Patent Auctions and Bidding Coalitions: Structuring the Sale of Club Goods

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Abstract

Auctioneers of patents are observed to allow joint bidding by coalitions of buyers. These auctions are distinguished from standard ones by the patents being non-rivalrous, but still excludable, in consumption–that is, they are club goods. This affects the way coalitions of bidders impact auction performance. We study the implications of these coalitions on second-price (or equivalently, ascending) auctions. Although the formation of coalitions per se can benefit the seller, we show that stable coalition profiles tend to consist of excessively large coalitions, to the detriment of auction revenue. Limiting the permitted coalition size confers benefits on the seller. Lastly, we compare patent auctions to multi-license auctions, and we find that the latter are superior in a large class of environments.

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1 Introduction

On June 27th, 2011 an auction began in which the patent portfolio of Nortel was put up for sale as part of its bankruptcy proceedings.[1] The auction started with an initial bid of $900 million dollars from Ranger (a subsidiary of Google). It proceeded in rounds, with each bidder being required to outbid the leading bid in each round to retain eligibility in subsequent rounds. The auction proceeded over 19 rounds of bidding, culminating in a winning bid of 4.5 billion dollars on June 30th, 2011.[2] When the Nortel auction began, it had five bidders: Ranger, Apple, Intel, Norpax (an affiliate of RPX Corporation, which is an membership-based aggregator including over 320 firms as of 2020), and Rockstar (a consortium of Research in Motion, EMC, Ericsson, Sony and Microsoft).[3] Importantly, both at the beginning and during the course of the auction, the seller explicitly accommodated the presence of bidding coalitions.[4]

This paper studies the incentives for coalitions, like those observed in the Nortel auction, to form in patent auctions. It then examines the impact that bidding coalitions have on auction revenues. It also identifies measures that sellers may implement so as to mitigate any adverse impact that such coalitions have on the ultimate sale price. We study these issues in the context of second-price auctions (or equivalently, ascending price auctions).

Central to the analytical approach are two attributes of intellectual property (IP), and patents in particular: First, patents are 'non-rivalrous,' in the sense that they can be productively used by multiple firms, although at some point negative externalities among users may emerge. Second, patents are 'excludable,' because additional users can be prevented from accessing a new technology protected by a patent. These attributes make patents an example of a 'club good'.[5] Hence, while we focus on the patent application, the analysis presented

[1] This account of the Nortel auction follows the report of Ernst & Young (2011).

[2] This result was described in subsequent court hearings as 'record breaking ... in the patent industry generally' (see Brickley, 2011).

[3] Norpax dropped out in round two. Rockstar did not submit a bid in round five resulting in a loss of eligibility. However, Rockstar regained eligibility in the same round after joining forces with Apple, with consent of the seller. Intel dropped out in round six. At the end of round six, two eligible bidders remained – Ranger and the reconstituted Rockstar. The seller at this point gave consent to Ranger and Rockstar to enter into partnership discussions with Intel and Norpax. After round eight, Ranger and Intel entered into a bidding partnership, with Ranger as the lead partner. In subsequent rounds, competition between Ranger and Rockstar drove the price of the Nortel patent portfolio up to the final price.

[4] The Nortel auction is a prominent example of the distinctive presence of bidding coalitions in patent auctions. While data on the patent market is limited (a consequence of the decentralized nature of the market) industry sources report RPX and AST (another membership-based aggregator) as the largest purchasers of patents in the U.S. in 2017 (see Richardson et al., 2018).

[5] See Buchanan (1965).

here applies to any other club good.[6] The non-rivalrous but excludable nature of club goods colors the incentives for coalitions to form in auctions, and the costs and benefits that the seller internalizes from their presence. In particular, the non-rivalrous nature of consumption means that coalition formation can increase bidders' valuations, and therefore revenues, at least over some range of coalition sizes. The narrative of coalition formation in the Nortel patent auction is suggestive of this feature. On the other hand, coalitions tend to depress the degree of competition on the market and may have a negative effect on the seller's revenue.

We explore these economic forces with a model in which a seller owns a patent that has multiple different applications. Let the number of applications be $\overline{n}$.[7] There is a pool of N potential firms interested in buying access to the underlying technology. Before the patent is sold in a second-price auction, firms are allowed to form coalitions, and the resulting coalitions of firms participate in the auction as individual bidders. Once the auction has taken place, the winning coalition allows its members to access the underlying technology (through, say, licensing).[8]

We account for the presence of potential negative externalities among patent users: once the patent's applications are implemented by $\overline{n}$ firms, distributing licenses further among members generates no additional value for the coalition.[9] Therefore, if the winning coalition is larger than $\overline{n}$, it does not distribute licenses to all of their members, but only to $\overline{n}$ of them.

We model how each coalition's value for the patent is formed in two alternative ways. In the first environment, which we name 'the limited-values case', after the coalition formation stage, each coalition larger than $\overline{n}$ arbitrarily chooses $\overline{n}$ members who then obtain a private and independent realization for the patent's value, drawn by the same distribution. Therefore, the valuation for the patent for each coalition is the sum of the realizations of the individual private valuations of its members, up to a maximum of $\overline{n}$ realizations. Observe that in the limited-values case, any coalition (weakly) larger than $\overline{n}$ has the same value distribution, and therefore the same probability of winning the auction.

[6] Our analysis also applies to auctions in which a bundle of goods, rather than a single one, are sold together, and coalitions of buyers (each with a single-unit demand) are allowed to participate as individual bidders.

[7] Note that patents are often sold in bundles, see Richardson et al. (2018).

[8] The Nortel auction has nuances that we abstract from for the sake of tractability. Notably, we do not not allow coalitions to form or change during the course of the auction. Rather, in our setting, coalitions form first, then firms' valuations for the patent are realized, and then the auction takes place.

[9] For example, if two firms commercialize the same application, some value is destroyed by market competition.

In the second environment, which we name 'the optimized-values case', if a coalition is larger than $\overline{n}$, all firms in the coalition obtain a value realization for the patent, and the coalition's value for the patent is the sum of the $\overline{n}$ highest realizations. If a coalition's size is less than or equal to $\overline{n}$, the limited-values case and the optimized-values case are equivalent.

We consider the process of coalition formation, which takes place before the patent values' realizations occur. Foreseeing each coalition's value, as well as the equilibrium outcome of the auction, firms endogenously form coalitions. Stable coalition profiles are ones that satisfy natural equilibrium constraints in the coalition-formation stage: a coalition profile is stable if no firm has a profitable unilateral deviation in joining a different coalition, or in participating in the auction as individual bidder.

Our first set of results provides a characterization of the stable coalition profiles in the complete information case, in which every firm's value for the patent is known and equal across all firms.[10] In this situation, any coalition profile in which there are at least two coalitions strictly larger than $\overline{n}$ is stable, and allows the seller to extract the maximum surplus. However, there are other stable coalition profiles (for example, the grand coalition), in which the seller's revenue can be lower, and potentially zero. Notably, we show that the maximum revenue that the seller can extract from the auction is non-monotonic in the number of applications $\overline{n}$–that is, sellers benefit from developing technologies with an intermediate number of applications.

Next, we address the case of independent and private valuations. We assume that firms' realizations of their private values occur once the coalition formation stage is over, and before the auction starts. Our first result illustrates the effect of increasing the concentration of bidders by considering a coalition profile and moving one firm i from a (weakly) smaller to a larger coalition, when both coalitions are strictly smaller than $\overline{n}$. Such move has several implications on the two coalitions' expected auction outcomes: (i) it changes the value distribution of the patent of both coalitions by adding firm i's realization to the valuation of the larger coalition and subtracting firm i's realization from the valuation of the smaller one; (ii) it changes the expected price paid by either of the two coalitions conditional on winning, and (iii) it increases the probability of winning the auction for the larger coalition, and it decreases the probability of winning for the smaller coalition. We show that the overall benefits generated by such move for the larger coalitions are larger than the expected costs generated for the smaller one, so that the move strictly increases the joint expected payoffs of the two coalitions in the auction.

[10]With complete information, the limited-values case and the optimized-values one are clearly equivalent.

In other words, *an individual firm generates strictly more additional value by joining a larger coalition than a smaller one.*[11]

This result has several important implications. First, it suggests that when coalitions of firms are allowed to participate in patent auctions there is a strong incentive for them to aggregate in coalitions large enough to exploit all the patent's applications. In particular, stable coalition profiles never include multiple coalitions strictly smaller than $\overline{n}$. However, this tendency toward aggregation is mitigated by the presence of negative externalities: once a coalitions becomes large enough (i.e. it already includes at least $\overline{n}$ firms), the additional value generated by additional members decreases, diminishing their ability to attract more firms. The result allows us to identify a set of necessary conditions for stable coalition profiles, which depends on the interplay between $\overline{n}$ and the total number of firms on the market, N.

Second, it allows us to derive implications on the seller's revenue under stable coalition profiles. These implications depend on the way coalitions form their valuation for the patent. In the limited-values case, we find that, once a coalition has already expanded to $\overline{n}$ firms, the addition of new firms to that coalition always (weakly) reduces the expected revenue of the seller in the auction. This result allows us to identify upper bounds for the seller's revenue: If the number of applications $\overline{n}$ is large relative to N, the expected revenue for the seller is very small. In particular, the expected second highest realization between the sum of $\overline{n}$ individual valuations, and the sum of $N - \overline{n}$ ones constitutes an upper bound for the expected revenue of the seller across all stable coalition profiles . As $\overline{n}$ approaches N, the seller's revenue converges to zero. On the other hand, if $\overline{n}$ is smaller, there can be enough firms in the industry to aggregate into multiple coalitions of size $\overline{n}$. In this case, the upper bound we identify for the seller's revenue is more significant, and it amounts to the expected second highest realization among $\lceil N/\overline{n} \rceil$ coalitions, all of size $\overline{n}$ except at most one. These results allow us to deliver a specific recommendation to allow the seller to be able to generate the previously identified revenue's upper bound in the limited-values case. In particular, we show that if the seller introduces a ceiling on the coalitions' size equal to $\overline{n}$, there is a uniquely stable coalition profile. As it turns out, such profile achieves the upper bound for the seller's revenue previously identified.

Next, we study the seller's revenue in the optimized-values case. In this case, even coalitions

[11] Since this result pertains to coalitions strictly smaller than $\overline{n}$, it holds both in the limited-values case and in the optimized-values one.

that are already of size $\overline{n}$ or larger can still increase their own expected value for the patent by adding new members (as they will be able to select the $\overline{n}$ highest among a larger number of draws). Therefore, *for any given coalition profile*, this setting induces large coalitions to bid more aggressively, hence improving the seller's revenue. On the other hand, we show that in this setting firms have an even stronger incentive to aggregate into large coalitions, ultimately driving the seller's revenue to zero. To illustrate this tendency, we show that even in the presence of very strong negative externalities ($\overline{n} = 1$), the grand coalition is the only stable coalition.

In addition to selling a patent through an auction, IP's owners typically have the option of retaining ownership of the patent and licensing it to multiple parties through a multi-object auction. In the last part of our analysis we compare the two auction formats from the seller's revenue perspective. Since in the perfect information case selling individual licenses always allows the seller to extract the maximum surplus from the IP, selling individual licenses clearly weakly dominates selling a patent. In the imperfect information case the comparison between the two selling mechanisms is more subtle, and it involves the following trade-off: in a multi-license auction, each firm participates in the auction as an individual bidder, and the revenue of the seller is determined by the highest realization of the new technology's value among the losers of the auction. For a relatively high number of licenses on sale, this realization can be rather low. On the other hand, in a patent's auction, bids are the sum of the coalition members' realized valuations.[12] Therefore, the winning bid is the highest sum, and the seller's revenue is the second highest sum, of randomly chosen realizations. We focus our attention on gamma-distributions and, by simulation, we find that multi-license auctions dominate patent auctions as selling mechanism across the vast majority of the parameter space we investigate. We also find that patent auctions have the best chance to perform well vis-à-vis multi-license auctions when the number of applications $\overline{n}$ is relatively large, and the distribution of the valuations is relatively right-skewed (i.e. the difference between the highest realizations and the others is relatively large).

