Supplementary Appendix to "Incentive Compatibility of Large Centralized Matching Markets"

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August 22, 2016

Abstract

First, we summarize definitions and related theorems of asymptotic statistics in Section A. We prove Theorems in Section B. Lastly, Section C contains additional simulation results.

A Asymptotic Statistics

We summarize some results of asymptotic statistics from (Serfling, 1980).

Let $X_1, X_2, \ldots$ and X be random variables on a probability space $(\Omega, \mathcal{A}, P)$. We say that X_n **converges in probability** to X if

$$\lim_{n\to\infty} P\left(|X_n - X| < \epsilon\right) = 1, \quad \text{every } \epsilon > 0.$$

This is written $X_n \xrightarrow{p} X$.

For $r > 0$, we say that X_n **converges in the r^{th} mean** (or in the L^r-norm) to X if

$$\lim_{n\to\infty} E\left(|X_n - X|^r\right) = 0.$$

This is written $X_n \xrightarrow{L^r} X$.

Fact 1. *If $X_n \xrightarrow{L^r} X$, then $X_n \xrightarrow{p} X$.*

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Fact 2. *Suppose that* $X_n \xrightarrow{p} X$, $|X_n| \leq |Y|$ *with probability 1 (for all* n*) and* $E(|Y|^r) < \infty$. *Then,* $X_n \xrightarrow{L^r} X$.

Remark 1. *In this paper, most random variables represent proportions, which are bounded above by 1 with probability 1. As such, convergence in probability and convergence in the* r^{th} *mean are equivalent.*

Fact 3. *Let* $\mathbf{X_1}, \mathbf{X_2}, \ldots$ *and* $\mathbf{X}$ *be random* k*-vectors defined on a probability space, and let* g *be a vector-valued Borel function defined on* $\mathbf{R}^k$. *If* g *is continuous with* $P_{\mathbf{X}}$*-probability 1, then*

$$\mathbf{X_n} \xrightarrow{p} \mathbf{X} \implies g(\mathbf{X_n}) \xrightarrow{p} g(\mathbf{X}).$$

Remark 2. *In particular, if* $X_n \xrightarrow{p} X$ *and* $Y_n \xrightarrow{p} Y$, *then* $X_n + Y_n \xrightarrow{p} X + Y$ *and* $X_n Y_n \xrightarrow{p} XY$.

Given a univariate distribution function F and $0 < q < 1$, we define q^{th} **quantile** ξ_q as

$$\xi_q \equiv \inf\{x : F(x) \geq q\}.$$

Consider an i.i.d sequence $\langle X_i \rangle$ with distribution function F. For each sample of size n, $\{X_1, X_2, \ldots, X_n\}$, a corresponding **empirical distribution function** F_n is constructed as

$$F_n(x) \equiv \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{X_i \leq x\}, \quad -\infty < x < \infty.$$

The **empirical** q^{th} **quantile** $\hat{\xi}_{q:n}$ is defined as the q^{th} quantile of the empirical distribution function. That is

$$\hat{\xi}_{q:n} \equiv \inf\{x : F_n(x) \geq q\}.$$

For each x, $F_n(x)$ is a random variable, and therefore $\hat{\xi}_{q:n}$ is also a random variable.

Fact 4. *Suppose that* q^{th} *quantile* ξ_q *is the unique solution* x *of* $F(x-) \leq q \leq F(x)$. *Then, for every* $0 < q < 1$ *and* $\epsilon > 0$,

$$P\left(\left|\hat{\xi}_{q:n} - \xi_q\right| > \epsilon\right) \leq 2e^{-2n\lambda_\epsilon^2}$$

for all n, *where* $\lambda_{1,\epsilon} = F(\xi_q + \epsilon) - q$, $\lambda_{2,\epsilon} = q - F(\xi_q - \epsilon)$, *and* $\lambda_\epsilon = \min\{\lambda_{1,\epsilon},\ \lambda_{2,\epsilon}\}$.

For each sample of size n, $\{X_1, X_2, \ldots, X_n\}$, the ordered sample values

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$$

are called the **order statistics**.

In view of

$$X_{k:n} = \hat{\xi}_{k/n:n}, \quad 1 \leq k \leq n, \tag{1}$$

we will carry out proofs in terms of empirical quantiles, even when variables are defined as order statistics.

B Proofs

We omit the proofs of Theorems 1, 2, and 7. The proof of Theorem 1 (in the one-to-one model) is subsumed by the proof of Theorem 3 (in the many-to-one model).

Theorem 2 and is immediate from Theorem 1. As ϵ is arbitrary, the lower and upper bounds of utilities from stable matchings can be arbitrarily close to each other. For the case of one-to-one matching, this observation implies vanishing incentives to manipulate stable mechanisms. Each agent's gain by manipulating a stable mechanism is bounded above by the gap between utilities from the best and the worst stable matching partners. A small proportion of agents with significant incentives may become better off by playing profitable truncation strategies. However, their truncations make other agents have less wedge between utilities from the extreme stable matchings.[1]

Lastly, Theorem 7 is immediate from Theorem 1 from the idea of convex combination.

B.1 Proof of Theorem 3

We show the assortative feature of stable matchings. We prove that for any $\epsilon > 0$ and $c \in (0, 1]$, firms with common values above c are most likely to achieve utilities higher than $\Phi \circ U(c, 1) - \epsilon$ (Proposition 1). We combine this observation with a similar result: an asymptotic lower bound on utilities of workers with common values above c. If the common values of some workers are much higher than c, they are most likely to match with firms with common values higher than c. Thus, firms with common values near c receive utilities that are asymptotically bounded above by $\Phi \circ U(c, 1) + \epsilon$ (Proposition 2).

An Asymptotic Utility Lower Bound The random bipartite graph theory provides the key technique to find an asymptotic utility lower bound. We prove that the proportion of

[1] This convenient proof would not be applicable for many-to-one matching.

firms with common values above c which underachieve in a stable matching converges to 0 in probability.

Proposition 1. *For every $\epsilon > 0$ and $c \in (0, 1]$,*

$$\frac{1}{n} \left|\{f \in F \mid C_f \geq c \text{ and } U_f^{\mu_W} \leq \Phi \circ U(c, 1) - \epsilon\}\right| \xrightarrow{p} 0 \quad as \quad n \to \infty.$$

To prove the proposition, take any $\hat{c} < c$, close to c, and a small $\hat{\epsilon} > 0$ such that

$$\Phi \circ U(\hat{c}, 1 - \hat{\epsilon}) \geq \Phi \circ U(c, 1) - \epsilon. \tag{2}$$

Also, take any $\tilde{c} \in (\hat{c}, c)$ such that

$$V(c, 1 - \hat{\epsilon}) \geq V(\tilde{c}, 1). \tag{3}$$

By continuity of U, V, and Φ, we can always find $\hat{c}, \hat{\epsilon}$, and $\tilde{c}$ satisfying the above two conditions.

We use a resulting partition $[0, \hat{c}], (\hat{c}, \tilde{c}], (\tilde{c}, c], (c, 1]$ to assign tiers to firms and workers. Agents with common values above c are in Tier 1, and agents with common values between $\tilde{c}$ and c are in Tier 2, etc.

