Polynomial Interpolation

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Overview

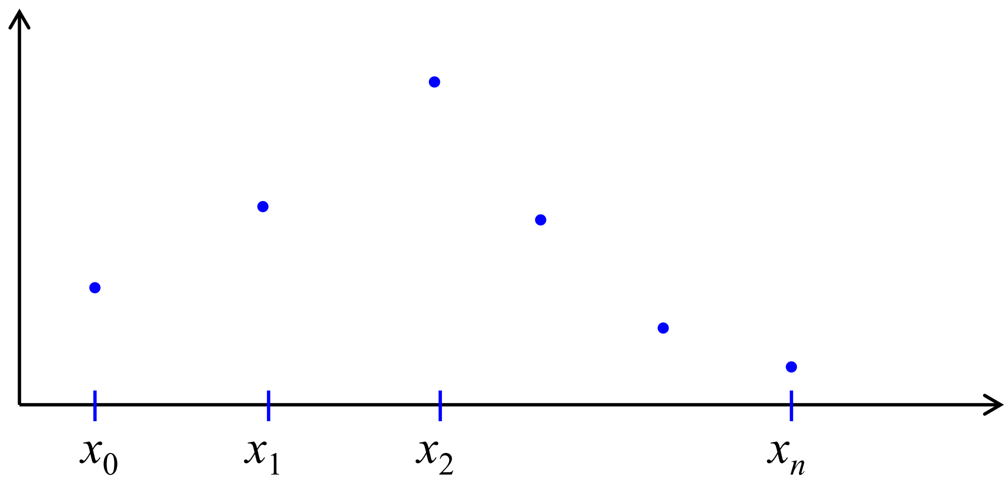
- Polynomial Interpolation
- Newtonian Approach
- Lagrangian Approach
- Polynomial Interpolation Error
- Runge’s Phenomenon
- Mini-Max Problem
- Chebyshev Polynomials

Interpolation

- Limited number of data points
- Interpolate = Estimate in between
- Interpolant = Function/Method
- Questions to be answered
  - How accurate is the interpolant
  - How expensive is the interpolant
  - How smooth is the interpolant
  - How many data points are needed

Interpolation Problem

x	x_0	x_1	x_2	$\cdots$	x_n
y	y_0	y_1	y_2	$\cdots$	y_n



Find function $f(x)$ such that $f(x_i) = y_i \quad 0 \le i \le n$

Example Problem

x_i	0	2	3
y_i	1	2	4

$$y = f(x) = a_2 x^2 + a_1 x + a_0$$

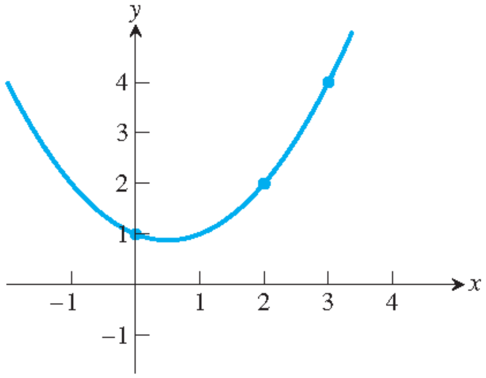
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$f(0) = a_2 0^2 + a_1 0^1 + a_0 = 1$$

$$f(2) = a_2 2^2 + a_1 2^1 + a_0 = 2$$

$$f(3) = a_2 3^2 + a_1 3^1 + a_0 = 4$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}$$



Other Interpolants

x_i	0	2	3
y_i	1	2	4

$$f(x) = a_2 e^x + a_1 \sin x + a_0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & \sin 2 & e^2 \\ 1 & \sin 3 & e^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$f(0) = a_2 e^0 + a_1 \sin 0 + a_0 = 1$$

$$f(2) = a_2 e^2 + a_1 \sin 2 + a_0 = 2$$

$$f(3) = a_2 e^3 + a_1 \sin 3 + a_0 = 4$$

Why Polynomials

Find polynomial $p(x)$ such that $p(x_i) = y_i \quad (0 \le i \le n)$

Taylor's theorem: $$f(x) = f(c) + \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k + E_n(x)$$

Weierstrass theorem: If f is continuous on $[a,b]$,

then for any $\varepsilon > 0$ there is a polynomial $p(x)$

satisfying $|f(x) - p(x)| < \varepsilon$ on the interval $[a,b]$.

Practical Considerations

- Polynomials have straightforward mathematical properties;
- There is a simple theory about interpolating polynomials;
- Polynomials are the most fundamental of functions for digital computers;
- Addition and multiplication are the only operations needed to evaluated a polynomial;
- CPUs have fast methods in hardware for adding and multiplying floating point numbers.
- Complicated functions can be approximated by polynomials to make them computable with these two hardware operations.

Fundamental Theorem of Algebra

- Every non-constant polynomial with complex coefficients has at least one complex root.

- Every non-zero, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots.

Interpolation Polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

We seek a polynomial p of lowest possible degree such that $p(x_i) = y_i \quad (0 \le i \le n)$

Theorem. If $x_0, x_1, \ldots x_n$ are distinct real numbers, then for arbitrary values $y_0, y_1, \ldots y_n$ there is a unique polynomial $p_n(x)$ of degree at most n such that $p_n(x_i) = y_i \ (0 \le i \le n)$.

Finding the Polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$\begin{aligned}
p_n(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n = y_0 \\
p_n(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n = y_1 \\
&\vdots \\
p_n(x_n) &= a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n = y_n
\end{aligned}$$

Vandermonde Matrix

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\det \mathbf{X} = \prod_{0 \le i < j \le n} \left(x_j - x_i \right) = \left(x_n - x_{n-1} \right) \cdot \ldots \cdot \left(x_1 - x_0 \right)$$

Avoid Vandermonde Matrix

$$n = 16 \qquad x_0 = 1/10 \qquad x_0^n = 10^{-16} \approx \varepsilon_M$$

$$fl\left(1 + x_0^n\right) = 1$$

$$\det \mathbf{X} = \prod_{0 \le i < j \le n} \left(x_j - x_i\right) = \left(x_n - x_{n-1}\right) \cdot \ldots \cdot \left(x_1 - x_0\right) \approx 0$$

