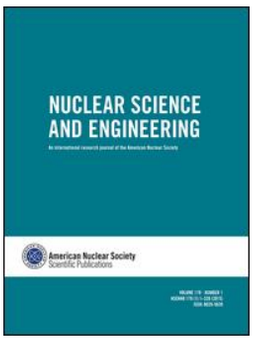

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An Upwind Numerical Method for Two-Fluid Two-Phase Flow Models

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Abstract—*An upwind numerical method for a six equation two-fluid model is presented based on a linearized Riemann solver. The construction of this approximate Riemann solver uses an extension of Roe's method that has been successfully used to solve gas dynamics equations in aerodynamics problems. As far as the two-fluid model is hyperbolic, this numerical method seems very efficient for the numerical solution of two-phase flow problems. In practice, this new numerical method has proved to be stable on nonstaggered grids and capable of generating accurate nonoscillating solutions for two-phase flow calculations. The scheme was applied both to shock tube problems and to standard tests for two-fluid codes.*

1. INTRODUCTION

Modeling and numerical simulation of two-phase flow phenomena keep causing complex problems for the development of computer codes dedicated to design and safety studies of nuclear reactors. Generally, engineering two-phase flow calculations requires the determination of the velocity, the pressure, and the energy fields for each phase. The usual way to establish physical modelings for two-phase flows is to start with a single continuous description for each phase given by Navier-Stokes equations. The presence of topologically complicated interfaces suggests the need for a multifield model of the flow. Various averaging techniques have been used to obtain practical and computable two-fluid models.[1] However, the averaging procedure leads to an ill-posed initial value problem and many differential terms as added mass and interfacial pressure are proposed to make the models hyperbolic.[2]

The model considered here is a first-order equal pressure six equation two-fluid model. If we take the exception of the interfacial pressure term, which contains partial derivatives, the other terms of mass and momentum transfer between phases are given by algebraic relationships. They will appear as source terms of the model. The resulting model is a nonconservative hyperbolic one, consisting in mass, momentum, and energy balance equations for each phase.

The most common numerical method used to solve these equations is a method based on staggered grids and donor-cell differencing.[3,4] This method, now almost universal in two-fluid codes like TRAC (Ref. 5), RELAP (Ref. 6), and CATHARE (Ref. 7), introduces a large amount of numerical diffusion. Moreover, Ransom and Mousseau[8] have shown that high frequency oscillations appear with a large number of cells. For numerical methods on nonstaggered grids, in Ref. 9 Ransom and Hicks have applied a Lax-Wendroff scheme to two-fluid models. In Ref. 10, a split matrix method has been presented for the numerical solution of two-phase flow equations. However, this method solves the flow equations in their characteristic form, and hence, it is not rigorously conservative.

In this paper, we build a numerical method in which the local characteristic structure, reached by solving a Riemann problem rather than by diagonalizing the Jacobian matrix, is used to define backward and forward differences to approximate the spatial derivatives. This method makes use of the characteristic information within the framework of a conservative method. It was

first proposed by Godunov.[11] For numerical efficiency, we use Godunov's method with the linearized approximate Riemann solver of Roe,[12] which has been widely used for gas dynamic calculations.[13] For two-phase fluid dynamics, a weak formulation of Roe's approximate Riemann solver has been introduced in Ref. 14. This weak formulation was applied to build a Roe-averaged Jacobian matrix for a conservative system governing a homogeneous equilibrium two-phase flow.[15] Here we extend this scheme to a hyperbolic nonconservative system that models a two-component two-phase flow.

In Sec. II, we present the two-phase flow model to which numerical solutions are considered, and we look at the hyperbolicity of this model. Section III is devoted to the construction of an approximate linearized Riemann solver for the two-fluid model. Following the method developed in Ref. 14, we construct a Roe-averaged matrix using the canonical path for a parameter vector. We also give an approximate diagonalization of the Roe-averaged matrix. This allows an easy implementation of the numerical scheme. In Sec. IV, we give some numerical results for shock tube problems and standard two-phase flow problems proposed as benchmarks for two-fluid computer codes.

II. TWO-FLUID TWO-PHASE FLOW MODEL

In this section, we consider a one-dimensional two-fluid model dealing with a two-component two-phase flow in a straight pipe.

II.A. Basic Two-Fluid Model

The basic system of equations for the two-fluid nonequilibrium model consists in two phasic mass equations, two phasic momentum equations, and two phasic energy equations. The development of such equations for the two-phase process has been recorded in the standard reference.[1]

The phasic mass equations are

$$\partial_t(\alpha_v\rho_v) + \partial_x(\alpha_v\rho_v u_v) = \Gamma_v \tag{1}$$

and

$$\partial_t(\alpha_l\rho_l) + \partial_x(\alpha_l\rho_l u_l) = \Gamma_l \ . \tag{2}$$

The phasic momentum equations are

$$\partial_t(\alpha_v\rho_v u_v) + \partial_x(\alpha_v\rho_v u_v^2) + \alpha_v\partial_x p + I = \alpha_v\rho_v B + F_v^w + F^i \tag{3}$$

and

$$\partial_t(\alpha_l\rho_l u_l) + \partial_x(\alpha_l\rho_l u_l^2) + \alpha_l\partial_x p - I = \alpha_l\rho_l B + F_l^w - F^i \ . \tag{4}$$

The phasic energy equations are

$$\partial_t\left[\alpha_v\rho_v\left(e_v + \frac{u_v^2}{2}\right)\right] + p\partial_t\alpha_v + \partial_x\left[\alpha_v\rho_v u_v\left(h_v + \frac{u_v^2}{2}\right)\right] = Q_v^w + Q_v^i + \Gamma_v h_v^* \tag{5}$$

and

$$\partial_t\left[\alpha_l\rho_l\left(e_l + \frac{u_l^2}{2}\right)\right] + p\partial_t\alpha_l + \partial_x\left[\alpha_l\rho_l u_l\left(h_l + \frac{u_l^2}{2}\right)\right] = Q_l^w + Q_l^i - \Gamma_l h_l^* \ . \tag{6}$$

In Eqs. (1) through (6), the subscripts v and l refer to the vapor and the liquid phases, respectively. The nomenclature is as follows:

α = void volume fraction ($\alpha_v + \alpha_l = 1$)

ρ = phasic density

u = phasic velocity

e = phasic internal energy

h = phasic enthalpy

p = common pressure.

We also define the phasic total enthalpy $H = h + u^2/2$ and the phasic total energy $E = e + u^2/2$.

On the right side of Eqs. (1) through (6),

Γ = interphase mass exchange

F^w = wall drag

F^I = interphase drag force

Q^w = wall heat flux

Q^i = interphase heat exchange

B = body forces (usually gravity).

The enthalpies associated with interphase mass transfer Γ_v in Eqs. (5) and (6) are defined in such a way that the interface energy jump conditions are satisfied. For more details on these latter terms, we refer for example to Ref. 16, where physical closure laws are given, particularly in the framework of nuclear safety studies.

To close the system, we need state functions for the liquid and the vapor phases. The liquid phase is assumed to be incompressible with constant mass density ρ_l, while the vapor mass density is given by the following state equation:

$$\rho_v = \rho_v(p, h_v) \ . \tag{7}$$

However, the results can be generalized to a compressible liquid phase.

II.B. Hyperbolicity of the Two-Fluid Model

It is well known that the basic two-fluid model leads to a nonhyperbolic system (the resulting system of equations possesses some complex eigenvalues). In the momentum Eqs. (3) and (4), two differential terms have not been described:

1. The virtual mass force term that describes inertial coupling of the phases in accelerating flows.[17,18]

2. An interface pressure term given as a spatial gradient of the void fraction.[16,19] The interface pressure term must be defined consistently with the physical situation of interest (stratified flows, bubbly flows...).

The inclusion is not discussed in what follows, but we refer to Ref. 20 for the application of an upwind scheme to a two-fluid model with a virtual mass force term. Here, we consider the following form for the interface pressure term

$$I = (p - p_i)\partial_x \alpha_v \tag{8}$$

with

$$p - p_i = \alpha_v \rho_l \delta (u_v - u_l)^2 \ , \tag{9}$$

where the value of the pressure coefficient δ will be defined in the following sections so as to have a hyperbolic system. A similar formulation has been used in Ref. 19.

We recall that the system is hyperbolic, if for any state belonging to the set Ω of physical states, all its eigenvalues are real. To determine these eigenvalues, we must find the six roots of a polynomial of degree six given by the characteristic determinant. By c_m we denote the speed of sound in the homogeneous two-phase mixture:

$$c_m = \left(\frac{\partial p}{\partial \rho}\right)_s^{1/2} = \left(\frac{\rho_v}{\alpha_l \rho}\right)^{1/2} c_v \ , \tag{10}$$

where $\rho = \rho_v \alpha_v + \rho_l \alpha_l$ is the mixture density and c_v is the speed of sound in the vapor phase. The speed c_m is the usual speed of sound in the mixture obtained for the homogeneous (equal phase velocity) model. For a two-fluid model including unequal phase velocity, the usual speed of sound is given by

$$a_m = \left[\frac{\rho(\alpha_v \rho_l + \alpha_l \rho_v)}{\rho_l \rho_v}\right]^{1/2} c_m \ . \tag{11}$$

We assume that the relative velocity between the two phases is much lower than the speed of sound of the two-phase mixture a_m. This is the case in many physically interesting configurations, for example, for steam and water in a vertical pipe. Consequently, we introduce the small parameter ξ, given by the dimensionless relative velocity:

$$\xi = \frac{(u_v - u_l)}{a_m} \ll 1 \ . \tag{12}$$

Then, it is shown in Ref. 21, using a perturbation method with the small parameter ξ, that the two-fluid model is hyperbolic provided that the interface pressure coefficient δ is greater than a minimum value δ_0. This value is explicitly derived in Sec. III.

II.C. Jacobian Matrix of the Two-Fluid Model

The discretization of the source terms, which does not contain partial derivatives, can easily be done using a centered scheme. Thus, these terms are not of any fundamental significance to our numerical scheme, and for the simplicity of the presentation, they are not discussed in what follows. Then, the system considered in this study takes the following form

$$\partial_t(\alpha_k \rho_k) + \partial_x(\alpha_k \rho_k u_k) = 0 \ ,$$

$$\partial_t(\alpha_k \rho_k u_k) + \partial_x(\alpha_k \rho_k u_k^2) + \alpha_k \partial_x p + (p - p_i)\partial_x \alpha_k = 0 \ ,$$

and

$$\partial_t\left[\alpha_k \rho_k \left(e_k + \frac{u_k^2}{2}\right)\right] + p\partial_t \alpha_k + \partial_x\left[\alpha_k \rho_k u_k \left(h_k + \frac{u_k^2}{2}\right)\right] = 0 \ , \tag{13}$$

where the subscript k refers to both the vapor phase ($k = v$) and the liquid phase ($k = l$). Since the liquid phase has a constant mass density ρ_l, we have

$$p\partial_t \alpha_l = -\frac{p}{\rho_l} \partial_x \alpha_l \rho_l u_l \ . \tag{14}$$

Substituting this equation in the energy equation of the system Eq. (13), we can write the two-fluid model in the following nonconservative form:

$$\partial_t \boldsymbol{U} + \mathbf{A}(\boldsymbol{U})\partial_x \boldsymbol{U} = 0 \ , \tag{15}$$

where $\boldsymbol{U}$ is the vector of conservative variables.

$$\boldsymbol{U} = \begin{vmatrix} \alpha_v \rho_v \\ \alpha_l \rho_l \\ \alpha_v \rho_v u_v \\ \alpha_l \rho_l u_l \\ \alpha_v \rho_v \left(e_v + \frac{u_v^2}{2}\right) \\ \alpha_l \rho_l \left(e_l + \frac{u_l^2}{2}\right) \end{vmatrix} \ . \tag{16}$$

A straightforward calculation gives the Jacobian matrix $\mathbf{A}(\boldsymbol{U})$ of the two-fluid model Eq. (13):

$$\mathbf{A}(\boldsymbol{U}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha_v p_u^1 - u_v^2 & \alpha_v p_u^2 - (p-p_i)/\rho_l & \alpha_v p_u^3 + 2u_v & \alpha_v p_u^4 & \alpha_v p_u^5 & \alpha_v p_u^6 \\ \alpha_l p_u^1 & \alpha_l p_u^2 + (p-p_i)/\rho_l - u_l^2 & \alpha_l p_u^3 & \alpha_l p_u^4 + 2u_l & \alpha_l p_u^5 & \alpha_l p_u^6 \\ (\alpha_v p_u^1 - H_v)u_v & (\alpha_v p_u^2 - p/\rho_l)u_v & \alpha_v p_u^3 u_v + H_v & \alpha_v p_u^4 u_v + p/\rho_l & \alpha_v p_u^5 u_v + u_v & \alpha_v p_u^6 u_v \\ \alpha_l p_u^1 u_l & (\alpha_l p_u^2 + p/\rho_l - H_l)u_l & \alpha_l p_u^3 u_l & \alpha_l p_u^4 u_l + H_l - p/\rho_l & \alpha_l p_u^5 u_l & \alpha_l p_u^6 u_l + u_l \end{bmatrix} . \tag{17}$$