For each market realization (of common values c_F, c_W and private values ζ, η), we define the set of firms with common values above c as

$$\bar{F}(c_F) \equiv \{f \in F \mid c_f \geq c\},$$

and the set of workers with common values above $\hat{c}$ as

$$\bar{W}(c_W) \equiv \{w \in W \mid c_w \geq \hat{c}\}.$$

Also, let $B_{\bar{F}}$ be the set of firms in $\bar{F}$ that do badly in a stable matching,

$$B_{\bar{F}}(c_F, c_W, \zeta, \eta) \equiv \{f \in \bar{F}(c_F) \mid u_f^{\mu_W} \leq \Phi \circ U(c, 1) - \epsilon\},$$

and $B_{\bar{W}}$ be the set of workers in $\bar{W}$ who do badly in all stable matchings

$$B_{\bar{W}}(c_F, c_W, \zeta, \eta) \equiv \{w \in \bar{W}(c_W) \mid v_w^{\mu_W} \leq V(\tilde{c}, 1)\}.$$

We complete the proof of Proposition 1 by showing that

$$\frac{|B_{\bar{F}}(C_F, C_W, \zeta, \eta)|}{n} \xrightarrow{p} 0, \quad \text{as} \quad n \to \infty.$$

Construct a bipartite graph with $\bar{F} \cup \bar{W}$ as a bipartition set of nodes. Two vertices $f \in \bar{F}$ and $w \in \bar{W}$ are joined by an edge if and only if

$$\zeta_{f,w} \leq 1 - \hat{\epsilon} \quad \text{or} \quad \eta_{f,w} \leq 1 - \hat{\epsilon}.$$

That is, every pair is joined by an edge independently with probability $1 - \hat{\epsilon}^2$.

It is important to notice that for each market realization $\langle F, W, u, v \rangle$, $B_{\bar{F}} \cup B_{\bar{W}}$ is a biclique, which is not necessarily balanced. To see why $B_{\bar{F}} \cup B_{\bar{W}}$ is a biclique, suppose a pair of $f \in B_{\bar{F}}$ and $w \in B_{\bar{W}}$ is *not* joined by an edge: i.e.,

$$\zeta_{f,w} > 1 - \hat{\epsilon} \quad \text{and} \quad \eta_{f,w} > 1 - \hat{\epsilon}.$$

Since $f \in \bar{F}$ and $w \in \bar{W}$, we must have

$$\begin{aligned} u_{f,w} &= U(c_w, \zeta_{f,w}) > U(\hat{c}, 1 - \hat{\epsilon}), \quad \text{and} \\ v_{f,w} &= V(c_f, \eta_{f,w}) > V(c, 1 - \hat{\epsilon}). \end{aligned}$$

Note from Equation (2) that

$$u_f^{\mu_W} \leq \Phi \circ U(c, 1) - \epsilon \leq \Phi \circ U(\hat{c}, 1 - \hat{\epsilon}).$$

That is, firm f receives an individual payoff less than or equal to $U(\hat{c}, 1 - \hat{\epsilon})$ from at least one worker. On the other hand, by Equation (3),

$$v_{f,w} > V(\tilde{c}, 1) \geq v_w^{\mu_W}.$$

Thus, if the pair (f, w) is not joined by an edge, the pair would be a blocking pair of μ_W, which contradicts that μ_W is stable.

Since $B_{\bar{F}} \cup B_{\bar{W}}$ is a biclique, the set contains a balanced biclique of size

$$\min \left\{ |B_{\bar{F}}(C_F, C_W, \zeta, \eta)|, |B_{\bar{W}}(C_F, C_W, \zeta, \eta)| \right\}.$$

Theorem 5 in the main paper implies that for $\beta_n = 2 \cdot \log(qn)/\log \frac{1}{1-\hat{\epsilon}^2}$,

$$P\left(\min\{|B_{\bar{F}}|, |B_{\bar{W}}|\} \leq \beta_n\right) \to 1 \quad \text{as} \quad n \to \infty.$$

Since $\beta_n/n \to 0$ as $n \to \infty$, we have

$$\min\left\{\frac{|B_{\bar{F}}|}{n}, \frac{|B_{\bar{W}}|}{qn}\right\} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty. \tag{4}$$

The size of $B_{\bar{W}}$ remains relatively large. $\bar{W}$ contains all workers with common values above $\hat{c}$. A large number of them have to match with firms with common values below $\tilde{c}$. Approximately $qn(\tilde{c} - \hat{c})$ of them should be in $B_{\bar{W}}$. Formally, by the weak law of large numbers (Remark 2),

$$\begin{aligned} \frac{|B_{\bar{W}}(C_F, C_W, \zeta, \eta)|}{qn} &\geq \frac{|\{w \in W \mid C_w \geq \hat{c}\}|}{qn} - \frac{|\{f \in F \mid C_f > \tilde{c}\}|}{qn} \\ &\xrightarrow{p} (1 - \hat{c}) - (1 - \tilde{c}) = \tilde{c} - \hat{c}. \end{aligned} \tag{5}$$

Therefore, for any $\epsilon' < \frac{\tilde{c}-\hat{c}}{2}$,

$$\begin{aligned} P\left(\frac{|B_{\bar{F}}(C_F, C_W, \zeta, \eta)|}{n} > \epsilon'\right) &\leq P\left(\min\left\{\frac{|B_{\bar{F}}(C_F, C_W, \zeta, \eta)|}{n}, \frac{|B_{\bar{W}}(C_F, C_W, \zeta, \eta)|}{qn}\right\} > \epsilon'\right) \\ &\quad + P\left(\frac{|B_{\bar{W}}(C_F, C_W, \zeta, \eta)|}{qn} \leq \epsilon'\right). \end{aligned}$$

Equations (4) and (5) imply that both terms on the right-hand side converge to 0 as n increases.

An Asymptotic Utility Upper Bound We find an asymptotic upper bound on firms' utilities from stable matchings.

Proposition 2. *Fix $\epsilon > 0$ and $c \in (0, 1]$,*

$$\frac{1}{n}\left|\{f \in F \mid C_f \leq c \ \text{ and } \ \Phi \circ U_f^{\mu_F} \geq U(c, 1) + \epsilon\}\right| \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

We only need to prove the case that $\Phi \circ U(c, 1) + \epsilon < \Phi \circ U(1, 1)$, otherwise the proposition holds trivially.

By continuity of $\Phi \circ U(.,.)$, we can find $\bar{c} \in (c, 1)$ such that

$$\Phi \circ U(\bar{c}, 1) \leq \Phi \circ U(c, 1) + \epsilon. \tag{6}$$

We claim that

$$\begin{aligned} &\left|\{f \in F \mid C_f \leq c \text{ and } U_f^{\mu_F} \geq \Phi \circ U(c,1) + \epsilon\}\right| \\ &\leq \left|\{w \in W \mid C_w \geq \bar{c} \text{ and } V_w^{\mu_F} \leq V(\bar{c},1) - \epsilon''\}\right| \end{aligned} \tag{7}$$

where $\epsilon'' = V(\bar{c}, 1) - V(c, 1)$.

