Numerical Integration

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Reactor Physics, KTH

Overview

- Quadrature
- Linearity of Integral → Composite
- Approximating Integrand
- Nodes and Weights
- Mid, Trapezium, Simpson
- Newton – Cotes Quadrature
- Richardson + Trapezoid = Romberg
- Adaptive Quadrature

Quadrature in Mathematics

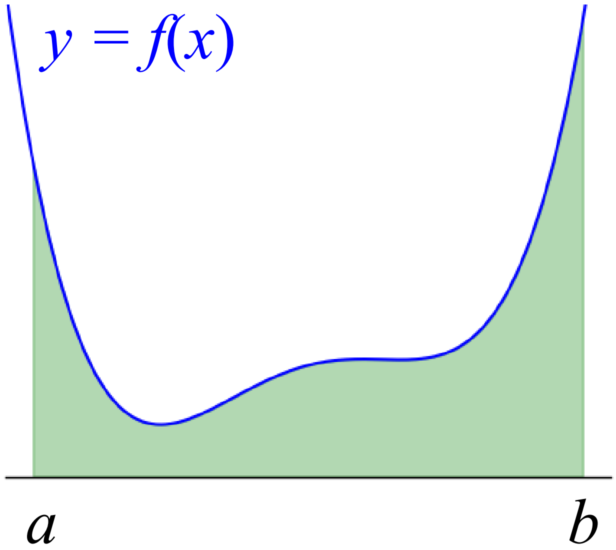
- Historical: The process of determining area;
- Pythagorus: Constructing a square with the same area;
- Differential equations: Solving an equation in terms of integrals;
- Numerical analysis: Evaluating definite integrals.

Riemann Integral

Georg Friedrich Bernhard Riemann,1826-1866 (39).

Presented at Göttingen University in 1854.

Published in a journal in 1868.



$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

$$\xi_i \in [x_i, x_{i+1}] \qquad h = \max_i (x_{i+1} - x_i)$$

$$\int_a^b f(x)dx \equiv \lim_{h \to 0} \sum_{i=0}^{n-1} f(\xi_i)(x_{i+1} - x_i)$$

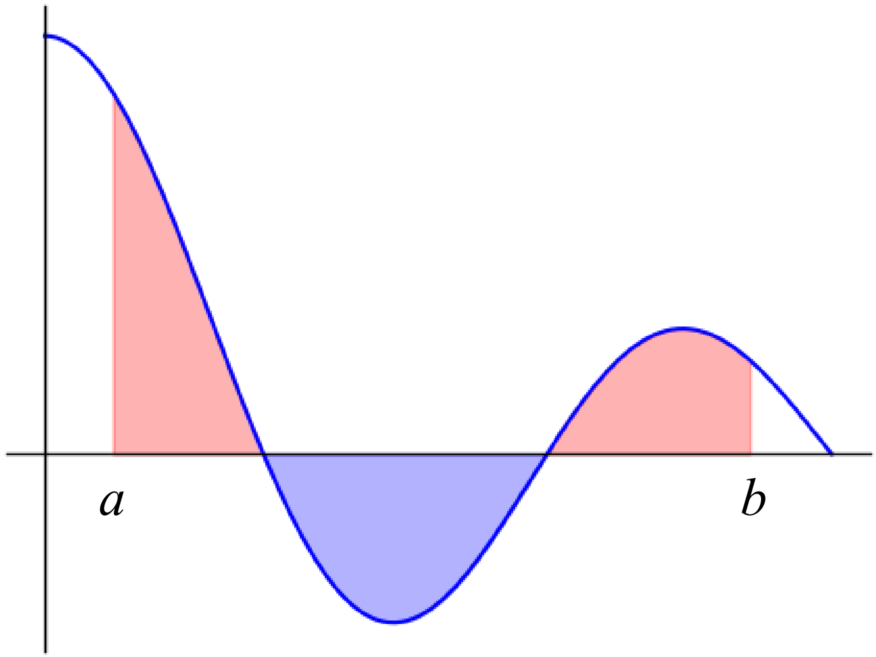
Fundamental Properties

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$F'(x) = f(x) \longrightarrow F(x) = \int f(x)\mathrm{d}x$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Signed Area



Quadrature

$$I(f) \equiv \int_a^b f(x)dx \approx Q(f)$$

Any formula or algorithm for calculating the numerical value of a definite integral, and by extension, the term is sometimes used to define the numerical solution of differential equations.

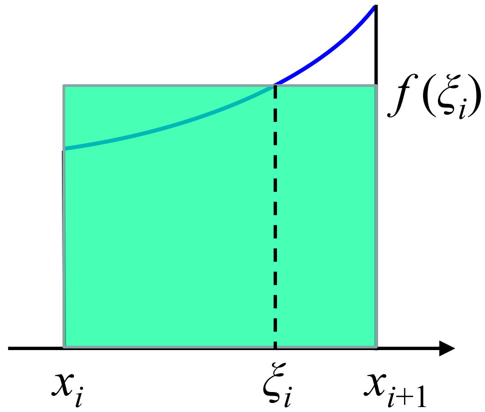
First Principles

$$y'(x) \equiv \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \approx \frac{y(x+h) - y(x)}{h}$$

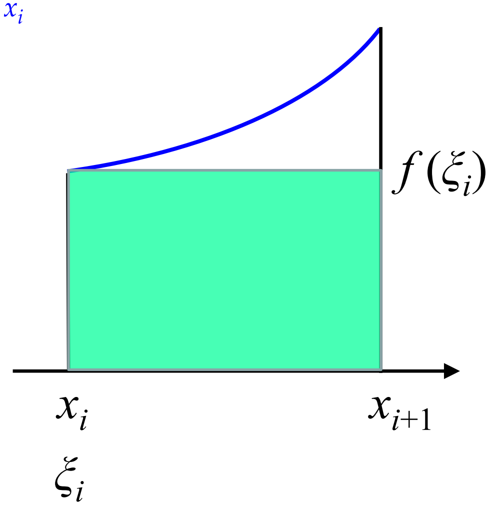
$$a = x_0 < x_1 < x_2 < \ldots < x_n = b \qquad \xi_i \in \left[x_i, x_{i+1} \right] \qquad h = \max_i \left(x_{i+1} - x_i \right)$$

