Partial Differential Equations

Vasily Arzhanov

Reactor Physics, KTH

Overview

- Classification of PDE
- DoD RoI
- Parabolic Equations
- Explicit vs. Implicit
- Crank-Nicolson Scheme
- Hyperbolic Equations
- Spectral Stability of FD Schemes
- FDS for Non-Linear PDE

PDE Definition

$$u(\mathbf{x}) = u(x_1,\ldots,x_n)$$

ODE

$$F\left(t, y, y', \ldots, y^{(m)}\right) = 0$$

PDE

$$F\left(x_i, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \ldots\right) = 0$$

Cauchy-Kovalevskaya

$$\frac{\partial^m u}{\partial t^m} = F\left(x_i, t, u, \frac{\partial^l}{\partial t^l} \frac{\partial^k u}{\partial x_i^k} \right)$$

F is analytic

Hans Lewy example

$F \in C^\infty$

Notation

$$u_x \equiv \frac{\partial u}{\partial x}$$

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$u_{xy} \equiv \frac{\partial^2 u}{\partial x \partial y}$$

$$\Delta = \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\dot{u} \equiv \frac{\partial u}{\partial t} \qquad \ddot{u} \equiv \frac{\partial^2 u}{\partial t^2}$$

1st Order PDE

Hom. Lin. $$a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$$

Non-Hom. $$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

Q.-L. $$a(x,y,u)u_x + b(x,y,u)u_y + c(x,y,u) = 0$$

W.E. Surface $$u_t^2 = c^2\left(u_x^2 + u_y^2 + u_z^2\right)$$

2^{nd} Order PDE

Linear $$a(x,y)u_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y)$$

Q.-Linear $$a(x,y,u)u_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y,u)$$

Discriminant $$D \equiv b^2 - 4ac = \begin{cases} < 0 & \text{Elliptic} \\ = 0 & \text{Parabolic} \\ > 0 & \text{Hyperbolic} \end{cases}$$

Conic Surface $$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

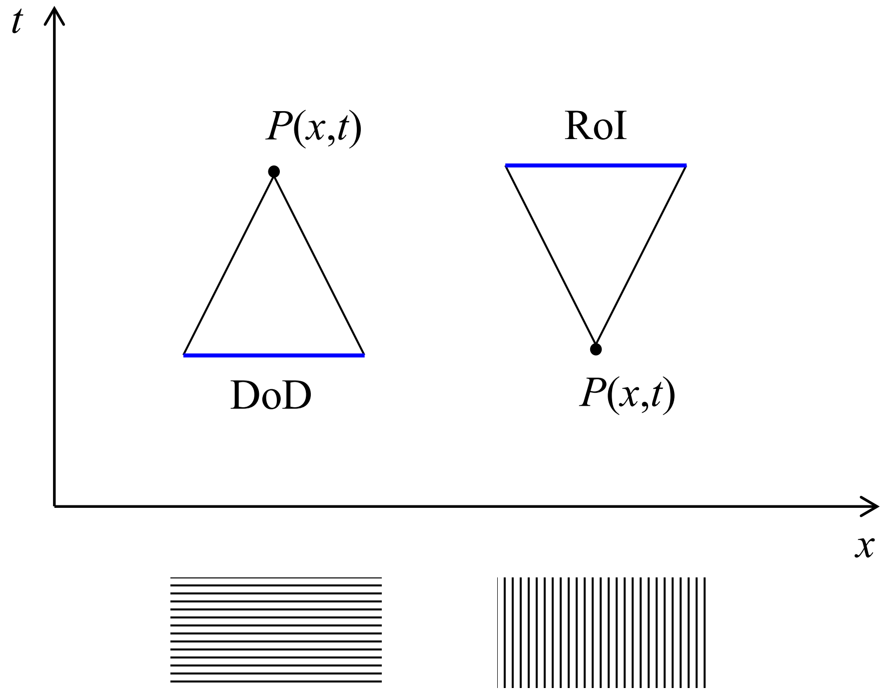
Quasi-Linear PDE

$$Lu(\mathbf{x}) \equiv \sum_{i,j} a_{ij}(\mathbf{x},u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_k b_k(\mathbf{x},u)\frac{\partial u}{\partial x_k} + cu = f$$

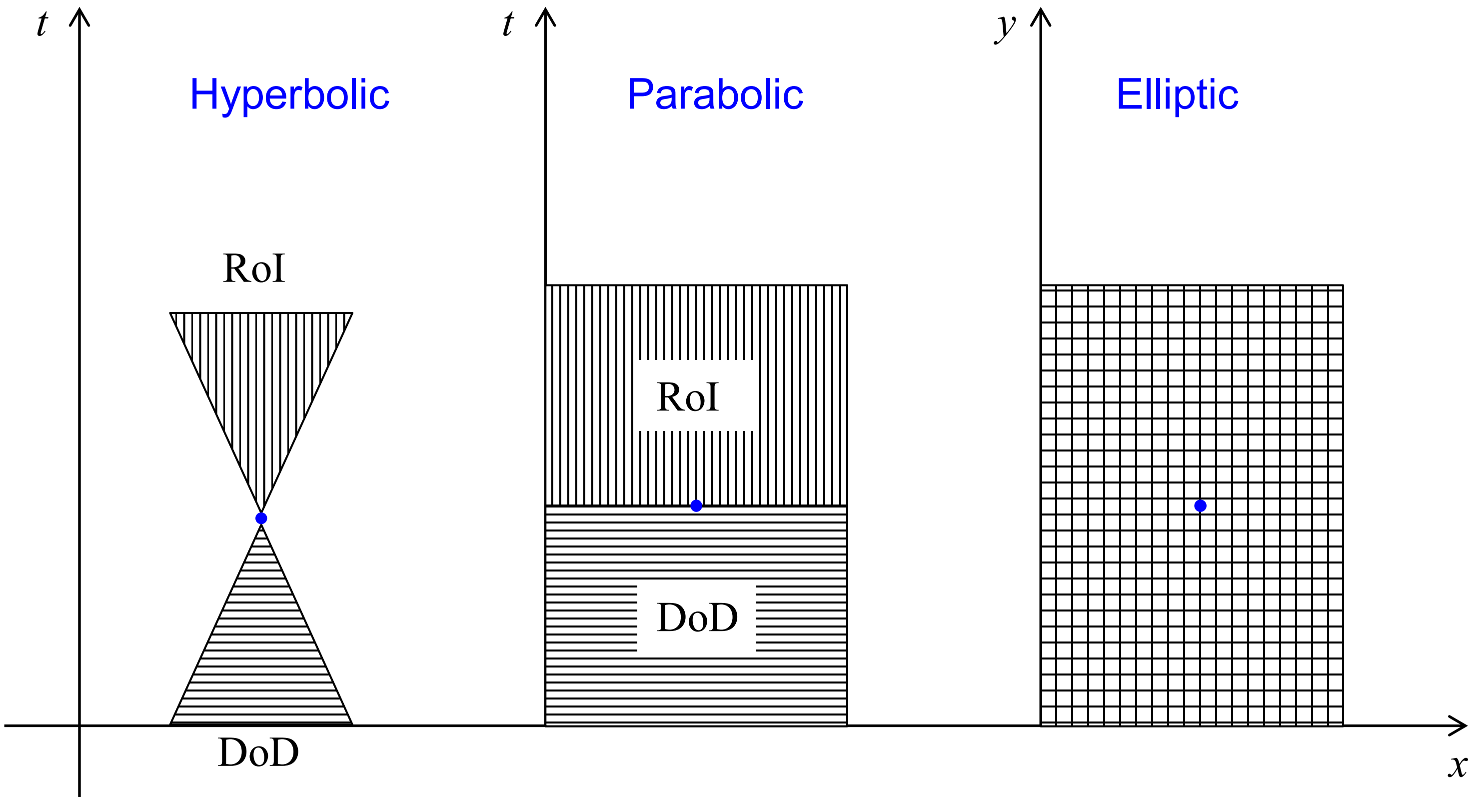
$$\mathbf{A} \equiv \left[a_{ij}\right] \qquad \mathbf{A}\mathbf{w}_i = \lambda_i \mathbf{w}_i$$

1) Elliptic: all $\lambda_i > 0$ or $\lambda_i < 0$
2) Parabolic: $\lambda_1 = 0$ the rest of the same sign
3) Hyperbolic: $\lambda_1 > 0$ the rest < 0 or vice versa