[12] This argument applies to both the limited-values and the optimized-values cases.

Related Literature. The question of how to sell IP has been examined in a previous literature starting from Katz and Shapiro (1985). This literature considers the strategic choice of selling, unilaterally exploiting, or licensing IP when the seller and buyers are competitors in the product market. In this literature transactions typically occur in the IP market via bargaining or take-it-or-leave it offers. Our approach is different as we focus on the performance and structure of auctions as selling mechanisms for IP and other club goods.

Related theoretical work has focused on framing the commercialization of licenses in the context of auctions with negative externalities (see Hoppe, Jehiel, and Moldovanu, 2006, Jehiel and Moldovanu, 1996, and Jehiel, Moldovanu and Stacchetti, 1996). Our work departs from this perspective because, in considering auctions for patents, we focus on the club-good nature of IPs, rather than the fact that the losing participants in an auction for a license may experience negative externalities on the final marketplace. In this sense, our work follows a relatively sparse literature on club goods started by Buchanan (1965), with more recent work by Deb and Razzolini (2001), Baik, Kim, and Na (2001), Norman (2004), and Fang and Norman (2010). In this literature, Loertscher and Marx (2017) study optimal pricing mechanisms for intermediaries of club goods (without the possibility for buyers to form coalitions which act as individual entities on the club-good marketplace).

This paper is also related to different strands of the auction literature. Our coalition formation process results in an auction populated by bidders with asymmetric value distributions. Asymmetric auctions have been studied, among others, by Maskin and Riley (2000a and 2000b) and Cantillon (2008), and, more recently, by Kirkegaard (2009, 2012).

Considering coalitions of buyers in auctions draws a natural parallel to the large literature on bidding rings in auctions (see the excellent overview provided by Marshall and Marx, 2012).[13] Most of this literature focuses on the fact that, since collusive behavior tends to hurt the seller and is deemed illegal under the antitrust laws, bidders' cartels need to be self-enforcing (see for example McAfee and McMillan, 1992). Our perspective is different, because in the case of an auction for a club good, coalition formation is not necessarily negative for the seller, and need not be presumptively anticompetitive. Therefore, we completely abstract from self-enforcement issues and we focus on the endogenous coalitions patterns we expect to see in an auction if coalitions are allowed to form.[14]

[13] For more recent contributions in this literature, see Chassang and Ortner (2019) and Che, Condorelli, and Kim (2018).

[14] There is also a parallel between our analysis and the question of how to optimally sell a bundle of identical

The notion that the set of players in a strategic situation is, in itself, endogenous is present in some of the work on club formation (see, e.g., Ellickson, Grodal, Scotchmer, and Zame, 1999, and Wooders, Cartwright, and Selten, 2006). More recently, Baccara and Yariv (2013 and 2016) have studied the (endogenous) homogeneity of players in the context of public-good games. Here we consider a different strategic interaction, namely an auction, and we analyze the set of bidders and the auction outcomes expected to emerge in equilibrium. In turn, this allows us to evaluate the auction as a selling mechanism from the seller's perspective, and to compare it with alternative mechanisms.

2 The Model

2.1 Set-Up

A seller owns a patent on a new technology, and several ex-ante identical firms are all potential adopters. Let $\mathcal{N}$ be the set of firms on the market, and $N \equiv |\mathcal{N}| > 1$.[15] The patent is made available for sale in a second-price auction, in which coalitions of firms can participate as individual bidders. If the patent is sold to a coalition of firms, the coalition can make it available to multiple members.

Each firm i's private valuation V_i is independently and identically distributed over $\mathbb{R}_+$. We consider two cases. In the complete information case, there exists $v > 0$ such that $\forall i$, $Pr(V_i = v) = 1$. That is, each firm's value distribution is degenerate at some $v > 0$, so all firms' valuations for the patent are known and equal.[16] In the incomplete information case, the value distribution is non-degenerate and has a cumulative distribution F over $\mathbb{R}_+$. We assume that $0 \in supp(F)$, $E(V_i) = v > 0$, and the distribution has a log-concave density f.[17]

goods in auctions where coalitions of bidders are allowed to form. Avery and Hendershott (2000) explore this issue, with the distinction that their setting does not allow the possibility of buyers' coalitions.

[15] Throughout the paper, we assume N to be exogenous and therefore abstract from firms' entry decisions in the auction. However, suppose that that is not the case and, for example, N grows in the number of applications $\overline{n}$. Our analysis still applies, with the only difference that, instead of N and $\overline{n}$ both being exogenous, our results will depend solely on $\overline{n}$ and the way in which $\overline{n}$ determines N.

[16] The complete information case also captures applications in which firms' valuations are realized after the auction takes place.

[17] That is, $\log f$ is concave on $supp(F)$. Examples of distributions with log-concave density functions include the uniform distribution (over any positive-length interval), the exponential distribution, the normal distribution (including any truncated one over $[a, b]$ for $-\infty \leq a < b \leq \infty$), the gamma distribution, the beta distribution, etc.

The game unfolds as follows:

(1) Firms form coalitions (see Section 2.3);

(2) In the incomplete information case, after the coalitions are formed, firms' valuations for the patent are realized. Coalitions determine their value for the patent according to either the limited-values or the optimized-value cases (see Section 2.2);

(3) Coalitions participate as individual bidders in a second-price auction (or, equivalently, an ascending auction) to buy the patent;

(4) Some or all members of the winning coalition gain access to the patent (see Section 2.2).

Before we discuss how coalitions determine their own valuations for the patent and form, some other assumptions merit discussion. The assumptions that the seller chooses a second-price auction (or, equivalently, an ascending auction) is made, in part, for tractability. In fact, as it will become apparent, the coalition formation stage yields an auction in which the bidders' valuations are drawn from different distributions. A second-price auction ensures bidding their own value to be each coalition's weakly dominant strategy, allowing us to characterize the equilibrium outcome of the asymmetric auction.[18,19]

An assumption underlying the coalition formation stage is that all firms observe the sizes of the other coalitions.[20] The assumption that valuations are realized after coalitions are formed allows us to abstract away from potential strategic communication issues at the time of coalition formation.[21]

[18]Equilibrium bidding behavior in first-price auction with asymmetric bidders is significantly more difficult to fully characterize, as highlighted, for example, by Maskin and Riley (2000a and 2000b) and Cantillon (2008).

[19]Note that we also ruling out the possibility of a secondary market for the patent. Allowing for secondary markets would require to add a third stage of the game in which, after the auction has taken place, the winner can resell the patent to another firm or coalition. This would make the coalitions' valuations for the patent a function of the ex-post market outcomes and, in turn, of the whole coalitions' profile. This possibility introduces non-trivial complications and it is outside the scope of this paper.

[20]This assumption simplifies the analysis of the coalition formation stage, but since we consider a second-price auction, any assumption on the information structure at the time of the auction (e.g., firms observe some other coalition's members' valuations, etc.) is irrelevant for the auction's outcome.

[21]Coalitions' formation with asymmetric information is known to be challenging and require numerous additional assumptions. See for example Dutta and Vohra (2005).

2.2 Coalitions' Values

In this section we illustrate how coalitions determine their own value for the patent. This value can be affected by the presence of potential negative externalities among the new technology's adopters. In general, we assume that if the patent is adopted by too many firms in the winning coalition, it might be the case that not all firms in the coalition are able to enjoy the its full benefits. For tractability, we represent such diminishing marginal value of the technology with an exogenously given $\overline{n}(\leq N)$ such that only up to $\overline{n}$ firms in the winning coalition can adopt the new technology. If more than $\overline{n}$ firms adopt it, any firm in excess to $\overline{n}$ obtains zero value from the adoption.

Therefore, in the complete information case, the total value of the patent for a coalition of size m is always $W \equiv \min\{m, \overline{n}\} \cdot v$. In the incomplete information case, we model the way in which the coalition determines its value in two alternative ways, which we term *limited-values* case and *optimized-values* case, respectively.

Limited-values Case: *In each coalition, at most $\overline{n}$ randomly selected firms obtain a realization of V and can adopt the patent. For a coalition of size m, this yields the value $W = \sum_{i=1}^{\min\{m,\overline{n}\}} V_i$, where, for $m > \overline{n}$, $i = 1, 2, \ldots,$ denote some randomly chosen firms in the coalition. Therefore, the value distribution of any coalition of size m is the convolution of $\min\{m, \overline{n}\}$ individual distributions F.*

Optimized-values Case: *Each firm on the market obtains a realization of V, and each coalition selects the highest $\min\{m, \overline{n}\}$ realizations among its members to determine the value of the patent. For a coalition of size m, this yields the value $W = \sum_{i=1}^{\min\{m,\overline{n}\}} V_{(m,i)}$, where we denote as $V_{(m,i)}$ the i-th highest value among m draws from the distribution F.*

Notice that for coalitions (weakly) smaller than $\overline{n}$, the coalition's value is equal to the sum of all members' realizations in both the limited- and the optimized-values cases. Therefore, for coalitions smaller than $\overline{n}$ the two cases are equivalent. Also, a fundamental difference between the limited values and optimized values cases is that, under the former, all coalitions of size $m \geq \overline{n}$ have the same value distribution. On the other hand, under the latter, coalitions of size $m > \overline{n}$ can still benefit from additional members as they will allow them to select the $\overline{n}$ highest from a larger pool of realizations.

The general externalities' structure we adopt can be also interpreted as follows. Suppose that the new technology has a maximum number $\overline{n}$ of distinct potential applications which are unrelated to each other. Hence, within each coalition, a realization of V represents the realized value of one application. Suppose that if more than $\overline{n}$ firms adopt the technology within a coalition, cumulative profits start decreasing because of increased competition in some applications' markets.[22] In other words, $\overline{n}$ could be interpreted as the number of adopting firms that maximizes the cumulative value of the patent for any coalition of size $m > \overline{n}$. In this case, the coalition would never find optimal to let more than $\overline{n}$ firms adopt the technology, and our assumption (any firm in excess to $\overline{n}$ obtains zero value from the adoption) would imply no loss of generality.[23]

The limited-values case applies to situations in which, after alliances among firms are formed, firms in each coalition start researching the applications of the patent in order to obtain a realization of V. However, suppose that coalitions act under the constraint that each application can be investigated by at most one member (this could be due, for example, to substantial research costs). Therefore, while larger coalitions have the breadth to explore more applications than smaller ones, they still cannot obtain more than $\overline{n}$ realizations of the patent's value. Thus, each coalition strictly larger than $\overline{n}$ randomly selects $\overline{n}$ firms that will explore the patent's applications. Such firms would be the only ones to obtain a realization of V within the coalition. After the research has been concluded, the firms that explored the applications learn their private valuations for the applications, and they make them public within the coalition. At that point, the coalition can assess its total value of the patent and participate as a bidder in the auction.

The optimized-values case corresponds to situations in which each firm on the market is able to conduct research to determine their individual valuation for the new technology. This corresponds to scenarios in which conducting research and obtaining value realizations is relatively cheap. Therefore, each coalition strictly larger than $\overline{n}$ is able to observe all the realizations obtained by its members and, upon winning the auction, to maximize the coalition's joint profits, it gives access to the patent to the $\overline{n}$ members associated with the highest valuations.

Finally, we assume that any coalition that does not win the auction gets a payoff of zero

[22] For example, consider a new type of lens, that could be applied to produce glasses, binoculars, and telescopes. However, if more than, say, one firm in the same product market adopts the new lens, joint profits in that market decline. Therefore, we have $\overline{n} = 3$.

[23] One could consider more general externalities' structures by allowing the adoption of the technology to yield strictly decreasing (rather than constant up to $\overline{n}$, and then equal to zero) marginal returns. This extension is discussed in Section 6.

regardless of the number of firms adopting the new technology (i.e. there are no negative externalities on non-adopting firms). This assumptions allows us to avoid the patent's value for a coalition to depend on the entire coalition profile.

