Answers:

Q1:

$$\begin{cases} 3x_1 - x_2 + x_3 = 1 \\ 3x_1 + 6x_2 + 2x_3 = 0 \text{-----------[1.1]} \\ 3x_1 + 3x_2 + 7x_3 = 4 \end{cases}$$

$x^{(0)} = (0,0,0)$

$x_1 = \frac{1}{3}\ [x_2 - \ x_3 + 1]$ $\qquad$ ${x_1}^{(1)} = \frac{1}{3}\left[x_2^{(0)} - x_3^{(0)} + 1\right] = \frac{1}{3}$

$x_2 = \frac{1}{6}\ [-3x_1 - \ 2x_3]$ $\qquad$ ${x_2}^{(1)} = \frac{1}{6}\left[3x_1^{(0)} - 2x_3^{(0)}\right] = 0$

$x_3 = \frac{1}{7}\ [-3x_1 - \ 3x_2 + 4]$ $\qquad$ ${x_3}^{(1)} = \frac{1}{7}\left[-3x_1^{(0)} - 3x_2^{(0)} + 4\right] = \frac{4}{7}$

${x_1}^{(2)} = \frac{1}{3}\left[x_2^{(1)} - x_3^{(1)} + 1\right] = 0.142$

${x_2}^{(2)} = \frac{1}{6}\left[3x_1^{(1)} - 2x_3^{(1)}\right] = -0.357$

${x_3}^{(2)} = \frac{1}{7}\left[-3x_1^{(1)} - 3x_2^{(1)} + 4\right] = 0.428$

$$x^{(2)} = (0.142, -0.357, 0.428)$$

Q2:

From [1.1] in Q1:

$x_1 = \frac{1}{3}\ [x_2 - \ x_3 + 1]$ $\qquad$ ${x_1}^{(1)} = \frac{1}{3}\left[x_2^{(0)} - x_3^{(0)} + 1\right] = \frac{1}{3}$

$x_2 = \frac{1}{6}\ [-3x_1 - \ 2x_3]$ $\qquad$ ${x_2}^{(1)} = \frac{1}{6}\left[3x_1^{(0)} - 2x_3^{(0)}\right] = \frac{1}{6}$

$x_3 = \frac{1}{7}\ [-3x_1 - \ 3x_2 + 4]$ $\qquad$ ${x_3}^{(1)} = \frac{1}{7}\left[-3x_1^{(0)} - 3x_2^{(0)} + 4\right] = 0.35$

${x_1}^{(2)} = \frac{1}{3}\left[x_2^{(1)} - x_3^{(1)} + 1\right] = 0.272$

${x_2}^{(2)} = \frac{1}{6}\left[3x_1^{(1)} - 2x_3^{(1)}\right] = 0.019$

${x_3}^{(2)} = \frac{1}{7}\left[-3x_1^{(1)} - 3x_2^{(1)} + 4\right] = 0.446$

$$x^{(2)} = (0.272,0.019,0.446)$$

Q3:

From [1.1] in Q1:

$$\begin{bmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$x_1{}^{(k)} = (1-\omega)x_1^{(k-1)} + \frac{\omega}{a_{11}}\left[b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)}\right]$$

$$x_2{}^{(k)} = (1-\omega)x_2^{(k-1)} + \frac{\omega}{a_{22}}\left[b_2 - a_{21}x_1^{(k-1)} - a_{23}x_3^{(k-1)}\right]$$

$$x_3{}^{(k)} = (1-\omega)x_3^{(k-1)} + \frac{\omega}{a_{33}}\left[b_3 - a_{31}x_1^{(k-1)} - a_{32}x_2^{(k-1)}\right]$$