The pressure derivatives with respect to the conservative variables, $p_u^i = (\partial p/\partial u_i)_{u_{j\neq i}}$, are calculated using the state function Eq. (7) and the vector of conservative variables Eq. (16). We obtain

$$\nabla_u p = (p_u^i) = \begin{bmatrix} -\left[h_v - \frac{u_v^2}{2} + \rho_v \Big/ \left(\frac{\partial \rho_v}{\partial h_v}\right)\right]\frac{\gamma-1}{\alpha_v} \\ -\frac{1}{\rho_l}\left[p + \rho_v^2 \Big/ \left(\frac{\partial \rho_v}{\partial h_v}\right)\right]\frac{\gamma-1}{\alpha_v} \\ -\frac{\gamma-1}{\alpha_v}\, u_v \\ 0 \\ \frac{\gamma-1}{\alpha_v} \\ 0 \end{bmatrix} \tag{18}$$

with the gamma coefficient

$$\gamma = \frac{\left(\frac{\partial \rho_v}{\partial p}\right)}{\left(\frac{\partial \rho_v}{\partial p}\right) + \frac{1}{\rho_v}\left(\frac{\partial \rho_v}{\partial h_v}\right)} . \tag{19}$$

II.D. Conservative Form of the Two-Fluid Model

The system of Eq. (15) is said to be in a conservative form if it can be written

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{f}(\boldsymbol{U}) = 0 , \tag{20}$$

where $\boldsymbol{f}(\boldsymbol{U})$ is a flux function. The system Eq. (15) cannot be written directly under a conservative form due to the presence of nonconservative products like $\alpha_k \partial_x p$ and $p_i \partial_x \alpha_v$ in the momentum equations. However, we prove that the basic two-fluid system, which is the system of Eq. (13) without the interface pressure term, has an equivalent conservative form for smooth solutions.

We introduce the mixture momentum ρu and the mixture total energy ρE, defined by

$$\rho u = \alpha_v \rho_v u_v + \alpha_l \rho_l u_l$$

and

$$\rho E = \alpha_v \rho_v \left(e_v + \frac{u_v^2}{2}\right) + \alpha_l \rho_l \left(e_l + \frac{u_l^2}{2}\right) . \tag{21}$$

These mixture quantities satisfy the following conservation equations

$$\partial_t \rho u + \partial_x(\alpha_v \rho_v u_v^2 + \alpha_l \rho_l u_l^2) + \partial_x p = 0 \tag{22}$$

and

$$\partial_t \rho E + \partial_x \left[\alpha_v \rho_v u_v \left(h_v + \frac{u_v^2}{2}\right) + \alpha_l \rho_l u_l \left(h_l + \frac{u_l^2}{2}\right)\right] = 0 . \tag{23}$$

Now we split the liquid phase momentum equation of the system of Eq. (13) into

$$u_l[\partial_t(\alpha_l \rho_l) + \partial_x(\alpha_l \rho_l u_l)] + \alpha_l \rho_l(\partial_t u_l + u_l \partial_x u_l) + \alpha_l \partial_x p = 0 . \tag{24}$$

Then, using the liquid phase mass conservation equation and dividing the previous equation by $\alpha_l \rho_l$ leads to the conservative equation:

$$\partial_t u_l + \partial_x \left(\frac{u_l^2}{2} + \frac{p}{\rho_l}\right) = 0 . \tag{25}$$

In the same way, we split the energy conservation equation of the liquid phase into

$$\partial_t(\alpha_l \rho_l e_l) + \partial_x(\alpha_l \rho_l h_l u_l) + p\partial_t \alpha_l + \frac{u_l}{2}[\partial_t(\alpha_l \rho_l u_l) + \partial_x(\alpha_l \rho_l u_l^2)] + \frac{\alpha_l \rho_l u_l}{2}(\partial_t u_l + u_l \partial_x u_l) = 0 . \tag{26}$$

Using the liquid phase momentum equations of the system and Eq. (25), we obtain

$$\partial_t(\alpha_l \rho_l e_l) + \partial_x(\alpha_l \rho_l h_l u_l) + p\partial_t \alpha_l - \alpha_l u_l \partial_x p = 0 , \tag{27}$$

and substituting the relationship $h_l = e_l + p/\rho_l$ in the previous equation leads to a conservation equation for the liquid phase internal energy:

$$\partial_t(\alpha_l\rho_l e_l) + \partial_x(\alpha_l\rho_l e_l u_l) = 0 \ . \tag{28}$$

Finally, for smooth solutions (without shock waves), the basic two-fluid model is equivalent to the conservative system

$$\partial_t \boldsymbol{V} + \partial_x \boldsymbol{g}(\boldsymbol{V}) = 0 \tag{29}$$

with

$$\boldsymbol{V} = \begin{bmatrix} \alpha_v\rho_v \\ \alpha_l\rho_l \\ \alpha_v\rho_v u_v + \alpha_l\rho_l u_l \\ u_l \\ \alpha_v\rho_v\left(e_v + \dfrac{u_v^2}{2}\right) + \alpha_l\rho_l\left(e_l + \dfrac{u_l^2}{2}\right) \\ \alpha_l\rho_l e_l \end{bmatrix} \tag{30}$$

and the flux function

$$\boldsymbol{g}(\boldsymbol{V}) = \begin{bmatrix} \alpha_v\rho_v u_v \\ \alpha_l\rho_l u_l \\ \alpha_v\rho_v u_v^2 + \alpha_l\rho_l u_l^2 + p \\ \dfrac{u_l^2}{2} + \dfrac{p}{\rho_l} \\ \alpha_v\rho_v u_v\left(h_v + \dfrac{u_v^2}{2}\right) + \alpha_l\rho_l u_l\left(h_l + \dfrac{u_l^2}{2}\right) \\ \alpha_l\rho_l e_l u_l \end{bmatrix} . \tag{31}$$

This conservative form of the basic two-fluid model, although derived for a smooth solution $\boldsymbol{U}$, will be useful to define jump conditions for the approximate Riemann solver. We call this system the basic conservative system.

III. NUMERICAL METHOD

III.A. Approximate Riemann Solver for Nonconservative Systems

Before presenting our numerical method for the two-fluid model, we review some of the basic theory of Godunov-type numerical schemes,[22] and we give an extension to hyperbolic nonconservative systems.

We discretize the x-t plane by choosing a mesh width Δx and a time step Δt, and we define the mesh points:

$$x_j = j\Delta x \quad \text{and} \quad x_{j+1/2} = x_j + \frac{\Delta x}{2} \ . \tag{32}$$

To make it simpler, we take a uniform mesh with Δx constant, although the methods discussed can be extended to variable meshes. Finite difference methods produce approximations $\boldsymbol{U}_j^n$ to the solutions $\boldsymbol{U}(x_j, n\Delta t)$ at the discrete grid points.

In Godunov's method,[11] the approximate solution $\boldsymbol{U}^{n+1}$, at time $(n+1)\Delta t$, is obtained by solving Riemann problems at cell interfaces:

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n + \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_j} \boldsymbol{U}_e\left(\frac{x - x_{j-1/2}}{\Delta t}, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n\right) dx + \frac{1}{\Delta x}\int_{x_j}^{x_{j+1/2}} \boldsymbol{U}_e\left(\frac{x - x_{j+1/2}}{\Delta t}, \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n\right) dx \ , \tag{33}$$

where

$$\boldsymbol{U}_e\left(\frac{x - x_{j-1/2}}{\Delta t}, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n\right)$$

is the exact solution of the Riemann problem

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{f}(\boldsymbol{U}) = 0 \ ,$$

$$\boldsymbol{U}(x,0) = \boldsymbol{U}_{j-1}^n \quad (x < x_{j-1/2}) \ ,$$

and

$$\boldsymbol{U}(x,0) = \boldsymbol{U}_j^n \quad (x > x_{j-1/2}) \ . \tag{34}$$

In practice, solving the aforementioned nonlinear Riemann problem may be difficult and time consuming because it typically requires some iterations for nonlinear equations. However, most of the structure of the exact solution to the Riemann problem is not used, due to the averaging process over each grid cell. This suggests using approximate Riemann solvers to build more efficient Godunov-type numerical methods. One of the most popular approximate Riemann solvers, currently in use in aerodynamics problems, is due to Roe.[12] To solve the nonlinear Riemann problem Eq. (34) for hyperbolic conservative systems, Roe introduces a local linearization

$$\partial_t \boldsymbol{U} + \mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)\partial_x \boldsymbol{U} = 0 \ , \tag{35}$$

where the matrix $\mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)$, known as the Roe-averaged matrix, is some Jacobian matrix depending on the initial states $\boldsymbol{U}_{j-1}^n$ and $\boldsymbol{U}_j^n$. The Roe-averaged matrix $\mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)$ is constructed to have the property

$$\boldsymbol{f}(\boldsymbol{U}_j^n) - \boldsymbol{f}(\boldsymbol{U}_{j-1}^n) = \mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)(\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n) \ . \tag{36}$$

This property guarantees that, when $\boldsymbol{U}_{j-1}^n$ and $\boldsymbol{U}_j^n$ are connected by a single shock wave, the approximate Riemann solution agrees with the exact solution. In this case, the Rankine-Hugoniot condition is satisfied

for some shock speed σ [equal to an eigenvalue of $\mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)$]:

$$\boldsymbol{f}(\boldsymbol{U}_j^n) - \boldsymbol{f}(\boldsymbol{U}_{j-1}^n) = \sigma(\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n) \ . \tag{37}$$

However, for our nonconservative system, modeling two-phase flows, Roe's method does not apply since the matrix $\mathbf{A}(\boldsymbol{U})$ is not the derivative of a flux function $\boldsymbol{f}(\boldsymbol{U})$. To overcome this difficulty, we use a weak formulation of Roe's approximate Riemann solver,[14] which has been applied to construct a Roe-averaged Jacobian matrix for the Euler equations with arbitrary equations of state. This weak formulation has also been used to build approximate Riemann solvers for both a homogeneous two-phase flow model,[14] and an isentropic two-fluid model.[20]

Precisely, we consider approximate solutions

$$\boldsymbol{U}_a\left(\frac{x - x_{j-1/2}}{\Delta t}, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n\right)$$

to the nonlinear Riemann problem, which are exact solutions to the following linear problem:

$$\partial_t \boldsymbol{U} + \mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)_\Phi \partial_x \boldsymbol{f}(\boldsymbol{U}) = 0 \ ,$$

$$\boldsymbol{U}(x,0) = \boldsymbol{U}_{j-1}^n \quad (x < x_{j-1/2}) \ ,$$

and

$$\boldsymbol{U}(x,0) = \boldsymbol{U}_j^n \quad (x > x_{j-1/2}) \ , \tag{38}$$

where the matrix $\mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)_\Phi$ is now depending also on a smooth path $\Phi(s, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)$ linking the two states $\boldsymbol{U}_{j-1}^n$ and $\boldsymbol{U}_j^n$ in the states space:

$$\Phi(0, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n) = \boldsymbol{U}_{j-1}^n$$

and

$$\Phi(1, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n) = \boldsymbol{U}_j^n \ . \tag{39}$$

This generalized Roe-averaged matrix must satisfy the jump condition

$$\int_0^1 \mathbf{A}[\Phi(s, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)] \frac{\partial \Phi}{\partial s} \, ds = \mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)_\Phi (\boldsymbol{U}_j^n - \boldsymbol{U}_{i-1}^n) \ , \tag{40}$$

which is a generalization of Roe's condition Eq. (36) to nonconservative systems. This condition shows that a single shock with speed σ satisfies the generalized Rankine-Hugoniot condition given in Ref. 23:

$$\int_0^1 \{\mathbf{A}[\Phi(s, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)] - \sigma \boldsymbol{I}\} \frac{\partial \Phi}{\partial s} \, ds = 0 \ . \tag{41}$$

We refer the reader to Ref. 14 for more details on this formulation of Roe's approximate solver and its application to conservative two-phase flow models.