2.3 Coalitions and Stability

Before values are realized and the auction takes place, firms can form coalitions. Consider $\sigma = \{\sigma_1, .., \sigma_J\}$ to be a partition of the set $\mathcal{N}$, or a *coalition profile.*[24] Within each coalition profile σ, each coalition $\sigma_j \in \sigma$ (which could be a set of multiple firms or an individual firm) is a bidder in the auction. Let $n_j \equiv |\sigma_j|$–that is, n_j is the size of coalition σ_j. Also, without loss of generality, let σ_1 be the largest coalition within σ, σ_2 the second largest, etc., and so on–that is, $n_1 \geq n_2 \geq ... \geq n_J$.

For coalition σ_j, we denote the total value of the patent as W_j, which is determined as described in Section 2.2. As two coalition profiles characterized by the same profile $\{n_1, .., n_J\}$ are ex-ante payoff-indistinguishable, we treat such coalition profiles as an equivalence class and denote the resulting collection of equivalence classes by Σ. For sake of simplicity, with a slight abuse of notation, we treat the elements of Σ as coalition profiles rather than equivalence classes of coalitions profiles.

Given a coalition profile σ, the payoff of coalition σ_j from winning the auction is the difference between the realization of W_j and the second highest bid. We denote the coalition's profit from an ex-ante perspective (i.e., expected at the time of the coalition formation, before private valuations realize) by $\pi(\sigma_j; \sigma)$. For any coalition profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_J)$, σ_j's expected payoff is

$$\pi(\sigma_j; \sigma) = \Pr\{\sigma_j \text{ wins}\} \times E\left[W_j - P_j \mid \sigma_j \text{ wins}\right],$$

where P_j is the price paid by σ_j conditional on winning. For standard arguments, it is immediate to see that in the second-price auction each coalition has a weakly dominant strategy in bidding their own value. In the rest of the paper, we focus on equilibria of the auction stage in which bidders bid their own patent's value. Therefore, $\pi(\sigma_j; \sigma)$ is well-defined for any σ and $\sigma_j \in \sigma$.[25]

[24] That is, (i) for each j, $\emptyset \neq \sigma_j \subseteq \mathcal{N}$, (ii) for each $j \neq k$, $\sigma_j \cap \sigma_k = \emptyset$, and (iii) $\bigcup_{j=1}^{J} \sigma_j = \mathcal{N}$. We do not allow the same firm to join two coalitions.

[25] Any tie-breaking rule we could specify for the auction is inconsequential in determining the payoff since such rule is applied only in events in which the payoffs of the winning firms are zero.

Next, we describe the equilibrium notion adopted in the analysis. We focus on coalition profiles that satisfy the stability notion identified by the following two definitions.

Definition (Profitable Deviation) *Consider a coalition profile $\sigma = \{\sigma_1, .., \sigma_J\}$. A firm i in coalition $\sigma_j \in \sigma$ has a profitable deviation if at least one of the following is true:*

1. *The coalition profile $\sigma' = (\sigma'_1, \ldots, \sigma'_{J+1})$ with $\sigma'_j = \sigma_j \backslash \{i\}$, $\sigma'_{J+1} = \{i\}$, and $\sigma'_k = \sigma_k$ for $k \neq j, J+1$ is such that*

$$\pi(\sigma_j; \sigma) < \pi(\sigma'_j; \sigma') + \pi(\{i\}; \sigma'); \tag{1}$$

2. *For some $k \neq j$, the coalition profile $\sigma' = (\sigma'_1, \ldots, \sigma'_J)$ with $\sigma'_j = \sigma_j \backslash \{i\}$, $\sigma'_k = \sigma_k \cup \{i\}$, and $\sigma'_h = \sigma_h$ for $h \neq j, k$ is such that*

$$\pi(\sigma_j; \sigma) + \pi(\sigma_k; \sigma) < \pi(\sigma'_j; \sigma') + \pi(\sigma'_k; \sigma'). \tag{2}$$

Definition (Stable Coalition Profile) *A coalition profile σ is stable if no firm has a profitable deviation.*

This notion of stability underlies the following two requirements for a coalition profile to be stable: first, (1) there exist no coalition σ_j and firm $i \in \sigma_j$ that would receive a strictly higher payoff as a singleton than the amount that her current coalition σ_j is willing to pay to make her stay, and (2) there exist no transfer that a coalition σ_k would be willing to pay a firm i belonging to a different coalition σ_j that is strictly higher than the amount that coalition σ_j is willing to pay to make firm i stay. Each firm, taking as given how other firms aggregate, and foreseeing the equilibrium played in the auction stage, unilaterally selects to be in the coalition where her own marginal value is the highest. Stability is, therefore, a natural equilibrium condition for the coalition-formation stage.[26] In what follows, for any given N and $\overline{n}(\leq N)$, we denote the set of stable coalition profiles by $\Sigma^*(N, \overline{n})$. In addition, for any coalition profile $\sigma \in \Sigma$, let $R(\sigma)$ be the seller's expected revenue after the coalition-formation

[26] We are agnostic with respect to the question of whether there is enough surplus within a coalition to compensate multiple members when they consider deviating simultaneously to other coalitions.

stage.[27] The seller's *maximum* expected revenue from a patent auction as a function of N and $\overline{n}$ is defined by

$$R^*(N, \overline{n}) \equiv \max_{\sigma \in \Sigma^*(N, \overline{n})} R(\sigma),$$

and $R^*(N, \overline{n}) \equiv 0$ if $\Sigma^*(N, \overline{n}) = \emptyset$.

The idea of a stable coalition profile is reminiscent of the notion of the core, which also requires a type of group stability. Nonetheless, there are some important distinctions. First, the setup is different – cooperative games normally specify exogenous group values, rather than group values that are *derived endogenously from a strategic interaction (specifically, an auction)*. Second, under a transferable utility assumption, the core coincides with the coalition profile that maximizes the sum of all coalitions' expected payoffs. In our game, however, each coalition's expected payoff in the auction depends on the entire coalition profile.[28] A deviation which is jointly profitable for the coalitions directly involved in it may impose negative externalities on other coalitions' profits. Therefore, a coalition profile that maximizes the sum of all coalitions' expected payoffs may not be stable.[29] Third, we take a unilateral-deviation approach in our stability notion, while cooperative solutions such as the core allow for arbitrary group deviations. As the auction stage of our game is non-cooperative, we adopt this notion of stability rather than a cooperative one to make the equilibrium notion in the game internally coherent. Finally, we note that similar notions of stability with respect to unilateral (or bilateral) deviations are common in the matching and network literature.[30]

3 Stable Coalition Profiles and Seller's Revenue with Complete Information

We first analyze the case in which all firms' valuations are known and equal to $v > 0$. This setup also applies when firms' private valuations are realized after the auction takes place and v is the individual expected patent's value. In the complete information case, the total value

[27] Since we focus on equilibria involving weakly dominant strategies in the auction stage, $R(\sigma)$ is well-defined.

[28] As it becomes apparent below, the entire coalition profile affects the probability of a coalition winning the auction as well as the expected price paid conditional on winning.

[29] More formally, for each coalition profile $\sigma = (\sigma_1, \ldots, \sigma_J)$, let the expected total welfare be $\Pi(\sigma) \equiv \sum_{j=1}^{J} \pi(\sigma_j; \sigma)$. A coalition profile $\sigma^* \in \arg\max_{\sigma \in \Sigma} \Pi(\sigma)$ may not be stable. In fact, a firm's unilateral deviation could increase the sum of the payoffs of the coalitions involved in (1) or (2), while at the same time decreasing the total welfare by decreasing the payoffs of coalitions not directly involved in the deviation.

[30] For a recent survey on group formation and stability concepts, see Ray and Vohra (2015).

of the patent for a coalition σ_j of size n_j is always $W_j = \min\{n_j, \overline{n}\} \cdot v$.

The following lemma describes the equilibrium payoff arising in the auction stage for any given coalition profile:

Lemma 1 (Coalitions' Payoffs with Complete Information) *For any coalition profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_J)$, the payoff of each coalition σ_j is*

$$\pi(\sigma_j; \sigma) = \begin{cases} (\min\{n_1, \overline{n}\} - \min\{n_2, \overline{n}\})v & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 follows immediately from the observation that, in the complete information case, the auction yields a strictly positive payoff to the winner only if $n_2 < \min\{n_1, \overline{n}\}$. In fact, if either $n_1 = n_2 < \overline{n}$ or $n_2 \geq \overline{n}$, the patent's valuation for the two largest coalitions is the same, yielding zero payoff for the winner.

The next result follows from Lemma 1 and characterizes the stable coalition profiles.

Proposition 1 (Stable Coalition Profiles with Complete Information) *With complete information, a coalition profile σ is stable if and only if it satisfies either (i) $n_1, n_2 > \overline{n}$, or (ii) $n_1 \geq \overline{n} \geq n_2 = n_3$.*

Proposition 1 guarantees the existence of a stable coalition profile with complete information: the grand coalition (i.e., $\sigma = \{N\}$) is always stable as it satisfies condition (ii). On the other hand, a stable coalition profile identified by condition (i) exists if and only if $\overline{n} \leq \frac{N}{2} - 1$. Thus, when $\overline{n}$ is large relative to N (i.e., when the new technology has a large number of applications), the set of stable coalition profiles tends to be smaller and may contain only the grand coalition.

Two additional observations follow from Proposition 1. First, if all firms bid individually without the option of forming coalitions, we have $\sigma_j = \{j\}$ and $\pi(\sigma_j; \sigma) = 0$ for any $j = 1, .., N$, and the seller's revenue would be v. Thus, as long as $n_2 > 1$, the seller can be better off in the presence coalitions than in their absence. Such an increase in the seller's revenue is due to the club-good nature of the patent. When the patent has potential multiple applications ($\overline{n} > 1$), a coalition can generate more value from it than what an individual firm would. However, for the seller to be able to extract the surplus $\overline{n}v$ in full through an auction, at least two coalitions larger than $\overline{n}$ need to form and compete for the patent, so we must have $\overline{n} \leq \frac{N}{2} - 1$.

Second, the fact that a patent (i.e. a club good) is the object being sold at the auction generates an additional incentive for bidders to cooperate. Consider the largest coalition in any given coalition profile. Even if this coalition is already larger than $\overline{n}$, the coalition might have an incentive to enlarge its size by adding firms belonging to the second-largest coalition. In fact, as long as $n_2 \leq \overline{n}$, by reducing the size of the second-largest coalition, the largest coalition lower its value, which is the price paid by the largest coalition. For example, consider a coalition profile such that $n_1 \geq \overline{n} \geq n_2 > n_3$. Such profile is clearly not stable because a firm $i \in \sigma_2$ has a profitable deviation in joining σ_1 as

$$\pi(\sigma_1 \cup \{i\}; \sigma') + \pi(\sigma_2 \backslash \{i\}; \sigma') = \pi(\sigma_1 \cup \{i\}; \sigma') = \overline{n}v - (n_2 - 1)v > \overline{n}v - n_2 v = \pi(\sigma_1; \sigma) + \pi(\sigma_2; \sigma),$$

where σ' corresponds to $\sigma'_1 = \sigma_1 \cup \{i\}$, $\sigma'_2 = \sigma_2 \backslash \{i\}$, and $\sigma'_i = \sigma_i$ for any $i \neq 1, 2$.

The next result illustrates the implications of Proposition 1 on the seller's revenue. Recall that we defined $R^*(N, \overline{n})$ as the maximum revenue achievable by a stable coalition profile for any given N and $\overline{n}$.