DoD and RoI



DoD/RoI Characterization

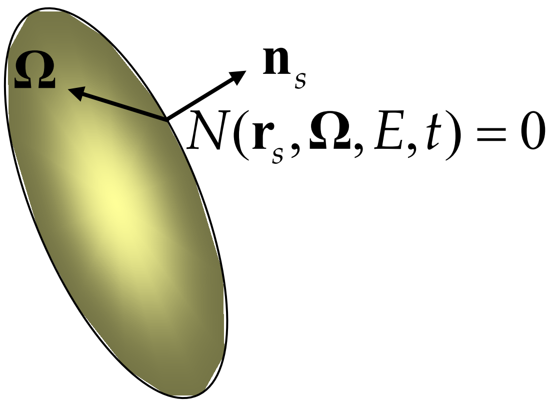


$$u_{tt} = c^2 u_{xx} \qquad u_t = \alpha u_{xx} \qquad \nabla^2 u = -\rho$$

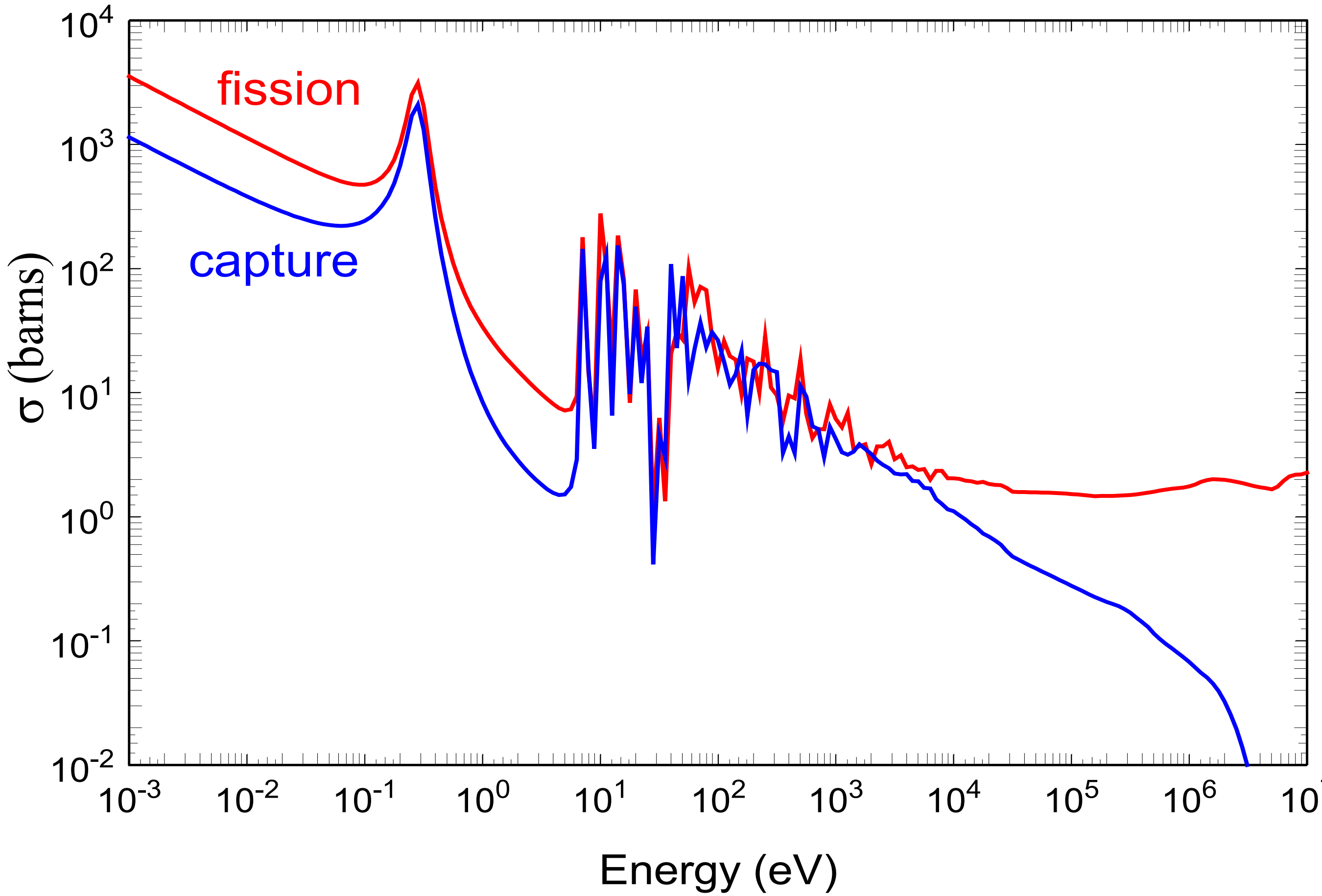
Neutron Transport Equation

$$\frac{dN}{dt} = \frac{\partial N\left(\mathbf{r},\mathbf{\Omega},E,t\right)}{\partial t} + \mathbf{v}\cdot\nabla N =$$
$$= -\Sigma_t vN + \int_0^\infty \int_{4\pi} \Sigma_s(\mathbf{r},\mathbf{\Omega}',E' \to \mathbf{\Omega},E)v'N(\mathbf{r},\mathbf{\Omega}',E',t)d\mathbf{\Omega}'dE' + Q$$

$$\begin{cases} N(\mathbf{r},\mathbf{\Omega},E,t=0) = N_0(\mathbf{r},\mathbf{\Omega},E) & : \text{Initial Condition} \\ N(\mathbf{r}_s,\mathbf{\Omega},E,t)\big|_{\mathbf{\Omega}\cdot\mathbf{n}_s<0} = 0 & : \text{BC (free surface)} \end{cases}$$



Microscopic X-Sections



Neutron Diffusion Equation

$$n(\mathbf{r},t) \equiv \int_{4\pi} \int_0^{\infty} N(\mathbf{r},\mathbf{\Omega},E,t)\, dE\, d\mathbf{\Omega} \qquad \phi(\mathbf{r},t) \equiv v n(\mathbf{r},t)$$

$$\frac{1}{v}\frac{\partial \phi(\mathbf{r},t)}{\partial t} = \nabla\left[D(\mathbf{r})\nabla\phi(\mathbf{r},t)\right] + \nu\Sigma_f(\mathbf{r})\phi(\mathbf{r},t) - \Sigma_a(\mathbf{r})\phi + S(\mathbf{r},t)$$

$$\begin{cases} \dfrac{1}{v_1}\dfrac{\partial \phi_1}{\partial t} = D_1\nabla^2\phi_1 - \Sigma_{a,1}\phi_1 - \Sigma_{1\to 2}\phi_1 + \nu\Sigma_{f,2}\phi_2 \\ \dfrac{1}{v_2}\dfrac{\partial \phi_2}{\partial t} = D_2\nabla^2\phi_2 - \Sigma_{a,2}\phi_2 + \Sigma_{1\to 2}\phi_1 \end{cases}$$

Parabolic Equations in 1D

$$\frac{1}{v}\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}\phi(x,t)\right] - \Sigma_a(x)\phi(x,t) + S(x,t)$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\kappa(x,t)\frac{\partial u}{\partial x}\right] + f(x,t) \qquad 0 \le x \le X \quad 0 \le t \le T$$

BC: $-\alpha_1 \frac{\partial u}{\partial x} + \beta_1 u = \psi_1(t) \qquad -\alpha_2 \frac{\partial u}{\partial x} + \beta_2 u = \psi_2(t)$

IC: $u(x,0) = u_0(x)$

Discretisation

$$u_t = \kappa u_{xx} \qquad 0 \le x \le X \quad 0 \le t \le T$$

BC: $u(0,t) = 0 \qquad u(X,t) = 0$

$$h = \frac{X}{N+1} \qquad x_i = i \cdot h \quad (i = 0,1,\ldots,N+1)$$

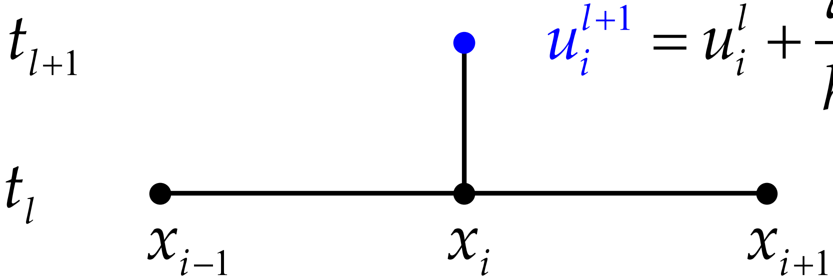
$$\tau = \frac{T}{M} \qquad t_l = l \cdot \tau \quad (l = 0,1,\ldots,M)$$

$$u(x_i, t_l) \approx u_i^l \longrightarrow \max_{i,l} \left| u(x_i, t_l) - u_i^l \right| \stackrel{?}{=} O(\tau + h^2)$$

