is guilty. It is not hard to verify that this function cannot be written in the form (13), and hence does not have SCED-X.[30] The logic of Theorem 0 implies that there will be choice problems—the prosecutor must choose from some given subset of experiments—in which two types of the prosecutor, say θ_l and θ_h, will both optimally choose one experiment, while an intermediate type $\theta_m \in (\theta_l, \theta_h)$ will choose a different experiment. The failure of SCED-X in this example is also at the core of Silbert (2018), who identifies the non-existence of a Condorcet Winner in a collective-choice version of this example.

On the other hand, consider a different example in which $|\Omega| = 2$ (for simplicity, so we can again view posteriors as $p \in [0, 1]$), $\Theta \subseteq \mathbb{R}$, and $v(p, \theta) = \theta g_1(p) + g_2(p) + f(\theta)p$, for some $g_1, g_2 : [0, 1] \to \mathbb{R}$ and $f : \Theta \to \mathbb{R}$. For instance, an agent chooses an experiment whose realization will be observed by two parties each of whom takes an action that depends on the posterior; the agent's weighted utility from each party's action is captured by each of the first two terms of v. The third, linear term captures the agent's own action-independent

[30] Consider four posteriors p_1, p_2, p_3, p_4 and three types $\theta_1, \theta_2, \theta_3$. Analogous to the discussion in the paragraph after Theorem 1, v cannot be written in the form (13) if there is an invertible 4×4 matrix M with elements $M_{i,j} \equiv \begin{cases} p_j & \text{if } i = 1 \\ v(p_j, \theta_{i-1}) - v(0, \theta_{i-1}) & \text{if } i \geq 2 \end{cases}$. In the current example, for any $0 < p_1 < \theta_1 < p_2 < \theta_2 < p_3 < \theta_3 < p_4$, this matrix is

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is invertible because it is upper triangular with a non-zero diagonal. We note that because of the simplicity of this example, it is also straightforward to directly find experiments such that the expected utility difference is not single-crossing in type.

preference over states (e.g., anticipatory utility over health or election outcomes). In general, this specification will not satisfy SCED but, by Proposition 3, it does satisfy SCED-X. Theorem 0 implies that, no matter the feasible set of experiments, the set of agent types that choose any experiment will be interval closed.

6. Discussion

6.1. Monotone Comparative Statics

There is a sense in which a choice correspondence over a set X inducing an interval structure is intimately related to monotone comparative statics holding with respect to *some* order over X. In light of Theorem 0, the connection is elucidated below by tying (S)SCD to monotone comparative statics. Our monotonicity theorems (Theorem 3 and Theorem 4 below) relate to, but are distinct from, the influential Theorem 4 and Theorem 4′ of Milgrom and Shannon (1994).

Choice Monotonicity. Throughout this subsection, we consider an ordered set of alternatives, $(X, \succeq)$, and, as earlier, an ordered set of types, $(\Theta, \leq)$. We maintain the assumption that Θ contains upper and lower bounds for every pair of its elements. Neither $\succeq$ nor $\leq$ need be complete. We are interested in comparative statics for a function $f : X \times \Theta \to \mathbb{R}$ such that X is **minimal** (with respect to f):

$$(\forall x \neq x')(\exists \theta)\, f(x, \theta) \neq f(x', \theta).$$

For any $x, y \in X$, let $x \vee y$ and $x \wedge y$ denote the usual join and meet respectively.[31] In general, neither need exist. Given any $Y, Z \subseteq X$, we say that Y dominates Z in the **strong set order**, denoted $Y \succeq_{SSO} Z$, if for every $y \in Y$ and $z \in Z$, (i) $y \vee z$ and $y \wedge z$ exist, and (ii) $y \vee z \in Y$ and $y \wedge z \in Z$.

Remark 2. The strong set order is neither reflexive nor transitive: for any $S \subseteq X$, it holds that $S \succeq_{SSO} \emptyset$ and $\emptyset \succeq_{SSO} S$, whereas $S \succeq_{SSO} S$ if and only if $(S, \succeq)$ is a lattice (i.e., each pair of elements in S has a join and meet in S). However, the strong set order is transitive on non-empty subsets of $(X, \succeq)$. While this transitivity is well-known when $(X, \succeq)$ is a lattice, it can be shown to be a general property. □

[31] $z \in X$ is the the join (or supremum) of $\{x, y\}$ if (i) $z \succeq x$ and $z \succeq y$, and (ii) if $w \succeq x$ and $w \succeq y$, then $w \succeq z$. The meet (or infimum) of $\{x, y\}$ is defined analogously.

Definition 8. $f : X \times \Theta \to \mathbb{R}$ has **monotone comparative statics (MCS)** on $(X, \succeq)$ if

$$(\forall S \subseteq X)\ (\forall \theta_l \leq \theta_h) \quad \arg\max_{x\in S} f(x, \theta_h) \succeq_{SSO} \arg\max_{x\in S} f(x, \theta_l).$$

Our definition of MCS is closely related to but not the same as Milgrom and Shannon (1994). We take $(X, \succeq)$ to be any ordered set while they require a lattice. We focus only on monotonicity of choice in θ but require the monotonicity to hold for every subset $S \subseteq X$; Milgrom and Shannon require monotonicity of choice jointly in the pair (θ, S), but this implicitly only requires choice monotonicity in θ to hold for every sub-lattice $S \subseteq X$.

Define binary relations $\succ_{SCD}$ and $\succeq_{SCD}$ on X as follows: $x \succ_{SCD} x'$ if $D_{x,x'}(\theta) \equiv f(x, \theta) - f(x', \theta)$ is single crossing *only* from below; $x \succeq_{SCD} x'$ if either $x \succ_{SCD} x'$ or $x = x'$. It is clear that $\succeq_{SCD}$ is reflexive and anti-symmetric. If $f : X \times \Theta \to \mathbb{R}$ has SCD, then $\succeq_{SCD}$ is also transitive.[32] Moreover, given SCD, the $\succeq_{SCD}$ order is incomplete only over pairs with "dominance": if $x \not\succeq_{SCD} x'$ and vice-versa, then either $(\forall\theta)\ D_{x,x'}(\theta) > 0$, or $(\forall\theta)\ D_{x,x'}(\theta) < 0$.

Given two orders $\succeq$ and $\succeq'$ on X, the order $\succeq'$ is a refinement of $\succeq$ if

$$(\forall x, x' \in X) \quad x \succeq x' \implies x \succeq' x'.$$

Theorem 3. *$f : X \times \Theta \to \mathbb{R}$ has monotone comparative statics on $(X, \succeq)$ if and only if f has SCD and $\succeq$ is a refinement of $\succeq_{SCD}$.*

Our definition of SCD does not require an order on the set of alternatives, whereas MCS does. Theorem 3 says that SCD is necessary and sufficient for there to exist an order that yields MCS. Moreover, the Theorem justifies viewing $\succeq_{SCD}$ as the prominent order for MCS: MCS does not hold with any order that either coarsens $\succeq_{SCD}$ or reverses a ranking by $\succ_{SCD}$. The argument for necessity in Theorem 3 only makes use of binary choice sets. If SCD fails, then there is no order $\succeq$ for which there is choice monotonicity for all binary choice sets. If SCD holds, then choice monotonicity on all binary choice sets requires $\succeq$ to refine $\succeq_{SCD}$.

Regarding sufficiency, for each $S \subseteq X$, let $C_f(S) \equiv \bigcup_{\theta\in\Theta} \arg\max_{x\in S} f(x, \theta)$. Given that f has SCD and X is minimal, the set $C_f(S)$ is completely ordered by $\succeq_{SCD}$ (as elaborated in the proof of Theorem 3). Since $\succeq$ is a refinement of $\succeq_{SCD}$, $\succeq$ coincides with $\succeq_{SCD}$ on $C_f(S)$, and the strong set orders generated by $\succeq$ and $\succeq_{SCD}$ on the collection of all subsets

[32] To confirm transitivity, take $x, y, z \in X$ such that $x \succeq_{SCD} y$ and $y \succeq_{SCD} z$. If $x = y$ or $y = z$, it is trivial that $x \succeq_{SCD} z$. If $x \neq y$ and $y \neq z$, then $x \succ_{SCD} y$ and $y \succ_{SCD} z$. Since f has SCD, $D_{x,z}$ is single crossing. As both $D_{x,y}$ and $D_{y,z}$ are single crossing only from below, $D_{x,z} = D_{x,y} + D_{y,z}$ is single crossing only from below. Thus, $x \succ_{SCD} z$.

of $C_f(S)$ also coincide. By definition of $\succeq_{SCD}$, f satisfies Milgrom and Shannon's (1994) single-crossing property in (x, θ) with respect to $\succeq_{SCD}$ and $\leq$. It follows from Milgrom and Shannon (1994, Theorem 4) that $\forall \theta_l \leq \theta_h$,

$$\arg\max_{x \in S} f(x, \theta_h) = \arg\max_{x \in C_f(S)} f(x, \theta_h) \succeq_{SSO} \arg\max_{x \in C_f(S)} f(x, \theta_l) = \arg\max_{x \in S} f(x, \theta_l).$$

A stronger notion of choice monotonicity is given by the next definition.

Definition 9. $f : X \times \Theta \to \mathbb{R}$ has **monotone selection (MS)** on $(X, \succeq)$ if for any $S \subseteq X$, every selection $s^*(\theta)$ from $\arg\max_{x \in S} f(x, \theta)$ is increasing in θ.[33]

Define binary relations $\succ_{SSCD}$ and $\succeq_{SSCD}$ on X as follows: $x \succ_{SSCD} x'$ if $D_{x,x'}$ is strictly single crossing only from below; $x \succeq_{SSCD} x'$ if either $x \succ_{SSCD} x'$ or $x = x'$. As before, if $f : X \times \Theta \to \mathbb{R}$ has SSCD, then $\succeq_{SSCD}$ is an order.

Theorem 4. *$f : X \times \Theta \to \mathbb{R}$ has monotone selection on $(X, \succeq)$ if and only if f has SSCD and $\succeq$ is a refinement of $\succeq_{SSCD}$.*

Remark 3. If $f : X \times \Theta \to \mathbb{R}$ has SCD (resp., SSCD), then a *complete* refinement $\succeq$ of $\succeq_{SCD}$ (resp., $\succeq_{SSCD}$) exists, and f has MCS (resp., MS) on $(X, \succeq)$. To construct one example of a completion of $\succeq_{SCD}$, define $\succ_{dom}$ on X as follows: $x \succ_{dom} x'$ if $(\forall \theta)\ f(x, \theta) > f(x', \theta)$. Given that f has SCD, a pair $x, x' \in X$ is related by $\succ_{dom}$ if and only if it is not related by $\succeq_{SCD}$. The relation $\succeq \equiv \succeq_{SCD} \cup \succ_{dom}$ is thus complete, reflexive, and anti-symmetric; it is not hard to check that $\succeq$ is also transitive. □

Choice Monotonicity Among Lotteries. We apply Theorem 3 and Theorem 4 to our context of choice among lotteries. Let $\succ_{SCED}$ and $\succeq_{SCED}$ be the $\succ_{SCD}$ and $\succeq_{SCD}$ relations defined on $\widetilde{\Delta}A$, with respect to the expected utility function V. That is, for $P, Q \in \widetilde{\Delta}A$, $P \succ_{SCED} Q$ if $D_{P,Q}$ is single crossing only from below, and $P \succeq_{SCED} Q$ if $P \succ_{SCED} Q$ or $P = Q$. If v has SCED, then the expected utility function V has SCD and $(\widetilde{\Delta}A, \succeq_{SCED})$ is an ordered set.

Corollary 3. *V has monotone comparative statics on $(\widetilde{\Delta}A, \succeq)$ if and only if v has SCED and $\succeq$ is a refinement of $\succeq_{SCED}$. If v has SCED, then there is a complete order with respect to which V has monotone comparative statics.*

Analogously, let $\succ_{SSCED}$ and $\succeq_{SSCED}$ be the $\succ_{SSCD}$ and $\succeq_{SSCD}$ relations defined on $\widetilde{\Delta}A$, with respect to the expected utility function V. That is, for $P, Q \in \widetilde{\Delta}A$, $P \succ_{SSCED} Q$ if $D_{P,Q}$

[33] $s^*(\theta) \equiv \emptyset$ if $\arg\max_{x \in S} f(x, \theta) = \emptyset$, and we extend $\succeq$ to $X \cup \{\emptyset\}$ by stipulating $x \succeq \emptyset \succeq x$ for every $x \in X$.

is strictly single crossing only from below, and $P \succeq_{SSCED} Q$ if $P \succ_{SSCED} Q$ or $P = Q$. If v has SSCED, then the expected utility function V has SSCD and $(\widetilde{\Delta} A, \succeq_{SSCED})$ is an ordered set.

Corollary 4. *V has monotone selection on $(\widetilde{\Delta} A, \succeq)$ if and only if v has SSCED and $\succeq$ is a refinement of $\succeq_{SSCED}$. If v has SSCED, then there is a complete order with respect to which V has monotone selection.*

6.2. Single Crossing vs. Monotonicity

We have characterized when $v : A \times \Theta \to \mathbb{R}$ has SCED. Viewing ΔA as a choice set and Θ as a parameter set, SCED is an *ordinal* property. Kushnir and Liu (2018), by contrast, study the following stronger *cardinal* property, which we term **monotonic expectational differences** (MED):[34]

$$(\forall P, Q \in \Delta A) \;\; D_{P,Q}(\theta) \text{ is monotonic in } \theta.$$

Analogous to Theorem 1, MED can be characterized as follows:

Theorem 5. *The function $v : A \times \Theta \to \mathbb{R}$ has MED if and only if it takes the form*

$$v(a, \theta) = g_1(a) f_1(\theta) + g_2(a) + c(\theta), \tag{14}$$

where $f_1 : \Theta \to \mathbb{R}$ is monotonic, $g_1, g_2 : A \to \mathbb{R}$, and $c : \Theta \to \mathbb{R}$.

There is a simple intuition for Theorem 5 based on the von Neumann-Morgenstern expected utility theorem. Suppose $\Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$ and $v_\theta(a, \theta)$, the partial derivative of v with respect to θ, exists and is continuous. Consider the following strengthening of MED: $(\forall P, Q \in \Delta A)$ $D_{P,Q}(\theta)$ is either a zero function or strictly monotonic. Then, for any P and Q, sign $\left[\int_A v_\theta(a, \theta) \mathrm{d}P - \int_A v_\theta(a, \theta) \mathrm{d}Q\right]$ is independent of θ. In other words, for all θ, $v_\theta(\cdot, \theta)$ is a von Neumann-Morgenstern representation of the same preferences over lotteries. The conclusion of Theorem 5 follows from the expected utility theorem's implication that $(\forall \theta', \theta'')$ $v_\theta(\cdot, \theta')$ must be a positive affine transformation of $v_\theta(\cdot, \theta'')$.[35] We are not aware of any related argument for Theorem 1's SCED characterization.

[34] As elaborated subsequently, Kushnir and Liu actually characterize a slightly stronger property, given some additional assumptions on the environment. Kushnir and Liu call their property "increasing differences over distributions".

[35] Pick any $\theta^* \in \Theta$. The expected utility theorem implies that for some $F_1 : \Theta \to \mathbb{R}_{++}$ and $C : \Theta \to \mathbb{R}$, $(\forall a, \theta)$ $v_\theta(a, \theta) = v_\theta(a, \theta^*) F_1(\theta) + C(\theta)$. Equation 14 follows from integrating up θ at each a, as $v(a, \theta) = \int_{\underline{\theta}}^{\theta} v_\theta(a, t) \mathrm{d}t + v(a, \underline{\theta})$.

Comparing Theorem 1 and Theorem 5, a function v with SCED has MED when the function f_2 in (8) is identically equal to one; f_1 and f_2 being ratio ordered is then equivalent to f_1 being monotonic. Typically when $v(a, \theta)$ has the form (14) with $f_1(\cdot) > 0$ and domain $\Theta \subseteq \mathbb{R}_{++}$, the function $\tilde{v}(a, \theta) \equiv \theta v(a, \theta)$ will satisfy SCED but violate MED. Furthermore, the set of preferences with SCED utility representations is larger than that with MED representations; see Example 2 in Appendix G.