For $\omega = 1.1, x^{(0)} = 0$:

$$a_{11} = 3, a_{12} = -1, a_{13} = 1,$$

$$a_{21} = 3, a_{22} = 6, a_{23} = 2,$$

$$a_{31} = 3, a_{32} = 3, a_{33} = 7$$

$$x_1{}^{(1)} \cong 0.366, x_2{}^{(1)} \cong -0.201, x_3{}^{(1)} \cong 0.55$$

Similarly, for second iteration:

$$x_1{}^{(2)} \cong 0.054, x_2{}^{(2)} \cong -0.211, x_3{}^{(2)} \cong 0.647$$

Q4:

$$\begin{cases} 2x_1 - x_2 + x_3 \quad = -1 \\ 2x_1 + 2x_2 + 2x_3 = 4 \\ -x_1 - x_2 + 2x_3 = -5 \end{cases} \text{-----------[4.1]}$$

$$(1,2,-1)^T = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$B \equiv B_J = D^{-1}(\mathcal{u} + L)$$

$$A = D - L - \mathcal{u}$$

$$B_{G-S} = (D - L)^{-1}\mathcal{u}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} \quad \mathcal{u} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(D - L) = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix} (D - L)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

$$B_J = \begin{bmatrix} 0 & 1/2 & -1/2 \\ -1 & 0 & -1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \qquad \rho(B_J) = \det(B_J) = \frac{\sqrt{5}}{2}$$

$$B_{G-S} = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & -1/2 \end{bmatrix} \qquad \rho(B_{G-S}) = \det(B_{G-S}) = \frac{1}{2}$$

Not convergent as we cannot see $\rho(B_{G-S}) = \rho^2(B_J) < 1$.

Q5:

$$\begin{cases} x_1 + 2x_2 - 2x_3 \quad = 7 \\ x_1 + x_2 + x_3 = 2 \\ 2x_1 + 2x_2 + x_3 = 5 \end{cases}$$ -----------[1]

$$(1,2,-1)^T = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$$

$$D = D\text{^}\{-1\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} \quad u = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B \equiv B_J = D^{-1}(u + L)$$

$$A = D - L - u$$

$$B_{G-S} = (D - L)^{-1} u$$

$$B_J = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \qquad \rho(B_J) = \det(B_J) = 0$$

$$B_{G-S} = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix} \qquad \rho(B_{G-S}) = \det(B_{G-S}) = \frac{1}{2}$$

Not convergent as we cannot see $\rho(B_{G-S}) = \rho^2(B_J) < 1$.

Q6:

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Jacobi Method
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Solution: x = [1.000000, -1.000000, 1.000000, -1.000000]'
Number of Iterations (Jacobi):     24


Gauss-Seidel Method
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Solution: x = [1.000000, -1.000000, 1.000000, -1.000000]'
Number of Iterations (Gauss-Seidel):     13


SOR Method
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Solution: x = [1.000000, -1.000000, 1.000000, -1.000000]'
Number of Iterations (SOR):      9


Hence, from the number of iterations and the error plot,
SOR method converges faster, followed by Gauss-Seidel.
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Q7:

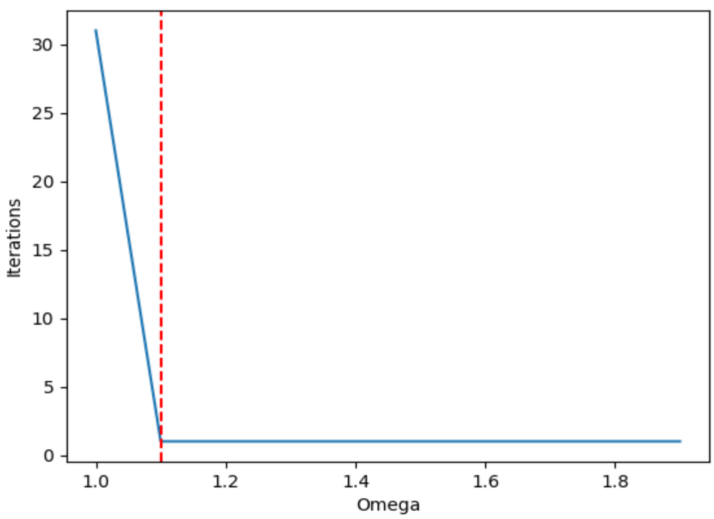

Fig 1: The number of iterations for convergence versus the value of ω graph.

Then the value of ω for which it results in the fastest convergence is 1.1 (from Fig 1).