

# Asset Reallocation in Markets with Intermediaries Under Selling Pressure\*

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## Abstract

We study a search model of investors' asset trading, intermediated by financial institutions that are at risk of selling under pressure. The pressure can lead to the development of a secondary market, where intermediaries bail each other out. Counterintuitively, an increase in competing intermediaries can lead to an increase of each intermediary's value: the enhanced benefits of secondary trades can be so large as to dominate the reduction in value from narrower buy-sell spreads due to more intense competition. The market exhibits search externality; suppressing investors' direct trading can improve welfare. We provide a calibration to the corporate acquisition market.

**Keywords:** Asset trading, OTC markets, Financial intermediation, Search and matching, Secondary market.

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# 1 Introduction

## 1.1 Overview

Financial intermediaries, which purchase assets and hold until resell, require external capital. The outside funding routinely put the intermediaries at the risk of selling assets under pressure. An example is a closed-ended private equity (PE) buyout or real estate fund that acquire portfolio assets and exit by selling them to provide liquidity to fund investors, all occurring within an average life span of 10-12 years.<sup>1</sup> When they approach the end of their life span, they sell assets under pressure, often to other PE funds in the secondary market (Arcot, Fluck, Gaspar, and Hege (2015)).

In general, dealers in bond, derivatives and mortgage-backed securities markets trade under roll-over risks and margin constraints. Refinancing of external capital may come at a higher cost - either in terms of direct interest rate or opportunity cost. Moreover, dealers often purchase security with borrowed money by pledging the security as a collateral. Dealers are unable to borrow the entire price of the security and must fund the difference between the price and collateral value with their own capital. When the price of the security goes down, they must ensure that additional capital becomes available to satisfy the margin requirements or alternatively, sell the security. In derivatives trading, dealers face the possibility of a hedge position expiring at a loss, necessitating a cash payment when the expiring hedge is replaced with a new one. In all the situations above, these intermediaries suffer from the risk of selling assets under pressure.

We build a search-theoretic model of asset reallocation, where intermediaries (hereafter, *funds*) are at such risk of selling under pressure. Our model is similar in spirit to Duffie, Gârleanu, and Pedersen (2005) and Hugonnier, Lester, and Weill (2018). The search-and-bargaining features are suitable for capturing intermediaries' attempt to sell under pressure in a decentralized market. In particular, while trading portfolio assets, PE funds often take

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<sup>1</sup>A PE fund consists of General Partners (GPs) who have specialties specific industries and operating expertise such finance and marketing. The GPs raise capital from outside investors, called Limited Partners (LPs). After raising capital, a typical PE fund usually has around 10 to 12 years of life after its inception. The rationale for a finite life is in that funds' invest in private firms whose market value is unknown. Only after an asset is sold, GPs and LPs observe gains of the fund and can determine GPs' management compensation and LPs' share.

several months to close a transaction. We explore the impact of market characteristics, such as search frictions and the number of funds, on welfare, fund valuations, trading volumes, and transaction prices. The model allows us to quantify how the characteristics of the funds' ecosystem determine their performance. Funds compete for intermediation opportunities but also provide each other with greater exit opportunities (i.e., secondary transactions). As such, funds may derive benefits from having additional funds in the ecosystem, in spite of lower spreads due to competition. A calibration of our model to the US corporate acquisition market shows that PE buyout funds benefit substantially from secondary transactions and the average fund valuation *increases* in the number funds.

Our model comprises a continuum of investors and funds who hold either one or zero assets. The assets are reallocated over time through (i) (inter-investor) direct trading, (ii) fund-investor trading, and (iii) (inter-fund) secondary trading, with heterogeneous search frictions. Assets are reallocated from low-type investors to high-type investors, sometimes through intermediated transactions. High-type investors can generate higher flow payoffs. Each investor's type changes over time by an exogenous shock that arrives according to a Poisson process. On the other hand, funds buy assets from low-type investors and sell to high-type investors through fund-investor transactions. While holding assets, funds may receive liquidity shocks that arrive according to a Poisson process. Due to liquidity constraints, funds incur a holding cost from owning assets, which puts them under selling pressure. The funds then try to sell the assets to high-type investors or other funds through secondary transactions.

We study a steady-state equilibrium in which all three markets are active. We set up a dynamic system of population measures and agents' payoffs, and prove that a unique steady-state solution exists. We find that funds enhance market efficiency by alleviating search frictions and providing greater liquidity to the market. The first channel, whereby intermediaries facilitate asset transfers and improve efficiency, is well-known. The second channel – the liquidity channel – is unique to our model and is related explicitly to funds' demand for liquidity due to their exit requirements.

Our first set of results focus on the inter-fund secondary market. Secondary transactions provide greater exit opportunities to funds, enhance fund value, and improve overall welfare (Proposition 3). Secondary transactions offer a channel through which funds can

complement each other (Proposition 4); funds sourcing deals provide liquidity to funds at the exit phase, and conversely funds at the exit phase allow fund buyers to acquire assets more quickly. The complementarity effect could potentially dominate the competition effect; as a result, perhaps surprisingly, funds' value can increase in the number of funds (Proposition 5). This mechanism explains that intermediaries can continue to perform well in spite of the consistently increasing number of them. Secondary transactions are sometimes criticized as opportunistic behavior among fund managers passing sub-par assets to their counterparts while both parties collect management fees.<sup>2</sup> However, our model suggests that funds generate high returns because of (and not in spite of) secondary transactions.

We study equilibrium properties of markets with many participants, which is common in markets for corporate acquisitions, bonds, derivatives, and mortgage-backed securities. A large (real) market corresponds to a fast-search market in our model, due to a normalization.<sup>3</sup> We derive various results on welfare, search externalities, and trade volumes (Proposition 7 and Proposition 8). Direct trades between low and high type investors clearly improves the corresponding owner's flow payoffs. Nevertheless, its welfare consequence is subtle. Investors' direct trading deprive funds of potential buyers, and funds find it harder to turn over their inventory quickly and intermediate the asset market efficiently. Slowing down investors' direct trading allow the exit-phase funds to off-load assets, reset the life cycle, and purchase new assets, all quickly, especially when funds have low search friction. This dynamics drive our results of asymptotic inefficiency and search externalities of a fast-search market equilibrium.

We calibrate our model to the US corporate acquisition market intermediated by PE buy-out funds. Our view of a corporate acquisition is a sale of assets among corporate investors and PE funds.<sup>4</sup> The estimated matching rate between funds for secondary transactions is

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<sup>2</sup>See also an article in *The Economist* at <https://www.economist.com/node/15580148>.

<sup>3</sup>If a type  $(i, j)$  pair of investors with population  $N_i$  and  $N_j$  meet at a Poisson rate  $L$ , the number of direct trading (say, per year), normalized by population  $N$ , is approximately  $(LN_iN_j)/N = (LN)(N_i/N)(N_j/N) = (LN)n_in_j$ .

<sup>4</sup>For example, General Electric became a conglomerate through a spate of acquisitions but has recently announced it will divest its healthcare and oil-service divisions, as it is no longer perceived to be generating the most value with these assets. An alternative view of a corporate acquisition is *two-sided matching* between corporate buyers and sellers. This literature studies various reasons leading to mergers. See Jovanovic and Rousseau (2002), Rhodes-Kropf, Robinson, and Viswanathan (2005), Rhodes-Kropf and Robinson (2008), Eisfeldt and Rampini (2008), David (2017), Almeida, Campello, and Hackbarth (2011), and the references

higher than the fund-investor (i.e., fund-corporation) trading rate, which in turn is higher than rate for direct trading among investors. Our estimates suggest that the number of high-type corporate investors is small relatively to the number of tradable assets. As such, some assets are held by low-type corporate investors and generate only low payoff flows. The secondary transactions among funds is estimated to increase each fund value by 26%. As Proposition 5 predicts fund value indeed increases in the number of funds. The number of PE funds is sufficiently small so that the benefit from having more PE funds through secondary transactions dominates the cost of increased competition for intermediation opportunities. Lastly, the calibrated model predicts the share of SBOs over the total PE fund exits to increase in the number of PE funds, which is consistent with historical evidence.

## 1.2 Related Literature

We discuss only closely related papers on OTC markets with intermediation and refer others to Nosal and Rocheteau (2011) and references therein.<sup>5</sup>

Hugonnier, Lester, and Weill (2018) also study an inter-dealer market and shares some similarities with our direct and secondary trading markets. There are several important distinctions. First, dealers in their model have ownership-independent type changes over time.<sup>6</sup> Such modeling approach does not suitably capture funds' experiencing selling pressure contingent on holding assets. Our model is tailored for funds intermediating with external capital; the selling pressure for a fund in our model arises due to asset ownership. This feature yields new results on the benefits of secondary transactions Proposition 3, Proposition 4, and Proposition 5. Second, their inter-dealer market is singled out such that investors can trade only through dealers. Our model, in contrast, allows concurrent operation of unrestricted interactions among investors and funds. The unrestricted interaction is relevant for some applications such as the corporate acquisition market and the real estate market (Phillips

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therein.

<sup>5</sup>This paper is also related to an extensive literature on the inter-bank network; see a survey by Allen and Babus (2009). Our model differs from interbank models where the focus is on lending and borrowing.

<sup>6</sup>In fact, Hugonnier, Lester, and Weill (2018) assume that a dealer's type does not change over time. In an earlier version (Hugonnier, Lester, and Weill (2014)), as also in Uslu (2019), a dealer's type changes over time independently from her ownership. A new type is drawn from a distribution independently from her current type.

and Zhdanov (2017)). Third, their results focus on trading patterns including intermediation chains, whereas our focus is on fund valuations and welfare.

Funds in our model choose to intermediate between low and high type investors, due to their moderate flow payoffs. This self-selection of intermediaries has been studied in Neklyudov (2012), Uslu (2019), Nosal, Wong, and Wright (2016), Shen, Wei, and Yan (2018), and Farboodi, Jarosch, and Shimer (2017). It is often the case that mid-type investors, which are similar to our funds, choose to intermediate with comparative advantages in search skills. Atkeson, Eisfeldt, and Weill (2015) is another study of an OTC market with endogenous intermediation. For derivative swap contracts, investors with risky endowments may be unable to share the risk fully, because of a size limit on bilateral trades. Buy and sell prices do not reflect the aggregate risk, and the price dispersion incentivizes some banks to act as intermediaries.<sup>7</sup>

The welfare effect of secondary transactions has been studied in other contexts. Gofman (2014) shows that better-connected intermediaries in financial markets can shorten intermediation chains and improve welfare. Pagano and Volpin (2012) study the bank-loan market where banks lend to consumers (primary issuance) while other banks provide liquidity by investing in the securitized loans (secondary market liquidity). High securitization activities in the secondary market yield high loan issuance in the primary market, driven by greater transparency in the secondary market. Finally, Hochberg, Ljungqvist, and Lu (2007) study the benefits of PE funds' networks in sharing information about assets.

Our paper focuses on intermediaries' selling pressure. However, in some applications, fund managers initiate secondary transactions, either under buying pressure when the fund is near the end of their investment phase with excess capital, or under selling pressure when the fund is close to the end of the fund's life (Arcot, Fluck, Gaspar, and Hege (2015), Degeorge, Martin, and Phalippou (2016), and Wang (2012)). In either case, the secondary market offers a channel through which funds can provide liquidity to each other.

The remainder of the paper is organized as follows. Section 2 introduces the formal model; Section 3 provides equilibrium properties; Section 4 discusses our main analysis of

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<sup>7</sup>An alternative question on endogenous intermediation is how much dealers would choose to intermediate. Yang and Zeng (2018) study a model in which heterogeneous dealers play a coordination game such that each dealer is more likely to intermediate (provide liquidity to customers by purchasing, holding, and selling more assets) if other dealers do so.

secondary transactions; Section 5 provides an analysis of a fast-search market; Section 6 presents a calibration of our model to the US corporate acquisition market; and Section 7 concludes. All proofs appear in the Appendix.

## 2 Model

Time runs continuously in  $t \in [0, \infty)$ . Over time, two kinds of agents, **investors** and **funds** trade **assets**. Initially, a fraction of investors and funds are endowed with assets. The measures of investors  $n_v$ , funds  $n_f$ , and tradable assets  $n_a$  remain constant. All agents are risk neutral and infinitely lived, with time preferences determined by a constant discount rate  $r$ . Each agent holds one or zero assets. Hence,  $n_a < n_v + n_f$ . We normalize the total measure of investors as  $n_v = 1$ .

An investor that holds an asset generates either a high payoff flow  $u_h$  or a low payoff flow  $u_l$  ( $< u_h$ ). An investor does not receive any payoff flow when not holding an asset. An investor's ability to create payoff flow switches from low to high with Poisson intensity  $\rho_u$ , or from high to low with intensity  $\rho_d$ . The arrival rate of this Poisson shock for each type of investor is independent of other investors. The set of investor types is  $\mathcal{T}_v \equiv \{ho, lo, hn, ln\}$ , where the letters  $h$  and  $l$  represent each investor's ability to generate payoffs, and the letters  $o$  and  $n$  denote whether an investor owns an asset or not.

A fund's life cycle consists of an investment phase, a harvesting phase, and an exit phase. A fund in the investment phase does not own assets and searches for an investor or a fund selling assets. After purchasing an asset, the fund enters the harvesting phase and creates payoff flow  $u_f$ . A fund in the harvesting phase sells its assets and starts a new life cycle (i.e., goes back to the investment phase),<sup>8</sup> or it receives a liquidity shock with intensity  $\rho_e$  and enters the exit phase. A fund in the exit phase incurs a holding cost and generates a lower payoff flow  $u_e$  ( $< u_f$ ). After selling the asset, the fund automatically starts a new life at the investment phase. We denote a fund in the investing phase by type  $fn$  (a fund non-owner), in the harvesting phase by type  $fo$  (a fund owner), and in the exiting phase by type  $fe$  (a fund that is exiting). The set of fund types is  $\mathcal{T}_f \equiv \{fn, fo, fe\}$ . We assume that

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<sup>8</sup>General partners of PE funds often start a new fund around the liquidation of an existing fund.

funds generate moderate payoff flows,  $u_l < u_e < u_f < u_h$ ,<sup>9</sup> such that funds play the role of intermediaries by purchasing assets from low-type investors and selling them to high-type investors.

Let  $\mathcal{T} \equiv \mathcal{T}_v \cup \mathcal{T}_f$  denote the set of types with typical elements  $i, j$ , etc. The measure of type  $i \in \mathcal{T}$  at time  $t \in [0, \infty)$  is denoted by  $\mu_i(t)$ . Then,

$$\begin{aligned}\mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) &= n_v (= 1), \\ \mu_{fn}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= n_f, \\ \mu_{ho}(t) + \mu_{lo}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= n_a.\end{aligned}\tag{1}$$

Agents meet each other over time and negotiate a trade. Two investors meet each other with intensity  $\lambda_d$  for **(inter-investor) direct trading**. An investor and a fund meet each other with intensity  $\lambda_f$  for a **fund-investor trading**. A fund in the exit phase ( $fe$ ) and a fund in the investment phase ( $fn$ ) meet each other with intensity  $\lambda_s$  for **(inter-fund) secondary trading**. The meeting rate between any pair of groups is linear in each group's population. That is, for any pair of investor types  $i, j \in \mathcal{T}_v$  with measures  $\mu_i$  and  $\mu_j$ , the total meeting rate is  $\lambda_d \mu_i \mu_j$ . Similarly, the total meeting rate between an investor type  $i \in \mathcal{T}_v$  and a fund type  $j \in \mathcal{T}_f$  is  $\lambda_f \mu_i \mu_j$ , and the total meeting rate in the secondary market is  $\lambda_s \mu_{fe} \mu_{fn}$ . When two agents meet each other, they trade an asset instantaneously if and only if the gain from trade (which we explain later) is positive. The assumption of immediate trading upon meeting follows the literature on bargaining without asymmetric information.

We will find an equilibrium in which all tradings denoted by  $\mathcal{M} \equiv \{lo-hn, lo-fn, fo-hn, fe-hn, fe-fn\}$  are active. That is, a  $lo$ -type investor sells an asset to a  $hn$ -type investor ( $lo-hn$  trade). Similarly, either a  $lo$ -type investor sells an asset to a  $fn$ -type fund ( $lo-fn$  trade), or a fund of type either  $fo$  or  $fe$  sells an asset to a  $hn$ -type investor (either  $fo-hn$  or  $fe-hn$  trade). In the secondary market, a  $fe$ -type fund sells to a  $fn$ -type fund ( $fe-fn$  trade). After all trades, the types change from 'o' to 'n' and vice versa. Overall, assets are transferred from low-type investors toward high-type investors, with a possible chain of

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<sup>9</sup>Funds in the corporate acquisition market add operational value through better corporate governance (reducing agency costs). However, funds are unable to improve operational value as much as corporate buyers, who can generate greater payoff through synergies. In financial markets, intermediating funds typically have lower cost of capital as compared to sellers but do not have hedging or portfolio diversification benefits.



trades among funds through secondary trades.

Figure 1 summarizes the model. Agent types are listed on the left column for owners and the right column for non-owners. An owner changes her type to one on the right column upon selling her asset; a non-owner changes her type to one on the left column upon purchasing an asset (a fund's type become  $fo$  after an asset purchase). The solid arrows represent how assets are transferred from sellers to buyers. The vertical dashed arrows represent the exogenous type changes: high vs. low for investors, or a liquidity shock to  $fo$ -type funds.

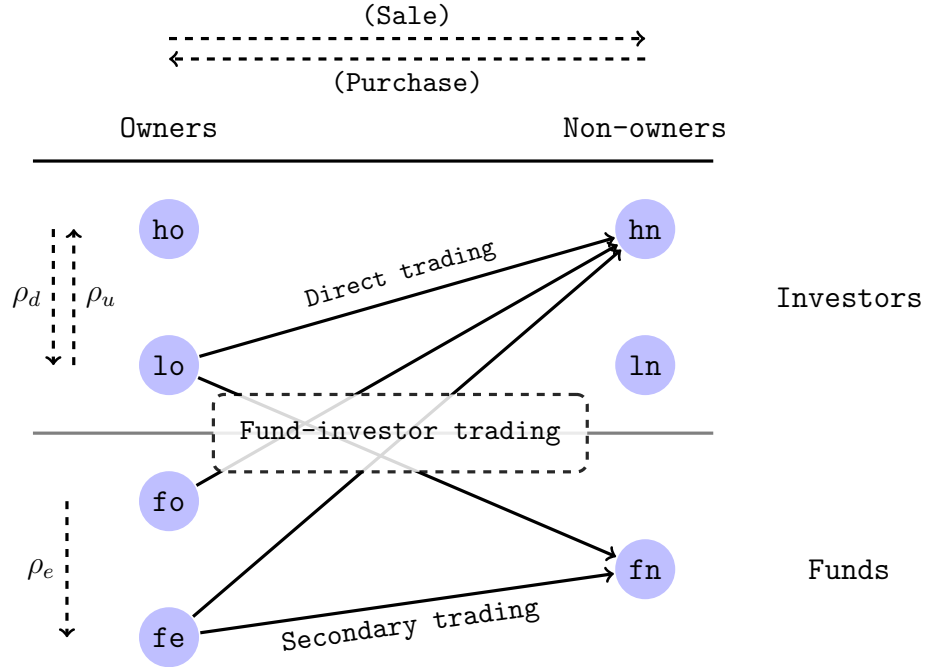


Figure 1: An asset market with fund intermediation.