Explicit FD Scheme

$$u_t = \kappa u_{xx}$$

$$\frac{u_i^{l+1} - u_i^l}{\tau} = \kappa \frac{u_{i-1}^l - 2u_i^l + u_{i+1}^l}{h^2}$$



$$u_i^{l+1} = u_i^l + \frac{\tau\kappa}{h^2}\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right)$$

Layer Matrix

$$u_i^{l+1} = u_i^l + \frac{\tau\kappa}{h^2}\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right) = \sigma u_{i-1}^l + \left(1-2\sigma\right)u_i^l + \sigma u_{i+1}^l$$

$$\sigma \equiv \frac{\tau\kappa}{h^2}; \quad \gamma \equiv 1-2\sigma; \quad \mathbf{u}^l \equiv \left[u_1^l, u_2^l, \ldots, u_N^l\right]^T; \qquad \mathbf{u}^{l+1} = \mathbf{A}\mathbf{u}^l$$

$$\mathbf{A} \equiv \begin{bmatrix} \gamma & \sigma & & & \\ \sigma & \gamma & \sigma & & \\ & \sigma & \gamma & \ddots & \\ & & \ddots & \ddots & \sigma \\ & & & \sigma & \gamma \end{bmatrix}$$

Error Propagation

$$\mathbf{u}^{l+1} = \mathbf{A}\mathbf{u}^{l} = \mathbf{A}^{2}\mathbf{u}^{l-1} = \ldots = \mathbf{A}^{l+1}\mathbf{u}^{0}$$

$$\tilde{\mathbf{u}}^{0} = \mathbf{u}^{0} + \mathbf{e}^{0} \longrightarrow \tilde{\mathbf{u}}^{l+1} = \mathbf{A}^{l+1}\mathbf{u}^{0} + \mathbf{A}^{l+1}\mathbf{e}^{0}$$

$$\tilde{\mathbf{u}}^{l+1} = \mathbf{u}^{l+1} + \mathbf{A}^{l+1}\mathbf{e}^{0} \longrightarrow \rho\left(\mathbf{A}\right) \leq 1$$

Tridiagonal Uniform Matrices

$$\mathbf{A} \equiv \begin{bmatrix} c & a & & & \\ a & c & a & & \\ & a & c & \ddots & \\ & & \ddots & \ddots & a \\ & & & a & c \end{bmatrix}$$

$$\lambda_k = c + 2a\cos\frac{k\pi}{n+1}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix}$$

$$u_{i,k} = \sin\left(i \cdot k\frac{\pi}{n+1}\right)$$

Stability Condition

$$\gamma = 1 - 2\sigma \longrightarrow \lambda_k = 1 - 4\sigma \sin^2 \frac{k\pi}{2(N+1)} < 1$$

$$-1 \le 1 - 4\sigma \sin^2 \frac{k\pi}{2(N+1)} \longrightarrow \sigma \sin^2 \frac{k\pi}{2(N+1)} \le \frac{1}{2}$$

$$\sin \frac{N\pi}{2(N+1)} \xrightarrow[N \to \infty]{} 1$$

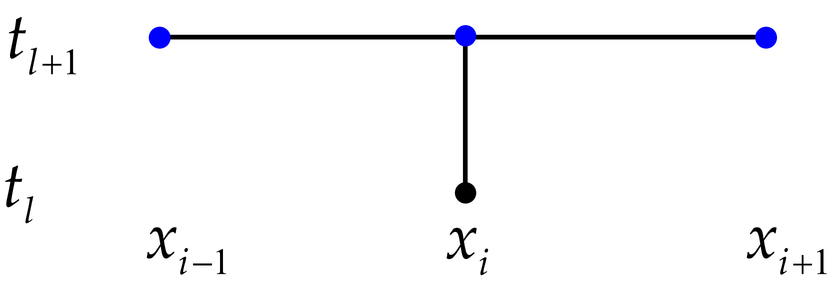
$$\sigma \equiv \frac{\tau\kappa}{h^2} \le \frac{1}{2}$$

Implicit FD Scheme

$$u_t = \kappa u_{xx}$$

$$\frac{u_i^{l+1} - u_i^l}{\tau} = \kappa \frac{u_{i-1}^l - 2u_i^l + u_{i+1}^l}{h^2}$$

$$\frac{u_i^{l+1} - u_i^l}{\tau} = \kappa \frac{u_{i-1}^{l+1} - 2u_i^{l+1} + u_{i+1}^{l+1}}{h^2}$$



$$-u_{i-1}^{l+1} + \left(2 + \frac{h^2}{\kappa\tau}\right) u_i^{l+1} - u_{i+1}^{l+1} = \frac{h^2}{\kappa\tau} u_i^l$$

Equivalent Matrix Equation

$$-u_{i-1}^{l+1} + \left(2 + \frac{h^2}{\kappa\tau}\right) u_i^{l+1} - u_{i+1}^{l+1} = \frac{h^2}{\kappa\tau} u_i^l$$

$$\begin{bmatrix} c & -1 & & & & \\ -1 & c & -1 & & & \\ & -1 & c & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & c & -1 \\ & & & & -1 & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \\ b_{N-1} \\ b_N \end{bmatrix}$$

Stability Analysis

$$-u_{i-1}^{l+1} + \left(2 + \frac{h^2}{\kappa\tau}\right) u_i^{l+1} - u_{i+1}^{l+1} = \frac{h^2}{\kappa\tau} u_i^l$$

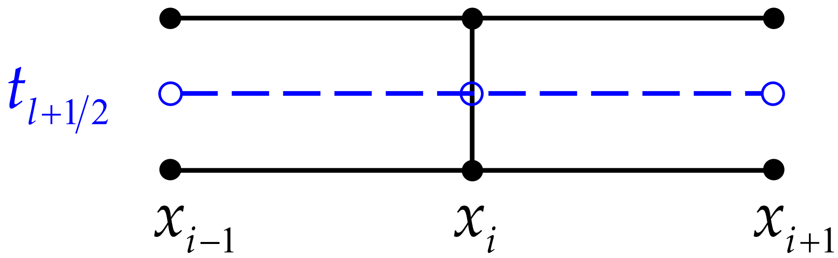
$$\mathbf{A}\mathbf{u}^{l+1} = \frac{h^2}{\kappa\tau}\mathbf{u}^l \longrightarrow \mathbf{u}^{l+1} = \frac{h^2}{\kappa\tau}\mathbf{A}^{-1}\mathbf{u}^l$$

$$\lambda_k(\mathbf{A}) = 2 + \frac{h^2}{\kappa\tau} - 2\cos\frac{k\pi}{N+1} = \frac{h^2}{\kappa\tau} + 4\sin^2\frac{k\pi}{2(N+1)}$$

$$\lambda_k\left(\frac{h^2}{\kappa\tau}\mathbf{A}^{-1}\right) = \frac{h^2/\kappa\tau}{h^2/\kappa\tau + 4\sin^2(\quad)} \rightarrow \rho\left(\frac{h^2}{\kappa\tau}\mathbf{A}^{-1}\right) < 1$$

Increasing Accuracy in Time

$$u_t = \kappa u_{xx}$$

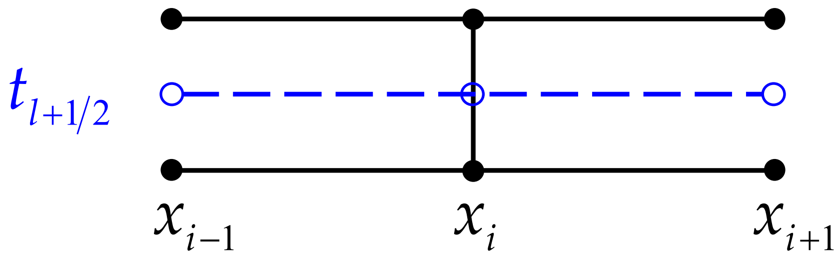


$$\frac{u_i^{l+1} - u_i^l}{\tau} = u_t\left(x_i, t_{l+1/2}\right) + O\left(\tau^2\right)$$

$$\frac{u_i^{l+1} - u_i^l}{\tau} = \kappa \frac{u_{i-1}^{l+1/2} - 2u_i^{l+1/2} + u_{i+1}^{l+1/2}}{h^2} \qquad O\left(h^2 + \tau^2\right)$$