Remark 4. Proposition 6 in Appendix G characterizes exactly when preferences with an SCED representation have an MED representation. It is when (a) some pair of types do not share the same strict preference over every pair of lotteries, or (b) there is a pair of lotteries over which all types share the same strict preference (e.g., lotteries over money). □

The MED characterization in Theorem 5 has largely been obtained by Kushnir and Liu (2018). They restrict attention to $\Theta \subseteq \mathbb{R}$, $A \subset \mathbb{R}^k$, and functions v that have some continuity. Modulo minor differences, Kushnir and Liu establish that for their environment, a strict version of MED (in fact the strengthening discussed in the paragraph after Theorem 5) is equivalent to the characterization in Theorem 5 with f_1 strictly monotonic.[36] Kushnir and Liu's focus is on the equivalence between Bayesian and dominant-strategy implementation. Their methodology requires certain functions to have MED rather than SCED, and they do not study the relationship between MED and SCED.

Theorem 5 and Remark 4 support suggestions put forward by Duggan (2014). Although his setting is formulated a little differently and he does not obtain a necessary condition, Duggan (2014, Section 4) discusses the difficulties of finding preferences that are not representable in the form of Theorem 5 and yet satisfy MED/SCED.

7. Conclusion

The main result of this paper is a full characterization of which von Neumann-Morgenstern utility functions of outcome and type satisfy single-crossing expectational differences (SCED): the difference in expected utility between every pair of probability distributions on outcomes is single crossing in type (Theorems 1 and 2). We have established that this property is, more or less, necessary and sufficient for two kinds of comparative statics: interval choice structure (Theorem 0) and monotone comparative statics with respect to some order (Theorems 3 and 4).

We close by suggesting directions for future research.

[36] The statement in Proposition 3 of their paper is that f_1 is strictly increasing; monotonicity vs. increasing is immaterial, as the direction of monotonicity can be reversed by flipping the sign of the function g_1.

Our characterizations in Theorems 1 and 2 lean on the requirement that the expectational difference must be single crossing for *all* pairs of distributions over outcomes. More precisely, a careful inspection of our proofs reveals that it suffices to have all pairs of distributions supported on at most three outcomes. Requiring single crossing for only a subset of (these) distributions would expand the set of utility functions satisfying the requirement—as seen in our extension to information design (Section 5)—but applications must then have enough stochastic structure to validate the restriction on distributions. Another possibility would be to weaken or alter the expected utility hypothesis.

Our results have direct bearing on problems in which all types of an agent face the same choice set of distributions. Such situations arise naturally, as illustrated in Section 4. But consider a variation of the cheap-talk application (Subsection 4.1) in which the sender's type is correlated with the receiver's type. Even though the receiver's type does not affect the sender's payoff, different sender types will generally have different beliefs about the distribution of the receiver's action that any message induces in equilibrium. Effectively, different sender types will be choosing from different sets of distributions. An approach that synthesizes the current paper's with that of, for example, Athey's (2002) may be useful for such problems.

Appendices

Appendix A contains a proof of Theorem 0 (Section 2); Appendix B proofs for our core results on single crossing and SCED (Subsection 3.1); Appendix C for strict single crossing and SSCED (Subsection 3.2); Appendix D for information design (Section 5); Appendix E for monotone comparative statics (Subsection 6.1); and Appendix F for MED (Subsection 6.2). Additional connections between SCED and MED are then provided in Appendix G. We elaborate on some related literature in Appendix H. Finally, Appendix I explains how our results can be extended to settings in which the type set is equipped with a preorder rather than an order (i.e., relaxing anti-symmetry).

Before turning to the proofs, we state a useful equivalence with the violation of single crossing; the result is obvious when $(\Theta, \leq)$ is a completely ordered set but also applies when it is not.

Claim 6. *A function $f : \Theta \to \mathbb{R}$ is not single crossing if and only if for some $\theta_l < \theta_m < \theta_h$, either*

$$\operatorname{sign}[f(\theta_l)] < \operatorname{sign}[f(\theta_m)] \textit{ and } \operatorname{sign}[f(\theta_m)] > \operatorname{sign}[f(\theta_h)], \quad \textit{or} \tag{15}$$

$$\operatorname{sign}[f(\theta_l)] > \operatorname{sign}[f(\theta_m)] \textit{ and } \operatorname{sign}[f(\theta_m)] < \operatorname{sign}[f(\theta_h)]. \tag{16}$$

Proof of Claim 6. The "if" direction of the claim is immediate. For the "only if" direction, suppose $f : \Theta \to \mathbb{R}$ is single crossing neither from below nor from above:

$$(\exists \theta_1 < \theta_2) \quad \operatorname{sign}[f(\theta_1)] < \operatorname{sign}[f(\theta_2)], \text{ and}$$
$$(\exists \theta_3 < \theta_4) \quad \operatorname{sign}[f(\theta_3)] > \operatorname{sign}[f(\theta_4)].$$

Let $\Theta_0 \equiv \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and $\overline{\theta}$ and $\underline{\theta}$ be upper and lower bounds of Θ_0. If $f(\underline{\theta}) = f(\overline{\theta}) = 0$, then $(\theta_l, \theta_m, \theta_h) = (\underline{\theta}, \theta_0, \overline{\theta})$ for some $\theta_0 \in \Theta_0$ with $f(\theta_0) \neq 0$ satisfies either (15) or (16). So assume $f(\overline{\theta}) \neq 0$; an similar argument applies if $f(\underline{\theta}) \neq 0$. If $f(\overline{\theta}) < 0$, then $(\theta_l, \theta_m, \theta_h) = (\theta_1, \theta_2, \overline{\theta})$ satisfies (15). If $f(\overline{\theta}) > 0$, then $(\theta_l, \theta_m, \theta_h) = (\theta_3, \theta_4, \overline{\theta})$ satisfies (16). *Q.E.D.*

A. Proof of Theorem 0 (Section 2)

Part 1. Suppose f has SCD, and consider any $S \subseteq X$, $x' \in S$, and $\theta_l, \theta_h \in \{\theta : x' \in \arg\max_{x \in S} f(x, \theta)\}$ with $\theta_l < \theta_h$. For any $x'' \in S$, $D_{x',x''}(\theta_l) \geq 0$ and $D_{x',x''}(\theta_h) \geq 0$, which imply that $D_{x',x''}(\theta_m) \geq 0$ for all θ_m with $\theta_l < \theta_m < \theta_h$. It follows that $\{\theta : x' \in \arg\max_{x \in S} f(x, \theta)\}$ is interval closed.

If f strictly violates SCD, then there exist $x', x'' \in X$ and $\theta_l < \theta_m < \theta_h$ such that

$\min\{D_{x',x''}(\theta_l), D_{x',x''}(\theta_h)\} > 0 > D_{x',x''}(\theta_m)$. Clearly, $\{\theta : x' \in \arg\max_{x\in\{x',x''\}} f(x,\theta)\}$ is not interval closed.

Part 2. Suppose $|\Theta| \geq 3$. A function $f : X \times \Theta \to \mathbb{R}$ does not have SSCD when there exist $x', x'' \in X$ such that $D_{x',x''}$ (and $D_{x'',x'}$) are not strictly single crossing. Alternatively, using Claim 6, f does not have SSCD if and only if $(\exists x', x'')$ $(\exists \theta_l < \theta_m < \theta_h)$ $D_{x',x''}(\theta_l) \geq 0$, $D_{x',x''}(\theta_m) \leq 0$, and $D_{x',x''}(\theta_h) \geq 0$. This condition is equivalent to $(\exists S \subseteq X$ with $x', x'' \in S)$ $(\exists \theta_l < \theta_m < \theta_h)$ $x' \in \bigcap_{\theta\in\{\theta_l,\theta\}} \arg\max_{x\in S} f(x,\theta)$ and $D_{x',x''}(\theta_m) \leq 0$, which holds if and only if some selection of choices does not induce an interval structure.

B. Proofs for Single Crossing (Subsection 3.1)

B.1. Proof of Lemma 1

When $|\Theta| \leq 2$, the proof is trivial as all functions are single crossing and every pair of functions are ratio ordered. Hereafter, we assume $|\Theta| \geq 3$.

$(\implies)$ It is clear that each function f_1 and f_2 is single crossing. We must show that f_1 and f_2 are ratio ordered.

To prove (6), we suppose towards contradiction that

$$\begin{aligned} &(\exists \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) < f_1(\theta_h) f_2(\theta_l), \text{ and} \\ &(\exists \theta' < \theta'') \quad f_1(\theta') f_2(\theta'') > f_1(\theta'') f_2(\theta'). \end{aligned} \tag{17}$$

Take any upper bound $\overline{\theta}$ of $\{\theta_l, \theta_h, \theta', \theta''\}$.

First, let $\alpha_l \equiv (f_2(\theta_l), -f_1(\theta_l))$. Then $(\alpha_l \cdot f)(\theta_l) = (f_2(\theta_l), -f_1(\theta_l)) \cdot (f_1(\theta_l), f_2(\theta_l)) = 0$, and $(\alpha_l \cdot f)(\theta_h) > 0$. Thus, $\alpha_l \cdot f$ is single crossing from below and $(\alpha_l \cdot f)(\overline{\theta}) > 0$.

Second, let $\alpha' \equiv (f_2(\theta'), -f_1(\theta'))$. Then $(\alpha' \cdot f)(\theta') = 0$ and $(\alpha' \cdot f)(\theta'') < 0$. Thus, $\alpha' \cdot f$ is single crossing from above and $(\alpha' \cdot f)(\overline{\theta}) < 0$.

Let $\overline{\alpha} = (f_2(\overline{\theta}), -f_1(\overline{\theta}))$. It follows that

$$\begin{aligned} &(\overline{\alpha} \cdot f)(\theta_l) = (f_2(\overline{\theta}), -f_1(\overline{\theta})) \cdot (f_1(\theta_l), f_2(\theta_l)) = -(\alpha_l \cdot f)(\overline{\theta}) < 0, \\ &(\overline{\alpha} \cdot f)(\theta') = -(\alpha' \cdot f)(\overline{\theta}) > 0, \text{ and} \\ &(\overline{\alpha} \cdot f)(\overline{\theta}) = 0. \end{aligned}$$

Therefore, $\overline{\alpha} \cdot f$ is not single crossing, a contradiction.

To prove (7), take any $\theta_l < \theta_m < \theta_h$.

First, we show that $f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l)$ implies $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$ and $f_1(\theta_m)f_2(\theta_l) = f_1(\theta_l)f_2(\theta_m)$. Assume f_1 is not a zero function on $\{\theta_l, \theta_m, \theta_h\}$, as otherwise the proof is trivial. Since f_1 is single crossing, either $f_1(\theta_l) \neq 0$ or $f_1(\theta_h) \neq 0$. We consider the case of $f_1(\theta_h) \neq 0$ (and omit the proof for the other case, as it is analogous). Let $\alpha_h \equiv (f_2(\theta_h), -f_1(\theta_h))$. Since $\alpha_h \cdot f$ is single crossing and $(\alpha_h \cdot f)(\theta) = 0$ for $\theta = \theta_l, \theta_h$, it holds that $(\alpha_h \cdot f)(\theta_m) = f_2(\theta_h)f_1(\theta_m) - f_1(\theta_h)f_2(\theta_m) = 0$. It follows immediately that $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$. As $(f_1(\theta_m), f_2(\theta_m))$ and $(f_1(\theta_h), f_2(\theta_h))$ are linearly dependent and $(f_1(\theta_h), f_2(\theta_h))$ is a non-zero vector, there exists $\lambda \in \mathbb{R}$ such that $f_i(\theta_m) = \lambda f_i(\theta_h)$ for $i = 1, 2$. Thus,

$$f_1(\theta_l)f_2(\theta_m) = \lambda f_1(\theta_l)f_2(\theta_h) = \lambda f_2(\theta_l)f_1(\theta_h) = f_2(\theta_l)f_1(\theta_m).$$

Next, we show that if $f_1(\theta_l)f_2(\theta_m) = f_1(\theta_m)f_2(\theta_l)$ and $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$, then $f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l)$. Let $\alpha \equiv (f_2(\theta_l) - f_2(\theta_h), -f_1(\theta_l) + f_1(\theta_h))$. It follows that

$$(\alpha \cdot f)(\theta_l) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_l) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_l) = f_1(\theta_h)f_2(\theta_l) - f_1(\theta_l)f_2(\theta_h),$$
$$(\alpha \cdot f)(\theta_h) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_h) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l) - f_1(\theta_l)f_2(\theta_h), \text{ and}$$
$$(\alpha \cdot f)(\theta_m) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_m) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_m) = 0.$$

As $\alpha \cdot f$ is single crossing, it follows that $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$, as we wanted to show.

$(\impliedby)$ Assume that f_1 and f_2 are each single crossing. We provide a proof for the case in which f_1 ratio dominates f_2, and omit the other case's analogous proof. For any $\alpha \in \mathbb{R}^2$, we prove that $\alpha \cdot f$ is single crossing. We may assume that $\alpha \neq 0$, as the result is trivial otherwise.

Suppose, towards contradiction, that $\alpha \cdot f$ is not single crossing. Claim 6 implies there exist $\theta_l < \theta_m < \theta_h$ such that either,

$$\operatorname{sign}[(\alpha \cdot f)(\theta_l)] < \operatorname{sign}[(\alpha \cdot f)(\theta_m)] \text{ and } \operatorname{sign}[(\alpha \cdot f)(\theta_m)] > \operatorname{sign}[(\alpha \cdot f)(\theta_h)], \quad \text{or} \tag{18}$$
$$\operatorname{sign}[(\alpha \cdot f)(\theta_l)] > \operatorname{sign}[(\alpha \cdot f)(\theta_m)] \text{ and } \operatorname{sign}[(\alpha \cdot f)(\theta_m)] < \operatorname{sign}[(\alpha \cdot f)(\theta_h)]. \tag{19}$$

First, we consider the case in which $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ for all $\theta \in \{\theta_l, \theta_m, \theta_h\}$ are non-zero vectors. Take any $\theta_1, \theta_2 \in \{\theta_l, \theta_m, \theta_h\}$ such that $\theta_1 < \theta_2$. As f_1 ratio dominates f_2, by Condition (6), $f(\theta_1)$ moves to $f(\theta_2)$ in a clockwise rotation with an angle less than or equal to 180 degrees. Let r_{12} be the clockwise angle from $f(\theta_1)$ to $f(\theta_2)$. The vector $\alpha \neq 0$ defines a partition of $\mathbb{R}^2$ into $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, $\mathbb{R}^2_{\alpha,0} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x = 0\}$, and $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$. In both cases (18) and (19), both $f(\theta_l)$ and $f(\theta_h)$ are not in the same part of the partition that $f(\theta_m)$ belongs to. Thus, $r_{lm} > 0$ and $r_{mh} > 0$. On the other hand, both $f(\theta_l)$ and $f(\theta_h)$ are in the same closed half-space, either $\mathbb{R}^2_{\alpha,+} \cup \mathbb{R}^2_{\alpha,0}$ or $\mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$, and $f(\theta_m)$ is in the other closed half-space, either $\mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$ or $\mathbb{R}^2_{\alpha,+} \cup \mathbb{R}^2_{\alpha,0}$, respectively. Thus, $r_{lh} \geq 180$. Since Condition (6) implies $r_{lh} \leq 180$, it follows that $r_{lh} = 180$. Hence, $f(\theta_l)$ and $f(\theta_m)$ are linearly independent ($0 < r_{lm} < 180$), and similarly for $f(\theta_m)$ and $f(\theta_h)$. However, $f(\theta_l)$ and $f(\theta_h)$ are linearly dependent ($r_{lh} = 180$). This contradicts (7).

Second, suppose either $f(\theta_l) = 0$ or $f(\theta_h) = 0$. We provide the argument assuming $f(\theta_l) = 0$; it is analogous if $f(\theta_h) = 0$. Under either (18) or (19), $f(\theta_m) \neq 0$. By Condition (7), $f(\theta_m)$ and $f(\theta_h)$ are linearly dependent. In particular, because $f(\theta_m) \neq 0$, there exists a unique $\lambda \in \mathbb{R}$ such that $f(\theta_h) = \lambda f(\theta_m)$. Under either (18) or (19), $\lambda \leq 0$, which contradicts the hypothesis that f_1 and f_2 are single crossing.

Last, suppose $f(\theta_l) \neq 0$, $f(\theta_m) = 0$, and $f(\theta_h) \neq 0$. By Condition (7), $f(\theta_l)$ and $f(\theta_h)$ are linearly dependent. Hence, there exists a unique $\lambda \in \mathbb{R}$ such that $f(\theta_l) = \lambda f(\theta_h)$. Under either (18) or (19), $\lambda > 0$, which contradicts the hypothesis that f_1 and f_2 are single crossing.