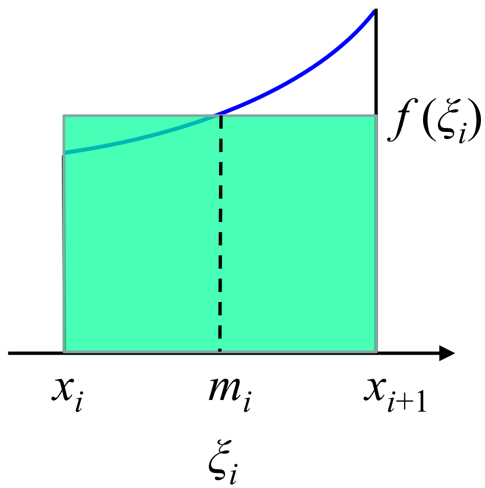
$$I\left(f\right) \equiv \int_a^b f(x)\,dx = \lim_{h \to 0} \sum_{i=0}^{n-1} f(\xi_i)\left(x_{i+1} - x_i\right) \approx \sum_{i=0}^{n-1} f(\xi_i)\left(x_{i+1} - x_i\right)$$

Riemann Approximation



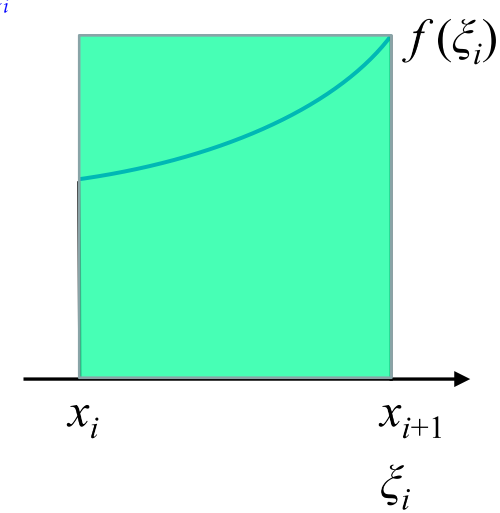
$$\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x \approx f(x_i)\left(x_{i+1} - x_i\right)$$



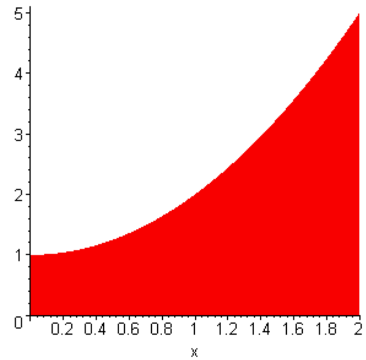


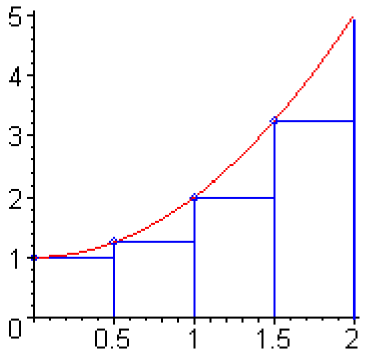
$$\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x \approx f(m_{i+1})\left(x_{i+1} - x_i\right)$$

$$\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x \approx f(x_{i+1})\left(x_{i+1} - x_i\right)$$

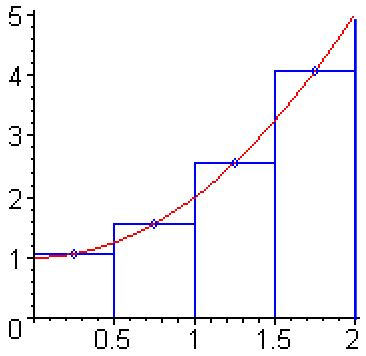


Riemann Sums

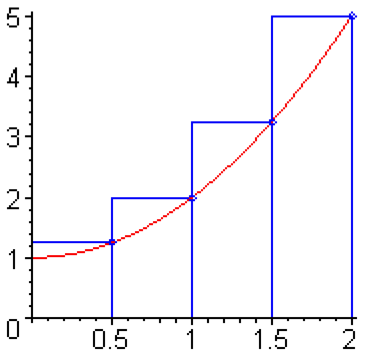




$$\sum_{i=0}^{n-1} f(x_i)\left(x_{i+1}-x_i\right)$$



$$\sum_{i=0}^{n-1} f(m_i)\left(x_{i+1}-x_i\right)$$



$$\sum_{i=0}^{n-1} f(x_{i+1})\left(x_{i+1}-x_i\right)$$

Possible Approaches

$$\int_a^b f(x)\mathrm{d}x = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x$$

1) Quadrature for $[\alpha, \beta]$ when $\beta - \alpha$ is small
2) Composite quadrature when $b - a$ is large

$$f(x) \approx f_n(x) \longrightarrow I(f) = \int_a^b f(x)dx \approx \int_a^b f_n(x)dx = I_n(f)$$

$$I(f) = \int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f\left(x_i\right) \equiv Q(f) \longleftrightarrow Q\left(p_m\right) = I\left(p_m\right)$$

weights

nodes

Norms of Functions

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|f\|_1 = \int_a^b |f(x)|\,dx$$

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \qquad \|f\|_2 = \left(\int_a^b f^2(x)\,dx \right)^{1/2}$$

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad \|f\|_p = \left(\int_a^b |f(x)|^p\,dx \right)^{1/p}$$

$$\|\mathbf{x}\|_\infty = \max_i |x_i| \qquad \|f\|_\infty = \max_x |f(x)|$$

Approximation Error

$$E_Q(f) \equiv I(f) - Q(f)$$

$$\left|E_n(f)\right| = \left|I(f) - I(f_n)\right| = \left|\int_a^b \left[f(x) - f_n(x)\right]dx\right| \le (b-a)\left\|f - f_n\right\|_\infty$$

$$Q(f) = I(f) \qquad \forall f \in P_n = \left\{p_n(x) = a_n x^n + \ldots + a_1 x + a_0\right\}$$

$$\text{Degree of exactness} = n$$

Sensitivity

$$\left|I(f)-I(\tilde{f})\right| = \left|\int_a^b \left[f(x)-\tilde{f}(x)\right]dx\right| \leq (b-a)\left\|f-\tilde{f}\right\|_\infty$$

$$\kappa_{abs}(f) \simeq (b-a)$$

$$\frac{\left|I(f)-I(\tilde{f})\right|}{\left|I(f)\right|} \leq (b-a)\frac{\left\|f\right\|_\infty}{\left|I(f)\right|}\frac{\left\|f-\tilde{f}\right\|_\infty}{\left\|f\right\|_\infty}$$

$$\kappa_\infty(f) \simeq (b-a)\frac{\left\|f\right\|_\infty}{\left|I(f)\right|}\frac{\left\|f-\tilde{f}\right\|_\infty}{\left\|f\right\|_\infty}$$

