Q7:

Answers

7a: Householder's method is a numerical algorithm for solving the nonlinear equation $f(x) = 0$. In this case, the cubic convergence rate of one real variable. The method consists of a sequence of iterations

$$x_{n+1} = x_n + d\,\frac{(1/f)^{(d-1)}(x_n)}{(1/f)^{(d)}(x_n)}$$

For $d = 1$, due to Newton's method from Householder's method:

$$x_{n+1} = x_n + 1\,\frac{(1/f)(x_n)}{(1/f)^{(1)}(x_n)}$$

$$= x_n + \frac{1}{f(x_n)} \cdot \left(\frac{-f'(x_n)}{f(x_n)^2}\right)^{-1}$$

$$= x_n - \frac{f(x_n)}{f'(x_n)}$$

7b: For $d = 2$, due to Halley's method from Householder's method:

$$(1/f)'(x) = -\frac{f'(x)}{f(x)^2}$$

And

$$(1/f)''(x) = -\frac{f''(x)}{f(x)^2} + 2\,\frac{f'(x)^2}{f(x)^3}$$

Therefore,

$$x_{n+1} = x_n + 2\,\frac{(1/f)'(x_n)}{(1/f)''(x_n)}$$

$$= x_n + \frac{-2f(x_n)f'(x_n)}{-f(x_n)f''(x_n) + 2[f'(x_n)]^2}$$

$$= x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - \frac{1}{2}f(x_n)f''(x_n)}$$

7c: The Lambert equation, $ye^y = x$, can be transformed to the general form, $f(y) = 0$, by defining $f(y)$ as:

$$f(y) = ye^y - x$$

The first derivative of $f(y)$ is:

$$f'(y) = (1 + y)e^y$$

The second derivative of f(y) is:

$$f''(y) = (2 + y)e^y$$

The ratio $f(y)/f'(y)$ is:

$$f(y)/f'(y) = (ye^y - x)/(1 + y)e^y$$

The ratio $f''(y)/f'(y)$ is:

$$f''(y)/f'(y) = (2 + y)e^y/(1 + y)e^y = (2 + y)/(1 + \mathrm{y})$$

These ratios are used to calculate the updated value of y in Halley's method, which is a Householder's method of the second order, $d = 2$.

7d:

We can rewrite the form of equation of Halley's method in terms of fraction of the derivatives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{2}}$$

$$= x_n - \frac{f(x_n)}{f'(x_n)}\left[1 - \frac{f(x_n)}{f'(x_n)} \cdot \frac{f''(x_n)}{2f'(x_n)}\right]^{-1}$$

From the values obtained in previous exercises, we can now show that

$$y_{n+1} = y_n - \frac{y_\mathrm{n}e^{y_\mathrm{n}} - x}{(1 + y_\mathrm{n})e^{y_\mathrm{n}}}\left[1 - \frac{(y_\mathrm{n}e^{y_\mathrm{n}} - x)}{(1 + y_\mathrm{n})e^{y_\mathrm{n}}}\frac{(2 + y_\mathrm{n})}{2(1 + y_\mathrm{n})}\right]^{-1}$$

$$= y_\mathrm{n} - \frac{2(1 + y_\mathrm{n})(y_\mathrm{n}e^{y_\mathrm{n}} - x)}{2e^{y_n}(1 + y_n)^2 - [(y_\mathrm{n}e^{y_\mathrm{n}} - x)(2 + y_\mathrm{n})]}$$