Power Method

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Overview

- Continuous/Discrete Eigen-problem
- Multiplication Factor, Neutron Generations
- Perron-Frobenius Theorem
- Direct Power Method
- Inverse Power Method, Shifted Inverse
- Normalisation
- A Priori Estimate (Gershgorin Theorem)
- A Posteriori Estimate

One-Speed Diffusion Equations

Homogeneous

$$\frac{1}{v}\frac{\partial \phi}{\partial t} = \nu\Sigma_f\phi - \Sigma_a\phi + \nabla \cdot D\nabla\phi$$

Time dependent

$$\frac{1}{v}\frac{\partial \phi}{\partial t} = \nu\Sigma_f\phi - \Sigma_a\phi + \nabla \cdot D\nabla\phi + S$$

Time independent

$$0 = \nu\Sigma_f\phi - \Sigma_a\phi + \nabla \cdot D\nabla\phi + S$$

Homogeneous stationary

$$0 = \nu\Sigma_f\phi - \Sigma_a\phi + \nabla \cdot D\nabla\phi$$

Eigenvalue

$$0 = \frac{\nu\Sigma_f}{k}\phi - \Sigma_a\phi + \nabla \cdot D\nabla\phi$$

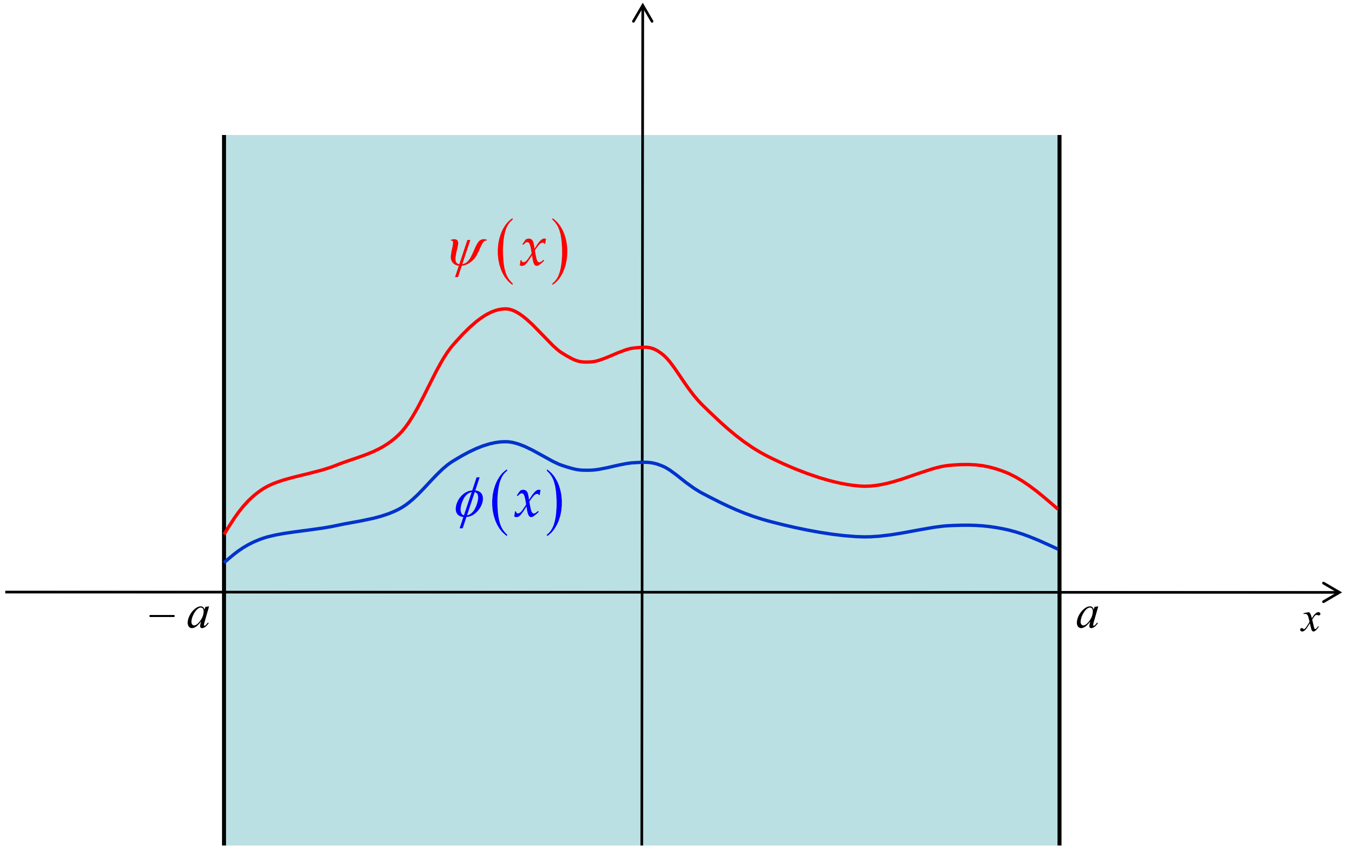
Critical Equation

$$\frac{\nu\Sigma_f(\mathbf{r})}{k}\phi(\mathbf{r}) - \Sigma_a(\mathbf{r})\phi(\mathbf{r}) + \nabla\cdot D(\mathbf{r})\nabla\phi(\mathbf{r}) = 0$$