B.2. Proof of Proposition 2

The result is trivial if $|X| = 1$ and it is equivalent to Lemma 1 if $|X| = 2$, so we may assume $|X| \geq 3$. The proof is also straightforward if all functions $f(x, \cdot) : \Theta \to \mathbb{R}$ are multiples of one function $f(x_1, \cdot)$, i.e., if there is x_1 such that $(\exists \lambda : X \to \mathbb{R})(\forall x) f(x, \cdot) = \lambda(x) f(x_1, \cdot)$. Thus, we further assume there exist x', x'' such that $f(x', \cdot) : \Theta \to \mathbb{R}$ and $f(x'', \cdot) : \Theta \to \mathbb{R}$ are linearly independent.

($\Longleftarrow$) Assume $f(x_1, \cdot)$ and $f(x_2, \cdot)$ are (i) each single crossing and (ii) ratio ordered, and that there are functions $\lambda_1, \lambda_2 : X \to \mathbb{R}$ such that $(\forall x)\ f(x, \cdot) = \lambda_1(x) f(x_1, \cdot) + \lambda_2(x) f(x_2, \cdot)$. Then, for any function $\mu : X \to \mathbb{R}$ with finite support,

$$\int_X f(x, \theta) \mathrm{d}\mu = \int_X \lambda_1(x) f(x_1, \theta) + \lambda_2(x) f(x_2, \theta) \mathrm{d}\mu = \sum_{i=1,2} \left(\int_X \lambda_i(x) \mathrm{d}\mu \right) f(x_i, \theta),$$

which is single crossing in θ by Lemma 1.

($\Longrightarrow$) Take any $x_1, x_2 \in X$ such that $f_1(\cdot) \equiv f(x_1, \cdot)$ and $f_2(\cdot) \equiv f(x_2, \cdot)$ are linearly independent. Then, by Lemma 1, f_1 and f_2 are each single crossing and ratio ordered, as their linear combinations are all single crossing.

For every θ', θ'', let

$$M_{\theta',\theta''} \equiv \begin{bmatrix} f_1(\theta') & f_2(\theta') \\ f_1(\theta'') & f_2(\theta'') \end{bmatrix}.$$

We first prove the following claim:

Claim 7. *There exists $\theta_l < \theta_h$ such that* $\mathrm{rank}[M_{\theta_l,\theta_h}] = 2$.

Proof of Claim 7. As f_1 and f_2 are linearly independent, there exists θ_0 such that $f_2(\theta_0) \neq 0$. Let $\lambda \equiv -\frac{f_1(\theta_0)}{f_2(\theta_0)}$. Then, for some θ_λ, $f_1(\theta_\lambda) + \lambda f_2(\theta_\lambda) \neq 0$ and $\mathrm{rank}[M_{\theta_0,\theta_\lambda}] = 2$.

The proof is complete if $\theta_0 > \theta_\lambda$ or $\theta_0 < \theta_\lambda$. If not, take a lower and upper bound, $\underline{\theta}$ and $\overline{\theta}$, of $\{\theta_0, \theta_\lambda\}$. Then $\mathrm{rank}[M_{\underline{\theta},\overline{\theta}}] = 2$. For otherwise, there exists $\alpha \in \mathbb{R}^2 \backslash \{0\}$ such that $M_{\underline{\theta},\overline{\theta}}\alpha = 0$. As θ_0 and θ_λ are between $\underline{\theta}$ and $\overline{\theta}$, and $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing, we have $M_{\theta_0,\theta_\lambda}\alpha = 0$, which contradicts $\mathrm{rank}[M_{\theta_0,\theta_\lambda}] = 2$. *Q.E.D.*

Now take any $x \in X$, the function $f_x(\cdot) \equiv f(x, \cdot)$, and θ_l, θ_h in Claim 7. As $\mathrm{rank}[M_{\theta_l,\theta_h}] = 2$, the system

$$\begin{bmatrix} f_x(\theta_l) \\ f_x(\theta_h) \end{bmatrix} = \begin{bmatrix} f_1(\theta_l) & f_2(\theta_l) \\ f_1(\theta_h) & f_2(\theta_h) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{20}$$

has a unique solution $\lambda \in \mathbb{R}^2$. We will show that $f_x = \lambda_1 f_1 + \lambda_2 f_2$.

Suppose, towards contradiction, there exists θ_λ such that

$$f_x(\theta_\lambda) \neq \lambda_1 f_1(\theta_\lambda) + \lambda_2 f_2(\theta_\lambda). \tag{21}$$

Let $\underline{\theta}$ and $\overline{\theta}$ respectively be a lower and an upper bound of $\{\theta_l, \theta_h, \theta_\lambda\}$. If $\mathrm{rank}[M_{\underline{\theta},\overline{\theta}}] < 2$, there is $\lambda' \in \mathbb{R}^2 \backslash \{0\}$ such that $\lambda'_1 f_1(\theta) + \lambda'_2 f_2(\theta) = 0$ for $\theta = \underline{\theta}, \overline{\theta}$. As $\lambda'_1 f_1 + \lambda'_2 f_2$ is single crossing, we have $\lambda'_1 f_1(\theta) + \lambda'_2 f_2(\theta) = 0$ for $\theta = \theta_l, \theta_h$, which contradicts $\mathrm{rank}[M_{\theta_l,\theta_h}] = 2$.[37]

If, on the other hand, $\mathrm{rank}[M_{\underline{\theta},\overline{\theta}}] = 2$, the system

$$\begin{bmatrix} f_x(\underline{\theta}) \\ f_x(\overline{\theta}) \end{bmatrix} = \begin{bmatrix} f_1(\underline{\theta}) & f_2(\underline{\theta}) \\ f_1(\overline{\theta}) & f_2(\overline{\theta}) \end{bmatrix} \begin{bmatrix} \lambda'_1 \\ \lambda'_2 \end{bmatrix}$$

has a unique solution $\lambda' \in \mathbb{R}^2$. As $f_x - \lambda'_1 f_1 - \lambda'_2 f_2$ is single crossing,

$$f_x(\theta_l) = \lambda'_1 f_1(\theta_l) + \lambda'_2 f_2(\theta_l) \quad \text{and} \quad f_x(\theta_h) = \lambda'_1 f_1(\theta_h) + \lambda'_2 f_2(\theta_h), \text{ and} \tag{22}$$

$$f_x(\theta_\lambda) = \lambda'_1 f_1(\theta_\lambda) + \lambda'_2 f_2(\theta_\lambda). \tag{23}$$

[37] The function $\lambda'_1 f_1 + \lambda'_2 f_2$ must be single crossing because we can consider $\mu : X \to \mathbb{R}$ such that $\mu(x_1) = \lambda'_1$, $\mu(x_2) = \lambda'_2$, and $\mu(x) = 0$ for any $x \neq x_1, x_2$. We use similar reasoning subsequently.

(22) implies that λ' solves (20). As the unique solution to (20) was λ, it follows that $\lambda' = \lambda$. But then (21) and (23) are in contradiction. Therefore, there exist $\lambda_1, \lambda_2 : X \to \mathbb{R}$ such that

$$(\forall x, \theta) \quad f(x,\theta) = \lambda_1(x) f(x_1,\theta) + \lambda_2(x) f(x_2,\theta).$$

B.3. Proof of Theorem 1

$(\impliedby)$ Suppose $v(a,\theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta) + c(\theta)$, with $f_1, f_2 : \Theta \to \mathbb{R}$ each single crossing and ratio ordered. Then, for any $P, Q \in \Delta A$,

$$D_{P,Q}(\theta) = \left[\int_A g_1(a)\mathrm{d}P - \int_A g_1(a)\mathrm{d}Q\right] f_1(\theta) + \left[\int_A g_2(a)\mathrm{d}P - \int_A g_2(a)\mathrm{d}Q\right] f_2(\theta),$$

which is single crossing by Lemma 1.

$(\implies)$ Assume, without loss of generality, that $|A| \geq 2$. Take any $a_0 \in A$, and define $A' \equiv A \setminus \{a_0\}$. Define $f : A \times \Theta \to \mathbb{R}$ as $f(a,\theta) \equiv v(a,\theta) - v(a_0,\theta)$. It is clear that $(\forall a \in A')\ f(a,\cdot)$ is single crossing: consider the expectational difference with probability distributions that put probability one on a and a_0 respectively.

We will show that, in some sense, every function $\mu' : A' \to \mathbb{R}$ can be represented as a multiple of the difference between two probability distributions $P, Q \in \Delta A$, and then apply Proposition 2.

For any function $\mu' : A' \to \mathbb{R}$, we define a function $\mu : A \to \mathbb{R}$ as an extension of μ':

$$\mu(a_0) \equiv -\sum_{a \in A'} \mu'(a), \text{ and } (\forall a \in A')\ \mu(a) \equiv \mu'(a).$$

In a sense, we let a_0 absorb the function values on A'. In particular, note that

$$\sum_{a \in A} \mu(a) = \mu(a_0) + \sum_{a \in A'} \mu(a) = 0.$$

We construct the Hahn-Jordan decomposition (μ_+, μ_-) of μ. That is, we define functions $\mu_+, \mu_- : A \to \mathbb{R}_+$ by $(\forall a \in A)\ \mu_+(a) \equiv \max\{\mu(a), 0\}$ and $\mu_-(a) \equiv -\min\{\mu(a), 0\}$. Then, μ_+ and μ_- are two positive functions with finite support such that $\mu = \mu_+ - \mu_-$. Let $M \equiv \sum_{a \in A} \mu_+(a) = \sum_{a \in A} \mu_-(a)$. If $M = 0$, pick an arbitrary $P \in \Delta A$ and let $Q = P$. If $M > 0$, define $P, Q \in \Delta A$ with probability mass functions p, q such that for any $a \in A$,

$$p(a) = \frac{\mu_+(a)}{M} \quad \text{and} \quad q(a) = \frac{\mu_-(a)}{M}.$$

Note that P and Q are probability distributions in ΔA: both are induced by positive real-valued functions μ_+ and μ_- with finite support, and $\sum_{a\in A} p(a) = \sum_{a\in A} p(a) = 1$.

It follows that

$$\begin{aligned}
\int_{A'} f(a,\theta)\mathrm{d}\mu' &= \int_A f(a,\theta)\mathrm{d}\mu \quad \text{(because } f(a_0,\theta)=0\text{)} \\
&= \int_A v(a,\theta)\mathrm{d}\mu - v(a_0,\theta)\mu(A) \\
&= \int_A v(a,\theta)\mathrm{d}\mu_+ - \int_A v(a,\theta)\mathrm{d}\mu_- \quad \text{(as } \mu(A)=0\text{)} \\
&= MD_{P,Q}(\theta).
\end{aligned}$$

Thus, if v has SCED, then $f : A' \times \Theta \to \mathbb{R}$ is linear combinations SC-preserving. By Proposition 2, there exist $a_1, a_2 \in A'$ and $\lambda_1, \lambda_2 : A' \to \mathbb{R}$ such that (i) $f(a_1,\theta)$ and $f(a_2,\theta)$ are each single crossing and ratio ordered, and (ii) $(\forall a \in A')$ $f(a,\cdot) = \lambda_1(a)f(a_1,\cdot) + \lambda_2(a)f(a_2,\cdot)$. Hence, there exist functions $g_1, g_2 : A \to \mathbb{R}$ with $g_1(a_0) = g_2(a_0) = 0$ such that $(\forall a \in A)$ $f(a,\cdot) = g_1(a)f(a_1,\cdot) + g_2(a)f(a_2,\cdot)$, or equivalently,

$$(\forall a,\theta) \quad v(a,\theta) = g_1(a)f(a_1,\theta) + g_2(a)f(a_2,\theta) + v(a_0,\theta).$$

B.4. Proof of Proposition 1

If $|\Theta| \le 2$, then the proof is trivial, so assume $|\Theta| \ge 3$. The "if" direction of the result follows directly from Theorem 1: if v has a positive affine transformation $\hat{v}$ of the form (9), then v, as a positive affine transformation of $\hat{v}$, has SCED.

For the "only if" direction, take any v that has SCED. Following the form given in Theorem 1, a positive affine transformation of v is

$$\hat{v}(a,\theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta),$$

where $f_1, f_2 : \Theta \to \mathbb{R}$ are each single crossing and ratio ordered.

First, we consider the case in which $f(\underline{\theta})$ and $f(\overline{\theta})$ are linearly dependent. Assume, with a positive affine transformation of $\hat{v}$, that the length of the vector $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ in $\mathbb{R}^2$ is either 0 or 1 for every θ. If $f(\underline{\theta}) = f(\overline{\theta}) = 0$, then because f_1 and f_2 are each single crossing, we have $(\forall\theta)$ $f_1(\theta) = f_2(\theta) = 0$ and $(\forall a,\theta)$ $\hat{v}(a,\theta) = 0$. We can easily now rewrite $\hat{v}$ in the form (9), with $\lambda : \Theta \to [0,1]$ increasing.

Suppose $f(\overline{\theta}) \neq 0$; we omit the analogous proof for the case of $f(\underline{\theta}) \neq 0$. By Condition

(7) of ratio ordering, for every θ, the vector $f(\theta) \in \mathbb{R}^2$ is linearly dependent of $f(\overline{\theta})$. As $(\forall\theta)\ \|f(\theta)\| \in \{0,1\}$, there exists $\lambda : \Theta \to \{-1,0,1\}$ such that $(\forall\theta)\ f(\theta) = \lambda(\theta)f(\overline{\theta})$. Note that λ is increasing because f_1 and f_2 are each single crossing. If either $\lambda(\underline{\theta}) = 0$ (and so $(\forall a)\ \hat{v}(a,\underline{\theta}) = 0$) or $\lambda(\underline{\theta}) = 1$ (and so $(\forall\theta)\ \lambda(\theta) = 1$), then

$$\hat{v}(a,\theta) = \lambda(\theta)\hat{v}(a,\overline{\theta}) + (1-\lambda(\theta))\hat{v}(a,\underline{\theta}),$$

with the last term equal to zero. If, on the other hand, $\lambda(\underline{\theta}) = -1$, then

$$\begin{aligned}\hat{v}(a,\theta) = \lambda(\theta)\hat{v}(a,\overline{\theta}) &= \frac{\lambda(\theta)+1}{2}\hat{v}(a,\overline{\theta}) + \frac{\lambda(\theta)-1}{2}\left(-\hat{v}(a,\underline{\theta})\right) \\ &= \frac{\lambda(\theta)+1}{2}\hat{v}(a,\overline{\theta}) + \left(1 - \frac{\lambda(\theta)+1}{2}\right)\hat{v}(a,\underline{\theta}).\end{aligned}$$

Next, suppose that the vectors $f(\underline{\theta}), f(\overline{\theta}) \in \mathbb{R}^2$ are linearly independent, so the angle between the vectors is strictly less than 180 degrees. As f_1 and f_2 are ratio ordered, for each θ there exists $\alpha(\theta), \beta(\theta) \in \mathbb{R}_+$ such that

$$f(\theta) = \alpha(\theta)f(\overline{\theta}) + \beta(\theta)f(\underline{\theta}),$$

or equivalently,

$$\hat{v}(a,\theta) = \alpha(\theta)\hat{v}(a,\overline{\theta}) + \beta(\theta)\hat{v}(a,\underline{\theta}).$$

By Condition (7), $f(\theta) \neq 0$, which implies that $\alpha(\theta)+\beta(\theta) > 0$. A positive affine transformation of dividing $\hat{v}(\cdot,\theta)$ by $\alpha(\theta)+\beta(\theta)$ results in the form (9), where $\lambda(\theta) \equiv \frac{\alpha(\theta)}{\alpha(\theta)+\beta(\theta)} \in [0,1]$.

To prove that the function λ is increasing, take θ_1, θ_2 such that $\underline{\theta} \leq \theta_1 \leq \theta_2 \leq \overline{\theta}$. To reduce notation below, let $\alpha_i \equiv \alpha(\theta_i)$ and $\beta_i \equiv \beta(\theta_i)$ for $i = 1,2$. We must show that $\frac{\alpha_1}{\alpha_1+\beta_1} \leq \frac{\alpha_2}{\alpha_2+\beta_2}$, or equivalently that $\alpha_1\beta_2 \leq \alpha_2\beta_1$. Suppose f_1 ratio dominates f_2; the other case is analogous. Then $f_1(\theta_1)f_2(\theta_2) \leq f_1(\theta_2)f_2(\theta_1)$, and hence

$$\left(\alpha_1 f_1(\overline{\theta}) + \beta_1 f_1(\underline{\theta})\right)\left(\alpha_2 f_2(\overline{\theta}) + \beta_2 f_2(\underline{\theta})\right) \leq \left(\alpha_2 f_1(\overline{\theta}) + \beta_2 f_1(\underline{\theta})\right)\left(\alpha_1 f_2(\overline{\theta}) + \beta_1 f_2(\underline{\theta})\right),$$

or equivalently,

$$(\alpha_1\beta_2 - \alpha_2\beta_1)\left(f_1(\overline{\theta})f_2(\underline{\theta}) - f_1(\underline{\theta})f_2(\overline{\theta})\right) \leq 0.$$

Note that $f_1(\overline{\theta})f_2(\underline{\theta}) - f_1(\underline{\theta})f_2(\overline{\theta}) > 0$ because f_1 ratio dominates f_2, and $f(\underline{\theta})$ and $f(\overline{\theta})$ are linearly independent. Hence, $\alpha_1\beta_2 \leq \alpha_2\beta_1$.