We denote **an asset market with fund intermediation** by a collection of exogenous variables  $\theta \equiv (n, r, u, \rho, \lambda)$ , where  $n \equiv (n_v, n_f, n_a)$ ,  $u \equiv (u_l, u_h, u_f, u_e)$ , and  $\lambda \equiv (\lambda_d, \lambda_f, \lambda_s)$ . All exogenously given parameters are strictly positive.<sup>10</sup> When necessary, we also include an initial population distribution  $\mu(0)$  in the model.

<sup>10</sup>We say that a real number  $x$  is *positive* if  $x \geq 0$ , and *strictly positive* if  $x > 0$ .

### 3 Equilibrium

We are interested in a steady-state equilibrium with active trades. That is, in addition to investors trading assets amongst themselves, funds actively intermediate by buying and selling assets. In this section, we provide a condition on the market under which such a steady-state equilibrium uniquely exists.

#### 3.1 Existence and Uniqueness

We first derive steady-state population measures. To ease our exposition, we denote the steady-state populations of high- and low-type investors by  $n_h$  and  $n_l$ . Since we normalized the total measure of investors as  $n_v = 1$ , from the rates of exogenous type changes,

$$n_h = \frac{\rho_u}{\rho_u + \rho_d} \quad \text{and} \quad n_l = \frac{\rho_d}{\rho_u + \rho_d}.^{11}$$

A  $hn$ -type investor switches its type to  $ho$  upon purchasing an asset from either a  $lo$ -type investor or a  $fo$ - or  $fe$ -type fund. As such,  $hn$ -type investors become  $ho$ -type at the rate of  $(\lambda_v\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})\mu_{hn}$ . On the other hand, as a result of exogenous type changes,  $hn$ -type investors switch their types to  $ln$  at the rate of  $\rho_d\mu_{hn}$ ; similarly,  $ln$ -type investors switch their types to  $hn$  at the rate of  $\rho_u\mu_{ln}$ . Thus,

$$\dot{\mu}_{hn}(t) = -(\lambda_d\mu_{lo}(t) + \lambda_f\mu_{fo}(t) + \lambda_f\mu_{fe}(t))\mu_{hn}(t) - \rho_d\mu_{hn}(t) + \rho_u\mu_{ln}(t). \quad (\mu\text{-hn})$$

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<sup>11</sup>Later, a reader can confirm the values of  $n_h$  and  $n_l$  by adding population equations ( $\mu$ -ho) and ( $\mu$ -hn), or ( $\mu$ -lo) and ( $\mu$ -ln), and applying  $n_h + n_l = n_v = 1$ .

The population measures for other types change over time by similar processes:

$$\begin{aligned}
\dot{\mu}_{ho}(t) &= (\lambda_d \mu_{lo}(t) + \lambda_f \mu_{fo}(t) + \lambda_f \mu_{fe}(t)) \mu_{hn}(t) - \rho_d \mu_{ho}(t) + \rho_u \mu_{lo}(t), & (\mu\text{-ho}) \\
\dot{\mu}_{ln}(t) &= (\lambda_d \mu_{hn}(t) + \lambda_f \mu_{fn}(t)) \mu_{lo}(t) - \rho_u \mu_{ln}(t) + \rho_d \mu_{hn}(t), & (\mu\text{-ln}) \\
\dot{\mu}_{lo}(t) &= -(\lambda_d \mu_{hn}(t) + \lambda_f \mu_{fn}(t)) \mu_{lo}(t) - \rho_u \mu_{lo}(t) + \rho_d \mu_{ho}(t), & (\mu\text{-lo}) \\
\dot{\mu}_{fn}(t) &= \lambda_f (\mu_{hn}(t) \mu_{fo}(t) + \mu_{hn}(t) \mu_{fe}(t) - \mu_{lo}(t) \mu_{fn}(t)), & (\mu\text{-fn}) \\
\dot{\mu}_{fo}(t) &= (\lambda_f \mu_{lo}(t) + \lambda_s \mu_{fe}(t)) \mu_{fn}(t) - \lambda_f \mu_{hn}(t) \mu_{fo}(t) - \rho_e \mu_{fo}(t), & (\mu\text{-fo}) \\
\dot{\mu}_{fe}(t) &= -(\lambda_f \mu_{hn}(t) + \lambda_s \mu_{fn}(t)) \mu_{fe}(t) + \rho_e \mu_{fo}(t). & (\mu\text{-fe})
\end{aligned}$$

Let  $P(\theta)$  denote the above system of population equations  $(\mu\text{-hn})$ - $(\mu\text{-fe})$ . A real-vector  $\mu \equiv (\mu_i)_{i \in \mathcal{T}}$  with each  $\mu_i \geq 0$  is a steady-state solution of  $P(\theta)$  if the right-hand sides of the equations, with  $\mu_i(t)$  replaced by  $\mu_i$  for each  $i \in \mathcal{T}$ , are equal to zero.

We observe that a steady-state population measure of every type is non-vanishing. As such, all kinds of meetings, hence all tradings, occur with strictly positive rates (subject to positive gains).

**Lemma 1.** *If  $\mu$  is a steady-state solution of  $P(\theta)$ , then  $\mu_i > 0$  for all  $i \in \mathcal{T}$ .*

The intuition of the lemma is simple. Strictly positive rates  $\lambda = (\lambda_d, \lambda_f, \lambda_s)$  and  $\rho = (\rho_u, \rho_d, \rho_e)$  allow the mass of investors and funds to *flow* across all types. Given  $0 < n_a < n_v + n_f$ , some fraction of agents are owners and others are non-owners. Investors who own assets may have their types changing between high and low exogenously and non-owners have similar probabilistic type changes. As transaction rates  $\lambda = (\lambda_d, \lambda_f, \lambda_s)$  are all strictly positive, some investors can buy or offload assets after their types change. However, not all of the investors can do so within any fixed time period. A similar idea holds for funds.

A steady-state solution of  $P(\theta)$  uniquely exists.<sup>12</sup> The solution is asymptotically stable in the sense that, if the population measure  $\mu(t)$  is perturbed locally from the steady-state solution  $\mu$ , then it converges back to  $\mu$ .

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<sup>12</sup>The uniqueness of a steady-state equilibrium is not universal in the literature on OTC, or more broadly search-and-bargain, markets with intermediation. A unique steady-state equilibrium appears in Duffie, Gârleanu, and Pedersen (2005), but multiple equilibria appear more commonly with financial market applications: e.g., Vayanos and Weill (2008) and Trejos and Wright (2016).

**Proposition 1.** (*Steady-state Population Measures*) Take any market  $\theta$ .

1. (*Existence and Uniqueness*) There exists a unique steady-state solution  $\mu$  of  $P(\theta)$ .
2. (*Asymptotic Stability*) Let  $\mu(t)$  be a dynamic solution of  $P(\theta)$  with initial condition  $\mu(0)$ . For any  $\epsilon > 0$ , there exists  $\delta > 0$  such that, if  $\|\mu(0) - \mu\| < \delta$ , then  $\|\mu(t) - \mu\| \leq \epsilon$  for all  $t$ , and  $\mu(t) \rightarrow \mu$  as  $t \rightarrow \infty$ .

The proof uses the Poincare-Hopf index theorem (Simsek, Ozdaglar, and Acemoglu, 2007), which is new to the literature on intermediated OTC markets. The index theorem generalizes the Intermediate Value Theorem.

To get an intuition, we set  $(\mu_i)_{i \in \mathcal{T}_v} \approx 0$ , while satisfying the population equations  $P(\theta)$  but violating  $n_v = 1$ . A small increase of  $\mu_{lo}$  (or  $\mu_{hn}$ ) increases the supply (resp., demand) of assets for investors' direct trading  $\lambda_d \mu_{lo} \mu_{hn}$ , and in turn the number of investors that are rightly holding or not-holding assets ( $\mu_{ho}$  and  $\mu_{ln}$ ). The increased populations of  $\mu_{ho}$  and  $\mu_{ln}$  lead to more inflow  $\rho_d \mu_{ho}$  of agents back to the aggregate supply and the inflow  $\rho_u \mu_{ln}$  to the aggregate demand for direct trading. That is, all four investor-type populations increase. We take into account how investor-type populations are related to fund-type populations (which is elaborated in Subsection A.2) and show that a unique supply  $\mu_{lo}$  and demand  $\mu_{hn}$  satisfy the constraint  $\mu_{lo} + \mu_{hn} + \mu_{ln} + \mu_{ho} = n_v = 1$ , by the index theorem. The second part of the proposition on stability is due to a classical result in dynamical systems (Hirsch and Smale, 1973). If all eigenvalues of the linearized system at the steady-state solution have negative real parts, then the solution is asymptotically stable.

We define a steady-state equilibrium via a recursive equation of certain values. The sources of value to all agents in our model are two-fold: (i) flow payoffs for the duration of assets held and (ii) gains from trade. Let  $v_{hn}$  denote the expected **value** of time-discounted future payoffs for a type- $hn$  investor. The value is defined implicitly by

$$rv_{hn} = \lambda_d \mu_{lo} g_{lo-hn} + \lambda_f \mu_{fo} g_{fo-hn} + \lambda_f \mu_{fe} g_{fe-hn} - \rho_d (v_{hn} - v_{ln}), \quad (\text{v-hn})$$

where each  $g_{lo-hn}$ ,  $g_{fo-hn}$ , and  $g_{fe-hn}$  denotes the investor's **gain from trade** (in fact, an equal share of the gain, which we define later). The meeting rate for a direct trading, taking

into account the population of sellers, is  $\lambda_d \mu_{lo}$ , and the gain from a trade is  $g_{lo-hn}$ . Two other terms are defined similarly for the cases of trading with either a  $f_o$ - or  $f_e$ -type fund. The investor changes its type from high to low with rate  $\rho_d$ , in which case, it loses value equivalent to  $v_{hn} - v_{ln}$ .

The values for other types are defined implicitly as follows:

$$rv_{ho} = u_h - \rho_d (v_{ho} - v_{lo}), \quad (\text{v-ho})$$

$$rv_{ln} = \rho_u (v_{hn} - v_{ln}), \quad (\text{v-ln})$$

$$rv_{lo} = u_l + \lambda_d \mu_{hn} g_{lo-hn} + \lambda_f \mu_{fn} g_{lo-fn} + \rho_u (v_{ho} - v_{lo}), \quad (\text{v-lo})$$

$$rv_{fn} = \lambda_f \mu_{lo} g_{lo-fn} + \lambda_s \mu_{fe} g_{fe-fn}, \quad (\text{v-fn})$$

$$rv_{fo} = u_f + \lambda_f \mu_{hn} g_{fo-hn} - \rho_e (v_{fo} - v_{fe}), \quad (\text{v-fo})$$

$$rv_{fe} = u_e + \lambda_f \mu_{hn} g_{fe-hn} + \lambda_s \mu_{fn} g_{fe-fn}. \quad (\text{v-fe})$$

A term representing flow payoffs is included for each owner type (the payoff flow is zero for non-owners).

We assume that buyers and sellers in each trading have equal bargaining power.<sup>13</sup> That is, the transaction prices (that we will characterize in the next subsection) will be set to ensure an equal division of gain from trade between a buyer and a seller. Each trading partner's gain is:

$$g_{lo-hn} \equiv (1/2)(v_{ho} + v_{ln} - v_{lo} - v_{hn}),$$

$$g_{lo-fn} \equiv (1/2)(v_{fo} + v_{ln} - v_{lo} - v_{fn}),$$

$$g_{fo-hn} \equiv (1/2)(v_{ho} + v_{fn} - v_{fo} - v_{hn}),$$

$$g_{fe-hn} \equiv (1/2)(v_{ho} + v_{fn} - v_{fe} - v_{hn}),$$

$$g_{fe-fn} \equiv (1/2)(v_{fo} + v_{fn} - v_{fe} - v_{fn}) = (1/2)(v_{fo} - v_{fe}).$$

Let  $V(\theta)$  denote the above system of value equations (v-hn)-(v-fe), with  $\mu$  being replaced

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<sup>13</sup>Our qualitative results do not depend on this assumption. The assumption is reasonable for the corporate-acquisition market, for which Ahern (2012) observes that the dollar gains of trades are often equally split between buyers and sellers.

by the unique steady-state solution of  $P(\theta)$ . The following proposition ensures that the values  $v \equiv (v_i)_{i \in \mathcal{T}}$  are well-defined:

**Proposition 2.** *There exists a unique solution of  $V(\theta)$ .*

We have characterized the unique steady-state population measures  $\mu$  and the values  $v$ , assuming that all tradings are active. If the unique steady-state solution  $(\mu, v)$  results in positive trade gains, we call it a **steady-state equilibrium**. We summarize the condition for positive trade gains with (21) and (22) in Subsection A.5.

Gains from certain trades are easily positive. For example,  $g_{fe-fn} \geq 0$  is immediate from  $u_f > u_e$ : secondary transactions bail out funds under liquidity constraints. The gains from a direct trading by investors and a fund-investor trading are related as  $g_{lo-hn} = g_{lo-fn} + g_{fo-hn}$ : a direct trade and indirect trade through fund intermediation result in the same total gains. In a similar vein, gains in a fund-investor trading and a secondary trading are related as  $g_{fe-hn} = g_{fo-hn} + g_{fe-fn}$ . Hence, it remains to make sure that two gains  $g_{lo-fn}$  and  $g_{fo-hn}$  are positive, which corresponds to (21) and (22), respectively.

The gains always positive if the differences  $u_f - u_l$  and  $u_h - u_f$  are sufficiently large. Also, a simple expression for (22) exists if  $\lambda_d < \lambda_f < \lambda_s$  and  $\rho = (\rho_u, \rho_d, \rho_e)$  is close to zero.<sup>14</sup> Suppose  $n_a > n_h$  (else, given rare exogenous type changes, almost all assets will be held by high-type investors and there will be little to no trade). Then,  $\mu_{hn} \approx 0$ , as there are hardly any type changes from  $ln$  to  $hn$ , and ensuring  $g_{lo-fn} \geq 0$  becomes equivalent to:

$$u_f + 0 \geq u_l + \lambda_s \mu_{fe} g_{fe-fn}.$$

Each side of the inequality considers a steady-state in which selling investors ( $lo$ ) and fund buyers ( $fn$ ) either trade or do not trade. On the left-hand side (i.e., trading in steady state), the total payoff flows after trading is likely to be  $u_f$  for a long time; a liquidity shock rarely arrives, and buying investors ( $hn$ ) for funds' assets are scarce. On the right-hand side (i.e., no trading in steady state), an  $lo$ -type investor receives payoff flow  $u_l$  for an extended period

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<sup>14</sup>(22) implies (21), when  $\lambda_d < \lambda_f < \lambda_s$  and  $\rho \rightarrow 0$ . For a proof, we take  $\rho \rightarrow 0$  in (20) (where  $g_1 \equiv g_{fo-hn}$  and  $g_2 \equiv g_{lo-fn}$ ) and show that  $g_{lo-fn} \geq 0$  implies  $g_{fo-hn} \geq 0$ . The inequalities over meeting rates are reasonable assumptions for the corporate-acquisition market; see our calibration in Section 6.

of time, as she does not trade with a fund buyer ( $fn$ ), and buying investors ( $hn$ ) are scarce. Fund buyers ( $fn$ ) purchase assets mostly from exiting funds ( $fe$ ) with the gain  $g_{fe-fn}$ .

The above condition for positive gains always when tradable assets are scarce ( $n_h + n_f > n_a$ ). Then,  $\mu_{fe} \approx 0$  because a non-negligible number of  $fn$ -type funds try to purchase assets, partly from  $fe$ -type funds, whereas liquidity shocks hardly arrive to  $fo$ -type funds. If the market has a surplus of tradable assets ( $n_a > n_h + n_f$ ), then the condition for positive gains holds if funds make a significant operational improvement (i.e., a large  $u_f - u_l$ ). Notice that  $\mu_{fe} \leq n_f$  and  $g_{fe-fn} \approx \frac{u_f - u_e}{2r}$ , as the total gains from secondary trades  $\frac{u_f - u_e}{r}$  is equally shared between buyers and sellers. Therefore, a simple expression for (22) is  $u_f - u_l \geq \lambda_s n_f \left( \frac{u_f - u_e}{2r} \right)$ .

### 3.2 Equilibrium Properties

The transaction prices are determined in a manner so that buyers and sellers equally share the gains from trades (the equal bargaining power assumption). For a direct trading,

$$p_{lo-hn} \equiv (1/2)(v_{ho} + v_{lo} - v_{hn} - v_{ln}),$$

so that a buyer's gain  $(v_{ho} - p_{lo-hn}) - v_{hn}$  and a seller's gain  $(v_{ln} + p_{lo-hn}) - v_{lo}$  are the same. The prices for other trades are similarly determined.<sup>15</sup> We establish below various relationships among transaction prices and spreads:

**Lemma 2.** (*Equilibrium Prices*)

1.  $p_{fo-hn} \geq p_{fe-hn} \geq p_{fe-fn}$ : funds sell at a lower price during the exit phase than in the harvesting phase, and at an even lower price in secondary trading.
2.  $p_{fo-hn} \geq p_{lo-hn} \geq p_{lo-fn}$ : funds buy assets at a lower price and sell at a higher price than investors.

A fund that manages to sell assets before receiving liquidity shocks can generate positive profits, from payoff flows  $u_f$  and the positive spread  $p_{fo-hn} - p_{lo-fn}$ . If a fund suffers a

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<sup>15</sup>To be more precise,  $p_{lo-fn} = (1/2)(v_{fo} + v_{lo} - v_{ln} - v_{fn})$ ,  $p_{fo-hn} = (1/2)(v_{ho} + v_{fo} - v_{fn} - v_{hn})$ ,  $p_{fe-hn} = (1/2)(v_{ho} + v_{fe} - v_{fn} - v_{hn})$ , and  $p_{fe-fn} = (1/2)(v_{fo} + v_{fe} - 2v_{fn})$ .

liquidity shock before finding a buyer, the fund may incur losses ex-post as the spread at the exit phase  $p_{fe-hn} - p_{lo-fn}$  can be negative.

We define welfare as  $W \equiv \sum_{i \in \mathcal{T}} \mu_i v_i$ . The investors' welfare and the funds' welfare are defined similarly as  $W_v \equiv \sum_{i \in \mathcal{T}_v} \mu_i v_i$  and  $W_f \equiv \sum_{i \in \mathcal{T}_f} \mu_i v_i$ . The welfare are naturally related to the investors' and funds' payoff flows and gains from trades as follows:

**Lemma 3.** (*Equilibrium Welfare*) For any market  $\theta$  with a steady-state equilibrium  $(\mu, v)$ ,

$$rW = \mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l, \quad (2)$$

$$rW_v = \mu_{ho}u_h + \mu_{lo}u_l + \underbrace{\lambda_f \mu_{lo} \mu_{fn} p_{lo-fn}}_{\text{sales to funds}} - \underbrace{\lambda_f \mu_{hn} (\mu_{fo} p_{fo-hn} + \mu_{fe} p_{fe-hn})}_{\text{purchases from funds}}, \quad \text{and} \quad (3)$$

$$rW_f = \mu_{fo}u_f + \mu_{fe}u_e + \underbrace{\lambda_f \mu_{fo} \mu_{hn} p_{fo-hn} + \lambda_f \mu_{fe} \mu_{hn} p_{fe-hn}}_{\text{sales to investors}} - \underbrace{\lambda_f \mu_{fn} \mu_{lo} p_{lo-fn}}_{\text{purchases from investors}}.$$

The first two terms for the investors' welfare  $W_v$  represent payoff flows to *ho*- and *lo*-type investors. The next term represents the inflow from selling assets to funds. Only *lo*-type investors sell assets to funds with the total rate  $\lambda_f \mu_{lo} \mu_{fn}$  and at price  $p_{lo-fn}$ . The last term represents the *hn*-type investors' payments to funds:  $p_{fo-hn}$  to *fo*-type funds with the aggregate rate of  $\lambda_f \mu_{hn} \mu_{fo}$ , or  $p_{fe-hn}$  to *fe*-type funds with the aggregate rate of  $\lambda_f \mu_{hn} \mu_{fe}$ . A similar interpretation explains the funds' welfare  $W_f$ .