Accuracy: Local vs. Global

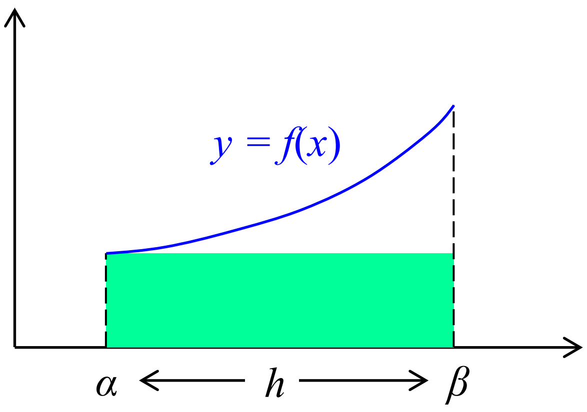
$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx \qquad h \equiv \frac{b-a}{n} \qquad x_i = i \cdot h$$

$$\int_\alpha^\beta f(x)dx = f(\xi)\cdot\left(\beta-\alpha\right) = O(h)$$

$$E^{[x_i,x_{i+1}]} \equiv I(f) - Q(f) = O\left(h^{m+1}\right)$$

$$E^{[a,b]} = \sum_{i=0}^{n-1} E^{[x_i,x_{i+1}]} = O\left(h^{m+1}\right)\cdot n = O\left(h^{m+1}\right)\cdot\frac{b-a}{h} = O\left(h^m\right)$$

Left Riemann Quadrature



$$I(f) = \int_{\alpha}^{\beta} f(x)\mathrm{d}x \approx (\beta - \alpha) f(\alpha) \equiv L(f)$$

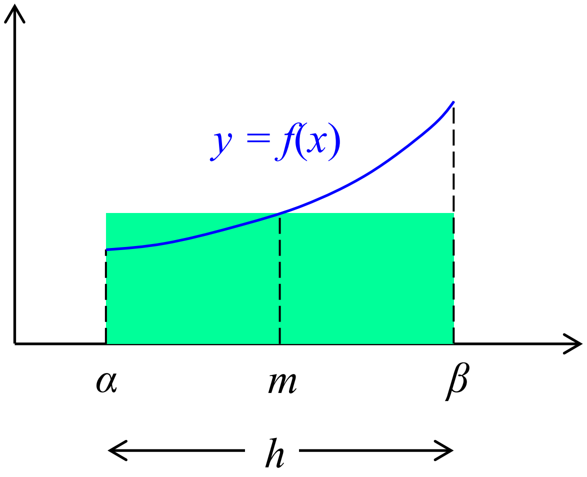
$$h = \beta - \alpha = (b - a)/n$$

$$f(x) = f(\alpha) + f'(\xi(x))(x - \alpha)$$

$$\int_{\alpha}^{\beta} f(x)\mathrm{d}x = f(\alpha)(\beta - \alpha) + \int_{\alpha}^{\beta} f'(\xi(x))(x - \alpha)\mathrm{d}x = L^{[\alpha,\beta]} + f'(\theta)\int_{\alpha}^{\beta}(x - \alpha)\mathrm{d}x$$

$$E_L^{[\alpha,\beta]}(f) = \frac{f'(\theta)}{2}h^2 \qquad E_L^{[a,b]}(f) = \frac{f'(\eta)(b-a)}{2}h$$

Midpoint (Rectangle) Rule



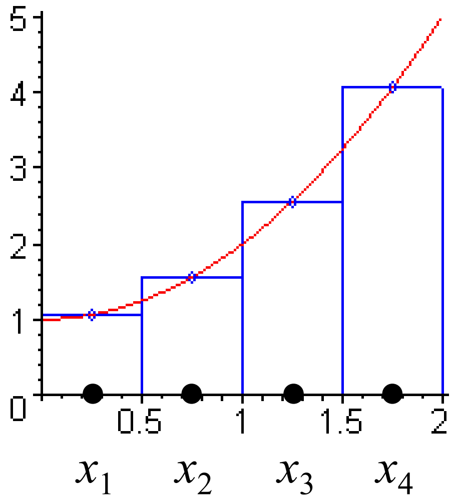
$$I(f) = \int_{\alpha}^{\beta} f(x)dx \approx (\beta - \alpha) f(m) \equiv M(f)$$

$$f(x) = f(m) + f'(m)(x - m) + \frac{f''(\xi(x))}{2}(x - m)^2$$

$$\int_{\alpha}^{\beta} f(x)dx = (\beta - \alpha) f(m) + \int_{\alpha}^{\beta} \frac{f''(\xi(x))}{2}(x - m)^2 \, dx$$

$$E_M^{[\alpha,\beta]}(f) = \frac{f''(\xi)}{24} h^3 \qquad E_M^{[a,b]}(f) = \frac{f''(\xi)(b - a)}{24} h^2$$