In practice, to construct such a linearized matrix, we introduce a parameter vector $\boldsymbol{W}$ and a canonical path defined by

$$\Phi(s, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n) = \Psi_0[\boldsymbol{W}_{j-1}^n + s(\boldsymbol{W}_j^n - \boldsymbol{W}_{j-1}^n)] \quad s \in [0,1] \ , \tag{42}$$

where Ψ is a smooth function such that $\Psi_0(\boldsymbol{W}) = \boldsymbol{U}$ and $\mathbf{A}_0(\boldsymbol{W}) = \partial \Psi_0 / \partial \boldsymbol{W}$ is a regular matrix for every state $\boldsymbol{W}$. From Eq. (42) we deduce

$$\frac{\partial \Phi}{\partial s}(s, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n) = \mathbf{A}_0[\boldsymbol{W}_{j-1}^n + s(\boldsymbol{W}_j^n - \boldsymbol{W}_{j-1}^n)] (\boldsymbol{W}_j^n - \boldsymbol{W}_{j-1}^n) \tag{43}$$

and

$$\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n = \left\{ \int_0^1 \mathbf{A}_0[\boldsymbol{W}_{j-1}^n + s(\boldsymbol{W}_j^n - \boldsymbol{W}_{j-1}^n)] \, ds \right\} \times (\boldsymbol{W}_j^n - \boldsymbol{W}_{j-1}^n) \ . \tag{44}$$

Using these equations and the condition Eq. (40), we get the definition of the Roe-averaged matrix associated with the path Eq. (41):

$$\mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)_\Phi = \tilde{\mathbf{C}}_{j-1/2} \tilde{\mathbf{B}}_{j-1/2}^{-1} \tag{45}$$

with

$$\tilde{\mathbf{B}}_{j-1/2} = \int_0^1 \mathbf{A}_0[\boldsymbol{W}_{j-1}^n + s(\boldsymbol{W}_j^n - \boldsymbol{W}_{j-1}^n)] \, ds \tag{46}$$

and

$$\tilde{\mathbf{C}}_{j-1/2} = \int_0^1 \mathbf{A}[\boldsymbol{W}_{j-1}^n + s(\boldsymbol{W}_j^n - \boldsymbol{W}_{j-1}^n)] \times \mathbf{A}_0[\boldsymbol{W}_{j-1}^n + s(\boldsymbol{W}_j^n - \boldsymbol{W}_{j-1}^n)] \, ds \ . \tag{47}$$

Once the Roe-averaged matrix has been constructed, the linear Riemann problem is relatively easy to solve by using its eigenvalues $\lambda_{j-1/2}^k$ and its eigenvectors $\boldsymbol{R}_{j-1/2}^k$. If we decompose

$$\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n = \sum_k \beta_{j-1/2}^k \boldsymbol{R}_{j-1/2}^k \ , \tag{48}$$

then the exact solution of the linear Riemann problem and the approximate solution of Eq. (34) are given by

$$\boldsymbol{U}_a(\xi, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n) = \boldsymbol{U}_{j-1}^n + \sum_{\lambda_{j-1/2}^k < \xi} \beta_{j-1/2}^k \boldsymbol{R}_{j-1/2}^k \ , \tag{49}$$

where the sum is over all the eigenvalues for which $\lambda_{j-1/2}^k < \xi$. Equivalently,

$$\boldsymbol{U}_a(\xi, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n) = \boldsymbol{U}_j^n - \sum_{\lambda_{j-1/2}^k > \xi} \beta_{j-1/2}^k \boldsymbol{R}_{j-1/2}^k \ . \tag{50}$$

Substituting this approximate solution into Eq. (33) and setting $\xi = (x - x_{j-1/2})/\Delta t$ in the first integral and $\xi = (x - x_{j+1/2})/\Delta t$ in the second one, leads to

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n + \frac{\Delta t}{\Delta x}\int_0^{\Delta x/2\Delta t} \boldsymbol{U}_a(\xi, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)\, d\xi + \frac{\Delta t}{\Delta x}\int_{-(\Delta x/2\Delta t)}^{0} \boldsymbol{U}_a(\xi, \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n)\, d\xi \ . \quad (51)$$

Thus from Eqs. (49) and (50), the expression of the numerical scheme follows:

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - \frac{\Delta t}{\Delta x}\left[\mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)_{\clubsuit}^{+}(\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n) + \mathbf{A}(\boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n)_{\clubsuit}^{-}(\boldsymbol{U}_{j+1}^n - \boldsymbol{U}_j^n)\right] \quad (52)$$

with the negative and the positive part of the Roe-averaged matrix defined by

$$\mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)_{\clubsuit}^{\pm} = \mathbf{R}_{j-1/2}\boldsymbol{\Lambda}_{j-1/2}^{\pm}\mathbf{R}_{j-1/2}^{-1} \ . \quad (53)$$

The value $\boldsymbol{\Lambda}_{j-1/2}^{\pm}$ is the diagonal matrix containing the negative and the positive parts of the eigenvalues:

$$\boldsymbol{\Lambda}_{j-1/2}^{\pm} = \mathrm{diag}(\lambda_{j-1/2}^{1\pm}, \ldots, \lambda_{j-1/2}^{m\pm}) \quad (54)$$

with

$$\lambda_{j-1/2}^{k-} = \min(0, \lambda_{j-1/2}^{k})$$

and

$$\lambda_{j-1/2}^{k+} = \max(0, \lambda_{j-1/2}^{k}) \ . \quad (55)$$

The numerical scheme of Eq. (52) is written in a nonconservative form. However, if the model has a conservation form, another expression can be derived for the numerical scheme:

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - \frac{\Delta t}{\Delta x}\left[\boldsymbol{F}(\boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n) - \boldsymbol{F}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)\right] \ , \quad (56)$$

where $\boldsymbol{F}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)$ is the numerical flux given by

$$\boldsymbol{F}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n) = \boldsymbol{f}(\boldsymbol{U}_j^n) - \mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)_{\clubsuit}^{+} \times (\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n) \ . \quad (57)$$

III.B. Parameter Vector for the Two-Fluid Model

To build an approximate Riemann solver for the two-fluid model, we have to find a parameter vector $\boldsymbol{W}$ and to apply the method just described. To derive the parameter $\boldsymbol{W}$, we look for a set of new variables w_i such that most of the conservative variables can be written as some quadratic function of the variables w_i. This leads to the following parameter vector:

$$\boldsymbol{W} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha_v \rho_v} \\ \sqrt{\alpha_l \rho_l} \\ \sqrt{\alpha_v \rho_v} u_v \\ \sqrt{\alpha_l \rho_l} u_l \\ \sqrt{\alpha_v \rho_v}\left(h_v + \dfrac{u_v^2}{2}\right) \\ \sqrt{\alpha_l \rho_l}\left(h_l + \dfrac{u_l^2}{2}\right) \end{bmatrix} \ . \quad (58)$$

Consequently, the function $\boldsymbol{\Psi}_0(\boldsymbol{W})$ is defined by

$$\boldsymbol{\Psi}_0(\boldsymbol{W}) = \begin{vmatrix} w_1^2 \\ w_2^2 \\ w_1 w_3 + w_2 w_4 \\ w_2 w_4 \\ w_1 w_5 - \alpha_v p \\ w_2 w_6 - \alpha_l p \end{vmatrix} \ . \quad (59)$$

Straightforward computations yield

$$\mathbf{A}_0(\boldsymbol{W}) = \begin{bmatrix} 2w_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2w_2 & 0 & 0 & 0 & 0 \\ w_3 & 0 & w_1 & 0 & 0 & 0 \\ 0 & w_4 & 0 & w_2 & 0 & 0 \\ w_5 - (\alpha_v p)_w^1 & -(\alpha_v p)_w^2 & -(\alpha_v p)_w^3 & -(\alpha_v p)_w^4 & w_1 - (\alpha_v p)_w^5 & -(\alpha_v p)_w^6 \\ -(\alpha_l p)_w^1 & w_2 - (\alpha_l p)_w^2 & -(\alpha_l p)_w^3 & -(\alpha_l p)_w^4 & -(\alpha_l p)_w^5 & w_2 - (\alpha_l p)_w^6 \end{bmatrix} \quad (60)$$

with the notation

$$(\alpha_k p)_w^i = \frac{\partial}{\partial w_i}(\alpha_k p) \ . \quad (61)$$

The matrix $\mathbf{A}_0(\boldsymbol{W})$ involves the partial derivatives of the pressure with respect to the components w_i of the parameter $\boldsymbol{W}$. By the chain rule for partial derivatives, we calculate

$$\nabla_w p = \left(\frac{\partial p}{\partial w_i}\right)_{i=1,6} = \begin{Bmatrix} 0 \\ \dfrac{\gamma-1}{\gamma}\dfrac{w_1}{\alpha_v} \\ 0 \\ -\dfrac{\gamma-1}{\gamma}\dfrac{w_1}{\alpha_v}u_v \\ -2\dfrac{\gamma-1}{\gamma}\dfrac{w_2}{\alpha_v}\left[\rho_v^2\Big/\left(\rho_l\dfrac{\partial \rho_v}{\partial h_v}\right)\right] \\ -\dfrac{\gamma-1}{\gamma}\dfrac{w_1}{\alpha_v}\left[h_v - \dfrac{u_v^2}{2} + 2\rho_v\Big/\left(\dfrac{\partial \rho_v}{\partial h_v}\right)\right] \end{Bmatrix} . \tag{62}$$

Then, using $\alpha_l = 1 - \alpha_v = w_2^2/\rho_l$, the Jacobian matrix $\mathbf{A}_0(\boldsymbol{W})$ becomes

$$\mathbf{A}_0(\boldsymbol{W}) = \begin{bmatrix} 2w_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2w_2 & 0 & 0 & 0 & 0 \\ w_3 & 0 & w_1 & 0 & 0 & 0 \\ 0 & w_4 & 0 & w_2 & 0 & 0 \\ w_5 - \alpha_v p_w^1 & -2\dfrac{w_2}{\rho_l}p - \alpha_v p_w^2 & -\alpha_v p_w^3 & 0 & w_1 - \alpha_v p_w^5 & 0 \\ -\alpha_l p_w^1 & w_2 + 2\dfrac{w_2}{\rho_l}p - \alpha_l p_w^2 & -\alpha_l p_w^3 & 0 & -\alpha_l p_w^5 & w_2 \end{bmatrix} . \tag{63}$$

III.C. Linearization of the Conservative Variables

In this section, we construct the average Jacobian matrix $\tilde{\mathbf{B}}_{j-1/2}$ of the conservative variables, using the canonical path for the parameter vector of Eq. (58) and also Eq. (46). Most of the integrals involved in the definition of the matrix $\tilde{\mathbf{B}}_{j-1/2}$ can be calculated explicitly. We have, for instance,

$$\int_0^1 [w_1^{j-1} + s(w_1^j - w_1^{j-1})]\, ds = w_1^{j-1} + \tfrac{1}{2}(w_1^j - w_1^{j-1}) = \overline{w}_1 , \tag{64}$$

where the overbar denotes the arithmetic mean value between cells $\boldsymbol{U}_{j-1}$ and $\boldsymbol{U}_j$. Then, using Eqs. (46) and (63), we obtain

$$\tilde{\mathbf{B}}_{j-1/2} = \begin{bmatrix} 2\overline{w}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\overline{w}_2 & 0 & 0 & 0 & 0 \\ \overline{w}_3 & 0 & \overline{w}_1 & 0 & 0 & 0 \\ 0 & \overline{w}_4 & 0 & \overline{w}_2 & 0 & 0 \\ \overline{w}_5 - \overline{\overline{\alpha_v p_w^1}} & -\dfrac{2}{\rho_l}\overline{\overline{w_2 p}} - \overline{\overline{\alpha_v p_w^2}} & -\overline{\overline{\alpha_v p_w^3}} & 0 & \overline{w}_1 - \overline{\overline{\alpha_v p_w^5}} & 0 \\ -\overline{\overline{\alpha_l p_w^1}} & \overline{w}_2 + \dfrac{2}{\rho_l}\overline{\overline{w_2 p}} - \overline{\overline{\alpha_l p_w^2}} & -\overline{\overline{\alpha_l p_w^3}} & 0 & -\overline{\overline{\alpha_l p_w^5}} & \overline{w}_2 \end{bmatrix} , \tag{65}$$

where the averages $\overline{\overline{\alpha_k p_w^i}}$ and $\overline{\overline{w_2 p}}$ are given by

$$\overline{\overline{\alpha_k p_w^i}} = \int_0^1 \alpha_k p_w^i [\boldsymbol{W}^{j-1} + s(\boldsymbol{W}^j - \boldsymbol{W}^{j-1})]\, ds \quad (66)$$

and

$$\overline{\overline{w_2 p}} = \int_0^1 [w_2^{j-1} + s(w_2^j - w_2^{j-1})] \times p[\boldsymbol{W}^{j-1} + s(\boldsymbol{W}^j - \boldsymbol{W}^{j-1})]\, ds \ . \quad (67)$$