Finally, we obtain an expression for the average time to sell for investors and funds.

**Lemma 4.** (*Time to Sell*) Let  $\tau_{sv}$  and  $\tau_{sf}$  denote the time to sell for investors and funds. Then,

$$E[\tau_{sv}] = \frac{1}{\lambda_d \mu_{hn} + \lambda_f \mu_{fn}}, \quad (4)$$

$$E[\tau_{sf}] = \frac{1}{\lambda_f \mu_{hn} + \rho_e} + \frac{\rho_e}{\lambda_f \mu_{hn} + \rho_e} \left( \frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}} \right). \quad (5)$$

Each seller-buyer meeting arrives according to a Poisson process, so the time until the first meeting by a selling investor follows an exponential distribution with parameter  $\lambda_d \mu_{hn} + \lambda_f \mu_{fn}$ . A similar, but more involved, calculation (see Supplemental Appendix) gives the expected time to sell for funds.



## 4 Secondary Market

In this section, we focus on the secondary transactions which can be a deal source and an exit channel for funds.

**Secondary trading provides liquidity and improves welfare.** Secondary trades meet funds' demand for liquidity, so a more liquid secondary market can attenuate the effects of fund liquidity shocks. A more liquid secondary market improves overall welfare as each secondary trade bails a fund out of liquidity constraints and offers fund buyers more transaction opportunities, at no cost to any other types of agents.

**Proposition 3.** *1. While liquidity shocks reduce funds' values ( $v_{fe} < v_{fo}$ ), their influence is mitigated by more liquid secondary market ( $\frac{\partial(v_{fo}-v_{fe})}{\partial\lambda_s} < 0$ ) and vanishes when the secondary market becomes completely liquid ( $\lim_{\lambda_s \rightarrow \infty}(v_{fo} - v_{fe}) = 0$ ).*

*2. The welfare increases in the secondary market liquidity ( $\frac{\partial W}{\partial\lambda_s} = \frac{\partial\mu_{fo}}{\partial\lambda_s}(\frac{u_f - u_e}{r}) > 0$ ).*

Intuitively, a faster secondary market allows  $fe$ -type funds to exit and re-enter as  $fn$ -type quickly, resulting in more funds in the investment phase ( $fn$ ) moving to the harvesting phase ( $fo$ ). The welfare increases by  $\frac{u_f - u_e}{r}$  for each unit measure of population shifted from  $\mu_{fe}$  to  $\mu_{fo}$ .

While each secondary trading benefits funds, it depletes potential fund-investor trades and may reduce investors' welfare. For instance, the value of  $fo$ -type funds is higher than the value of  $fe$ -type funds ( $v_{fo} > v_{fe}$ ), so investors pay a higher price to  $fo$ -type funds than to  $fe$ -type funds:  $p_{fo-hn} > p_{fe-hn}$ . An increase in secondary trading shifts some measure of  $\mu_{fe}$  to  $\mu_{fo}$ , without affecting other types' populations. Then, investors may pay more on purchases from funds (Equation 3) and suffers from lower welfare.<sup>16</sup>

**Through secondary trades, funds complement each other.** Although funds compete for intermediation opportunities, self-interested funds help each other out through secondary trades. When there is a greater number of funds in the market, exiting funds under pressure can sell more quickly. At the same time, funds looking to buy assets benefit from additional

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<sup>16</sup>A formal proof would be complicated as transaction prices also change.

trade opportunities from exiting funds. The mutual benefits of secondary trades among funds can be so large as to dominate the reduction in value from narrower buy-sell spreads due to the increased competition.

**Proposition 4.** *There exists  $\bar{n}_f$  such that if  $n_f < \bar{n}_f$ , then there is a complementarity between the secondary market liquidity and the number of funds ( $\frac{\partial^2 v_{fn}}{\partial \lambda_s \partial n_f} > 0$ ).*

The proof of Proposition 4 in Subsection B.2 obtains the closed-form expression of the complementarity ( $\frac{\partial^2 v_{fn}}{\partial \lambda_s \partial n_f}$ ) at  $n_f \approx 0$ . The complementarity is strong if there is an oversupply of assets relative to the number of buying investors; for otherwise, exiting funds could easily sell assets through the secondary market, and the benefits of secondary trades would be insignificant. Moreover, the complementarity is stronger if the liquidity shocks are more severe either in intensity ( $u_f - u_e$ ) or frequency ( $\rho_e$ ).

**Proposition 5.** *There exists  $\bar{n}_f$  and a function  $\bar{\lambda}_s(n_f)$  such that, if there are not too many funds ( $n_f < \bar{n}_f$ ) and the secondary market is liquid enough ( $\lambda_s > \bar{\lambda}_s(n_f)$ ), then funds' value increases in their number ( $\frac{\partial v_{fn}}{\partial n_f} > 0$ ).*

When the number of funds increases, competition amongst them in chasing intermediation opportunities becomes stiffer, and the value reduction due to competition may ultimately cancel out the benefit from complementarities. Figure 2 shows how the value of  $fn$ -type fund (i.e., a direct competitor of a new entrant) responds to  $n_f$  and  $\lambda_s$ , while other parameters are fixed at the values calibrated in Section 6 for the corporate acquisition market. The vertical line at  $n_f = n_f^*$  and the curve for  $\lambda_s = \lambda_s^*$  correspond to the observed number of funds and the calibrated search rate for secondary trades.<sup>17</sup> The figure confirms Proposition 5. At the calibrated values of parameters, the benefit from complementarity among funds due to secondary trades supersedes competitive pressures. However, for large  $n_f$ , lower spreads due to increased competition for intermediation opportunities turn out to be the dominant force impacting fund valuation.

**The share of fund exits by secondary trading** Our calibration analysis in Section 6 shows that secondary trading plays an increasingly important role in the growth of fund

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<sup>17</sup>The calibrated search rates  $\lambda = (\lambda_d, \lambda_f, \lambda_s)$  tend to be very large due to our normalization of  $n_v = 1$  and motivates us to study a fast-search market. See Section 5 for detailed discussions.

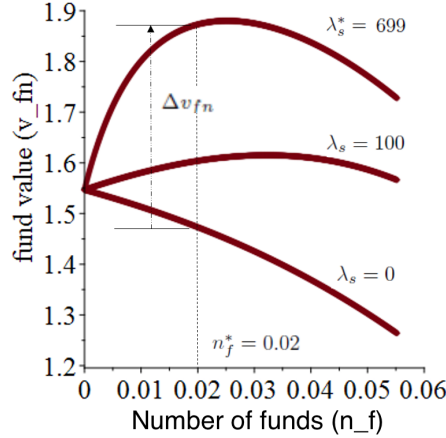


Figure 2: The y-axis shows  $v_{fn}$  for various values of  $n_f$  at calibrated values of all other market parameters (see Section 6). The topmost curve is for the calibrated value of  $\lambda_s$  while the bottom two curves are based on the counter-factual parameter values. The observed number of funds is  $n_f^* = 0.02$  (normalized by the number of investors). The increase in fund valuation  $\Delta v_{fn}$  represents the contribution by secondary trades, i.e., an increase of  $\lambda_s$  from 0 to the calibrated value  $\lambda_s^* = 699$ .

intermediations in asset trading. As we increase the number of funds, holding other parameters at calibrated values, we find that an increasing fraction of funds exit through secondary trading rather than sales of assets to investors (Figure 3). While it is obvious that the number of secondary trades increase with the growth in funds, our model explains a tandem increase in the *share* of exits through secondary trades. This prediction of the model is consistent with a conspicuous increase of fund intermediations in the corporate acquisition market (see Section 6 for details).

## 5 Large Market Analysis: Welfare and Trading Properties

We are interested in quantitative equilibrium properties, such as welfare and trading patterns, given parameter values that represent real markets. It is common that markets for corporate acquisitions, bonds, derivatives, and mortgage-backed securities have many buyers, sellers,

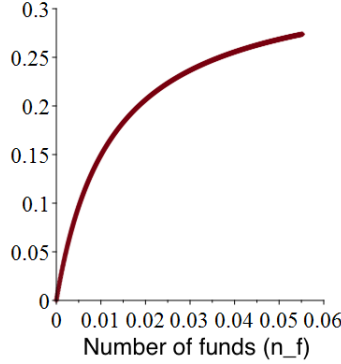


Figure 3: The share of fund exits by secondary trading  $\left(\frac{\lambda_s \mu_{fe} \mu_{fn}}{\lambda_f \mu_{fe} \mu_{hn} + \lambda_s \mu_{fe} \mu_{fn}}\right)$  for various values of  $n_f$  at calibrated values of all other market parameters (see Section 6).

and intermediaries. As such, we consider a set of parameter values that represents a large number of agents. We find closed-form expressions for the steady-state equilibrium and show inefficiencies and search externalities.

A market with many participants corresponds to *fast-search* – large search rates  $\lambda = (\lambda_d, \lambda_f, \lambda_s)$  – in our model. The reason is that we normalized the total number of investors as  $n_v = 1$  and proportionally re-scaled the number of funds and trade volumes. Let  $N_v$  be the total number of investors before normalization, with  $N_i$  for  $i \in \mathcal{T}_v$  being the number of type- $i$  investors. If each pair of investors meet at a Poisson rate  $l_d$ , the total number of direct trading (say, per year), with normalization, would be  $(l_d N_{lo} N_{hn})/N_v = (l_d N_v)(N_{lo}/N_v)(N_{hn}/N_v) = (l_d N_v) \mu_{lo} \mu_{hn}$ .

We set up a formal fast-search market as follows. Given any exogenous parameters  $\theta \equiv (n, r, u, \rho, \lambda)$ , we increase meeting rates  $(\lambda_d, \lambda_f, \lambda_s)$ , while preserving the relative ratios. That is, we consider a sequence of markets  $\theta^\kappa \equiv (n, r, u, \rho, \kappa\lambda)$ , where  $\kappa\lambda = (\kappa\lambda_d, \kappa\lambda_f, \kappa\lambda_s)$ . We analyze the steady-state solution  $(\mu^\kappa, v^\kappa)$  in the limit as  $\kappa$  increases to infinity. To ease expositions, we assume a **regular** environment:  $n_a \notin \{n_h, n_h + n_f\}$ . The fast-search market is well-defined as the steady-state population measures  $\mu^\kappa$  converge (the convergence of  $v^\kappa$ , as a linear function of  $\mu^\kappa$ , follows immediately (see (16)). The speed of convergence is  $O(1/\kappa)$  which gives a precise sense of how closely a fast-search equilibrium would approximate an equilibrium of the calibrated market (in our calibration,  $\kappa \approx 103,000$ ):

**Proposition 6.** (*Convergence and Convergence Speed*) Take any regular environment  $\theta$ . For any  $i \in \mathcal{T}$ , the population limit  $\mu_i^* \equiv \lim_{\kappa \rightarrow \infty} \mu_i^\kappa$  and the convergence speed  $\mu_i^{**} \equiv \lim_{\kappa \rightarrow \infty} \kappa(\mu_i^\kappa - \mu_i^*)$  exist.

The appendix has the closed-form expressions for the limit populations and the convergence speed.

## 5.1 Inefficiency and Search Externality

We compare the fast-search equilibrium welfare  $W^*$  against two extreme situations: an autarkic economy with no functioning market or fund intermediation, and a centralized economy with a planner moving assets across agents without search friction.<sup>18</sup> In the autarkic economy, an  $n_a$  fraction among  $n_v (= 1)$  corporations hold assets with no trades, resulting in the welfare  $\underline{W}$  such that  $r\underline{W} = n_a(n_h u_h + n_l u_l)$ . In the centralized economy, a planner solves

$$\begin{aligned} r\overline{W} &\equiv \max_{\mu \in \mathbb{R}_+^T} \mu_{ho} u_h + \mu_{fo} u_f + \mu_{fe} u_e + \mu_{lo} u_l, \\ \text{subject to } &\mu_{ho} + \mu_{hn} = n_h, \quad \mu_{lo} + \mu_{ln} = n_l, \quad \text{and} \quad (1). \end{aligned}$$

The maximum welfare  $\overline{W}$  takes into account exogenous type changes  $\rho_u$  and  $\rho_d$  but ignores search frictions. It is also independent of the liquidity shock  $\rho_e$ , because assets can be transferred between funds instantaneously. The population measures  $\bar{\mu}$  that achieves the maximum welfare is such that  $\bar{\mu}_{ho} = \min\{n_a, n_h\}$ ,  $\bar{\mu}_{fo} = \min\{(n_a - n_h)^+, n_f\}$ ,  $\bar{\mu}_{lo} = (n_a - n_f - n_h)^+$ , and  $\bar{\mu}_i = 0$  for  $i \neq ho, fo, lo$ . In essence, assets are allocated to high-type investors up to their steady-state population  $n_h$ ; any remaining assets are given to funds up to  $n_f$ ; and the still-remaining assets are given to low-type investors. The maximum welfare satisfies  $r\overline{W} = \bar{\mu}_{ho} u_h + \bar{\mu}_{fo} u_f + \bar{\mu}_{lo} u_l$ .<sup>19</sup>

<sup>18</sup>The welfare of  $W^\kappa$  converges to  $W^*$  at the same speed  $O(1/\kappa)$  as  $\mu^\kappa$ , because the welfare  $W^\kappa$  is a linear aggregation of the population measures  $(\mu_i^\kappa)_{i \in \mathcal{T}}$ .

<sup>19</sup>The maximum welfare  $\overline{W}$  is also achieved by a planner who is under the search friction, like agents, but can choose not to execute some transactions. The planner's problem is  $rW_p(\lambda) \equiv \sup_{0 \leq \lambda_p \leq \lambda} \mu_{ho} u_h + \mu_{fo} u_f + \mu_{fe} u_e + \mu_{lo} u_l$ , subject to  $\mu$  being a solution of  $P(n, r, u, \rho, \lambda_p)$ . A large search rate allows the planner to achieve the maximum welfare approximately: i.e.,  $\overline{W} = \lim_{\lambda \rightarrow \infty} W_p(\lambda)$ . An intuition will become clear after Proposition 7. The planner can slow down the investors' direct trading, eliminate search externalities, and increase the welfare to the maximum.

|                    | A. $n_a < n_h$ | B. $n_h < n_a < n_h + n_f$ | C. $n_h + n_f < n_a$ |
|--------------------|----------------|----------------------------|----------------------|
| $\bar{\mu}_{ho} =$ | $n_a$          | $n_h$                      | $n_h$                |
| $\bar{\mu}_{fo} =$ | 0              | $n_a - n_h$                | $n_f$                |
| $\bar{\mu}_{lo} =$ | 0              | 0                          | $n_a - n_f - n_h$    |
| $\bar{\mu}_{fe} =$ | 0              | 0                          | 0                    |
| $\mu_{ho}^* =$     | $n_a$          | $n_h$                      | $n_h$                |
| $\mu_{fo}^* =$     | 0              | $n_a - n_h$                | $< n_f$              |
| $\mu_{lo}^* =$     | 0              | 0                          | $n_a - n_f - n_h$    |
| $\mu_{fe}^* =$     | 0              | 0                          | $> 0$                |

Table 1: The population under fast search ( $\mu^*$ ) and the efficient allocation ( $\bar{\mu}$ ).

We compare the efficient allocation  $\bar{\mu}$  (the upper part of Table 1) with the fast-search equilibrium population  $\mu^*$  (the lower part of Table 1).

**Proposition 7.** (*Fast-search Market: Welfare*) As  $\kappa \rightarrow \infty$ ,<sup>20</sup>

- A. If  $n_a < n_h$ , then  $W^* = \bar{W}$ , which is independent of  $u_f, u_e, \lambda_s$ , and  $\lambda_d$ .
- B. If  $n_h < n_a < n_h + n_f$ , then  $W^* = \bar{W}$ , which is strictly increasing in  $u_f$  and independent of  $u_e, \lambda_s$ , and  $\lambda_d$ .
- C. If  $n_h + n_f < n_a$ , then  $W^*$  is strictly less than  $\bar{W}$ , strictly increasing in  $u_f, u_e$ , and  $\lambda_s$ , and strictly decreasing in  $\lambda_d$ .

The characterization of the fast-search equilibrium welfare depends on the number of assets ( $n_a$ ) relative to the number of potential buyers ( $n_h, n_f$ ). A sufficiently large number of assets ( $n_a > n_h + n_f$ ), which is indeed the case of our calibration in Section 6, results in an inefficient fast-search equilibrium.

Suppose that the fast-search market has sufficiently many potential buyers ( $n_a < n_h + n_f$ ) as in Cases A and B (the first two columns in Table 1). Fast search allows investors and funds to quickly transfer assets from low-type investors ( $lo$ ) and exiting funds ( $fe$ ) to high-type investors ( $hn$ ) and, in Case B, also to funds at the investment phase ( $fn$ ). Accordingly,

<sup>20</sup> The fast-search market population  $\mu^* = \lim_{\kappa \rightarrow \infty} \mu^\kappa$  is constructed from Lemma 5 and Lemma 6 in Subsection C.1. In Case C,  $\mu_{fe}^* = \frac{n_f}{1 + \frac{\rho u \mu_{ln}^*}{\rho e \mu_{lo}^*} \frac{\lambda_f \mu_{lo}^* + \lambda_s n_f}{\lambda_d \mu_{lo}^* + \lambda_f n_f}}$  and  $\mu_{fo}^* = n_f - \mu_{fe}^*$ .

the steady-state population  $\mu^*$  equals the efficient allocation  $\bar{\mu}$  and achieves the maximum welfare ( $W^* = \bar{W}$ ).

The comparative statics of the welfare becomes trivial: the maximum welfare is dependent on payoff flows (e.g.,  $u_f$ ) only if the corresponding type's population (resp.,  $\mu_{fo}$ ) is non-zero. In either case, the impact of a liquidity shock is zero. Funds transfer assets without holding any inventory (Case A) just like, e.g., in Rubinstein and Wolinsky (1987); or, they hold assets, but funds under liquidity constraints transfer assets to others through speedy secondary trades (Case B), as is the case in Duffie, Gârleanu, and Pedersen (2005). Under a surplus of tradable assets relative to potential buyers ( $n_h + n_f < n_a$ ), as in Case C (the third column in Table 1), the equilibrium is more interesting because, counter-intuitively, slowing down investors' direct trading *improves* the welfare ( $\frac{\partial W^*}{\partial \lambda_d} < 0$ ). Since investors or funds on demands can quickly find sellers and purchase assets, there is negligible left-over high-type investors or fund non-owners. Hence, a significant fraction of exiting funds ( $fe$ ) will find it difficult to offload their assets, and the welfare loss is  $r(\bar{W} - W^*) = \mu_{fe}^*(u_f - u_e) > 0$ .