Corollary 1 (Seller's Revenue with Complete Information) *With complete information, if $\overline{n} \leq \frac{N}{2} - 1$, $R^*(N, \overline{n}) = \overline{n}v$, and if $\overline{n} > \frac{N}{2} - 1$, $R^*(N, \overline{n}) = \left\lfloor \frac{N - \overline{n}}{2} \right\rfloor v (\leq \overline{n}v)$. Therefore, $R^*(N, \overline{n})$ is non-monotonic in $\overline{n}$.*

Corollary 1 follows from the observation that in a stable coalition profile characterized by condition (i) in Proposition 1, the seller's revenue is $\overline{n}v$; in a stable coalition profile described by condition (ii), the seller's revenue is $n_2 v$, which is (weakly) less than $\overline{n}v$, and can potentially be zero in the grand coalition case. Corollary 1 also considers the seller's equilibrium revenue as a function of $\overline{n}$, which represents the scope of the patent's applications. If $\overline{n} \leq \frac{N}{2} - 1$, all the coalition profiles described in Proposition 1 are stable. Thus, at least two coalitions larger than $\overline{n}$ can be sustained in a stable coalition profile, generating a seller's revenue of $\overline{n}v$, which is increasing in $\overline{n}$, and maximized at $\overline{n} = \left\lfloor \frac{N}{2} - 1 \right\rfloor$. If $\overline{n} > \frac{N}{2} - 1$, the only stable coalition profiles are the ones characterized by (ii) of Proposition 1. The maximum possible revenue for the seller is achieved when n_2 and n_3, which have to be of the same, are as large as possible. This happens when $n_2 = n_3 = \left\lfloor \frac{N - \overline{n}}{2} \right\rfloor (\leq \overline{n} \leq n_1)$, yielding a seller's revenue of

$\left\lfloor \frac{N-\overline{n}}{2} \right\rfloor v$, which is decreasing in $\overline{n}$. This implies that the maximum revenue achievable for the seller $R^*(N,\overline{n})$ is non-monotonic in $\overline{n}$, and patents are potentially most profitable when they have an intermediate number of applications relative to the total market size N (specifically, when $\overline{n} = \left\lfloor \frac{N}{2} - 1 \right\rfloor$).

4 Stable Coalition Profiles and Seller's Revenue with Incomplete Information

In this section we analyze the incomplete information case: the firms' individual values for the patent are independent random draws from a non-degenerate distribution F. Values are realized after the coalition formation stage and before the auction takes place. Specifically, once coalitions are formed, they determine their value for the patent as described in Section 2.2: in the limited-values model, the total valuation for a coalition σ_j of size n_j is the sum of $\min\{n_j, \overline{n}\}$ value-contributing firms' private valuations, or $W_j \equiv \sum_{i=1}^{\min\{n_j,\overline{n}\}} V_i$; on the other hand, in the optimized-values model, the total valuation for a coalition σ_j of size n_j is the sum of the highest $\min\{n_j, \overline{n}\}$ among n_j firms' private valuations, or $W_j \equiv \sum_{i=1}^{\min\{n_j,\overline{n}\}} V_{(n_j,i)}$ (recall that $V_{(n,i)}$ is the i-th highest value among n draws from the distribution F, and we let $v_{(n,i)}$ be its expected value).

Therefore, any given coalition profile $\sigma = \{\sigma_1, .., \sigma_J\}$ is associated with a valuation vector $(W_1, ..., W_J)$. The winner of the patent is the bidder with the highest realized valuation, and the revenue raised is the second-order statistic associated with the vector $(W_1, ..., W_J)$. If we denote the m-th order statistics associated to the vector $(W_1, ..., W_J)$ as $W_{(\sigma,m)}$, and by $w_{(\sigma,m)}$ its expected value, the expected revenue of the seller given a set of bidders σ is $R(\sigma) = w_{(\sigma,2)}$.

4.1 Stable Coalition Profiles

We start the analysis by addressing the existence of a stable coalition profile in our setting. The next result shows that the grand coalition profile $\sigma = \{N\}$ is stable under a mild assumption on the distribution F in the limited-values model, and it is always stable in the optimized-values model.

Lemma 2 (Existence of a Stable Coalition Profile)

(a) *In the limited-values case, if* $3v_{(\overline{n}+1,2)} + \sum_{h=3}^{\overline{n}+1} v_{(\overline{n}+1,h)} \geq v_{(\overline{n}+1,1)}$*, then the grand coalition* $\sigma = \{\mathcal{N}\}$ *is stable;*

(b) *In the optimized-values case, the grand coalition* $\sigma = \{\mathcal{N}\}$ *is always stable.*

Part (a) of Lemma 2 provides a sufficient condition on F for the existence of a stable coalition profile under the limited values model. This condition is satisfied for a large class of distributions F.[31] Intuitively, consider an individual firm i that may deviate from the grand coalition and form a coalition $\{i\}$ by itself. The singleton coalition can win the auction and realize a positive profit only in the scenario in which firm i's realization is the highest among $\overline{n} + 1$ draws from the distribution F, and the sum of all other $\overline{n}$ draws is lower than firm i's realization. In such scenario, the price paid by coalition $\{i\}$ is the sum of the other $\overline{n}$ realizations. Therefore, the grand coalition tends to be stable if $v_{(\overline{n}+1,1)}$ is low relative to the sum all other order statistics $v_{(\overline{n}+1,h)}$ for $h = 2, ..., \overline{n} + 1$. Part (b) of Lemma 2 guarantees that, in the optimized-values model, the grand coalition is always stable. This follows immediately from the fact that the grand coalition always achieves the maximum feasible payoff in this market, i.e., the sum of the $\overline{n}$ highest valuations across all N firms on the market (i.e. $\sum_{i=1}^{\overline{n}} V_{(N,i)}$).

Next, in Proposition 2, we describe an important and general property of the coalitions' auction equilibrium payoffs. In the proof, which we present immediately following the result, we exploit some tools drawn from the literature on stochastic orders. For ease of reference, we summarize the relevant results in Section 7.1 (for more detail, see Shaked and Shanthikunar, 2007).

[31] For example, if each firm i's private valuation's distribution is $V_i \sim U[0,1]$, the condition in Part (a) of Lemma 2 becomes $\frac{\overline{n}+1}{\overline{n}+2} \leq 3\frac{\overline{n}}{\overline{n}+2} + \frac{\overline{n}-1}{\overline{n}+2} + \cdots + \frac{1}{\overline{n}+2}$, which holds for every $\overline{n} \geq 1$. Also, suppose that $V_i \sim \exp(\beta)$. In this case, we have $V_{(\overline{n}+1,\overline{n}+1)} \sim \exp((\overline{n}+1)\beta)$ and $v_{(\overline{n}+1,\overline{n}+1)}=\frac{1}{\beta}\frac{1}{\overline{n}+1}$. Since the exponential distribution is memoryless, the difference between the next smallest value $V_{(\overline{n}+1,\overline{n})}$ and the minimum $V_{(\overline{n}+1,\overline{n}+1)}$ follows an independent exponential with parameter $\overline{n}\beta$, which results in $v_{(\overline{n}+1,\overline{n})} = \frac{1}{\beta}(\frac{1}{\overline{n}+1} + \frac{1}{\overline{n}})$. Similarly, $v_{(\overline{n}+1,i)} = \frac{1}{\beta}(\frac{1}{\overline{n}+1} + \frac{1}{\overline{n}} + \cdots + \frac{1}{i})$. Therefore, the condition in Part (a) of Lemma 2 holds for every $\overline{n} \geq 1$ since $2v_{(\overline{n}+1,2)} + (v_{(\overline{n}+1,2)} - v_{(\overline{n}+1,1)}) = \frac{1}{\beta}[2(\frac{1}{\overline{n}+1} + \cdots + \frac{1}{2}) - 1] > 0$. In fact, for any distribution F with a bounded support, the condition is always satisfied for large enough $\overline{n}$, because the difference $v_{(\overline{n}+1,1)} - v_{(\overline{n}+1,2)}$ is strictly decreasing in $\overline{n}$.

Proposition 2 (Effect of Concentration on Coalitions' Payoffs) *In both the limited-values and optimized-values cases, let* $\sigma = (\sigma_1, \ldots, \sigma_J)$ *be a coalition profile such that* $0 < n_j \leq n_k < \overline{n}$. *For any firm* $i \in \sigma_j$, *let* $\sigma' = (\sigma'_1, \ldots, \sigma'_J)$ *be such that* $\sigma'_j = \sigma_j \backslash \{i\}$, $\sigma'_k = \sigma_k \cup \{i\}$, *and* $\sigma'_h = \sigma_h$ *for* $h \neq j, k$. *Then,*

$$\pi(\sigma_j; \sigma) + \pi(\sigma_k; \sigma) < \pi(\sigma'_j; \sigma') + \pi(\sigma'_k; \sigma').$$

Proof of Proposition 2: *Step 1*: We define

$$X \equiv \sum_{a \in \sigma_j \backslash \{i\}} V_a, \quad Y \equiv \sum_{a \in \sigma_k} V_a, \quad \text{and} \quad Z \equiv \max_{h \neq j,k} \sum_{a \in \sigma_h} V_a,$$

and observe that

$$\pi(\sigma_j; \sigma) + \pi(\sigma_k; \sigma) = \max\{X + V_i, Y, Z\} - \max\{\min\{X + V_i, Y\}, Z\}, \quad \text{and}$$
$$\pi(\sigma'_j; \sigma') + \pi(\sigma'_k; \sigma') = \max\{X, Y + V_i, Z\} - \max\{\min\{X, Y + V_i\}, Z\}.$$

Additionally, for any $x, y, v_i, z \geq 0$, we define

$$\phi(x, y; v_i, z) \equiv \max\{x + v_i, y, z\} - \max\{\min\{x + v_i, y\}, z\}.$$

In the following Steps 2-3, we show that

$$E[\phi(X, Y; V_i, Z)] < E[\phi(Y, X; V_i, Z)], \tag{3}$$

which guarantees the claim of Proposition 2.

Step 2: First, we use Theorems 1 and 2 (see Section 7.1) to obtain a weak version of inequality (3). As $n_k > n_j - 1$, by Theorem 1, we have $X \leq_{lr} Y$. To use Theorem 2, observe that, whenever $x \leq y$,

$$\phi(x, y; v_i, z) = \max\{x + v_i, y, z\} - \max\{\min\{x + v_i, y\}, z\} \quad \text{and}$$
$$\phi(y, x; v_i, z) = \max\{y + v_i, z\} - \max\{x, z\}.$$

Hence, we have $\phi(x, y; v_i, z) \leq \phi(y, x; v_i, z)$. Then, for every realization of $V_i = v_i$ and $Z = z$,

$$E_{X,Y}[\phi(X, Y; v_i, z)] \leq E_{X,Y}[\phi(Y, X; v_i, z)],$$

which implies that

$$E[\phi(X, Y; V_i, Z)] \leq E[\phi(Y, X; V_i, Z)].$$

Step 3: We now find some realizations $V_i = v^*$ and $Z = 0$ such that

$$E_{X,Y}[\phi(X, Y; v^*, 0)] < E_{X,Y}[\phi(Y, X; v^*, 0)].$$

Take any v^* such that $v^* > 0$, $f(v^*) > 0$, and $Pr(Y \geq X + v^*) \geq 1/2$.[32] Then,

$$\phi(x, y; v^*, 0) = \max\{x + v^*, y\} - \min\{x + v^*, y\} = |y - x - v^*| \quad \text{and}$$
$$\phi(y, x; v^*, 0) = \max\{y + v^*, x\} - \min\{y + v^*, x\} = |y - x + v^*|.$$

Therefore,

$$\phi(y, x; v^*, 0) - \phi(x, y; v^*, 0) = \begin{cases} 2v^* & \text{if } y - x \geq v^*, \\ -2v^* & \text{if } y - x \leq -v^*, \\ 2(y - x) \quad (> -2v^*) & \text{if } -v^* < y - x < v^*. \end{cases}$$

Last, $Pr(-v^* < Y - X < v^*) > 0$ since $0 \in supp(F)$, so that the support of the distribution of $Y - X$ contains 0. Therefore,

$$E[\phi(Y, X; v^*, 0) - \phi(X, Y; v^*, 0)] > 2v^* \left(Pr(Y - X \geq v^*) - Pr(Y - X < v^*)\right) \geq 0.$$

As ϕ is a continuous function of v_i and z, the above strict inequality holds for every realization of V_i and Z near v^* and 0, respectively. Inequality (3) follows from the log-concavity of f, which guarantees that distribution functions of V_i and Z are continuous. ■

First of all, note that Proposition 2 pertains to coalitions that are strictly smaller than $\overline{n}$. Since for such coalitions the limited-values and optimized-values cases coincide, Proposition 2 holds in both cases. Moreover, note that Proposition 2 holds without the additional assumption on F imposed in Part (a) of Lemma 2.