Newtonian Approach

$$
\begin{aligned}
p_n(x) = c_0 &+ \\
& c_1(x - x_0) + \\
& c_2(x - x_0)\cdot(x - x_1) + \\
& \vdots \\
& c_k(x - x_0)\cdot\ldots\cdot(x - x_{k-1}) + \\
& \vdots \\
& c_n(x - x_0)\cdot\ldots\cdot(x - x_{k-1})\cdot\ldots\cdot(x - x_{n-1})
\end{aligned}
$$

Newton's Basis

$$\left\{\pi_0(x), \pi_1(x), \ldots, \pi_n(x)\right\} \qquad \left\{1, x, x^2, \ldots, x^n\right\}$$

$$\pi_0(x) = 1$$

$$\pi_1(x) = (x - x_0)$$

$$\pi_2(x) = (x - x_0)(x - x_1)$$

$$\vdots$$

$$\pi_n(x) = (x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

$$p_n(x) = c_0\pi_0(x) + c_1\pi_1(x) + \ldots + c_n\pi_n(x)$$

Newton's Coefficients

$$p_n(x_0) = f(x_0) = c_0$$

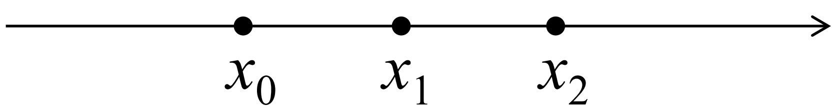
$$p_n(x_1) = f(x_1) = c_0 + c_1(x_1 - x_0) \rightarrow c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$p_n(x_2) = f(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1)$$

Coefficient c_2

$$c_2 = \frac{f(x_2) - c_0 - c_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$c_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



1^{st} Order Divided Difference

$$f\left[x_1, x_2\right] \equiv \frac{f\left(x_2\right) - f\left(x_1\right)}{x_2 - x_1}$$

$$c_2 = \frac{f\left[x_1, x_2\right] - f\left[x_0, x_1\right]}{x_2 - x_0}$$

$$c_1 = \frac{f\left(x_1\right) - f\left(x_0\right)}{x_1 - x_0} = f\left[x_0, x_1\right]$$

2nd Order Divided Difference

$$f[x_0, x_1, x_2] \equiv \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Order 2: $c_2 = f[x_0, x_1, x_2]$

Order 1: $c_1 = f[x_0, x_1]$

Order 0: $c_0 = f[x_0] \equiv f(x_0)$

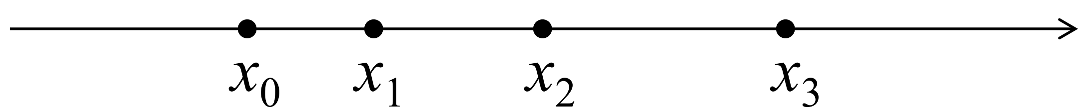
High-Order Divided Differences

$$c_3 = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} \equiv f[x_0, x_1, x_2, x_3]$$

$$\vdots$$

$$c_k = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0} \equiv f[x_0, x_1, \ldots, x_{k-1}, x_k]$$

c_3:



Property 1

Theorem 1. The divided difference is a symmetric function of its arguments. Thus, if $(z_0, z_1, \ldots z_n)$ is a permutation of $(x_0, x_1, \ldots x_n)$ then

$$f\left[z_0, z_1, \ldots, z_n\right] = f\left[x_0, x_1, \ldots, x_n\right]$$

$$f\left[x_0, x_1\right] \equiv \frac{f\left(x_1\right) - f\left(x_0\right)}{x_1 - x_0}$$

Property 2

Theorem 2. Let $p_n(x)$ be the polynomial of degree at most n that interpolates $f(x)$ at a set of $n+1$ distinct nodes, $x_0, x_1, \ldots x_n$. If x is a point different from the nodes, then

$$f(x) - p_n(x) = f\left[x_0, x_1, \ldots, x_n, x\right] \prod_{j=0}^{n} \left(x - x_j\right)$$

Property 3

Theorem 3. If f is n times continuously differentiable on $[a,b]$ and if $x_0, x_1, \ldots x_n$ are distinct points in $[a,b]$, then there exists a point ξ in (a,b) such that

$$f\left[x_0, x_1, \ldots, x_n\right] = \frac{1}{n!} f^{(n)}(\xi)$$

$$f\left[x_0, x_1\right] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi)$$

Comparing with Taylor

$$f(x)-p_n(x)=f\left[x_0,x_1,\ldots,x_n,x\right]\prod_{j=0}^{n}\left(x-x_j\right)$$

$$f\left[x_0,x_1,\ldots,x_n,x\right]=\frac{1}{(n+1)!}f^{(n+1)}(\xi)$$

$$f(x)-p_n(x)=\frac{1}{(n+1)!}f^{(n+1)}(\xi)\prod_{j=0}^{n}\left(x-x_j\right)$$

$$f(x)-t_n(x)=E_n(x)=\frac{f^{(n+1)}\left(\xi\right)}{\left(n+1\right)!}\left(x-c\right)^{n+1}$$

Newton's Form

$$\begin{aligned}p_n(x) &= f[x_0] + \overset{f'(\xi)}{f[x_0,x_1]}(x-x_0) + \\ &\quad + \overset{f''(\xi)/2!}{f[x_0,x_1,x_2]}(x-x_0)(x-x_1) + \\ &\quad \vdots \\ &\quad + \overset{f^{(n)}(\xi)/n!}{f[x_0,x_1,\cdots,x_n]}(x-x_0)\cdots(x-x_{n-1})\end{aligned}$$

Taylor

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots \frac{f^{(n)}(c)}{n!}(x-c)^n +$$

Table of Divided Differences

x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Nested Multiplication

$$p_3(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2)$$

$$p_3(x) = c_0 + (x - x_0)\Big[c_1 + (x - x_1)\big[c_2 + c_3(x - x_2)\big]\Big]$$