Any solution? $\phi(\mathbf{r}) = ?$

$$\psi(\mathbf{r}) \equiv 2\cdot\phi(\mathbf{r})$$

Neutron Flux Profile

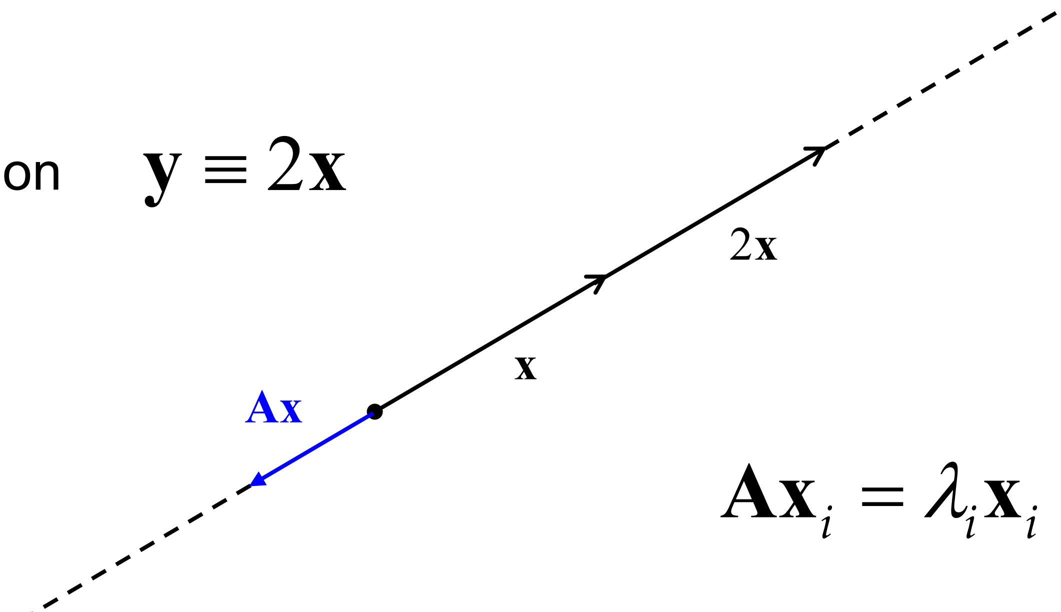


Matrix Eigen-Problem

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Trivial solution $\mathbf{x} = \mathbf{0}$

Another solution $\mathbf{y} \equiv 2\mathbf{x}$



$$\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i$$

Continuous Eigen-Problem

$$\frac{\nu\Sigma_f(\mathbf{r})}{k}\phi(\mathbf{r}) - \Sigma_a(\mathbf{r})\phi(\mathbf{r}) + \nabla\cdot D(\mathbf{r})\nabla\phi(\mathbf{r}) = 0$$

$$\frac{\nu\Sigma_f(\mathbf{r})}{k}\phi(\mathbf{r}) - \left[\Sigma_a(\mathbf{r}) - \nabla\cdot D(\mathbf{r})\nabla\right]\phi(\mathbf{r}) = 0$$

$$\left[\Sigma_a(\mathbf{r}) - \nabla\cdot D(\mathbf{r})\nabla\right]^{-1}\nu\Sigma_f(\mathbf{r})\phi(\mathbf{r}) = k\phi(\mathbf{r})$$

Transpose of Matrix

$\mathbf{A}$ is a real $n \times n$ matrix:

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

$$\mathbf{y}_j^T \mathbf{A} = \mu_j \mathbf{y}_j^T \longrightarrow \mathbf{A}^T \mathbf{y}_j = \mu_j \mathbf{y}_j$$

Theorem for $\mathbf{A}^T \neq \mathbf{A}$

1) $\mu_i = \lambda_i$

2) $\mathbf{y}_j \neq \mathbf{x}_i \qquad \forall \lambda_j \neq \lambda_i$

3) $\mathbf{y}_j^T \cdot \mathbf{x}_i = 0 \qquad \forall \lambda_j \neq \lambda_i$

Commute of Matrices

$$(\mathbf{AB})\mathbf{x} = \lambda\mathbf{x} \longleftarrow ? \longrightarrow (\mathbf{BA})\mathbf{y} = \mu\mathbf{y}$$

$$\mathbf{A}^T = \mathbf{A} \qquad \mathbf{B}^T = \mathbf{B}$$

$$\mathbf{BA} = \mathbf{B}^T\mathbf{A}^T = \left(\mathbf{AB}\right)^T$$

$$\mu_i = \lambda_i$$

Change of Variables

Expression
$$\left[\Sigma_a(\mathbf{r}) - \nabla \cdot D(\mathbf{r}) \nabla\right]^{-1} Q(\mathbf{r})$$

New dependent variable
$$\phi(\mathbf{r}) \equiv \left[\Sigma_a(\mathbf{r}) - \nabla \cdot D(\mathbf{r}) \nabla\right]^{-1} Q(\mathbf{r})$$

NDE
$$\left[\Sigma_a(\mathbf{r}) - \nabla \cdot D(\mathbf{r}) \nabla\right] \phi(\mathbf{r}) = Q(\mathbf{r})$$

???
$$\left[\Sigma_s(\mathbf{r}) - \nabla \cdot D(\mathbf{r}) \nabla\right] \psi(\mathbf{r}) = Q(\mathbf{r})$$

Physical Processes

$$\left[\Sigma_a(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla\right]\phi(\mathbf{r}) = Q(\mathbf{r})$$