Crank Nicolson's Idea

$$u_t = \kappa u_{xx}$$



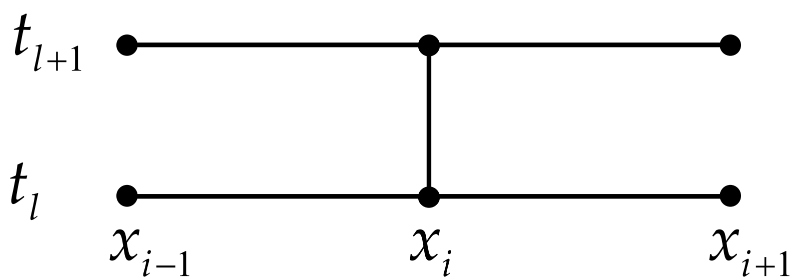
$$u_i^{l+1/2} \approx \frac{1}{2}\left(u_i^l + u_i^{l+1}\right)$$

$$u_{xx}\left(x_i, t_{l+1/2}\right) \approx \frac{1}{2}\left(\nabla_h^2 u_i^l + \nabla_h^2 u_i^{l+1}\right) =$$

$$= \frac{1}{2}\left(\frac{u_{i-1}^l - 2u_i^l + u_{i+1}^l}{h^2} + \frac{u_{i-1}^{l+1} - 2u_i^{l+1} + u_{i+1}^{l+1}}{h^2}\right)$$

Crank Nicolson Scheme

$$u_t = \kappa u_{xx}$$



$$\sigma \equiv \frac{\tau\kappa}{2h^2}$$

$$\frac{u_i^{l+1} - u_i^l}{\tau} = \kappa \frac{1}{2}\left(\frac{u_{i-1}^l - 2u_i^l + u_{i+1}^l}{h^2} + \frac{u_{i-1}^{l+1} - 2u_i^{l+1} + u_{i+1}^{l+1}}{h^2} \right)$$

$$-\sigma u_{i-1}^{l+1} + \left(1 + 2\sigma\right) u_i^{l+1} - \sigma u_{i+1}^{l+1} = u_i^l + \sigma\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right)$$

Parabolic Equations in 2D

$$\frac{\partial u(x,y,t)}{\partial t} = \nabla \cdot (\kappa \nabla u) + f(x,y,t)$$

$$u_t = \kappa (u_{xx} + u_{yy}) \qquad 0 \le x \le X, \quad 0 \le y \le Y, \quad 0 \le t \le T$$

BC: $u(0,y,t) = u(X,y,t) = 0; \qquad u(x,0,t) = u(x,Y,t) = 0.$

IC: $u(x,y,0) = u_0(x,y)$

Space-Time Mesh in 2D

$$h_x = \frac{X}{N_x + 1} \qquad x_i = i \cdot h_x \quad \left(i = 0, 1, \ldots, N_x + 1\right)$$

$$h_y = \frac{Y}{N_y + 1} \qquad y_j = j \cdot h_y \quad \left(j = 0, 1, \ldots, N_y + 1\right)$$

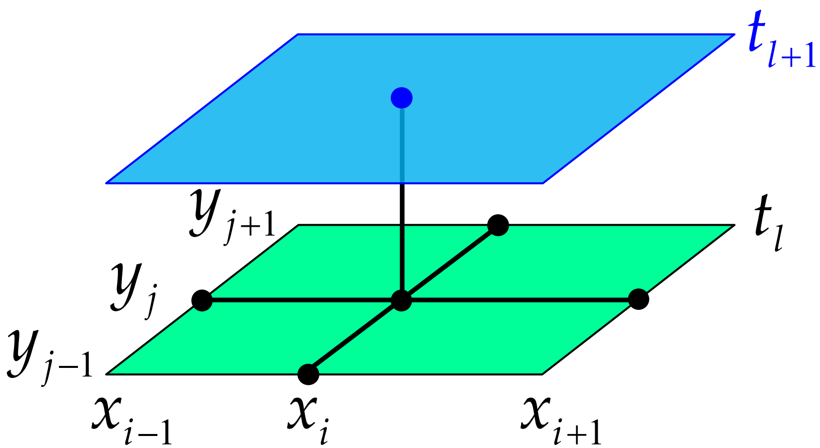
$$\tau = \frac{T}{M} \qquad t_l = l \cdot \tau \quad \left(l = 0, 1, \ldots, M\right)$$

$$u_{i,j}^{l} \approx u\left(x_i, y_j, t_l\right)$$

Explicit FD Scheme in 2D

$$u_t = \kappa\left(u_{xx} + u_{yy}\right)$$

$$\frac{u_{i,j}^{l+1} - u_{i,j}^{l}}{\tau} = \kappa\left[\frac{u_{i-1,j}^{l} - 2u_{i,j}^{l} + u_{i+1,j}^{l}}{h_x^2} + \frac{u_{i,j-1}^{l} - 2u_{i,j}^{l} + u_{i,j+1}^{l}}{h_y^2}\right]$$



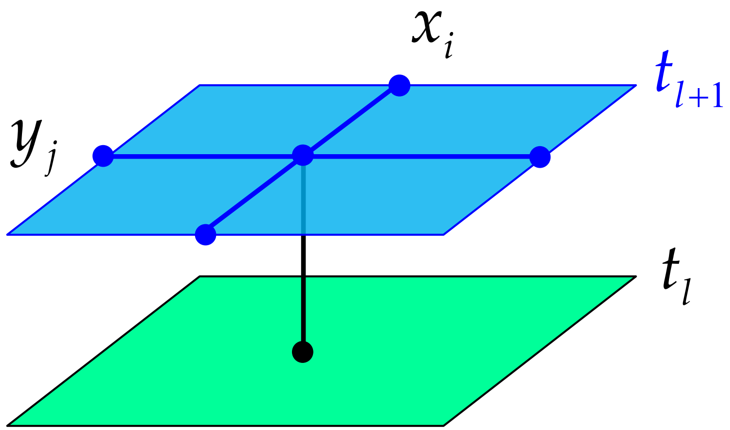
$$\tau \le \frac{1}{2}\frac{h^2}{\kappa}$$

$$\tau \le \frac{1}{8}\frac{h_x^2 + h_y^2}{\kappa} = \frac{1}{4}\frac{h^2}{\kappa}$$

Implicit FD Scheme in 2D

$$u_t = \kappa\left(u_{xx} + u_{yy}\right)$$

$$\frac{u_{i,j}^{l+1} - u_{i,j}^{l}}{\tau} = \kappa\left[\frac{u_{i-1,j}^{l+1} - 2u_{i,j}^{l+1} + u_{i+1,j}^{l+1}}{h_x^2} + \frac{u_{i,j-1}^{l+1} - 2u_{i,j}^{l+1} + u_{i,j+1}^{l+1}}{h_y^2}\right]$$