For a realization c_F, c_W, ζ, and η, we take a firm $f \in \{f \in F \mid c_f \leq c \text{ and } u_f^{\mu_F} \geq \Phi \circ U(c,1) + \epsilon\}$. Since $u_f^{\mu_F} = u_{f,w} \geq \Phi \circ U(c,1) + \epsilon \geq \Phi \circ U(\bar{c}, 1)$, there exists at least one worker w who matches to f in μ_F such that $u_{f,w} \geq U(\bar{c}, 1)$. In particular, the common value of w must be at least $\bar{c}$. In addition, firm f has common value below c, so the worker w receives $v_w^{\mu_F} = v_{f,w} \leq V(c,1) = V(\bar{c},1) - \epsilon''$. Thus, $w \in \{w \in W \mid c_w \geq \bar{c} \text{ and } v_w^{\mu_F} \leq V(\bar{c},1) - \epsilon''\}$. Then, Proposition 2 is immediate from the above inequality (7) and the counterpart result of Proposition 1 for workers: for every $\epsilon > 0$ and $c \in (0, 1]$,

$$\frac{1}{n} \left|\{w \in W \mid C_w \geq c \text{ and } V_w^{\mu_F} \leq V(c,1) - \epsilon\}\right| \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

Completing the Proof of Theorem 3 We prove that for any $\epsilon, \theta > 0$,

$$P\left(\frac{|F \backslash A_F^M(\epsilon; U, V)|}{n} > \theta\right) \to 0, \quad \text{as} \quad n \to \infty. \tag{8}$$

Take a positive integer K such that $1/K < \theta$ and for any $c, c' \in [0, 1]$ with $|c - c'| < 1/K$,

$$|\Phi \circ U(c, 1) - \Phi \circ U(c', 1)| < \epsilon/2.$$

We claim the following inequality:

$$\begin{aligned} \frac{|F \backslash A_F^M(\epsilon; U, V)|}{n} \leq\ & \sum_{k=1}^{K} \frac{1}{n} \left|\{f \in F \mid C_f \geq k/K \text{ and } U_f^{\mu_W} \leq \Phi \circ U(k/K, 1) - \epsilon/2\}\right| \\ +\ & \sum_{k=1}^{K} \frac{1}{n} \left|\{f \in F \mid C_f \leq k/K \text{ and } U_f^{\mu_F} \geq \Phi \circ U(k/K, 1) + \epsilon/2\}\right| \\ +\ & \frac{1}{n} \left|\{f \in F | C_f \leq 1/K\}\right|. \end{aligned} \tag{9}$$

For a realization c_F, c_W, ζ and η (i.e., a realization u, v), take any firm $f \in A_F(\epsilon; U, V)$. If $c_f \le \frac{1}{K}$, the firm is trivially in one of the sets on the right-hand side of (9). If not, there exists $k \in \{1, 2, \ldots, K-1\}$ such that $\frac{k}{K} \le c_f \le \frac{k+1}{K}$.

Note that

$$u_f^{\mu_W} \le \Phi \circ U(c_f, 1) - \epsilon < \Phi \circ U((k+1)/K, 1) - \epsilon/2,$$

or

$$u_f^{\mu_F} \ge \Phi \circ U(c_f, 1) + \epsilon > \Phi \circ U(k/K, 1) + \epsilon/2.$$

Thus, the firm f must be in one of the sets on the right-hand side of (9).

From Propositions 1 and 2 and by the weak law of large numbers, the right-hand side of Equation (9) converges in probability to $1/K$ which is strictly less than θ. Therefore, we obtain the convergence (8) and complete the proof of Theorem 1.

B.2 Proof of Theorem 4

We first find an ϵ-Nash equilibrium for any fixed ϵ and a market realization $\langle F, W, u, v \rangle$. We interactively update firms' strategies, starting from the truth-telling. Let all firms that have more than ϵ gain, conditioned on others' truth-telling, play truncation strategies and capacity under-reporting. Next, let firms that have more than ϵ gain, conditioned on the strategy profile of the previous iterative step, truncate their preferences further and report even fewer capacities. This iteration will terminate, because all firms become weakly better off in each step, and firms' payoffs in a feasible matching are bounded. The outcome is a strategy profile of firms, in which none can obtain additional ϵ utility gain: i.e., an ϵ-Nash equilibrium.

Let $D(\epsilon; u, v)$ be the set of firms that are not truth-telling in this equilibrium. We prove that an equivalent result of Theorem 4: for any $\epsilon > 0$,

$$\frac{|D(\epsilon; U, V)|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

Take a positive integer K such that for any $c, c' \in [0, 1]$ with $|c - c'| < 1/K$,

$$|\Phi \circ U(c, 1) - \Phi \circ U(c', 1)| < \epsilon/4.$$

Similar to the proof of the previous theorem, we assign tiers to firms and workers based on their common values.

As K is finite, it is enough to prove that for every $k = 1, 2, \ldots, K$,

$$\frac{|\{f \in F \mid \frac{k-1}{K} \leq C_f \leq \frac{k}{K}\} \cap D(\epsilon; U, V)|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty. \tag{10}$$

We prove the convergence inductively from $k = K$ to $k = 1$.

First, fix $k = K, K-1$, or $K-2$. For each market realization, take any firm f with $c_f \geq \frac{k-1}{K}$ that has achieved at least ϵ gain in equilibrium compared to all agents' truth-telling. It must be that

$$u_f^{\mu_W} \leq \Phi \circ U(1,1) - \epsilon < \Phi \circ U((k-1)/K, 1) - \epsilon/4.$$

Then, Equation (10) for k is immediately from Proposition 1 as

$$\frac{\left|\{f \in F \mid C_f \geq (k-1)/K \ \text{ and } \ U_f^{\mu_W} \leq \Phi \circ U((k-1)/K, 1) - \epsilon/4\}\right|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

Next, fix any $k < K-2$. Assuming that the convergence holds for $k+1, \ldots, K$, we prove

$$\frac{|\{f \in F \mid \frac{k-1}{K} \leq C_f \leq \frac{k}{K}\} \cap D(\epsilon; U, V)|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

For each market realization, take a firm $f \in D(\epsilon; u, v)$ with $\frac{k-1}{K} \leq c_f < \frac{k}{K}$. Let μ'_W be the equilibrium outcome: i.e., worker-optimal stable matching after misreporting. It must be that either

$$u_f^{\mu_W} \leq \Phi \circ U((k-1)/K, 1) - \epsilon/4$$

or

$$u_f^{\mu'_W} \geq \Phi \circ U((k-1)/K, 1) + 3\epsilon/4.$$

We know from Proposition 1 that

$$\frac{\left|\{f \in F \mid (k-1)/K \leq C_f \ \text{ and } \ U_f^{\mu_W} \leq \Phi \circ U((k-1)/K, 1) - \epsilon/4\}\right|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

Thus, we only need to show that

$$\frac{\left|\{f \in F \mid (k-1)/K \leq C_f \leq k/K \ \text{ and } \ U_f^{\mu'_W} \geq \Phi \circ U((k-1)/K, 1) + 3\epsilon/4\}\right|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

We prove the following sufficient condition.

$$\frac{\left|\{f \in F \mid C_f \leq k/K \ \text{ and } \ U_f^{\mu'_W} \geq \Phi \circ U((k+2)/K, 1)\}\right|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty. \tag{11}$$

The idea behind the proof is simple. A firm f, which achieves a payoff higher than $\Phi \circ U((k+2)/K, 1)$, must match at least one worker with a common value higher than $\frac{k+2}{K}$. However, many workers with such high common values would not be achievable. The worker-optimal stable matching on true preferences is assortative, and most workers with such high common values match to firms with similarly high common values. Moreover, all firms will be weakly better off in equilibrium than in the true worker-optimal stable matching. Only workers with high common values can support those firms' increased payoffs.