$\Sigma_a(\mathbf{r})$ Absorption

$D(\mathbf{r}) = \dfrac{1}{3\Sigma_s(\mathbf{r})}$ Scattering = Diffusion

Material Composition

$$\left[\Sigma_a(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla\right]\phi(\mathbf{r}) = Q(\mathbf{r})$$

$$\Sigma_a(\mathbf{r}) = \Sigma_{a,M}(\mathbf{r}) + \Sigma_{a,F}(\mathbf{r})$$ Moderator + Fuel

$$\Sigma_a(\mathbf{r}) = \Sigma_{a,M}(\mathbf{r})$$ Moderator

Two Eigen Problems

$$\left[\Sigma_a(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla\right]^{-1} \nu\Sigma_f(\mathbf{r})\phi(\mathbf{r}) = k\phi(\mathbf{r})$$

$$f(\mathbf{r}) \equiv \nu\Sigma_f(\mathbf{r})\phi(\mathbf{r})$$

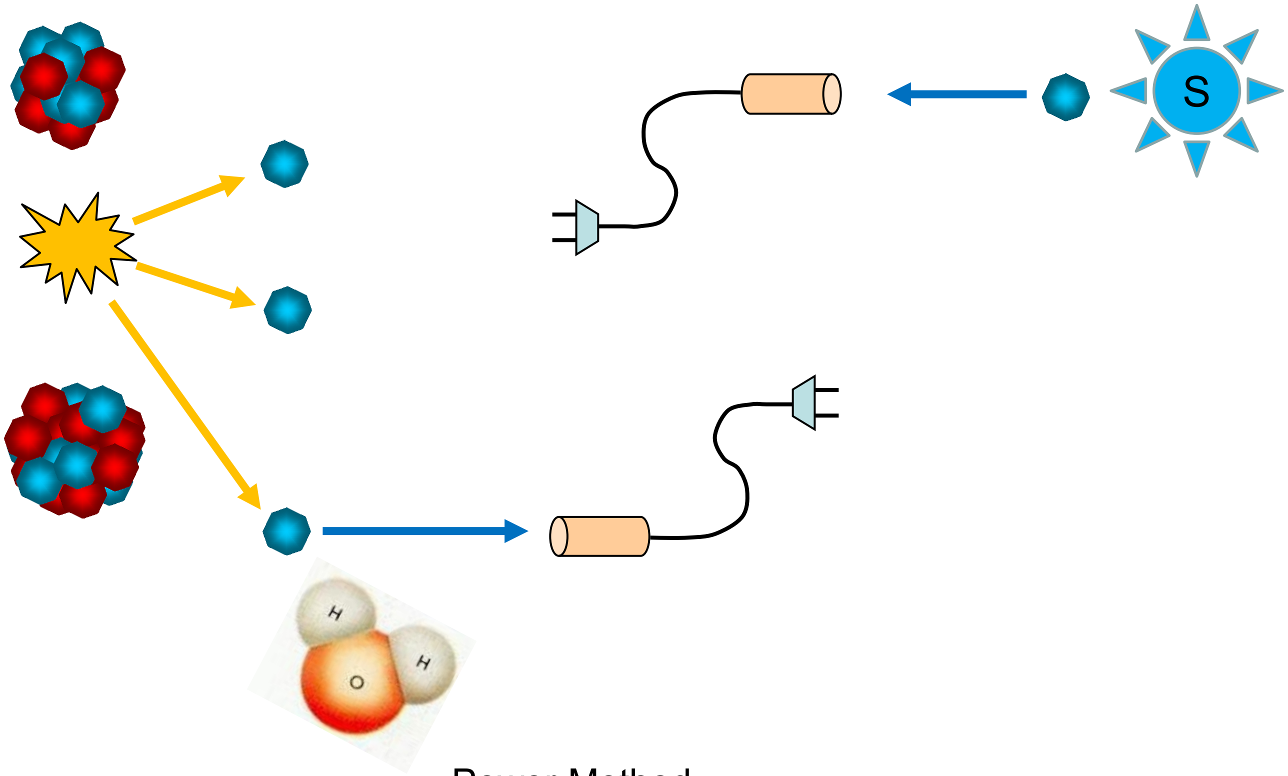
$$\left[\Sigma_a(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla\right]^{-1} f(\mathbf{r}) = k\phi(\mathbf{r})$$

$$\nu\Sigma_f(\mathbf{r})\left[\Sigma_a(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla\right]^{-1} f(\mathbf{r}) = kf(\mathbf{r})$$

Multiplication Factor

$$k = \frac{N_{j+1}}{N_j}$$

Can we distiguish neutron generations ?



Time-Independent Problem

$$\Sigma_a(\mathbf{r})\phi(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla\phi(\mathbf{r}) = S(\mathbf{r}) + \nu\Sigma_f(\mathbf{r})\phi(\mathbf{r})$$

$$\Sigma_a(\mathbf{r}) = \Sigma_{a,M}(\mathbf{r}) + \Sigma_{a,F}(\mathbf{r}) \qquad D(\mathbf{r}) = \frac{1}{3\Sigma_{tr}(\mathbf{r})}$$

$$\Sigma_a(\mathbf{r})\phi_1(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla\phi_1(\mathbf{r}) = S(\mathbf{r})$$

$$\Sigma_a(\mathbf{r})\phi_2(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla\phi_2(\mathbf{r}) = \nu\Sigma_f(\mathbf{r})\phi(\mathbf{r})$$