Composite Midpoint



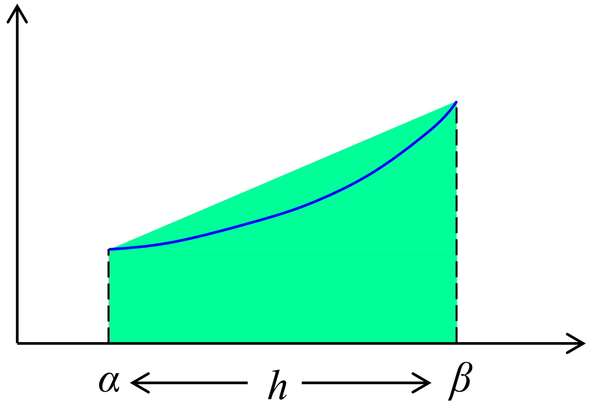
$$h = (b-a)/n$$

$$x_i = a + (i - 1/2)h \qquad i = 1, \cdots, n$$

$$M(f) = h\sum_{i=1}^{n} f(x_i)$$

$$|E_M(f)| \le \frac{\|f''\|_\infty (b-a)}{24} h^2$$

Trapezoidal Rule

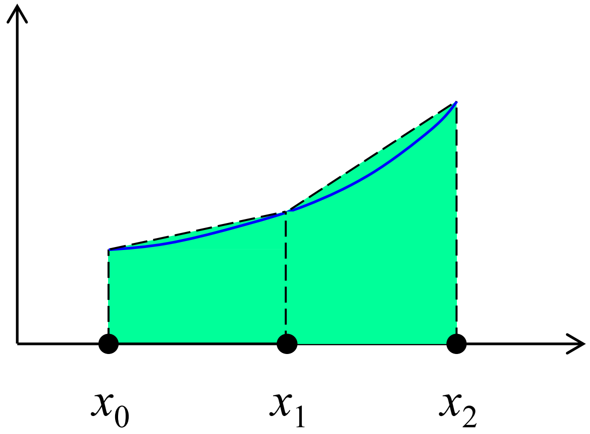


$$I(f) = \int_{\alpha}^{\beta} f(x)\mathrm{d}x \approx \frac{\beta-\alpha}{2}\left[f(\alpha)+f(\beta)\right] \equiv T(f)$$

$$E_T^{[\alpha,\beta]}(f) = -\frac{f''(\xi)}{12}h^3 \qquad E_T^{[a,b]}(f) = -\frac{f''(\xi)(b-a)}{12}h^2$$

Degree of exactness = 1 for both midpoint and trapezoidal quadrature!!

Composite Trapezoidal Rule



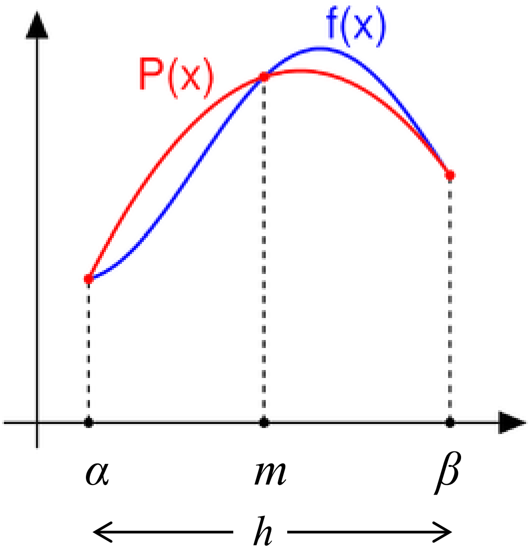
$$h = (b-a)/n$$

$$x_i = a + ih \qquad i = 0, 1, \cdots, n$$

$$T(f) = \frac{h}{2}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right] \qquad \left|E_T(f)\right| \le \frac{\left\|f''\right\|_\infty (b-a)}{12} h^2$$

$$T(f) = h\left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]$$

Simpson's Rule



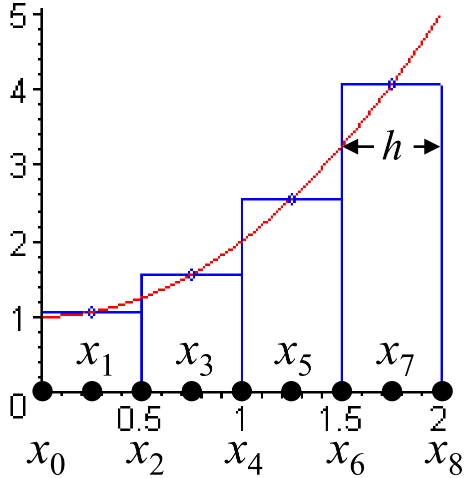
$$I(f)=\int_{\alpha}^{\beta} f(x)\mathrm{d}x \approx \frac{\beta-\alpha}{6}\left[f(\alpha)+4f(m)+f(\beta)\right] \equiv S(f)$$

$$E_S^{[\alpha,\beta]}(f) = -\frac{f^{(4)}(\xi)}{2880}h^5 \qquad E_S^{[a,b]}(f) = -\frac{f^{(4)}(\xi)(b-a)}{2880}h^4$$

$$P(x)=f(\alpha)\frac{(x-m)(x-\beta)}{(\alpha-m)(\alpha-\beta)}+f(m)\frac{(x-\alpha)(x-\beta)}{(m-\alpha)(m-\beta)}+f(\beta)\frac{(x-\alpha)(x-m)}{(\beta-\alpha)(\beta-m)}$$

Thomas Simpson, 1710 – 1761; Johannes Kepler 1571 – 1630.