To understand the intuition behind Proposition 2, consider a unilateral move of a firm i from coalition σ_j to σ_k, such that $n_j \leq n_k < \overline{n}$–that is σ_k is (weakly) larger than σ_j, and they are both strictly smaller than $\overline{n}$. This implies that the distribution of the coalition σ_k's valuation in the auction (the convolution of n_k identical distributions F) 'increases' (in a first-order stochastic sense) by one additional distribution F, while the distribution of coalition σ_j's valuation in the auction (the convolution of n_j identical distributions F) 'decreases' by one distribution F. Such move has several implications on the expected payoffs of coalitions σ_k (now σ'_k) and σ_j (now σ'_j). For convenience, denote $X \equiv \sum_{a \in \sigma_j \setminus \{i\}} V_a$, and $Y \equiv \sum_{a \in \sigma_k} V_a$.

[32] Such v^* exists as $Y \geq_{lr} X + V_i$ which implies that $Pr(Y \geq X + V_i) = \int Pr(Y \geq X + v) f(v) dv \geq 1/2$.

First, (i) *the move of firm i from coalition σ_j to σ_k increases the two coalitions' total expected valuation, conditional on the event that either of them wins.* To see why, for simplicity, assume that σ_j and σ_k are the only two coalitions on the market, so one of them is the winner for sure. The winning coalition's valuations before and after the move are $\max\{X + V_i, Y\}$ and $\max\{X, Y + V_i\}$, respectively. Intuitively, adding V_i to the value that is already more likely to be larger must contribute more to the maximum between the two variables. In fact, $\max\{X, Y + V_i\}$ first-order stochastically dominates $\max\{X + V_i, Y\}$.

Second, (ii) *the move of firm i from coalition σ_j to σ_k decreases the price paid by the winning coalition, conditional on the event that either of them wins.* Again, for simplicity assume that σ_j and σ_k are the only two coalitions on the market, so one of them wins for sure, and the price is the realized valuation of the other coalition. Observe that the prices paid by the winner before and after the move are $\min\{X + V_i, Y\}$ and $\min\{X, Y + V_i\}$, respectively. Along the same lines as before, adding the realization of V_i to the coalition that is less likely to have the lower realization tends to reduce the minimum between the two values. In particular, $\min\{X, Y + V_i\}$ is first-order stochastically dominated by $\min\{X + V_i, Y\}$.

Third (iii), in the presence of other coalitions on the market, *the move of firm i from coalition σ_j to σ_k increases the probability of the event that either of the two coalitions wins.* This is because, as we already observed, $\max\{X, Y + V_i\}$ first-order stochastically dominates $\max\{X + V_i, Y\}$. Therefore, $\max\{X, Y + V_i\}$ has a better chance to generate the highest realization on the market when other bidders are present.

Therefore, the move of firm i from coalition σ_j to coalition σ_k increases the two coalitions' total expected valuation and decreases the price paid by the winning coalition conditional on either of them winning the auction (effects (i) and (ii)). The move also increases the probability of the event that either of the two firms wins (effect (iii)). Hence, it must be the case that the move increases the joint expected payoffs of the two coalitions in the auction, as Proposition 2 indeed guarantees.[33,34]

[33] Moreover, even if it is not relevant to our argument, the move for firm i from σ_j to σ_k changes the probabilities of winning the auction of all other coalitions $\sigma_h \neq \sigma_j, \sigma_k$, as well as the expected price paid conditional on any of them winning.

[34] The assumption $0 \in supp(F)$ ensures that small coalitions can still win the auction with positive probability. Otherwise, small coalitions may get zero expected payoffs regardless of how a firm may deviate from one coalition to another.

Proposition 2 implies that if two coalitions are both strictly smaller than $\overline{n}$, by moving one firm from the (weakly) smaller between the two coalition to the (weakly) larger one, the sum of the expected payoffs of the two coalitions from the auction strictly increases, making the initial coalition profile not stable. Corollaries 2 and 3 look at the implications of Proposition 2 on the set of stable coalition profiles. They illustrate some necessary conditions for any coalition profile to be stable which we use in the next sections to study the seller's revenue.

Corollary 2 (Small Coalitions) *In both the limited-values and optimized-values cases, any stable coalition profile can include at most one coalition strictly smaller than $\overline{n}$.*

Corollary 3 (Stable Coalition Profiles with Incomplete Information) *In both the limited-values and optimized-values cases, for any stable coalition $\sigma = (\sigma_1, \ldots, \sigma_J)$,*

(a) *If $\overline{n} = N$, then $\sigma = \{\mathcal{N}\}$ (the grand coalition);*

(b) *If $\frac{N}{2} \leq \overline{n} < N$, then either (i) $\overline{n} \leq n_1 < N$, and $n_2 = N - n_1$; or (ii) $\sigma = \{\mathcal{N}\}$ (the grand coalition);*

(c) *If $\overline{n} < \frac{N}{2}$, then either (i) n_1,.. $n_J \geq \overline{n}$; or (ii) n_1,.. $n_{J-1} \geq \overline{n}$ and $n_J < \overline{n}$.*

By Corollary 2, a stable coalition profile cannot have more than one coalition strictly smaller than $\overline{n}$. Thus, if $\overline{n} = N$, only the grand coalition can be a stable coalition profile (Lemma 2 identifies sufficient conditions for it to be stable). If $\frac{N}{2} \leq \overline{n} < N$, there are not enough firms to form three coalitions and still satisfy Corollary 2. Therefore, a stable coalition profile can contain at most two coalitions, the (weakly) larger one must contain at least $\overline{n}$ firms, and the smaller one the remaining ones. Finally, if $\overline{n} < \frac{N}{2}$, stable coalition profiles can contain three coalitions or more, as long as $n_{J-1} \geq \overline{n}$.

4.2 Seller's Revenue in the Limited-Values Case

Next, we focus on the limited-values case, and we address the implications of Proposition 2 on the seller's expected revenue. Consider the effect of moving one firm from a smaller to a larger coalition on the seller's expected revenue, when both coalitions are strictly smaller than $\overline{n}$. The move may benefit the seller. To see this, consider an example in which N and $\overline{n}$ are both very large, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_2 = \{i\}$, $\sigma_3 = \{i'\}$, and $\sigma_1 = N \backslash \{i, i'\}$. For

N and $\overline{n}$ large enough, σ_1 wins the auction almost for sure and the seller's revenue is very likely to be $\max\{V_i, V_{i'}\}$. However, upon a deviation of i' from σ_3 to σ_2 we have $\sigma_1' = \sigma_1$, $\sigma_2' = \{i, i'\}$, and the seller's revenue is very likely to increase to $V_i + V_{i'}$. However, the next result guarantees that moving firms from smaller coalitions toward sizable ones (i.e. larger than $\overline{n}$) always reduces the seller's expected revenue.

Lemma 3 (Effect of Concentration on Seller's Revenue) *In the limited-values case, let $\sigma = (\sigma_1, \ldots, \sigma_J)$ be a coalition profile such that $0 < n_j \leq \overline{n} \leq n_k$. For any firm $i \in \sigma_j$, let $\sigma' = (\sigma'_1, \ldots, \sigma'_J)$ be such that $\sigma'_j = \sigma_j \backslash \{i\}$, $\sigma'_k = \sigma_k \cup \{i\}$, and $\sigma'_h = \sigma_h$ for $h \neq j, k$. Then, $R(\sigma) > R(\sigma')$.*

If a firm i is moved from a coalition σ_j (weakly) smaller than $\overline{n}$ to a coalition σ_k larger than $\overline{n}$, the value distribution of coalition σ_j decreases (in a first-order stochastic sense), but, as coalition σ_k already includes $\overline{n}$ members, the value distribution of coalition σ_k is not affected by the move. Therefore, firm i's contribution effectively disappears from the auction, and the seller's expected revenue strictly decreases. Hence, from the seller's perspective, profiles involving some coalitions strictly larger than $\overline{n}$ are dominated by coalition profiles in which the firms in excess of $\overline{n}$ are moved toward smaller coalitions, making the coalitions' sizes profile more even.

Next, we exploit the necessary condition of stable coalition profiles characterized in Corollary 3, together with Lemma 3, to find the maximum revenue a seller can raise through a second-price (or ascending) auction of a patent. For each N and $\overline{n}(\leq N)$, recall that $\Sigma^*(N, \overline{n})$ is defined as the set of stable coalition profiles, and $R^*(N, \overline{n})$ as the seller's *maximum* expected revenue generated by the coalition profiles in $\Sigma^*(N, \overline{n})$ (where $R^*(N, \overline{n}) \equiv 0$ if $\Sigma^*(N, \overline{n}) = \emptyset$).

Proposition 3 (Seller's Revenue with Incomplete Information) *In the limited-values case, (i) if $\overline{n} = N$, $R^*(N, \overline{n}) = 0$; (ii) if $\overline{n} < N$, let $K \equiv \lceil N/\overline{n} \rceil$, and consider the coalition profile $\widetilde{\sigma} \equiv \{\sigma_1, .., \sigma_K\}$ with $n_1 = ... = n_{K-1} = \overline{n}$ and $n_K = N - \overline{n}(K-1)$. Then, $R^*(N, \overline{n}) \leq \overline{R}^*(N, \overline{n}) \equiv w_{(\widetilde{\sigma}, 2)}$.*

If $\overline{n} = N$, Proposition 3 translates part (a) of Corollary 3 (the grand coalition is the only coalition profile that can be stable) into the fact that the grand coalition generates zero revenue for the seller. If $\overline{n} < N \leq 2\overline{n}$, by construction we have $K = 2$, and Proposition 3 is obtained observing that Lemma 3 guarantees that among all the coalition profiles described in part (b) of Corollary 3, the profile $\widetilde{\sigma} = \{\sigma_1, \sigma_2\}$ in which $n_1 = \overline{n}$ and $n_2 = N - n_1$ is the one maximizing the seller's revenue. If $N \geq 2\overline{n} + 1$, Lemma 3 guarantees tha, among all coalition profiles described in part (c) of Corollary 3, the one yielding the maximum potential revenue for the seller is formed by the maximum possible number $(K - 1)$ of coalitions of size $\overline{n}$, and the remaining firms allocated in one smaller coalition. This profile corresponds to $\widetilde{\sigma}$ in Proposition 3, and the revenue generated by it is $w_{(\widetilde{\sigma},2)}$.Therefore, $\overline{R}^*(N, \overline{n}) \equiv w_{(\widetilde{\sigma},2)}$ represents an upper bound for the revenue that the seller can generate across all stable coalition profiles.

Finally, Propositions 2 and 3 allow us to derive a precise auction design recommendation that allows the seller to obtain the upper bound identified in Proposition 3, $w_{(\widetilde{\sigma},2)}$. In particular, when $\overline{n} < N$, if the seller imposes a ceiling equal to $\overline{n}$ to the bidding coalitions' size, the upper bound identified in Proposition 3 can be uniquely obtained by the seller as auction's revenue.[35]

Corollary 4 (Ceiling on Coalitions' Size) *In the limited-values case, let $\overline{n} < N$. If the seller imposes a ceiling $\overline{n}$ on the bidding coalitions' size, $\widetilde{\sigma}$ is uniquely stable and it generates the revenue $\overline{R}^*(N, \overline{n})$.*

4.3 Seller's Revenue in the Optimized-Values Case

We now turn to the implications of Proposition 2, and Corollaries 2 and 3 on the seller's expected revenue in the optimized-values case.