B.5. Proof of Corollary 1

It is clear from Theorem 1 that $v(a,\theta) = -|a-\theta|^2 = -a^2+2a\theta-\theta^2$ has SCED, as $f_1(\theta) = -1$ and $f_2(\theta) = 2\theta$ are each single crossing and ratio ordered, and we take $g_1(a) = a^2$, $g_2(a) = a$, and $c(\theta) = -\theta^2$.

For the converse, it is sufficient to prove the following claim.

Claim 8. *If there exist $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ and $f_1, f_2, c : \Theta \to \mathbb{R}$ such that*

$$v(a,\theta) \equiv -|a-\theta|^z = g_1(a)f_1(\theta) + g_2(a)f_2(\theta) + c(\theta),$$

then $z = 2$.

Proof of Claim 8. Fix $a_0 \in \mathbb{R}$ and define $\tilde{v}(a,\theta) \equiv v(a,\theta) - v(a_0,\theta) = \tilde{g}_1(a)f_1(\theta) + \tilde{g}_2 f_2(\theta)$, where $\tilde{g}_1(a) \equiv g_1(a) - g_1(a_0)$ and $\tilde{g}_2 \equiv g_2(a) - g_2(a_0)$. Fix any $\theta_l < \theta_m < \theta_h$. There exists $(\lambda_l, \lambda_m, \lambda_h) \in \mathbb{R}^3 \backslash \{0\}$ such that

$$\begin{bmatrix} f_1(\theta_l) & f_1(\theta_m) & f_1(\theta_h) \\ f_2(\theta_l) & f_2(\theta_m) & f_2(\theta_h) \end{bmatrix} \begin{bmatrix} \lambda_l \\ \lambda_m \\ \lambda_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, for every $a \in \mathbb{R}$,

$$\begin{aligned} h(a) &\equiv \lambda_l \tilde{v}(a,\theta_l) + \lambda_m \tilde{v}(a,\theta_m) + \lambda_h \tilde{v}(a,\theta_h) \\ &= \begin{bmatrix} \tilde{g}_1(a) & \tilde{g}_2(a) \end{bmatrix} \begin{bmatrix} f_1(\theta_l) & f_1(\theta_m) & f_1(\theta_h) \\ f_2(\theta_l) & f_2(\theta_m) & f_2(\theta_h) \end{bmatrix} \begin{bmatrix} \lambda_l \\ \lambda_m \\ \lambda_h \end{bmatrix} = 0. \end{aligned}$$

We hereafter consider $\lambda_l \neq 0$ (and omit the proofs for the other two cases, $\lambda_m \neq 0$ and $\lambda_h \neq 0$, which are analogous). The previous equation implies that for any $a \in \mathbb{R}$,

$$\tilde{v}(a,\theta_l) = -\frac{\lambda_m}{\lambda_l}\tilde{v}(a,\theta_m) - \frac{\lambda_h}{\lambda_l}\tilde{v}(a,\theta_h). \tag{24}$$

At any $a < \theta$, $\tilde{v}(a,\theta) = -(\theta - a)^z - v(a_0,\theta)$ is differentiable in a, and hence (24) implies that the partial derivative $\tilde{v}_a(a,\theta_l)$ exists at $a = \theta_l$. Thus, the right partial derivative $\lim_{\varepsilon\downarrow 0} \frac{\tilde{v}(\theta_l+\varepsilon,\theta_l)-\tilde{v}(\theta_l,\theta_l)}{\varepsilon} = -\lim_{\varepsilon\downarrow 0} \varepsilon^{z-1}$ must equal the left partial derivative $\lim_{\varepsilon\downarrow 0} \frac{\tilde{v}(\theta_l-\varepsilon,\theta_l)-\tilde{v}(\theta_l,\theta_l)}{-\varepsilon} = \lim_{\varepsilon\downarrow 0} \varepsilon^{z-1}$, which implies $\lim_{\varepsilon\downarrow 0} \varepsilon^{z-1} = 0$, and thus $z > 1$.

Now suppose to contradiction that $z \neq 2$. At any $a > \theta_h$, (24) and $\tilde{v}(a,\theta) = -(a-\theta)^z -$

$v(a_0,\theta)$ imply

$$-\lambda_l(a-\theta_l)^z = \lambda_m(a-\theta_m)^z + \lambda_h(a-\theta_h)^z + (\lambda_m+\lambda_h-\lambda_l)v(a_0,\theta),$$

and hence, differentiating with respect to a and simplifying using $z>1$ and $z\neq 2$:

$$-\lambda_l(a-\theta_l)^{z-1} = \lambda_m(a-\theta_m)^{z-1} + \lambda_h(a-\theta_h)^{z-1}, \tag{25}$$
$$-\lambda_l(a-\theta_l)^{z-2} = \lambda_m(a-\theta_m)^{z-2} + \lambda_h(a-\theta_h)^{z-2}, \tag{26}$$
$$-\lambda_l(a-\theta_l)^{z-3} = \lambda_m(a-\theta_m)^{z-3} + \lambda_h(a-\theta_h)^{z-3}. \tag{27}$$

It follows that $\lambda_m\lambda_h \neq 0$: if, for example, $\lambda_m = 0$, then (25) implies $\lambda_h \neq 0$ (as $\lambda_l \neq 0$), and then (25) and (26) imply $a-\theta_l = a-\theta_h$ for all $a>\theta_h$, contradicting $\theta_l<\theta_h$. Since $\left((a-\theta_l)^{z-2}\right)^2 = (a-\theta_l)^{z-1}(a-\theta_l)^{z-3}$, we manipulate the right-hand sides of (25)–(27) to obtain

$$2\lambda_m\lambda_h(a-\theta_m)^{z-2}(a-\theta_h)^{z-2} = \lambda_m\lambda_h\left((a-\theta_m)^{z-1}(a-\theta_h)^{z-3} + (a-\theta_m)^{z-3}(a-\theta_h)^{z-1}\right),$$

which simplifies, using $\lambda_m\lambda_h\neq 0$, to

$$2 = \frac{a-\theta_h}{a-\theta_m} + \frac{a-\theta_m}{a-\theta_h}.$$

Therefore, $a-\theta_h = a-\theta_m$ for all $a>\theta_h$, contradicting $\theta_m<\theta_h$. *Q.E.D.*

B.6. Proof of Corollary 2

We only prove part (1) of the corollary, as it implies part (2).

If f is monotonic, it ratio dominates any positive constant function. It follows from Theorem 1 that $v((q,t),\theta)$ has SCED.

To prove the converse, suppose, towards contradiction, that f is not monotonic. Then there exist $\theta_l<\theta_m<\theta_h$ such that either $f(\theta_m)>\max\{f(\theta_l),f(\theta_h)\}$ or $f(\theta_m)<\min\{f(\theta_l),f(\theta_h)\}$. Let us assume the first of these two cases; the argument is analogous for the other case. Take any $z\in\mathbb{R}$ such that $f(\theta_m)>z>\max\{f(\theta_l),f(\theta_h)\}$, any $q_1,q_2\in\mathbb{R}$ such that $g(q_1)-g(q_2)>0$, and $t_1,t_2\in\mathbb{R}$ such that $(g(q_1)-g(q_2))z-(t_1-t_2)=0$. The expectational difference between degenerate lotteries on $a_1=(q_1,t_1)$ and $a_2=(q_2,t_2)$,

$$D_{1,2}(\theta)\equiv v(a_1,\theta)-v(a_2,\theta) = (g(q_1)-g(q_2))f(\theta)-(t_1-t_2)$$

is not single crossing as $D_{1,2}(\theta_m)>0>\max\{D_{1,2}(\theta_l),D_{1,2}(\theta_h)\}$.

C. Proofs for Strict Single Crossing (Subsection 3.2)

Similar to the proof of Theorem 1, our proof of Theorem 2 requires conditions ensuring that arbitrary linear combinations of functions are strictly single crossing. We state and discuss the analogs of Lemma 1 and Proposition 2 below in Appendix C.1; their proofs are in Appendix C.2 and Appendix C.3 respectively. The proof of Theorem 2 then follows in Appendix C.4.

C.1. Aggregating Strictly Single-Crossing Functions

Lemma 2. *Let $f_1, f_2 : \Theta \to \mathbb{R}$. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is strictly single crossing $\forall \alpha \in \mathbb{R}^2 \setminus \{0\}$ if and only if f_1 and f_2 are strictly ratio ordered.*

Besides the change to strict single crossing and, correspondingly, strict ratio ordering, Lemma 2 has two other differences from Lemma 1. First, we rule out $(\alpha_1, \alpha_2) = 0$; this is unavoidable because a zero function is not strictly single crossing. Second, and more important, there is no explicit mention in Lemma 2 that f_1 and f_2 are each strictly single crossing. It turns out—as elaborated in the Lemma's proof—that when two functions are strictly ratio ordered, each of them must be strictly single crossing.

To extend Lemma 2 to more than two functions, we say that $f : X \times \Theta \to \mathbb{R}$ is **linear combinations SSC-preserving** if $\int_X f(x, \theta) \mathrm{d}\mu$ is either a zero function or strictly single crossing in θ for every function $\mu : X \to \mathbb{R}$ with finite support. Parallel to Proposition 2:

Proposition 4. *Let $f : X \times \Theta \to \mathbb{R}$ for some set X, and assume there exist $x_1, x_2 \in X$ such that $f(x_1, \cdot) : \Theta \to \mathbb{R}$ and $f(x_2, \cdot) : \Theta \to \mathbb{R}$ are linearly independent. The function f is linear combinations SSC-preserving if and only if there exist $\lambda_1, \lambda_2 : X \to \mathbb{R}$ such that*

1. *$f(x_1, \cdot) : \Theta \to \mathbb{R}$ and $f(x_2, \cdot) : \Theta \to \mathbb{R}$ are strictly ratio ordered, and*

2. *$(\forall x)\ f(x, \cdot) = \lambda_1(x) f(x_1, \cdot) + \lambda_2(x) f(x_2, \cdot)$.*

For the "if" direction of Proposition 4, the existence of a pair of linearly independent functions need not be assumed, because strict ratio ordering implies linear independence. However, without that hypothesis, the "only if" direction would fail: given $X = \{x_1, x_2\}$, and $f(x_1, \cdot) = 2f(x_2, \cdot)$ with $f(x_1, \cdot)$ strictly single crossing, the function f is linear combinations SSC-preserving even though $f(x_1, \cdot)$ and $f(x_2, \cdot)$ are not strictly ratio ordered.

C.2. Proof of Lemma 2

When $|\Theta| \leq 2$.

If $|\Theta| = 1$, the proof is trivial as all functions are strictly single crossing and every pair of f_1, f_2 satisfy strict ratio ordering. So assume $|\Theta| = 2$ and denote $\Theta = \{\theta_l, \theta_h\}$; without loss, we may assume $\theta_h > \theta_l$ because of our maintained assumption that upper and lower bounds exist for all pairs.

$(\implies)$ Either $(f_1(\theta_l), f_2(\theta_l)) \neq 0$ or $(f_1(\theta_h), f_2(\theta_h)) \neq 0$: otherwise, for every $\alpha \in \mathbb{R}^2 \backslash \{0\}$, $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$, and hence $\alpha \cdot f$ is a zero function, which is not strictly single crossing. Assume $(f_1(\theta_l), f_2(\theta_l)) \neq 0$; the proof for the other case is analogous. Let $\alpha_l \equiv (f_2(\theta_l), -f_1(\theta_l))$ and consider $(\alpha_l \cdot f)(\theta) = f_2(\theta_l) f_1(\theta) - f_1(\theta_l) f_2(\theta)$. We have $(\alpha_l \cdot f)(\theta_l) = 0$ and, by strict single crossing of $\alpha_l \cdot f$, $(\alpha_l \cdot f)(\theta_h) \neq 0$. That is, $f_2(\theta_l) f_1(\theta_h) \neq f_1(\theta_l) f_2(\theta_h)$, which means that f_1 and f_2 are strictly ratio ordered.

$(\impliedby)$ For any $\alpha \in \mathbb{R}^2 \backslash \{0\}$, $\alpha \cdot f$ is not strictly single crossing if and only if $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$. This implies $\alpha_1 f_1(\theta_l) = -\alpha_2 f_2(\theta_l)$ and $\alpha_1 f_1(\theta_h) = -\alpha_2 f_2(\theta_h)$, and hence

$$\begin{aligned} \alpha_1 f_1(\theta_l) f_2(\theta_h) &= -\alpha_2 f_2(\theta_l) f_2(\theta_h) = \alpha_1 f_1(\theta_h) f_2(\theta_l) \quad \text{and} \\ \alpha_2 f_1(\theta_l) f_2(\theta_h) &= -\alpha_1 f_1(\theta_l) f_1(\theta_h) = \alpha_2 f_1(\theta_h) f_2(\theta_l). \end{aligned}$$

As $(\alpha_1, \alpha_2) \neq 0$, $f_1(\theta_l) f_2(\theta_h) = f_1(\theta_h) f_2(\theta_l)$, contradicting strict ratio ordering of f_1 and f_2.

When $|\Theta| \geq 3$.

$(\implies)$ Suppose, towards contradiction, that f_1 and f_2 are not strictly ratio ordered:

$$\begin{aligned} &(\exists \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) \leq f_1(\theta_h) f_2(\theta_l) \quad \text{and} \\ &(\exists \theta' < \theta'') \quad f_1(\theta') f_2(\theta'') \geq f_1(\theta'') f_2(\theta'). \end{aligned} \tag{28}$$

Take any upper bound $\overline{\theta}$ of $\{\theta_l, \theta_h, \theta', \theta''\}$. Letting $\alpha_l \equiv (f_2(\theta_l), -f_1(\theta_l))$, it holds that $\alpha_l \cdot f$ is strictly single crossing only from below, as $(\alpha_l \cdot f)(\theta_l) = (f_2(\theta_l), -f_1(\theta_l)) \cdot (f_1(\theta_l), f_2(\theta_l)) = 0$ and by (28), $(\alpha_l \cdot f)(\theta_h) \geq 0$. Hence $(\alpha_l \cdot f)(\overline{\theta}) \geq 0$. Analogously, letting $\alpha' \equiv (f_2(\theta'), -f_1(\theta'))$, we conclude that $(\alpha' \cdot f)(\overline{\theta}) \leq 0$. Now let $\overline{\alpha} \equiv (f_2(\overline{\theta}), -f_1(\overline{\theta}))$. It follows that

$$\begin{aligned} &(\overline{\alpha} \cdot f)(\theta_l) = (f_2(\overline{\theta}), -f_1(\overline{\theta})) \cdot (f_1(\theta_l), f_2(\theta_l)) = -(\alpha_l \cdot f)(\overline{\theta}) \leq 0, \\ &(\overline{\alpha} \cdot f)(\theta') = (f_2(\theta'), -f_1(\theta')) \cdot (f_1(\theta'), f_2(\theta')) = -(\alpha' \cdot f)(\overline{\theta}) \geq 0, \text{ and} \\ &(\overline{\alpha} \cdot f)(\overline{\theta}) = 0. \end{aligned}$$

Therefore, $\overline{\alpha} \cdot f$ is not strictly single crossing.

$(\Longleftarrow)$ We provide a proof for the case in which f_1 strictly ratio dominates f_2, and omit the other case's analogous proof. For any $\alpha \in \mathbb{R}^2 \backslash \{0\}$, we prove that $\alpha \cdot f$ is single crossing. The argument is very similar to that used in proving Lemma 1, but note that here we do not assume that f_1 and f_2 are each strictly single crossing.

As f_1 strictly ratio dominates f_2,

$$(\forall \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) < f_1(\theta_h) f_2(\theta_l). \tag{29}$$

Suppose, towards contradiction, that $\alpha \cdot f$ is not strictly single crossing.

