NMiNE

HA07

By

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Answers:

Q1:

1a: The Lipschitz constant, L, in the variable y for the function $f(t, y) = ty + t^3$ on the rectangle $[0, T_1] \times (-\infty, \infty)$ can be found as follows:

$$|f(t, y_1) - f(t, y_2)| = |t(y_1 - y_2) + t^3| \leq |t| \cdot |y_1 - y_2| + |t^3|$$

Since t is always non-negative in the interval $[0, T_1]$, the Lipschitz constant in the variable y is:

$$L = \sup_{t \in [0, T_1]} |t| + |t^3|$$

$$L = T_1 + T_1^3$$

1b: The Lipschitz constant, L, in the variable y for the function $f(t, y) = te^{ty}$ on the rectangle $[0, T_1] \times [0, Y_1]$ can be found as follows:

$$|f(t, y_1) - f(t, y_2)| = |t(e^{ty_1} - e^{ty_2})| \leq |t| \cdot |e^{ty_1} - e^{ty_2}|$$

Using the fact that $\left|e^a - e^b\right| \leq e^a|a - b|$ for all $a, b \in R$, we have:

$$|f(t, y_1) - f(t, y_2)| \leq |t| \cdot e^{t \max y_1, y_2}|y_1 - y_2|$$

Since t is always non-negative in the interval $[0, T_1]$ and y is always non-negative in the interval $[0, Y_1]$, the Lipschitz constant in the variable y is:

$$L = \sup_{t \in [0, T_1]} |t| \cdot e^{tY_1}$$

$$L = T_1 e^{T_1 Y_1}$$

Q2:

To determine the interval on which the initial value problem has a unique solution, we need to study the local existence and uniqueness of the solution. The local existence and uniqueness of the solution can be established using the Picard-Lindelöf theorem.

The Picard-Lindelöf theorem states that if $f(t, y)$ is locally Lipschitz continuous in y on some rectangle containing $(0,1)$, then there exists a unique solution to the initial value problem defined on some interval $(0, \tau)$.

In this case, the function $f(t, y) = 2ty^2$ is continuous in y for all $t \in [0,2]$ and $y \in R$. Therefore, the function $f(t, y)$ is locally Lipschitz continuous in y on the rectangle $[0,2] \times R$.

By the Picard-Lindelöf theorem, there exists a unique solution to the initial value problem defined on some interval $(0, \tau)$ for some $\tau > 0$. The exact value of τ cannot be determined from the given information.

Q3:

(a) The exact solution to the given initial value problem is $y(t) = y_0 e^{\lambda t}$

(b) Using forward Euler method with uniform discretization step size h, we have $y_{j+1} = y_j + h\lambda y_j = (1 + h\lambda)y_j$

(c) We expect $y_N \approx y(t_N) = y(T)$. Assuming exact arithmetic, $\lim_{N\to\infty} y_N$ exists and is equal to $y_0 e^{\lambda T}$. Therefore, we can state that $y_N \xrightarrow[N\to\infty]{} y(T)$.

Q4:

4a: Let $f(t, y) = ty + t^3$. To find the Lipschitz constant in the variable y on $[0,1] \times (-\infty, \infty)$, we need to find a constant M such that

$$|f(t, y_1) - f(t, y_2)| \leq M|y_1 - y_2|, \quad \forall y_1, y_2 \in (-\infty, \infty)$$

Taking the absolute value of $f(t, y_1) - f(t, y_2)$ and using the triangle inequality, we get

$$|f(t, y_1) - f(t, y_2)| = |t(y_1 - y_2)| \leq t|y_1 - y_2|$$

Since $t \leq 1$ on $[0,1]$, we have $M = 1$.