The inefficiency is a result of investors' search externalities on funds. A direct trading by investors take away selling opportunities from exit-phase funds and leads them to suffer from liquidity constraints for a longer period of time. If the investors' direct trading were absent, an exit-phase fund could off-load an asset, reset its type, and purchase another asset, all quickly under fast search. This alternative scenario results in a more efficient asset allocation.

## 5.2 Trading Patterns

In this section, we characterize the transaction volumes and how they respond to exogenous parameters, such as search frictions and transition rates. For each submarket  $m \in \mathcal{M}$ , with a seller's type  $s$  and the buyer's type  $b$ , the **trade volume** is  $\eta_m^\kappa \equiv (\kappa \lambda_m) \mu_s^\kappa \mu_b^\kappa$ .

**Proposition 8.** (*Fast-search Market: Trade Volumes*) For each submarket  $m \in \mathcal{M}$ , trade volume in the limit  $\eta_m^* \equiv \lim_{\kappa \rightarrow \infty} \eta_m^\kappa$  is given by:

|                    | A. $n_a < n_h$                       | B. $n_h < n_a < n_h + n_f$           | C. $n_h + n_f < n_a$                 |
|--------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| $\eta_{lo-hn}^* =$ | $\lambda_d \mu_{lo}^{**} \mu_{hn}^*$ | 0                                    | $\lambda_d \mu_{lo}^* \mu_{hn}^{**}$ |
| $\eta_{lo-fn}^* =$ | $\lambda_f \mu_{lo}^{**} \mu_{fn}^*$ | $\lambda_f \mu_{lo}^{**} \mu_{fn}^*$ | $\lambda_f \mu_{lo}^* \mu_{fn}^{**}$ |
| $\eta_{fo-hn}^* =$ | $\lambda_f \mu_{fo}^{**} \mu_{hn}^*$ | $\lambda_f \mu_{fo}^* \mu_{hn}^{**}$ | $\lambda_f \mu_{fo}^* \mu_{hn}^{**}$ |
| $\eta_{fe-hn}^* =$ | 0                                    | 0                                    | $\lambda_f \mu_{fe}^* \mu_{hn}^{**}$ |
| $\eta_{fe-fn}^* =$ | 0                                    | $\lambda_s \mu_{fe}^{**} \mu_{fn}^*$ | $\lambda_s \mu_{fe}^* \mu_{fn}^{**}$ |

where (i)  $\mu^*$  denotes the population limit, and (ii) for type  $i$  with  $\mu_i^* = 0$ ,  $\mu_i^{**} \equiv \lim_{\kappa \rightarrow \infty} \kappa \mu_i^\kappa$  denotes the convergence speed.<sup>21</sup>

With a large number of high-type investors (Case A), secondary trades are unnecessary for funds; if the deficit of high-type investors is supplemented by funds (Case B), selling investors resort to fund buyers and there are no investors' direct trading; with an excess supply of tradable assets (Case C), there are transactions in all submarkets.

The trade volumes under fast search follow from the convergence and the convergence speed of population measures (Proposition 6). For each submarket  $m \in \mathcal{M}$ , because of fast search, the steady-state measure of either buyers or sellers vanishes: i.e.,  $\mu_b^* = 0$  or  $\mu_s^* = 0$ . If  $\mu_s^* = 0$ , then  $\eta_m^\kappa = (\kappa \lambda_m) \mu_b^\kappa \mu_s^\kappa = \lambda_m \mu_b^\kappa (\kappa \mu_s^\kappa) \rightarrow \lambda_m \mu_b^* \mu_s^{**}$  as  $\kappa \rightarrow \infty$ .

Proposition 8 suggests that all submarkets are active under fast search only if the market has an excess supply of tradable assets ( $n_a > n_h + n_f$ ). In a calibration of our model (Section 6), we observe non-negligible transaction volumes in all submarkets, which suggests Case C best approximates the US corporate acquisition market.

The trade volumes also identify the main drivers of the convergences of certain population measures. For example,  $\mu_{fo}^* = 0$  in Case A could be due to the fact that (i) funds can rarely purchase assets because of a vanishingly small number of selling investors ( $lo$ ), or (ii) funds do acquire assets, but quickly re-sell to buying investors ( $hn$ ). Proposition 8 implies the latter case; funds buy/sell a significant number of assets from/to investors in the fast-search market and there are no secondary transactions (like middlemen in Rubinstein and Wolinsky (1987)). Similarly, the vanishing number of selling investors ( $lo$ ) and buying investors ( $hn$ ) in Case B is the result of an efficient fund-investor trading rather than an efficient market

<sup>21</sup>See Lemma 6 in Subsection C.1 and Section S.1 in Supplemental Appendix for the closed-form expressions.



for investors' direct trading – the number of investors' direct transactions ( $\eta_{lo-hn}^*$ ) is indeed vanishingly small.

Table 2 summarizes a comparative static analysis for trade volumes relative to search frictions and transition rates.<sup>22</sup>

|                  | $\lambda_d$ | $\lambda_f$ | $\lambda_s$ | $(\rho_u, \rho_d)$ (with a fixed ratio) | $\rho_e$ |
|------------------|-------------|-------------|-------------|---|----------|
| $\eta_{lo-hn}^*$ | +           | –           | 0           | +                                       | 0        |
| $\eta_{lo-fn}^*$ | –           | +           | 0           | +                                       | 0        |
| $\eta_{fo-hn}^*$ | –           | +           | +           | +                                       | –        |
| $\eta_{fe-hn}^*$ | –           | +           | –           | +                                       | +        |
| $\eta_{fe-fn}^*$ | –           | –           | +           | +                                       | +        |

Table 2: Comparative statics of trade volumes. Each + (or –) indicates the corresponding volume to be non-decreasing (resp., non-increasing) in the parameter, and 0 indicates that the volume is independent of the parameter.

Most results are intuitive. If investors' types are more volatile (i.e., larger  $\rho_u$  and  $\rho_d$  with a fixed ratio), assets will be transferred across agents frequently, ultimately from low-type investors to high-type investors, with possible fund intermediations. Fast search among investors (i.e., higher  $\lambda_d$ ) allows them to transact directly (i.e., higher  $\eta_{lo-hn}^*$ ), resulting in fewer intermediation opportunities for funds. The parameters for the secondary market ( $\lambda_s$  and  $\rho_e$ ) only shift the population measures between  $fo$  and  $fe$ . Therefore, these parameters do not affect the volume of investors' direct trading ( $\eta_{lo-hn}^*$ ) and only shift volumes between two kinds of fund-investor transactions:  $\eta_{fo-hn}^*$  and  $\eta_{fe-hn}^*$ .

The positive response of  $\eta_{fe-hn}^*$  to  $\lambda_f$  is perhaps surprising. On one hand, fast search between funds and investors (i.e., higher  $\kappa\lambda_f$ ) orchestrates more transactions between exiting funds ( $fe$ ) and buying investors ( $hn$ ). On the other hand, as funds are able to sell assets before receiving liquidity shocks, fewer funds enter the exit phase, which could potentially reduce the trade volume between exiting funds and buying investors. It turns out that the former effect of  $\lambda_f$  dominates the latter.

<sup>22</sup>The proofs are tedious and follow directly from  $\mu^*$  and  $\mu^{**}$  in Table 1 and Section S.1 of Supplemental Appendix, so we omit. As an example, take  $\eta_{lo-hn}^*$  and  $\lambda_d$ . Note that  $\eta_{lo-hn}^* = \lambda_d \mu_{lo}^* \mu_{hn}^{**}$ . In Cases A and B,  $\mu_{lo}^* = 0$ , so  $\eta_{lo-hn}^*$  is independent of  $\lambda_s$ . In Case C,  $\mu_{hn}^{**} = \frac{\rho_u \mu_{ln}^*}{\lambda_d \mu_{lo}^* + \lambda_f (\mu_{fo}^* + \mu_{fe}^*)} = \frac{\rho_u \mu_{ln}^*}{\lambda_d \mu_{lo}^* + \lambda_f n_f}$ . Note that  $\mu_{lo}^*$  and  $\mu_{ln}^*$  are independent of  $\lambda_d$ . Thus, the volume  $\eta_{lo-hn}^*$  is increasing in  $\lambda_d$  because  $\frac{\partial}{\partial \lambda_d} \left( \frac{\lambda_d}{\lambda_d \mu_{lo}^* + \lambda_f n_f} \right) = \lambda_f n_f > 0$ .

## 6 Calibration

We perform a calibration exercise to evaluate the quantitative significance of our theoretical results applied to the mid-size US corporate acquisition market. A typical transaction involves a sale of assets – either all the corporation assets or a subdivision – by corporate investors or by private equity (PE) buyout funds. A subset of corporations have assets (the tradable assets in our model), and assets create higher flow payoffs in the hands of high-type investors. The corporate acquisition market lends itself naturally to a search framework. Corporate sellers take about a year to find an appropriate buyer and close a transaction (Boone and Mulherin (2011)).

A typical PE buyout fund acquires a small number of portfolio firms, holds the firms as inventory and adds operational value through better management, and exits by selling their portfolio firms to provide liquidity to fund investors.<sup>23</sup> PE funds divest the acquired firms to corporate investors through primary buyouts (PBOs), or to other PE funds through secondary buyouts (SBOs), each of which accounted for 52% and 42% of PE exits in 2017. PE buyout funds have become pivotal players in the US corporate acquisition market. In 2017, their investment represents \$538 billion out of the \$2.1 trillion (or 4,053 out of 10,769 deals) with a compounded annual growth rate of 7.5% since 2011.<sup>24</sup>

We address various quantitative questions such as: (i) Is there an over or undersupply of tradable assets relative to the market demand (i.e., the number of high-type investors and PE funds)? (ii) How are fund valuations affected by liquidity provision through SBOs and fund managers' operational improvement? (iii) What is the impact of PE entry on fund valuation and transaction prices? (iv) What are the welfare losses associated with search frictions?

Our model has 14 exogenous parameters:  $n = (n_v, n_f, n_a)$ ,  $u = (u_l, u_h, u_f, u_e)$ ,  $\rho = (\rho_u, \rho_d, \rho_e)$ ,  $\lambda = (\lambda_d, \lambda_f, \lambda_s)$  and  $r$ . Some parameters  $(n, u)$  are either directly observed or obtained from other empirical studies, and the remaining seven parameters  $(\rho, \lambda, r)$  are estimated to fit the model statistics in Table 3 to the data. The calibrated parameters are

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<sup>23</sup>We distinguish buyout funds from other PE funds such as venture capital funds, which usually invest in fractional equity stakes of start-ups and early-stage firms.

<sup>24</sup>The data on PE activities in the corporate acquisition market from PitchBook Data, Inc., can be found at <https://pitchbook.com/news/reports/2q-2018-ma-report>.

reported in Table 4.

**Data and Methodology** We focus on acquisitions by mid-size US companies, defined as companies with annual revenues between \$20 million and \$1,000 million. We exclude small companies, for which there is no reliable data on acquisition activities, and large companies to maintain an overall homogeneity in our sample set. According to the latest available U.S. Economic Census Data (2012), there are approximately 102,626 mid-size companies. We normalize  $n_v = 1$  and apply the same rescaling to the number of PE funds, assets, trading volumes, etc. dividing by the number of companies.

The primary data is from the 2018 US PE middle market report by Pitchbook Data Inc. The data includes corporate acquisition deals with transaction values between \$25 million and \$1,000 million. There are 1,893 PE funds targeting the middle market (thus  $n_f \simeq 0.02$  with normalization).<sup>25</sup> The dataset also provides the transaction volumes for various kinds of trading, summarized in Table 3, along with other model statistics. The number of direct transactions among companies is 9,626 per-year from 2007 to 2017. In the same period, PE funds acquired an average of 1,799 firms per year, of which 359 ( $\approx 20\%$ ) are through SBOs and 1,440 are through PBOs. The steady state of our model implies that the number of PE buyouts must be equal to the number of exits (i.e.,  $\eta_{fe-hn} + \eta_{fo-hn} = \eta_{lo-fn}$ ). Then, the total number of deals equals the sum  $\eta_{lo-hn} + 2\eta_{lo-fn} + \eta_{fe-fn}$ , which gives us an estimate of 6,387 direct transactions ( $\eta_{lo-hn}$ ) per year. The trade volumes allow us to identify the search intensities ( $\lambda_d, \lambda_f, \lambda_s$ ).

For the average time to sell by corporate investors and PE funds ( $E[\tau_{sv}]$  and  $E[\tau_{sf}]$  in Lemma 4), we resort to various reports available online and take their average. According to reports prepared by selling agents such as business brokers or investment bankers (see Table S.1 in Supplemental Appendix), selling a firm takes an average of 11 months (0.91 years) for PE funds and 15 months (1.25 years) for corporate investors.<sup>26</sup>

<sup>25</sup>Pitchbook reports the number of PE funds raised in each year targeting the middle market. We cumulate these numbers for 2007-2018, as the average lifespan of PE funds is 12 years (Metrick and Yasuda (2010)).

<sup>26</sup>We include both time taken in the preparation process and the listing-to-sale process. The preparation process for PE funds takes only an average of 2 months – much shorter than an average of 6 months for corporate investors. Portfolio firms of PE funds are usually in a better state of readiness to approach the market due to high-quality governance, accounting, and information systems. The listing-to-sale process takes an average of 9 months for selling agents.

| Description                              | Data       | Model Statistic<br>(for normalized values)   |
|--|------------|--|
| Corporate (Direct) acquisitions          | 6387       | $\eta_{lo-hn} = \lambda_d \mu_{lo} \mu_{hn}$ |
| Primary Buyouts (PBOs)                   | 1440       | $\eta_{lo-fn} = \lambda_f \mu_{lo} \mu_{fn}$ |
| Secondary Buyouts (SBOs)                 | 359        | $\eta_{fn-fe} = \lambda_s \mu_{fn} \mu_{fe}$ |
| Avg time to sell for corporate investors | 1.25 years | $E[\tau_{sv}]$ (eq. (4))                     |
| Avg time to sell for PE funds            | 0.91 years | $E[\tau_{sf}]$ (eq. (5))                     |
| Fund performance (PME)                   | 1.01       | PME (eq. (6))                                |
| Price multiple (EV/EBITDA)               | 9.0        | $P_{lo-hn}/u_l$                              |

Table 3: Key Statistics on Corporate Acquisitions and Private Equity

A common performance measure of PE funds is the Public Market Equivalent (PME), introduced by Kaplan and Schoar (2005) (see also Sorensen and Jagannathan (2015)) and defined as

$$PME = \frac{\text{Present value of distributions to fund investors}}{\text{Present value of capital calls made by fund investors}}. \quad (6)$$

PME is the ratio of cash outflows over cash contributions, both discounted at the public market total return (e.g., S&P 500 index) after subtracting management fees paid to the fund managers – see Supplemental Appendix for a derivation of the closed-form expression.<sup>27</sup> A PME of one is indicative of a fund’s performance in line with the public market, and a lower PME indicates underperformance. We take the average PME of 1.01 from various estimates.<sup>28</sup>

The most widely used metric for the relative acquisition prices is the EV/EBITDA multiple. This transaction multiple, which is matched to our model’s  $P_{lo-hn}/u_l$ , is on average 9.0 from 2005 to 2017.<sup>29</sup>

<sup>27</sup>According to Metrick and Yasuda (2010), the management fees are usually 2% of the committed capital and paid from the inception of the fund until its liquidation.

<sup>28</sup>Kaplan and Schoar (2005) estimate an average PME of 0.93 for PE funds in the period 1980-1994, while Phalippou and Gottschalg (2008), using similar dataset but different methodology, report an average PME of 0.88. Harris, Jenkinson, and Kaplan (2014), on the other hand, report significantly better performance with an average PME of 1.22 for the period 1984-2008. The estimates of PME by PitchBook Data, Inc. yields an average of 1.00 for the period 2006-15.

<sup>29</sup>See a recent report on EV/EBITDA by FactSet Research Systems Inc. at [https://www.factset.com/hubfs/mergerstat\\_em/monthly/US-Flashwire-Monthly.pdf](https://www.factset.com/hubfs/mergerstat_em/monthly/US-Flashwire-Monthly.pdf).

|             | Parameters                       | Variable    | Value |
|-------------|----------------------------------|-------------|-------|
| (Observed)  | No. of corporate investors       | $n_v$       | 1.0   |
|             | No. of PE funds                  | $n_f$       | 0.02  |
|             | No. of assets                    | $n_a$       | 0.5   |
|             | Flow Payoff low type             | $u_l$       | 1     |
|             | Flow payoff high type            | $u_h$       | 1.4   |
|             | Flow payoff PE (harvesting)      | $u_f$       | 1.3   |
|             | Flow payoff PE (exiting)         | $u_e$       | 1.1   |
| (Estimated) | Low valuation shock              | $\rho_d$    | 0.23  |
|             | High valuation shock             | $\rho_u$    | 0.16  |
|             | Liquidity shock                  | $\rho_e$    | 0.38  |
|             | Match intensity (direct trading) | $\lambda_d$ | 46.1  |
|             | Match intensity (PBO)            | $\lambda_f$ | 61.7  |
|             | Match intensity (SBO)            | $\lambda_s$ | 699   |
|             | Discount rate                    | $r$         | 11.8% |

Table 4: Fitted Parameters of Calibration

We now turn to the payoff parameters. Low-type corporate investors' flow payoff is normalized as  $u_l = 1$ . For high-type corporate investors' flow payoff, we use Muscarella and Vetsuypens (1990), Opler (1992) and Andrade and Kaplan (1998). Each of these papers estimate the increase in operating profits (or cash flows) for firms after fund buyouts as 23.5%, 16.5%, and 52.9%, respectively. We take the average and set  $u_f = 1.3$  (i.e., 30% increase from  $u_l$ ), which is also close to the average 28.5% of the takeover premium paid by PE acquirers (Bargeron, Schlingemann, Stulz, and Zutter, 2008). For the flow payoff net of liquidity cost  $u_e$ , Nadauld, Sensoy, Vorkink, and Weisbach (2016) find that fund investors under liquidity shocks sell their PE ownership to other fund investors at a 13.8% discount. This observation motivates our choice of  $u_e = (1 - 0.138) \times u_f \simeq 1.1$ . Last, it is difficult to observe the payoff improvement from acquisitions ( $u_h - u_l$ ) because the target's and acquirer's operations often blend together. As such, we seek an indirect evidence from the premium paid by acquirers. Betton, Eckbo, and Thorburn (2008) reports an average 43% takeover premium over 4,880 acquisitions during 1980-2002, and Bargeron, Schlingemann, Stulz, and Zutter (2008) find that the takeover premium paid by a private acquirer is 40.9%. From these estimates, we choose  $u_h = 1.4$ .

We do not directly observe the number of assets. In our benchmark analysis, we set  $n_a = 0.5$ . The estimates of the parameter values in Table 4 with  $n_a = 0.25$  and  $n_a = 0.75$  change insignificantly, except for  $\rho_d$  and  $\rho_u$ .