Lagrange's Approach

x	x_0	x_1	x_2	$\dots$	x_n
y	y_0	y_1	y_2	$\dots$	y_n

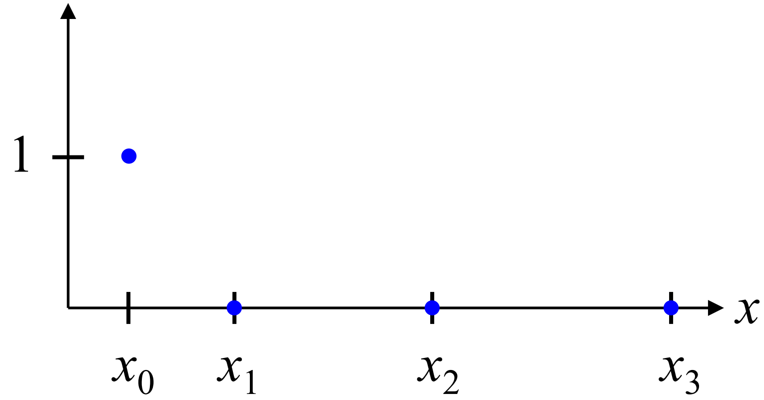
$$p(x) = y_0\ell_0(x) + y_1\ell_1(x) + \ldots + y_n\ell_n(x)$$

$\ell_k(x)$ is a polynomial of degree at most n that depends only on the nodes $x_0, x_1, \ldots x_n$ but not on the ordinates $y_0, y_1, \ldots y_n$.

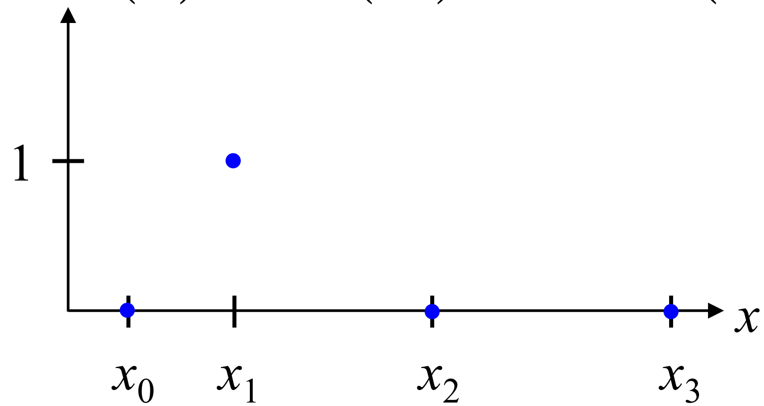
$$\ell_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Example

$$\ell_0(x): \quad \ell_0(x_0) = 1 \qquad \ell_0(x_i) = 0 \quad i \neq 0$$



$$\ell_1(x): \quad \ell_1(x_1) = 1 \qquad \ell_1(x_i) = 0 \quad i \neq 1$$



Interpolant

$$\ell_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

$$p(x) = y_0\ell_0(x) + y_1\ell_1(x) + \ldots + y_n\ell_n(x)$$

$$p(x_0) = y_0\ell_0(x_0) + y_1\ell_1(x_0) + \ldots + y_n\ell_n(x_0) = y_0$$

$$p(x_1) = y_0\ell_0(x_1) + y_1\ell_1(x_1) + \ldots + y_n\ell_n(x_1) = y_1$$

Cardinal Functions

$$\ell_0(x) = 0 \quad x = x_1, x_2, \ldots, x_n \longrightarrow \ell_0(x) = c(x - x_1)\cdots(x - x_n)$$

$$\ell_0(x_0) = 1 = c\prod_{j=1}^{n}(x_0 - x_j) \longrightarrow c = \prod_{j=1}^{n}(x_0 - x_j)^{-1}$$

$$\ell_0(x) = \prod_{j=1}^{n}\frac{x - x_j}{x_0 - x_j}; \qquad \ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n}\frac{x - x_j}{x_i - x_j} \quad (0 \le i \le n)$$

Polynomial Interpolation Error

Theorem. Let f be a function in $C^{n+1}[a,b]$, and let p_n be the polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, \ldots x_n$ in the interval $[a,b]$. To each x in $[a,b]$ there corresponds a point ξ_x in (a,b) such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

Evaluating Error

Example. Let $f(x) = \sin x$. It is interpolated by $p_9(x)$ at 10 points in (0,1). How large is the error?

$$\left|f^{(n+1)}(\xi_x)\right| \leq 1; \quad \left|\prod_{i=0}^{n}(x - x_i)\right| \leq 1$$

$$\left|f(x) - p_n(x)\right| = \frac{\left|f^{(n+1)}(\xi_x)\right|}{(n+1)!} \cdot \left|\prod_{i=0}^{n}(x - x_i)\right| \leq \frac{1}{10!} < 2.8 \times 10^{-7}$$

Convergence

$$\left|\sin x - p_n(x)\right| = \frac{\left|\sin^{(n+1)}(\xi_x)\right|}{(n+1)!}\left|\prod_{i=0}^{n}(x - x_i)\right| \le \frac{1}{(n+1)!} \xrightarrow[n\to\infty]{} 0$$

$$\forall f(x) \in C^m[a,b] \qquad p_n(x) \xrightarrow[n\to\infty]{} f(x) \quad ??$$

$$f_n(x) \xrightarrow[n\to\infty]{} f(x): \qquad \begin{cases} \forall x \in [a,b] \qquad f_n(x) \xrightarrow[n\to\infty]{} f(x) \\ \\ \left\| f - f_n \right\| \xrightarrow[n\to\infty]{} 0 \qquad \text{(uniformly)} \end{cases}$$