The implications of optimized values on the set of stable coalition profiles are two-fold. First, consider a firm deviating from coalition σ_j to a larger coalition σ_k such that $n_j \leq \overline{n} \leq n_k$. In the limited-values case, such deviation does not change σ_k's valuation, while it decreases σ_j's valuation as well as its probability of winning the auction. As σ_j is less likely to win, all other coalitions, including σ_k, are more likely to win (and they pay a lower price conditional on winning). On the other hand, in the optimized-values case, such a deviation does increases σ_k's valuation. In fact, if coalitions have the ability to select the highest ones among all their

[35] While Lemma 3 guarantees that in the limited-values case setting the ceiling at $\overline{n}$ clearly dominates setting it at any higher level, setting a ceiling below $\overline{n}$ could generate even higher expected revenues for the seller. Indeed, a ceiling below $\overline{n}$ generates a larger number of smaller bidding coalitions, each with a lower value distribution (in a first-order stochastic sense). Whether this turns out to be beneficial for the seller depends on the specific distribution F.

members' realizations, even a coalition already larger than $\overline{n}$ can still increase its own value distribution (in a first-order stochastic sense) by adding more members (this is because by adding more members the coalition will be able to select the $\overline{n}$ highest realizations among a larger number of draws, therefore increasing their sum). Also, as σ_k's valuation increases, the chances of σ_k winning the auction increase more than in the limited-value case. Therefore, deviations from a coalition to another which is larger than $\overline{n}$ are more likely to be profitable than in the limited-value case. *Therefore, firms have an even stronger incentive to concentrate in larger coalitions than in the limited-values case.*

Second, if a firm is considering deviating from a coalition σ_j such that $n_j \geq \overline{n}$ to a singleton, such a deviation is less likely to be profitable than in the limited-values case, as the original coalition σ_j's value distribution is higher in this case. Both of these forces tend to make coalition profiles that include a small number of large coalitions relatively stable.

Similarly to the limited-values case, the incentives for firms to concentrate into large coalitions are stronger for larger $\overline{n}$. In fact, if $\overline{n}$ is small, coalitions tend to benefit less from the acquisition of additional members. The following result shows that even if $\overline{n} = 1$, and therefore *the incentives to aggregate in larger coalitions are minimal*, in the optimized-values case, the grand coalition $\sigma = \{\mathcal{N}\}$ is uniquely stable, guaranteeing zero revenue for the seller.

Proposition 4 *In the optimized-values case, assume $\overline{n} = 1$, and let F have a continuous and strictly positive density f. Then, the grand coalition $\sigma = \{\mathcal{N}\}$ is the only stable coalition profile.*

Proof of Proposition 4: *Let $\sigma = (\sigma_1, \sigma_2, \dots)$ be a coalition profile such that $n_1 \geq n_2 \geq 1$. For any $i \in \sigma_2$, let $\sigma' = (\sigma'_1, \sigma'_2, \dots)$ be such that $\sigma'_1 = \sigma_1 \cup \{i\}$, $\sigma'_2 = \sigma_2 \backslash \{i\}$, and $\sigma'_k = \sigma_k$ for $k \neq 1, 2$. Then,*

$$\pi(\sigma_1; \sigma) + \pi(\sigma_2; \sigma) < \pi(\sigma'_1; \sigma') + \pi(\sigma'_2; \sigma') \tag{4}$$

is sufficient to show that $\sigma = \{\mathcal{N}\}$ is the only stable coalition profile. To show (4), we apply a modified proof of Theorem 1 (see Section 7.1). Define

$$X \equiv \max\{V_j \mid j \in \sigma_2 \backslash \{i\}\}, \quad Y \equiv \max\{V_k \mid k \in \sigma_1\}, \quad \text{and} \quad Z \equiv \max\{V_k \mid k \notin \sigma_1 \cup \sigma_2\}.$$

Let $X' \equiv \max\{X, V_i\}$ and $Y' \equiv \max\{Y, V_i\}$. Also, for any $x, y, v_i, z \geq 0$, define $x' \equiv \max\{x, v_i\}$ and

$$\phi(x, y; v_i, z) \equiv \max\{x', y, z\} - \max\{\min\{x', y\}, z\}.$$

Then,

$$\pi(\sigma_1;\sigma)+\pi(\sigma_2;\sigma)=E[\phi(X,Y;V_i,Z)],\ \text{and}$$
$$\pi(\sigma_1';\sigma')+\pi(\sigma_2';\sigma')=E[\phi(Y,X;V_i,Z)].$$

Note that, for any $V_i=v_i$ and $Z=z$, we have

$$\phi(x,y;v_i,z)=\max\{x',y,z\}-\max\{\min\{x',y\},z\},$$
$$\phi(y,x;v_i,z)=\max\{y',x,z\}-\max\{\min\{y',x\},z\},$$

and whenever $x\leq y$, we have

$$\phi(y,x;v_i,z)-\phi(x,y;v_i,z)=\max\{\min\{x',y\},z\}-\max\{x,z\}\geq 0. \tag{5}$$

If $X=0$ (i.e., $n_2=1$), then

$$\phi(y,0;v_i,z)-\phi(0,y;v_i,z)=\max\{\min\{v_i,y\},z\}-z\geq 0,$$

with a strict inequality for some y,v_i,z. In this case, (4) follows from the continuity of ϕ.

Suppose instead that $X\neq 0$ (i.e., $n_2>1$). Note that the density functions of X and Y are

$$F_X(x)\equiv Pr(V_{(n_2-1,1)}\leq x)=F^{n_2-1}(x)\quad\Longrightarrow\ f_X(x)=(n_2-1)F^{n_2-2}(x)f(x),$$
$$F_Y(y)\equiv Pr(V_{(n_1,1)}\leq y)=F^{n_1}(y)\quad\Longrightarrow\ f_Y(y)=n_1F^{n_1-1}(y)f(y).$$

The likelihood ratio $\frac{f_Y(x)}{f_X(x)}=\frac{n_1}{n_2-1}F^{n_1-n_2+1}(x)$ is strictly increasing in $x\in supp(F)$, i.e.,

$$f_X(x)f_Y(y)>f_X(y)f_Y(x),\quad \text{for all } x<y.$$

Note that, for any $V_i=v_i$ and $Z=z$,

$$\begin{aligned}&E_{X,Y}[\phi(Y,X;v_i,z)]-E_{X,Y}[\phi(X,Y;v_i,z)]\\ =&\int_y\int_{y\geq x}(\phi(y,x;v_i,z)-\phi(x,y;v_i,z))f_X(x)f_Y(y)dxdy\\ &+\int_y\int_{y\leq x}(\phi(y,x;v_i,z)-\phi(x,y;v_i,z))f_X(x)f_Y(y)dxdy.\end{aligned}$$

We can rewrite the last term as

$$
\begin{aligned}
&\int_y \int_{y \le x} (\phi(y, x; v_i, z) - \phi(x, y; v_i, z)) f_X(x) f_Y(y) dx dy \\
&= \int_x \int_{y \le x} (\phi(y, x; v_i, z) - \phi(x, y; v_i, z)) f_X(x) f_Y(y) dy dx \\
&= \int_y \int_{x \le y} (\phi(x, y; v_i, z) - \phi(y, x; v_i, z)) f_X(y) f_Y(x) dx dy.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&E_{X,Y}[\phi(Y, X; v_i, z)] - E_{X,Y}[\phi(X, Y; v_i, z)] \\
&= \int_y \int_{y \ge x} (\phi(y, x; v_i, z) - \phi(x, y; v_i, z))(f_X(x) f_Y(y) - f_X(y) f_Y(x)) dx dy \ge 0.
\end{aligned} \tag{6}
$$

For some x, y, v_i and z, the inequality (5) holds strictly. Since ϕ is continuous, (6) holds strictly in a neighborhood of (v_i, z), which guarantees (4) to hold. ■

To summarize the implications of optimized values on the seller's revenue, observe that, since any coalition (weakly) larger than $\overline{n}$ can still increase their value distribution (in a first-order stochastic sense) by adding more members, for any *given* coalition profile σ the expected bids of large coalitions are going to be higher than in the limited-values case. Therefore, if several coalitions larger than $\overline{n}$ coexist in a stable coalition profile, the seller may obtain a higher revenue. However, since the firms' incentives to consolidate in large coalitions are even stronger than in the limited-value case, it is likely the case that the grand coalition becomes uniquely stable (as Proposition 4 suggests), *yielding zero revenue to the seller.*

As in the limited-value case, setting a ceiling on the coalitions' size at $\overline{n}$ can surely avoid the grand coalition and guarantee a positive revenue for the seller. In fact, similarly to Corollary 4, such ceiling would make the profile $\widetilde{\sigma}$ uniquely stable. However, in the optimized-value case such ceiling could be dominated by both lower and higher ones. To see this, consider for example $N = 102$ and $\overline{n} = 50$. Setting a ceiling at 50 results in the coalition profile $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ with $n_1 = n_2 = 50$ and $n_3 = 2$ being uniquely stable. However, setting a ceiling at 51 may result in a coalition profile $\sigma' = \{\sigma'_1, \sigma'_2\}$ with $n'_1 = n'_2 = 51$ being stable as well. As the value distributions of σ'_1 and σ'_2 are higher (in a first-order stochastic sense) than the ones of σ_1 and σ_2, depending on the underlying distribution F, the profile σ' could increase the expected revenue of the seller.

5 Auctions of Multiple Licenses

In this section we study the implications of the IP's owner auctioning off multiple licenses rather than one patent. We compare the revenue generated by a multi-license auction to the one generated by a second-price auction of a patent. We focus on the limited-values case since the analysis in Sections 4.2 and 4.3 suggests that the seller's revenue tends to be greater in this setting.

If the seller opts for an auction of multiple licenses, he is able to choose the number of licenses to sell, as well as the auction format. We assume that each firm can only participate in the auction individually, since each license is going to be issued to one specific winning firm.[36] The seller auctions off at most $\overline{n}$ licenses, as any additional license would be valued zero by all firms. For each choice of selling $k \leq \overline{n}$ licenses, we assume that the seller holds a $(k+1)$-th price auction, which is well-known to be revenue-equivalent to a large class of standard multi-unit auctions (see Harris and Raviv, 1981, Weber, 1983, and Maskin and Riley, 1989).[37] The expected revenue from selling $k(\leq \overline{n})$ licenses is $kv_{(N,k+1)}$. Therefore, the seller's revenue from a multi-license auction is $R^{lic} \equiv \max_{k \leq \overline{n}} kv_{(N,k+1)}$. In the complete information case, it is optimal to sell exactly $\overline{n}$ licenses, and $R^{lic} = \overline{n}v$. In the incomplete information case, we have $R^{lic} \geq \overline{n}v_{(N,\overline{n}+1)}$–that is, the expected revenue generated by selling exactly $\overline{n}$ licenses at the uniform price of the $(\overline{n}+1)$-th order statistic among N draws from the distribution F represents a lower bound for the revenue generated by a multi-license auction.

We now compare the seller's revenue raised by a multi-license ($(\overline{n}+1)$-th price) auction to the one generated by a second-price auction of a patent. The following result is an immediate implication of Corollary 1.

Proposition 5 *In the complete information case, a multi-license auction dominates a single-patent auction from the seller's perspective.*

[36] Firms may still decide to collude illegally to manipulate the auction's price. We rule out this possibility because this paper focuses on transparent agreements to share patents legally. Considering the possibility of such illicit agreements would naturally lead the revenue generated by multiple licenses to decrease. We also assume that licenses are non-transferable.