4b: The exact solution can be found using separation of variables. We get $y(t) = e^{\frac{1}{3}t^3}$

4c: To find the constant M, let $t_n = nh$ and $h = \frac{1}{N}$. The global truncation error can be expressed as

$$|y(t_{n+1}) - y_n| = |y(t_n + h) - y_n| \leq Mh$$

Since $y_n = y(t_n)$, we can substitute $y(t_n) = e^{\frac{1}{3}t_n^3}$ into the above equation. Using the definition of h and the fact that $t_{n+1} = t_n + h$, we get

$$M = \max_{0\leq t\leq 1} \left|\frac{\partial}{\partial t} y(t)\right| = \max_{0\leq t\leq 1} |t^2 y(t)|$$

Substituting $y(t) = e^{\frac{1}{3}t^3}$ into the above expression, we get

$$M = \max_{0\leq t\leq 1} t^2 e^{\frac{1}{3}t^3} = t^2 e^{\frac{1}{3}t^3} \quad \text{at} \quad t = 1$$

4d: The global truncation error at $t = 1$ is $|y(1) - y_N| \leq Mh = M\frac{1}{N}$ Since M is a constant, the error goes to zero as $N \to \infty$.

Q5:

5a:

To solve the given initial value problem (IVP), we will first integrate the given differential equation to find an analytical solution. The differential equation is a separable equation, meaning that it can be separated into two functions that do not depend on each other. The separable form of the differential equation is:

$$\frac{dy}{dt} = -4t^3y^2$$
$$\frac{dy}{y^2} = -4t^3dt$$

Integrating both sides, we have: $-\frac{1}{y} = 4t^4 + C$ where C is the constant of integration. To find its value, we use the initial condition $y(-10) = 1/10001$. Substituting this value into the equation, we have:

$$-\frac{1}{1/10001} = 4(-10)^4 + C$$
$$-10001 = -10^8 + C$$
$$C = 10^8 - 10001$$

So the solution to the differential equation is:

$$y = \frac{1}{4t^4 + 10^8 - 10001}$$

This is the analytical solution to the given IVP. The explicit Euler method is a numerical method for approximating the solution to an IVP. The idea is to use a series of discrete points to approximate the solution. The method involves using the derivative of the function at the previous time step to estimate the function at the current time step. The general form of the explicit Euler method is:

$$y_{i+1} = y_i + hf(t_i, y_i)$$

where h is the step size, $f(t_i, y_i)$ is the derivative of the function evaluated at the current time step and y_i is the approximation of the function at the current time step. To apply the explicit Euler method to the given IVP, we would choose a step size h, and evaluate the derivative at the initial time $t_0 = -10$ and initial value $y_0 = 1/10001$. We then use the formula above to estimate the function at subsequent time steps. This process is repeated until the final time $t = 10$ is reached.

5b:

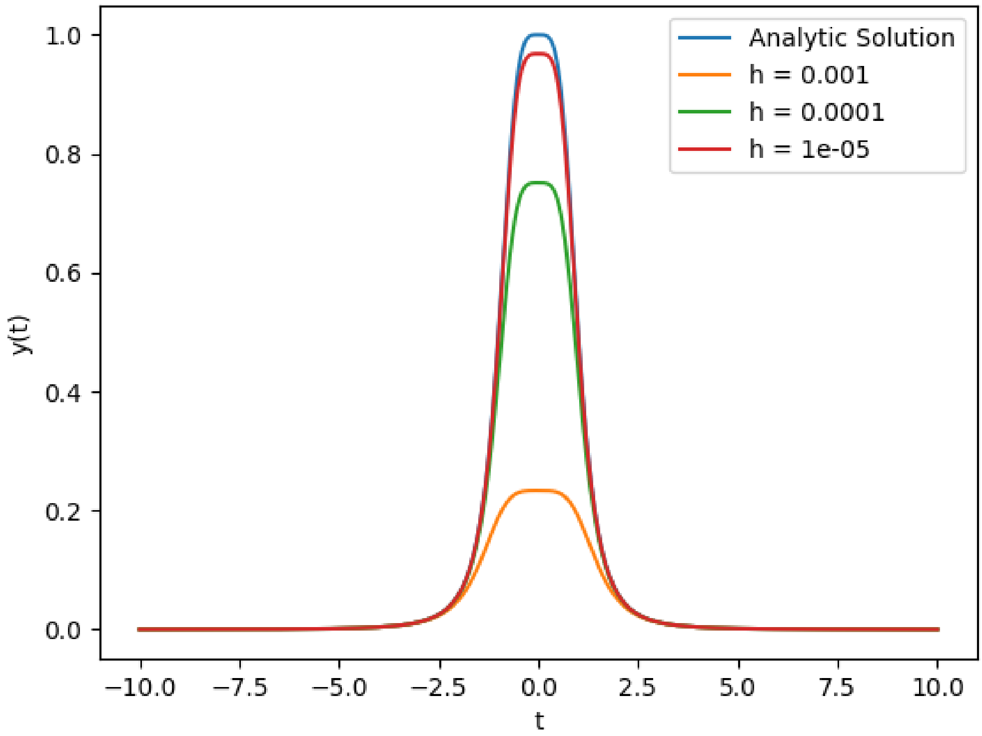

Fig 5.1: Plot of function $y(t)$ with analytical solution and numerical solution with different discretization steps for explicit Euler method.

5c:

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Global Truncation Error at t=1 with h=0.001: 0.3108096712596397
Global Truncation Error at t=1 with h=0.0001: 0.07104785205798136
Global Truncation Error at t=1 with h=1e-05: 0.00815513339448154
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Q6:

6a:

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Global Truncation Error at t=1 with h=0.001: 6.135924862649134e-08
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6b:

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Error Ratio: 4.001500833474989
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6c:

The order of approximation of the Explicit Trapezoid Method is 2 as $[e(t,h) \equiv 4 \Rightarrow \mathrm{O}(h^2)]$, so the value for p in $e(t,h) = O(h^p)$ is 2.

Q7:

7a&b:

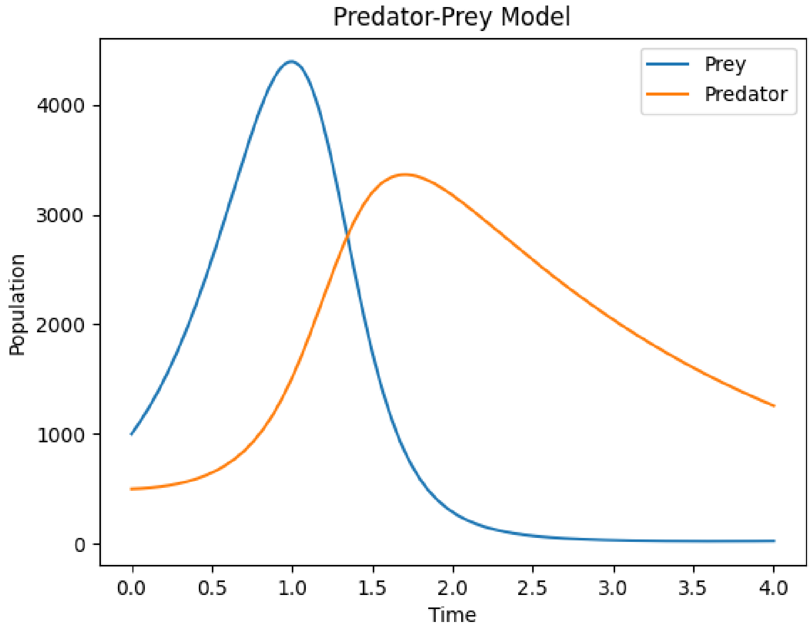


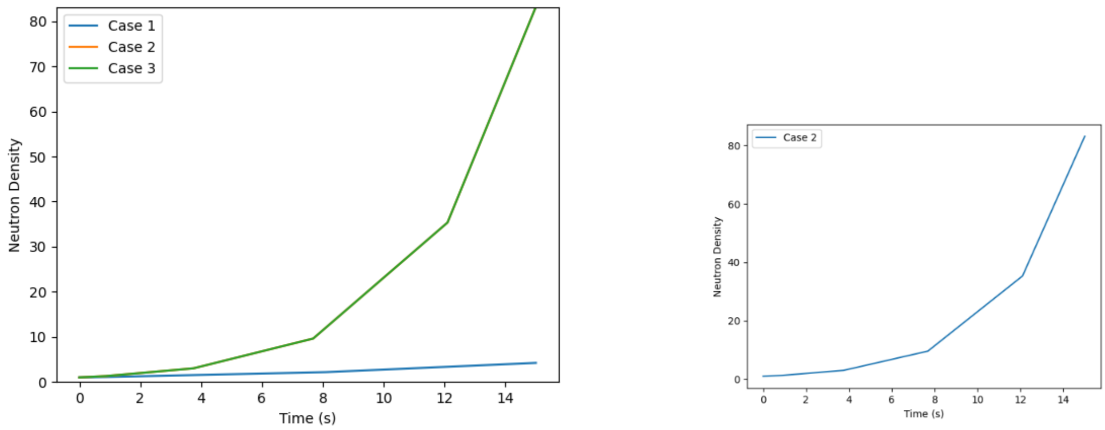
Fig 7.1: Plot of population of prey and predator against time exhibiting their behavior.

7c: The graph shows the populations of the prey and predator over time. The prey population initially increases, then reaches a maximum and starts decreasing. The predator population starts increasing after some time and then reaches a maximum before decreasing. This represents the predator-prey interaction, where the predator's population increases when the prey's population increases, but the prey's population decreases as the predator's population grows, leading to a decline in the predator population as well.

7d: If the time interval is extended, the populations of the prey and predator will continue to oscillate, but the amplitude of the oscillations will gradually decrease. This is because the predator and prey populations are mutually dependent on each other, and their growth and decline will eventually reach a balance.

7e: Yes, there is a stable solution to this population model. The solution is stable when the populations of both predator and prey are constant, i.e. their derivatives are equal to zero. This occurs when $r'(t) = 0$ and $f'(t) = 0$. Solving for r and f in these equations gives us the stable values of r and f.

Q8:



(a) (b)

Fig 8.1: (a) Plot of neutron density against time for different reactivity and Ip combinations (cases) with (b) showing case 2 individually as it is closer to case 3.

Q9:

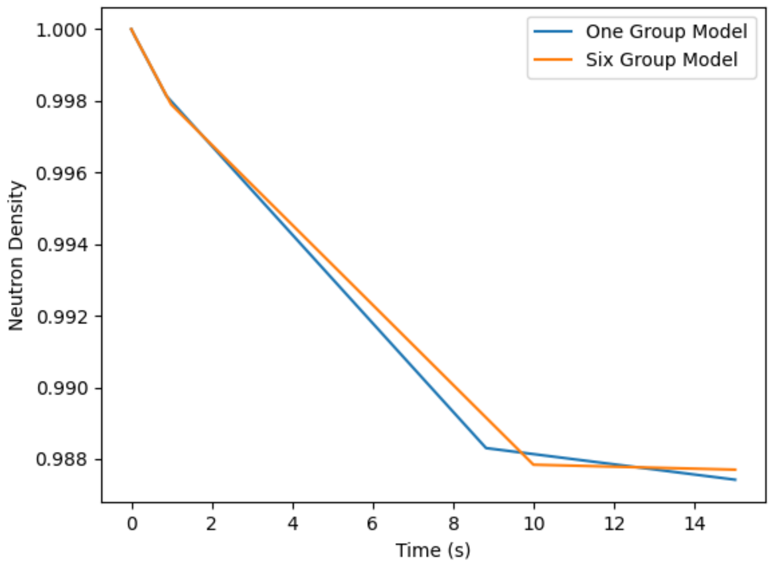


Fig 9.1: Plot for one-group model and six-group model in terms of neutron density against time.