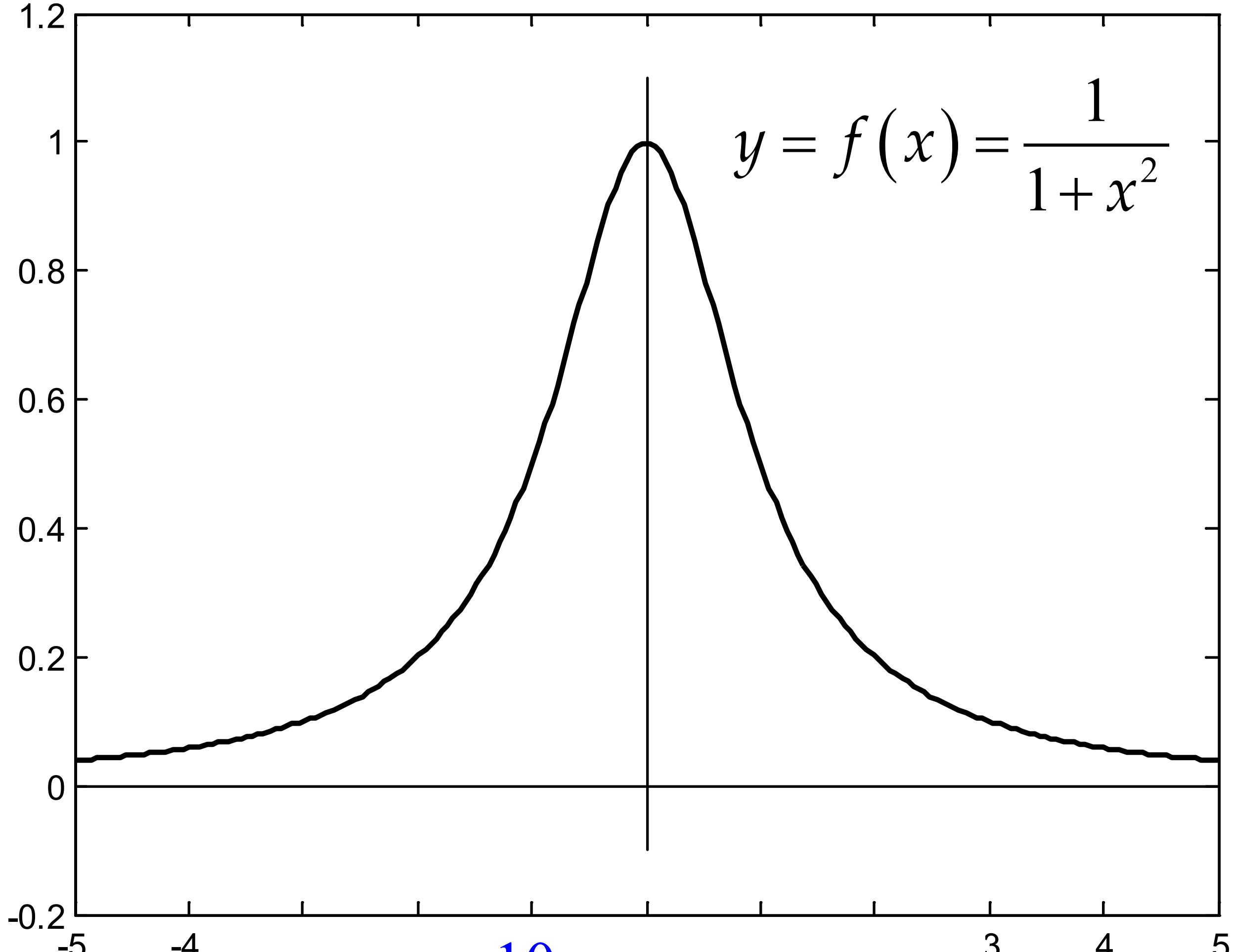
Convergence of p_n

$$\left\|\sin - p_n\right\|_\infty = \max_{0\le x\le 1}\left|\sin x - p_n(x)\right| \xrightarrow[n\to\infty]{} 0$$

$$\left\|f - p_n\right\|_\infty = \max_{a\le x\le b}\left|f(x) - p_n(x)\right| \not\xrightarrow[n\to\infty]{} 0$$

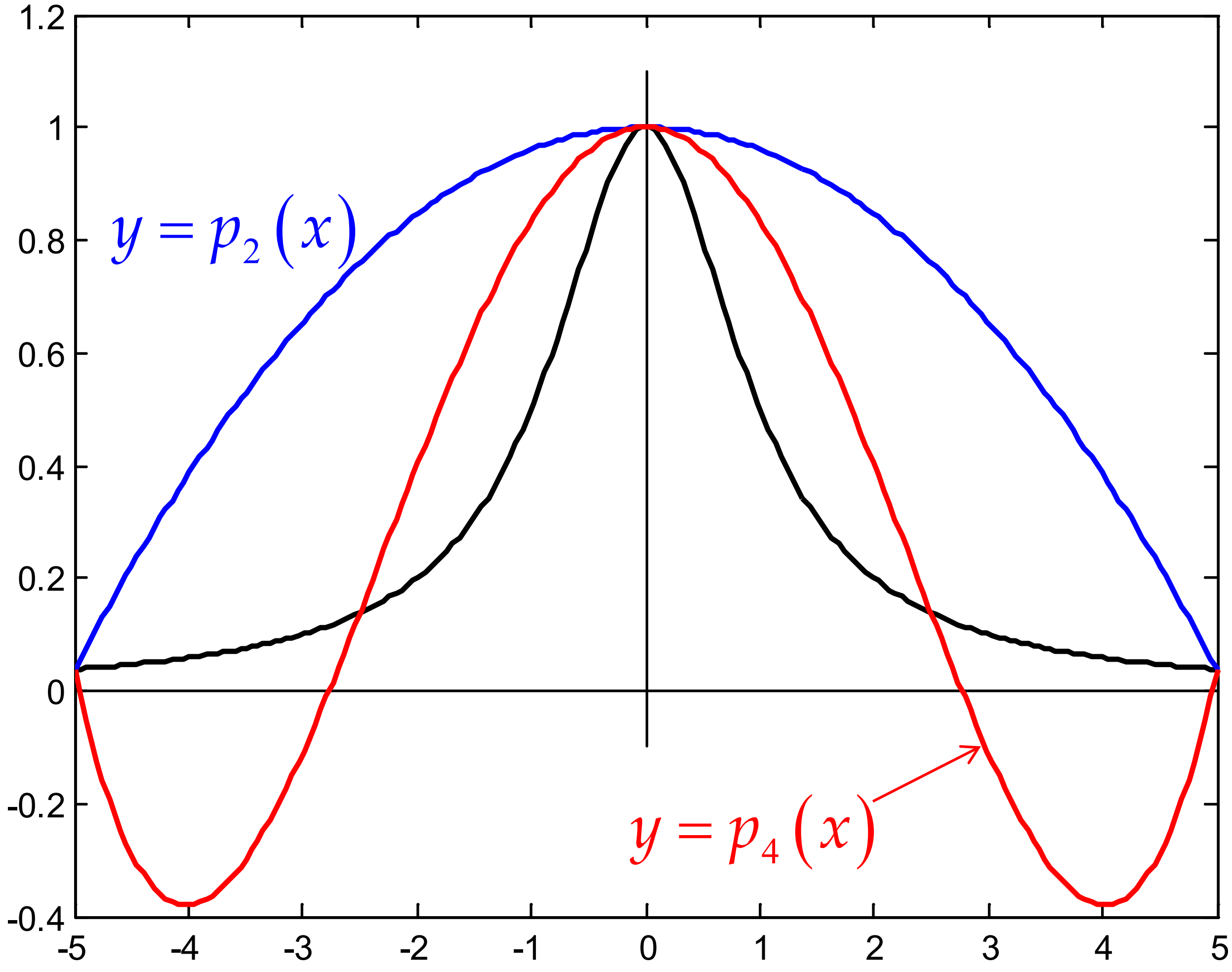
(For most continuous functions!!)