Flux Generations

$$\left[\Sigma_a(\mathbf{r}) - \nabla\cdot D(\mathbf{r})\nabla\right]^{-1} \nu\Sigma_f(\mathbf{r})\phi(\mathbf{r}) = k\phi(\mathbf{r})$$

$$f(\mathbf{r}) \equiv \nu\Sigma_f(\mathbf{r})\phi(\mathbf{r})$$

$$\left[\Sigma_a(\mathbf{r}) - \nabla\cdot D(\mathbf{r})\nabla\right]^{-1} f(\mathbf{r}) \equiv \phi_1(\mathbf{r})$$

$$f(\mathbf{r}) = \left[\Sigma_a(\mathbf{r}) - \nabla\cdot D(\mathbf{r})\nabla\right]\phi_1(\mathbf{r})$$

$$\phi_1(\mathbf{r}) = k\phi(\mathbf{r})$$

Fission Generations

$$\nu\Sigma_f(\mathbf{r})\underbrace{\left[\Sigma_a(\mathbf{r})-\nabla\cdot D(\mathbf{r})\nabla\right]^{-1} f(\mathbf{r})}_{\phi(\mathbf{r})}=kf(\mathbf{r})$$

$$\phi(\mathbf{r})\equiv\left[\Sigma_a(\mathbf{r})-\nabla\cdot D(\mathbf{r})\nabla\right]^{-1} f(\mathbf{r})$$

$$\left[\Sigma_a(\mathbf{r})-\nabla\cdot D(\mathbf{r})\nabla\right]\phi(\mathbf{r})=f(\mathbf{r})$$

$$f_1(\mathbf{r})=\nu\Sigma_f(\mathbf{r})\phi(\mathbf{r})=kf(\mathbf{r})$$

Discrete Eigen-Problem

$$-\frac{d}{dx}\left[D(x)\frac{d}{dx}\phi(x)\right]+\Sigma_a(x)\phi(x)=\frac{\nu\Sigma_f(x)}{k}\phi(x)$$

$$a\le x\le b \longrightarrow a\le x_i\le b \quad (i=1,\ldots,N)$$

$$-a_{i-1}\phi_{i-1}+c_i\phi_i-a_i\phi_{i+1}=q_i=\frac{1}{k}\nu\Sigma_{f,i}\phi_i$$

Matrix Eigen-Problem

$$\frac{1}{k}\nu\Sigma_{f,i}\phi_i = -a_{i-1}\phi_{i-1} + c_i\phi_i - a_i\phi_{i+1}$$

$$\frac{1}{k}\mathbf{F}\boldsymbol{\phi} = \mathbf{A}\boldsymbol{\phi} \longrightarrow \mathbf{A}^{-1}\mathbf{F}\boldsymbol{\phi} = k\boldsymbol{\phi}$$

$$\mathbf{F}\mathbf{A}^{-1}\mathbf{F}\boldsymbol{\phi} = k\mathbf{F}\boldsymbol{\phi} \longrightarrow \mathbf{F}\mathbf{A}^{-1}\mathbf{f} = k\mathbf{f}$$

$$\mathbf{f} \equiv \mathbf{F}\boldsymbol{\phi} \qquad f_i \equiv \nu\Sigma_{f,i}\phi_i \qquad \mathbf{F}\mathbf{A}^{-1} \geq 0$$