We estimate the remaining parameters  $\beta \equiv (\rho, \lambda, r)$  that best explain the key statistics in Table 3. Each choice of the remaining parameters' values, together with the directly observed parameters  $(n, u)$ , defines a market  $\theta = (n, r, u, \rho, \lambda)$ . We compute the statistics  $Y_i(\beta; n, u)$  for each row  $i = 1, \dots, 7$  in Table 3 and compare them with the observed data  $Y_i^{obs}$ . The estimate of  $\beta$  minimizes the sum of squared residuals (SSR), subject to positive trade gains in the unique steady-state solution of the market  $(\beta, n, u)$ :

$$\min_{\beta} SSR(\beta; n, u) \equiv \sum_{i=1}^7 \left( \frac{Y_i(\beta; n, u) - Y_i^{obs}}{Y_i^{obs}} \right)^2$$

subject to  $g_m(\beta; n, u) \geq 0$ , for each  $m \in \mathcal{M}$ .

The lower section of Table 4 summarizes the parameter estimates. Our model fits the observed data with a high degree of accuracy: the minimum SSR is approximately  $2.9 \times 10^{-5}$ .

The parameter estimates are of reasonable magnitudes. The estimated type transition rates  $\rho_u$  and  $\rho_d$  suggest that the type transitions (from high to low, and vice versa) happen within 6.3 to 4.3 years on average. The meeting rates  $\lambda = (\lambda_d, \lambda_f, \lambda_s)$  are of a much higher order, due to normalization, as discussed in Section 5. The meeting rate for funds is higher than corporate investors ( $\lambda_f > \lambda_d$ ). The calibrated inter-PE meeting rate  $\lambda_s$  is much higher than the other two, reflecting the high volume of SBOs relative to the small fraction of PE funds. Lastly, the estimated discount rate  $r = 11.8\%$ , although high, seems reasonable given that assets represent stakes in mid-size private firms.

**Results** The calibration results in Table 5 answer the quantitative questions (i)-(iv) at the beginning of this section.

First, there is an oversupply of assets. The calibrated  $\rho_u$  and  $\rho_d$  imply  $n_h \equiv \frac{\rho_u}{\rho_u + \rho_d} = 0.40$  and an excess supply of assets  $n_a > n_h + n_f$ . The result, together with large meeting rates, suggests that the US corporate acquisition market is close to Case C of the fast-search market

|       | Description  | Symbol   | Value |
|-------|--|--|-------|
| (i)   | Oversupply of assets   | $n_a - (n_h + n_f)$                              | 0.08  |
| (ii)  | Impact of SBOs on fund valuation<br>(changing of $\lambda_s$ from 0 to 699)                      | $\frac{\Delta v_{fn}}{v_{fn}}$                   | 26.2% |
|       | Sensitivity of fund value to operations  | $\frac{\partial \log v_{fn}}{\partial \log u_f}$ | 4.22  |
|       | Sensitivity of fund value to liquidity shocks  | $\frac{\partial \log v_{fn}}{\partial \log u_e}$ | 0.52  |
| (iii) | Impact of PE entry on fund valuation<br>(changing the number of funds from $n_f$ to $2n_f$ )     | $\frac{\Delta v_{fn}}{v_{fn}}$                   | 0.7%  |
|       | Impact of PE entry on transaction prices<br>(changing the number of funds from $n_f$ to $2n_f$ ) | $\frac{\Delta P_{lo-hn}}{P_{lo-hn}}$             | -0.9% |
| (iv)  | Welfare gain by asset reallocations<br>(relative to the welfare with no asset reallocations)     | $\frac{W-W}{W}$                                  | 13.6% |
|       | Welfare gain by asset reallocations<br>(relative to the maximum welfare gain)                    | $\frac{W-W}{\bar{W}-W}$                          | 92.1% |
|       | corporate investors' share of the welfare gain   | $\frac{W_v-W}{W-W}$                              | 73.3% |
|       |  |  |       |

Table 5: Calibration Results

(Proposition 7).<sup>30</sup>

Second, SBOs significantly improve fund valuations, as explained with Figure 2. PE fund values with SBOs are 26.2% higher than without SBOs, highlighting the complementarities among PE funds. While we acknowledge the criticism against SBOs, maybe it is because of (and not in spite of) SBOs that PE funds generate high returns. An improvement in firms' operation  $u_f$  by 1% leads to a significant 4.22% increase in fund value  $v_{fn}$ . However, a similar improvement of  $u_e$  is attenuated by a vibrant SBO market and has a negligible influence on fund value: sensitivity is 0.52.

Third, when the number of PE funds increases, the percentage of SBOs relative to the total fund exits would increase (Figure 3). This pattern is indeed observed in the data. The share of firms sold by PE funds through SBOs has increased from 13% in the 1980s, 19% in 2009, to 42% in 2017. Doubling the number of PE funds would increase the average fund value by 0.7%, suggesting that the complementarities among funds plays a stronger role than

<sup>30</sup>The calibration result on the oversupply of assets is not sensitive to our choice of  $n_a = 0.5$ . While a choice of  $n_a = 0.25$  or  $0.75$  results in different estimates of  $\rho_u$  and  $\rho_d$ , the oversupply remains about the same ( $n_a - (n_h + n_f) \approx 0.08$ ).

the competitive effect. Doubling the number of funds leads to only a 0.9% decrease in direct transaction prices. This is because more purchases by funds empower selling investors and more sales by funds increase buying investors' bargaining position.

Finally, the welfare gain by asset reallocations is 13.6%, relative to the autarkic situation welfare  $\underline{W}$  (see page 21). This welfare gain ( $W - \underline{W}$ ) attains 92.1% of the best possible gain ( $\bar{W} - \underline{W}$ ). This fraction is lower than the gain in OTC markets for municipal bonds as described in Hugonnier, Lester, and Weill (2018), likely due to higher search frictions in the corporate acquisition market. The corporate investors' percentage share of this welfare gain is 73.3%, which leaves 26.7% to PE funds. The PE funds' welfare share is very large relative to their small number  $n_f = 0.02$ .

## 7 Conclusion

We provide a search-based model of asset trading with fund intermediation. Funds in our model intermediate between buyers and sellers at risk of selling assets under pressure, possibly to other funds. The model shows that secondary transactions can make substantial contribution to fund values. Our paper offers a novel explanation on persistent intermediators' returns in the corporate acquisition market, despite increased competition. A well-lubricated private market for corporate acquisitions can partly explain the recent shift in firm ownership from public to private, predicated by Jensen (1991). Lower liquidity costs in a market for buying and selling private firms help make PE ownership a form of governance that may indeed eclipse public companies.

## A Proofs for Section 3

### A.1 Proof of Lemma 1:

Take any market  $\theta = (n, r, u, \rho, \lambda)$  with  $n_f > 0$ . Let  $\mu \in \mathbb{R}_+^T$  be a steady-state solution of  $P(\theta)$ .

First, we show that  $\mu_{ho} > 0$  and  $\mu_{lo} > 0$ . It is clear that  $\mu_{ho} = 0$  if and only if  $\mu_{lo} = 0$ . If  $\mu_{ho} = 0$ , then  $\mu_{lo} = 0$ , as only  $ho$ -type investors flow in type  $lo$ ; conversely, if  $\mu_{lo} = 0$ ,



then  $\mu_{ho} = 0$  as the inflow to the type  $lo$  must be zero. Suppose, toward contradiction, that  $\mu_{ho} = \mu_{lo} = 0$ . That is, all investors are non-owners. The in-flow from  $hn$ -type investors to type  $ho$  must be zero, so it must be that  $\mu_{fo} = \mu_{fe} = 0$ . Then,  $\mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = 0$ , a contradiction to  $n_a > 0$ .

Second, we show that  $\mu_{ln} > 0$  and  $\mu_{hn} > 0$ . As before, it is clear that  $\mu_{ln} = 0$  if and only if  $\mu_{hn} = 0$ . If  $\mu_{ln} = 0$ , then  $\mu_{hn} = 0$  as only  $ln$ -type investors can flow in type  $hn$ ; conversely, if  $\mu_{hn} = 0$ , then  $\mu_{ln} = 0$  as the inflow to the type  $hn$  must be zero. Suppose, toward contradiction, that  $\mu_{ln} = \mu_{hn} = 0$ . That is, all investors are owners, which implies that some funds are non-owners:  $\mu_{fn} = n_v + n_f - n_a > 0$ . Since  $\lambda_f \mu_{lo} \mu_{fn} > 0$ , some  $lo$ -type investors change their types and flow into type  $ln$  by trading with PE funds, a contradiction to  $\mu_{ln} = 0$ .

Lastly, we consider funds. Suppose that  $\mu_{fn} = 0$ . As the inflow to type  $fo$  becomes zero, it must be that  $\mu_{fo} = 0$ , which in turn leads to no inflow by liquidity shocks to type  $fe$ : i.e.,  $\mu_{fe} = 0$ . Such case contradicts to  $n_f > 0$ . When  $\mu_{fn} > 0$ , given strictly positive population  $\mu_{lo}$ , the inflow of type- $fn$  funds to type  $fo$  is strictly positive:  $\lambda_f \mu_{lo} \mu_{fn} > 0$ . As such,  $\mu_{fo} > 0$ , which in turn creates a strictly positive inflow by liquidity shocks to type  $fe$ :  $\mu_{fe} > 0$ .

## A.2 Proof for Part 1 of Proposition 1

We reduce the number of variables and population equations in  $P(\theta)$  by imposing some necessary conditions for a steady-state solution. Note that any steady-state solution  $\mu$  must satisfy  $\mu_{ho} + \mu_{hn} = n_h \equiv \frac{\rho_u}{\rho_u + \rho_d}$  and  $\mu_{lo} + \mu_{ln} = n_l \equiv \frac{\rho_d}{\rho_u + \rho_d}$  (which we can obtain by adding  $(\mu\text{-}ho)$  and  $(\mu\text{-}hn)$ , or  $(\mu\text{-}lo)$  and  $(\mu\text{-}ln)$ , and apply  $n_v = 1$ ). If we substitute  $\mu_{ho} = n_h - \mu_{hn}$  and  $\mu_{ln} = n_l - \mu_{lo}$  into  $(\mu\text{-}ho)\text{--}(\mu\text{-}fe)$ , then we are left with the following three linearly

independent equations:<sup>31</sup>

$$\begin{aligned}
(\lambda_d \mu_{hn} + \lambda_f \mu_{fn}) \mu_{lo} + \rho_u \mu_{lo} - \rho_d \mu_{ho} &= 0, & (\text{from } (\mu\text{-lo})) \\
(\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe}) \mu_{hn} + \rho_d \mu_{hn} - \rho_u \mu_{ln} &= 0, & (\text{from } (\mu\text{-hn})) \\
-(\lambda_f \mu_{hn} + \lambda_s \mu_{fn}) \mu_{fe} + \rho_e \mu_{fo} &= 0. & (\text{from } (\mu\text{-fe}))
\end{aligned}$$

We re-write the first two equations with respect to  $\mu_{lo}$  and  $\mu_{hn}$ . Since

$$\mu_{fo} + \mu_{fe} = n_a - \mu_{ho} - \mu_{lo} = n_a - (n_h - \mu_{hn}) - \mu_{lo} \quad \text{and} \quad (7)$$

$$\mu_{fn} = n_f - (\mu_{fo} + \mu_{fe}) = n_f - n_a + n_h - \mu_{hn} + \mu_{lo}, \quad (8)$$

we have

$$(\lambda_d \mu_{hn} + \lambda_f (n_f - n_a + n_h - \mu_{hn} + \mu_{lo})) \mu_{lo} + \rho_u \mu_{lo} - \rho_d (n_h - \mu_{hn}) = 0, \quad (9)$$

$$(\lambda_d \mu_{lo} + \lambda_f (n_a - n_h + \mu_{hn} - \mu_{lo})) \mu_{hn} + \rho_d \mu_{hn} - \rho_u (n_l - \mu_{lo}) = 0. \quad (10)$$

We show below that there exists a unique solution  $(\mu_{lo}, \mu_{hn})$  of (9)-(10) such that (i)  $0 \leq \mu_{lo} \leq n_l$ , (ii)  $0 \leq \mu_{hn} \leq n_h$ , and (iii)  $n_a - n_f - n_h \leq \mu_{lo} - \mu_{hn} \leq n_a - n_h$  (for  $0 \leq \mu_{fn} \leq n_f$ ). Other population measures will be determined by  $\mu_{ho} = n_h - \mu_{hn}$ ,  $\mu_{ln} = n_l - \mu_{lo}$ , and

$$\mu_{fn} = n_f - n_a + n_h - \mu_{hn} + \mu_{lo}. \quad (11)$$

We find the last two populations  $(\mu_{fe}, \mu_{fo})$  by solving

$$\begin{aligned}
\mu_{fo} &= -\mu_{fe} + (n_f - \mu_{fn}), & (\text{from } \mu_{fn} + \mu_{fo} + \mu_{fe} = n_f) \\
\mu_{fo} &= \frac{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}}{\rho_e} \mu_{fe}. & (\text{from } (\mu\text{-fe}))
\end{aligned}$$

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<sup>31</sup>Any other equation in  $P(\theta)$  is redundant, as it depends linearly on  $(\mu\text{-lo})$ ,  $(\mu\text{-hn})$ , and  $(\mu\text{-fe})$ . Each sum of the right-hand sides of  $(\mu\text{-ho})$  and  $(\mu\text{-hn})$ , or  $(\mu\text{-lo})$  and  $(\mu\text{-ln})$  equals zero, which allow us to delete  $(\mu\text{-ho})$  and  $(\mu\text{-ln})$  without changing the solution set. The sum of the right-hand sides of  $(\mu\text{-fn})$ ,  $(\mu\text{-fo})$ , and  $(\mu\text{-fe})$  equals zero, so we can delete  $(\mu\text{-fn})$ . Last, the sum of the right-hand sides of  $(\mu\text{-ho})$ ,  $(\mu\text{-lo})$ ,  $(\mu\text{-fo})$ , and  $(\mu\text{-fe})$  equals zero, so we can delete  $(\mu\text{-fo})$ .

The unique solution is

$$\mu_{fe} = \frac{\rho_e(n_f - \mu_{fn})}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + \rho_e}, \quad \text{and} \quad \mu_{fo} = \frac{(\lambda_f \mu_{hn} + \lambda_s \mu_{fn})(n_f - \mu_{fn})}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + \rho_e}. \quad (12)$$

Therefore, it remains to prove the following claim:

**Claim 1.** *Let*

$$X(\theta) \equiv \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq n_l, 0 \leq x_2 \leq n_h, 0 \leq g_{fn}(x) \leq n_f\}.$$

where  $g_{fn}(x) \equiv n_a - n_h + x_2 - x_1$ . Also, define  $F \equiv (F_{lo}, F_{hn}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$\begin{aligned} F_{lo}(x) &\equiv (\lambda_d x_2 + \lambda_f(n_f - g_{fn}(x)))x_1 + \rho_u x_1 - \rho_d(n_h - x_2), \\ F_{hn}(x) &\equiv (\lambda_d x_1 + \lambda_f g_{fn}(x))x_2 + \rho_d x_2 - \rho_u(n_l - x_1). \end{aligned}$$

Then, there exists a unique solution of  $F(x) = 0$  in  $X(\theta)$ .

We apply the Poincare-Hopf index theorem, a version in Simsek, Ozdaglar, and Acemoglu (2007, p.194); see also Hirsch (2012).

First,  $X(\theta)$  is non-empty, compact, and convex.<sup>32</sup> The boundary of  $X(\theta)$  is

$$\partial X(\theta) \equiv \{(x_1, x_2) \in X(\theta) : x_1 = 0, x_1 = n_l, x_2 = 0, x_2 = n_h, g_{fn}(x) = 0, \text{ or } g_{fn}(x) = n_f\}.$$

Second, the function  $F(x)$  is continuously differentiable at every  $x \in \mathbb{R}^2$ .

Third, the determinant of the Jacobian matrix of  $F$  is strictly positive for every interior

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<sup>32</sup>The Poincare-Hopf index theorem also requires  $X(\theta)$  to be a 2-dimensional smooth manifold, which a reader can easily verify by applying the identify function to the definition of a smooth manifold in Simsek, Ozdaglar, and Acemoglu (2007, p.193).

point of  $X(\theta)$ : for each  $x \in \mathbb{R}^2$ ,

$$\begin{aligned}\nabla F(x) &\equiv \begin{bmatrix} \frac{\partial F_{lo}}{\partial x_1} & \frac{\partial F_{lo}}{\partial x_2} \\ \frac{\partial F_{hn}}{\partial x_1} & \frac{\partial F_{hn}}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} (\lambda_d x_2 + \lambda_f(n_f - g_{fn}(x))) + \lambda_f x_1 + \rho_u & (\lambda_d - \lambda_f)x_1 + \rho_d \\ (\lambda_d - \lambda_f)x_2 + \rho_u & (\lambda_d x_1 + \lambda_f g_{fn}(x)) + \lambda_f x_2 + \rho_d \end{bmatrix},\end{aligned}$$

and for any interior point  $x \in X(\theta) \setminus \partial X(\theta)$ ,

$$\begin{aligned}\det(\nabla F(x)) &\geq (\lambda_d x_2 + \lambda_f x_1)(\lambda_d x_1 + \lambda_f x_2) + \rho_u \lambda_d x_1 + \rho_d \lambda_d x_2 \\ &\quad - (\lambda_d - \lambda_f)^2 x_1 x_2 - (\lambda_d - \lambda_f)(\rho_d x_2 + \rho_u x_1) \\ &= \lambda_d \lambda_f (x_1^2 + x_2^2) + 2\lambda_d \lambda_f x_1 x_2 + \lambda_f(\rho_d x_2 + \rho_u x_1) \\ &> 0.\end{aligned}\tag{13}$$

Last, we show that, for every boundary point  $x \in \partial X(\theta)$ , the vector  $F(x) \in \mathbb{R}^2$  points strictly outward of  $X(\theta)$ . We partition the boundary  $\partial X(\theta)$  into six *faces* (i.e., flat surfaces) of  $X(\theta)$ . For each face, we find an outward normal vector  $\mathbf{n} \in \mathbb{R}^2$  and show that the angle between  $\mathbf{n}$  and  $F(x)$  is acute (i.e.,  $\leq 90$ ) at any point  $x$  in the face:

1. ( $x_1 = 0$  and  $0 \leq x_2 < n_h$ )  $\mathbf{n} = (-1, 0)$  is an outward normal vector, and  $\mathbf{n} \cdot F(x) = \rho_d(n_h - x_2) > 0$ .
2. ( $x_2 = 0$  and  $0 \leq x_1 < n_l$ )  $\mathbf{n} = (0, -1)$  is an outward normal vector, and  $\mathbf{n} \cdot F(x) = \rho_u(n_l - x_1) > 0$ .
3. ( $x_1 = n_l$  and  $0 \leq x_2 \leq n_h$ )  $\mathbf{n} = (1, 0)$  is an outward normal vector, and

$$\begin{aligned}\mathbf{n} \cdot F(x) &= (\lambda_d x_2 + \lambda_f(n_f - g_{fn}(x)))n_l + \rho_u n_l - \rho_d n_h + \rho_d x_2 \\ &\geq (\lambda_d x_2 + \lambda_f(n_f - g_{fn}(x)))n_l \quad (\text{as } \rho_u n_l = \rho_d n_h) \\ &\geq \min\{\lambda_d x_2 n_l, \lambda_f(n_f - g_{fn}(x))n_l\}.\end{aligned}$$

As either  $x_2 > 0$  or  $x_2 = 0$ , we have  $n_f - g_{fn}(x) = n_v + n_f - n_a > 0$ , and  $\mathbf{n} \cdot F(x) > 0$ .