Runge's Function

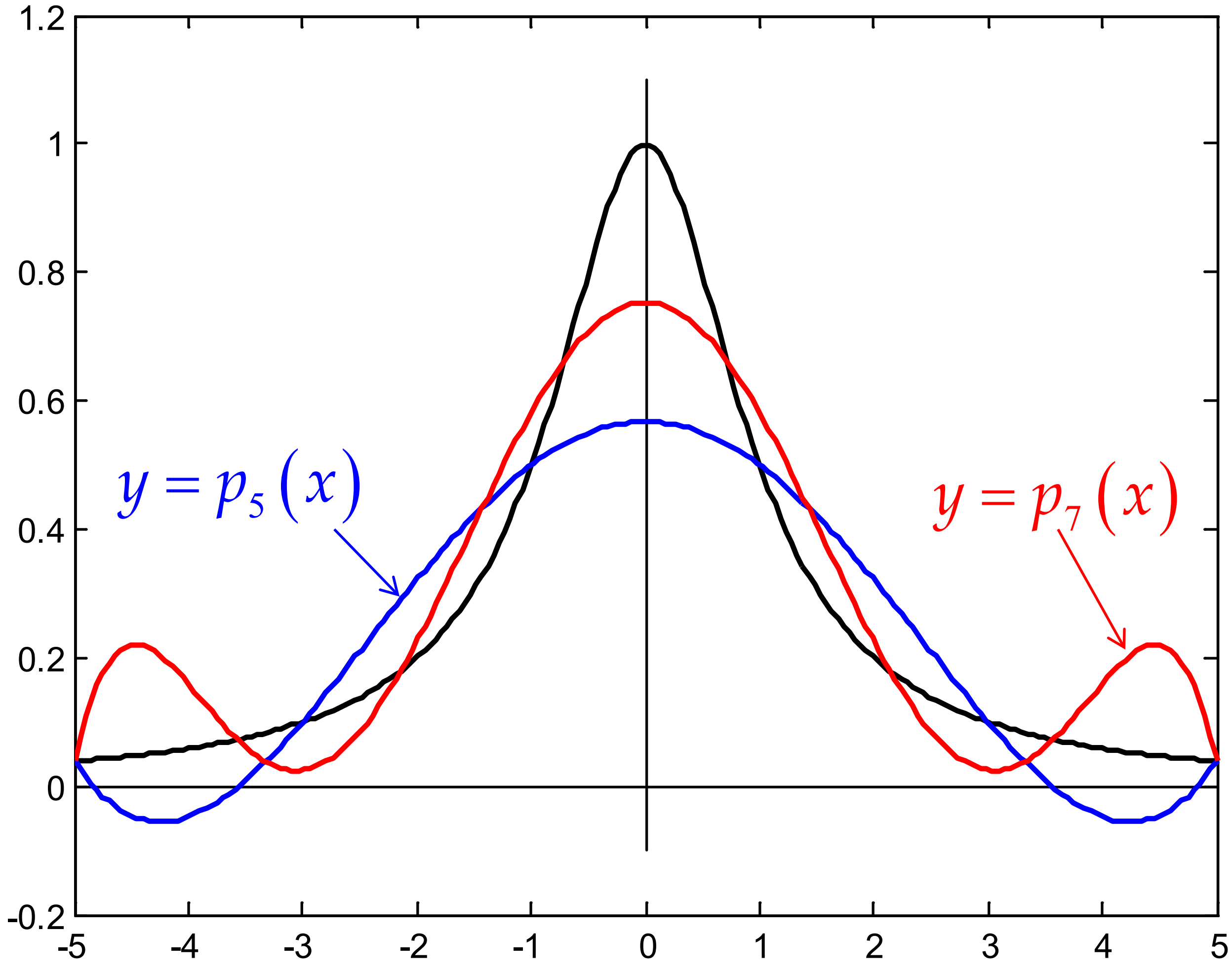


$$x_i = -5 + i\frac{10}{n} \qquad (i = 0,1,\ldots,n)$$

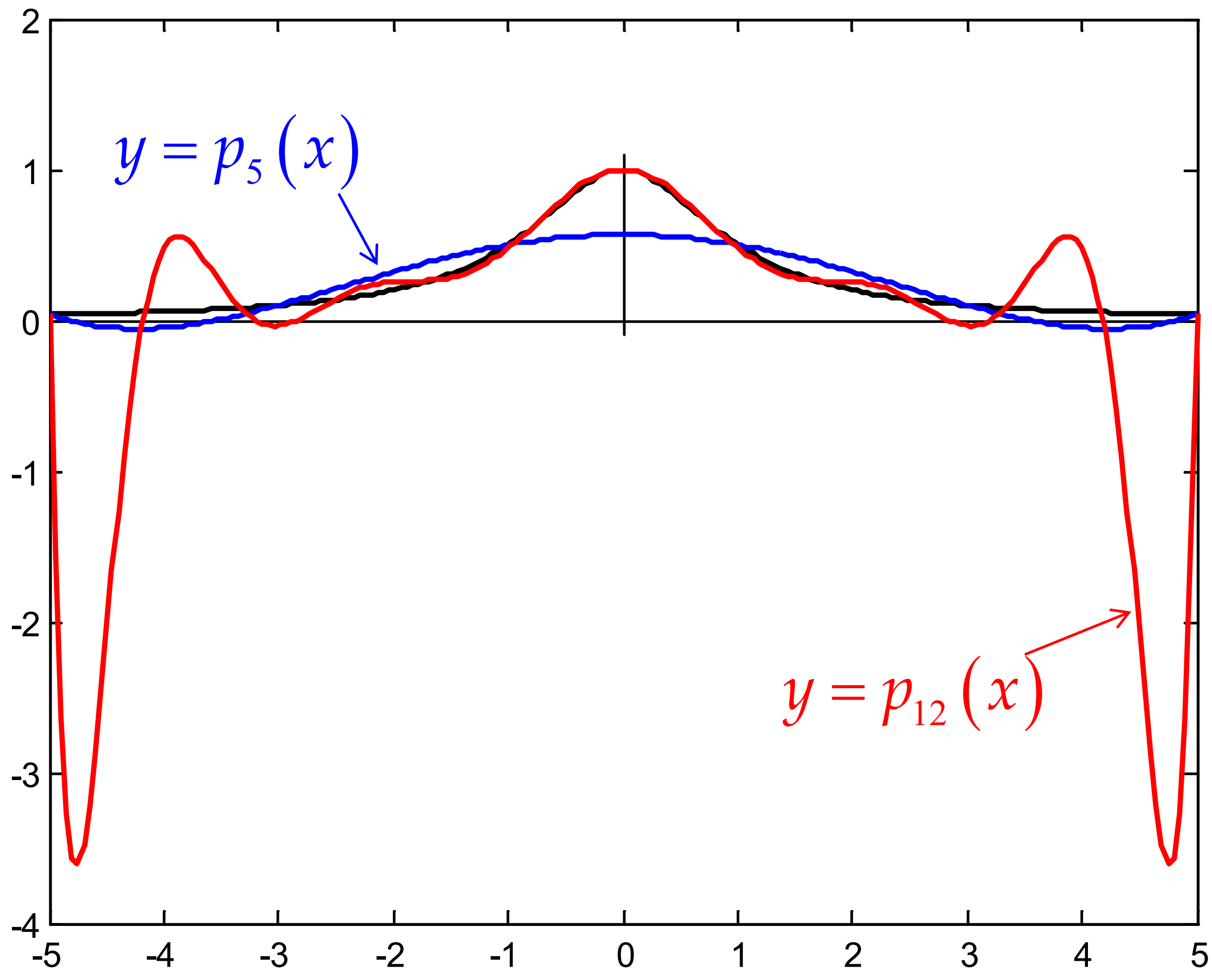
Runge's Example



Medium Degree Polynomials



Runge's Phenomenon



Negative Result

Theorem. For any prescribed system of nodes

$$a \le x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} \le b$$

$$\left(x_i = i \cdot \frac{b-a}{n} \qquad 0 \le i \le n \right)$$

there exists a continuous function f on $[a,b]$ such that the interpolating polynomials for f using these nodes fail to converge uniformly to f, i.e. $\left\| f - p_n \right\| \not\to 0$

Positive Result

Theorem. If f is a continuous function on $[a,b]$ then there exists a system of nodes

$$a \le x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} \le b$$

Such that the polynomials p_n of interpolation to f at these nodes converge uniformly to f i.e.

$$\left\| f - p_n \right\|_\infty = \max_{a \le x \le b} \left| f(x) - p_n(x) \right| \xrightarrow[n \to \infty]{} 0$$

Interpolation Error

$$f \in C^{n+1}[a,b] \longrightarrow f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$$\min_{x_i} \max_{x} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