For a formal proof of (11), we show that few workers with common values higher than $\frac{k+2}{K}$ would match firms with common values lower than k/K in equilibrium.

$$\frac{|\{w \in W \mid C_w \geq (k+2)/K \quad \text{and} \quad V_w^{\mu'_W} \leq V(k/K, 1)\}|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

We construct another bipartite graph. For each market realization, let

$$\bar{W} \equiv \{w \in W \mid c_w \geq (k+2)/K\}$$

and

$$\bar{F} \equiv \{c_f \geq (k+1)/K\}$$

Also, define two sets of agents who do badly *in equilibrium* as

$$B_{\bar{W}}(u, v) \equiv \left\{w \in \bar{W} \mid v_w^{\mu'_W} \leq V(k/K, 1)\right\},$$

and

$$B_{\bar{F}}(u, v) \equiv \left\{f \in \bar{F} \mid u_f^{\mu'_W} \leq \Phi \circ U((k+1)/K, 1)\right\}.$$

Next, fix any $\hat{\epsilon} > 0$ such that

$$\begin{aligned} &U((k+2)/K, 1-\hat{\epsilon}) > U((k+1)/K, 1), \quad \text{and} \\ &V((k+1)/K, 1-\hat{\epsilon}) > V(k/K, 1). \end{aligned}$$

We construct a bipartite graph with a bi-partitioned set of nodes $\bar{W} \cup \bar{F}$. Two vertices

$f \in \bar{F}$ and $w \in \bar{W}$ are joined by an edge if and only if

$$\zeta_{f,w} \leq 1 - \hat{\epsilon} \quad \text{or} \quad \eta_{f,w} \leq 1 - \hat{\epsilon}.$$

We claim that $B_{\bar{W}} \cup (B_{\bar{F}} \backslash D)$ is a biclique, which may not be balanced. Take a pair of $w \in B_{\bar{W}}$ and $f \in B_{\bar{F}} \backslash D$. That is, both f and w are truth-telling and doing badly in the equilibrium. Suppose that the pair is *not* joined by an edge: i.e.,

$$\zeta_{f,w} > 1 - \hat{\epsilon} \quad \text{and} \quad \eta_{f,w} > 1 - \hat{\epsilon}.$$

It must be that

$$\begin{aligned} u_{f,w} &> U((k+2)/K, 1 - \hat{\epsilon}) > U((k+1)/K, 1), \quad \text{and} \\ v_{f,w} &> V((k+1)/K, 1 - \hat{\epsilon}) > V(k/K, 1). \end{aligned}$$

Thus, the pair (f, w) would be a blocking pair of μ'_W.

Since $B_{\bar{W}} \cup (B_{\bar{F}} \backslash D)$ is a biclique, the set contains a balanced biclique of size $\min\{|B_{\bar{W}}|, |B_{\bar{F}} \backslash D|\}$. By the result of random bipartite graph theory, we have

$$\min\left\{\frac{|B_{\bar{W}}|}{qn}, \frac{|B_{\bar{F}} \backslash D|}{n}\right\} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

The size of $B_{\bar{F}} \backslash D$ remains relatively large. $\bar{F}$ contains all firms with common values above $\frac{k+1}{K}$. A large number of firm must employ at least one worker with common value below $\frac{k+2}{K}$ in any feasible matching. That is, approximately n/K firms should be in $B_{\bar{F}}$. Finally, by the induction hypothesis, only a small fraction of firms in $\bar{F}$ are misreporting their preferences and capacities: the size of $B_{\bar{F}} \backslash D$ is large. Therefore,

$$\frac{|B_{\bar{W}}(U, V)|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty,$$

which completes the proof of the convergence (12).

B.3 Proof of Theorem 6

We study the convergence speeds in Theorem 3 and Theorem 1). As explained in the main paper, we consider the case of linear utilities with deterministic common values. Since the proof of the main theorems is based on the random bipartite graph theory, we first study the convergence speed of a result cited from Dawande et al. (2001). We ease notations by stating proofs for the case of one-to-one matching (i.e., omitting Φ and q). The proof extends to the many-to-one model.

The speed in Dawande et al. (2001) We go back to Theorem 2.6 of Dawande et al. (2001). Consider a random bipartite graph $G(V_1 \cup V_2, p)$ of size $n \times n$. Let $\beta_n = \frac{2\log n}{\log \frac{1}{p}}$ and Z_b be the number of bicliques of size $b \times b$. In the proof of the theorem (on page 396), it is shown that for any $n > 1$ and $b \geq \beta_n$,

$$P\left(Z_b \geq 1\right) \leq \frac{1}{(b!)^2}.$$

Denote the size of a maximal balanced biclique of G by by $B \times B$. Clearly, if $B \geq \beta_n$, there exists at least one biclique of size $\lceil \beta_n \rceil \times \lceil \beta_n \rceil$.[2] Thus

$$P\left(B \geq \beta_n\right) \leq \frac{1}{(\lceil \beta_n \rceil !)^2}.$$

From Stirling's formula, we get

$$P\left(B \geq \beta_n\right) \leq \frac{1}{\left(\sqrt{2\pi\beta_n}\left(\frac{\beta_n}{e}\right)^{\beta_n}\right)^2} = \frac{e^{\beta_n}}{2\pi\beta_n^{\beta_n+1}} \quad \text{for every } n.^3 \tag{13}$$

An increasing number of tiers For the proof of the main theorems, we considered a finite (i.e., fixed) number of tiers that is determined by a fixed θ. As we now consider vanishing θ_n, we need to define a sequence of tier structures, in which the number of tiers increases. Notice that we should increase the number of tiers slower than n so that the number of agents in each tier increases.

Let $K_n = \lceil n^{1/4} \rceil$. We partition the set of firms F_n into K_n tiers. For each $k = 1, \ldots, K_n$, the set of firms in tier-k is

$$F_{k;n} \equiv \{f_i \in F_n \mid (k-1)n^{3/4} < i \leq kn^{3/4}\}.$$

[2] For $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer not less than x.

[3] Stirling's formula: for all $n \geq 1$,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{r_n} \quad \text{with} \quad \frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

Note that a firm f_i has a deterministic common value $c_{f_i} = 1 - \frac{i}{n}$, so it is in tier-$k$ if and only if

$$1 - kn^{-1/4} \leq c_{f_i} < 1 - (k-1)n^{-1/4}.$$

We assign workers to tiers similarly.

The speeds in Proposition 1 The key part of the proof applies (13) to the asymptotic utility lower bound.