Composite Simpson's Rule



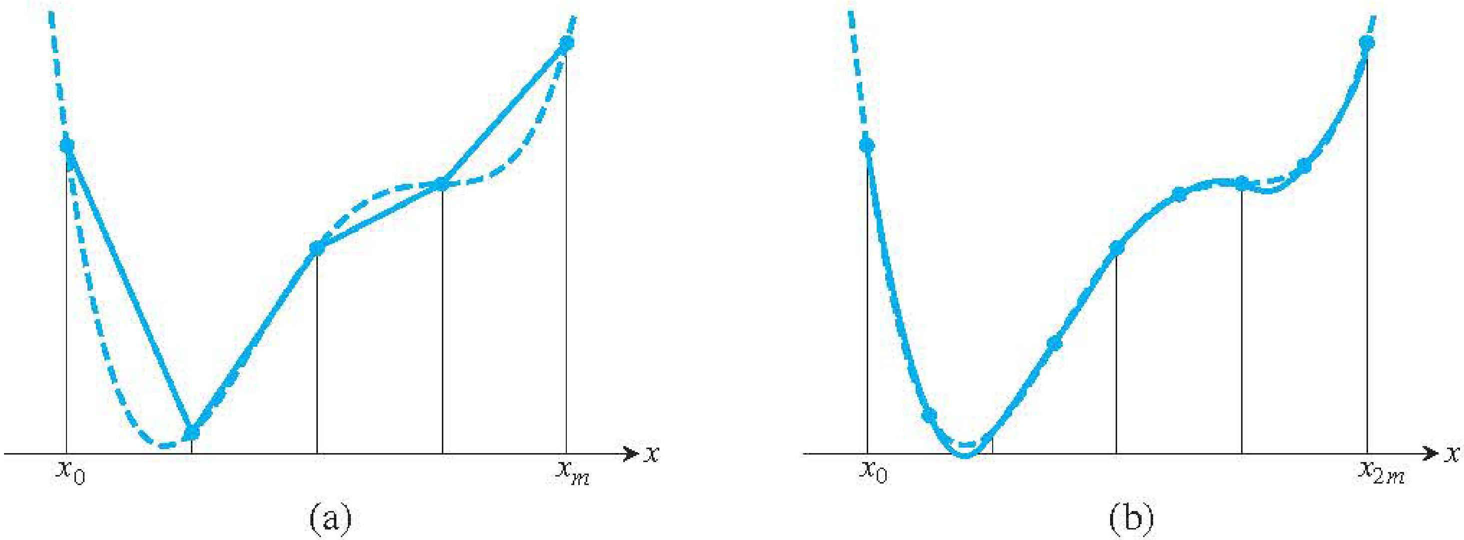
$$h=(b-a)/n$$

$$x_i=a+\frac{i}{2}h \qquad i=0,1,\cdots,2n$$

$$S(f)=\frac{h}{6}\left[f(x_0)+2\sum_{i=1}^{n-1}f(x_{2i})+4\sum_{i=0}^{n-1}f(x_{2i+1})+f(x_{2n})\right]$$

$$\left|E_S(f)\right|\le\frac{\left\|f^{(4)}\right\|_\infty (b-a)}{2880}h^4 \qquad S=\frac{2M+T}{3}$$

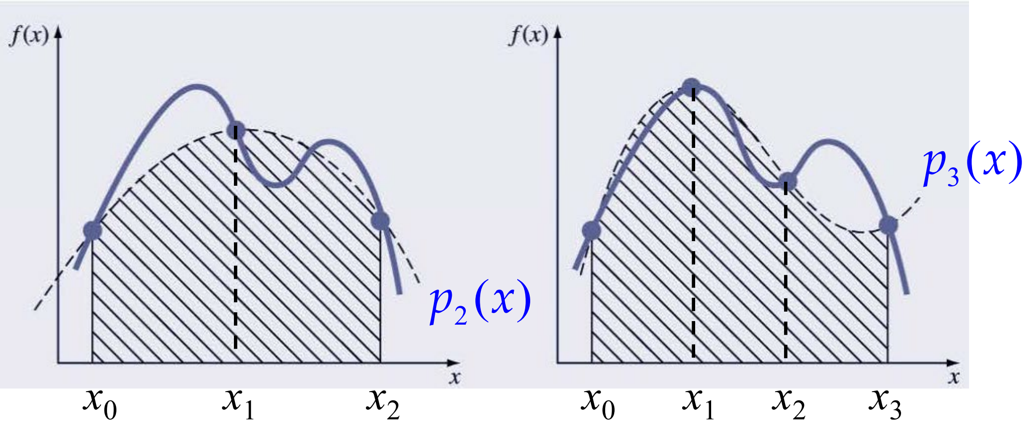
Trapezoid vs Simpson



$$\left|E_T(f)\right| \le \frac{\left\|f''\right\|_\infty (b-a)}{12} h^2$$

$$\left|E_S(f)\right| \le \frac{\left\|f^{(4)}\right\|_\infty (b-a)}{2880} h^4$$

Simpson 1/3 and 3/8



h h

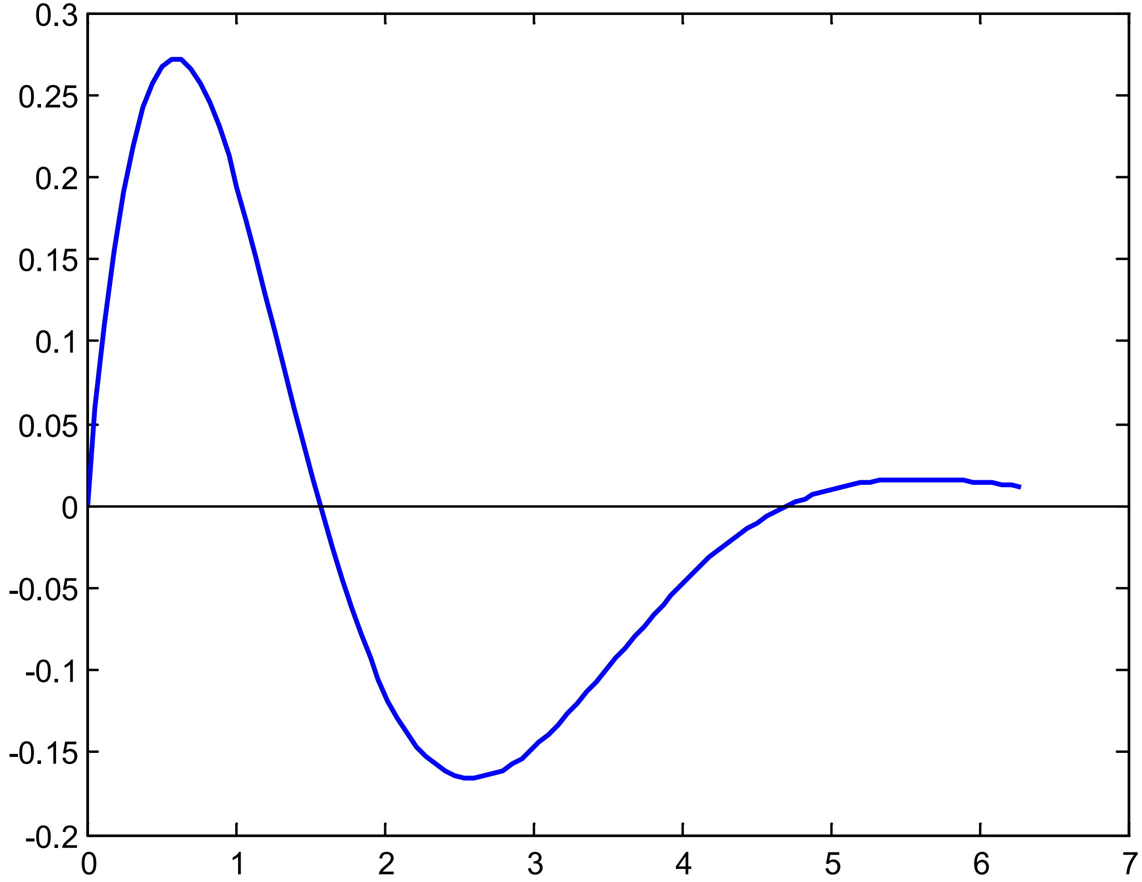
$h = x_1 - x_0$

$$S_{1/3}(f) \equiv \frac{1}{3}h\left[f(x_0)+4f(x_1)+f(x_2)\right]$$

$$S_{3/8}(f) \equiv \frac{3}{8}h\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]$$