Claim: There exist $\theta_l, \theta_m, \theta_h$ with $\theta_l < \theta_m < \theta_h$ such that

$$(\alpha \cdot f)(\theta_l) \leq 0, (\alpha \cdot f)(\theta_m) \geq 0, \text{ and } (\alpha \cdot f)(\theta_h) \leq 0, \quad \text{or} \tag{30}$$

$$(\alpha \cdot f)(\theta_l) \geq 0, (\alpha \cdot f)(\theta_m) \leq 0, \text{ and } (\alpha \cdot f)(\theta_h) \geq 0. \tag{31}$$

Proof of claim: Since $\alpha \cdot f$ is not strictly single crossing either from below or from above,

$$(\exists \theta_1 < \theta_2) \quad (\alpha \cdot f)(\theta_1) \geq 0 \geq (\alpha \cdot f)(\theta_2), \text{ and}$$
$$(\exists \theta_3 < \theta_4) \quad (\alpha \cdot f)(\theta_3) \leq 0 \leq (\alpha \cdot f)(\theta_4).$$

Let $\Theta_0 \equiv \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and let $\overline{\theta}$ and $\underline{\theta}$ be an upper and lower bound of Θ_0, respectively. Either $(\alpha \cdot f)(\underline{\theta}) \neq 0$ or $(\alpha \cdot f)(\overline{\theta}) \neq 0$, as otherwise $f_1(\underline{\theta}) f_2(\overline{\theta}) = f_2(\underline{\theta}) f_1(\overline{\theta})$, contradicting (29). Suppose $(\alpha \cdot f)(\overline{\theta}) \neq 0$. If $(\alpha \cdot f)(\overline{\theta}) < 0$, then we choose $(\theta_l, \theta_m, \theta_h) = (\theta_3, \theta_4, \overline{\theta})$, which satisfies (30). If $(\alpha \cdot f)(\overline{\theta}) > 0$, then we choose $(\theta_l, \theta_m, \theta_h) = (\theta_1, \theta_2, \overline{\theta})$, which satisfies (31). A similar argument applies when $(\alpha \cdot f)(\underline{\theta}) \neq 0$. $\|$

Condition (29) implies that $f(\theta) \equiv (f_1(\theta), f_2(\theta)) \neq 0$ for all $\theta \in \{\theta_l, \theta_m, \theta_h\}$. Take any $\theta_1, \theta_2 \in \{\theta_l, \theta_m, \theta_h\}$ such that $\theta_1 < \theta_2$. By (29), $f(\theta_1)$ moves to $f(\theta_2)$ in a clockwise rotation with an angle $r_{12} \in (0, 180)$. Suppose (30) holds; the argument is analogous if (31) holds. It follows from $0 < r_{lh} < 180$, $(\alpha \cdot f)(\theta_l) \leq 0$, and $(\alpha \cdot f)(\theta_h) \leq 0$ that $\{f(\theta_l), f(\theta_h)\} \subseteq \mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$ with $\{f(\theta_l), f(\theta_h)\} \not\subseteq \mathbb{R}^2_{\alpha,0}$.[38] This, together with $r_{lm} > 0$ and $r_{mh} > 0$, implies $f(\theta_m) \in \mathbb{R}^2_{\alpha,-}$, which contradicts (30).

C.3. Proof of Proposition 4

[38] Recall that $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, $\mathbb{R}^2_{\alpha,0} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x = 0\}$, and $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$.

Appendix B.2 proved Proposition 2 assuming certain functions are linearly independent. Essentially the same proof can be used for Proposition 4, replacing statements involving "single crossing" with "either a zero function or strictly single crossing".

C.4. Proof of Theorem 2

The utility function $v : A \times \Theta \to \mathbb{R}$ has SSCED if and only if $(\forall P, Q \in \Delta A)$ $D_{P,Q}$ is either a zero function or strictly single crossing. Most statements in the proof of Theorem 1 go through for Theorem 2 when we replace "single crossing" with "either a zero function or strictly single crossing".

We need only to rewrite the proof of the "only if" part in the following two special cases:

1. $(\forall a', a'')(\forall \theta)\ v(a', \theta) = v(a'', \theta)$, or
2. $(\exists a', a'')$ such that (i) $v(a'', \theta) - v(a', \theta)$ is not a zero function of θ, and (ii) $(\forall a)\ v(a, \theta) - v(a', \theta)$ and $v(a'', \theta) - v(a', \theta)$ are linearly dependent functions of θ.

In the first case, we can write $v(a, \theta)$ in form of (8) where g_1, g_2 are zero functions, $c(\theta) \equiv v(a_0, \theta)$ for any a_0, $(\forall \theta)\ f_1(\theta) = 1$, and $f_2(\theta)$ is any strictly decreasing function of θ. Then,

$$(\forall \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) = f_2(\theta_h) < f_2(\theta_l) = f_1(\theta_h) f_2(\theta_l).$$

In the second case, for every a, there exists $\lambda \in \mathbb{R}^2 \backslash \{0\}$ such that $\lambda_1 \left(v(a, \cdot) - v(a', \cdot)\right) + \lambda_2 \left(v(a'', \cdot) - v(a', \cdot)\right)$ is a zero function. Note that $\lambda_1 \neq 0$, as otherwise $v(a'', \cdot) - v(a', \cdot)$ would be a zero function. It follows that there exists $\lambda : A \to \mathbb{R}$ such that

$$(\forall a, \theta) \quad v(a, \theta) - v(a', \theta) = \lambda(a) \left(v(a'', \theta) - v(a', \theta)\right),$$

or equivalently,

$$(\forall a, \theta) \quad v(a, \theta) = \lambda(a) \left(v(a'', \theta) - v(a', \theta)\right) + v(a', \theta).$$

Note that $v(a'', \theta) - v(a', \theta)$ is a strictly single-crossing function of θ: consider the expectational difference with distributions that put probability one on a'' and a' respectively. If the difference is strictly single crossing from below, we can write $v(a, \theta)$ in the form of (8) where $g_1(a) = \lambda(a)$, $g_2(a) = 0$, $f_1(\theta) = v(a'', \theta) - v(a', \theta)$, and $c(\theta) = v(a', \theta)$. If the difference is strictly single crossing only from above, we let $g_1(a) = -\lambda(a)$ and $f_1(\theta) = v(a', \theta) - v(a'', \theta)$.

Now take any strictly increasing function $h : \Theta \to \mathbb{R}$ and define

$$\hat{h}(\theta) \equiv \begin{cases} -e^{h(\theta)} & \text{if } f_1(\theta) \le 0 \\ e^{-h(\theta)} & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(\theta) \equiv \begin{cases} \hat{h}(\theta) f_1(\theta) & \text{if } f_1(\theta) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

To verify that f_1 and f_2 are strictly ratio ordered, take any $\theta_l < \theta_h$. There are three possibilities to consider:

1. If $f_1(\theta_l) f_1(\theta_h) > 0$, then

$$f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_h) < f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_l) = f_1(\theta_h) f_2(\theta_l),$$

as $\hat{h}(\theta)$ is strictly decreasing over $\{\theta \,|\, f_1(\theta) < 0\}$ and $\{\theta \,|\, f_1(\theta) > 0\}$.

2. If $f_1(\theta_l) f_1(\theta_h) < 0$, then as $f_1(\theta)$ is strictly single crossing from below, we have $f_1(\theta_l) < 0 < f_1(\theta_h)$. Hence,

$$f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_h) < 0 < f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_l) = f_1(\theta_h) f_2(\theta_l).$$

3. If $f_1(\theta_l) f_1(\theta_h) = 0$, then because f_1 is strictly single crossing from below, we have either (i) $f_1(\theta_l) < 0 = f_1(\theta_h)$, which results in $f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) < 0 = f_1(\theta_h) f_2(\theta_l)$, or (ii) $f_1(\theta_l) = 0 < f_1(\theta_h)$, which results in $f_1(\theta_l) f_2(\theta_h) = 0 < f_1(\theta_h) = f_1(\theta_h) f_2(\theta_l)$.

D. Proof of Proposition 3 (Section 5)

$(\Longleftarrow)$ For any $Q \in \Delta\Delta\Omega$,

$$V(Q, \theta) = \left(\int_{\Delta\Omega} g_1(p) dQ \right) f_1(\theta) + \left(\int_{\Delta\Omega} g_2(p) dQ \right) f_2(\theta) + \int_{\Delta\Omega} \left(\sum_{\omega \in \Omega} v(\delta_\omega, \theta) p(\omega) \right) dQ.$$

The last term on the right-hand side is equal to

$$\sum_{\omega \in \Omega} v(\delta_\omega, \theta) \left(\int_{\Delta\Omega} p(w) dQ \right) = \sum_{\omega \in \Omega} v(\delta_\omega, \theta) \overline{Q}(\omega).$$

Thus, for any $Q, R \in \Delta\Delta\Omega$ with $\overline{Q} = \overline{R}$,

$$D_{Q,R}(\theta) = \left(\int_{\Delta\Omega} g_1(p) dQ - \int_{\Delta\Omega} g_1(p) dR \right) f_1(\theta) + \left(\int_{\Delta\Omega} g_2(p) dQ - \int_{\Delta\Omega} g_2(p) dR \right) f_2(\theta),$$

which is single crossing in θ by Lemma 1.

($\implies$) For each posterior $p \in \Delta\Omega$, consider two experiments: $\delta_p \in \Delta\Delta\Omega$ yields p with certainty, and $Q_p \in \Delta\Delta\Omega$ yields each degenerate posterior $\delta_\omega \in \Delta\Omega$ with probability $p(\omega)$. For every θ,

$$V(\delta_p, \theta) = v(p, \theta) \quad \text{and} \quad V(Q_p, \theta) = \sum_{\omega \in \Omega} v(\delta_\omega, \theta) p(\omega).$$

The average of both experiments is the posterior p: $(\forall \omega)\ \overline{\delta}_p(\omega) = \overline{Q}_p(\omega) = p(\omega)$. Thus,

$$f(p, \theta) \equiv V(\delta_p, \theta) - V(Q_p, \theta) = v(p, \theta) - \sum_{\omega \in \Omega} v(\delta_\omega, \theta) p(\omega)$$

is single crossing in θ.

The remaining part of the proof is similar to the proof of Proposition 2 in Appendix B.2. If there exists $p_1 \in \Delta\Omega$ such that

$$(\exists \lambda : \Delta\Omega \to \mathbb{R})(\forall p, \theta) \quad f(p, \theta) = \lambda(p) f(p_1, \theta),$$

then v is in the form of (13) with $f_1(\theta) = f(p_1, \theta)$, $f_2(\theta) = 0$, $g_1(p) = \lambda(p)$, and $g_2(p) = 0$. Otherwise, there exist p_1 and p_2 such that $f(p_1, \cdot), f(p_2, \cdot) : \Theta \to \mathbb{R}$ are linearly independent. The following claim, together with Lemma 1, then implies that $f(p_1, \cdot)$ and $f(p_2, \cdot)$ are ratio ordered.

Claim 9. *For all $\alpha \in \mathbb{R}^2$, the linear combination $\alpha_1 f(p_1, \theta) + \alpha_2 f(p_2, \theta)$ is single crossing in θ.*

Proof. Assume that $\alpha \in \mathbb{R}^2 \backslash \{0\}$; otherwise the linear combination is a zero function, which is clearly single crossing. Moreover, without loss assume that $\alpha_1 > 0$. (If $\alpha_1 < 0$, multiply the linear combination by -1, while if $\alpha_1 = 0$, swap indices $i = 1, 2$; neither modification affects whether the linear combination is single crossing or not.)

If $\alpha_2 \geq 0$, let $M \equiv \alpha_1 + \alpha_2$ and define $R_1, R_2 \in \Delta\Delta\Omega$ by

$$R_1 \equiv \frac{\alpha_1}{M} \delta_{p_1} + \frac{\alpha_2}{M} \delta_{p_2}, \quad \text{and} \quad R_2 \equiv \frac{\alpha_1}{M} Q_{p_1} + \frac{\alpha_2}{M} Q_{p_2}.$$

Note that $\overline{R}_1 = \overline{R}_2 = \frac{\alpha_1}{M} p_1 + \frac{\alpha_2}{M} p_2$. Thus,

$$\begin{aligned} \alpha_1 f(p_1, \theta) + \alpha_2 f(p_2, \theta) &= \alpha_1 \left(V(\delta_{p_1}, \theta) - V(Q_{p_1}, \theta) \right) + \alpha_2 \left(V(\delta_{p_2}, \theta) - V(Q_{p_2}, \theta) \right) \\ &= M \left(V(R_1, \theta) - V(R_2, \theta) \right) \end{aligned}$$

is single crossing in θ.

If, on the other hand, $\alpha_2 < 0$, let $M \equiv \alpha_1 - \alpha_2$ and define $R_1, R_2 \in \Delta\Delta\Omega$ by

$$R_1 \equiv \frac{\alpha_1}{M}\delta_{p_1} - \frac{\alpha_2}{M}Q_{p_2}, \quad \text{and} \quad R_2 \equiv \frac{\alpha_1}{M}Q_{p_1} - \frac{\alpha_2}{M}\delta_{p_2}.$$

As before, $\overline{R}_1 = \overline{R}_2$, and

$$\alpha_1 f(p_1, \theta) + \alpha_2 f(p_2, \theta) = M\left(V(R_1, \theta) - V(R_2, \theta)\right)$$

is single crossing in θ. *Q.E.D.*

We can now follow the proof of Proposition 2 in Appendix B.2 and show that there exist $\lambda_1, \lambda_2 : \Delta\Omega \to \mathbb{R}$ such that $(\forall p)\ f(p, \cdot) = \lambda_1(p) f(p_1, \cdot) + \lambda_2(p) f(p_2, \cdot)$. By definition of $f(p, \theta)$, the function v satisfies (13).

E. Proofs for Monotone Comparative Statics (Subsection 6.1)

E.1. Proof of Theorem 3

$(\implies)$ Suppose $f : X \times \Theta \to \mathbb{R}$ has MCS on X with some order $\succeq$. We first prove the following claim.

Claim 10. *For every $x', x'' \in X$, if $\exists \theta_l < \theta_h$ such that $\operatorname{sign}[D_{x',x''}(\theta_l)] < \operatorname{sign}[D_{x',x''}(\theta_h)]$, then $x' \succ x''$.*

Proof. Consider $S = \{x', x''\}$. Since $\operatorname{sign}[D_{x',x''}(\theta_l)] \neq \operatorname{sign}[D_{x',x''}(\theta_h)]$, we have

$$\arg\max_{x \in S} f(x, \theta_l) \neq \arg\max_{x \in S} f(x, \theta_h).$$

Thus, either (i) $x' \in \arg\max_{x \in S} f(x, \theta_l)$ and $x'' \in \arg\max_{x \in S} f(x, \theta_h)$, or (ii) $x'' \in \arg\max_{x \in S} f(x, \theta_l)$ and $x' \in \arg\max_{x \in S} f(x, \theta_h)$. Since f has MCS on $(X, \succeq)$, we have $\arg\max_{x \in S} f(x, \theta_h) \succeq_{SSO} \arg\max_{x \in S} f(x, \theta_l)$. Therefore, $x' \wedge x'' \in \arg\max_{x \in S} f(x, \theta_l)$ and $x' \vee x'' \in \arg\max_{x \in S} f(x, \theta_h)$, which implies that either $x' \succeq x''$ or $x'' \succeq x'$. Since $x' \neq x''$, we have either $x' \succ x''$ or $x'' \succ x'$. If $x'' \succ x'$, then $x'' = x' \vee x'' \in \arg\max_{x \in S} f(x, \theta_h)$, contradicting $\operatorname{sign}[D_{x',x''}(\theta_l)] < \operatorname{sign}[D_{x',x''}(\theta_h)]$. Thus, $x' \succ x''$. *Q.E.D.*

To show that f has SCD on X, suppose not, per contra. Claim 6 implies there exist

$x', x'' \in X$ and $\theta_l < \theta_m < \theta_h$ such that either

$$\text{sign}[D_{x',x''}(\theta_l)] < \text{sign}[D_{x',x''}(\theta_m)] \quad \text{and} \quad \text{sign}[D_{x',x''}(\theta_m)] > \text{sign}[D_{x',x''}(\theta_h)], \quad \text{or} \tag{32}$$

$$\text{sign}[D_{x',x''}(\theta_l)] > \text{sign}[D_{x',x''}(\theta_m)] \quad \text{and} \quad \text{sign}[D_{x',x''}(\theta_m)] < \text{sign}[D_{x',x''}(\theta_h)]. \tag{33}$$

Given either (32) or (33), Claim 10 implies $x' \succ x''$ and $x'' \succ x'$, a contradiction.

To show that $\succeq$ is a refinement of $\succeq_{SCD}$, it suffices to show that

$$(\forall x', x'' \in X) \quad x' \succ_{SCD} x'' \implies x' \succ x'', \tag{34}$$

because both $\succeq$ and $\succeq_{SCD}$ are anti-symmetric. Take any $x', x'' \in X$ such that $x' \succ_{SCD} x''$. As $D_{x',x''}$ is single crossing only from below, $\exists \theta_l < \theta_h$ such that $\text{sign}[D_{x',x''}(\theta_l)] < \text{sign}[D_{x',x''}(\theta_h)]$. Claim 10 implies $x' \succ x''$, which proves (34).