Algebraic Eigen-Problem

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

$$\sigma(\mathbf{A}) \equiv \{\lambda \;|\; \lambda \text{ is an eigenvalue}\}$$

$$\rho(\mathbf{A}) \le \|\mathbf{A}\|$$

$$|\lambda| \le \|\mathbf{A}\| \quad \forall \lambda \in \sigma(\mathbf{A})$$

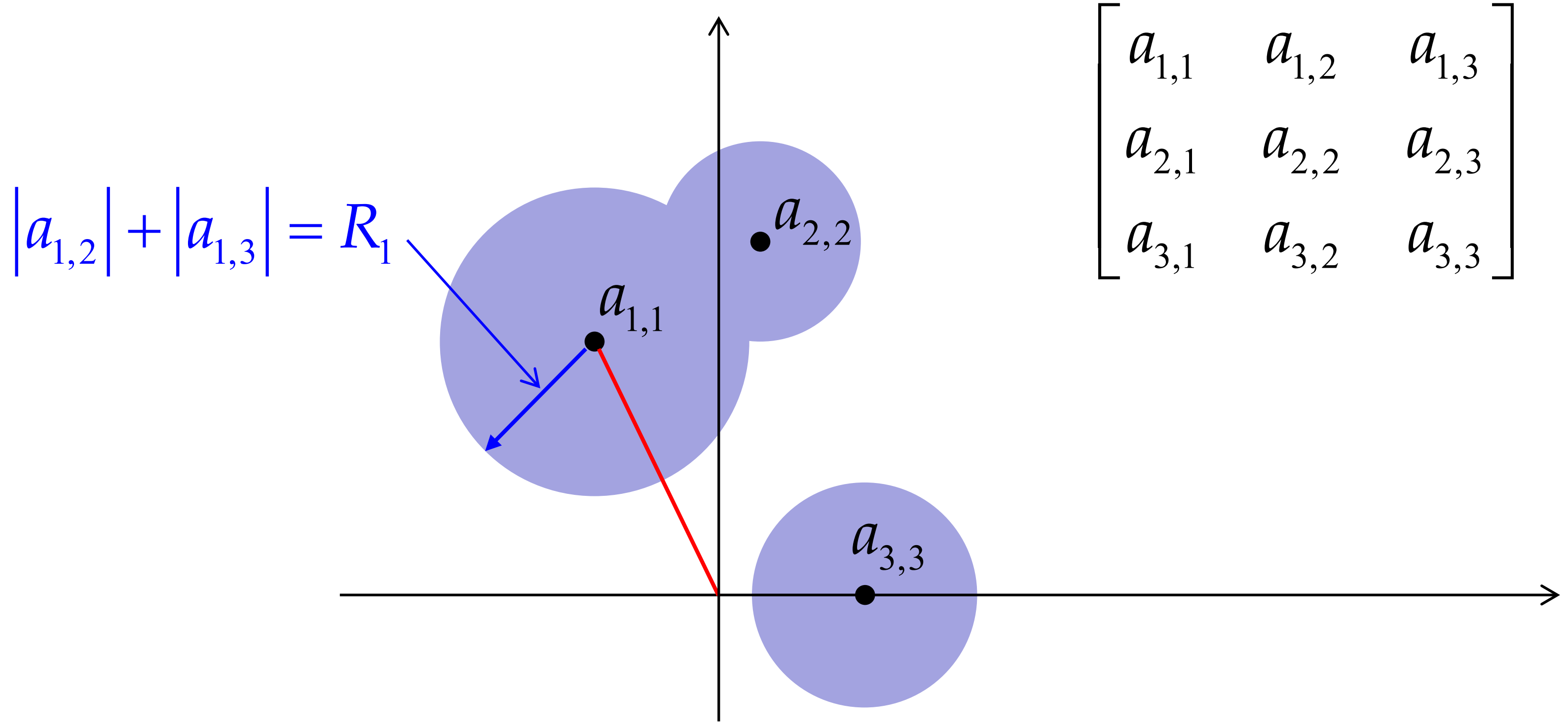
Gershgorin Theorem

$$\mathbf{A} \in \mathbb{C}^{n \times n}$$

$$G_i \equiv \left\{ z \in C \ : \ \left| z - a_{ii} \right| \leq R_i = \sum_{j \neq i}^{n} \left| a_{ij} \right| \right\}$$

$$\sigma\left(\mathbf{A}\right) \subseteq G \equiv \bigcup_{i=1}^{n} G_i$$

Gershgorin Circles



$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

A Posteriori Estimate

Hermitian: $\mathbf{A} \in \mathbb{C}^{n\times n}$ $\qquad \mathbf{A}\mathbf{x} = \lambda \mathbf{x} \longrightarrow \mathbf{A}\tilde{\mathbf{x}} \approx \tilde{\lambda}\tilde{\mathbf{x}}$

$$\tilde{\mathbf{r}} = \mathbf{A}\tilde{\mathbf{x}} - \tilde{\lambda}\tilde{\mathbf{x}}$$

$$\min_i \left|\tilde{\lambda} - \lambda_i\right| \le \frac{\left\|\tilde{\mathbf{r}}\right\|_2}{\left\|\tilde{\mathbf{x}}\right\|_2}$$

Power Method Concept

$$\mathbf{A} \in \mathbb{C}^{n\times n} \qquad \mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i \qquad |\lambda_1| > |\lambda_2| \geq \ldots$$

$$\forall \mathbf{z}^{(0)} \qquad \mathbf{z}^{(k)} = \mathbf{A}\,\mathbf{z}^{(k-1)} = \mathbf{A}^k \mathbf{z}^{(0)}$$

$$\mathbf{z}^{(0)} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \ldots$$

$$\mathbf{z}^{(k)} = c_1 \lambda_1^k \mathbf{x}_1 + c_2 \lambda_2^k \mathbf{x}_2 + \ldots \approx c_1 \lambda_1^k \mathbf{x}_1$$

Sufficient Conditions

$\mathbf{A} \in \mathbb{C}^{n\times n}$ Diagonalizable

$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i \quad \left|\lambda_1\right| > \left|\lambda_2\right| \geq \ldots \geq \left|\lambda_n\right|$

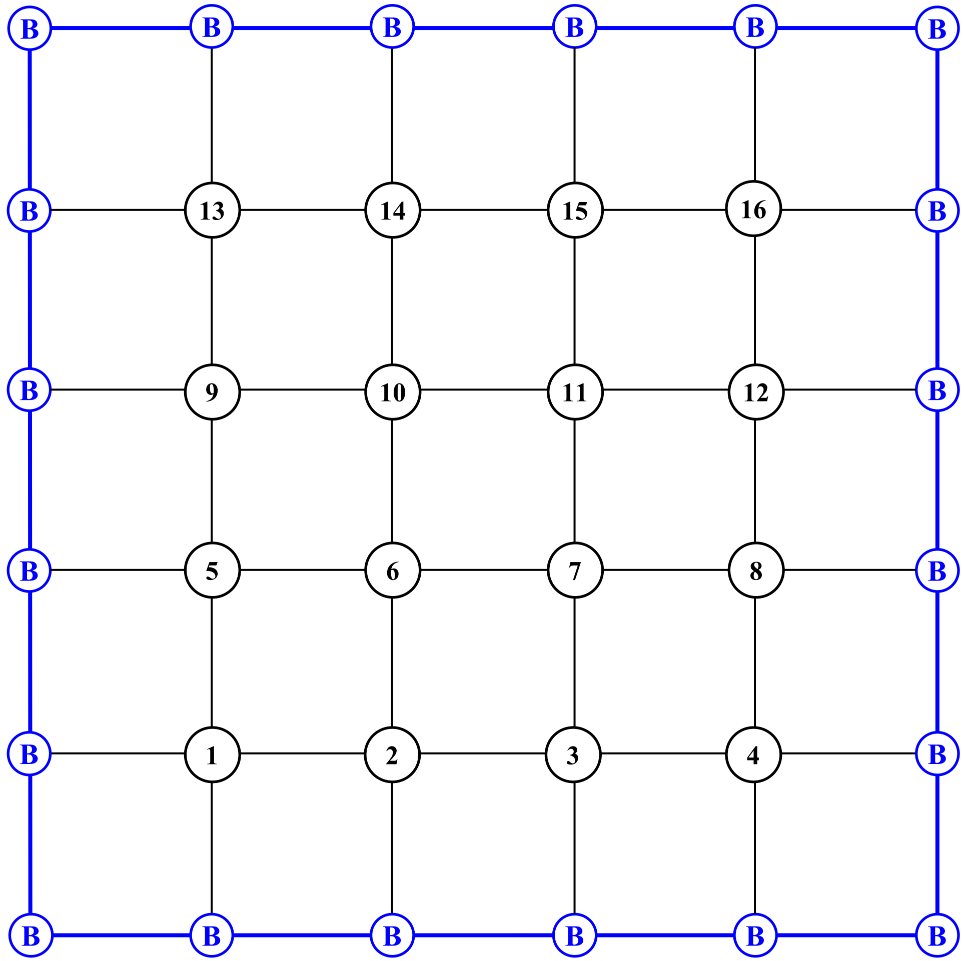
$m\left(\lambda_1\right) = 1$ Multiplicity