In addition, we note that the aforementioned averages are not independent since

$$\begin{aligned}\Delta(\alpha_k p) &= (\alpha_k p)^j - (\alpha_k p)^{j-1} \\ &= \int_0^1 \frac{d}{ds}\{\alpha_k p[\boldsymbol{W}^{j-1} + s(\boldsymbol{W}^j - \boldsymbol{W}^{j-1})]\, ds\} \\ &= \left\{\int_0^1 \frac{d}{d\boldsymbol{W}}(\alpha_k p) \times [\boldsymbol{W}^{j-1} + s(\boldsymbol{W}^j - \boldsymbol{W}^{j-1})]\, ds\right\} \\ &\quad \times (\boldsymbol{W}^j - \boldsymbol{W}^{j-1}) \ , \end{aligned} \quad (68)$$

which gives the following necessary condition

$$\Delta(\alpha_k p) = \overline{\overline{\alpha_k p_w^1}}\Delta w_1 + \left(\overline{\overline{\alpha_k p_w^2}} \pm \frac{2}{\rho_l}\overline{\overline{w_2 p}}\right)\Delta w_2 + \overline{\overline{\alpha_k p_w^3}}\Delta w_3 + \overline{\overline{\alpha_k p_w^5}}\Delta w_5 \ . \quad (69)$$

The drawback of this formulation is the evaluation of the integrals in Eq. (69), which might be expensive to compute. As noted in Ref. 14, while an exact definition of the averages $\overline{\overline{\alpha_k p_w^i}}$ are given by Eq. (66), for practical implementation it is possible to replace them by some approximations, as long as the condition Eq. (69) is met precisely. Consequently, we choose the following approximations

$$\overline{\overline{\alpha_k p_w^i}} = \tilde{\alpha}_k \tilde{p}_w^i \quad (70)$$

and

$$\overline{\overline{w_2 p}} = \overline{w}_2 \tilde{p} \ , \quad (71)$$

where $\tilde{\alpha}_l = 1 - \tilde{\alpha}_v$ and $\tilde{p}$ are average void fraction and average pressure, respectively, satisfying the condition

$$\Delta(\alpha_l p) = \tilde{\alpha}_l \Delta p + \tilde{p}\Delta\alpha_l \ . \quad (72)$$

Proposition 1 (Sec. III.D) gives a complete definition of these averages.

On the other hand, the coefficients $\tilde{p}_w^i$ are approximations of the pressure derivatives Eq. (62), which must satisfy the equation

$$\Delta(p) = \sum_{i=1}^{6} \tilde{p}_w^i \Delta w_i \quad (73)$$

to meet the condition Eq. (69). A natural choice for these approximations is given by

$$\tilde{p}_w^i = \int_0^1 \frac{\partial p}{\partial w_i}[\boldsymbol{W}^{j-1} + s(\boldsymbol{W}^j - \boldsymbol{W}^{j-1})]\, ds \ . \quad (74)$$

However, other formulas can be chosen (see Ref. 14 for a definition of these coefficients requiring only the integral of the γ coefficient following the canonical path).

Finally, the matrix $\tilde{\mathbf{B}}_{j-1/2}$ is given by

$$\tilde{\mathbf{B}}_{j-1/2} = \begin{bmatrix} 2\overline{w}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\overline{w}_2 & 0 & 0 & 0 & 0 \\ \overline{w}_3 & 0 & \overline{w}_1 & 0 & 0 & 0 \\ 0 & \overline{w}_4 & 0 & \overline{w}_2 & 0 & 0 \\ \overline{w}_5 - \tilde{\alpha}_v\tilde{p}_w^1 & -\frac{2}{\rho_l}\overline{w}_2\tilde{p} - \tilde{\alpha}_v\tilde{p}_w^2 & -\tilde{\alpha}_v\tilde{p}_w^3 & 0 & \overline{w}_1 - \tilde{\alpha}_v\tilde{p}_w^5 & 0 \\ -\tilde{\alpha}_l\tilde{p}_w^1 & \overline{w}_2 + \frac{2}{\rho_l}\overline{w}_2\tilde{p} - \tilde{\alpha}_l\tilde{p}_w^2 & -\tilde{\alpha}_l\tilde{p}_w^3 & 0 & -\tilde{\alpha}_l\tilde{p}_w^5 & \overline{w}_2 \end{bmatrix} . \quad (75)$$

III.D. Linearization of the Convective Flux

The convective part of the system Eq. (13) is written in a nonconservative form. However, the linearized matrix $\tilde{\mathbf{C}}_{j-1/2}$ of the convective flux does not require a flux function $\boldsymbol{f}(\boldsymbol{U})$ and, hence, may be calculated for our nonconservative system using the same method as for the conservative variables. In this case the matrix $\tilde{\mathbf{C}}_{j-1/2}$, the Roe-averaged matrix $\mathbf{A}(\boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)_{\Phi}$, and the Rankine-Hugoniot conditions will be dependent on the path Φ.

For conservative systems [i.e., $\mathbf{A}(\boldsymbol{U}) = \partial \boldsymbol{f}/\partial \boldsymbol{U}$], the Rankine-Hugoniot conditions are independent from the path Φ since

$$\int_0^1 \mathbf{A}[\Phi(s, \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n)] \frac{\partial \Phi}{\partial s}\, ds = \boldsymbol{f}(\boldsymbol{U}_j^n) - \boldsymbol{f}(\boldsymbol{U}_{j-1}^n) \ . \quad (76)$$

Then, the path Φ is only useful to linearize the Jacobian matrix. Conversely, for nonconservative systems,

it is not surprising to have path-dependent jump conditions (this is discussed in Ref. 14).

Here, in order to preserve consistency with the basic conservative system Eq. (29), we proceed as follows:

1. First, we choose an average void fraction $\tilde{\alpha}_v$ and an average pressure $\tilde{P}$ so as to get a linearization of the nonconservative products and hence a conservative system at each cell interface. This conservative system must have the same jump conditions as the basic conservative system.

2. Then, we apply this method to construct the corresponding linearized Jacobian matrix.

By choosing averages for α_k and p in the nonconservative product of the basic two-fluid model, we obtain the linearized conservative system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}, \tilde{\alpha}_v, \tilde{\alpha}_l, \tilde{p}) = 0 \tag{77}$$

with the flux function

$$\boldsymbol{F}(\boldsymbol{U}, \tilde{\alpha}_v, \tilde{\alpha}_l, \tilde{p}) = \begin{bmatrix} \alpha_v \rho_v u_v \\ \alpha_l \rho_l u_l \\ \alpha_v \rho_v u_v^2 + \tilde{\alpha}_v p \\ \alpha_l \rho_l u_l^2 + \tilde{\alpha}_l p \\ \alpha_v \rho_v u_v \left(h_v + \dfrac{u_v^2}{2} \right) + \tilde{p} \alpha_l u_l \\ \alpha_l \rho_l u_l \left(h_l + \dfrac{u_l^2}{2} \right) - \tilde{p} \alpha_l u_l \end{bmatrix} . \tag{78}$$

The following proposition completes the definition of this flux function:

Proposition 1: Let $\boldsymbol{U}$ be a shock wave solution of the system Eq. (77), defined by

$$\boldsymbol{U}(x,t) = \boldsymbol{U}_L (x < \sigma t) \quad \text{and} \quad \boldsymbol{U}(x,t) = \boldsymbol{U}_R (x > \sigma t) . \tag{79}$$

Let $\tilde{\alpha}_v$, $\tilde{\alpha}_l$, and $\tilde{p}$ be defined by

$$\frac{1}{\tilde{\alpha}_l} = \frac{1}{2} \left[\frac{1}{\alpha_l(\boldsymbol{U}_L)} + \frac{1}{\alpha_l(\boldsymbol{U}_R)} \right] ,$$

$$\tilde{\alpha}_v = 1 - \tilde{\alpha}_l ,$$

and

$$\tilde{p} = \frac{\alpha_l p(\boldsymbol{U}_L) + \alpha_l p(\boldsymbol{U}_R)}{\alpha_l(\boldsymbol{U}_L) + \alpha_l(\boldsymbol{U}_R)} . \tag{80}$$

Then, $\boldsymbol{U}$ is a shock wave solution of the basic conservative system Eq. (31).

Proof: We have to show that the systems Eqs. (31) and (77) have the same Rankine-Hugoniot conditions. The total momentum and energy equations of the conservative system Eq. (31) are obtained by summing the phasic momentum and energy equations of the linearized conservative system Eq. (77). Then, we need only to compare the Rankine-Hugoniot conditions for the momentum and energy equations of the liquid phase. From Eq. (78), we obtain for the system Eq. (77):

$$[\alpha_l \rho_l u_l^2] + \tilde{\alpha}_l [p] = \sigma [\alpha_l \rho_l u_l] , \tag{81}$$

where $[.] = (.)_R - (.)_L$. Using the mass conservation for the liquid phase, we get

$$M = [\alpha_l \rho_l (\sigma - u_l)]_L = [\alpha_l \rho_l (\sigma - u_l)]_R \tag{82}$$

and

$$\tilde{\alpha}_l [p] = M [u_l] . \tag{83}$$

On the other hand, the Rankine-Hugoniot condition for the system Eq. (31) gives

$$\left[\frac{u_l^2}{2} + \frac{p}{\rho_l} \right] = \sigma [u_l] . \tag{84}$$

From Eqs. (83) and (84) we deduce

$$\frac{M}{\rho_l \tilde{\alpha}_l} = \sigma - \frac{1}{2} (u_l^L + u_l^R) . \tag{85}$$

Finally, using Eqs. (82) and (85), we obtain the average void fraction given by Eq. (80).

For the liquid energy equations, the jump conditions for the systems Eqs. (31) and (77) are given, respectively, by

$$[\alpha_l \rho_l e_l u_l] = \sigma [\alpha_l \rho_l e_l]$$

and

$$\left[\alpha_l \rho_l u_l \left(e_l + \frac{p}{\rho_l} + \frac{u_l^2}{2} \right) \right] - \tilde{p} [\alpha_l u_l] = \sigma \left[\alpha_l \rho_l \left(e_l + \frac{u_l^2}{2} \right) \right] . \tag{86}$$

These conditions are equivalent if

$$[\tfrac{1}{2} \alpha_l \rho_l (u_l - \sigma) u_l^2] + [\alpha_l u_l p] - \tilde{p} [\alpha_l u_l] = 0 . \tag{87}$$

Substituting Eqs. (82) and (83) into the previous equation, we obtain

$$-\tilde{\alpha}_l \bar{u}_l [p] + [\alpha_l u_l p] - \tilde{p} [\alpha_l u_l] = 0 . \tag{88}$$

This last condition is satisfied if $\tilde{p}$ is given by the expression Eq. (80). This settles the proof of proposition 1.

We can now use the canonical path Eq. (42) to get a linearization of the nonlinear Jacobian matrix

$$\mathbf{C}(\boldsymbol{W}, \tilde{\alpha}_v, \tilde{\alpha}_l, \tilde{p}) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{W}} (\boldsymbol{W}, \tilde{\alpha}_v, \tilde{\alpha}_l, \tilde{p}) . \tag{89}$$

To construct this matrix, we begin by rewriting the flux function $\boldsymbol{F}(\boldsymbol{U}, \tilde{\alpha}_v, \tilde{\alpha}_l, \tilde{p})$ as a function of the parameter $\boldsymbol{W}$:

$$\boldsymbol{F}(\boldsymbol{U},\tilde{\alpha}_v,\tilde{\alpha}_l,\tilde{p}) = \begin{bmatrix} w_1 w_3 \\ w_2 w_4 \\ w_3^2 + \tilde{\alpha}_v p(\boldsymbol{W}) \\ w_4^2 + \tilde{\alpha}_l p(\boldsymbol{W}) \\ w_3 w_5 + \dfrac{\tilde{p}}{\rho_l} w_2 w_4 \\ w_4 w_6 - \dfrac{\tilde{p}}{\rho_l} w_2 w_4 \end{bmatrix} . \quad (90)$$