$(\impliedby)$ Suppose that $f : X \times \Theta \to \mathbb{R}$ has SCD, and $\succeq$ is a refinement of $\succeq_{SCD}$. For any $S \subseteq X$, define $C_f(S) \equiv \bigcup_{\theta \in \Theta} \arg\max_{x \in S} f(x, \theta)$. It is clear that

$$(\forall \theta) \quad \arg\max_{x \in S} f(x, \theta) = \arg\max_{x \in C_f(S)} f(x, \theta).$$

We claim that $C_f(S)$ is completely ordered by $\succeq_{SCD}$. To see why, take any pair $x', x'' \in C_f(S)$ with $x' \neq x''$. As f has SCD, $D_{x',x''}$ is single crossing in θ. As X is minimal, $D_{x',x''}$ is not a zero function. Also, as $x', x'' \in C_f(S)$, $\text{sign}[D_{x',x''}]$ is not a constant function with value either 1 or -1. Thus, $D_{x',x''}$ is single crossing either only from below, or only from above. It follows that $x' \succ_{SCD} x''$ or $x'' \succ_{SCD} x'$.

Since $\succeq$ is a refinement of $\succeq_{SCD}$, $\succeq$ coincides with $\succeq_{SCD}$ on $C_f(S)$, and the strong set orders generated by $\succeq$ and $\succeq_{SCD}$ on the collection of all subsets of $C_f(S)$ also coincide. By definition of $\succeq_{SCD}$, f satisfies Milgrom and Shannon's single-crossing property in (x, θ) with respect to $\succeq_{SCD}$ and $\leq$.[39] It follows from Milgrom and Shannon (1994, Theorem 4)

[39] Given $(X, \succeq)$ that is completely ordered, $(\Theta, \leq)$ that is (partially) ordered, and $f : X \times \Theta \to \mathbb{R}$, Milgrom and Shannon's single-crossing property in (x, θ) is equivalent to

$$(\forall x' \succ x'')(\forall \theta_l < \theta_h) \quad f(x', \theta_l) \geq (>) f(x'', \theta_l) \implies f(x', \theta_h) \geq (>) f(x'', \theta_h).$$

that $\forall \theta_l < \theta_h$,

$$\arg\max_{x \in S} f(x, \theta_h) = \arg\max_{x \in C_f(S)} f(x, \theta_h) \succeq_{SSO} \arg\max_{x \in C_f(S)} f(x, \theta_l) = \arg\max_{x \in S} f(x, \theta_l).$$[40]

E.2. Proof of Theorem 4

The proof is similar to the proof of Theorem 3 in Appendix E.1.

($\Longrightarrow$) Suppose $f : X \times \Theta \to \mathbb{R}$ has MS on $(X, \succeq)$.

To show that f has SSCD on X, suppose not, per contra. As we have shown in the proof of Theorem 0,

$$(\exists x', x'' \text{ with } x' \neq x'')(\exists \theta_l < \theta_m < \theta_h) \quad D_{x',x''}(\theta_l) \geq 0, D_{x',x''}(\theta_m) \leq 0, \text{ and } D_{x',x''}(\theta_h) \geq 0.$$

Let $S \equiv \{x', x''\}$ and consider a selection $s^*(\theta)$ from $\arg\max_{x \in S} f(x, \theta)$ such that $s^*(\theta_l) = s^*(\theta_h) = x'$ and $s^*(\theta_m) = x''$. Since f has MS on $(X, \succeq)$, we must have $x' \succeq x''$ and $x'' \succeq x'$, a contradiction to anti-symmetry of $\succeq$.

To show that $\succeq$ is a refinement of $\succeq_{SSCD}$, it suffices to show that

$$(\forall x', x'' \in X) \quad x' \succ_{SSCD} x'' \implies x' \succ x'',$$

because both $\succeq$ and $\succeq_{SSCD}$ are anti-symmetric. Take any $x', x'' \in X$ such that $x' \succ_{SSCD} x''$. As $D_{x',x''}$ is strictly single crossing only from below, $\exists \theta_l < \theta_h$ such that $\text{sign}[D_{x',x''}(\theta_l)] < \text{sign}[D_{x',x''}(\theta_h)]$, which implies that $\text{sign}[D_{x',x''}(\theta_l)] \leq 0$ and $\text{sign}[D_{x',x''}(\theta_h)] \geq 0$. Consider a selection $x'' \in \arg\max_{x \in \{x', x''\}} f(x, \theta_l)$ and $x' \in \arg\max_{x \in \{x', x''\}} f(x, \theta_h)$. Since f has MS on $(X, \succeq)$, we have $x' \succeq x''$. Since $x' \neq x''$ and $\succeq$ is anti-symmetric, it must be that $x' \succ x''$.

($\Longleftarrow$) For any $S \subseteq X$, define $C_f(S) \equiv \bigcup_{\theta \in \Theta} \arg\max_{x \in S} f(x, \theta)$. First, we claim that $C_f(S)$ is completely ordered by $\succeq_{SSCD}$. To see this, take any pair $x', x'' \in C_f(S)$ with $x' \neq x''$. As f has SSCD , $D_{x',x''}$ is strictly single crossing in θ. As $x', x'' \in C_f(S)$, $\text{sign}[D_{x',x''}]$ is not a constant function with value either 1 or -1. Thus, $D_{x',x''}$ is strictly single crossing either only from below or only from above. It follows that $x' \succ_{SSCD} x''$ or $x'' \succ_{SSCD} x'$. Next, since $\succeq$ is a refinement of $\succeq_{SSCD}$, $\succeq$ coincides with $\succeq_{SSCD}$ on $C_f(S)$. By definition of $\succeq_{SSCD}$,

[40] Milgrom and Shannon (1994, Theorem 4) identify their single-crossing property and quasi-supermodularity as jointly necessary and sufficient for their monotone comparative statics. (On a lattice $(X, \succeq)$, $h : X \to \mathbb{R}$ is quasisupermodular if $h(x) \geq (>) h(x \wedge x') \implies h(x \vee x') \geq (>) h(x')$.) When the choice set is completely ordered, the quasi-supermodularity holds trivially.

f satisfies Milgrom and Shannon's strict single-crossing property in (x, θ) with respect to $\succeq_{SSCD}$ and $\leq$.[41] It follows from Milgrom and Shannon (1994, Theorem 4′) that any selection $s^*(\theta)$ from $\arg\max_{x \in C_f(S)} f(x, \theta) (= \arg\max_{x \in S} f(x, \theta))$ is increasing in θ.

F. Proofs for Monotonic Expectational Differences (Subsection 6.2)

We begin with the analogs of Lemma 1 and Proposition 2 below in Appendix F.1; their proofs are in Appendix F.2 and Appendix F.3 respectively. The proof of Theorem 5 then follows in Appendix F.4.

F.1. Aggregating Monotonic Functions

Lemma 3. *Let $f_1, f_2 : \Theta \to \mathbb{R}$ be monotonic functions. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is monotonic $\forall \alpha \in \mathbb{R}^2$ if and only if either f_1 or f_2 is an affine transformation of the other, i.e., there exists $\lambda \in \mathbb{R}^2$ such that either $f_2 = \lambda_1 f_1 + \lambda_2$ or $f_1 = \lambda_1 f_2 + \lambda_2$.*

We say that $f : X \times \Theta \to \mathbb{R}$ is **linear combinations monotonicity-preserving** if $\sum_{x \in X} f(x, \theta)\mu(x)$ is a monotonic function of θ for every function $\mu : X \to \mathbb{R}$ with finite support.

Proposition 5. *Let $f : X \times \Theta \to \mathbb{R}$ for some set X. The function f is linear combinations monotonicity-preserving if and only if there exist $x' \in X$ and $\lambda_1, \lambda_2 : X \to \mathbb{R}$ such that (i) $f(x', \cdot)$ is monotonic, and (ii) $(\forall x)\ f(x, \cdot) = \lambda_1(x) f(x', \cdot) + \lambda_2(x)$.*

F.2. Proof of Lemma 3

$(\Longleftarrow)$ Suppose there exist $\lambda \in \mathbb{R}^2$ such that $f_2 = \lambda_1 f_1 + \lambda_2$. Then, for any $\alpha \in \mathbb{R}^2$,

$$(\alpha \cdot f)(\theta) = \alpha_1 f_1(\theta) + \alpha_2(\lambda_1 f_1(\theta) + \lambda_2) = (\alpha_1 + \alpha_2 \lambda_1) f_1(\theta) + \lambda_2,$$

which is monotonic.

[41] Given $(X, \succeq)$ that is completely ordered, $(\Theta, \leq)$ that is (partially) ordered, and $f : X \times \Theta \to \mathbb{R}$, Milgrom and Shannon's strict single-crossing property in (x, θ) is equivalent to

$$(\forall x' \succ x'')(\forall \theta_l < \theta_h) \quad f(x', \theta_l) \geq f(x'', \theta_l) \implies f(x', \theta_h) > f(x'', \theta_h).$$

($\Longrightarrow$) The proof is trivial if both f_1 and f_2 are constant functions. So we suppose that at least one function, say f_1, is not constant:

$$(\exists\theta', \theta'') \quad f_1(\theta') \neq f_1(\theta''). \tag{35}$$

This implies that $\text{rank}[M_{\theta',\theta''}] = 2$, where

$$M_{\theta',\theta''} \equiv \begin{bmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{bmatrix}.$$

Hence, the system

$$\begin{bmatrix} f_2(\theta') \\ f_2(\theta'') \end{bmatrix} = \begin{bmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{36}$$

has a unique solution $\lambda^* \in \mathbb{R}^2$. We will show that $f_2 = \lambda_1^* f_1 + \lambda_2^*$.

Suppose, towards contradiction, there exists θ^* such that

$$f_2(\theta^*) \neq \lambda_1^* f_1(\theta^*) + \lambda_2^*. \tag{37}$$

Let $\underline{\theta}$ and $\overline{\theta}$ be a lower and upper bound of $\{\theta', \theta'', \theta^*\}$. If $\text{rank}[M_{\underline{\theta},\overline{\theta}}] < 2$, then $f_1(\overline{\theta}) = f_1(\underline{\theta})$. As θ' and θ'' are in between $\underline{\theta}$ and $\overline{\theta}$ and f_1 is monotonic, we have $f_1(\theta') = f_1(\theta'')$, which contradicts (35). If, on the other hand, $\text{rank}[M_{\underline{\theta},\overline{\theta}}] = 2$, then the system

$$\begin{bmatrix} f_2(\underline{\theta}) \\ f_2(\overline{\theta}) \end{bmatrix} = \begin{bmatrix} f_1(\underline{\theta}) & 1 \\ f_1(\overline{\theta}) & 1 \end{bmatrix} \begin{bmatrix} \lambda_1' \\ \lambda_2' \end{bmatrix}$$

has a unique solution $\lambda' \in \mathbb{R}^2$. As θ', θ'', and θ^* are in between $\underline{\theta}$ and $\overline{\theta}$ and $f_2 - \lambda_1' f_1$ is monotonic, we have

$$\begin{bmatrix} f_2(\theta') \\ f_2(\theta'') \end{bmatrix} = \begin{bmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{bmatrix} \begin{bmatrix} \lambda_1' \\ \lambda_2' \end{bmatrix} \text{ and} \tag{38}$$

$$f_2(\theta^*) = \lambda_1' f_1(\theta^*) + \lambda_2'. \tag{39}$$

Equation 38 implies that λ' solves (36). As the unique solution to (36) was λ^*, it follows that $\lambda' = \lambda^*$. But then (37) and (39) are in contradiction.

F.3. Proof of Proposition 5

($\Longleftarrow$) We omit the proof as it is similar to the proof of Proposition 2 in Appendix B.2.

$(\implies)$ For the proof of necessity, if $(\forall x)\ f(x,\theta)$ is a constant function of θ, then we let $\lambda_1(x)=0$ and $\lambda_2(x)=f(x,\theta)$. If there exists $x'\in X$ such that $f(x',\theta)$ is not a constant function of θ, then Lemma 3 implies $(\forall x,\theta)\ f(x,\theta)=\lambda_1(x)f(x',\theta)+\lambda_2(x)$, with $\lambda_1,\lambda_2:X\to\mathbb{R}$.

F.4. Proof of Theorem 5

$(\impliedby)$ We omit the proof as it is similar to the proof of Theorem 1 in Appendix B.3.

$(\implies)$ The proof is trivial if $(\forall a,\theta)\ v(a,\theta)=0$, so assume there exists a_0 such that $v(a_0,\cdot):\Theta\to\mathbb{R}$ is not a zero function. Define $f:A\times\Theta\to\mathbb{R}$ by $f(a,\theta)\equiv v(a,\theta)-v(a_0,\theta)$. Note that $(\forall a)\ f(a,\theta)$ is a monotonic function of θ: consider the expectational difference with distributions that put probability one on a and a_0 respectively.

Let $A'\equiv A\backslash\{a_0\}$. As in the proof of Theorem 1 in Appendix B.3, for every real-valued function μ' over A' with finite support, there exist $P,Q\in\Delta A$ such that $\int_{A'}f(a,\theta)\mathrm{d}\mu'$ is monotonic if and only if $D_{P,Q}$ is monotonic. By Proposition 5, there exist $a'\in A\backslash a_0$ and $\lambda_1,\lambda_2:A\backslash\{a_0\}\to\mathbb{R}$ such that $(\forall a,\theta)\ f(a,\theta)=\lambda_1(a)f(a',\theta)+\lambda_2(a)$. Hence, there exist functions $g_1,g_2:A\to\mathbb{R}$ with $g_1(a_0)=g_2(a_0)=0$ such that $f(a,\theta)=g_1(a)f(a',\theta)+g_2(a)$, or equivalently, $v(a,\theta)=g_1(a)f(a',\theta)+g_2(a)+v(a_0,\theta)$.

G. Further Results Comparing MED and SCED

This appendix identifies when the preferences defined by a utility function with SCED also have a utility representation satisfying MED. Appendix G.1 states and discusses the characterization result, Proposition 6; the proof is provided in Appendix G.2; and Appendix G.3 applies the characterization.

G.1. SCED and MED Representations

Let $\succeq_\Theta\equiv\{\succeq_\theta:\ \theta\in\Theta\}$ be a family of type-dependent preferences (i.e., complete, reflexive, and transitive binary relations) over ΔA. We say that $v:A\times\Theta\to\mathbb{R}$ **represents** $\succeq_\Theta$ (in the expected utility form) if

$$(\forall\theta)(\forall P,Q\in\Delta A)\quad P\succeq_\theta Q\iff\int_A v(a,\theta)\mathrm{d}P\geq\int_A v(a,\theta)\mathrm{d}Q. \tag{40}$$

For any $v:A\times\Theta\to\mathbb{R}$ and $\succeq_\Theta$ defined by (40), a function $v':A\times\Theta\to\mathbb{R}$ is a positive affine transformation of v if and only if v' represents $\succeq_\Theta$.

Proposition 6. *Let $v : A \times \Theta \to \mathbb{R}$ have SCED:*

$$v(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + c(\theta),$$

where $f_1, f_2 : \Theta \to \mathbb{R}$ are each single crossing and ratio ordered, $g_1, g_2 : A \to \mathbb{R}$, and $c : \Theta \to \mathbb{R}$. Let $\succeq_\Theta$ be the family of preferences over ΔA defined by (40). Then, $\succeq_\Theta$ can be represented by a function with MED if and only if (i) either g_1 is an affine transformation of g_2 or vice-versa, or (ii) f_1 and f_2 are linearly dependent, or (iii) there exists $\lambda \in \mathbb{R}^2 \backslash \{0\}$ such that $(\forall \theta)$ $(\lambda \cdot f)(\theta) > 0$.

We can interpret Proposition 6 as follows: given $\succeq_\Theta$ with an SCED representation, there is an MED representation if and only if either

(a) there is a pair of types that do not share the same strict preference over any pair of lotteries (i.e., $(\exists \theta', \theta'')$ $(\forall P, Q \in \Delta A)$ $D_{P,Q}(\theta') D_{P,Q}(\theta'') \leq 0$), or

(b) there is a pair of lotteries over which all types share the same strict preference (i.e., $(\exists P, Q \in \Delta A)$ $(\forall \theta)$ $D_{P,Q}(\theta) > 0$).