Irreducible Matrices

Reducible: $\mathbf{M} = \mathbf{PAP}^T = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{M}_{12} & \mathbf{M}_{22} \end{bmatrix}$

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{M}_{12} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

2D FD Mesh



Perron-Frobenius Theorem

- Let **A** be *n*x*n*, non-negative and irreducible then

$$\exists \quad \lambda > 0 \ \text{ such that } \ |\lambda_i| < \lambda$$

$$\exists^1 \quad \mathbf{x} > 0 \ \text{ such that } \ \mathbf{Ax} = \lambda \mathbf{x}$$

$$\text{multiplicity} \quad m(\lambda) = 1$$

Oscar Perron, 1907, (1880 -1975) Georg Frobenius, 1912, (1849 -1917)

Power Method

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \qquad \forall \mathbf{q}^{(0)} \in \mathbb{C}^n \qquad \left\| \mathbf{q}^{(0)} \right\|_2 = 1$$

$$\mathbf{z}^{(k)} = \mathbf{A}\mathbf{q}^{(k-1)} \qquad \mu^{(0)} = 1$$

$$\mathbf{q}^{(k)} = \mathbf{z}^{(k)} \Big/ \left\| \mathbf{z}^{(k)} \right\|_2 \qquad \mathbf{q}^{(k)} = \mathbf{z}^{(k)} \Big/ \mu^{(k-1)}$$

$$\mu^{(k)} = \left\langle \mathbf{q}^{(k)}, \mathbf{A}\mathbf{q}^{(k)} \right\rangle \qquad \mu^{(k)} = \left\langle \mathbf{q}^{(k)}, \mathbf{A}\mathbf{q}^{(k)} \right\rangle \Big/ \left\langle \mathbf{q}^{(k)}, \mathbf{q}^{(k)} \right\rangle$$

$$\mu^{(k)} \xrightarrow[k \to \infty]{} \lambda_1 \qquad \frac{\langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} : \text{Rayleigh quotient}$$

Power Method Convergence

$$\mathbf{q}^{(0)} = \sum_{i=1}^{n} c_i \mathbf{x}_i \qquad \mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i \qquad \mathbf{q}^{(k)} = c_1 \lambda_1^k \mathbf{x}_1 + c_2 \lambda_2^k \mathbf{x}_2 + \ldots$$

$$\mathbf{q}^{(k)} = \mathbf{A}^k \mathbf{q}^{(0)} = c_1 \lambda_1^k \left[\mathbf{x}_1 + \sum_{i=2}^{n} \frac{c_i}{c_1} \left(\frac{\lambda_i}{\lambda_1} \right)^k \mathbf{x}_i \right] = c_1 \lambda_1^k \left[\mathbf{x}_1 + \mathbf{y}^{(k)} \right]$$

$$\mathbf{q}^{(k)} = \frac{c_1 \lambda_1^k \left(\mathbf{x}_1 + \mathbf{y}^{(k)} \right)}{\left\| c_1 \lambda_1^k \left(\mathbf{x}_1 + \mathbf{y}^{(k)} \right) \right\|_2} = \pm \frac{\left(\mathbf{x}_1 + \mathbf{y}^{(k)} \right)}{\left\| \left(\mathbf{x}_1 + \mathbf{y}^{(k)} \right) \right\|_2} \xrightarrow[k \to \infty]{} \pm \frac{\mathbf{x}_1}{\left\| \mathbf{x}_1 \right\|_2}$$

Rate of Convergence

$$\mathbf{q}^{(k)} \sim \mathbf{x}_1 + \sum_{i=2}^{n} \frac{c_i}{c_1} \left(\frac{\lambda_i}{\lambda_1} \right)^k \mathbf{x}_i$$

Dominance ratio $\left| \dfrac{\lambda_2}{\lambda_1} \right|$

Real and Symmetric

$$\left|\mu^{(k)} - \lambda_1\right| \le \left|\lambda_1 - \lambda_n\right| \tan^2\left(\theta_0\right) \left|\frac{\lambda_2}{\lambda_1}\right|^{2k}$$

$$\cos\left(\theta_0\right) = \left|\left\langle \mathbf{x}_1, \mathbf{q}^{(0)} \right\rangle\right|$$

$$\left|\lambda_1 - \lambda_n\right| \le \left|\lambda_1\right| \approx \left|\mu^{(k)}\right|$$

$$\cos\left(\theta_0\right) \approx \left|\left\langle \mathbf{q}^{(k)}, \mathbf{q}^{(0)} \right\rangle\right|$$