Monic polynomials: $$\prod_{i=0}^{n} (x - x_i) = x^{n+1} + a_n x^n + \cdots + a_0$$

Mini-Max Problem

If $f \in C^{n+1}[-1,1]$ then $f(x) - p_n(x) = \dfrac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$

$$\max_{|x|\le 1} |f(x) - p_n(x)| \le \frac{1}{(n+1)!} \max_{|x|\le 1} \left|f^{(n+1)}(x)\right| \max_{|x|\le 1} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

Theorem. The minimum of the maximum is attained when the nodes x_i are chosen to be the roots of the Chebyshev polynomial $T_{n+1}(x)$ then

$$\min_{|x_i|\le 1} \max_{|x|\le 1} \left| \prod_{i=0}^{n} (x - x_i) \right| = \frac{1}{2^n}$$

Chebyshev Polynomials

$$T_n(x) = \begin{cases} \cos\left(n \cos^{-1} x\right) & (-1 \le x \le 1) \\ \cosh\left(n \cosh^{-1} x\right) & (x > 1) \\ (-1)^n \cosh\left(n \cosh^{-1}(-x)\right) & (x < -1) \end{cases}$$

$$\begin{cases} T_0(x) = 1 \qquad T_1(x) = x \\ \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \end{cases}$$

First 6 Polynomials

$$T_0(x) = 1 \qquad T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T_n(x) = 2^{n-1}x^n + \cdots$$