4. ( $x_2 = n_h$  and  $0 \leq x_1 \leq n_l$ )  $\mathbf{n} = (0, 1)$  is an outward normal vector, and

$$\begin{aligned}\mathbf{n} \cdot F(x) &= (\lambda_d x_1 + \lambda_f g_{fn}(x))n_h + \rho_d n_h - \rho_u n_l + \rho_u x_1 \\ &\geq \min\{\lambda_d x_1 n_h, \lambda_f g_{fn}(x) n_h\}.\end{aligned}$$

As either  $x_1 > 0$  or  $x_1 = 0$ , we have  $g_{fn}(x) = n_a > 0$ , and  $\mathbf{n} \cdot F(x) > 0$ .

5. ( $g_{fn}(x) = 0$  and  $x_1 > 0$ )  $\mathbf{n} = (1, -1)$  is an outward normal vector, and  $\mathbf{n} \cdot F(x) = F_{lo}(x) - F_{hn}(x) = \lambda_f n_f x_1 > 0$ .
6. ( $g_{fn}(x) = n_f$  and  $x_2 > 0$ )  $\mathbf{n} = (-1, 1)$  is an outward normal vector, and  $\mathbf{n} \cdot F(x) = F_{hn}(x) - F_{lo}(x) = \lambda_f n_f x_2 > 0$ .

We are ready to apply the Poincare-Hopf index theorem in Simsek, Ozdaglar, and Acemoglu (2007, p.194). The Euler characteristic of  $X(\theta)$  is 1; see their definition on p.193 for the case of non-empty and convex sets. Claim 1 follows immediately from the index theorem, which completes the proof for Part 1 of Proposition 1.

### A.3 Proof for Part 2 of Proposition 1

We first reduce the system  $P(\theta)$ . For any market  $\theta$  and an initial condition  $\mu(0)$ , any dynamic solution  $\mu : [0, \infty) \rightarrow \mathbb{R}^T$  of the system  $P(\theta)$  satisfies, for every  $t \in [0, \infty)$ ,

$$\begin{aligned}\mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) &= n_v (= 1), \\ \mu_{fn}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= n_f, \quad \text{and} \\ \mu_{ho}(t) + \mu_{lo}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= n_a.\end{aligned}$$

As such, without changing the set of dynamic solutions, we can reduce the system  $P(\theta)$  for  $x(t) \equiv (\mu_{ho}(t), \mu_{hn}(t), \mu_{lo}(t), \mu_{fo}(t))$  by

$$\dot{x} = F(x) \equiv (F_{ho}(x), F_{hn}(x), F_{lo}(x), F_{fo}(x)), \quad (14)$$

where

$$\begin{aligned}
F_{ho}(x) &\equiv (\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe}(x)) \mu_{hn} - \rho_d \mu_{ho} + \rho_u \mu_{lo}, \\
F_{hn}(x) &\equiv -(\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe}(x)) \mu_{hn} - \rho_d \mu_{hn} + \rho_u \mu_{ln}(x), \\
F_{lo}(x) &\equiv -(\lambda_d \mu_{hn} + \lambda_f \mu_{fn}(x)) \mu_{lo} - \rho_u \mu_{lo} + \rho_d \mu_{ho}, \\
F_{fo}(x) &\equiv (\lambda_f \mu_{lo} + \lambda_s \mu_{fe}(x)) \mu_{fn}(x) - \lambda_f \mu_{hn} \mu_{fo} - \rho_e \mu_{fo},
\end{aligned}$$

and

$$\begin{aligned}
\mu_{ln}(x) &\equiv 1 - \mu_{ho} - \mu_{hn} - \mu_{lo}, \\
\mu_{fe}(x) &\equiv n_a - \mu_{ho} - \mu_{lo} - \mu_{fo}, \\
\mu_{fn}(x) &\equiv n_f - \mu_{fo} - \mu_{fe}(x) = n_f - n_a + \mu_{ho} + \mu_{lo}.
\end{aligned} \tag{15}$$

The above reduction of the system  $P(\theta)$  does not change the set of dynamic solutions.<sup>33</sup> If  $\mu$  is a dynamic (either steady-state or not) solution of  $P(\theta)$ , then  $x \equiv (\mu_{ho}, \mu_{hn}, \mu_{lo}, \mu_{fo})$  solves  $F(x; \theta) = 0$ ; conversely, for any dynamic solution  $x$  of  $F(x; \theta) = 0$ , we can find a dynamic solution  $\mu$  of  $P(\theta)$ , from  $x$  and the induced  $\mu_{ln}$ ,  $\mu_{fe}$ , and  $\mu_{fn}$ . Therefore, a dynamic solution  $\mu$  of  $P(\theta)$  is asymptotically stable if and only if  $x \equiv (\mu_{ho}, \mu_{hn}, \mu_{lo}, \mu_{fo})$  is asymptotically stable.

A steady-state solution  $x$  of  $F(x; \theta) = 0$  is asymptotically stable if all eigenvalues of the Jacobian matrix of  $F(x; \theta)$  at the steady-state solution  $x$  have strictly negative real parts (Hirsch, 2012). The Jacobian matrix is

$$\begin{aligned}
\nabla F(x) &\equiv \left[ \frac{\partial F_i(x)}{\partial x_j} \right]_{i,j \in \{ho, hn, lo, fo\}} \\
&= \left[ \begin{array}{ccc|c}
-\lambda_f \mu_{hn} - \rho_d & \lambda_v \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe} & (\lambda_d - \lambda_f) \mu_{hn} + \rho_u & 0 \\
\lambda_f \mu_{hn} - \rho_u & -(\lambda_v \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe}) - \rho_d - \rho_u & (\lambda_f - \lambda_d) \mu_{hn} - \rho_u & 0 \\
-\lambda_f \mu_{lo} + \rho_d & -\lambda_d \mu_{lo} & -\lambda_f (\mu_{fn} + \mu_{lo}) - \lambda_d \mu_{hn} - \rho_u & 0 \\
\lambda_f \mu_{lo} + \lambda_s (\mu_{fe} - \mu_{fn}) & -\lambda_f \mu_{fo} & \lambda_f (\mu_{fn} + \mu_{lo}) + \lambda_s (\mu_{fe} - \mu_{fn}) & -\lambda_f \mu_{hn} - \lambda_s \mu_{fn} - \rho_e
\end{array} \right]
\end{aligned}$$

where we omit the dependency of  $\mu_{fn}$  and  $\mu_{fe}$  on  $x$  to simplify the expression.

<sup>33</sup>In the proof of Part 1 of Proposition 1, we reduced  $F(x; \theta) = 0$  further as a system of only two equations in Claim 1. The reduction requires  $\mu_{hn} + \mu_{ho} = n_h \equiv \frac{\rho_u}{\rho_u + \rho_d}$  and  $\mu_{lo} + \mu_{ln} = n_l \equiv \frac{\rho_d}{\rho_u + \rho_d}$ , which hold in a steady-state but may not hold on a path of  $\mu(t)$  after a perturbation.

Due to the block structure, one eigenvalue is  $-\lambda_f\mu_{hn} - \lambda_s\mu_{fn} - \rho_e < 0$ . The other eigenvalues are the eigenvalue of the  $3 \times 3$  sub-matrix with the first three rows and columns. A direct calculation shows that the other three eigenvalues are also strictly negative, which completes the proof.

## A.4 Proof of Proposition 2

First, we simplify expositions:

$$\begin{aligned} g_1 &\equiv g_{fo-hn} = (1/2)(v_{ho} + v_{fn} - v_{fo} - v_{hn}), \\ g_2 &\equiv g_{lo-fn} = (1/2)(v_{fo} + v_{ln} - v_{lo} - v_{fn}), \\ g_3 &\equiv g_{fe-fn} = (1/2)(v_{fo} + v_{fn} - v_{fe} - v_{fn}) = (1/2)(v_{fo} - v_{fe}), \end{aligned}$$

so that

$$\begin{aligned} g_{lo-hn} &= (1/2)(v_{ho} + v_{ln} - v_{lo} - v_{hn}) = g_2 + g_1 \quad \text{and} \\ g_{fe-hn} &= (1/2)(v_{ho} + v_{fn} - v_{fe} - v_{hn}) = g_1 + g_3. \end{aligned}$$

The matrix representations of the value equations (v-hn)-(v-fe) are:

$$\begin{aligned} \begin{bmatrix} v_{ho} \\ v_{lo} \end{bmatrix} &= \begin{bmatrix} r + \rho_d & -\rho_d \\ -\rho_u & r + \rho_u \end{bmatrix}^{-1} \begin{bmatrix} u_h \\ u_l + \lambda_d\mu_{hn}(g_1 + g_2) + \lambda_f\mu_{fn}g_2 \end{bmatrix}, \\ \begin{bmatrix} v_{hn} \\ v_{ln} \end{bmatrix} &= \begin{bmatrix} r + \rho_d & -\rho_d \\ -\rho_u & r + \rho_u \end{bmatrix}^{-1} \begin{bmatrix} \lambda_d\mu_{lo}(g_1 + g_2) + \lambda_f\mu_{fo}g_1 + \lambda_f\mu_{fe}(g_1 + g_3) \\ 0 \end{bmatrix}, \quad \text{and} \\ \begin{bmatrix} v_{fn} \\ v_{fo} \\ v_{fe} \end{bmatrix} &= \frac{1}{r} \begin{bmatrix} \lambda_f\mu_{lo}g_2 + \lambda_s\mu_{fe}g_3 \\ u_f + \lambda_f\mu_{hn}g_1 - 2\rho_eg_3 \\ u_e + \lambda_f\mu_{hn}(g_1 + g_3) + \lambda_s\mu_{fn}g_3 \end{bmatrix}, \end{aligned} \tag{16}$$

where the inverse matrix is well-defined: i.e.,  $(r + \rho_d)(r + \rho_u) - \rho_d\rho_u > 0$ . As in the case of  $n_f = 0$ , we compute the gains  $g_1, g_2$ , and  $g_3$ . Then, the solution  $v$  will be uniquely determined by the above matrix equations.

First,

$$2rg_3 = r(v_{fo} - v_{fe}) = (u_f - u_e) - 2\rho_e g_3 - \lambda_f \mu_{hn} g_3 - \lambda_s \mu_{fn} g_3,$$

which results in

$$g_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}} > 0. \quad (17)$$

Next,

$$\begin{aligned} 2(g_1 + g_2) &= v_{ho} + v_{ln} - v_{lo} - v_{hn} = (1, -1) \cdot (v_{ho} - v_{hn}, v_{lo} - v_{ln}) \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} r + \rho_d & -\rho_d \\ -\rho_u & r + \rho_u \end{bmatrix}^{-1} \begin{bmatrix} u_h - \lambda_d \mu_{lo}(g_1 + g_2) - \lambda_f \mu_{fo} g_1 - \lambda_f \mu_{fe}(g_1 + g_3) \\ u_l + \lambda_d \mu_{hn}(g_1 + g_2) + \lambda_f \mu_{fn} g_2 \end{bmatrix}. \end{aligned}$$

Since

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} r + \rho_d & -\rho_d \\ -\rho_u & r + \rho_u \end{bmatrix}^{-1} = \frac{1}{r + \rho_u + \rho_d} \begin{bmatrix} 1 & -1 \end{bmatrix},$$

we have

$$\begin{aligned} (2(r + \rho_u + \rho_d) + \lambda_d(\mu_{lo} + \mu_{hn}))(g_1 + g_2) + \lambda_f(\mu_{fo} + \mu_{fe})g_1 + \lambda_f \mu_{fn} g_2 \\ = (u_h - u_l) - \lambda_f \mu_{fe} g_3. \end{aligned} \quad (18)$$

On the other hand, by (v-lo), (v-ln), (v-fn), and (v-fo),

$$\begin{aligned} 2rg_2 &= r(v_{fo} - v_{fn}) - r(v_{lo} - v_{ln}) \\ &= (u_f + \lambda_f \mu_{hn} g_1 - 2\rho_e g_3 - \lambda_f \mu_{lo} g_2 - \lambda_s \mu_{fe} g_3) \\ &\quad - (u_l + \lambda_d \mu_{hn}(g_1 + g_2) + \lambda_f \mu_{fn} g_2) + \rho_u(v_{ho} - v_{lo} + v_{ln} - v_{hn}). \end{aligned}$$



As  $v_{ho} - v_{lo} + v_{ln} - v_{hn} = 2(g_1 + g_2)$ ,

$$\begin{aligned} (2\rho_u + \lambda_d\mu_{hn})(g_1 + g_2) - \lambda_f\mu_{hn}g_1 + (2r + \lambda_f\mu_{lo} + \lambda_f\mu_{fn})g_2 \\ = (u_f - u_l) - (2\rho_e + \lambda_s\mu_{fe})g_3. \end{aligned} \quad (19)$$

The linear system of equations (18) and (19) is summarized as follows:

$$\begin{bmatrix} c_1 + \lambda_f(\mu_{fo} + \mu_{fe}) & c_1 + \lambda_f\mu_{fn} \\ c_2 - \lambda_f\mu_{hn} & c_2 + 2r + \lambda_f(\mu_{lo} + \mu_{fn}) \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} u_h - u_l - \lambda_f\mu_{fe}g_3 \\ u_f - u_l - 2\rho_e g_3 - \lambda_s\mu_{fe}g_3 \end{bmatrix} \quad (20)$$

where  $c_1 \equiv 2(r + \rho_u + \rho_d) + \lambda_d(\mu_{lo} + \mu_{hn}) > 0$  and  $c_2 \equiv 2\rho_u + \lambda_d\mu_{hn} > 0$ .

The determinant of the coefficient matrix is bounded below by

$$2rc_1 + \lambda_f\mu_{fn}(c_1 - c_2) > 4r^2 + \lambda_f\mu_{fn}(2r + 2\rho_d + \lambda_d\mu_{lo}) > 0.$$

Thus, the above linear system has a unique solution  $(g_1, g_2)$ . This solution, together with  $g_3$ , determined the unique solution  $v$  of  $V(\theta)$ . ■

## A.5 Condition for positive trade gains

The gains from trade are all positive if and only if

$$\begin{aligned} g_1 \equiv g_{fo-hn} \geq 0 & \iff (c_2 + 2r + \lambda_f(\mu_{lo} + \mu_{fn}))((u_h - u_l) - \lambda_f\mu_{fe}g_3) \\ & - (c_1 + \lambda_f\mu_{fn})((u_f - u_l) - (2\rho_e + \lambda_s\mu_{fe})g_3) \geq 0. \end{aligned} \quad (21)$$

$$\begin{aligned} g_2 \equiv g_{lo-fn} \geq 0 & \iff - (c_2 - \lambda_f\mu_{hn})((u_h - u_l) - \lambda_f\mu_{fe}g_3) \\ & + (c_1 + \lambda_f(\mu_{fo} + \mu_{fe}))((u_f - u_l) - (2\rho_e + \lambda_s\mu_{fe})g_3) \geq 0, \end{aligned} \quad (22)$$

Note that both expressions depends on the steady-state population measure  $\mu$ .

## A.6 Proof of Lemma 2

For Part 1:

$$\begin{aligned} 2(p_{fo-hn} - p_{fe-hn}) &= v_{fo} - v_{fe} = 2g_{fe-fn} \geq 0, \\ 2(p_{fe-hn} - p_{fe-fn}) &= (v_{ho} - v_{hn}) - (v_{fo} - v_{fn}) = 2g_{fo-hn} \geq 0. \end{aligned}$$

For Part 2:

$$\begin{aligned} 2(p_{fo-hn} - p_{lo-hn}) &= (v_{ho} - v_{hn} + v_{fo} - v_{fn}) - (v_{lo} - v_{ln} + v_{fo} - v_{fn}) \\ &= (v_{ho} - v_{hn}) - (v_{lo} - v_{ln}) = 2g_{lo-hn} \geq 0, \\ 2(p_{lo-hn} - p_{lo-fn}) &= (v_{ho} - v_{hn}) - (v_{fo} - v_{fn}) = 2g_{fo-hn} \geq 0. \end{aligned}$$

## A.7 Proof of Lemma 3

First, from (v-hn)-(v-ln),

$$\begin{aligned} rW_v &\equiv r(\mu_{ho}v_{ho} + \mu_{hn}v_{hn} + \mu_{lo}v_{lo} + \mu_{ln}v_{ln}) \\ &= \mu_{ho}(u_h + \rho_d(v_{lo} - v_{ho})) + \mu_{hn}(\lambda_d\mu_{lo}g_{lo-hn} + \lambda_f\mu_{fo}g_{fo-hn} + \lambda_f\mu_{fe}g_{fe-hn} + \rho_d(v_{ln} - v_{hn})) \\ &\quad + \mu_{lo}(u_l + \lambda_d\mu_{hn}g_{lo-hn} + \lambda_f\mu_{fn}g_{lo-fn} + \rho_u(v_{ho} - v_{lo})) + \mu_{ln}\rho_u(v_{hn} - v_{ln}). \end{aligned}$$

We rewrite the above expression in terms of the investors' expected values and their payments to or received from funds. That is, we substitute  $g_{fo-hn} = v_{ho} - v_{hn} - p_{fo-hn}$ ,  $g_{fe-hn} = v_{ho} - v_{hn} - p_{fe-hn}$ ,  $g_{lo-fn} = p_{lo-fn} - v_{lo} - v_{ln}$ , and  $g_{lo-hn} = (1/2)(v_{ho} + v_{ln} - v_{lo} + v_{hn})$ . Then, (3) follows from

$$\begin{aligned} rW_v &- (\mu_{ho}u_h + \mu_{lo}u_l + \lambda_f\mu_{lo}\mu_{fn}p_{lo-fn} - \lambda_f\mu_{hn}(\mu_{fo}p_{fo-hn} + \mu_{fe}p_{fe-hn})) \\ &= (\rho_u\mu_{lo} - \rho_d\mu_{ho})(v_{ho} - v_{lo}) + (\rho_u\mu_{ln} - \rho_u\mu_{hn})(v_{hn} - v_{ln}) \\ &\quad + \mu_{hn}(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})(v_{ho} - v_{hn}) + \mu_{lo}(\lambda_d\mu_{hn} + \lambda_f\mu_{fn})(v_{ln} - v_{lo}). \end{aligned}$$

The combined coefficient of  $v_{ho}$  on the right-hand side of the above equation is  $-\rho_d\mu_{ho} + \rho_u\mu_{lo} + \mu_{hn}(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})$ , which equals the right-hand side of the population

equation ( $\mu$ -ho), so it is zero. We can similarly verify that the combined coefficient of  $v_i$  for  $i = hn, lo, ln$  are all equal to zero.

Second, we obtain (2) from all population equations ( $\mu$ -hn)-( $\mu$ -fe) such that

$$\begin{aligned}
& rW - (\mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l) \\
&= (\rho_u\mu_{lo} - \rho_d\mu_{ho})(v_{ho} - v_{lo}) + (\rho_u\mu_{ln} - \rho_d\mu_{hn})(v_{hn} - v_{ln}) + \rho_e\mu_{fo}(v_{fe} - v_{fo}) \\
&+ \mu_{hn}(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})(v_{ho} - v_{hn}) + \mu_{lo}(\lambda_d\mu_{hn} + \lambda_f\mu_{fn})(v_{ln} - v_{lo}) \\
&+ ((\lambda_f\mu_{lo} + \lambda_s\mu_{fe})\mu_{fn} - \lambda_f\mu_{hn}\mu_{fo})v_{fo} \\
&+ \lambda_f(\mu_{hn}\mu_{fo} + \mu_{hn}\mu_{fe} - \mu_{lo}\mu_{fn})v_{fn} - (\lambda_f\mu_{hn} + \lambda_s\mu_{fn})\mu_{fe}v_{fe}.
\end{aligned}$$

As before, we can verify that the combined coefficients of  $v_i$  for each  $i \in \mathcal{T}$  equals the right-hand side of the type's population equation, so it is zero.