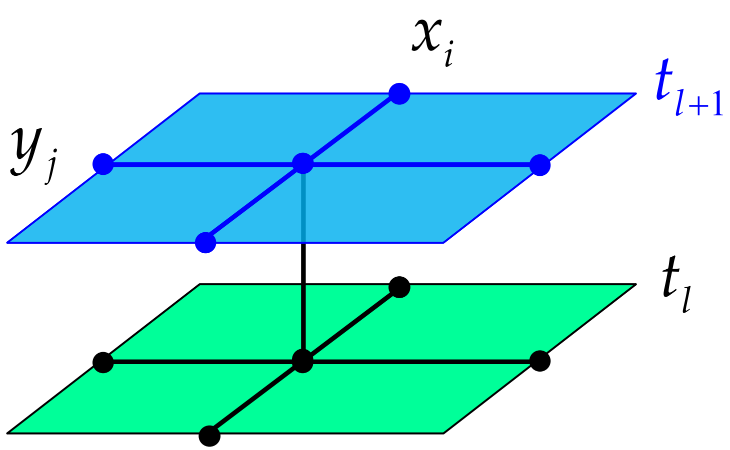


$$O\left(\tau + h^2\right)$$

CN FD Scheme in 2D

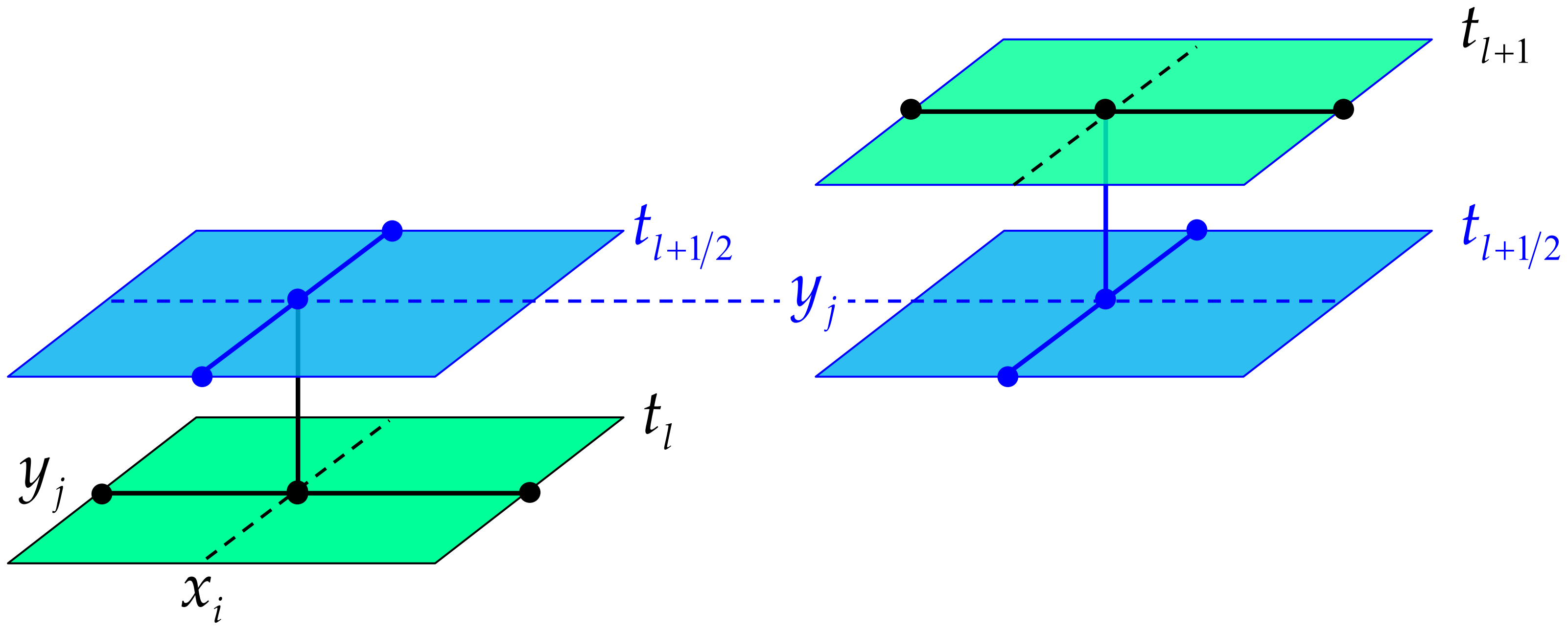
$$u_t = \kappa\left(u_{xx} + u_{yy}\right) \longrightarrow \frac{u_{i,j}^{l+1} - u_{i,j}^{l}}{\tau} = \kappa \nabla_h^2 u_{i,j}^{l+1}$$

$$\frac{u_{i,j}^{l+1} - u_{i,j}^{l}}{\tau} = \kappa \frac{1}{2}\left(\nabla_h^2 u_{i,j}^{l} + \nabla_h^2 u_{i,j}^{l+1}\right)$$



$O\left(\tau^2 + h^2\right)$

Alternating-Direction Implicit



ADI Scheme in 2D

$$u_t = \kappa\left(u_{xx} + u_{yy}\right) + f\left(x, y, t\right)$$

$$\frac{u_{i,j}^{l+1/2} - u_{i,j}^{l}}{\tau/2} = \kappa\left[\frac{u_{i-1,j}^{l} - 2u_{i,j}^{l} + u_{i+1,j}^{l}}{h_x^2} + \frac{u_{i,j-1}^{l+1/2} - 2u_{i,j}^{l+1/2} + u_{i,j+1}^{l+1/2}}{h_y^2}\right] + \frac{1}{2} f_{i,j}^{l+1/2}$$

$$\frac{u_{i,j}^{l+1} - u_{i,j}^{l+1/2}}{\tau/2} = \kappa\left[\frac{u_{i-1,j}^{l+1} - 2u_{i,j}^{l+1} + u_{i+1,j}^{l+1}}{h_x^2} + \frac{u_{i,j-1}^{l+1/2} - 2u_{i,j}^{l+1/2} + u_{i,j+1}^{l+1/2}}{h_y^2}\right] + \frac{1}{2} f_{i,j}^{l+1/2}$$

Wave Equation

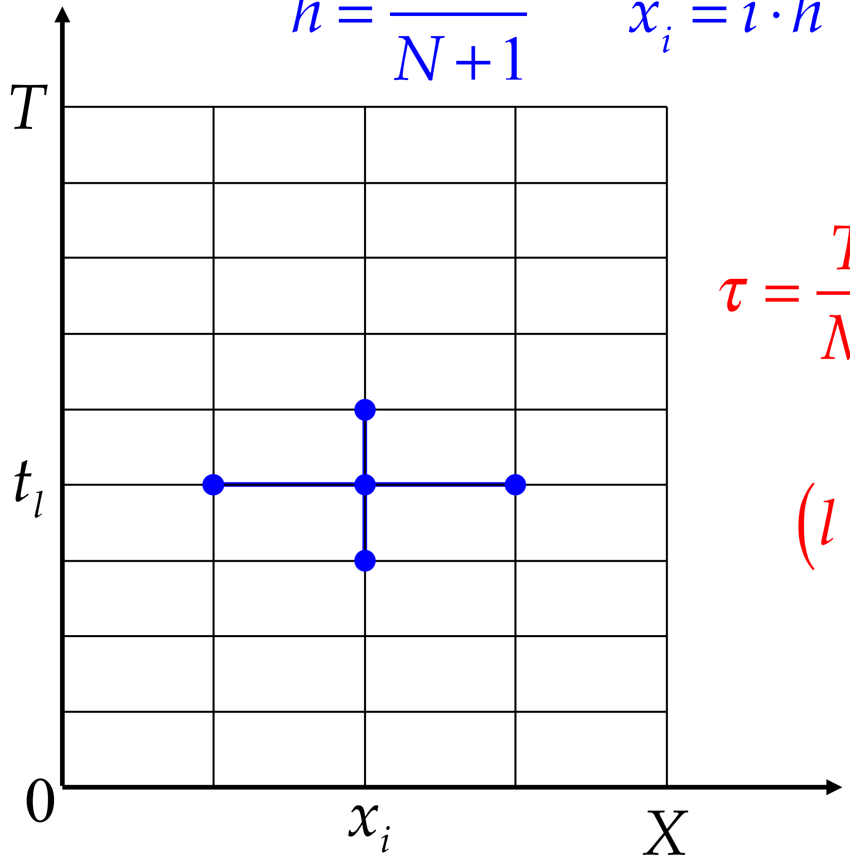
$$\begin{cases} u_{tt} = c^2 u_{xx} \qquad 0 \le x \le X \qquad 0 \le t \le T \\ u(0,t) = u(X,t) = 0 \\ u(x,0) = f(x) \quad u_t(x,0) = g(x) \end{cases}$$

$$\frac{u_i^{l+1} - 2u_i^l + u_i^{l-1}}{\tau^2} = c^2 \frac{u_{i-1}^l - 2u_i^l + u_{i+1}^l}{h^2}$$

$$LTE = O\left(\tau^2 + h^2\right) \longrightarrow Err = O\left(\tau^2 + h^2\right)$$