Some Properties

$$\left|T_n(x)\right| \le 1 \qquad (-1 \le x \le 1)$$

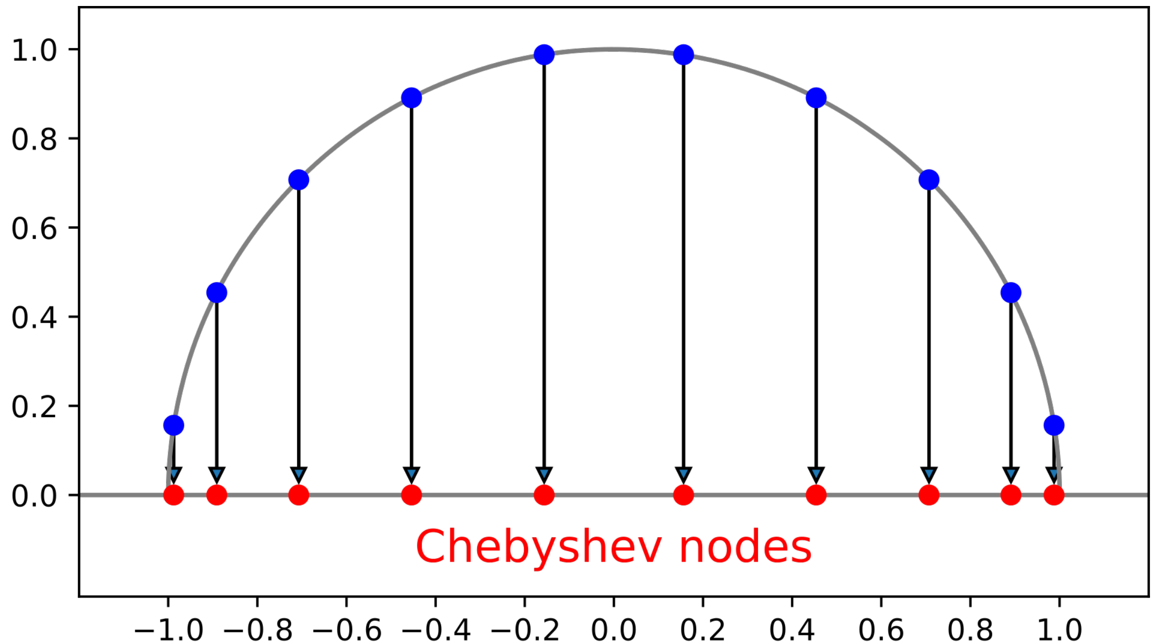
$$T_n(1) = 1 \qquad T_n(-1) = (-1)^n$$

$$T_n(m_j) = (-1)^j \qquad m_j = \cos\frac{j\pi}{n} \qquad (0 \le j \le n)$$

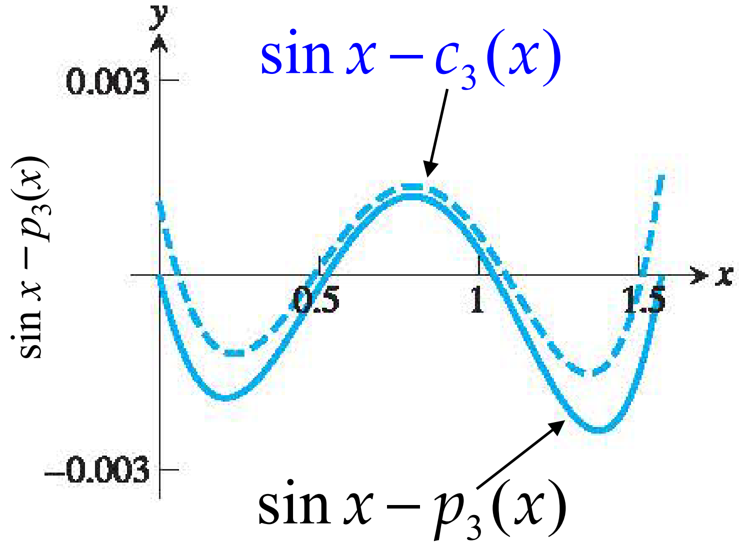
$$T_n(z_j) = 0 \qquad z_j = \cos\frac{(j - 1/2)\pi}{n} \qquad (1 \le j \le n)$$

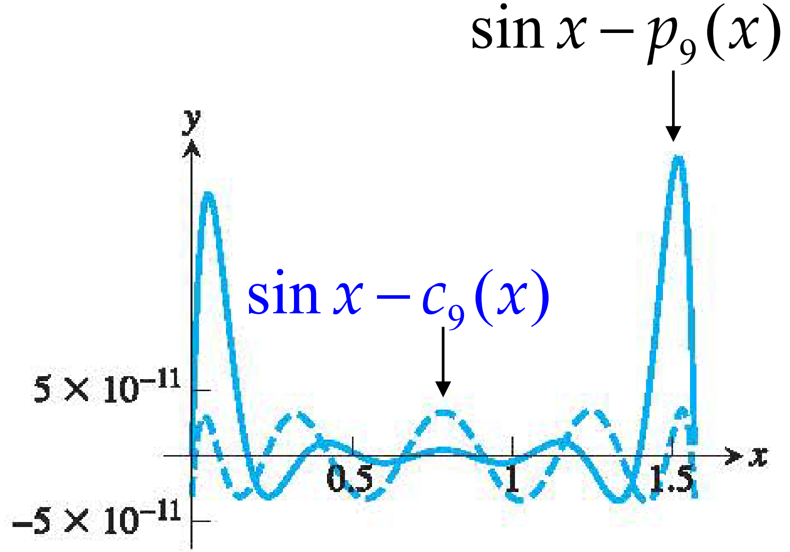
Chebyshev Zeros

$$z_j = \cos\frac{(j-1/2)\pi}{n} = \cos\frac{(2j-1)\pi}{2n} \qquad (1 \le j \le n)$$