For each market realization, define the sets of firms that do badly in a stable matching: for every $k = 1, \ldots, K_n$,

$$B_{F_{k;n}}(\zeta, \eta) \equiv \left\{ f \in F_{k;n} \mid u_f^{\mu_W} \leq \lambda(1 - kn^{-1/2}) + (1-\lambda) - (2/5)\epsilon_n \right\},$$

where $\epsilon_n = 6\lambda n^{-1/4}$.

We show that for every $k \in \{1, 2, \ldots, K_n - 3\}$,

$$P\left(\left|B_{F_{k;n}}\right| \geq \phi_n\right) \leq \psi_n, \tag{14}$$

where $\phi_n = O((\log n)n^{1/2})$ and $\psi_n = o(e^{-\phi_n})$.

Define

$$\bar{W}_{k;n} \equiv W_{k;n} \cup W_{k+1;n} \cup W_{k+2;n}.$$

For each market realization, let $B_{\bar{W}_{k;n}}$ be the set of workers in $\bar{W}_{k;n}$ who do badly in *all* stable matching:

$$\begin{aligned} B_{\bar{W}_{k;n}}(\zeta, \eta) &\equiv \left\{ w \in \bar{W}_{k;n} \mid v_w^{\mu_W} \leq \lambda(1 - kn^{-1/4}) + (1-\lambda) - \lambda n^{-1/4} \right\} \\ &= \left\{ w \in \bar{W}_{k;n} \mid v_w^{\mu_W} \leq \lambda(1 - (k+1)n^{-1/4}) + (1-\lambda) \right\}. \end{aligned}$$

Consider the bipartite graph with a bipartition set of nodes $F_{k;n} \cup \bar{W}_{k;n}$. Each pair of vertices $f \in F_{k;n}$ and $w \in \bar{W}_{k;n}$ is joined by an edge if and only if

$$\zeta_{f,w} \leq 1 - \epsilon_{F;n} \quad \text{or} \quad \eta_{f,w} \leq 1 - \epsilon_{W;n},$$

where $\epsilon_{F;n} \equiv \frac{2(\epsilon_n - 5\lambda n^{-1/4})}{5(1-\lambda)}$ and $\epsilon_{W;n} \equiv \frac{\lambda n^{-1/4}}{1-\lambda}$.

We claim that $B_{F_{k;n}} \cup B_{\bar{W}_{k;n}}$ is a biclique. Suppose a pair of $f \in B_{F_{k;n}}$ and $w \in B_{\bar{W}_{k;n}}$ is not joined by an edge. Then,

$$\begin{aligned} u_{f,w} &= \lambda c_w + (1-\lambda)\zeta_{f,w} \\ &> \lambda\left(1-(k+2)n^{-1/4}\right) + (1-\lambda)(1-\epsilon_{F;n}) \\ &= \lambda\left(1-kn^{-1/4}\right) + (1-\lambda) - (2/5)\epsilon_n, \quad \text{and} \\ v_{f,w} &= \lambda c_f + (1-\lambda)\eta_{f,w} \\ &> \lambda(1-kn^{-1/4}) + (1-\lambda)(1-\epsilon_{W;n}) \\ &= \lambda(1-(k+1)n^{-1/4}) + (1-\lambda). \end{aligned}$$

The two utilities $u_{f,w}$ and $v_{f,w}$ are higher than utilities that the payoffs for f and w, respectively, in μ_W. The pair would have blocked μ_W, which contradicts that μ_W is stable.

The above bipartite graph is drawn with firms in one tier and workers in three tiers, so the size is not larger than $3n^{3/4}$. In addition, every pair is joined by an edge independently with probability

$$p_n = 1 - \epsilon_{F;n}\epsilon_{W;n} = 1 - \frac{2\lambda^2 n^{-1/2}}{5(1-\lambda)^2}.$$

By applying (13), we obtain

$$P\left(\left|B_{F_{k;n}}\right| \geq \phi_n\right) \leq \psi_n, \tag{15}$$

where $\phi_n = \frac{2\log(3n^{3/4})}{\log\frac{1}{p_n}}$ and $\psi_n = \frac{1}{2\pi}\frac{1}{\phi_n}\left(\frac{e}{\phi_n}\right)^{\phi_n}$.

Next, we find the order of ϕ_n and δ_n. Note that $\log p_n \leq p_n - 1$, so

$$\phi_n = \frac{2\log(3n^{3/4})}{\log\frac{1}{p_n}} = \frac{2\log(3n^{3/4})}{-\log p_n} \leq \frac{2\log 3 + (3/2)\log n}{1-p_n}.$$

Moreover, as p_n approaches 1, for any $c_2 > 1$ and every sufficiently large n, we have $\log p_n > c_2(p_n - 1)$. Thus,

$$\phi_n = \frac{2\log(3n^s)}{-\log p_n} > \frac{2(\log 3 + s\log n)}{c_2(1-p_n)} > \frac{(3/2)\log n}{c_2(1-p_n)} \tag{16}$$

Using the definition of p_n, we find $\phi_n = O((\log n)n^{1/2})$.

Next, notice that

$$\log\psi_n = -\log(2\pi) - \log(\phi_n) + \phi_n(1 - \log(\phi_n)).$$

As $\phi_n \to \infty$, for any $0 < c_3 < 1$ and every sufficiently large n, we have $\log \psi_n < -c_3 \phi_n \log \phi_n$. Thus, $\psi_n < e^{-c_3 \phi_n \log \phi_n}$, which implies that $\psi_n = o(e^{-\phi_n})$.

The Convergence Speed in Theorems 1 and 3 It is clear that

$$\begin{aligned} |F \backslash A_F(\epsilon; U, V)| \leq & \left|\left\{f \in F_n \mid U_f^{\mu_W} \leq \lambda c_f + (1-\lambda) - (3/5)\epsilon_n\right\}\right| \\ & + \left|\left\{f \in F_n \mid U_f^{\mu_F} \geq \lambda c_f + (1-\lambda) + (2/5)\epsilon_n\right\}\right|. \end{aligned} \tag{17}$$

If a firm f in a market realization receives

$$u_f^{\mu_F} \geq \lambda c_f + (1-\lambda) + (2/5)\epsilon_n,$$

we have

$$u_f^{\mu_F} \geq \lambda c_f + (1-\lambda) + (2/5)\epsilon_n = \lambda\left(c_f + (2\epsilon_n/5\lambda)\right) + (1-\lambda).$$

So the firm matches a worker w with $c_w \geq c_f + \frac{2\epsilon_n}{5\lambda}$. The payoff for the worker is

$$\begin{aligned} v_w^{\mu_F} & \leq \lambda c_f + (1-\lambda) \leq \lambda\left(c_w - (2\epsilon_n/5\lambda)\right) + (1-\lambda) \\ & = \lambda c_w + (1-\lambda) - (2/5)\epsilon_n. \end{aligned}$$

Thus we can rewrite (17) as

$$\begin{aligned} |F \backslash A_F(\epsilon; U, V)| \leq & \left|\left\{f \in F_n \mid U_f^{\mu_W} \leq \lambda c_f + (1-\lambda) - (3/5)\epsilon_n\right\}\right| \\ & + \left|\left\{w \in W_n \mid V_w^{\mu_F} \leq \lambda c_w + (1-\lambda) - (2/5)\epsilon_n\right\}\right| \end{aligned} \tag{18}$$