Lastly, the expression for  $W_f$  follows from  $W_f = W - W_v$ .

## A.8 Proof of Lemma 4

First, consider the path of a  $lo$ -type investor in a steady-state equilibrium. This investor can sell its asset upon meeting either a buying investor ( $hn$ ) or a fund buyer ( $fn$ ). Each kind of meeting arrives with Poisson rate  $\lambda_c\mu_{hn}$  or  $\lambda_f\mu_{fn}$ . The time until the first meeting of each kind, denoted by  $\tau_{lo-hn}$  and  $\tau_{lo-fn}$ , follows the exponential distributions. Thus, the time until selling  $\tau_{sc} \equiv \min\{\tau_{lo-hn}, \tau_{lo-fn}\}$  follows an exponential distribution with parameter  $\lambda_c\mu_{hn} + \lambda_f\mu_{fn}$ :

$$E[\tau_{sc}] = \frac{1}{\lambda_c\mu_{hn} + \lambda_f\mu_{fn}}.$$

Second, consider the path of a  $fo$ -type fund in a steady-state equilibrium. The fund sells its asset before receiving a liquidity shock to a buying investor ( $hn$ ), or receives a liquidity shock and enters the exit phase (after which it can sell to either a buying investor ( $hn$ ) or a fund buyer ( $fn$ )). We denote by  $\tau_{fo}$  this period for which a fund maintain its type as  $fo$ . The time  $\tau_{fo}$  follows an exponential distribution with parameter  $\lambda_f\mu_{hn} + \rho_e$ :

$$E[\tau_{fo}] = \frac{1}{\lambda_f\mu_{hn} + \rho_e}.$$

Finally, we evaluate the path of an  $fe$  type fund (an outcome of an  $fo$  type fund receiving a liquidity shock before meeting a buying investor with probability  $\frac{\rho_e}{\lambda_f \mu_{hn} + \rho_e}$ ). The  $fe$  type fund maintains its type until it sells its portfolio asset either to a buying investor ( $hn$ ) or a fund buyer ( $fn$ ). Thus, the fund maintains its type for the time period  $\tau_{fe}$ , which follows an exponential distribution with parameter  $\lambda_f \mu_{hn} + \lambda_s \mu_{fn}$ :

$$E[\tau_{fe}] = \frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}}.$$

As a result, the overall expected time for a fund to sell an asset is:

$$E[\tau_{sf}] = \frac{1}{\lambda_f \mu_{hn} + \rho_e} + \frac{\rho_e}{\lambda_f \mu_{hn} + \rho_e} \left( \frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}} \right).$$

## B Proofs for Section 4

### B.1 Proof of Proposition 3

#### B.1.1 Part 1

Let  $(\mu(\theta), v(\theta))$  be the unique steady-state solution of population and value for each market  $\theta$ . We compute the comparative static derivatives with respect to  $\lambda_s$ . It is intuitive that the unique steady-state measure of each investor type  $(\mu_i)_{i \in \mathcal{T}_v}$  and the measure  $\mu_{fn}$  are independent of  $\lambda_s$ . Through a secondary trade, one fund changes its type from  $fe$  to  $fn$ , replacing another fund of type changed from  $fn$  to  $fo$ .

To confirm the intuition, from the proof of Proposition 1, take the unique steady-state solution  $x(\theta) \equiv (\mu_{lo}(\theta), \mu_{hn}(\theta))$  of  $F(x) \equiv (F_{lo}(x), F_{hn}(x)) = 0$ , where

$$\begin{aligned} F_{lo}(x) &\equiv (\lambda_d x_2 + \lambda_f (n_f - g_{fn}(x)))x_1 + \rho_u x_1 - \rho_d (n_h - x_2), \\ F_{hn}(x) &\equiv (\lambda_d x_1 + \lambda_f g_{fn}(x))x_2 + \rho_d x_2 - \rho_u (n_l - x_1). \end{aligned}$$

By Implicit function theorem,  $x(\theta)$  is differentiable in  $\lambda_s$ , and

$$\frac{\partial x(\theta)}{\partial \lambda_s} = -[\nabla_x F(x(\theta); \theta)]^{-1} \frac{\partial F(x(\theta); \theta)}{\partial \lambda_s}.$$

We denoted the domain of  $F(x)$  by  $X(\theta)$  in the proof of Claim 1. The unique solution  $x(\theta)$  of  $F(x; \theta) = 0$  is an interior point of  $X(\theta)$ , as shown in Lemma 1 for the case of  $n_f > 0$  and in the proof of Part 1 of Proposition 1 for the case of  $n_f = 0$ . As a result, we have shown in the proof of Claim 1, the Jacobian matrix  $\nabla F(x)$  at the unique solution  $x(\theta)$  is invertible.

It follows from

$$\frac{\partial F(x(\theta); \theta)}{\partial \lambda_s} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

that

$$\frac{\partial x(\theta)}{\partial \lambda_s} = \begin{bmatrix} \partial \mu_{lo}(\theta) / \partial \lambda_s \\ \partial \mu_{hn}(\theta) / \partial \lambda_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (23)$$

For other types, it follows from  $\mu_{ho}(\theta) + \mu_{hn}(\theta) = n_h$  and  $\mu_{lo}(\theta) + \mu_{ln}(\theta) = n_l$  that  $\frac{\partial \mu_{ln}(\theta)}{\partial \lambda_s} = \frac{\partial \mu_{ho}(\theta)}{\partial \lambda_s} = 0$ , and from  $\mu_{ho}(\theta) + \mu_{lo}(\theta) + (n_f - \mu_{fn}(\theta)) = n_f$  that  $\frac{\partial \mu_{fn}(\theta)}{\partial \lambda_s} = 0$ . Lastly, from (12) and  $\mu_{fo} + \mu_{fe} + \mu_{fn} = n_f$ ,

$$\frac{\partial \mu_{fe}(\theta)}{\partial \lambda_s} = \frac{-\rho_e(n_f - \mu_{fn}(\theta))\mu_{fn}(\theta)}{(\rho_e + \lambda_f \mu_{hn}(\theta) + \lambda_s \mu_{fn}(\theta))^2} = -\frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s}. \quad (24)$$

From the definition of  $g_3$  on p. 38 and Equation 17,

$$v_{fo} - v_{fe} = 2g_{fe-fn} = 2g_3 = \frac{2(u_f - u_e)}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}}.$$

The comparative static derivatives show that  $\mu_{hn}$  and  $\mu_{fn}$  are independent of  $\lambda_s$ . Thus,  $\frac{\partial (v_{fo} - v_{fe})}{\partial \lambda_s} < 0$  and  $\lim_{\lambda_s \rightarrow \infty} (v_{fo} - v_{fe}) = 0$ .

### B.1.2 Part 2

The above comparative static derivatives with respect to  $\lambda_s$  show that  $\frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s} > 0$ ,  $\frac{\partial \mu_{fe}(\theta)}{\partial \lambda_s} < 0$ , and  $\frac{\partial \mu_i(\theta)}{\partial \lambda_s} = 0$ , for all  $i \neq fo, fe$ . Thus,

$$\begin{aligned} r \frac{\partial W(\theta)}{\partial \lambda_s} &= \frac{\partial \mu_{ho}(\theta)}{\partial \lambda_s} u_h + \frac{\partial \mu_{lo}(\theta)}{\partial \lambda_s} u_l + \frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s} u_f + \frac{\partial \mu_{fe}(\theta)}{\partial \lambda_s} u_e \\ &= \frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s} (u_f - u_e) + \frac{\partial (\mu_{fo} + \mu_{fe})(\theta)}{\partial \lambda_s} u_e \\ &= \frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s} (u_f - u_e) - \frac{\partial \mu_{fn}(\theta)}{\partial \lambda_s} u_e \\ &= \frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s} (u_f - u_e) > 0. \end{aligned}$$

## B.2 Proof of Proposition 4 and Proposition 5

Recall from Claim 1 that the steady-state population is determined by a solution of  $F(x; n_f) \equiv (F_{lo}(x; n_f), F_{hn}(x; n_f)) = 0$  where

$$\begin{aligned} F_{lo}(x; n_f) &\equiv (\lambda_d x_2 + \lambda_f (n_f - g_{fn}(x))) x_1 + \rho_u x_1 - \rho_d (n_h - x_2), \\ F_{hn}(x; n_f) &\equiv (\lambda_d x_1 + \lambda_f g_{fn}(x)) x_2 + \rho_d x_2 - \rho_u (n_l - x_1), \end{aligned}$$

and  $g_{fn}(x) \equiv n_a - n_h + x_2 - x_1$ . We extend the system  $F(x; n_f) = 0$  such that  $n_f$  can be any real number and  $x$  can be any real vector of length 2. Each solution  $x = (x_1, x_2)$  defines a vector  $\mu = (\mu_i)_{i \in \mathcal{T}}$  as  $(\mu_{lo}, \mu_{hn}, \mu_{ln}, \mu_{ho}) = (x_1, x_2, n_l - x_1, n_h - x_2)$  and  $(\mu_{fn}, \mu_{fo}, \mu_{fe})$  by (11) and (12). According to Claim 1, if  $n_f > 0$ , a solution exists in certain domain (denoted by  $X(\theta)$  in the claim) such that the resulting vector  $\mu$  is a steady-state population. In general, without any restrictions on  $n_f$ , the vector  $\mu$  may not even be positive.

The proof consists of three steps. First, for  $n_f = 0$ , we find a population measure  $\hat{\mu}$  such that  $\hat{x} \equiv (\hat{\mu}_{lo}, \hat{\mu}_{hn})$  solves  $F(x; n_f) = 0$ . Second, by Implicit Function Theorem, we differentiate a solution function  $x(n_f)$  defined in the neighborhood of  $n_f = 0$  and  $x = \hat{x}$ , and obtain the comparative static derivative  $\mu'_i \equiv \frac{\partial \mu_i}{\partial n_f} \Big|_{n_f=0}$  for each  $i \in \mathcal{T}$ . Last, we prove the following claim:

**Claim 2.** *There exist  $\beta_1 > 0$  and  $\beta_2$ , each being independent of  $\lambda_s$ , such that*

$$\left. \frac{\partial v_{fn}}{\partial n_f} \right|_{n_f=0} = \beta_1 \lambda_s + \beta_2.$$

Then, Proposition 4 and Proposition 5 follow immediately.

### B.2.1 A benchmark model ( $n_f = 0$ )

We set  $\hat{\mu}_i = 0$  for every fund type  $i \in \mathcal{T}_f$ , and impose

$$\begin{aligned} \hat{\mu}_{ho} &= n_h - \hat{\mu}_{hn}, & \hat{\mu}_{lo} &= n_a - \hat{\mu}_{ho} = n_a - n_h + \hat{\mu}_{hn}, & \text{and} \\ \hat{\mu}_{ln} &= n_l - \hat{\mu}_{lo} = n_v - n_a - \hat{\mu}_{hn}. \end{aligned}$$

By substituting the above expressions of  $\hat{\mu}_{lo}$  and  $\hat{\mu}_{ln}$  in

$$\lambda_d \hat{\mu}_{lo} \hat{\mu}_{hn} + \rho_d \hat{\mu}_{hn} - \rho_u \hat{\mu}_{ln} = 0, \quad (\mu\text{-hn})$$

we obtain

$$\hat{\mu}_{hn} = \frac{1}{2} \left( \sqrt{(R + n_a - n_h)^2 + 4R \cdot n_h (1 - n_a)} - (R + n_a - n_h) \right), \quad (25)$$

where  $R \equiv \frac{\rho_u + \rho_d}{\lambda_d}$ . It is clear that  $\hat{x} = (\hat{\mu}_{lo}, \hat{\mu}_{hn})$  solves the system  $F(x; n_f) = 0$ .

### B.2.2 Comparative static derivatives of $\mu$ with respect to $n_f$

We apply Implicit Function Theorem.  $F(x; n_f)$  is an infinitely differentiable function of  $x \in \mathbb{R}^2$  and  $n_f \in \mathbb{R}$ , and the Jacobian matrix  $\nabla_x F(\hat{x}; 0)$  is invertible (see Equation 13). As such, there is a differentiable function  $x(n_f)$  defined in a neighborhood of  $n_f = 0$  and  $x = \hat{x}$  such that  $F(x(n_f); n_f) = 0$ . It is important to note that the derivative of  $x(n_f)$  at any  $n_f > 0$  is independent of the choice of the function  $x(n_f)$ ; Claim 1 ensures that any choice of function  $x(n_f)$  gives the same value of  $x$  for each  $n_f > 0$ .

As explained above, the function  $x(n_f)$ , together with (11) and (12), defines  $\mu(n_f) =$

$(\mu_i(n_f))_{i \in \mathcal{T}}$ , which is also differentiable. Let  $\mu'_i \equiv \frac{\partial \mu}{\partial n_f} \Big|_{n_f=0}$  for each  $i \in \mathcal{T}$ . Then,

$$\begin{aligned} \begin{bmatrix} \mu'_{lo} \\ \mu'_{hn} \end{bmatrix} &= -[\nabla_x F(\hat{x}; 0)]^{-1} \frac{\partial F(\hat{x}; 0)}{\partial n_f} \\ &= - \begin{bmatrix} \lambda_d \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo} + \rho_u & (\lambda_d - \lambda_f) \hat{\mu}_{lo} + \rho_d \\ (\lambda_d - \lambda_f) \hat{\mu}_{hn} + \rho_u & \lambda_d \hat{\mu}_{lo} + \lambda_f \hat{\mu}_{hn} + \rho_d \end{bmatrix}^{-1} \begin{bmatrix} \lambda_f \hat{\mu}_{lo} \\ 0 \end{bmatrix}. \end{aligned} \quad (26)$$

Also,  $\mu'_{ho} = -\mu'_{hn}$  and  $\mu'_{ln} = -\mu'_{lo}$ .

From (11),

$$\begin{aligned} \mu'_{fn} &= 1 - \mu'_{hn} + \mu'_{lo} \\ &= 1 - \frac{1}{\det(\nabla_x F(\hat{x}; 0))} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_d \hat{\mu}_{lo} + \lambda_f \hat{\mu}_{hn} + \rho_d & * \\ -(\lambda_d - \lambda_f) \hat{\mu}_{hn} - \rho_u & * \end{bmatrix} \begin{bmatrix} \lambda_f \hat{\mu}_{lo} \\ 0 \end{bmatrix} \\ &= 1 - \frac{\lambda_f \hat{\mu}_{lo} (\lambda_d \hat{\mu}_{lo} + \lambda_d \hat{\mu}_{hn} + \rho_d + \rho_u)}{\det(\nabla_x F(\hat{x}; 0))}, \end{aligned}$$

where

$$\begin{aligned} \det(\nabla_x F(\hat{x}; 0)) &= (\lambda_d \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo} + \rho_u) (\lambda_d \hat{\mu}_{lo} + \lambda_f \hat{\mu}_{hn} + \rho_d) \\ &\quad - ((\lambda_d - \lambda_f) \hat{\mu}_{hn} + \rho_u) ((\lambda_d - \lambda_f) \hat{\mu}_{lo} + \rho_d) \\ &= (\lambda_d \hat{\mu}_{hn} + \rho_u) (\lambda_f \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo}) + (\lambda_f \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo}) (\lambda_d \hat{\mu}_{lo} + \rho_d) \\ &= (\lambda_f \hat{\mu}_{hn} + \lambda_f \hat{\mu}_{lo}) (\lambda_d \hat{\mu}_{lo} + \lambda_d \hat{\mu}_{hn} + \rho_d + \rho_u). \end{aligned}$$

It follows that

$$\mu'_{fn} = \frac{\hat{\mu}_{hn}}{\hat{\mu}_{hn} + \hat{\mu}_{lo}} > 0. \quad (27)$$

Next, from (12),

$$\mu_{fo} = \frac{(\lambda_f \mu_{hn} + \lambda_s \mu_{fn})(n_f - \mu_{fn})}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + \rho_e}.$$



Thus,

$$\begin{aligned}\mu'_{fo} &= \frac{(1 - \mu'_{fn})(\lambda_f \hat{\mu}_{hn})}{\rho_e + \lambda_f \hat{\mu}_{hn}} = \frac{\lambda_f \hat{\mu}_{hn} \hat{\mu}_{lo}}{(\rho_e + \lambda_f \hat{\mu}_{hn})(\hat{\mu}_{hn} + \hat{\mu}_{lo})} > 0, \\ \mu'_{fe} &= 1 - \mu'_{fn} - \mu'_{fo} = \frac{\rho_e \hat{\mu}_{lo}}{(\rho_e + \lambda_f \hat{\mu}_{hn})(\hat{\mu}_{hn} + \hat{\mu}_{lo})} > 0.\end{aligned}\tag{28}$$

### B.2.3 Proof of Claim 2

From (16),

$$rv_{fn} = \lambda_f \mu_{lo} g_2 + \lambda_s \mu_{fe} g_3,$$

where  $g_2$  and  $g_3$  are determined by (17) and (20), respectively.

Let  $v'_{fn} \equiv \frac{\partial v_{fn}}{\partial n_f} \Big|_{n_f=0}$  and  $g'_m \equiv \frac{\partial g_m}{\partial n_f} \Big|_{n_f=0}$  for  $m = 2, 3$ . Then

$$rv'_{fn} = \lambda_f (\hat{g}_2 \mu'_{lo} + \hat{\mu}_{lo} g'_2) + \lambda_s (\hat{g}_3 \mu'_{fe} + \hat{\mu}_{fe} g'_3).$$

We find the value of each variable on the right-hand side of the above equation. For certain variables that we will use later, we remark whether the values are strictly positive and/or independent of  $\lambda_s$ .

We have observed the following properties:

1. (from (25)) the population  $\hat{\mu} = (\mu_i)_{i \in \mathcal{T}}$  is strictly positive for corporate types, zero for fund types, and independent of  $\lambda_s$ ,
2. (from (26), (27), and (28)) the derivative  $\mu'$  is independent of  $\lambda_s$ ,
3. (from (17)) As  $g_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}}$ , we have  $\hat{g}_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}} > 0$  and  $g'_3 = -\frac{(\lambda_f \mu'_{hn} + \lambda_s \mu'_{fn}) \hat{g}_3}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}}$ , which are independent of  $\lambda_s$ .

It remains to find the values of  $\hat{g}_2$  and  $g'_2$ .