Five Point Stencil for WE

$$h = \frac{X}{N+1} \qquad x_i = i \cdot h \qquad (i = 0, 1, \ldots, N+1)$$



$$\tau = \frac{T}{M} \qquad t_l = l \cdot \tau$$

$$(l = 0, 1, \ldots, M)$$

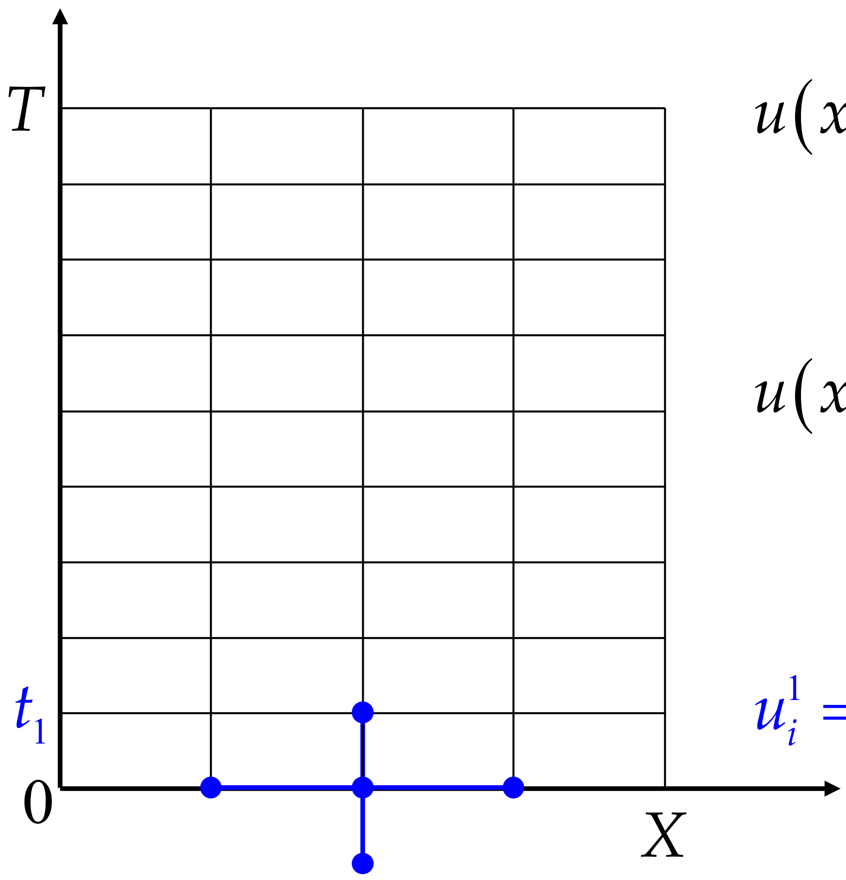
Explicit Scheme for WE

$$\frac{u_i^{l+1} - 2u_i^l + u_i^{l-1}}{\tau^2} = c^2 \frac{u_{i-1}^l - 2u_i^l + u_{i+1}^l}{h^2}$$

$$u_i^{l+1} = 2u_i^l - u_i^{l-1} + \frac{c^2\tau^2}{h^2}\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right)$$

$$LTE = O\left(\tau^2 + h^2\right) \longrightarrow Err = O\left(\tau^2 + h^2\right)$$

First Time Layer



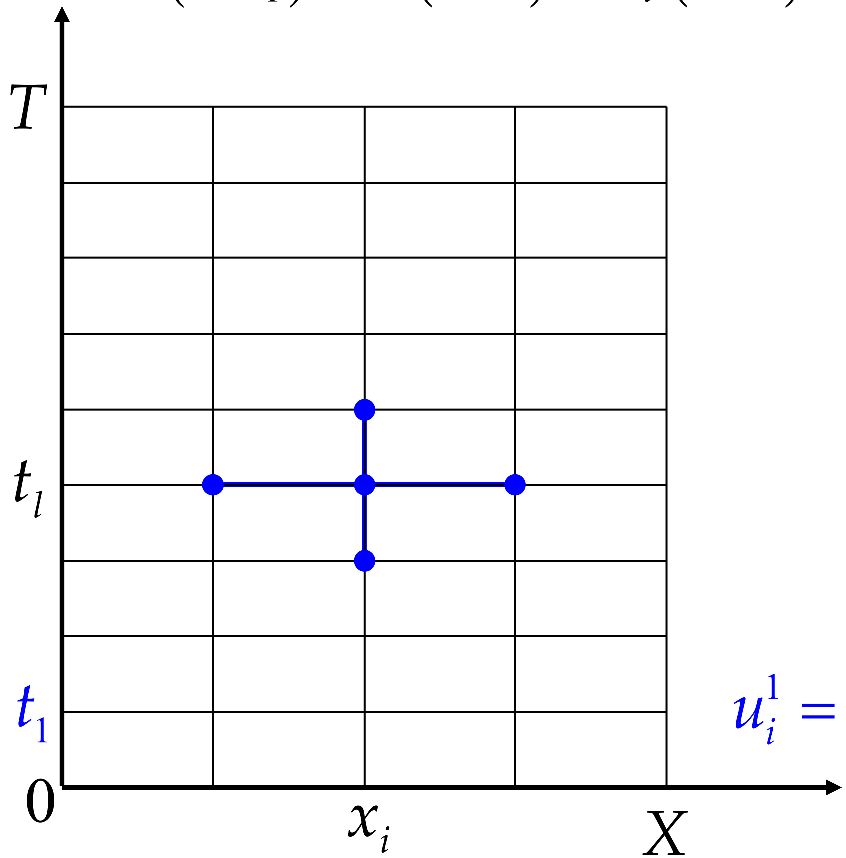
$$u(x,t_1)=u(x,0)+u_t(x,0)\tau+\frac{u_{tt}(x,\xi)}{2!}\tau^2$$

$$u(x,t_1)=f(x)+g(x)\tau+\frac{u_{tt}(x,\xi)}{2!}\tau^2$$

$$u_i^1=f(x_i)+g(x_i)\tau\rightarrow Err=O(\tau+h^2)$$

Improving IC

$$u(x,t_1) = u(x,0) + u_t(x,0)\tau + \frac{u_{tt}(x,0)}{2!}\tau^2 + \frac{u_{ttt}(x,\xi)}{3!}\tau^3$$



$$u_{tt} = c^2 u_{xx}$$

$$u_{tt}(x,0) = c^2 u_{xx}(x,0) = c^2 f''(x)$$

$$u_i^1 = f(x_i) + g(x_i)\tau + f''(x_i)\frac{c^2}{2}\tau^2$$

Explicit Scheme for WE

$$\frac{u_i^{l+1} - 2u_i^l + u_i^{l-1}}{\tau^2} = c^2 \frac{u_{i-1}^l - 2u_i^l + u_{i+1}^l}{h^2}$$

$$u_i^{l+1} = 2u_i^l - u_i^{l-1} + \frac{c^2\tau^2}{h^2}\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right)$$

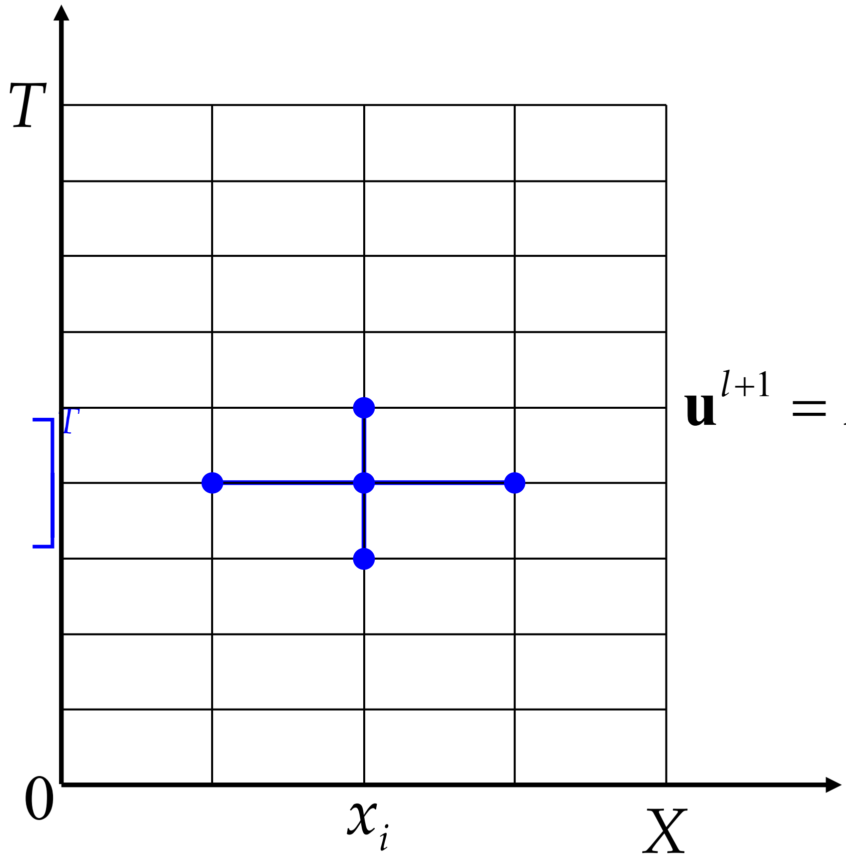
$$\sigma \equiv \left(\frac{c\tau}{h}\right)^2$$

$$u_i^{l+1} = \sigma u_{i-1}^l + 2\left(1-\sigma\right)u_i^l + \sigma u_{i+1}^l - u_i^{l-1}$$