To see this interpretation, suppose Case (i) or (ii) holds in Proposition 6. Then, there are functions $\hat{g}_1$, $\hat{f}_1$, and $\hat{c}$ such that $v(a, \theta) = \hat{g}_1(a) \hat{f}_1(\theta) + \hat{c}(\theta)$, with $\hat{f}_1$ single crossing. If $\hat{g}_1$ is constant or $\hat{f}_1$ is a zero function, every type is indifferent across all lotteries, and (a) holds. Suppose $\hat{g}_1$ is not constant. If $\hat{f}_1 > 0$ (or < 0), then (b) holds.[42] If $\hat{f}_1(\theta)$ is single crossing only from below or only from above, then for some θ' and θ'', $\hat{f}_1(\theta') \hat{f}_1(\theta'') \leq 0$; the pair θ' and θ'' does not share the same strict preference over any two lotteries, and so (a) holds. On the other hand, if Proposition 6's Case (iii) applies and Case (i) does not, then (b) holds, because $(\exists P, Q \in \Delta A, M \in \mathbb{R}_{++})$ $(\forall \theta)$ $M D_{P,Q}(\theta) = (\lambda \cdot f)(\theta) > 0$; see (41).[43]

Here is some geometric intuition for the "if" direction of Proposition 6. For Case (i) or (ii), $v(a, \theta) = \hat{g}_1(a) \hat{f}_1(\theta) + \hat{c}(\theta)$ with $\hat{f}_1$ single crossing, as already noted. We can rescale $\hat{f}_1(\theta)$ using a function $b : \Theta \to \mathbb{R}_{++}$ such that $b(\theta) \hat{f}_1(\theta)$ is monotonic. Thus, $v'(a, \theta) \equiv b(\theta) v(a, \theta)$ represents $\succeq_\Theta$ and has MED. For Case (iii), assume without loss of generality that $\|\lambda\| = 1$. Let $b(\theta) \equiv \frac{1}{(\lambda \cdot f)(\theta)}$. It follows that $(\forall \theta)$ $(\lambda \cdot (bf))(\theta) = 1$, i.e., the function b adjusts the lengths of vectors $\{f(\theta) \in \mathbb{R}^2 : \theta \in \Theta\}$ while maintaining their directions, as illustrated in Figure 6. The vector $(bf)(\theta)$ rotates monotonically as θ increases, while staying on the hyperplane

[42] Consider two degenerate lotteries over a' and a'' such that $\hat{g}_1(a') \neq \hat{g}_2(a'')$.

[43] Conversely, (a) implies either Case (i) or (ii) of Proposition 6. If g_1 and g_2 are affinely independent (i.e., a violation of (i)), then $(\forall \lambda \in \mathbb{R}^2 \backslash \{0\})$ $(\exists P, Q \in \Delta A, M \in \mathbb{R}_{++})$ $(\forall \theta)$ $(\lambda \cdot f)(\theta) = M D_{P,Q}(\theta)$; see (41). Then, (a) implies that for some θ', θ'' and for some $\beta \leq 0$, $f(\theta') = \beta f(\theta'')$, which, together with Condition (7) of ratio ordering, implies that f_1 and f_2 are linearly dependent. Moreover, (b) implies Case (iii) of the Proposition: letting $\lambda_i = \int_A g_i(a) \mathrm{d}[P - Q]$ for $i = 1, 2$, it holds that $\lambda_1 f_1(\theta) + \lambda_2 f_2(\theta) = D_{P,Q}(\theta) > 0$ for all θ.

$\{x \in \mathbb{R}^2 \,|\, \lambda \cdot x = 1\}$. Let $e_1 \equiv (1,0)$ and $e_2 \equiv (0,1)$. Suppose $\{e_1, \lambda\}$ is a basis for $\mathbb{R}^2$ (an analogous argument would hold if instead $\{e_2, \lambda\}$ were a basis). Then, for every θ, the vector $(bf)(\theta)$ is represented as $(b(\theta)f_1(\theta), 1)$ with respect to the new basis. We define $\tilde{f}_1(\theta) \equiv b(\theta)f_1(\theta)$ and

$$v'(a,\theta) \equiv b(\theta)v(a,\theta) = \tilde{g}_1(a)\tilde{f}_1(\theta) + \tilde{g}_2(a) + b(\theta)c(\theta),$$

with appropriately defined functions $\tilde{g}_1$ and $\tilde{g}_2$. Since $(bf)(\theta)$ rotates monotonically, $\tilde{f}_1(\theta)$ is monotonic. It follows that v' has MED.

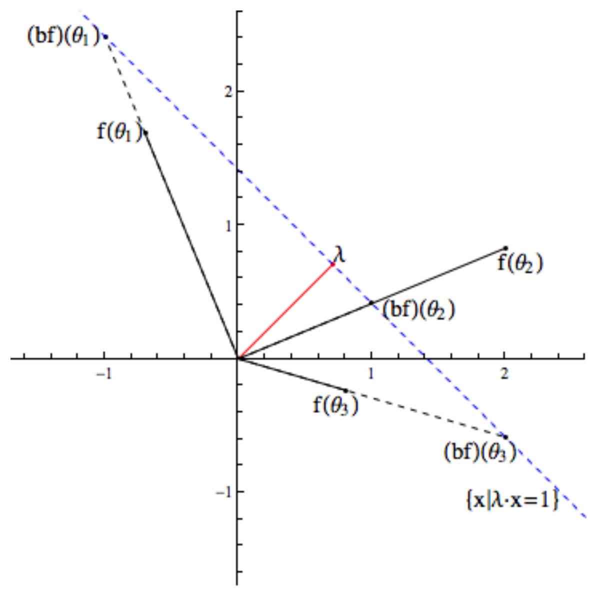


Figure 6: A geometric intuition for sufficiency of (iii) in Proposition 6.

G.2. Proof of Proposition 6

($\Longleftarrow$) First we prove that if either (i) or (ii) holds, then we can write v as $v(a,\theta) = \hat{g}_1(a)\hat{f}_1(\theta) + \hat{c}(\theta)$, with $\hat{f}_1$ single crossing.

Suppose (i) holds; without loss, assume $(\exists d \in \mathbb{R}^2)$ $(\forall a)$ $g_2(a) = d_1 g_1(a) + d_2$. Then, $\hat{g}_1(a) = g_1(a)$, $\hat{f}_1(\theta) = f_1(\theta) + d_1 f_2(\theta)$, and $\hat{c}(\theta) = d_2 f_2(\theta) + c(\theta)$. Next suppose (ii) holds; without loss, assume $(\exists d \in \mathbb{R})$ $(\forall \theta)$ $f_2(\theta) = d f_1(\theta)$. Then, $\hat{g}_1(a) = g_1(a) + d g_2(a)$, $\hat{f}_1(\theta) = f_1(\theta)$, and $\hat{c}(\theta) = c(\theta)$. In either case, $\hat{f}_1$ is a linear combination of f_1 and f_2, so it is single crossing.

Define

$$b(\theta) \equiv \begin{cases} \frac{1}{|\hat{f}_1(\theta)|} & \text{if } \hat{f}_1(\theta) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since $(\forall\theta)\ b(\theta) > 0$, $v'(a,\theta) \equiv b(\theta)v(a,\theta) = \hat{g}_1(a)b(\theta)\hat{f}_1(\theta) + b(\theta)\hat{c}(\theta)$ is a positive affine transformation of v, and hence it also represents $\succeq_\Theta$. As $\hat{f}_1$ is single crossing, $b(\theta)\hat{f}_1(\theta) = \mathrm{sign}[\hat{f}_1(\theta)]$ is monotonic, and v' has MED.

Now suppose (iii) holds: $(\exists\lambda \in \mathbb{R}^2\backslash\{0\})\ (\forall\theta)\ (\lambda\cdot f)(\theta) > 0$. Define $b(\theta) \equiv \frac{1}{(\lambda\cdot f)(\theta)}$. Then $(\lambda\cdot(bf))(\theta) = 1$, so that either $(bf_1)(\theta)$ is an affine transformation of $(bf_2)(\theta)$, or vice-versa: $(\exists\gamma,\omega \in \mathbb{R})$ such that either $b(\theta)f_1(\theta) = \gamma b(\theta)f_2(\theta) + \omega$ or $b(\theta)f_2(\theta) = \gamma b(\theta)f_1(\theta) + \omega$.

We consider the case in which $b(\theta)f_2(\theta) = \gamma b(\theta)f_1(\theta) + \omega$ and omit the other case's analogous proof. If $\lambda_2 \geq (\leq) 0$, then as f_1 ratio dominates f_2,

$$
\begin{aligned}
(\forall\theta' < \theta'') \qquad & f_1(\theta')f_2(\theta'') \leq f_1(\theta'')f_2(\theta') \\
\implies \quad & \lambda_1 f_1(\theta')f_1(\theta'') + \lambda_2 f_1(\theta')f_2(\theta'') \leq (\geq) \lambda_1 f_1(\theta')f_1(\theta'') + \lambda_2 f_1(\theta'')f_2(\theta') \\
\implies \quad & f_1(\theta')(\lambda\cdot f)(\theta'') \leq (\geq) f_1(\theta'')(\lambda\cdot f)(\theta') \\
\implies \quad & b(\theta')f_1(\theta') \leq (\geq) b(\theta'')f_1(\theta'').
\end{aligned}
$$

Thus, regardless of whether $\lambda_2 \geq 0$ or $\lambda_2 \leq 0$, $(bf_1)(\theta)$ is monotonic in θ. It follows that

$$
\begin{aligned}
v'(a,\theta) &\equiv b(\theta)v(a,\theta) = g_1(a)b(\theta)f_1(\theta) + g_2(a)b(\theta)f_2(\theta) + b(\theta)c(\theta) \\
&= g_1(a)b(\theta)f_1(\theta) + g_2(a)(\gamma b(\theta)f_1(\theta) + \omega) + b(\theta)c(\theta) \\
&= (g_1(a) + \gamma g_2(a))b(\theta)f_1(\theta) + \omega g_2(a) + b(\theta)c(\theta)
\end{aligned}
$$

has MED.

$(\implies)$ We prove that if neither (i) or (ii) holds, then (iii) holds.

We first show that when (i) does not hold,

$$(\forall\lambda \in \mathbb{R}^2\backslash\{0\})\ (\exists P,Q \in \Delta A, M \in \mathbb{R}_{++})(\forall\theta) \quad (\lambda\cdot f)(\theta) = MD_{P,Q}(\theta). \tag{41}$$

As (i) does not hold, the three functions g_1, g_2, and 1 (where 1 represents the constant function whose value is 1) are linearly independent. For otherwise, either g_1 and g_2 are linearly dependent, or $(\exists\alpha \in \mathbb{R}^2\backslash\{0\})\ (\forall a)\ \alpha_1 g_1(a) + \alpha_2 g_2(a) = 1$; in either case, either g_1 would be an affine transformation of g_2 or vice-versa.

By the aforementioned linear independence, there exist $a_0, a_1, a_2 \in A$ such that

$$\text{rank} \begin{bmatrix} g_1(a_0) & g_1(a_1) & g_1(a_2) \\ g_2(a_0) & g_2(a_1) & g_2(a_2) \\ 1 & 1 & 1 \end{bmatrix} = 3.$$

It follows that for any $\lambda \in \mathbb{R}^2 \backslash \{0\}$ there exists $(d_0, d_1, d_2) \in \mathbb{R}^3$ such that

$$\begin{bmatrix} g_1(a_0) & g_1(a_1) & g_1(a_2) \\ g_2(a_0) & g_2(a_1) & g_2(a_2) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{bmatrix}.$$

Take a sufficiently large $M \in \mathbb{R}_{++}$ such that $|d_i| \leq M/3$ for $i = 0, 1, 2$, and define $P, Q \in \Delta A$ with probability mass functions p, q such that $p(a_i) = 1/3$ and $q(a_i) = 1/3 - d_i/M$ for $i = 0, 1, 2$. Then, both P and Q are probability distributions on $\{a_0, a_1, a_2\}$, and

$$\begin{aligned}(\lambda \cdot f)(\theta) &= (d_0 g_1(a_0) + d_1 g_1(a_1) + d_2 g_1(a_2))\, f_1(\theta) + (d_0 g_2(a_0) + d_1 g_2(a_1) + d_2 g_2(a_2))\, f_2(\theta) \\ &= \sum_{i \in \{0,1,2\}} d_i \left(g_1(a_i) f_1(\theta) + g_2(a_i) f_2(\theta)\right) \\ &= M D_{P,Q}(\theta).\end{aligned}$$

Suppose $\succeq_\Theta$ is represented by a function $v' : A \times \Theta \to \mathbb{R}$ with MED. As both v and v' represent $\succeq_\Theta$ in the expected utility form, v' is a positive affine transformation of v. It follows that there exist $b : \Theta \to \mathbb{R}_{++}$ and $d : \Theta \to \mathbb{R}$ such that

$$v'(a, \theta) = b(\theta) v(a, \theta) + d(\theta) = g_1(a) \hat{f}_1(\theta) + g_2(a) \hat{f}_2(\theta) + \hat{c}(\theta),$$

where $\hat{f}_1(\theta) = b(\theta) f_1(\theta)$, $\hat{f}_2(\theta) = b(\theta) f_2(\theta)$, and $\hat{c}(\theta) = b(\theta) c(\theta) + d(\theta)$.

Given (41), for any $\lambda \in \mathbb{R}^2 \backslash \{0\}$, there exist $P, Q \in \Delta A$ and $M \in \mathbb{R}_{++}$ such that

$$(\lambda \cdot \hat{f})(\theta) = b(\theta)(\lambda \cdot f)(\theta) = b(\theta) M D_{P,Q}(\theta) = M \int_A v'(a, \theta) d[P - Q],$$

which is monotonic.

We find $\lambda \in \mathbb{R}^2 \backslash \{0\}$ such that $(\forall \theta)\ (\lambda \cdot \hat{f})(\theta) = 1$. For any θ', θ'', let

$$M_{\theta'\theta''} \equiv \begin{bmatrix} f_1(\theta') & f_1(\theta'') \\ f_2(\theta') & f_2(\theta'') \end{bmatrix} \quad \text{and} \quad \hat{M}_{\theta'\theta''} \equiv \begin{bmatrix} \hat{f}_1(\theta') & \hat{f}_1(\theta'') \\ \hat{f}_2(\theta') & \hat{f}_2(\theta'') \end{bmatrix}.$$

As f_1 and f_2 are linearly independent, there exist θ_1, θ_2 such that $\mathrm{rank}[M_{\theta_1,\theta_2}] = 2$, which implies that $\mathrm{rank}[\hat{M}_{\theta_1,\theta_2}] = 2$. Let $\lambda^* \in \mathbb{R}^2\backslash\{0\}$ be the unique solution of $\hat{M}_{\theta_1,\theta_2}\lambda = (1,1)$. Take any θ_0, and let $\underline{\theta}$ and $\overline{\theta}$ be a lower and upper bound of $\{\theta_0, \theta_1, \theta_2\}$. It must be that $\mathrm{rank}[\hat{M}_{\underline{\theta},\overline{\theta}}] = 2$. If otherwise, there exists $\lambda \in \mathbb{R}^2\backslash\{0\}$ such that $(\lambda \cdot \hat{f})(\underline{\theta}) = (\lambda \cdot \hat{f})(\overline{\theta}) = 0$. By (41), $\lambda \cdot \hat{f}$ is monotonic, so $(\lambda \cdot \hat{f})(\theta_1) = (\lambda \cdot \hat{f})(\theta_2) = 0$, which contradicts $\mathrm{rank}[\hat{M}_{\theta_1,\theta_2}] = 2$. Let λ^{**} be the unique solution of $\hat{M}_{\underline{\theta},\overline{\theta}}\lambda = (1,1)$. By monotonicity of $\lambda^{**} \cdot \hat{f}$, $\hat{M}_{\theta_1,\theta_2}\lambda^{**} = (1,1)$, which implies that $\lambda^{**} = \lambda^*$. It follows that $(\lambda^* \cdot \hat{f})(\theta_0) = 1$. As θ_0 is arbitrary, we have $(\forall\theta)$ $(\lambda^* \cdot \hat{f})(\theta) = 1$.

Finally, $(\forall\theta)$ $(\lambda^* \cdot f)(\theta) = \frac{(\lambda^* \cdot \hat{f})(\theta)}{b(\theta)} > 0$.

G.3. An Example of SCED-but-not-MED Preferences

We can use Proposition 6 to provide an example of type-dependent preferences representable by an SCED function that are not representable by any MED function.

Example 2. Let $\Theta \equiv (-1, 1] \subset \mathbb{R}$ and $A \equiv \{a_0, a', a''\}$. Consider $v(a, \theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta)$, with

1. $g_1(a_0) = g_2(a_0) = 0$, $g_1(a') = g_2(a') = 1$, $g_1(a'') = 2$, $g_2(a'') = 3$, and
2. $f_1(\theta) = \theta$, $f_2(\theta) = 1 - \theta^2$.