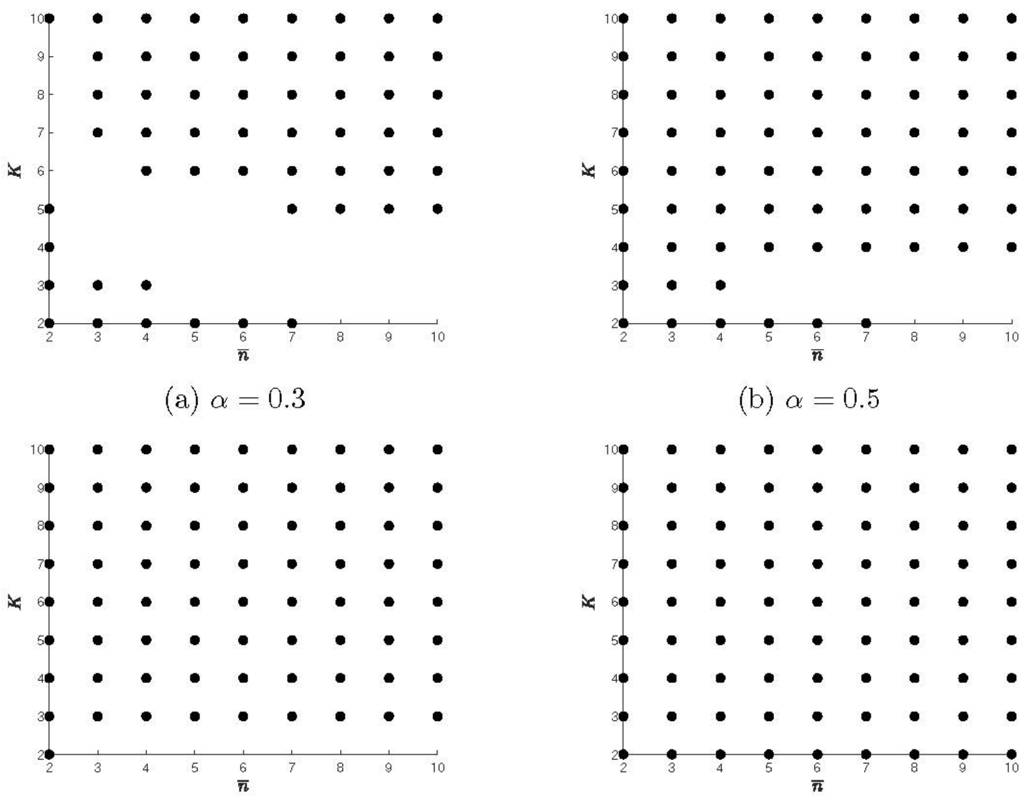
[37] Conditional on bidders' i.i.d. private valuations and single-unit demands, revenue equivalence holds for any standard multi-unit auction, either sequential or sealed-bid, uniform-price or discriminatory (in the case of uniform-price either k-th price or $(k+1)$-th price). Notice that our externalities' structure (there are no negative externalities to non-adopters) guarantees that firms have a single-unit demand for licences. Otherwise, firms may have an incentive to bid for multiple licenses to preemptively limit the adoption of the IP and decrease negative externalities.

In the incomplete information case, the comparison is less straightforward, and in general depends on the distribution F. To get some intuition, suppose that $N = K\overline{n}$ for some $K \in \mathbb{Z}_+$. As seen above, a lower bound for the revenue raised by a multi-license auction is identified by $\overline{n}v_{(N,\overline{n}+1)}$. The upper bound of the patent auction revenue identified in point (ii) of Proposition 3 is the seller's expected revenue when firms form K coalitions of size $\overline{n}$. If the $\overline{n}$ firms with the highest ex-post realizations happen to be consolidated in one group, the seller's revenue can be at most $\sum_{i=1}^{\overline{n}} V_{(N,\overline{n}+i)}$, which is clearly below the lower bound of the multi-license auction revenue. When the firms with the ex-post highest realizations happen to be separated into different coalitions, the revenue's comparison depends on the distribution F. Consider for example the case $N = 4$ and $\overline{n} = 2$. If the seller decides to opt for a multi-license auction, the revenue is at least $2V_{(4,3)}$. In the patent-auction case, consider two coalitions of size $\overline{n} = 2$. Ex-post, the two firms with highest realizations are equally likely to be in either one coalition or separate coalitions. In the first case, the seller's revenue will be $V_{(4,3)} + V_{(4,4)}$, which is strictly lower than $2V_{(4,3)}$. In the second case, the seller's revenue is equally likely to be either $\min\{V_{(4,1)} + V_{(4,4)}, V_{(4,2)} + V_{(4,3)}\}$ or $\min\{V_{(4,1)} + V_{(4,3)}, V_{(4,2)} + V_{(4,4)}\} = V_{(4,2)} + V_{(4,4)}$. The seller's revenue from the patent auction would dominate $2V_{(4,3)}$ when the distribution F is right-skewed, so that $V_{(4,1)}$ and $V_{(4,2)}$ tend to be significantly larger than $V_{(4,3)}$.[38] In addition, the patent auction is less likely to be optimal for large $K(= N/\overline{n})$, because firms with relatively high ex-post realizations tend to be more spread out in coalitions with mid- to low- valuation firms, decreasing the second-order statistics among the coalitions' total value realizations, and therefore the seller's revenue.

These considerations suggest that *(i) for a large set of parameters and distributions F, the seller is better off selecting a multi-license auction over a patent auction; (ii) a patent auction has the best chance to be optimal if the value distribution is relatively right-skewed, and if $K(= N/\overline{n})$ is relatively small ($\overline{n}$ is large relative to N).*

To illustrate these intuitions numerically, we consider the case of valuations distributed according to a Gamma distribution–that is, $V_i \sim Gamma(\alpha, \beta)$ for each $i \in N$. In the case of a Gamma distribution, the convolution of $\overline{n}$ random variables V_i is another gamma distribution $\sum_{i=1}^{\overline{n}} V_i \sim Gamma(\overline{n}\alpha, \beta)$. Therefore, the expected revenue from the a patent auction in

[38] Recall however that we are comparing a lower bound of the multi-license auction revenue and an upper bound of a patent auction: if F is severely right-skewed, the seller can decide to sell less than $\overline{n}$ licenses in a multi-license auction, therefore increasing revenue of the multi-licence auction with respect to the lower bound.



(a) $\alpha = 0.3$ (b) $\alpha = 0.5$

(c) $\alpha = 3$ (d) $\alpha = 7$

Figure 1: Relative Performance of Patent and Multi-License Auctions for $V_i \sim Gamma(\alpha, \beta)$.

which bidders form K coalitions of size $\overline{n}$, each with valuation $\sum_{i=1}^{\overline{n}} V_i$, can be expressed as an expectation of the second order statistics of K draws from the distribution $Gamma(\overline{n}\alpha, \beta)$. To allow for comparative statics with respect to the skewness of the distribution, we choose $\beta = \sqrt{\alpha}$, hold $Var[V_i] = \frac{\alpha}{\beta^2}$ fixed, and increase α to decrease the skewness $\frac{2}{\sqrt{\alpha}}$.[39]

Figure 1 summarizes the revenue comparison for a range of different parameter values. The dots in the figure identify markets (each defined by $(\overline{n}, N = K\overline{n})$) in which $\overline{n}v_{(N,\overline{n}+1)}$ (the revenue generated by an auction to sell $\overline{n}$ licenses) is (weakly) larger than the expected second order statistic of K draws from the distribution $Gamma(\overline{n}\alpha, \beta)$ (the revenue generated by a patent auction with K coalitions of size $\overline{n}$). The results support the intuitions (i) and (ii) above. Observe that the revenue generated by a multi-license auction dominates the one

[39] While the expectation $E[V_i] = \frac{\alpha}{\beta} = \frac{\alpha}{\sqrt{\alpha}} = \sqrt{\alpha}$ is also increasing, the scale of the private valuations is irrelevant in the revenue comparison across different auction schemes.

generated by a patent auction in most markets. When the distribution of private valuations is relatively right-skewed (i.e., α is smaller) the difference between the highest realizations and the other realizations is relatively large. In this scenarios, a patent auction has the best chance to generate larger revenues than the multi-license auction, particularly when $K(= N/\overline{n})$ is small.

6 Discussion and Conclusion

In this paper we study and compare alternative ways to sell IP rights or other goods that are non-rivalrous but excludable ("club goods"). If a patent is sold in a second-price (equivalently, ascending) auction, we analyze the potential emergence of coalitions of firms intended to participate in the auction as individual bidders. We illustrate the impact of such coalitions on the seller's revenue and we compare the auction outcomes to the revenue generated by selling multiple licenses instead.

Several extensions are left to further research. It would be interesting to explore the potential implementation of a first-price auction, and compare both the stable coalition profiles and the seller's revenue of the two auction formats.

In addition, one can consider an alternative timing in which coalitions form first, and then the seller chooses the auction mechanism based on the observed coalition profile. Further exploration in this direction would require firms to forecast the optimal auction mechanism selected by the seller in the coalition-formation stage. A small literature has explored the optimal auction mechanism based on particular profiles of (asymmetric) bidders (see Bulow and Roberts, 1989, and Maskin and Riley, 2000a). In particular, Kirkegaard (2012) ranks auction formats based on the properties of the values' distributions in the case of two asymmetric bidders. However, characterizing the optimal mechanism for a wide range of potential coalition profiles with multiple bidders generates non-trivial complexities, that are beyond the scope of this paper.

Finally, it would be interesting to expand the range of externalities among the users of the new technology. In this paper, we considered the simple case in which no externalities are present if the new technology is adopted by up to $\overline{n}$ firms (and no additional value is generated if the adopters are more than $\overline{n}$). In other words, the marginal returns of the new technology are constant in the number of adopters up to $\overline{n}$, and they are zero afterwards. One could instead consider decreasing marginal returns in the number of adopters. In this case, the tendency of firm to form large coalitions would depend on the rate at which the marginal returns decrease. Along the lines of Proposition 2, if the rate of decrease of marginal returns is not too high, it would still be beneficial for a firm to move from a smaller coalition to a larger one. However, if the marginal returns decrease rapidly, that could not be the case and we should observe a larger number of small coalitions coexisting in stable coalition profiles (resembling our setting in the case of a low $\overline{n}$).

7 Appendix

7.1 References from Shaked and Shanthikunar (2007)

Some of the proofs presented in the main text and in Section 7.2 below require some tools from the statistical literature on stochastic orders. For ease of reference, here we provide a summary of these results (see Shaked and Shanthikunar, 2007, for more detail).

Let X and Y be two continuous random variables such that $P(X \geq x) \leq P(Y \geq x)$ for all x. Then X is smaller than Y in the usual stochastic order (denoted $X \leq_{st} Y$). Similarly, if X and Y have densities g and h, respectively, X is smaller than Y in the likelihood-ratio order (denoted by $X \leq_{lr} Y$) if $g(x)h(y) \geq g(y)h(x)$ for all $x \leq y$. It is well known that likelihood-ratio dominance implies stochastic dominance. With these definitions, we can state the following theorems:

Theorem 1 (Shaked and Shanthikunar, 2007, Theorem 1.C.11) *Let X_i, $i = 1, 2, ...$ be a sequence of non-negative independent random variables with log-concave densities. Let M and N be two discrete positive integer-valued random variables such that $M \leq_{lr} N$, and assume that M and N are independent of X_i, $i = 1, 2, ...$ Then,*

$$\sum_{i=1}^{M} X_i \leq_{lr} \sum_{i=1}^{N} X_i.$$

Next, define $\mathcal{G}_{lr} = \{\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \phi(x, y) \leq \phi(y, x) \text{ whenever } x \leq y\}$. Then,

Theorem 2 (Shaked and Shanthikunar, 2007, Theorem 1.C.20) *Let X and Y be independent random variables. Then, $X \leq_{lr} Y$ if and only if*

$$\phi(X, Y) \leq_{st} \phi(Y, X) \text{ for all } \phi \in \mathcal{G}_{lr}.$$

Theorem 3 (Shaked and Shanthikunar, 2007, Theorem 1.A.3) *Let $X_1, X_2, \ldots, X_m$ be a set of independent random variables, and let $Y_1, Y_2, \ldots, Y_m$ be another set of independent random variables. If $X_i \leq_{st} Y_i$ for $i = 1, 2, \ldots, m$, then, for any increasing function $\psi : R^m \to R$, one has*

$$\psi(X_1, X_2, \ldots, X_m) \leq_{st} \psi(Y_1, Y_2, \ldots, Y_m).$$

7.2 Proofs

Proof of Proposition 1: Consider a coalition profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_J)$ and let $n_j \equiv 0$ for $j > J$. Consider the following cases:

(1) If $n_1 < \overline{n}$, a firm $i \in \sigma_2$ has a profitable deviation in joining σ_1. As $\overline{n} \leq N$ and $\overline{n} > n_1 \geq n_2 > 0$,

$$\pi(\sigma_1; \sigma) + \pi(\sigma_2; \sigma) = (n_1 - n_2)v < [(n_1 + 1) - (n_2 - 1)]v = \pi(\sigma_1'; \sigma') + \pi(\sigma_2'; \sigma')$$

with $\sigma_1' = \sigma_1 \cup \{i\}$, $\sigma_2' = \sigma_2 \backslash \{i\}$, and $\sigma_k' = \sigma_k$ for $k \neq 1, 2$.