Layer Equation

$$u_i^{l+1} = \sigma u_{i-1}^l + 2(1-\sigma)u_i^l + \sigma u_{i+1}^l - u_i^{l-1}$$

$$\mathbf{u}^l \equiv \begin{bmatrix} u_1^l & u_2^l & \cdots & u_N^l \end{bmatrix}^T$$



$$\mathbf{u}^{l+1} = \mathbf{A}\mathbf{u}^l - \mathbf{u}^{l-1}$$

Spectral Stability

- Analytically doable (relatively simple)
- Widely spread
- **Does not give exact answer!**
- Filters out vast majority of unstable FDS
- Spectrally stable FDS are stable very often
- Real FDS is simplified
  - Linear, homogeneous, constant coefficients
  - Extending to full space (removing BC)

Example of Simplification

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\kappa(x,t,u)\frac{\partial u}{\partial x}\right] + f(x,t,u) \qquad 0 \le x \le X \quad 0 \le t \le T$$

BC: $$-\alpha_1 \frac{\partial u}{\partial x} + \beta_1 u = \psi_1(t) \qquad -\alpha_2 \frac{\partial u}{\partial x} + \beta_2 u = \psi_2(t)$$

IC: $$u(x,0) = u_0(x)$$

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \alpha u \qquad -\infty \le x \le \infty \quad 0 \le t \le T$$

Partial Solutions

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad -\infty \le x \le \infty \quad 0 \le t \le T$$

$$u(x,t) = \sum_k C_k u_k(x,t) \quad u_k(x,t) = e^{-\kappa k^2 t} e^{-ikx}$$

$$\frac{u_m^{l+1} - u_m^l}{\tau} = \kappa \frac{u_{m+1}^l - 2u_m^l + u_{m-1}^l}{h^2} + a u_m^l \qquad m = 0, \pm 1, \pm 2, \cdots$$

$$u_m^l = \lambda^l \cdot e^{im\varphi} \qquad 0 \le \varphi \le 2\pi$$

Spectrally Stable FDS

$$u_m^l = \lambda^l \cdot e^{im\varphi}$$

(Spectral function)

$$0 \le \varphi \le 2\pi \qquad \lambda = \lambda(h, \tau, FDS, \varphi) = \lambda(\varphi)$$

Spectrally Stable $\quad \left|\lambda(\varphi)\right| \le 1 + C\tau; \quad \forall \varphi \in [0, 2\pi]$

Unstable $\quad \exists q > 1 \ \& \ \varphi_0 \in [0, 2\pi] \to \left|\lambda(\varphi_0)\right| \ge q > 1$

Explicit FDS for PE

$$u_t = \kappa u_{xx}$$

$$u_m^l = \lambda^l \cdot e^{im\varphi} \rightarrow \quad \frac{u_m^{l+1} - u_m^l}{\tau} = \kappa \frac{u_{m-1}^l - 2u_m^l + u_{m+1}^l}{h^2}$$

$$\frac{\lambda^{l+1} \cdot e^{im\varphi} - \lambda^l \cdot e^{im\varphi}}{\tau} = \kappa \frac{\lambda^l \cdot e^{i(m-1)\varphi} - 2\lambda^l \cdot e^{im\varphi} + \lambda^l \cdot e^{i(m+1)\varphi}}{h^2}$$

$$\frac{\lambda - 1}{\tau} = \kappa \frac{e^{-i\varphi} - 2 + e^{i\varphi}}{h^2}$$

Stability for Explicit FDS

$$\frac{\lambda - 1}{\tau} = \kappa \frac{e^{-i\varphi} - 2 + e^{i\varphi}}{h^2} = \kappa \frac{-4\sin^2 \dfrac{\varphi}{2}}{h^2} \rightarrow \lambda(\varphi) = 1 - \frac{\kappa\tau}{h^2} 4\sin^2 \frac{\varphi}{2}$$

$$\lambda(\pi) = 1 - 4\frac{\kappa\tau}{h^2} \le \lambda(\varphi) \le 1 = \lambda(0)$$

$$-1 \le 1 - 4\frac{\kappa\tau}{h^2} \longrightarrow \tau \le \frac{h^2}{2\kappa}$$

Implicit FDS for PE

$$u_t = \kappa u_{xx}$$

$$\frac{u_m^{l+1} - u_m^l}{\tau} = \kappa \frac{u_{m-1}^{l+1} - 2u_m^{l+1} + u_{m+1}^{l+1}}{h^2} \qquad \leftarrow u_m^l = \lambda^l \cdot e^{im\varphi}$$

$$\frac{\lambda^{l+1} \cdot e^{im\varphi} - \lambda^l \cdot e^{im\varphi}}{\tau} = \kappa \frac{\lambda^{l+1} \cdot e^{i(m-1)\varphi} - 2\lambda^{l+1} \cdot e^{im\varphi} + \lambda^{l+1} \cdot e^{i(m+1)\varphi}}{h^2}$$

$$\frac{\lambda - 1}{\tau} = \kappa \frac{\lambda\left(e^{-i\varphi} - 2 + e^{i\varphi}\right)}{h^2} \longrightarrow \lambda(\varphi) = \frac{1}{1 + 4\frac{\kappa\tau}{h^2}\sin^2\frac{\varphi}{2}}$$

Spectral Function for WE FDS

$$\frac{u_m^{l+1} - 2u_m^l + u_m^{l-1}}{\tau^2} = c^2 \frac{u_{m-1}^l - 2u_m^l + u_{m+1}^l}{h^2} \leftarrow u_m^l = \lambda^l \cdot e^{im\varphi}$$

$$\frac{\lambda^{l+1} e^{im\varphi} - 2\lambda^l e^{im\varphi} + \lambda^{l-1} e^{im\varphi}}{\tau^2} = c^2 \frac{\lambda^l e^{i(m-1)\varphi} - 2\lambda^l e^{im\varphi} + \lambda^l e^{i(m+1)\varphi}}{h^2}$$

$$\frac{\lambda^2 - 2\lambda + 1}{\tau^2} = c^2 \frac{\lambda\left(e^{-i\varphi} - 2 + e^{i\varphi}\right)}{h^2}$$

$$\lambda^2 - 2\left(1 - 2\frac{c^2\tau^2}{h^2}\sin^2\frac{\varphi}{2}\right)\lambda + 1 = 0$$

Stability Condition for WE

$$\lambda^2 - 2\left(1 - 2\frac{c^2\tau^2}{h^2}\sin^2\frac{\varphi}{2}\right)\lambda + 1 = 0 \longrightarrow \lambda_1\lambda_2 = 1$$

- Real roots → FDS is spectrally unstable
- Complex conjugate roots → $\left|\lambda_1\right| = \left|\lambda_2\right| = 1$

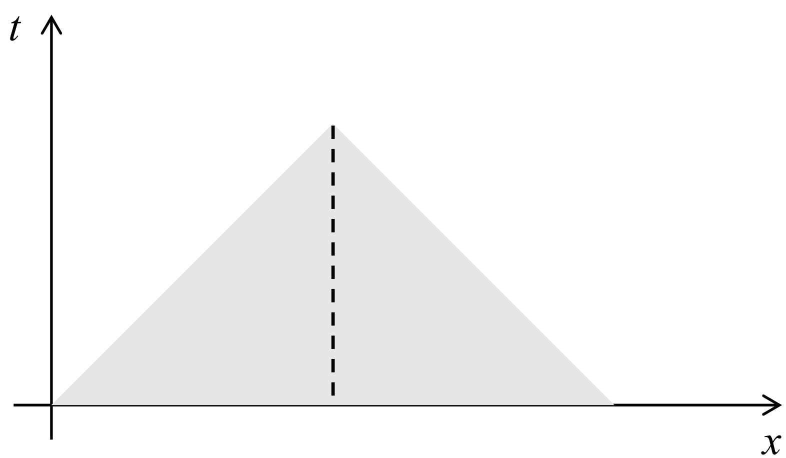
$$D = \left(1 - 2\frac{c^2\tau^2}{h^2}\sin^2\frac{\varphi}{2}\right)^2 - 1 = 4\frac{c^2\tau^2}{h^2}\sin^2\frac{\varphi}{2}\left(\frac{c^2\tau^2}{h^2}\sin^2\frac{\varphi}{2} - 1\right) \le 0$$