Observe that f_1 ratio dominates f_2: if $\theta' < \theta''$, then $\theta'\theta'' < 1$, and hence $f_1(\theta')f_2(\theta'') < f_1(\theta'')f_2(\theta')$. It follows that v has SCED.

We claim the family of type-dependent preferences $\succeq_\Theta$ represented by v is not representable by any MED function. It is easy to verify that neither g_1 is an affine transformation of g_2 nor vice-versa, and that f_1 and f_2 are linearly independent. By Proposition 6, it suffices to show that $\nexists\lambda \in \mathbb{R}^2$ such that $(\forall\theta)$ $(\lambda \cdot f)(\theta) > 0$. Take any $\lambda \in \mathbb{R}^2\backslash\{0\}$. If $\lambda_1 = 0$, then $(\lambda \cdot f)(1) = 0$. If, on the other hand, $\lambda_1 \neq 0$, then $\mathrm{sign}[(\lambda \cdot f)(1)] = \mathrm{sign}[\lambda_1]$ and $\lim_{\theta\to-1} \mathrm{sign}[(\lambda \cdot f)(\theta)] = -\mathrm{sign}[\lambda_1]$, and so $(\exists\theta)(\lambda \cdot f)(\theta) < 0$. $\square$

H. Relationship to Signed-Ratio Monotonicity and the Variation Diminishing Property

H.1. Signed-Ratio Monotonicity

Quah and Strulovici (2012) establish that for any two functions $f_1 : \Theta \to \mathbb{R}$ and $f_2 : \Theta \to \mathbb{R}$ that are each single crossing from below, $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing from below for all

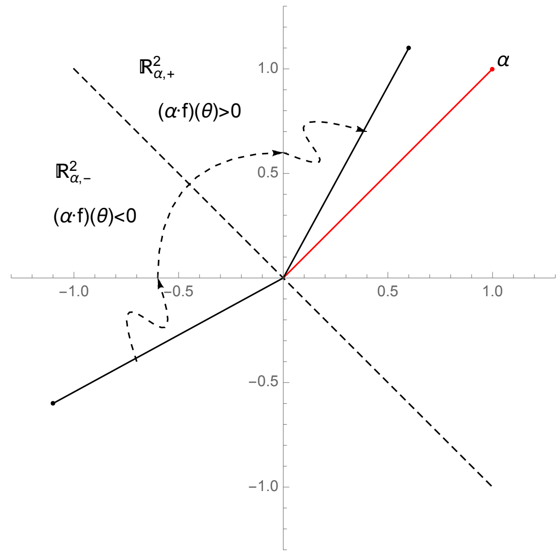


Figure 7: Signed-ratio monotonicity and single crossing of a convex combination.

$\alpha \in \mathbb{R}^2_+$ if and only if f_1 and f_2 satisfy **signed-ratio monotonicity**: for all $i, j \in \{1,2\}$,

$$(\forall \theta_l < \theta_h) \quad f_j(\theta_l) < 0 < f_i(\theta_l) \implies f_i(\theta_h) f_j(\theta_l) \leq f_i(\theta_l) f_j(\theta_h). \tag{42}$$

Given our discussion in Subsection 3.1.1 of a graphical interpretation of ratio ordering, one can see that Condition (42) implies that the vector $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ rotates clockwise as θ increases within the upper-left quadrant (i.e., when $f_1(\cdot) < 0 < f_2(\cdot)$), while it rotates counterclockwise within the lower-right quadrant (i.e., when $f_1(\cdot) > 0 > f_2(\cdot)$); there are no restrictions in the other two quadrants.[44] The dashed curve with arrowheads in Figure 7 provides a depiction. Note that if f_1 and f_2 are both single crossing from below (or both from above), then there cannot exist $\theta_l < \theta_h$ such that one of $f(\theta_l)$ and $f(\theta_h)$ is in the upper-left quadrant and the other in the lower-right quadrant. It follows that if f_1 and f_2 are both single crossing from below, then ratio ordering implies signed-ratio monotonicity; more generally, however, the implication is not valid.

Figure 7 also illustrates Quah and Strulovici's (2012) result, analogous to Figure 4 for Lemma 1. Any linear combination $\alpha \in \mathbb{R}^2_+ \backslash \{0\}$ defines two open half spaces, $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$ and $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, as indicated in Figure 7. If the vector $f(\theta)$ rotates monotonically as θ increases from $\mathbb{R}^2_{\alpha,-}$ to $\mathbb{R}^2_{\alpha,+}$, or either half space contains the vector $f(\theta)$ for all θ, then $\alpha \cdot f \equiv \alpha_1 f_1 + \alpha_2 f_2$ is single crossing from below. Conversely, if $f(\theta)$ does not rotate monotonically in the upper-left or lower-right quadrant, then there

[44] To be precise: by "quadrant" we mean the interiors, i.e., excluding the axes.

exists $\alpha \in \mathbb{R}^2_+\backslash\{0\}$ such that $\alpha \cdot f$ is not single crossing from below.

H.2. Variation Diminishing Property

Karlin (1968) assumes a completely ordered domain, so consider a completely ordered Θ, with $|\Theta| \geq 3$ to avoid trivialities. Let $K(i, \theta) \equiv K_i(\theta)$ for $i = 1, 2$ and some functions $K_i : \Theta \to \mathbb{R}$. The function K is said to be totally positive of order two, abbreviated TP_2, if K_1 and K_2 are both non-negative functions and $(\forall \theta_l < \theta_h)\ K_1(\theta_l)K_2(\theta_h) \leq K_1(\theta_h)K_2(\theta_l)$. The variation diminishing property of Karlin (1968, Theorem 3.1 in Chapter 5) implies that if—and, more or less, only if—K is TP_2, then any linear combination of K_1 and K_2 is single crossing. There are, however, single-crossing functions f_1 and f_2 that are ratio ordered such that there is no TP_2 function K with

$$\{\alpha_1 f_1 + \alpha_2 f_2 : \alpha \in \mathbb{R}^2\} = \{\alpha_1 K_1 + \alpha_2 K_2 : \alpha \in \mathbb{R}^2\}. \tag{43}$$

When f_1 and f_2 are each single crossing, ratio ordered, and linearly independent, a TP_2 function K satisfying (43) exists if and only if the set $\{f(\theta) : \theta \in \Theta\}$ lies in an open half space of $\mathbb{R}^2$ defined by a hyperplane that passes through the origin. (A proof is available from the authors on request.) Ratio ordering does imply that the set lies in a half space, as noted earlier, but the half space need not be open.

I. Relaxing Anti-Symmetry

This appendix shows how anti-symmetry of $\leq$ over Θ can be dropped by appropriately generalizing the definition of single crossing. This extension is useful, for example, because rankings over Θ based on norms (say, when $\Theta \subseteq \mathbb{R}^n$) generally violate anti-symmetry.

Assume $(\Theta, \leq)$ is a preordered set, i.e., $\leq$ is a binary relation that is reflexive and transitive, but not necessarily anti-symmetric. We write $\theta' \cong \theta''$ when $\theta' \geq \theta''$ and $\theta' \leq \theta''$.

Definition 10. When $(\Theta, \leq)$ is a preordered set, a function $f : \Theta \to \mathbb{R}$ is **single crossing** if $\mathrm{sign}[f]$ is monotonic and

$$(\forall \theta' \cong \theta'') \quad \mathrm{sign}[f(\theta')] = \mathrm{sign}[f(\theta'')]. \tag{44}$$

The relation $\cong$ partitions the set Θ into equivalence classes. Definition 10 augments our maintained definition of single crossing (Definition 1) by requiring the function's sign to be constant within each equivalence class. Thus, Theorem 0 and its proof remain the same.

Lemma 1 is modified as follows:

Lemma 4. *Let $(\Theta, \leq)$ be a preordered set and $f_1, f_2 : \Theta \to \mathbb{R}$. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is single crossing $\forall \alpha \in \mathbb{R}^2$ if and only if f_1 and f_2 are (i) each single crossing, (ii) ratio ordered, and (iii) $(\forall \theta' \cong \theta'')$ $f_1(\theta')f_2(\theta'') = f_1(\theta'')f_2(\theta')$.*

The rest of our main results—in particular, Proposition 2, Theorem 1, and Theorem 3—and their proofs remain the same.

The proof of Lemma 4 consists of establishing Claim 11 and Claim 12 below.

Claim 11. *For any $\alpha \in \mathbb{R}^2$,*

$$\begin{aligned} \textit{either } (\forall \theta' < \theta'') \quad & \operatorname{sign}[(\alpha \cdot f)(\theta')] \leq \operatorname{sign}[(\alpha \cdot f)(\theta'')], \\ \textit{or } (\forall \theta' < \theta'') \quad & \operatorname{sign}[(\alpha \cdot f)(\theta')] \geq \operatorname{sign}[(\alpha \cdot f)(\theta'')] \end{aligned}$$

if and only if (i) $\operatorname{sign}[f_1]$ *and* $\operatorname{sign}[f_2]$ *are monotonic, and (ii) for either $i = 1$ and $j = 2$, or vice-versa,*

$$\begin{aligned} &(\forall \theta_l < \theta_h) \quad f_i(\theta_l)f_j(\theta_h) \leq f_i(\theta_h)f_j(\theta_l) \quad \textit{and} \\ &(\forall \theta_l < \theta_m < \theta_h) \quad f_i(\theta_l)f_j(\theta_h) = f_i(\theta_h)f_j(\theta_l) \iff \begin{cases} f_i(\theta_l)f_j(\theta_m) = f_i(\theta_m)f_j(\theta_l), \\ f_i(\theta_m)f_j(\theta_h) = f_i(\theta_h)f_j(\theta_m). \end{cases} \end{aligned}$$

Proof. The proof is the same as the proof of Lemma 1, as that proof does not use anti-symmetry of $\leq$ over Θ. *Q.E.D.*

Claim 12. *For any $\alpha \in \mathbb{R}^2$,*

$$(\forall \theta' \cong \theta'') \quad \operatorname{sign}[(\alpha \cdot f)(\theta')] = \operatorname{sign}[(\alpha \cdot f)(\theta'')] \tag{45}$$

if and only if (i) f_1 and f_2 each satisfy (44), and (ii) $(\forall \theta' \cong \theta'')$ $f_1(\theta')f_2(\theta'') = f_1(\theta'')f_2(\theta')$.

Proof. ($\implies$) Part (i) holds trivially: consider $\alpha = (1, 0)$ or $\alpha = (0, 1)$. For Part (ii), suppose, towards contradiction, that $(\exists \theta' \cong \theta'')$ $f_1(\theta')f_2(\theta'') \neq f_1(\theta'')f_2(\theta')$. Then, $\alpha' \equiv (-f_2(\theta'), f_1(\theta')) \neq 0$. It follows that $(\alpha' \cdot f)(\theta') = 0 \neq (\alpha' \cdot f)(\theta'')$, contradicting (45).

($\impliedby$) Take any $\alpha \in \mathbb{R}^2$ and $\theta' \cong \theta''$. If $f_1(\theta')f_2(\theta'') = f_1(\theta'')f_2(\theta') \neq 0$, then all four function values are non-zero. Thus,

$$\begin{aligned} \operatorname{sign}[(\alpha \cdot f)(\theta')] &= \operatorname{sign}\left[\frac{f_1(\theta'')}{f_1(\theta')}\left(\alpha_1 f_1(\theta') + \alpha_2 f_2(\theta')\right)\right] \quad (\text{because } \operatorname{sign}[f_1(\theta')] = \operatorname{sign}[f_1(\theta'')] \neq 0) \\ &= \operatorname{sign}\left[\alpha_1 f_1(\theta'') + \alpha_2 f_2(\theta'')\right] \quad (\text{using } f_2(\theta'') = f_2(\theta')f_1(\theta'')/f_1(\theta')) \\ &= \operatorname{sign}[(\alpha \cdot f)(\theta'')]. \end{aligned}$$

If, on the other hand, $f_1(\theta')f_2(\theta'') = f_1(\theta'')f_2(\theta') = 0$, then at least one function value, say $f_1(\theta')$, equals zero. It follows from (44) that $f_1(\theta') = f_1(\theta'') = 0$. Thus,

$$\mathrm{sign}[(\alpha \cdot f)(\theta')] = \mathrm{sign}[\alpha_2 f_2(\theta')] = \mathrm{sign}[\alpha_2 f_2(\theta'')] = \mathrm{sign}[(\alpha \cdot f)(\theta'')]. \qquad \textit{Q.E.D.}$$

References

ABBAS, A. E. AND D. E. BELL (2011): "One-switch independence for multiattribute utility functions," *Operations Research*, 59, 764–771.

AMADOR, M. AND K. BAGWELL (2013): "The Theory of Optimal Delegation with an Application to Tariff Caps," *Econometrica*, 81, 1541–1599.

ATHEY, S. (2002): "Monotone Comparative Statics under Uncertainty," *Quarterly Journal of Economics*, 117, 187–223.

BERGEMANN, D. AND S. MORRIS (2017): "Information Design: A Unified Perspective," Cowles Foundation Discussion Paper No. 2075R2.

BLUME, A., O. J. BOARD, AND K. KAWAMURA (2007): "Noisy talk," *Theoretical Economics*, 2, 395–440.

CARROLL, G. (2012): "When are Local Incentive Constraints Sufficient?" *Econometrica*, 80, 661–686.

CELIK, G. (2015): "Implementation by Gradual Revelation," *RAND Journal of Economics*, 46, 271–296.

CHO, I.-K. AND J. SOBEL (1990): "Strategic Stability and Uniqueness in Signaling Games," *Journal of Economic Theory*, 50, 381–418.

CHOI, M. AND L. SMITH (2016): "Ordinal Aggregation Results via Karlin's Variation Diminishing Property," *Journal of Economic Theory*, 168, 1–11.

CRAWFORD, V. AND J. SOBEL (1982): "Strategic Information Transmission," *Econometrica*, 50, 1431–1451.

DE MEYER, B. AND P. MONGIN (1995): "A note on affine aggregation," *Economics Letters*, 47, 177–183.

DOWNS, A. (1957): *An Economic Theory of Democracy*, New York: Harper.

DUGGAN, J. (2014): "Majority voting over lotteries: Conditions for existence of a decisive voter," *Economics Bulletin*, 34, 263–270.

FISHBURN, P. C. (1977): "Approximations of two-attribute utility functions," *Mathematics of Operations Research*, 2, 30–44.

GANS, J. S. AND M. SMART (1996): "Majority Voting with Single-crossing Preferences," *Journal of Public Economics*, 59, 219–237.

HARSANYI, J. C. (1955): "Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility," *Journal of Political Economy*, 63, 309–321.

JAFFRAY, J.-Y. (1975): "Semicontinuous Extension of a Partial Order," *Journal of Mathematical Economics*, 2, 395–406.

KAMENICA, E. AND M. GENTZKOW (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615.

KARLIN, S. (1968): *Total Positivity*, vol. I, Stanford, California: Stanford University Press.

KUSHNIR, A. AND S. LIU (2018): "On the Equivalence of Bayesian and Dominant Strategy Implementation for Environments with Non-Linear Utilities," Forthcoming in *Economic Theory*.

LIU, S. AND H. PEI (2017): "Monotone Equilibria in Supermodular Signalling Games," Available at SSRN.

MELUMAD, N. D. AND T. SHIBANO (1991): "Communication in Settings with No Transfers," *RAND Journal of Economics*, 22, 173–198.

MILGROM, P. AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180.

MILGROM, P. R. (1981): "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 12, 380–391.

——— (2004): *Putting Auction Theory to Work*, Cambridge University Press.

OK, E. A. (2007): *Real Analysis with Economic Applications*, vol. 10, Princeton University Press.

QUAH, J. K.-H. AND B. STRULOVICI (2012): "Aggregating the Single Crossing Property," *Econometrica*, 80, 2333–2348.

SEIDMANN, D. J. (1990): "Effective Cheap Talk with Conflicting Interests," *Journal of Economic Theory*, 50, 445–458.

SHEPSLE, K. A. (1972): "The Strategy of Ambiguity: Uncertainty and Electoral Competition," *American Political Science Review*, 66, 555–568.

SILBERT, J. (2018): "Persuasion with a Committee of Senders," Manuscript in preparation.

SMITH, L. (2011): "Frictional Matching Models," *Annual Review of Economics*, 3, 319–338.

SPENCE, M. (1973): "Job Market Signaling," *Quarterly Journal of Economics*, 87, 355–374.

ZECKHAUSER, R. (1969): "Majority Rule with Lotteries on Alternatives," *Quarterly Journal of Economics*, 83, 696–703.