Test Function

$$I(f)=\int_0^{2\pi} xe^{-x}\cos(2x)\,dx=\frac{3\left(e^{-2\pi}-1\right)-10\pi e^{-2\pi}}{25}\approx -0.122122$$



Efficiency

$$\frac{\|f''\|_\infty (b-a)}{24} h^2 \qquad \frac{\|f''\|_\infty (b-a)}{12} h^2 \qquad \frac{f^{(4)}(\xi)(b-a)}{2880} h^4$$

n	E_M	R_M	E_T	R_T	E_S	R_S
1	0.98		0.159		0.703	
2	1.04	0.94	0.567	0.28	0.502	1.400
4	0.12	8.49	0.234	2.42	3.14×10^{-3}	160.0
8	2.98×10^{-2}	4.10	5.64×10^{-2}	4.17	1.09×10^{-3}	2.892
16	6.75×10^{-3}	4.42	1.33×10^{-2}	4.25	7.38×10^{-5}	14.70
32	1.64×10^{-3}	4.12	3.26×10^{-3}	4.07	4.68×10^{-6}	15.77
64	4.07×10^{-4}	4.03	8.12×10^{-4}	4.02	2.94×10^{-7}	15.95
128	1.01×10^{-4}	4.01	2.03×10^{-4}	4.00	1.84×10^{-8}	15.99
256	2.54×10^{-5}	4.00	5.07×10^{-5}	4.00	1.15×10^{-9}	16.00

$$R_{2n} \equiv |E_n| / |E_{2n}|$$

Lagrange Polynomials

$$\left.\begin{matrix} x_0, x_1, \cdots, x_n \\ y_0, y_1, \cdots, y_n \end{matrix}\right\} \longrightarrow L(x_i) = y_i \qquad i = 0, 1, \cdots, n$$

$$L(x) = \sum_{i=0}^{n} y_i l_i(x) \qquad l_i(x_j) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$l_i(x) = \frac{x - x_0}{x_i - x_0} \cdot \frac{x - x_1}{x_i - x_1} \cdot \ldots \cdot \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdot \ldots \cdot \frac{x - x_n}{x_i - x_n}$$

Lagrange Quadrature

$$a = x_0 < x_1 < \ldots < x_n = b \qquad f(x) \approx L(x)$$

$$I(f) \equiv \int_a^b f(x)\,dx \approx \int_a^b L(x)\,dx \equiv LG(f)$$

$$LG(f) = \int_a^b \sum_{i=0}^{n} f(x_i)\, l_i(x)\,dx = \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x)\,dx$$

$$w_i = \int_a^b l_i(x)\,dx \longrightarrow LG(f) = \sum_{i=0}^{n} w_i f(x_i)$$

Newton-Cotes Quadrature

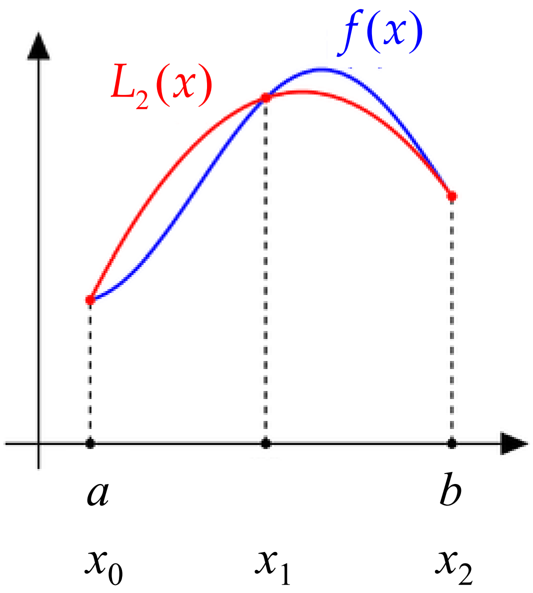
$$x_i = a + ih \qquad i = 0, 1, \cdots, n \qquad h = \left(b - a\right)/n \qquad x_0 = a \quad x_n = b$$

$$NC\left(f\right) = \begin{cases} \displaystyle\sum_{i=0}^{n} w_i f\left(x_i\right) & \text{Closed} \qquad f(x) \approx L_n(x) \\ \\ \displaystyle\sum_{i=1}^{n-1} w_i f\left(x_i\right) & \text{Open} \qquad f(x) \approx L_{n-2}(x) \end{cases}$$

NCQ = LG + Equally spaced nodes

$$w_i = \int_a^b l_i(x)dx$$

Closed vs. Open



$n = 1$

Closed: $f(x) \approx L_2(x) \longrightarrow$ Simpson's

Open: $f(x) \approx L_0(x) \longrightarrow$ Midpoint

$$I(f) = \int_0^1 \frac{\sin x}{x} dx$$

$$I(f) = \int_0^1 \frac{dx}{\sqrt{x}}$$

Newton-Cotes Weights

$$l_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x - x_k}{x_i - x_k} \qquad x_k = a + k \cdot h \qquad x(t) = a + t \cdot h$$