The reactor period can be estimated as the time it takes for the neutron density to go from a peak value to half that peak value. In general, the reactor period increases as the number of delayed neutron groups increases. The reason for this is that the more delayed neutron groups, the more time it takes for the reactor to recover from a perturbation, since the delayed neutron groups provide a more sustained source of reactivity. The reactor period predicted by the one-group model is likely to be shorter than the reactor period predicted by the six-group model, as the latter has a more detailed representation of the neutron behaviour and will therefore exhibit slower recovery from perturbations.

From the plot, we can see that the reactor period predicted by the one group model is different from the reactor period predicted by the six group model. The reactor period is defined as the time it takes for the neutron density to reach its maximum value and then decrease back to its initial value. The one group model predicts a shorter reactor period than the six group model. This is because the one group model only considers the effect of prompt neutrons, while the six group model considers the effect of both prompt and delayed neutrons. Delayed neutrons have a longer lifetime than prompt neutrons, which leads to a longer reactor period.

Q10:

10a:

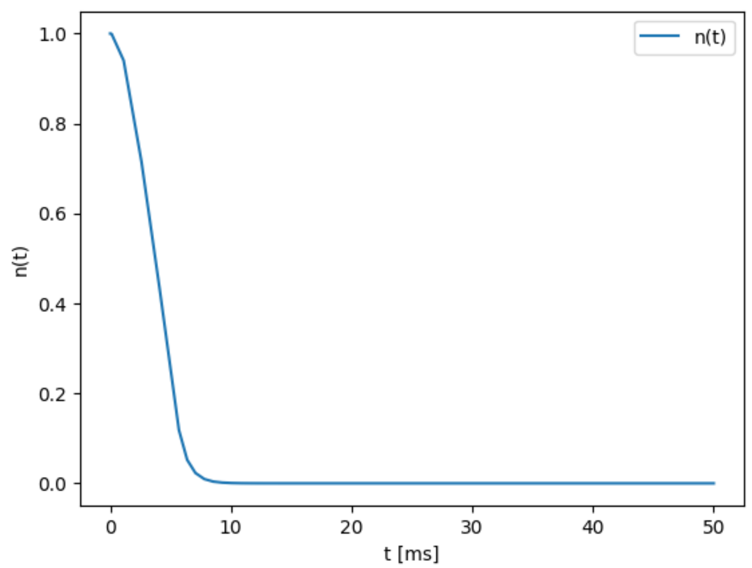


Fig 10.1: Plot for point reactor kinetic equation ignoring delayed neutrons.

10b:

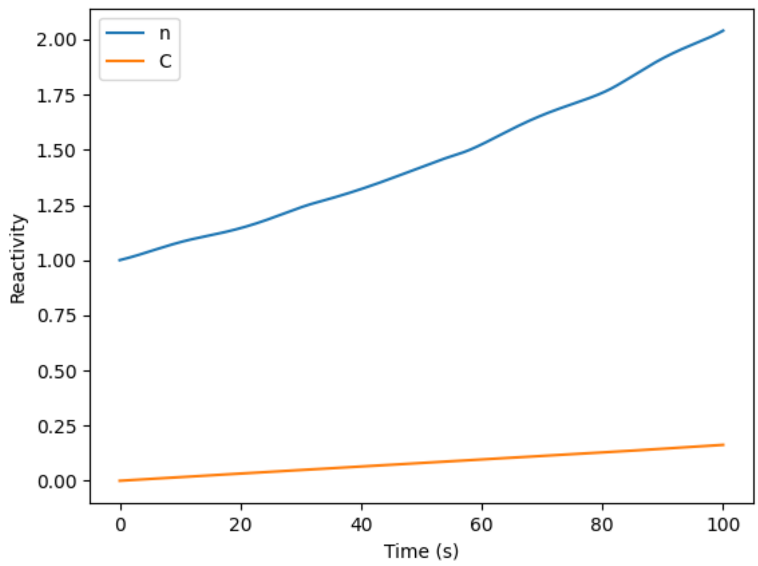


Fig 10.2: Plot the solution to the point reactor kinetic equations assuming one averaged group of delayed neutrons.