Notice that if a firm f in Tier k receives payoff $u_f^{\mu_W} \leq \lambda c_f + (1-\lambda) - (3/5)\epsilon_n$, we have

$$\begin{aligned} u_f^{\mu_W} & \leq \lambda\left(1 - (k-1)n^{-1/4}\right) + (1-\lambda) - (3/5)\epsilon_n \\ & \leq \lambda\left(1 - kn^{-1/4}\right) + (1-\lambda) - (2/5)\epsilon_n. \end{aligned}$$

Similarly, if a worker w in Tier k receives payoff $v_w^{\mu_F} \leq \lambda c_w + (1-\lambda) - (3/5)\epsilon_n$, we have

$$v_w^{\mu_F} \leq \lambda\left(1 - kn^{-1/4}\right) + (1-\lambda) - (2/5)\epsilon_n.$$

Thus (18) implies that

$$|F \backslash A_F(\epsilon; U, V)| \le \sum_{k=1}^{K_n-3} \left(\left| B_{F_{k;n}} \right| + \left| B_{W_{k;n}} \right| \right) + \sum_{k=K_n-2}^{K_n} \left(|F_{k;n}| + |W_{k;n}| \right).$$

In particular,

$$\begin{aligned} \sum_{k=K_n-2}^{K_n} |W_{k;n}| = \sum_{k=K_n-2}^{K_n} |F_{k;n}| \quad &= \left| \{ f_i \in F_n \mid (K_n - 3) n^{3/4} < i \le n \} \right| \\ &\le \left| \{ f_i \in F_n \mid n - 3n^{3/4} < i \le n \} \right| \quad \le 3n^{3/4}, \end{aligned}$$

which leads to

$$|F \backslash A_F(\epsilon; U, V)| \le \sum_{k=1}^{K_n-3} \left| B_{F_{k;n}} \right| + \sum_{k=1}^{K_n-3} \left| B_{W_{k;n}} \right| + 6n^{3/4}.$$

From (15), we can show that

$$\begin{aligned} & P \left(\sum_{k=1}^{K_n-3} \left| B_{F_{k;n}} \right| + \sum_{k=1}^{K_n-3} \left| B_{W_{k;n}} \right| > 2(K_n - 3)\phi_n \right) \\ \le \quad & 1 - \Pi_{k=1}^{K_n-3} \left(1 - P \left(\left| B_{F_{k;n}} \right| > \phi_n \right) \right) \times \Pi_{k=1}^{K_n-3} \left(1 - P \left(\left| B_{W_{k;n}} \right| > \phi_n \right) \right) \\ \le \quad & 1 - (1 - \psi_n)^{2(K_n-3)}. \end{aligned}$$

Therefore,

$$P \left(|F \backslash A_F(\epsilon; U, V)| > 2(K_n - 3)\phi_n + 6n^{3/4} \right) \le 1 - (1 - \psi_n)^{2(K_n-3)}.$$

By taking into account $K_n \le n^{1/4}$, we have

$$P \left(\frac{|F \backslash A_F(\epsilon; U, V)|}{n} > \frac{2}{n}(n^{1/4} - 3)\phi_n + 6n^{-1/4} \right) \le 1 - (1 - \psi_n)^{2(n^{1/4}-3)}.$$

First,

$$\theta_n \equiv \frac{2}{n} \left(n^{1/4} - 3 \right) \phi_n + 6n^{-1/4} = O(n^{-1/4}).$$

Second,

$$\delta_n \equiv 1 - (1 - \psi_n)^{2(n^{1/4}-3)} = 1 - (1 - \psi_n)^{2(n^{1/4}-3)} = 1 - (1 - \psi_n)^{\frac{1}{\psi_n} 2\psi_n (n^{1/4}-3)}.$$

Since $\psi_n \to 0$, we have $(1-\psi_n)^{\frac{1}{\psi_n}} \to e^{-1}$. So, for any $0 < c_4 < e^{-2}$ and any sufficiently large n,

$$\delta_n < 1 - c_4^{\psi_n(n^s-3)} < 1 - c_4^{\psi_n n^s}.$$

As $1 - x \le -\log x$, it must be that $\delta_n < \log(1 - c_4^{\psi_n n^{1/4}}) = (-\log c_4) n^{1/4} \psi_n$. Therefore,

$$\delta_n = o(n^{1/4} e^{-\phi_n}) = o(e^{(1/4)\log n - \phi_n}).$$

Since $(1/4)\log n - \phi_n < -n^{-1/2}$ for any sufficiently large n, we have $\delta_n = o(e^{-n^{1/2}})$.

B.4 Proof of Theorem 8

We prove the theorem in three steps. We construct a strategy profile that agents best respond to others' truth-telling approximately. It is most likely that all agent tell the truth in the strategy profile. That is, it is most likely that every agent has correct beliefs on her opponents' strategies. As such, the strategy profile is an ϵ-Bayesian-Nash equilibrium.

Constructing a strategy profile The utilities $U = [U_{f,w}]$ and $V = [V_{f,w}]$ are i.i.d. draws from continuous distributions over bounded intervals. We assume without loss of generality that the utilities are drawn from the uniform distribution over $[0, 1]$.

Fix any stable matching mechanism. We have an induced game in which each participant submits her preference list after learning her own utilities only. A (pure) strategy s_f for a firm $f \in F$ is a mapping from realizations of $\{U_{f,w}\}_{w \in W}$ to preference lists over workers, and a (pure) strategy s_w for worker $w \in W$ is a mapping from realizations of $\{V_{f,w}\}$ to preference lists over firms.

Fix a market realization. For each $a \in F \cup W$, let $\Delta^E(a; u, v)$ be the maximum expected gain by manipulation, conditioned on others' truth-telling. We consider the following strategy profile $s^* = \big((s_f^*)_{f \in W}, (s_w^*)_{w \in W}\big)$:

$$s_f^*(\{u_{f,w}\}_{w \in W}) = \begin{cases} \succ_f & \text{if} \quad \Delta^E(f; u, v) \le \epsilon/3 \\ BR_f\left(\succ_{-f} \;\mid \{u_{f,w}\}_{w \in W}\right) & \text{if} \quad \Delta^E(f; u, v) > \epsilon/3, \end{cases}$$

and

$$s_w^*(\{v_{f,w}\}_{f \in F}) = \begin{cases} \succ_w & \text{if} \quad \Delta^E(w; u, v) \le \epsilon/3 \\ BR_w\left(\succ_{-w} \;\mid \{v_{f,w}\}_{f \in F}\right) & \text{if} \quad \Delta^E(w; u, v) > \epsilon/3, \end{cases}$$

where $\succ_a$ denotes the true preference list of agent a.

Later we will show that most likely *all* agents in a large market has a small expected gain from manipulation, conditioned on others' truth-telling. Formally, for any $\epsilon > 0$,

$$P_n \equiv P\left(\max_{a \in F_n \cup W_n} \Delta^E(a; U, V) \leq \epsilon/3\right) \to 1, \quad \text{as} \quad n \to \infty. \tag{19}$$

An immediate implication is that, with probability P_n converging to one, all agents tell the truth by playing the strategy profile s^*.