$$h = \frac{b-a}{n} \qquad 0 \le k \le n \qquad 0 \le t \le n$$

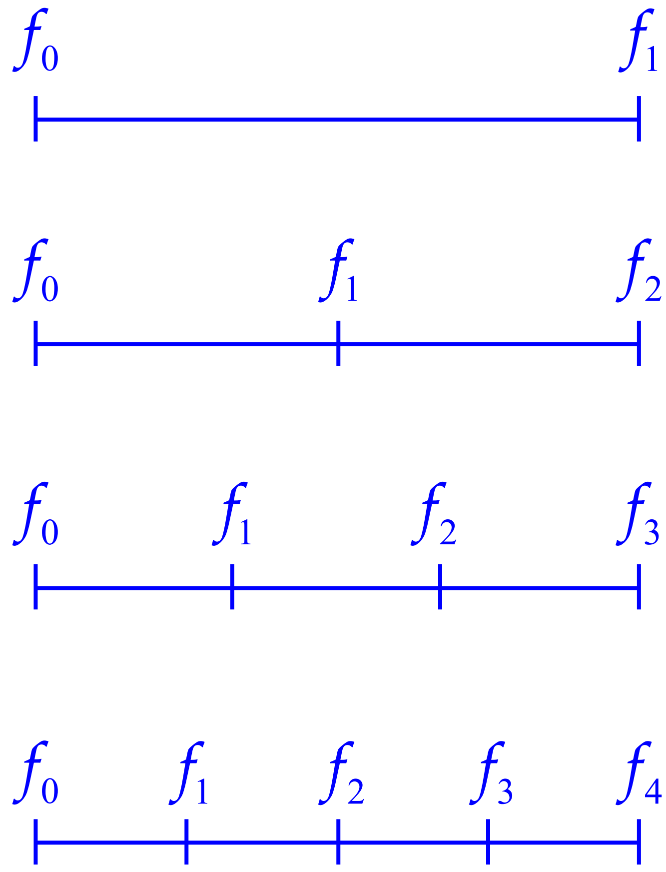
$$l_i(x) \sim \frac{(a+th) - (a+kh)}{(a+ih) - (a+kh)} = \frac{t-k}{i-k}$$

$$l_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{t-k}{i-k} \equiv \varphi_i(t) \longrightarrow w_i = \int_a^b l_i(x)dx = h\int_0^n \varphi_i(t)dt$$

NC weights do not depend on [a, b] !!

Closed Newton-Cotes Nodes

Com. name	**n**
Trapezoid	1
Simpson	2
3/8 Simpson	3
Boole	4



Closed N-C Formulas

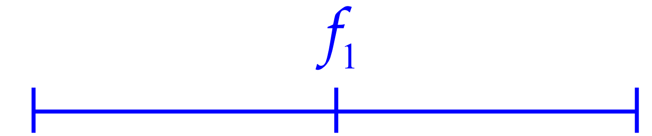
$$NC_n(f) = h\sum_{i=0}^{n} w_i f(x_i) \qquad w_i = \int_0^n \varphi_i(t)dt \qquad w_i = w_{n-i} \qquad h = x_1 - x_0$$

Com. name	n	Formula	Error
Trapezoid	1	$h\left(\frac{1}{2}f_0 + \frac{1}{2}f_1\right)$	$-\frac{(b-a)^3}{12}f^{(2)}(\xi)$
Simpson	2	$h\left(\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right)$	$-\frac{(b-a)^5}{2880}f^{(4)}(\xi)$
3/8 Simpson	3	$h\left(\frac{3}{8}f_0 + \frac{9}{8}f_1 + \frac{9}{8}f_2 + \frac{3}{8}f_3\right)$	$-\frac{(b-a)^5}{6480}f^{(4)}(\xi)$
Boole	4	$h\left(\frac{14}{45}f_0 + \frac{64}{45}f_1 + \frac{8}{15}f_2 + \frac{64}{45}f_3 + \frac{14}{45}f_4\right)$	$-\frac{(b-a)^7}{1935360}f^{(6)}(\xi)$

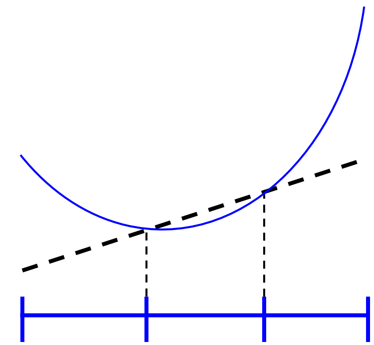
Open Newton-Cotes Nodes

Com. name **n**

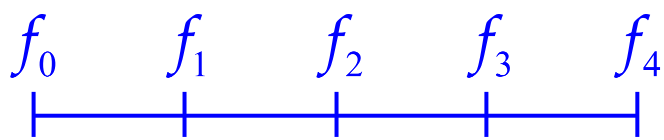
Mid-point 2



Trapezoid 3



Milne 4



Open N-C Formulas

$$NC_n(f) = h\sum_{i=1}^{n-1} w_i f(x_i) \qquad w_i = \int_{-1}^{n+1} \varphi_i(t)dt \qquad w_i = w_{n-i} \qquad h = (b-a)/n$$

Common name	n	Formula	Error
Midpoint	2	hf_1	$\frac{(b-a)^3}{24} f^{(2)}(\xi)$
Trapezoid	3	$h\left(\frac{3}{2}f_1 + \frac{3}{2}f_2\right)$	$\frac{(b-a)^3}{36} f^{(2)}(\xi)$
Milne	4	$h\left(\frac{8}{3}f_1 - \frac{4}{3}f_2 + \frac{8}{3}f_3\right)$	$\frac{7(b-a)^5}{23040} f^{(4)}(\xi)$
No name	5	$h\left(\frac{55}{24}f_1 + \frac{5}{24}f_2 + \frac{5}{24}f_3 + \frac{55}{24}f_4\right)$	$\frac{19(b-a)^5}{90000} f^{(4)}(\xi)$

Error of Trapezoidal Rule

$$h=(b-a)/n \qquad x_i=a+ih \qquad i=0,1,\cdots,n$$

$$T(f)=h\left[\frac{1}{2}f(x_0)+f(x_1)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right]$$

$f(x)\in C^2[a,b]$ $\qquad I(f)=T(f)-\frac{f''(\xi)(b-a)}{12}h^2$

$f(x)\in C^4[a,b]$ $\qquad I(f)=T_h(f)+C_1h^2+C_2h^4$

$f(x)\in C^\infty[a,b]$ $\qquad I(f)=T_h(f)+C_1h^2+C_2h^4+C_2h^6+\cdots$

Richardson Extrapolation

$$I(f) = T_h(f) + C_1 h^2 + C_2 h^4 + C_2 h^6 + \cdots$$

$$I(f) = T_{h/2}(f) + \frac{1}{4} C_1 h^2 + \frac{1}{16} C_2 h^4 + \frac{1}{64} C_2 h^6 + \cdots$$

$$3I(f) = 4T_{h/2}(f) - T_h(f) + \frac{1}{4} C_2 h^4 + \frac{1}{16} C_2 h^6 + \cdots$$