From Eq. (90), we calculate the integral of the Jacobian matrix Eq. (89) following the canonical path:

$$\tilde{\mathbf{C}}_{j-1/2} = \begin{bmatrix} \overline{w}_3 & 0 & \overline{w}_1 & 0 & 0 & 0 \\ 0 & \overline{w}_4 & 0 & \overline{w}_2 & 0 & 0 \\ \tilde{\alpha}_v \tilde{p}_w^1 & \tilde{\alpha}_v \tilde{p}_w^2 & \tilde{\alpha}_v \tilde{p}_w^3 + 2\overline{w}_3 & 0 & \tilde{\alpha}_v \tilde{p}_w^5 & 0 \\ \tilde{\alpha}_l \tilde{p}_w^1 & \tilde{\alpha}_l \tilde{p}_w^2 & \tilde{\alpha}_l \tilde{p}_w^3 & 2\overline{w}_4 & \tilde{\alpha}_l \tilde{p}_w^5 & 0 \\ 0 & \overline{w}_4 \dfrac{\tilde{p}}{\rho_l} & \overline{w}_5 & \overline{w}_2 \dfrac{\tilde{p}}{\rho_l} & \overline{w}_3 & 0 \\ 0 & -\overline{w}_4 \dfrac{\tilde{p}}{\rho_l} & 0 & \overline{w}_6 - \overline{w}_2 \dfrac{\tilde{p}}{\rho_l} & 0 & \overline{w}_4 \end{bmatrix} . \quad (91)$$

III.E. Roe-Averaged Matrix for the Two-Fluid Model

We are now able to calculate the Roe-averaged matrix for the two-fluid model Eq. (13) by using Eq. (14) and the matrices $\tilde{\mathbf{B}}_{j-1/2}$ and $\tilde{\mathbf{C}}_{j-1/2}$ obtained in the previous sections. To simplify the expression of the Roe-averaged matrix, we focus on the pressure derivatives with respect to the conservative variables given by Eq. (18). By the chain rule for partial derivatives, we have

$$\nabla_u p = \left(\frac{\partial p}{\partial u_i}\right)_{i=1,6} = \frac{1}{w_1 - \alpha_v p_w^5} \begin{bmatrix} \dfrac{1}{2}\left(p_w^1 - \dfrac{w_3}{w_1} p_w^3 - \dfrac{w_5}{w_1} p_w^5\right) \\ \dfrac{w_1}{2w_2} p_w^2 - \dfrac{p}{\rho_l} p_w^5 \\ p_w^3 \\ 0 \\ p_w^5 \\ 0 \end{bmatrix} . \quad (92)$$

We note that these relations between p_u^i and p_w^i, while holding pointwise, do not necessarily hold for the average state quantities, which are functions of the two states $\boldsymbol{U}_{j-1}$ and $\boldsymbol{U}_j$. Then, a number of choices can be made for the approximations $\tilde{p}_u^i$, but it is clear that the most natural choice is to take

$$\nabla_u \tilde{p} = (\tilde{p}_u^i)_{i=1,6} = \frac{1}{\overline{w}_1 - \tilde{\alpha}_v \tilde{p}_w^5} \begin{bmatrix} \dfrac{1}{2}\left(\tilde{p}_w^1 - \dfrac{\overline{w}_3}{\overline{w}_1} \tilde{p}_w^3 - \dfrac{\overline{w}_5}{\overline{w}_1} \tilde{p}_w^5\right) \\ \dfrac{\overline{w}_1}{2\overline{w}_2} \tilde{p}_w^2 - \dfrac{\tilde{P}}{\rho_l} \tilde{p}_w^5 \\ \tilde{p}_w^3 \\ 0 \\ \tilde{p}_w^5 \\ 0 \end{bmatrix} . \quad (93)$$

We define also the average phasic velocity and the average phasic enthalpy

$$\tilde{u}_v = \frac{\overline{w}_3}{\overline{w}_1} = \frac{(\sqrt{\alpha_v \rho_v} u_v)_{j-1} + (\sqrt{\alpha_v \rho_v} u_v)_j}{(\sqrt{\alpha_v \rho_v})_{j-1} + (\sqrt{\alpha_v \rho_v})_j} ,$$

$$\tilde{u}_l = \frac{\overline{w}_4}{\overline{w}_2} = \frac{(\sqrt{\alpha_l \rho_l} u_l)_{j-1} + (\sqrt{\alpha_l \rho_l} u_l)_j}{(\sqrt{\alpha_l \rho_l})_{j-1} + (\sqrt{\alpha_l \rho_l})_j} ,$$

$$\tilde{H}_v = \frac{\overline{w}_5}{\overline{w}_1} = \frac{(\sqrt{\alpha_v \rho_v} H_v)_{j-1} + (\sqrt{\alpha_v \rho_v} H_v)_j}{(\sqrt{\alpha_v \rho_v})_{j-1} + (\sqrt{\alpha_v \rho_v})_j} ,$$

and

$$\tilde{H}_l = \frac{\overline{w}_6}{\overline{w}_2} = \frac{(\sqrt{\alpha_l \rho_l} H_l)_{j-1} + (\sqrt{\alpha_l \rho_l} H_l)_j}{(\sqrt{\alpha_l \rho_l})_{j-1} + (\sqrt{\alpha_l \rho_l})_j} . \quad (94)$$

Finally, substituting Eq. (92) into Eq. (82) together with the definitions of Eq. (94) leads to

$$\mathbf{A}(\boldsymbol{U}_{j-1}^{n},\boldsymbol{U}_{j}^{n})_{\clubsuit}=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \tilde{\alpha}_v\tilde{p}_u^1-\tilde{u}_v^2 & \tilde{\alpha}_v\tilde{p}_u^2 & \tilde{\alpha}_v\tilde{p}_u^3+2\tilde{u}_v & 0 & \tilde{\alpha}_v\tilde{p}_u^5 & 0 \\ \tilde{\alpha}_l\tilde{p}_u^1 & \tilde{\alpha}_l\tilde{p}_u^2-\tilde{u}_l^2 & \tilde{\alpha}_l\tilde{p}_u^3 & 2\tilde{u}_l & \tilde{\alpha}_l\tilde{p}_u^5 & 0 \\ (\tilde{\alpha}_v\tilde{p}_u^1-\tilde{H}_v)\tilde{u}_v & \left(\tilde{\alpha}_v\tilde{p}_u^2-\dfrac{\tilde{p}}{\rho_l}\right)\tilde{u}_v & \tilde{\alpha}_v\tilde{p}_u^3\tilde{u}_v+\tilde{H}_v & \dfrac{\tilde{p}}{\rho_l} & \tilde{\alpha}_v\tilde{p}_u^5\tilde{u}_v+\tilde{u}_v & 0 \\ \tilde{\alpha}_l\tilde{p}_u^1\tilde{u}_l & \left(\tilde{\alpha}_l\tilde{p}_u^2+\dfrac{\tilde{p}}{\rho_l}-\tilde{H}_l\right)\tilde{u}_l & \tilde{\alpha}_l\tilde{p}_u^3\tilde{u}_l & \tilde{H}_l-\dfrac{\tilde{p}}{\rho_l} & \tilde{\alpha}_l\tilde{p}_u^5\tilde{u}_l & \tilde{u}_l \end{bmatrix} . \quad (95)$$

We remark that this Roe-averaged matrix is equal to the exact Jacobian matrix $\mathbf{A}(\boldsymbol{U})$ of the basic two-fluid model, evaluated at some average state $\tilde{\boldsymbol{U}}$ defined by $\tilde{\alpha}_k$, $\tilde{u}_k$, $\tilde{H}_k$, and $\tilde{p}_u^i$:

$$\mathbf{A}(\boldsymbol{U}_{j-1}^{n},\boldsymbol{U}_{j}^{n})_{\clubsuit}=\mathbf{A}[\tilde{\boldsymbol{U}}(\tilde{\alpha}_k,\tilde{u}_k,\tilde{H}_k,\tilde{p}_u^i)] . \quad (96)$$

The linearized matrix $\mathbf{A}(\tilde{\boldsymbol{U}})$ has generally two complex eigenvalues. As for the exact Jacobian matrix $\mathbf{A}(\boldsymbol{U})$, the inclusion of the interface pressure term will lead to a hyperbolic linearized system. The linearized form of the interface pressure term is derived from Eqs. (8) and (9) using the average state $\tilde{\boldsymbol{U}}$:

$$\tilde{\boldsymbol{I}}=\boldsymbol{I}(\tilde{\boldsymbol{U}})=\tilde{\alpha}_v\tilde{\delta}(\tilde{u}_v-\tilde{u}_l)^2\partial_x\alpha_l\rho_l . \quad (97)$$

Finally, we obtain the Roe-averaged matrix for the complete two-fluid model by including the contribution of this latter term in Eq. (95).

III.F. Diagonalization of the Roe-Averaged Matrix

The diagonalization of the matrix $\mathbf{A}(\boldsymbol{U}_{j-1}^{n},\boldsymbol{U}_{j}^{n})_{\clubsuit}$ is necessary for the numerical scheme to calculate the negative and the positive part of the Roe-averaged matrix. This diagonalization may be done using a numerical method. Since computational speed is a major requirement, it is preferable to have analytical expressions for some approximations of the eigenvalues and the eigenvectors.

To obtain these approximations, we assume that the average mixture velocity is small compared with the average speed of sound,

$$\xi=\frac{(\tilde{u}_v-\tilde{u}_l)}{\tilde{a}_m}\ll 1 , \quad (98)$$

and we look for a first-order approximation of the eigenvalues and the eigenvectors.

For clarity of presentation, and to simplify the algebraic calculations, the vapor phase is assumed to be an ideal gas governed by the state function

$$P=(\gamma-1)\rho_v e_v \quad (99)$$

with a constant coefficient $\gamma>1$. Using this ideal state function, the average pressure derivatives get the following simple form:

$$\nabla_u\tilde{p}=(\tilde{p}_u^i)_{i=1,6}=\begin{bmatrix} \dfrac{\gamma-1}{\tilde{\alpha}_v}\tilde{u}_v^2 \\ \dfrac{\tilde{p}}{\tilde{\alpha}_v\rho_l} \\ -\dfrac{\gamma-1}{\tilde{\alpha}_v}\tilde{u}_v \\ 0 \\ \dfrac{\gamma-1}{\tilde{\alpha}_v} \\ 0 \end{bmatrix} . \quad (100)$$

Substituting these expressions of the pressure derivatives into Eq. (95), we obtain the Roe-averaged matrix for the ideal vapor phase case:

$$\tilde{\mathbf{A}}_{j-1/2}=\begin{Bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ (\gamma-3)\tilde{u}_v^2 & \dfrac{\tilde{p}}{\rho_l}-\tilde{\delta}\tilde{\alpha}_v(\tilde{u}_v-\tilde{u}_l)^2 & (3-\gamma)\tilde{u}_v & 0 & \gamma-1 & 0 \\ (\gamma-1)\dfrac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\dfrac{\tilde{u}_v^2}{2} & \dfrac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\dfrac{\tilde{p}}{\rho_l}-\tilde{u}_l^2+\tilde{\delta}\tilde{\alpha}_v(\tilde{u}_v-\tilde{u}_l)^2 & (1-\gamma)\dfrac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\tilde{u}_v & 2\tilde{u}_l & (\gamma-1)\dfrac{\tilde{\alpha}_l}{\tilde{\alpha}_v} & 0 \\ \left[(\gamma-1)\dfrac{\tilde{u}_v^2}{2}-\tilde{H}_v\right]\tilde{u}_v & 0 & \tilde{H}_v-(\gamma-1)\tilde{u}_v^2 & \dfrac{\tilde{p}}{\rho_l} & \gamma\tilde{u}_v & 0 \\ (\gamma-1)\dfrac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\dfrac{\tilde{u}_v^2}{2}\tilde{u}_l & \left(\dfrac{\tilde{p}}{\tilde{\alpha}_v\rho_l}-\tilde{H}_l\right)\tilde{u}_l & (1-\gamma)\dfrac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\tilde{u}_v\tilde{u}_l & \tilde{H}_l-\dfrac{\tilde{p}}{\rho_l} & (\gamma-1)\dfrac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\tilde{u}_l & \tilde{u}_l \end{Bmatrix} . \quad (101)$$

To determine the eigenvalues of the matrix $\tilde{\mathbf{A}}_{j-1/2}$, we must find the six roots of the characteristic polynomial $\det(\tilde{\mathbf{A}}_{j-1/2} - \lambda \mathbf{I})$. A straightforward computation of this determinant leads to the following polynomial:

$$\tilde{D}_A = (\tilde{u}_v - \lambda)(\tilde{u}_l - \lambda)\tilde{P}_A(\lambda) \tag{102}$$

with

$$\begin{aligned}\tilde{P}_A(\lambda) = {} & (\tilde{u}_v - \lambda)^2(\tilde{u}_l - \lambda)^2 - (\gamma - 1)\left(\tilde{H}_v - \frac{\tilde{u}_v^2}{2}\right) \\ & \times [(\tilde{u}_l - \lambda)^2 + \tilde{\delta}(\tilde{u}_v - \tilde{u}_l)^2] \\ & - \left[\gamma \frac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\frac{\tilde{P}}{\rho_l} - \tilde{\delta}\tilde{\alpha}_v(\tilde{u}_v - \tilde{u}_l)^2\right](\tilde{u}_v - \lambda)^2 \\ & - (\gamma - 1)\tilde{\delta}\tilde{\alpha}_l\tilde{u}_v(\tilde{u}_v - \tilde{u}_l)^2(\tilde{u}_v - \lambda) \ . \end{aligned} \tag{103}$$