(2) If $n_1 \geq n_2 > \overline{n}$, any coalition profile is stable. No unilateral deviation by any firm to any coalition changes the payoff of the winning coalition and the losing coalitions as they all remain equal to zero.

(3) If $n_1 \geq \overline{n} \geq n_2 > n_3$, a firm i in σ_2 has a profitable deviation in joining σ_1 because

$$\pi(\sigma_1; \sigma) + \pi(\sigma_2; \sigma) = (\overline{n} - n_2)v < [\overline{n} - (n_2 - 1)]v = \pi(\sigma_1'; \sigma') + \pi(\sigma_2'; \sigma'),$$

with $\sigma_1' = \sigma_1 \cup \{i\}$, $\sigma_2' = \sigma_2 \backslash \{i\}$, and $\sigma_k' = \sigma_k$ for $k \neq 1, 2$.

(4) If $n_1 \geq \overline{n} \geq n_2 = n_3$, no firm $i \in \sigma_j$ has a profitable deviation in deviating to be a singleton. Since the coalition $\{i\}$ never wins the patent auction with a strictly positive profit, we have

$$\pi(\sigma_j; \sigma) = \begin{cases} (\overline{n} - n_2)v & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

cannot be strictly lower than

$$\pi(\sigma_j'; \sigma') + \pi(\{i\}; \sigma') = \pi(\sigma_j'; \sigma') = \begin{cases} (\min\{n_1 - 1, \overline{n}\} - n_2)v & \text{if } j = 1 \text{ and } n_1 > n_2 \\ 0 & \text{otherwise.} \end{cases}$$

with $\sigma_j' = \sigma_j \backslash \{i\}$, $\sigma_{J+1}' = \{i\}$, and $\sigma_k' = \sigma_k$ for $k \neq j$.

Next, consider a unilateral deviation by a firm $i \in \sigma_j$ to join another coalition σ_k, to form $\sigma_j' = \sigma_j \backslash \{i\}$, $\sigma_k' = \sigma_k \cup \{i\}$, and $\sigma_h' = \sigma_h$ for $h \neq j, k$. If $j, k \neq 1$, the deviation would not be profitable, as since $n_1 \geq \overline{n}$, neither σ_j or σ_k, and neither σ_j' or σ_k', win the patent auction with a strictly positive payoff. On the other hand, if $j = 1$ or $k = 1$, then the winning coalition's payment to the seller can only be (weakly) increased by the deviation due to the fact that $n_2 = n_3$. As the winning coalition's valuation is bounded above by $\overline{n}v$, we have

$$\pi(\sigma_j; \sigma) + \pi(\sigma_k; \sigma) = \pi(\sigma_1; \sigma) = \overline{n}v - n_2 v \geq \pi(\sigma_j'; \sigma') + \pi(\sigma_k'; \sigma'). \quad \blacksquare$$

Proof of Lemma 2: (a) In the limited values model, consider a firm i, let $\sigma \equiv \{\mathcal{N}\}$ be the grand coalition and $\sigma' \equiv \{\mathcal{N}\backslash\{i\}, \{i\}\}$. First, consider the case in which $\overline{n} = N$. The payoff of the winning coalition of the patent auction under the profile σ' is $\max\left\{\sum_{a\in\mathcal{N}\backslash\{i\}} V_a, V_i\right\} - V_{(\sigma',2)}$. Since

$$\max\left\{\sum_{a\in\mathcal{N}\backslash\{i\}} V_a, V_i\right\} - V_{(\sigma',2)} \leq \max\left\{\sum_{a\in\mathcal{N}\backslash\{i\}} V_a, V_i\right\} \leq \sum_{a\in\mathcal{N}} V_a,$$

firm i would not unilaterally deviate from the grand coalition.

Suppose now that $\overline{n} < N$. The payoff of the grand coalition σ is the sum of $\overline{n}$ value-contributing firms' private valuations. We generate the grand coalition's payoff in two steps: adding $\overline{n}+1$ draws from the distribution F and excluding one of the draws with the same probability among all of them. Recall that we denote by $V_{(n,m)}$ the m-th highest value among n draws from the distribution F, and that we denote by $v_{(n,m)}$ the expected value of $V_{(n,m)}$. We have

$$\pi(\mathcal{N};\sigma) = \frac{1}{\overline{n}+1}\sum_{i=1}^{\overline{n}+1}\left(v_{(\overline{n}+1,1)} + v_{(\overline{n}+1,2)} + \dots v_{(\overline{n}+1,\overline{n}+1)} - v_{(\overline{n}+1,i)}\right).$$

Consider now the coalition profile σ'. To obtain the coalitions' payoffs associated with σ', let us consider $\overline{n}+1$ draws from the distribution F and select among them, with the same probability, one to be the valuation of firm i. Consider the case in which coalition $\{i\}$'s realization is the highest among all draws (which happens with probability $\frac{1}{\overline{n}+1}$). Conditional on this case, if coalition $\{i\}$ wins the patent auction, its payoff is $V_{(\overline{n}+1,1)} - [V_{(\overline{n}+1,2)} + \dots + V_{(\overline{n}+1,\overline{n})}]$, while if coalition $\mathcal{N}\backslash\{i\}$ wins the auction its payoff is $[V_{(\overline{n}+1,2)} + \dots + V_{(\overline{n}+1,\overline{n})}] - V_{(\overline{n}+1,1)}$. Therefore, the winning coalition's expected payoff is bounded above by $\sum_{j\neq 2} V_{(\overline{n}+1,j)} - V_{(\overline{n}+1,2)}$. If coalition $\{i\}$ realizes the h-th highest valuation among all draws, with $h \neq 1$, coalition $\mathcal{N}\backslash\{i\}$ wins the auction and the winner's expected payoff is $\sum_{j\neq h} v_{(\overline{n}+1,j)} - v_{(\overline{n}+1,h)}$. Coalition $\{i\}$ realizes each of the h-th highest valuation among all draws, with $h \neq 1$ with probability $\frac{1}{\overline{n}+1}$. Thus,

$$\pi(N\backslash\{i\};\sigma') + \pi(\{i\};\sigma') < \frac{1}{\overline{n}+1}\left[\sum_{j\neq 2} v_{(\overline{n}+1,j)} - v_{(\overline{n}+1,2)}\right] + \frac{1}{\overline{n}+1}\sum_{h=2}^{\overline{n}+1}\left[\sum_{j\neq h} v_{(\overline{n}+1,j)} - v_{(\overline{n}+1,h)}\right].$$

Therefore, exploiting the condition in Lemma 2's claim, we have

$$\pi(\mathcal{N};\sigma) - [\pi(\mathcal{N}\backslash\{i\};\sigma') + \pi(\{i\};\sigma')] > \frac{1}{\overline{n}+1}\left(\sum_{h=3}^{\overline{n}+1} v_{(\overline{n}+1,h)} - v_{(\overline{n}+1,1)} + 3v_{(\overline{n}+1,2)}\right) > 0,$$

which guarantees that $\sigma \equiv \{\mathcal{N}\}$ is stable. (b) In the optimized values model the grand coalition $\sigma \equiv \{\mathcal{N}\}$ always achieves the maximum feasible payoff in this market, i.e., the sum of the $\overline{n}$ highest valuations across all N firms on the market (i.e. $\sum_{i=1}^{\overline{n}} V_{(N,i)}$). Therefore, it has to be stable. ■

Proof of Lemma 3: In the limited-values case, consider a coalition profile $\sigma = (\sigma_1, .., \sigma_J)$ with $n_j \leq \overline{n} \leq n_k$. Let $(W_1, W_2, \dots, W_J)$ denote the associated coalitions' valuations. Firm i's move from coalition σ_j to σ_k results in another coalition profile $\sigma' = (\sigma'_1, \sigma'_2, \dots, \sigma'_J)$ with $\sigma'_j = \sigma_j \backslash \{i\}$, $\sigma'_k = \sigma_k \cup \{i\}$, and $\sigma'_h = \sigma_h$ for $h \neq j, k$. Coalition σ'_k's valuation W'_k is the sum of $\overline{n}$ members' valuations, and the distribution of W'_k is identical to the distribution of W_k. It follows that, for the coalition profile σ', the coalitions' valuations $(W'_1, W'_2, \dots, W'_J)$ are such that $W'_j = W_j - V_i$, and $W'_h = W_h$ for every $h \neq j$. As such, since every coalition bids their own valuation, by Theorem 3, the equilibrium bids under σ' are (weakly) lower than the equilibrium bids under σ, and strictly lower for coalition j with probability $P(V_i > 0) > 0$. Lemma 3 follows. ■

Proof of Proposition 3: (i) If $\overline{n} = N$, only the grand coalition can be stable; (ii) If $\overline{n} < N \leq 2\overline{n}$, from Corollary 3 it is immediate to see that $\sigma = \{\sigma_1, \sigma_2\}$ with $n_1 = \overline{n}$, *and* $n_2 = N - \overline{n}$ yields the maximum potential revenue for the seller. Therefore, $R^*(N, \overline{n}) \leq w_{(\widetilde{\sigma},2)}$, and the bound is attained if σ is stable; If $N \geq 2\overline{n} + 1$, for any stable coalition profile $\sigma' = (\sigma'_1, \dots, \sigma'_J)$, Corollary 3 implies that $n_1 \geq \dots \geq n_{J-1} \geq \overline{n}$. The total value of each coalition σ'_j is $W'_j \equiv \sum_{i=1}^{\min\{n_j \overline{n}\}} V_i$. Let $W'_j \equiv 0$ for $j = J+1, J+2, \dots, K$. Similarly, let W_k denote the value of each coalition of σ_k in the coalition profile $\widetilde{\sigma}$ defined in the claim. Then, for every $k = 1, 2, \dots, K$, we have $W'_k \leq_{lr} W_k$, which implies $W'_k \leq_{st} W_k$. Observe that a function that finds the second highest number from a vector of integers is increasing in each argument. It follows from Theorem 3 that $W_{(\sigma',2)} \leq_{st} W_{(\widetilde{\sigma},2)}$. Therefore, $R^*(N, \overline{n}) \leq w_{(\widetilde{\sigma},2)}$, and the bound is attained if $\widetilde{\sigma}$ is stable ■

Proof of Corollary 4: Points (b) and (c) of Corollary 3 imply that in the presence of a ceiling the coalition profile $\widetilde{\sigma}$ is the only remaining candidate to be stable. For it to be stable, we need to guarantee that (i) if $n_K < \overline{n}$, no firm belonging to a coalition of size $\overline{n}$ has a profitable deviation in joining σ_K; (ii) if $n_K < \overline{n}$, no firm belonging to σ_K has a profitable deviation in participating in the auction as individual bidder, and (iii) no firm belonging to a coalition of size $\overline{n}$ has a profitable deviation in participating in the auction as individual bidder. However, in the proof of Proposition 2, we defined

$$X \equiv \sum_{a \in \sigma_j \setminus \{i\}} V_a, \quad Y \equiv \sum_{a \in \sigma_k} V_a, \quad Z \equiv \max_{h \neq j,k} \sum_{a \in \sigma_h} V_a,$$

and for any $x, y, v_i, z \geq 0$,

$$\phi(x, y; v_i, z) \equiv \max\{x + v_i, y, z\} - \max\{\min\{x + v_i, y\}, z\}.$$

We showed that

$$E[\phi(X, Y; V_i, Z)] < E[\phi(Y, X; V_i, Z)].$$

This result holds for any $n_j > 0$,and in particular for $n_j = 1$ (in this case, note that $X = 0$).The result guarantees that (i), (ii), and (iii) do not represent profitable deviations in the coalition profile $\widetilde{\sigma}$. Therefore, in the presence of a coalitions' size ceiling $\overline{n}$, the coalition profile $\widetilde{\sigma}$ is uniquely stable and the upper bound for the seller's expected revenue $w_{(\widetilde{\sigma},2)}$ is uniquely obtained. ■

8 References

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