$$I(f) = \frac{4T_{h/2}(f) - T_h(f)}{3} + C_2' h^4 + C_2' h^6 + \cdots =$$

$$= T_{h/2}(f) + \frac{T_{h/2}(f) - T_h(f)}{3} + C_2' h^4 + C_2' h^6 + \cdots$$

Romberg Method

$$n=1 \quad R_{1,1}=T_1(f)$$

$$n=2 \quad R_{2,1}=T_2(f) \longrightarrow R_{2,2}=R_{2,1}+\frac{1}{3}\left(R_{2,1}-R_{1,1}\right)$$

$$n=4 \quad R_{3,1}=T_4(f) \longrightarrow R_{3,2}=R_{3,1}+\frac{1}{3}\left(R_{3,1}-R_{2,1}\right) \longrightarrow R_{3,3}=R_{3,2}+\frac{1}{15}\left(R_{3,2}-R_{2,2}\right)$$

$$n=8 \quad R_{4,1}=T_8(f) \longrightarrow R_{4,2}=R_{4,1}+\frac{1}{3}\left(R_{4,1}-R_{3,1}\right) \longrightarrow R_{4,3}=R_{4,2}+\frac{1}{15}\left(R_{4,2}-R_{3,2}\right)$$

$$n=16 \quad R_{5,1}=T_{16}(f) \longrightarrow R_{5,2}=R_{5,1}+\frac{1}{3}\left(R_{5,1}-R_{4,1}\right) \longrightarrow R_{5,3}=R_{5,2}+\frac{1}{15}\left(R_{5,2}-R_{4,2}\right)$$

Romberg Step

column = generation

$$R_{k,j-1}$$

row

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1}-1}\left(R_{k,j-1} - R_{k-1,j-1}\right)$$

$$k = j, j+1, \ldots$$

Example, O(h^2)

$$I(\sin x) = \int_0^\pi \sin x dx = 2$$

$$R_{1,1} = \frac{\pi}{2}[\sin 0 + \sin \pi] = 0$$

$$R_{2,1} = \frac{\pi}{4}\left[\sin 0 + 2\sin\frac{\pi}{2} + \sin \pi\right] = 1.5708$$

$$R_{3,1} = \frac{\pi}{8}\left[\sin 0 + 2\left(\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4}\right) + \sin \pi\right] = 1.896$$

Example, O(h^4)

$$R_{2,2} = R_{2,1} + \frac{1}{3}\left(R_{2,1} - R_{1,1}\right) = 2.09439511$$

$$R_{3,2} = R_{3,1} + \frac{1}{3}\left(R_{3,1} - R_{2,1}\right) = 2.00455976$$

$$R_{4,2} = R_{4,1} + \frac{1}{3}\left(R_{4,1} - R_{3,1}\right) = 2.00026917$$

Example, Table

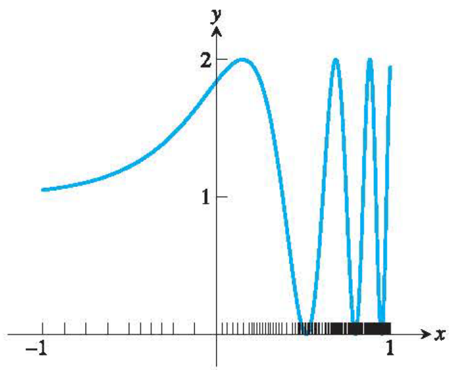
0				
1.57079633	2.09439511			
1.89611890	2.00455976	1.99857073		
1.97423160	2.00026917	1.99998313	2.00000555	
1.99357034	2.00001659	1.99999975	2.00000001	1.99999999

Strongly Varying Functions

$$f(x) = 1 + \sin\left(e^{3x}\right) \qquad I(f) = \int_{-1}^{1} f(x)dx$$

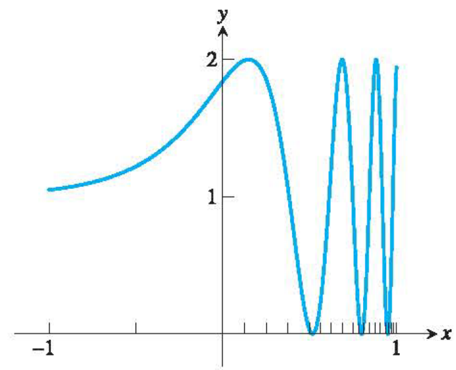
Two problems:
- Equal step size
- Evaluating error

$$I(f) = T_{[\alpha,\beta]} - h^3 \frac{f''(\xi)}{12}$$



(a)

Adaptive Trapezoid: 140



(b)

Adaptive Simpson: 20

Adaptive Quadrature

(1) Step h $$I(f) = T_{[\alpha,\beta]} - h^3 \frac{f''(\xi)}{12}$$

$$\frac{w_1 g(\xi_1) + w_2 g(\xi_2)}{w_1 + w_2} = g(\xi)$$

(2) Step $h/2$ $$I(f) = T_{[\alpha,\gamma]} - \frac{h^3}{8}\frac{f''(\xi_1)}{12} + T_{[\gamma,\beta]} - \frac{h^3}{8}\frac{f''(\xi_2)}{12}$$

$$I(f) = T_{[\alpha,\gamma]} + T_{[\gamma,\beta]} - \frac{1}{4}h^3 \frac{f''(\xi_3)}{12} = T_{[\alpha,\gamma]} + T_{[\gamma,\beta]} - Err$$

(1) – (2) $$T_{[\alpha,\beta]} - \left(T_{[\alpha,\gamma]} + T_{[\gamma,\beta]}\right) = -\frac{h^3}{4}\frac{f''(\xi_3)}{12} + h^3 \frac{f''(\xi)}{12} \approx \frac{3}{4}h^3 \frac{f''(\xi_3)}{12} = 3 \times Err$$

$$1)\ tol = 1e-8; \quad 2)\ Err = \frac{1}{3}\left[T_{[\alpha,\beta]} - \left(T_{[\alpha,\gamma]} + T_{[\gamma,\beta]}\right)\right]; \quad 3)\ Err \le tol\ ?$$

Important

- Quadrature
- Linearity of Integral $\rightarrow$ Composite
- Approximating Integrand
- Nodes and Weights
- Mid, Trapezium, Simpson
- Newton – Cotes Quadrature
- Richardson + Trapezoid = Romberg
- Adaptive Quadrature