This characteristic polynomial has two trivial roots $\lambda = \tilde{u}_v$ and $\lambda = \tilde{u}_l$, which correspond to the characteristic velocities of the phasic energy equations. The four other roots are given by the polynomial $\tilde{P}_A(\lambda)$. We expect to find two roots in the neighborhood of the phasic velocities and two roots depending on an average speed of sound. First, to give a first-order approximation of these roots, we discuss the case of a homogeneous two-fluid model, where the parameter $\xi = 0$. In this case, the two phases have the same velocity $\tilde{u}$ and the characteristic polynomial has the following limit:

$$\begin{aligned}\lim_{\xi \to 0} \tilde{D}_A(\lambda) = (\tilde{u} - \lambda)^4 & \left[(\tilde{u} - \lambda)^2 - (\gamma - 1)\left(\tilde{H}_v - \frac{\tilde{u}_v^2}{2}\right)\right. \\ & \left. - \gamma \frac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\frac{\tilde{P}}{\rho_l}\right] . \end{aligned} \tag{104}$$

This latter polynomial has six real roots

$$\tilde{\lambda}_1^0 = \tilde{u} - \tilde{a}_m \ ,$$

$$\tilde{\lambda}_2^0 = \tilde{\lambda}_3^0 = \tilde{\lambda}_4^0 = \tilde{\lambda}_5^0 = \tilde{u} \ ,$$

and

$$\tilde{\lambda}_6^0 = \tilde{u} + \tilde{a}_m \ , \tag{105}$$

where $\tilde{a}_m$ is the average speed of sound in the two-phase mixture, given by

$$\tilde{a}_m^2 = (\gamma - 1)\left(\tilde{H}_v - \frac{\tilde{u}_v^2}{2}\right) + \gamma \frac{\tilde{\alpha}_l}{\tilde{\alpha}_v}\frac{\tilde{P}}{\rho_l} \ . \tag{106}$$

We deduce that the linearized system is always hyperbolic if the two phases have the same velocity (even without corrective terms like the interface pressure term). We prove now that the simple roots $\tilde{\lambda}_1^0$ and $\tilde{\lambda}_6^0$ remain real, and the multiple roots $\tilde{\lambda} = \tilde{u}$ split into four distinct real roots.

It will be convenient to introduce the dimensionless variable

$$z = \frac{1}{\tilde{a}_m}\left(\frac{\tilde{u}_v + \tilde{u}_l}{2} - \lambda\right) \tag{107}$$

and to rewrite the polynomial $\tilde{P}_A$ as follows:

$$\frac{\tilde{P}_A(\lambda)}{\tilde{a}_m^4} = \tilde{P}_0(z) + \xi\tilde{P}_1(z) + \frac{\xi^2}{2}\tilde{P}_2(z) + \xi^2\tilde{P}_r(z,\xi) \tag{108}$$

with

$$\tilde{P}_0(z) = z^2(z - 1) \ ,$$

$$\tilde{P}_1(z) = \left(1 - 2\gamma \frac{\tilde{\alpha}_l\tilde{P}}{\tilde{\alpha}_v\rho_l\tilde{a}_m^2}\right)z \ ,$$

$$\begin{aligned}\tilde{P}_2(z) = -z^2 - \frac{1}{2} + 2\tilde{\delta}\left[1 - \gamma \frac{\tilde{\alpha}_l\tilde{P}}{\tilde{\alpha}_v\rho_l\tilde{a}_m^2}\right. \\ \left. + (\gamma - 1)\tilde{\alpha}_l \frac{\tilde{u}_v}{\tilde{a}_m} z - \tilde{\alpha}_v z^2\right] , \end{aligned} \tag{109}$$

and

$$\lim_{\xi \to 0} \tilde{P}_r(z,\xi) = 0 \ . \tag{110}$$

We look for the roots of the polynomial $\tilde{P}_A$ in the neighborhood of a root z_0 of the polynomial $\tilde{P}_0(z)$. We distinguish two cases, depending on whether z_0 is a simple root or a double root of $\tilde{P}_A$ (Refs. 24 and 25):

First case: z_0 is a simple root, $z_0 = \pm 1$. Then, the first-order approximation, defined for $\xi \neq 0$ close enough to zero, is given by

$$z(\xi) = z_0 - \frac{\tilde{P}_1(z_0)}{\tilde{P}_0'(z_0)}\xi + o(\xi) \ . \tag{111}$$

This case leads to the first-order approximations:

$$z_1 = 1 - \frac{1}{2}\left(1 - 2\gamma \frac{\tilde{\alpha}_l\tilde{P}}{\tilde{\alpha}_v\rho_l\tilde{a}_m^2}\right)\xi$$

and

$$z_6 = -1 - \frac{1}{2}\left(1 - 2\gamma \frac{\tilde{\alpha}_l\tilde{P}}{\tilde{\alpha}_v\rho_l\tilde{a}_m^2}\right)\xi \ . \tag{112}$$

Second case: z_0 is a double root of $\tilde{P}_0(z)$, and $\tilde{P}_1(z_0) = 0$. Then, at first order in ξ, the double root splits into two simple roots, defined for $\xi \neq 0$ close enough to zero and given by

$$z^{\pm} = z_0 + \xi z_1^{\pm} \ , \tag{113}$$

where $z_1^{\pm}$ are the two roots of the equation

$$\tilde{P}_0''(z_0)z_1^2 + 2\tilde{P}_1'(z_0)z_1 + \tilde{P}_2(z_0) = 0 \ . \qquad (114)$$

Consequently, the roots remain real at first order, and the linearized system is hyperbolic if the condition $\tilde{P}_1'(z_0)^2 - \tilde{P}_0''(z_0)\tilde{P}_2(z_0) \geq 0$ is satisfied. Using Eq. (109), this condition is equivalent to

$$\tilde{\delta} \geq \delta_0 = \gamma \frac{\tilde{\alpha}_l \tilde{P}}{\tilde{\alpha}_v \rho_l \tilde{a}_m^2} \qquad (115)$$

and

$$z^{\pm} = \tfrac{1}{2}\xi(1 - 2\delta_0) \ . \qquad (116)$$

Finally, by substituting Eq. (107) into Eq. (111) and Eq. (116), we obtain the first-order approximations in ξ of the eigenvalues of the Roe-averaged matrix:

$$\tilde{\lambda}_1^0 \approx (1 - \delta_0)\tilde{u}_v + \delta_0 \tilde{u}_l - \tilde{a}_m \ ,$$

$$\tilde{\lambda}_2^0 \approx \delta_0 \tilde{u}_v + (1 - \delta_0)\tilde{u}_l - [(1 - \delta_0)(\delta - \delta_0)]^{1/2}(\tilde{u}_v - \tilde{u}_l) \ ,$$

$$\tilde{\lambda}_3^0 \approx \delta_0 \tilde{u}_v + (1 - \delta_0)\tilde{u}_l + [(1 - \delta_0)(\delta - \delta_0)]^{1/2}(\tilde{u}_v - \tilde{u}_l) \ ,$$

$$\tilde{\lambda}_4^0 = \tilde{u}_v \ ,$$

$$\tilde{\lambda}_5^0 = \tilde{u}_l \ ,$$

and

$$\tilde{\lambda}_6^0 \approx (1 - \delta_0)\tilde{u}_v + \delta_0 \tilde{u}_l + \tilde{a}_m \ . \qquad (117)$$

Following the same method, we obtain after a lengthy calculation, a first-order approximation of the eigenvectors of the Roe-averaged matrix:

Acoustic waves: eigenvectors associated with eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_6$

$$\tilde{\boldsymbol{R}}_1 = \begin{Bmatrix} \tilde{\alpha}_v\left(1 - \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right) \\ \tilde{\alpha}_l\left(1 + \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right) \\ \tilde{\alpha}_v\left(1 - \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)\tilde{\lambda}_1 \\ \tilde{\alpha}_l\left(1 + \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)\tilde{\lambda}_1 \\ \tilde{\alpha}_v\left(1 - \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)[\tilde{H}_v + \tilde{u}_v(\tilde{\lambda}_1 - \tilde{u}_v)] + \tilde{\alpha}_l\left(1 + \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)\dfrac{\tilde{p}}{\rho_l} \\ \tilde{\alpha}_l\left(1 + \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)[\tilde{H}_l + \tilde{u}_l(\tilde{\lambda}_1 - \tilde{u}_l)] - \tilde{\alpha}_l\left(1 + \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)\dfrac{\tilde{p}}{\rho_l} \end{Bmatrix} \qquad (118)$$

and

$$\tilde{\boldsymbol{R}}_6 = \begin{Bmatrix} \tilde{\alpha}_v\left(1 + \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right) \\ \tilde{\alpha}_l\left(1 - \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right) \\ \tilde{\alpha}_v\left(1 + \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)\tilde{\lambda}_6 \\ \tilde{\alpha}_l\left(1 - \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)\tilde{\lambda}_6 \\ \tilde{\alpha}_v\left(1 + \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)[\tilde{H}_v + \tilde{u}_v(\tilde{\lambda}_6 - \tilde{u}_v)] + \tilde{\alpha}_l\left(1 - \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)\dfrac{\tilde{p}}{\rho_l} \\ \tilde{\alpha}_l\left(1 - \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)[\tilde{H}_l + \tilde{u}_l(\tilde{\lambda}_6 - \tilde{u}_l)] - \tilde{\alpha}_l\left(1 - \dfrac{\tilde{u}_v - \tilde{u}_l}{\tilde{a}_m}\right)\dfrac{\tilde{p}}{\rho_l} \end{Bmatrix} \qquad (119)$$

Void fraction and pressure waves: eigenvectors associated to eigenvalues $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$

$$\tilde{\boldsymbol{R}}_{2,3} = \begin{bmatrix} 1 \\ \dfrac{\tilde{\alpha}_l(\delta_0 - 1)}{\tilde{\alpha}_v \delta_0} \\ \tilde{\lambda}_{2,3} \\ \dfrac{\tilde{\alpha}_l(\delta_0 - 1)}{\tilde{\alpha}_v \delta_0} \tilde{\lambda}_{2,3} \\ \dfrac{1}{\gamma}\left(\tilde{H}_v - \dfrac{\tilde{u}_v^2}{2}\right) - \tilde{u}_v\left(\dfrac{\tilde{u}_v}{2} - \tilde{\lambda}_{2,3}\right) \\ \dfrac{\tilde{\alpha}_l(\delta_0 - 1)}{\tilde{\alpha}_v \delta_0}\left(\tilde{H}_l - \dfrac{\tilde{P}}{\rho_l}\right) \end{bmatrix} . \quad (120)$$

Entropy waves: eigenvectors associated to eigenvalues $\tilde{\lambda}_4$ and $\tilde{\lambda}_5$

$$\tilde{\boldsymbol{R}}_4 = \begin{bmatrix} 1 \\ 0 \\ \tilde{u} \\ 0 \\ \dfrac{\tilde{u}_v^2}{2} \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{\boldsymbol{R}}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} . \quad (121)$$

IV. NUMERICAL RESULTS

In this section, we present some test problems and show the numerical results obtained, using the approximate Riemann solver built in Sec. III. The vapor was assumed to behave as an ideal gas with specific heat ratio $\gamma = 1.0931$. The liquid was assumed to have a constant density $\rho_l = 720\ \text{kg/m}^3$.

IV.A. Problem 1: Comparison with Lax-Friedrichs Scheme

The purpose of this first test is to compare the numerical method described in Sec. III with the standard Lax-Freidrichs scheme[26] on a Riemann problem for the two-fluid model. The Riemann problem consists of an initial pressure discontinuity at the middle of the pipe. The left and right states are defined by Table I.

It is impossible to construct an exact solution for the Riemann problem in a general case. However, for two states $\boldsymbol{U}_L$ and $\boldsymbol{U}_R$, which are close enough, we know that the exact solution is composed of six waves (either shock, rarefaction, or contact discontinuities) separated by constant states.[27] Because no analytical solution is available, the objective of this test is to compare the results obtained with this new numerical method with results of calculations that use the standard Lax-Friedrichs numerical scheme on the same system Eq. (77).

TABLE I

Initial States for Problem 1

Left State U_L	Right State U_R
$\alpha_L = 0.25$	$\alpha_R = 0.25$
$P_L = 20$ MPa	$P_R = 15$ MPa
$u_v^L = 0$ m/s	$u_v^R = 0$ m/s
$u_l^L = 0$ m/s	$u_l^R = 0$ m/s
$h_v^L = 3092.7$ kJ/kg	$h_v^R = 3092.7$ kJ/kg
$h_l^L = 1338.2$ kJ/kg	$h_l^R = 1338.2$ kJ/kg

The calculations have been done using 200 cells and the same time step. Figures 1 and 2 give a comparison between the two schemes on the linearized conservative system Eq. (77) for the pressure and the liquid velocity.