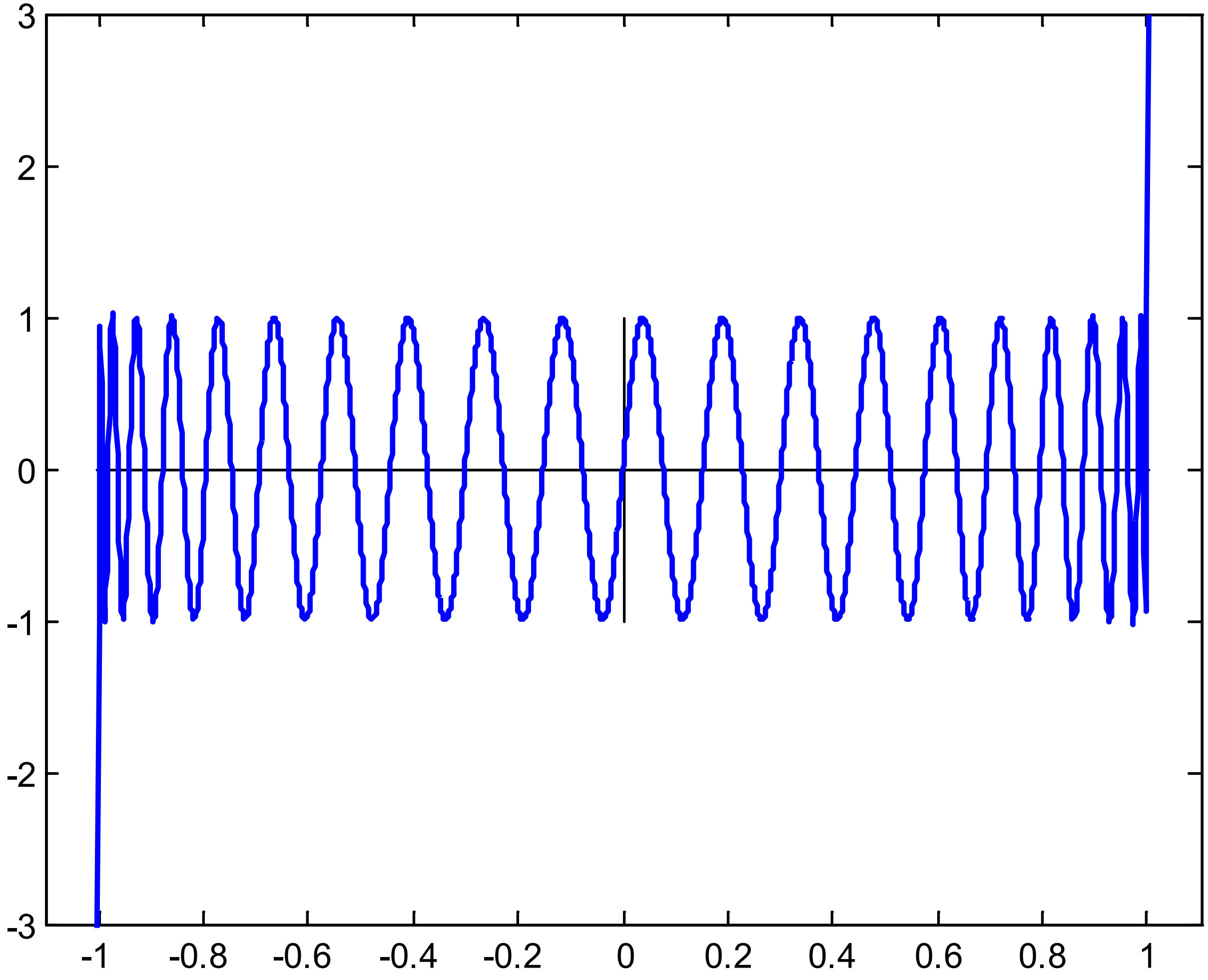


Approximation Error





Plotting $T_{41}(x)$



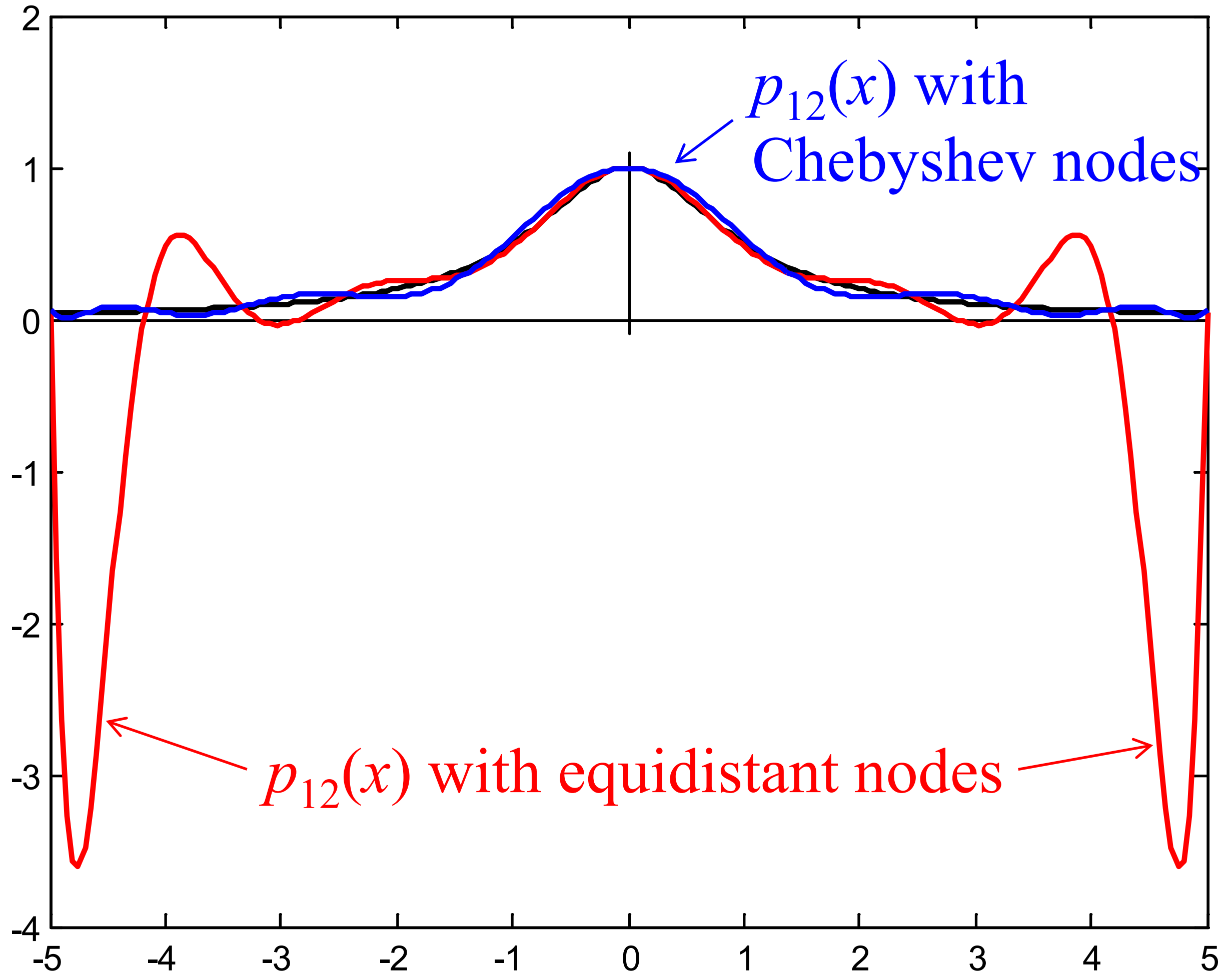
Best Estimate

$$x_i = z_j = \cos\frac{\left(j - 1/2\right)\pi}{n+1} \qquad \left(1 \le j \le n+1\right)$$

$$\left\|f - p_n\right\|_\infty \le \frac{1}{\left(n+1\right)!2^n} \max_{|x|\le 1}\left|f^{(n+1)}\left(x\right)\right|$$

$$x = \frac{b-a}{2}z + \frac{b+a}{2}: \quad \left[-1,1\right] \to \left[a,b\right]$$

Chebyshev Interpolation



Important

- Polynomial Interpolation
- Newton's Approach
- Lagrange Approach
- Polynomial Interpolation Error
- Runge's Phenomenon
- Mini-Max Problem
- Chebyshev Polynomials