To state how  $g_2$  is determined by (20), let  $c_1 \equiv 2(r + \rho_u + \rho_d) + \lambda_d(\mu_{lo} + \mu_{hn})$ ,  $c_2 \equiv 2\rho_u + \lambda_d \mu_{hn}$ , and

$$D \equiv \begin{bmatrix} c_1 + \lambda_f(\mu_{fo} + \mu_{fe}) & c_1 + \lambda_f \mu_{fn} \\ c_2 - \lambda_f \mu_{hn} & c_2 + 2r + \lambda_f(\mu_{lo} + \mu_{fn}) \end{bmatrix}.$$

Also, let  $\alpha_1 \equiv \frac{-D_{21}}{\det(\hat{D})}$  and  $\alpha_2 \equiv \frac{D_{11}}{\det(\hat{D})}$ . Then,

$$g_2 = \alpha_1 (u_h - u_l - \lambda_f \mu_{fe} g_3) + \alpha_2 (u_f - u_l - 2\rho_e g_3 - \lambda_s \mu_{fe} g_3).$$

Note that, when  $n_f = 0$ ,

$$\begin{aligned} \hat{c}_1 &= 2(r + \rho_u + \rho_d) + \lambda_d(\hat{\mu}_{lo} + \hat{\mu}_{hn}) > 0, \\ \hat{c}_2 &= 2\rho_u + \lambda_d \hat{\mu}_{hn} > 0, \\ \hat{D} &= \begin{bmatrix} \hat{c}_1 & \hat{c}_1 \\ \hat{c}_2 - \lambda_f \hat{\mu}_{hn} & \hat{c}_2 + 2r + \lambda_f \hat{\mu}_{lo} \end{bmatrix} \quad (\text{with a strictly positive determinant}), \\ \hat{\alpha}_1 &= \frac{-\hat{c}_2 + \lambda_f \hat{\mu}_{hn}}{\det(\hat{D})} \quad (\text{the exact value is unnecessary for our proof}), \quad \text{and} \\ \hat{\alpha}_2 &= \frac{\hat{c}_1}{\det(\hat{D})} = \frac{1}{2r + \lambda_f(\hat{\mu}_{lo} + \hat{\mu}_{hn})} > 0, \end{aligned} \tag{29}$$

which are all independent of  $\lambda_s$ . It follows that

$$\hat{g}_2 = \hat{\alpha}_1(u_h - u_l) + \hat{\alpha}_2(u_f - u_l - 2\rho_e \hat{g}_3)$$

is independent of  $\lambda_s$ .

Last, let  $c'_1$ ,  $c'_2$ ,  $\alpha'_1$ , and  $\alpha'_2$  be the corresponding variables' derivatives: e.g.,  $c'_1 \equiv \frac{\partial c_1}{\partial n_f} \Big|_{n_f=0}$ . The derivatives are all independent of  $\lambda_s$ , because  $\hat{\mu}$  and  $\mu'$  are independent of  $\lambda_s$ . Therefore,

$$\begin{aligned} g'_2 &= \alpha'_1(u_h - u_l - \lambda_f \hat{\mu}_{fe} \hat{g}_3) - \hat{\alpha}_1 \lambda_f (\mu'_{fe} \hat{g}_3 + \hat{\mu}_{fe} g'_3) \\ &\quad + \alpha'_2(u_f - u_l - 2\rho_e \hat{g}_3 - \lambda_s \hat{\mu}_{fe} \hat{g}_3) - \hat{\alpha}_2(2\rho_e g'_3 + \lambda_s \mu'_{fe} \hat{g}_3 + \lambda_s \hat{\mu}_{fe} g'_3) \\ &= \alpha'_1(u_h - u_l) - \hat{\alpha}_1 \lambda_f \mu'_{fe} \hat{g}_3 + \alpha'_2(u_f - u_l - 2\rho_e \hat{g}_3) - \hat{\alpha}_2(2\rho_e g'_3 + \lambda_s \mu'_{fe} \hat{g}_3). \quad (\text{as } \hat{\mu}_{fe} = 0) \end{aligned}$$

Only the last term  $-\hat{\alpha}_2(2\rho_e g'_3 + \lambda_s \mu'_{fe} \hat{g}_3)$  is (affinely) dependent on  $\lambda_s$ , through  $-\hat{\alpha}_2 \mu'_{fe} \hat{g}_3 \lambda_s$  and  $g'_3 = -\frac{(\lambda_f \mu'_{hn} + \lambda_s \mu'_{fn}) \hat{g}_3}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}}$ . As such,  $g'_2 = \gamma_1 \lambda_s + \gamma_2$ , for  $\gamma_1 = \hat{\alpha}_2 \hat{g}_3 \left( \frac{2\rho_e \mu'_{fn}}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}} - \mu'_{fe} \right)$  and some  $\gamma_2$  which aggregates all remaining terms. Both  $\gamma_1$  and  $\gamma_2$  are independent of  $\lambda_s$ .

Finally,

$$\begin{aligned}
rv'_{fn} &= \lambda_f (\hat{g}_2 \mu'_{lo} + \hat{\mu}_{lo} g'_2) + \lambda_s (\hat{g}_3 \mu'_{fe} + \hat{\mu}_{fe} g'_3) \\
&= \lambda_f (\hat{g}_2 \mu'_{lo} + \hat{\mu}_{lo} (\gamma_1 \lambda_s + \gamma_2)) + \lambda_s \hat{g}_3 \mu'_{fe} \quad (\text{as } \hat{\mu}_{fe} = 0) \\
&= (\lambda_f \hat{\mu}_{lo} \gamma_1 + \hat{g}_3 \mu'_{fe}) \lambda_s + (\lambda_f \hat{g}_2 \mu'_{lo} + \lambda_f \hat{\mu}_{lo} \gamma_2),
\end{aligned}$$

where the coefficient of  $\lambda_s$  and the last term are both independent of  $\lambda_s$ .

It remains to show that the coefficient of  $\lambda_s$  is strictly positive:

$$\begin{aligned}
\lambda_f \hat{\mu}_{lo} \gamma_1 + \hat{g}_3 \mu'_{fe} &= \lambda_f \hat{\mu}_{lo} \hat{\alpha}_2 \hat{g}_3 \left( \frac{2\rho_e \mu'_{fn}}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}} - \mu'_{fe} \right) + \hat{g}_3 \mu'_{fe} \\
&> -\lambda_f \hat{\mu}_{lo} \hat{\alpha}_2 \hat{g}_3 \mu'_{fe} + \hat{g}_3 \mu'_{fe} \quad (\text{as } \hat{\mu}_{lo}, \hat{\alpha}_2, \hat{g}_3, \mu'_{fn}, \hat{\mu}_{hn} \text{ are strictly positive}) \\
&= \mu'_{fe} \hat{g}_3 (1 - \lambda_f \hat{\alpha}_2 \hat{\mu}_{lo}) \\
&= \mu'_{fe} \hat{g}_3 \left( 1 - \frac{\lambda_f \hat{\mu}_{lo}}{2r + \lambda_f (\hat{\mu}_{lo} + \hat{\mu}_{hn})} \right) \quad (\text{from (29)}) \\
&> 0. \quad (\text{as } \mu'_{fe} \text{ and } \hat{g}_3 \text{ are strictly positive})
\end{aligned}$$

## C Proofs for Section 5

For any regular environment  $\theta \equiv (n, r, u, \rho, \lambda)$ , we consider a sequence  $\theta^\kappa \equiv (n, r, u, \rho, \kappa\lambda)$  with  $\kappa \rightarrow \infty$ . Let  $\mu^\kappa$  be the unique steady-state solution of  $P(\theta^\kappa)$  and  $v^\kappa$  be the unique solution of  $V(\theta^\kappa)$  with  $\mu(t)$  being replaced by  $\mu^\kappa$ .

### C.1 Proof of Proposition 6

We prove Part 1 and relegate the proof of Part 2 to the Supplemental Appendix.

In solving  $P(\theta^\kappa)$ , it is more convenient to take  $z \equiv 1/\kappa$  and define another market  $\psi^z \equiv (n, r, u, z\rho, \lambda)$ . (i.e., low type-change rates, instead of high search rates) and solve  $P(\psi^z)$ . It is easy to verify that the unique steady-state solution  $\mu^\kappa$  of  $P(\theta^\kappa)$  also uniquely solves  $P(\psi^z)$ . Last, define  $\psi^0 \equiv (n, r, u, 0, \lambda)$ .

**Lemma 5.**  $\mu^0 \in \mathbb{R}^\mathcal{T}$  is a steady-state solution of  $P(\psi^0)$  if and only if

1. (when  $n_f + n_h > n_a$ )  $\mu_{ho}^0 = \min\{n_a, n_h\}$ ,  $\mu_{hn}^0 = n_h - \mu_{ho}^0$ ,  $\mu_{lo}^0 = 0$ ,  $\mu_{ln}^0 = n_l$ ,  $\mu_{fo}^0 = \max\{0, n_a - n_h\}$ ,  $\mu_{fe}^0 = 0$ , and  $\mu_{fn}^0 = n_f - \mu_{fo}^0 - \mu_{fe}^0$ .
2. (when  $n_f + n_h < n_a$ )  $\mu_{ho}^0 = n_h$ ,  $\mu_{hn}^0 = 0$ ,  $\mu_{lo}^0 = n_a - n_h - n_f$ ,  $\mu_{ln}^0 = n_l - \mu_{lo}^0$ ,  $\mu_{fn}^0 = 0$ , and  $\mu_{fo}^0 + \mu_{fe}^0 = n_f$ .

The problem  $P(\psi^0)$  has multiple steady-state solutions in Case 2 ( $n_f + n_h < n_a$ ), where many possible combinations of  $(\mu_{fo}, \mu_{fe})$  satisfy  $\mu_{fo} + \mu_{fe} = n_f$ .

**(Proof)** The problem  $P(\psi^0)$  consists of

$$\begin{aligned}
(\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe}) \mu_{hn} &= 0, & (\text{from } (\mu\text{-ho})) \\
(\lambda_d \mu_{hn} + \lambda_f \mu_{fn}) \mu_{lo} &= 0, & (\text{from } (\mu\text{-ln})) \\
\lambda_f (\mu_{lo} \mu_{fn} - \mu_{hn} \mu_{fo}) + \lambda_s \mu_{fn} \mu_{fe} &= 0, & (\text{from } (\mu\text{-fo}))
\end{aligned}$$

and the following four conditions that replace  $(\mu\text{-hn})$ ,  $(\mu\text{-lo})$ ,  $(\mu\text{-fe})$ , and  $(\mu\text{-fn})$ :

$$\mu_{ho} + \mu_{hn} = n_h, \quad \mu_{lo} + \mu_{ln} = n_l, \quad \mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = n_a, \quad \text{and} \quad \mu_{fn} + \mu_{fo} + \mu_{fe} = n_f.$$

It follows from  $\lambda_d, \lambda_f, \lambda_s > 0$  that

$$\mu_{lo} \mu_{hn} = \mu_{fo} \mu_{hn} = \mu_{fe} \mu_{hn} = \mu_{lo} \mu_{fn} = \mu_{fn} \mu_{fe} = 0.$$

Suppose that  $n_f + n_h > n_a$ . If  $\mu_{lo} > 0$  or  $\mu_{fe} > 0$ , then  $\mu_{hn} = 0$  and  $\mu_{fn} = 0$ , which results in a contradiction:  $\mu_{ho} + (\mu_{fo} + \mu_{fe}) = n_h + n_f > n_a$ . As  $\mu_{lo} = \mu_{fe} = 0$ , either  $\mu_{fo} = 0$  or  $\mu_{hn} = 0$ . As  $\mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = n_a > 0$ , if  $\mu_{fo} = 0$ , then  $\mu_{ho} = n_a$ ; for otherwise  $\mu_{hn} = 0$  implies that  $\mu_{ho} = n_h$  and  $\mu_{fo} = n_a - n_h$ . On the other hand, if  $n_f + n_h < n_a$ , then  $\mu_{lo} > 0$ , which implies that  $\mu_{hn} = \mu_{fn} = 0$ . Thus,  $\mu_{ho} = n_h$ ,  $\mu_{fo} + \mu_{fe} = n_f$ , and  $\mu_{lo} = n_a - n_h - n_f$ . ■

The following lemma implies that  $\lim_{\kappa \rightarrow \infty} \mu^\kappa$  exists in  $\mathbb{R}_+^T$ :

**Lemma 6.** *There exists a solution  $\mu^*$  of  $P(\psi^0)$  such that  $\mu^* \equiv \lim_{\kappa \rightarrow \infty} \mu^\kappa$ .*

**(Proof)** For each  $z \equiv 1/\kappa$ , let  $F(\mu, \psi^z)$  denote the right-hand sides of the population

equations  $(\mu\text{-hn})$ - $(\mu\text{-fe})$  for a market  $\psi^z \equiv (n, r, u, z\rho, \lambda)$ . Define

$$f(\mu, z) \equiv -\|F(\mu, \psi^z)\|,$$

where  $\|\cdot\|$  denotes the Euclidean norm. It is clear that  $\mu^\kappa$  with  $\kappa = 1/z$  is the unique maximizer of  $f$  with the maximum value equals zero. Let  $M(z) \equiv \{\mu^{1/z}\}$ .

We similarly define  $F(\mu, \psi^0)$  as the right-hand sides of the population equations for the market  $\psi^0$  and  $f(\mu, 0)$ . Let  $M(0)$  be the solution set of  $\max_\mu f(\mu, 0)$ . According to Lemma 5, the solution set  $M(0)$  is singleton if  $n_h + n_f > n_a$ ; for otherwise,  $M(0)$  contains multiple solutions, each being different from others only in  $(\mu_{fo}, \mu_{fe})$  under the constraint  $\mu_{fo} + \mu_{fe} = n_f$ .

The function  $f$  is continuous in  $\mu$  and  $z$  because the equations  $F$  are continuous. It follows from Berge's Maximum Theorem that the correspondence  $M(\cdot)$  is upper hemicontinuous at  $z = 0$ :

1. (when  $n_h + n_f > n_a$ )  $\mu^\kappa$  converges to the unique solution of  $P(\psi^0)$ .
2. (when  $n_h + n_f < n_a$ ) for each type  $i \neq fo, fe$ , the population  $\mu_i^\kappa$  converges to  $\mu_i^0$  given in Lemma 5, and  $\mu_{fo}^\kappa + \mu_{fe}^\kappa$  converges to  $n_f$ .

It remains to show that, when  $n_h + n_f > n_a$ , the sequence  $\mu_{fe}^\kappa$  converges. (The convergence of  $\mu_{fo}^\kappa$  follows immediately from  $\lim_{\kappa \rightarrow \infty} (\mu_{fo}^\kappa + \mu_{fe}^\kappa) = n_f$ .)

For every  $\kappa > 0$ ,  $\kappa(\lambda_f \mu_{hn}^\kappa + \lambda_s \mu_{fn}^\kappa) \mu_{fe}^\kappa = \rho_e (n_f - \mu_{fn}^\kappa - \mu_{fe}^\kappa)$  (from  $(\mu\text{-fe})$ ), or equivalently that

$$\mu_{fe}^\kappa = \frac{\rho_e (n_f - \mu_{fn}^\kappa)}{\rho_e + \kappa(\lambda_f \mu_{hn}^\kappa + \lambda_s \mu_{fn}^\kappa)}. \quad (30)$$

We find  $\lim_{\kappa \rightarrow \infty} \kappa(\lambda_f \mu_{hn}^\kappa + \lambda_s \mu_{fn}^\kappa)$  from

$$\begin{aligned} \kappa \mu_{hn}^\kappa (\lambda_d \mu_{lo}^\kappa + \lambda_f \mu_{fo}^\kappa + \lambda_f \mu_{fe}^\kappa) &= -\rho_d \mu_{hn}^\kappa + \rho_u \mu_{ln}^\kappa, & (\text{from } (\mu\text{-hn})) \\ \kappa (\lambda_v \mu_{hn}^\kappa + \lambda_f \mu_{fn}^\kappa) \mu_{lo}^\kappa &= -\rho_u \mu_{lo}^\kappa + \rho_d \mu_{ho}^\kappa. & (\text{from } (\mu\text{-lo})) \end{aligned}$$

By the convergence of  $\mu_i^\kappa$  for  $i \neq fo, fe$ , and the convergence of  $\mu_{fe}^\kappa + \mu_{fo}^\kappa$  to  $n_f$ ,

$$\begin{aligned} \lim_{\kappa \rightarrow \infty} \kappa \mu_{hn}^\kappa &= \frac{\rho_u \mu_{ln}^*}{\lambda_d \mu_{lo}^* + \lambda_f n_f}, \quad \text{and} \\ \lim_{\kappa \rightarrow \infty} \kappa (\lambda_v \mu_{hn}^\kappa + \lambda_f \mu_{fn}^\kappa) &= \frac{\rho_d \mu_{ho}^* - \rho_u \mu_{lo}^*}{\mu_{lo}^*} = \frac{\rho_d n_h - \rho_u \mu_{lo}^*}{\mu_{lo}^*} = \frac{\rho_u \mu_{ln}^*}{\mu_{lo}^*}. \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{\kappa \rightarrow \infty} \kappa (\lambda_f \mu_{hn}^\kappa + \lambda_s \mu_{fn}^\kappa) &= \frac{\lambda_s \rho_u \mu_{ln}^*}{\lambda_f \mu_{lo}^*} + \left( \lambda_f - \frac{\lambda_d \lambda_s}{\lambda_f} \right) \frac{\rho_u \mu_{ln}^*}{\lambda_d \mu_{lo}^* + \lambda_f n_f} \\ &= \frac{\rho_u \mu_{ln}^*}{\mu_{lo}^*} \frac{\lambda_f \mu_{lo}^* + \lambda_s n_f}{\lambda_d \mu_{lo}^* + \lambda_f n_f} > 0. \end{aligned}$$

Therefore,

$$\mu_{fe}^* \equiv \lim_{\kappa \rightarrow \infty} \mu_{fe}^\kappa = \frac{n_f}{1 + \frac{\rho_u \mu_{ln}^*}{\rho_e \mu_{lo}^*} \frac{\lambda_f \mu_{lo}^* + \lambda_s n_f}{\lambda_d \mu_{lo}^* + \lambda_f n_f}}, \quad \text{and} \quad \mu_{fo}^* = n_f - \mu_{fe}^*.$$

■

## C.2 Proof of Proposition 7

By Lemma 5 and Lemma 6, if  $n_a < n_h + n_f$ , then  $\mu_{ho}^* = \min\{n_a, n_h\}$ ,  $\mu_{fo}^* = \max\{0, n_a - n_h\}$ ,  $\mu_{fe}^* = 0$ , and  $\mu_{lo}^* = 0$ . Since  $\mu^*$  coincides with the efficient asset allocation  $\bar{\mu}$ , we have  $W^* = \bar{W}$ . The independence of  $W^*$  on  $u_f$  and  $u_e$  is trivial as  $\mu_{fo}^* = \mu_{fe}^* = 0$ . The independence of  $W^*$  on  $\lambda_d$  also follows from  $\bar{W}$ 's independence of any search friction. When  $n_h < n_a < n_h + n_f$ , we have  $\mu_{fo}^* > 0$ , so  $W^* = \bar{W}$  is strictly increasing in  $u_f$ .

If  $n_a > n_h + n_f$ , then  $rW^* = r\bar{W} - \mu_{fe}^*(u_f - u_e)$ . We have  $W^* < \bar{W}$  because

$$\mu_{fe}^* = \frac{n_f}{1 + \frac{\rho_u \mu_{ln}^*}{\rho_e \mu_{lo}^*} \frac{\lambda_f \mu_{lo}^* + \lambda_s n_f}{\lambda_d \mu_{lo}^* + \lambda_f n_f}} > 0,$$

The welfare  $W^*$  is increasing in  $u_f$  and  $u_e$  as  $\mu_{fo}^*$  and  $\mu_{fe}^*$  are strictly positive. Moreover,  $\mu_{fe}^*$  is decreasing in  $\lambda_s$  and increasing in  $\lambda_d$ . Thus, the welfare  $W^*$  is increasing in  $\lambda_s$  and decreasing in  $\lambda_d$ .

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