As expected, the Lax-Friedrichs scheme has much more numerical diffusion than the approximate Riemann solve, but the results obtained with the two methods are in good agreement.

IV.B. Problem 2: Two-Phase Shock-Tube Problem

This problem consists of a Riemann problem for the two-fluid model with the left and right states as defined in Table II. The computations have been done with 300 nodes and using an interface pressure term $\delta = \sigma\delta_0$ with $\sigma = 1.01$ and $\sigma = 10$. Figure 3 shows the pressure profile for the two cases. We note that the pressure is constant through the waves associated with the eigenvalues λ_2 to λ_5. Moreover, the characteristic velocities of the first and the sixth wave are not very sensitive to the interface pressure term.

Figure 4 shows the liquid velocity for the two values of the coefficient σ. For this variable, both the intermediate states and the characteristic velocities are

TABLE II

Initial States for Problem 2

Left State U_L	Right State U_R
$\alpha_L = 0.25$	$\alpha_R = 0.10$
$P_L = 20$ MPa	$P_R = 10$ MPa
$u_v^L = 0$ m/s	$u_v^R = 0$ m/s
$u_l^L = 1$ m/s	$u_l^R = 1$ m/s
$h_v^L = 3092.7$ kJ/kg	$h_v^R = 3099.9$ kJ/kg
$h_l^L = 1338.2$ kJ/kg	$h_l^R = 1343.4$ kJ/kg

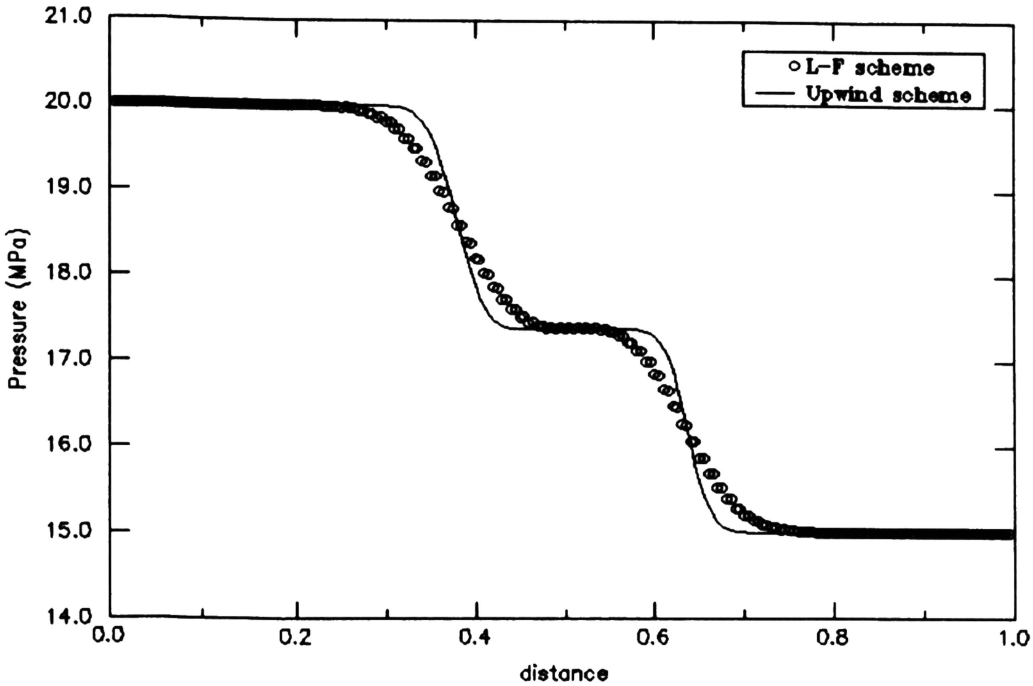

Fig. 1. Comparison with the Lax-Friedrichs scheme (pressure).

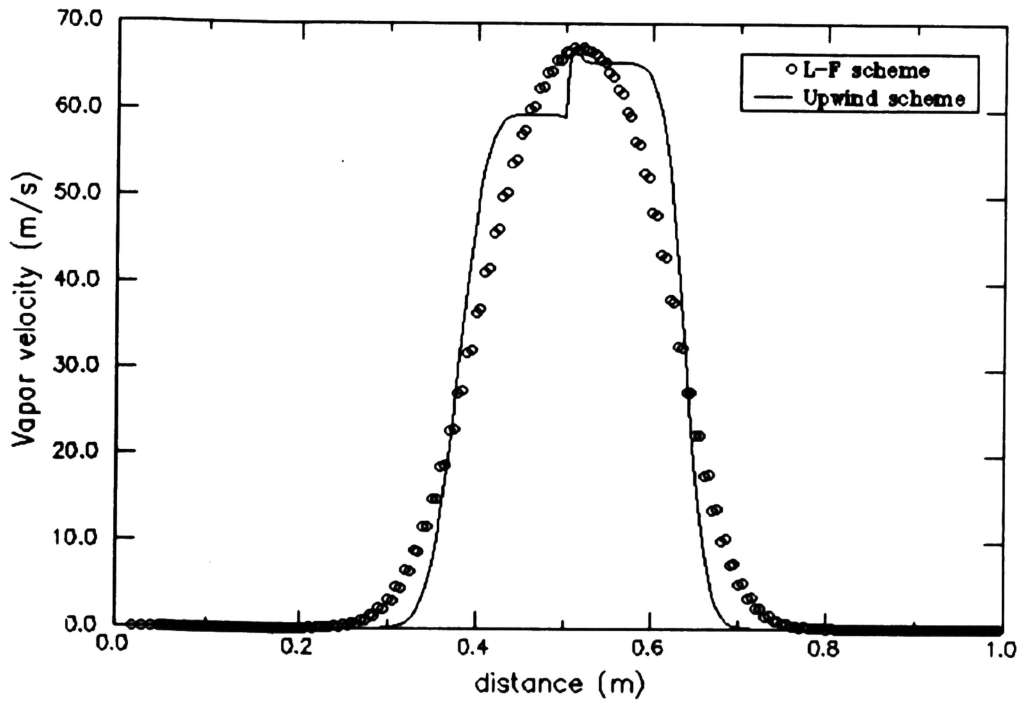

Fig. 2. Comparison with the Lax-Friedrichs scheme (vapor velocity).

strongly dependent on the interface pressure term. Figure 5 shows that the total energy profile is composed of seven constant states separated by six waves. However, the propagation velocities of the second and third waves being close to each other for δ close to δ_0, these waves are not well separated for $\sigma = 1.01$.

IV.C. Problem 3: Gravity-Dominated Flow Problem

This test proposed by Ransom[28] consists of a vertical water jet, contained within a cylindrical channel, that is accelerated under the action of gravity. At the

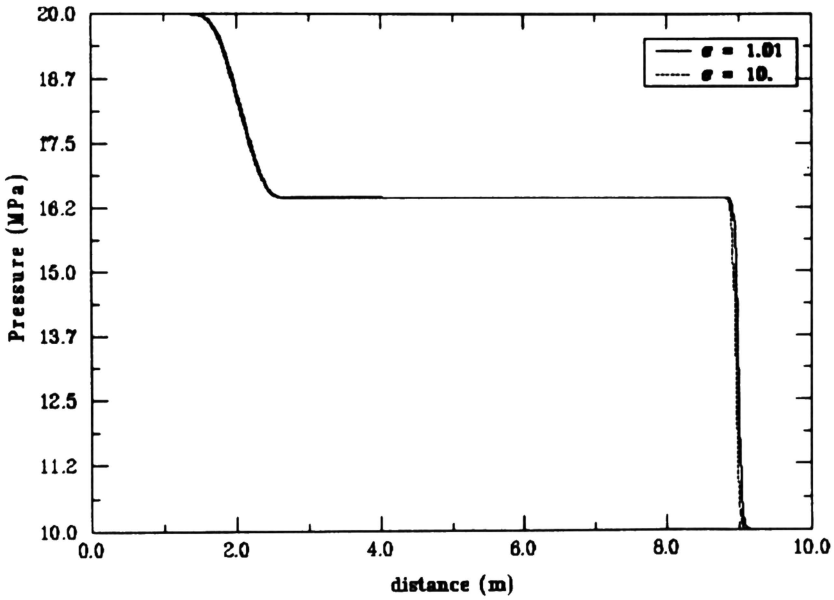

Fig. 3. Two-phase shock tube problem (pressure).

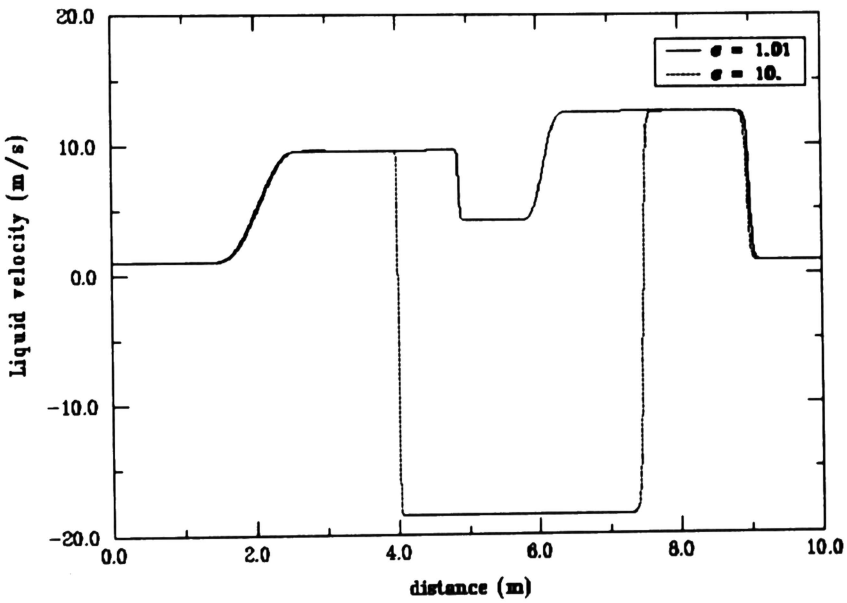

Fig. 4. Two-phase shock tube problem (liquid velocity).

initial state, the pipe is filled with a uniform column of water surrounded by stagnant vapor, such that the void fraction is 0.2 and the column has a uniform velocity of 10 m/s and a uniform pressure of 10^5 Pa.

The boundary conditions are specified velocities of 10 m/s for the liquid and 0 m/s for the vapor at the inlet and constant pressure at the outlet. The water faucet problem can be solved analytically by making some further idealizations.[28] This analytical solution was used as a code test in Ref. 8. As illustrated on Fig. 6, a void wave develops and is propagated at liquid velocity. Once the void wave exits the pipe, a steady void profile is established. The calculation was carried out until a steady state was reached, with 96 nodes and a constant Courant-Friedrichs-Levy (CFL) number equal to 0.9. The interface pressure term was equal to $\delta = \sigma\delta_0$ with $\sigma = 1.01$. Figure 7 shows the vapor void fraction profile at various times. To test the convergence and the stability character of the scheme, computations have been done using a discretization with 12, 24, 48, 96, 192,

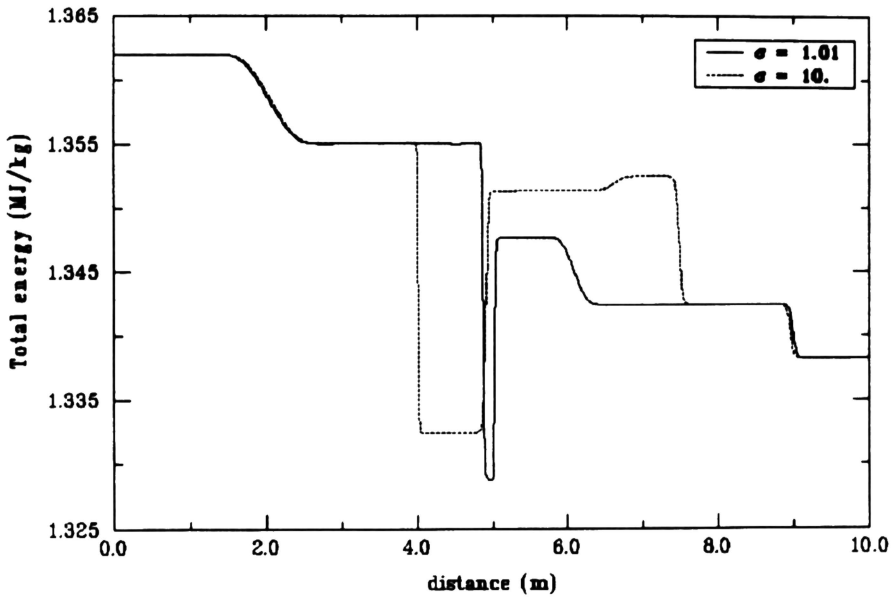

Fig. 5. Two-phase shock tube problem (total energy).

and 768 cells, but constant CFL numbers equal to 0.9. Figure 8 gives a comparison of the void fraction profile for the various discretizations and the analytical solution at time $t = 0.5$ s. An interesting feature of the results shown in Fig. 8 is that there are no oscillations at the discontinuity of the void fraction when the meshing is refined. These results clearly demonstrate the ability of the numerical scheme to capture discontinuities.