Stopping Criterion

$$\left|\lambda_1 - \mu^{(k)}\right| \simeq \frac{\left\|\mathbf{r}^{(k)}\right\|_2}{\left|\cos(\theta)\right|} \qquad \mathbf{r}^{(k)} \equiv \mathbf{A}\mathbf{q}^{(k)} - \mu^{(k)}\mathbf{q}^{(k)}$$

$$\cos(\theta) \equiv \frac{\langle \mathbf{y}_1, \mathbf{x}_1 \rangle}{\left\|\mathbf{y}_1\right\| \cdot \left\|\mathbf{x}_1\right\|} \qquad \mathbf{y}_1^T \mathbf{A} = \lambda_1 \mathbf{y}_1^T \qquad \mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$$

$$\left|\lambda_1 - \mu^{(k)}\right| \le \left\|\mathbf{r}^{(k)}\right\|_2 \qquad \mathbf{A}^H = \mathbf{A}$$

Iterating Inverse Matrix

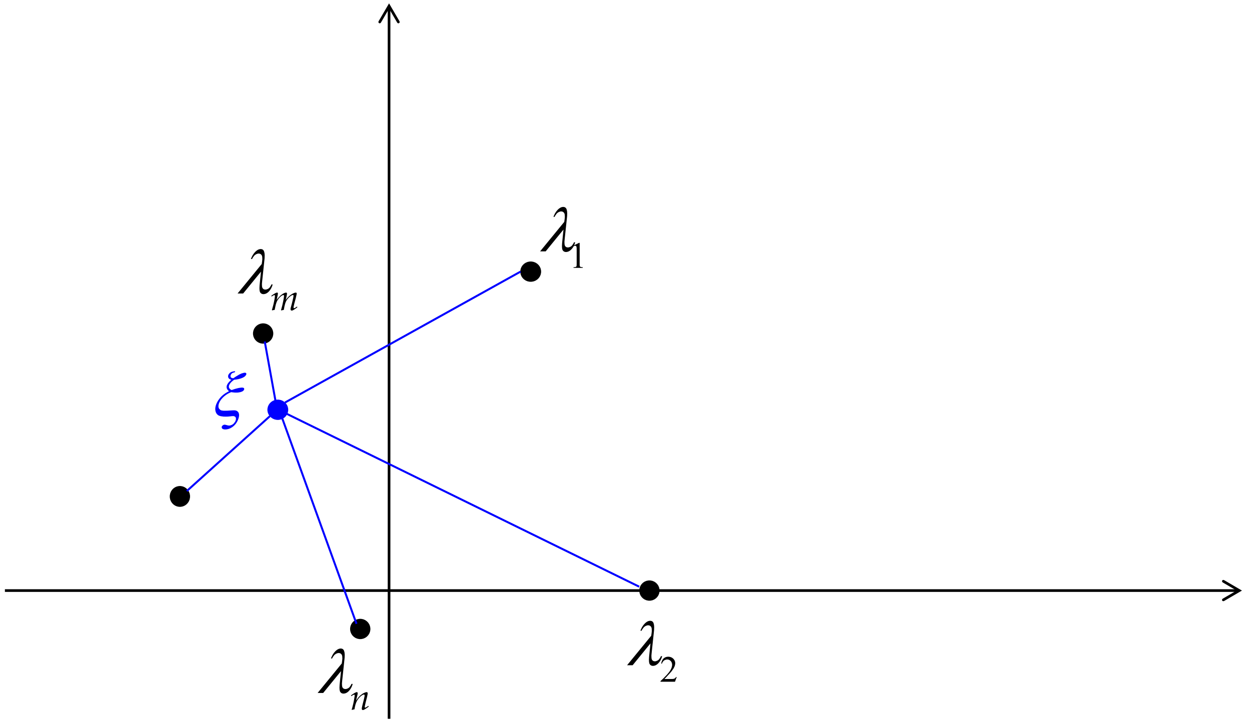
$$\forall \mathbf{z}^{(0)} \qquad \mathbf{z}^{(k)} = \mathbf{A}\,\mathbf{z}^{(k-1)} = \mathbf{A}^{k}\mathbf{z}^{(0)} \longrightarrow \left(\lambda_1, \mathbf{x}_1\right)$$

$$\forall \mathbf{z}^{(0)} \qquad \mathbf{z}^{(k)} = \mathbf{A}^{-1}\,\mathbf{z}^{(k-1)} = \left(\mathbf{A}^{-1}\right)^{k}\mathbf{z}^{(0)} \longrightarrow ??$$

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i \longrightarrow \mathbf{A}^{-1}\mathbf{x}_i = \lambda_i^{-1}\mathbf{x}_i$$

$$\mathbf{z}^{(k)} = \mathbf{A}^{-1}\,\mathbf{z}^{(k-1)} \longrightarrow \left(\lambda_n, \mathbf{x}_n\right)$$

Other Eigenvalues



$$|\lambda_m - \xi| < |\lambda_i - \xi| \qquad \forall i \neq m$$

Shifted Inverse

$$\left|\lambda_m - \xi\right| < \left|\lambda_i - \xi\right| \qquad \forall i \neq m \quad \mathbf{q}^{(0)} \in \mathbb{C}^n$$