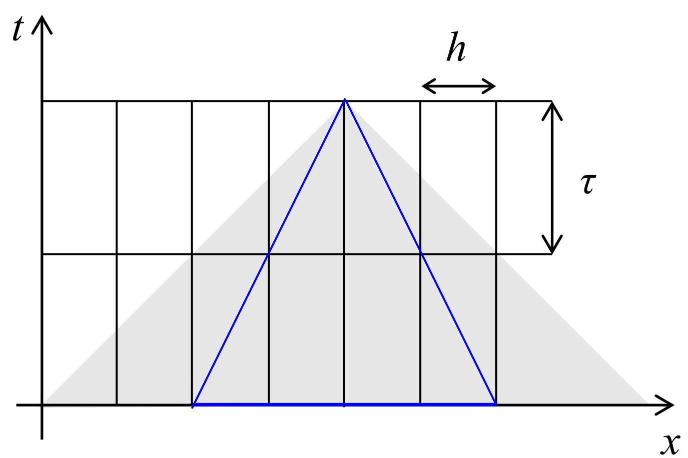
$$\frac{c\tau}{h} \le 1 \qquad c\tau \le h$$

DoD

$$u_{tt} = u_{xx} \qquad c = 1$$

$$\frac{u_m^{l+1} - 2u_m^l + u_m^{l-1}}{\tau^2} = \frac{u_{m-1}^l - 2u_m^l + u_{m+1}^l}{h^2}$$





$$c = 1 \longrightarrow \tau \le h$$

Non-Linear FD Schemes

$$c(x,t,u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\kappa(x,t,u)\frac{\partial u}{\partial x}\right] + f(x,t,u)$$

$$c_i^l \frac{u_i^{l+1} - u_i^l}{\tau} = \frac{1}{h}\left[\kappa_{i+1/2}^l \frac{u_{i+1}^l - u_i^l}{h} - \kappa_{i-1/2}^l \frac{u_i^l - u_{i-1}^l}{h}\right]$$

$$c_i^l \equiv c\left(x_i, t_l, u_i^l\right) \quad \kappa_{i+1/2}^l \equiv \kappa\left(x_{i+1/2}, t_l, \frac{u_i^l + u_{i+1}^l}{2}\right) \quad f_i^l \equiv f\left(x_i, t_l, u_i^l\right)$$

Major Steps in FD Method

- Discretisation of Domain
- Approximation of Derivatives, LTE
- Finite-Difference Scheme
- Residual
- Stability
- Convergence

Discretisation

$$\begin{cases} u''(x) = f(x) & x \in Dom = [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

$$Dom_h = \{x_i, \quad i = 0,1,\ldots,N+1\}$$

$$h = \frac{1}{N+1} \qquad x_i = i \cdot h \quad (i = 0,1,\ldots,N+1)$$

$$Dom_h = \{0, h, 2h, \ldots, Nh, 1\}$$

Approximation of Derivatives

$$\frac{y(x-h)-2y(x)+y(x+h)}{h^2}=y''(x)+\frac{y^{(4)}(\xi)}{12}h^2$$

$$LTE \equiv \frac{y^{(4)}(\xi)}{12}h^2 = O\left(h^2\right)$$

Make sure consistency: LTE $\rightarrow$ 0

Finite-Difference Scheme

$$\frac{y(x-h)-2y(x)+y(x+h)}{h^2}\approx y''(x)$$

$$\begin{cases} \dfrac{y_{i-1}-2y_i+y_{i+1}}{h^2}=f_i=f(x_i) \\ y_0=y_{N+1}=0 \end{cases}$$

$$\nabla_h^2\mathbf{y}=\mathbf{f}$$

$$\mathbf{y}\equiv\left[y_1,y_2,\ldots,y_N\right]^T$$

$$\left|u(x_i)-y_i\right|\xrightarrow[h\to 0]{}0 \quad ??$$

Finite-Difference Matrix

$$\nabla_h^2 \mathbf{y} = \mathbf{f}$$

$$\nabla_h^2 = \frac{1}{h^2} \underbrace{\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix}}_{N \simeq 1/h} = \frac{1}{h^2} \mathbf{A}(h) \simeq N^2 \mathbf{A}_{N \times N}$$

$$\left(\nabla_h^2\right)^{-1} = h^2 A^{-1}(h) \longrightarrow \left\| \left(\nabla_h^2\right)^{-1} \right\| \leq C \neq C(h)$$

Residual

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f_i$$

$$\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = u''(x_i) + \frac{u^{(4)}(\xi_i)}{12}h^2$$

$$\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = f_i + r_i$$

$$e_i \equiv u(x_i) - y_i \qquad \frac{e_{i-1} - 2e_i + e_{i+1}}{h^2} = r_i$$

Two Vectors

$$\mathbf{u} \equiv \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{bmatrix} \qquad \mathbf{y} \equiv \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$\left\| \mathbf{u} - \mathbf{y} \right\| \xrightarrow[h \to 0]{} 0$$

Stability

$$\nabla_h^2 \mathbf{y} = \mathbf{f}$$

$$\nabla_h^2 \tilde{\mathbf{y}} = \tilde{\mathbf{f}}$$

$$\mathbf{y} - \tilde{\mathbf{y}} = \left(\nabla_h^2\right)^{-1}\left(\mathbf{f} - \tilde{\mathbf{f}}\right) \qquad \left\|\left(\nabla_h^2\right)^{-1}\right\| \le C \ne C(h)$$

$$\left\|\mathbf{y} - \tilde{\mathbf{y}}\right\| \le \left\|\left(\nabla_h^2\right)^{-1}\right\| \cdot \left\|\mathbf{f} - \tilde{\mathbf{f}}\right\| \le C\left\|\mathbf{f} - \tilde{\mathbf{f}}\right\|$$

Convergence

$$\nabla_h^2 \mathbf{u} = \mathbf{f} + \mathbf{r}$$

$$\nabla_h^2 \mathbf{y} = \mathbf{f}$$

$$\mathbf{u} - \mathbf{y} = \left(\nabla_h^2\right)^{-1} \mathbf{r}$$

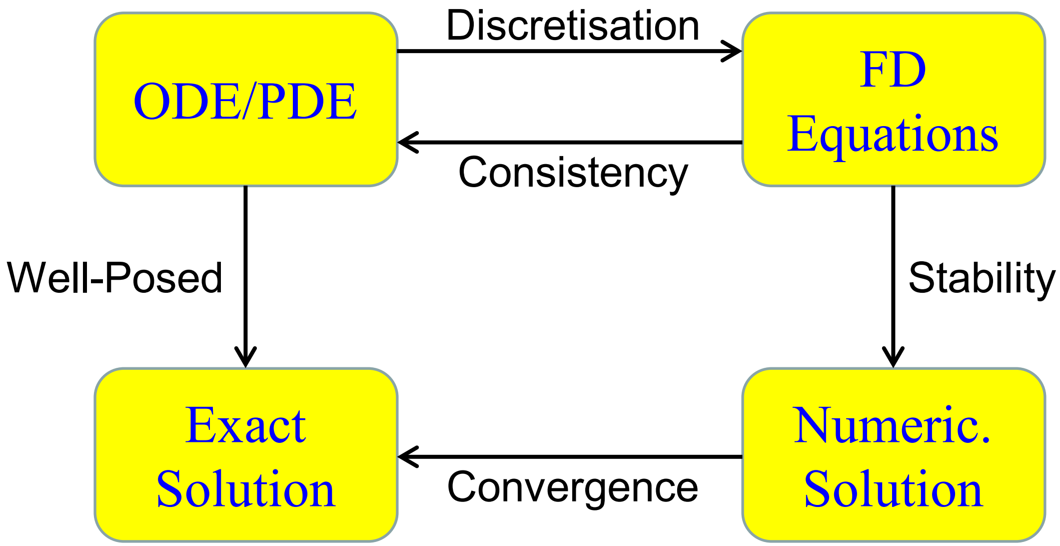
$$\left\|\mathbf{u} - \mathbf{y}\right\| \le \left\|\left(\nabla_h^2\right)^{-1}\right\| \cdot \left\|\mathbf{r}\right\| \le C\left\|\mathbf{r}\right\| \xrightarrow[h \to 0]{} 0$$

Lax Equivalence Theorem

Fundamental theorem in theory of FD method for ODE/PDE.

For a consistent FD method for a well-posed linear

IVP/BVP, the FD scheme is convergent iff it is stable.



Important

- Classification of PDE
- DoD RoI
- Parabolic Equations
- Explicit vs. Implicit
- Crank-Nicolson Scheme
- Hyperbolic Equations
- Spectral Stability of FD Schemes
- FDS for Non-Linear PDE