IV.D. Problem 4: Edwards Pipe Blowdown Experiment

This standard test for transient two-phase codes concerns the prediction of the blowdown of an initially subcooled liquid from a pipe of 4-m length.[29] This test contains features similar to a pressurized water reactor loss-of-coolant accident. The water in the pipe has an initial pressure of 7.0 MPa. The transient is initiated by opening the right side of the pipe. The pressure of the environment is atmospheric pressure.

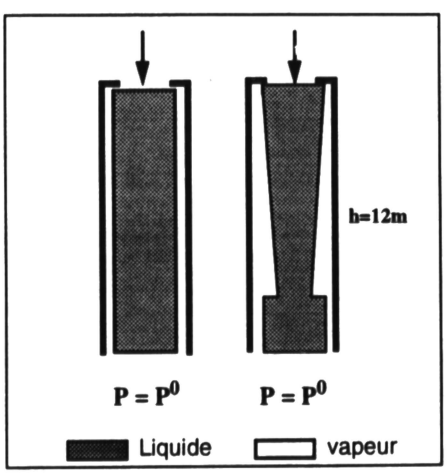

Fig. 6. Schematic of the water faucet problem.

In this test, we have introduced source terms in the mass and energy equations to model the mass and energy transfer between phases. The interfacial heat transfer into phase k is given by

$$Q_k^i = \alpha_v \alpha_l \rho_k \frac{(h_k^{sat} - h_k)}{\tau} ,$$

where the parameter τ can be considered as a constant rate for the interfacial heat transfer process. This expression was chosen not because of any expected physical validity, although it should be adequate as a rough approximation, but for its simplicity. The calculations have been done with 50 cells for the spatial discretization.

During the first 10 ms of the transient, a rarefaction wave propagates from the open end into the pipe, leading to a fast depressurization and the onset of flashing of the subcooled liquid. Figure 9 shows a comparison between experimental and predicted values using various rates τ of the interfacial heat transfer process, for the pressure history at the closed end. The results obtained with a simplified model show a reasonable agreement for $\tau = 5 \times 10^{-2}$.

IV.E. Problem 5: Two-Phase Blowdown Problem

The two-phase blowdown problem, proposed by Ransom and Hicks,[9] tests the ability of the model to simulate rapid transients. The blowdown calculations

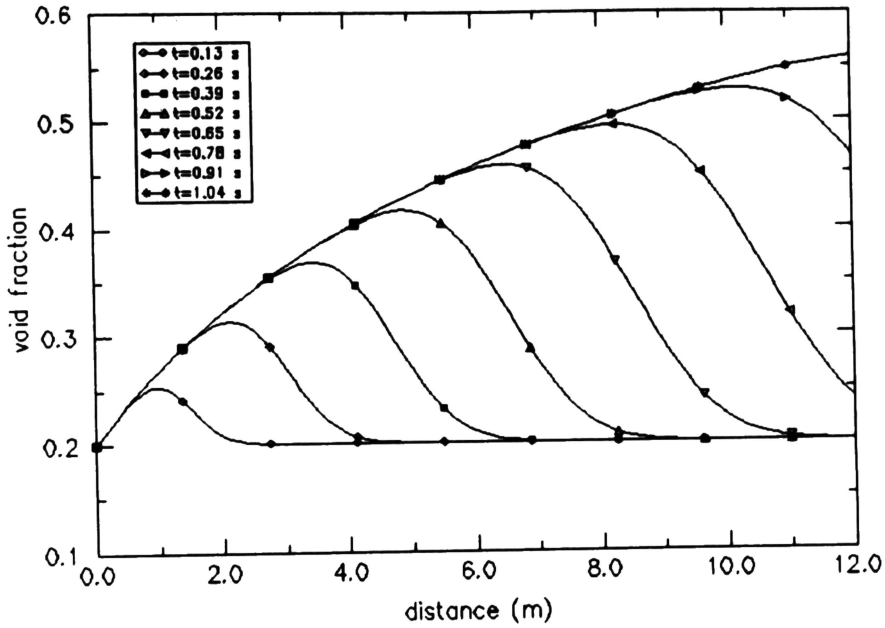

Fig. 7. Water faucet problem (transient solutions).

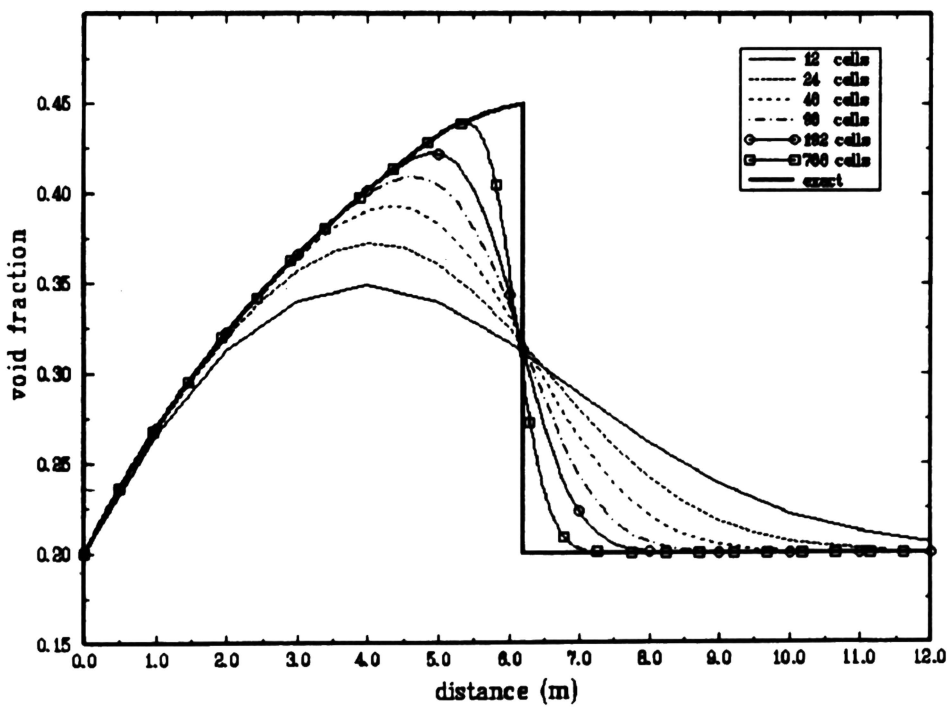

Fig. 8. Water faucet problem at time $t = 0.5$ s (convergence).

were made for a pipe 3 m long, discretized with 50 cells, and initially pressurized to 0.3 MPa. Boundary conditions were zero velocity at the closed end and constant ambient pressure at the open end. Figure 10 plots pressure versus time at the closed end of the pipe. This result agrees with the one obtained by Ransom and Hicks for the eight equation two pressure model.

V. CONCLUSION

We have presented here an extension to two-phase flow calculations of an upwind numerical method, based on approximate Riemann solvers, which turned out to be very efficient for fluid dynamics calculations. This numerical method seems very promising for the

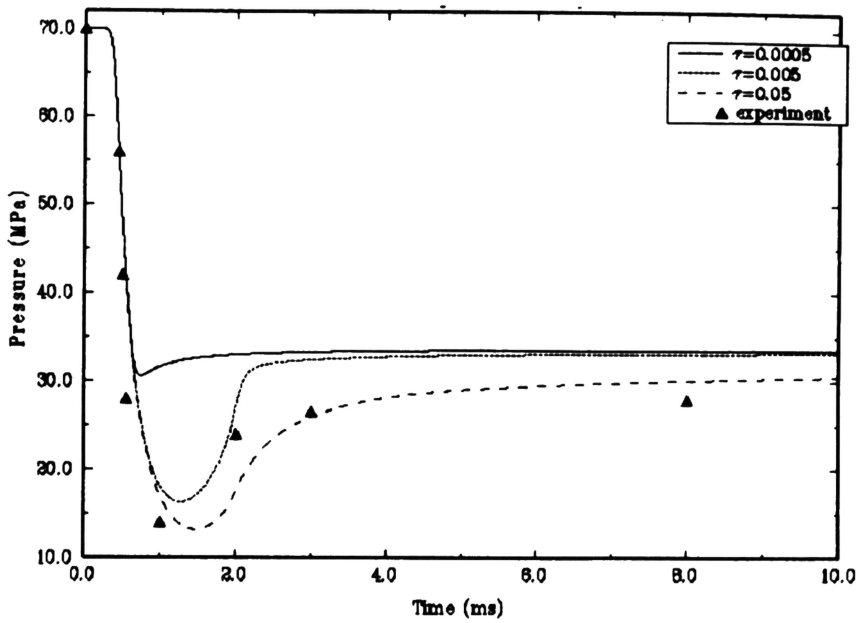

Fig. 9. Edwards pipe blowdown experiment (closed end pressure versus time).

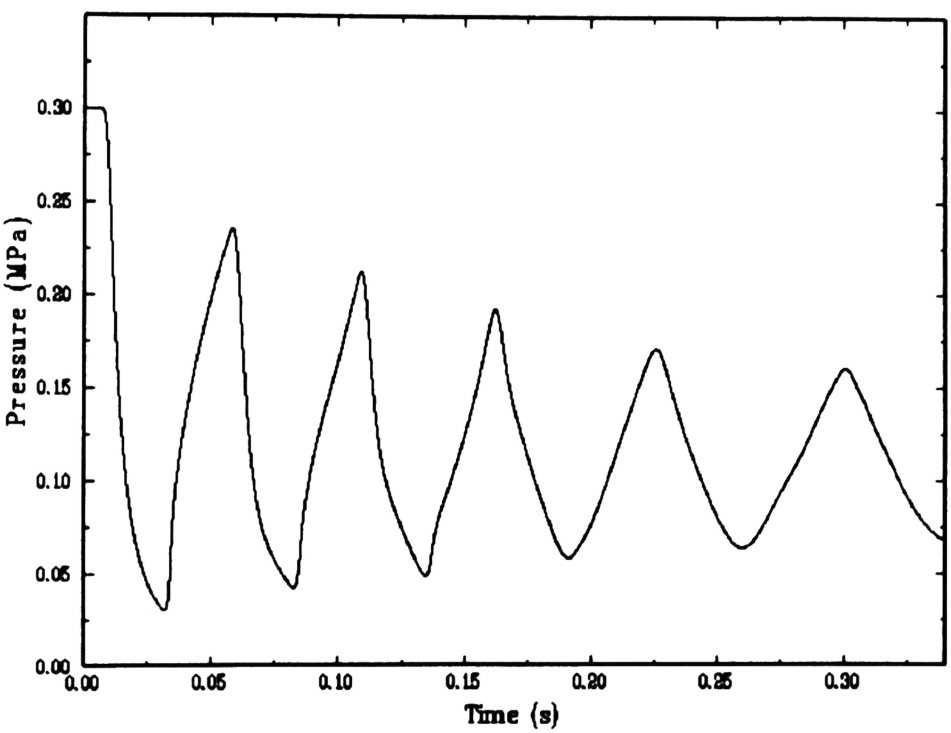

Fig. 10. Two-phase blowdown problem (closed end pressure versus time).

numerical solution of two-phase flows as is shown by the numerical results for standard two-phase flow problems.

Compared with standard numerical methods for two-fluid models, the new approach has the following advantages:

1. Its theoretical basis leads to upwind numerical fluxes, based on the characteristic information, without any heuristic for the upstream or downstream differencing. However, it requires a mathematical analysis for each new system to construct a good approximate Riemann solver.

2. It avoids the use of staggered grids to ensure stability. Consequently, this method can be easily extended to multidimensional nonstructured meshing.

3. It has low numerical diffusion, and it is designed to calculate strong discontinuities without oscillations.

This work is a first step in the study of more complete two-fluid models including sophisticated constitutive equations. Moreover, further work is planned to extend this numerical method to a second-order fully implicit algorithm, which is expected to provide an efficient way to calculate more realistic two-phase flow problems like those arising in design and safety studies of nuclear reactors.

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