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i \longrightarrow \left(\mathbf{A} - \xi\mathbf{I}\right)\mathbf{x}_i = \left(\lambda_i - \xi\right)\mathbf{x}_i$$

$$\left(\mathbf{A} - \xi\mathbf{I}\right)\mathbf{z}^{(k)} = \mathbf{q}^{(k-1)}$$

$$\mathbf{q}^{(k)} = \mathbf{z}^{(k)} \Big/ \left\|\mathbf{z}^{(k)}\right\|_2 \xrightarrow[m\to\infty]{} \mathbf{x}_m$$

$$\mu^{(k)} = \left\langle \mathbf{q}^{(k)}, \mathbf{A}\mathbf{q}^{(k)} \right\rangle \xrightarrow[m\to\infty]{} \lambda_m$$

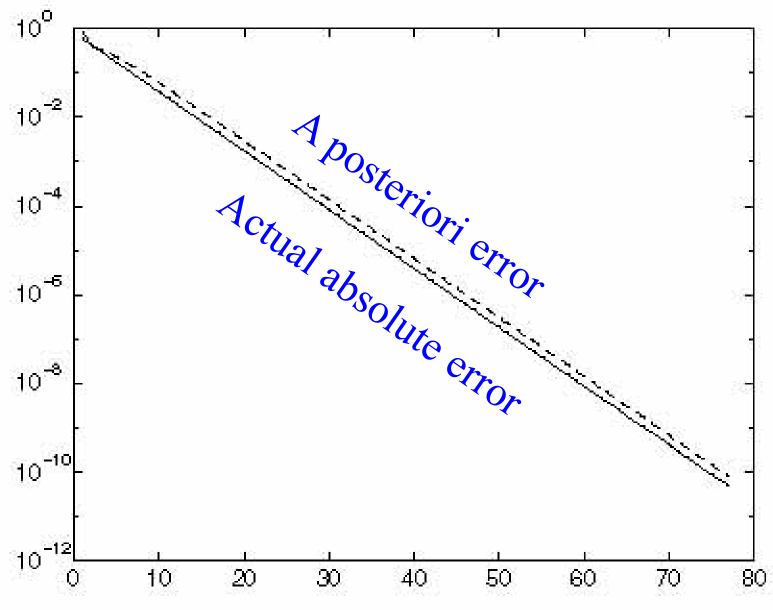
Implementation Issues

1) $\lambda_1 = \lambda_2 \longrightarrow \mathbf{A}^k \mathbf{q}^{(0)} \simeq \lambda_1^k \left(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 \right)$

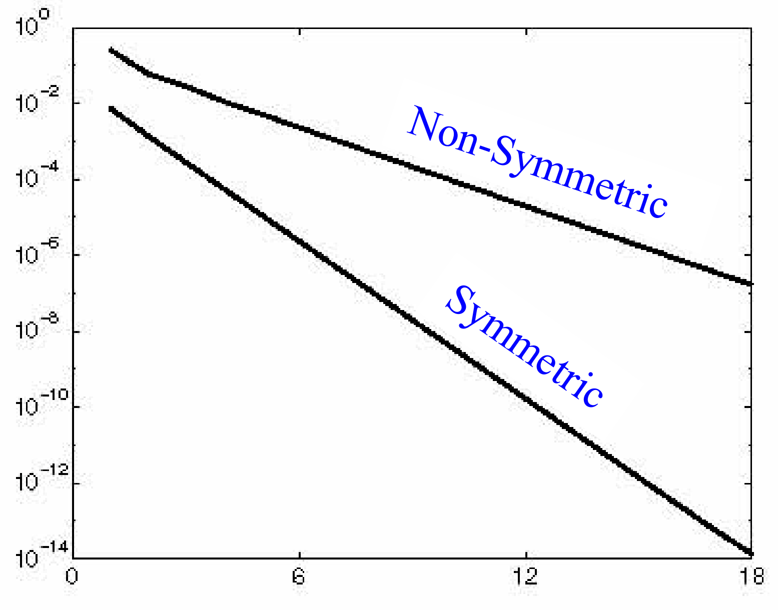
2) $\lambda_1 = -\lambda_2 \longrightarrow \mathbf{B} \equiv \mathbf{A}^2 \qquad \lambda_i \left(\mathbf{A}^2 \right) = \left[\lambda_i \left(\mathbf{A} \right) \right]^2$

3) $\lambda_2 = \overline{\lambda_1} \longrightarrow$ Undamped oscillations of $\mathbf{q}^{(k)}$

Example



A posteriori error
Actual absolute error



Non-Symmetric
Symmetric

Power Method for NDE

$$\mathbf{F}\mathbf{A}^{-1}\mathbf{f} = k\mathbf{f}$$

$$k^{(0)} = 1; \quad \mathbf{f}^{(0)} = \mathbf{1}$$

for $n = 0,1,\ldots$

$$\mathbf{f}^{(n+1)} = \mathbf{F}\mathbf{A}^{-1}\mathbf{f}^{(n)}$$

$$\mathbf{A}\boldsymbol{\phi}^{(n+1)} = \mathbf{f}^{(n)}$$

$$\mathbf{f}^{(n+1)} = \nu\Sigma_f\boldsymbol{\phi}^{(n+1)}$$

$$\mathbf{f}^{(n+1)} = \mathbf{f}^{(n+1)} \big/ k^{(n)}$$

$$k^{(n+1)} = k^{(n)} \left\langle \mathbf{f}^{(n+1)}, \mathbf{f}^{(n)} \right\rangle \Big/ \left\langle \mathbf{f}^{(n)}, \mathbf{f}^{(n)} \right\rangle$$

end

Important

- Continuous/Discrete Eigen-problem
- Multiplication Factor, Neutron Generations
- Perron-Frobenius Theorem
- Direct Power Method
- Inverse Power Method, Shifted Inverse
- A Priori Estimate (Gershgorin Theorem)
- A Posteriori Estimate
- Outer/Inner Iterations