The strategy profile s^* is an ϵ-Bayesian-Nash equilibrium when P_n is sufficiently close to 1. Take any firm f. The strategy s_f^* (i.e., truth-telling) is an approximate best-response for firm f. With probability at least P_n, all other agents tell the truth according to their strategies s_{-f}^*. Thus, the expected utility gain from any deviation does not exceed $P_n \cdot (\epsilon/3) + (1 - P_n) \cdot 1$.

Proof of (19) We take any $f \in F_n$ and prove that

$$P\left(\Delta^E(f; U, V) \leq 6\epsilon\right) \geq 1 - 4e^{-2n\epsilon^2}. \tag{20}$$

Then, we have

$$\begin{aligned} P\left(\max_{f \in F_n} \Delta^E(f; U, V) \leq \epsilon/2\right) &= P\left(\Delta^E(f; U, V) \leq \epsilon/2\right)^n \\ &\geq \left(1 - 4e^{-2n(\epsilon/12)^2}\right)^n = \left(1 - \frac{4}{e^{2n(\epsilon/12)^2}}\right)^n \to 1, \end{aligned}$$

so the convergence (19) is immediate from

$$\max_{a \in F_n \cup W_n} \Delta^E(a; U, V) = \max\left\{\max_{f \in F_n} \Delta^E(f; U, V), \max_{w \in W_n} \Delta^E(w; U, V)\right\} \xrightarrow{p} 0.$$

We now prove (20).

Take a firm f from a realized market $\langle F, W, u, v \rangle$. Recall the information of the firm f is $\pi_f = \{u_{f,w}\}_{w \in W}$. A sufficient condition of $\Delta^E(f; u, v) \leq 6\epsilon$ is $E\left[U_f^{\mu_W} \mid \pi_f\right] \geq 1 - 6\epsilon$. Thus, it is sufficient to show that

$$P\left(E\left[U_f^{\mu_W} \mid \Pi_f\right] \geq 1 - 6\epsilon\right) \geq 1 - 4e^{-2n\epsilon^2}.$$

Firm f's realized private information π_f is decomposed into an ordinal preference list over workers and utilities associated with rank numbers. That is, each realization $\pi_f =$

$\{u_{f,w_j}\}_{j=1}^n$ is uniquely determined by a permutation σ_n of $\{1, 2, \ldots, n\}$ and a realization of n-order statistics $\{u_{1;n}, u_{2;n}, \ldots, u_{n;n}\}$ (see Section A for the definition and properties of order statistics). Thus

$$E\left[U_f^{\mu_W} \mid \pi_f\right] = E\left[U_f^{\mu_W} \mid \sigma_n, \{u_{k;n}\}_{k=1}^n\right],$$

where σ_n denotes a permutation of $\{1, 2, \ldots, n\}$, drawn uniformly at random.

Let $R_f^{\mu_W}$ be the rank number of firm f's worker-optimal stable matching partner such that, e.g., $R_f^{\mu_W} = 1$ if the firm matches to the most preferred worker. Since the utility from the worker-optimal stable matching is determined by rank numbers,

$$E\left[U_f^{\mu_W} \mid \sigma_n, \{u_{k;n}\}_{k=1}^n\right] = E\left[U_{(n-R_f^{\mu_W}+1);n} \mid \sigma_n, \{u_{k;n}\}_{k=1}^n\right].$$

In the pure private value model, all agents are symmetric so that each agent's preference list contains no information about the rank number of worker-optimal stable matching partner. Therefore,

$$E\left[U_{(n-R_f^{\mu_W}+1);n} \mid \sigma_n, \{u_{k;n}\}_{k=1}^n\right] = E\left[U_{(n-R_f^{\mu_W}+1);n} \mid \{u_{k;n}\}_{k=1}^n\right].$$

In the end, we are enough to show that

$$P\left(E\left[U_{(n-R_f^{\mu_W}+1);n} \mid \{U_{k;n}\}_{k=1}^n\right] < 1 - 6\epsilon\right) \leq 4e^{-2n\epsilon^2}. \tag{21}$$

Let $p_1 = 1 - \epsilon$ and $p_2 = 1 - 5\epsilon$. Also, let $\hat{\xi}_{p_1;n}$ and $\hat{\xi}_{p_2;n}$ be the realized empirical p_1-th and p_2-th quantiles of $\{u_{f,w}\}_{w \in W}$ (see Section A for the definition of empirical quantile). We utilize independence between the order statistics ($\{u_{k;n}\}_{k=1}^n$) and the random preference order (σ_n) and find that

$$P\left(u_{(n-R_f^{\mu_W}+1);n} \geq \hat{\xi}_{p_1;n} \mid \{u_{k;n}\}_{k=1}^n\right) = P\left(\frac{n - R_f^{\mu_W} + 1}{n} \geq p_1\right) \geq P\left(\frac{R_f^{\mu_W}}{n} \leq 1 - p_1\right).$$

Pittel (1989) shows that $\frac{\sum_{f \in F} R_f^{\mu_W}}{n^2 \log^{-1} n} \xrightarrow{p} 1$, and we have $\frac{R_f^{\mu_W}}{n} \xrightarrow{p} 0$ by symmetry of agents. Thus for any sufficiently large n,

$$P\left(\frac{R_f^{\mu_W}}{n} \leq 1 - p_1\right) \geq 1 - \frac{2\epsilon}{1 - 4\epsilon}.$$

In addition, if the realized empirical quantiles satisfy $\hat{\xi}^f_{p_1;n} - \hat{\xi}^f_{p_2;n} \geq 2\epsilon$,

$$E\left[U_{(n-R_f^{\mu_W}+1);n} \,|\, \{u_{k;n}\}_{k=1}^n\right] \geq \left(1 - \frac{2\epsilon}{1-4\epsilon}\right)\hat{\xi}_{p_1;n} \geq \left(1 - \frac{2\epsilon}{1-4\epsilon}\right)\left(\hat{\xi}_{p_2;n} + 2\epsilon\right).$$

Therefore,

$$\begin{aligned} & P\left(E\left[U_{(n-R_f^{\mu_W}+1);n} \,|\, \{U_{k;n}\}_{k=1}^n\right] < 1-6\epsilon\right) \\ \leq \;& P\left(\hat{\xi}^f_{p_1;n} - \hat{\xi}^f_{p_2;n} < 2\epsilon \quad \text{or} \quad \left(1 - \frac{2\epsilon}{1-4\epsilon}\right)\left(\hat{\xi}_{p_2;n} + 2\epsilon\right) < 1-6\epsilon\right) \\ = \;& P\left(\hat{\xi}^f_{p_1;n} - \hat{\xi}^f_{p_2;n} < 2\epsilon \quad \text{or} \quad \hat{\xi}_{p_2;n} < 1-6\epsilon\right). \end{aligned}$$

We study the two conditions in the last probability. First, by the triangular inequality,

$$|\xi_{p_1} - \hat{\xi}^f_{p_1;n}| + |\hat{\xi}^f_{p_1;n} - \hat{\xi}^f_{p_2;n}| + |\hat{\xi}^f_{p_2;n} - \xi_{p_2}| \geq |\xi_{p_1} - \xi_{p_2}| = 4\epsilon.$$

If $\hat{\xi}^f_{p_1;n} - \hat{\xi}^f_{p_2;n} < 2\epsilon$, we have $|\xi_{p_1} - \hat{\xi}^f_{p_1;n}| + |\hat{\xi}^f_{p_2;n} - \xi_{p_2}| > 2\epsilon$, so either $|\xi_{p_1} - \hat{\xi}^f_{p_1;n}| > \epsilon$ or $|\hat{\xi}^f_{p_2;n} - \xi_{p_2}| > \epsilon$ must be true. Second, $\hat{\xi}_{p_2;n} < 1-6\epsilon$ implies that $|\hat{\xi}_{p_2;n} - \xi_{p_2;n}| = |\hat{\xi}_{p_2;n} - (1-5\epsilon)| > \epsilon$. As such,

$$P\left(E\left[U_{(n-R_f^{\mu_W}+1);n} \,|\, \{U_{k;n}\}_{k=1}^n\right] < 1-6\epsilon\right) \leq P\left(|\hat{\xi}^f_{p_1;n} - \xi_{p_1}| > \epsilon\right) + P\left(|\hat{\xi}^f_{p_2;n} - \xi_{p_2}| > \epsilon\right).$$

The sum of the last two terms is bounded above by $4e^{-2n\epsilon^2}$ by Fact 4 in Section A.

C Additional Simulations

We demonstrate that the assumption of short preference lists may leave most agents in a large market unmatched in stable matchings.

We simulate one-to-one matching markets. A firm's utility is defined as $U_{f,w} = \lambda\, C_w + (1-\lambda)\, \zeta_{f,w}$ with $C_w, \zeta_{f,w} \sim U[0,1]$ and $\lambda \in [0,1]$, and similarly for a worker. Each worker considers only thirty of the most preferred firms acceptable. Figure 1 shows increasing proportions of firms (and workers) that remain unmatched in stable matchings, averaged over ten simulations for each market size.

It is worth noting that random preferences in the above simulations were generated by the setup of our model, rather than by the previous studies' model. We show similar effects of short preference lists with simulations based on the model in the previous studies. Let

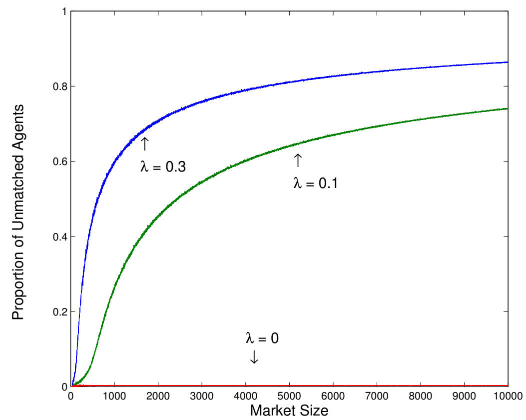


Figure 1: Proportion of unmatched agents in stable matchings.

L be the maximum number of firms that each worker considers acceptable. We generate random preferences following the previous model, in particular, Immorlica and Mahdian (2005), which studies the one-to-one matching model. For each market size n, a market is given two underlying distributions, one for firms and the other for workers, called *popularity distributions*.[4] A worker's preference list is constructed by sequentially sampling L firms from the popularity distribution without replacement. The firm chosen first is the most preferred, and the next chosen firm becomes the second most preferred. We similarly construct firms' preferences, except that firms' preferences are of length n: i.e., all workers are acceptable.

We use two classes of popularity distributions.

1. Normalized geometric distribution

   For each market size n, we define the normalized geometric distribution as:

$$\text{PDF}: p_k = \frac{(1-q)^k}{\sum_{k'=1}^{n}(1-q)^{k'}}, \quad (0 \leq q < 1,\ k = 1, 2, \ldots, n).$$

   Consider a pair of firms, f_{k_1} and f_{k_2} ($k_1 < k_2 \leq n$). For each worker, the probability of choosing f_{k_1} before f_{k_2}, conditioned on at least one of the firms chosen, equals to

$$\frac{(1-q)^{k_1}}{(1-q)^{k_1} + (1-q)^{k_2}} = \frac{1}{1 + (1-q)^{k_2 - k_1}}.$$

[4] Immorlica and Mahdian (2005) construct random preferences only for workers: firms' preferences are arbitrarily given. In our simulation, however, we generate firms' preferences also randomly so that we can measure the *likely* proportions of unmatched agents.

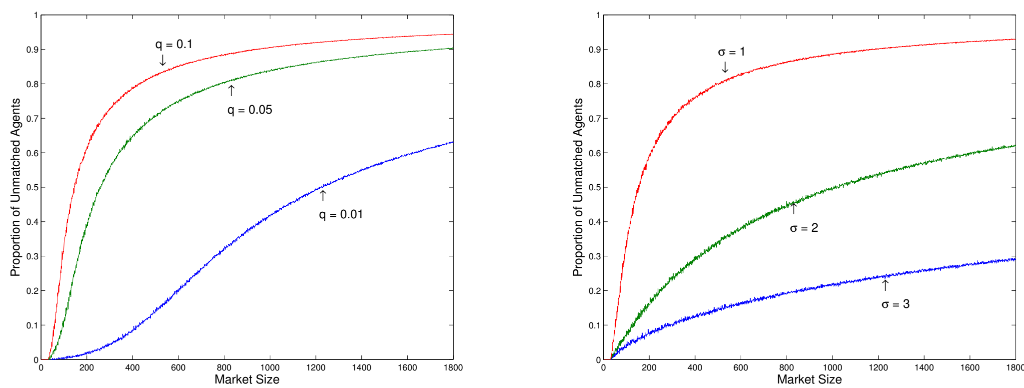
which is independent of the market size n. If $q = 0$, we have the uniform popularity distribution over firms, so all firms have an equal chance of being chosen before another. As q becomes close to 1, more popular firms have higher chances of being chosen before other firms, which generates a commonality of preferences among workers.

2. Normalized log-normal distribution

   Let $F(\,\cdot\,; \mu, \sigma)$ be the cumulative distribution function of a log-normal distribution. For each market size n, we define the normalized log-normal distribution as:

$$\text{PDF} : p_k = \frac{F(\,k\,; \mu, \sigma) - F(\,k-1\,; \mu, \sigma)}{F(\,n\,; \mu, \sigma)}, \quad (\mu, \sigma \in \mathbb{R},\ k = 1, 2, \ldots, n).$$

   Given $\mu = 1$, as σ increases, firms have similar probabilities to be chosen. This generates a weaker commonality of workers' preferences.



(a) Normalized geometric distribution

(b) Normalized log-normal distribution

Figure 2: Proportions of unmatched agents in stable matchings.

Figure 2 shows that the proportions of unmatched agents in stable matchings increase as markets increase. Each graph represents the proportion of unmatched agents when workers consider only thirty most preferred firms acceptable. The proportions are averaged over ten repetitions.

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