Basic Iterative Methods

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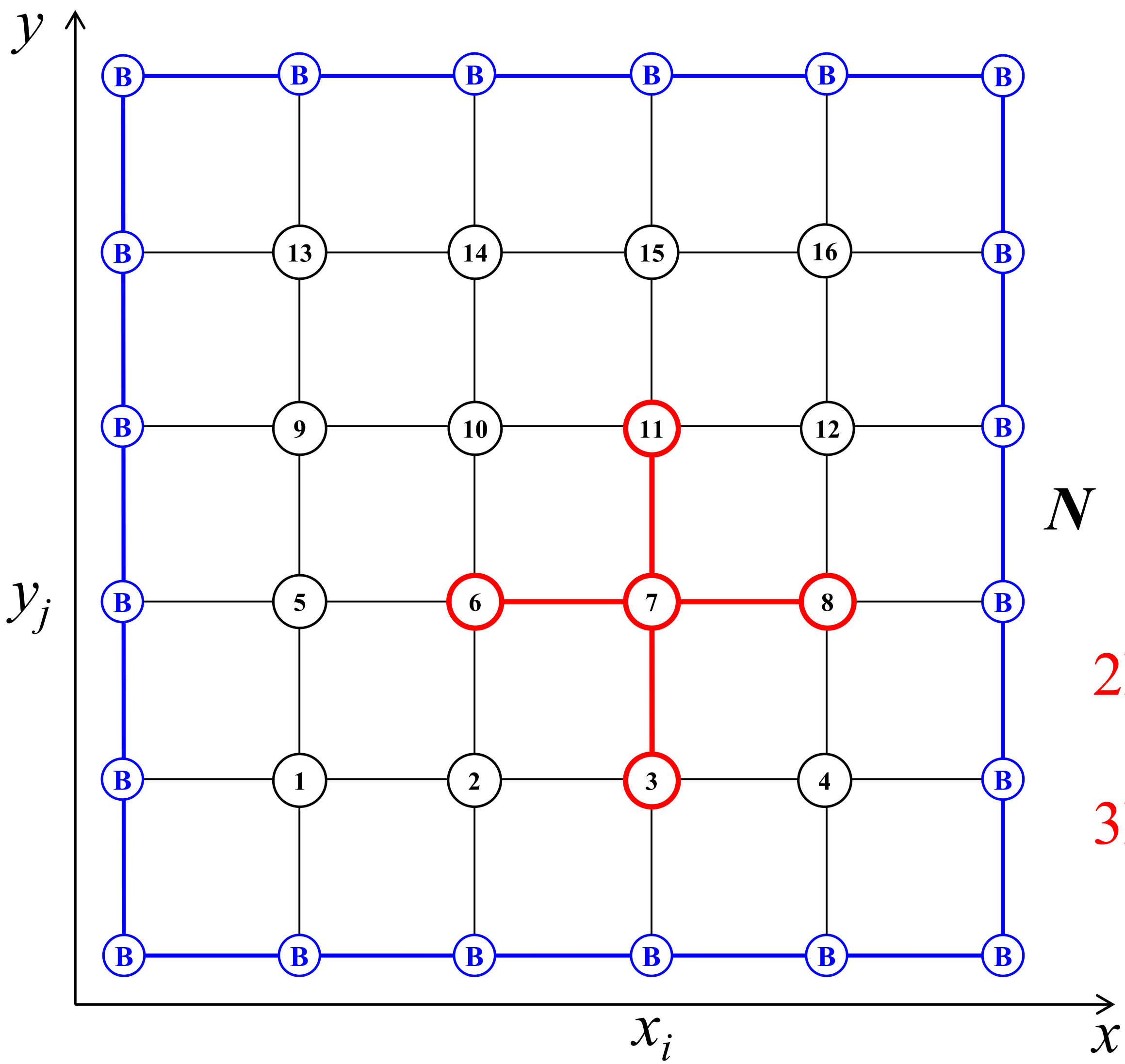
Overview

- Classification of iterative methods
- Linear 1-st order iterative methods
- Definitions, convergence theorems
- Jacobi and Jacobi Over-Relaxation
- Gauss-Seidel
- Successive Over-Relaxation
- Convergence conditions
- Examples

2D FD Mesh



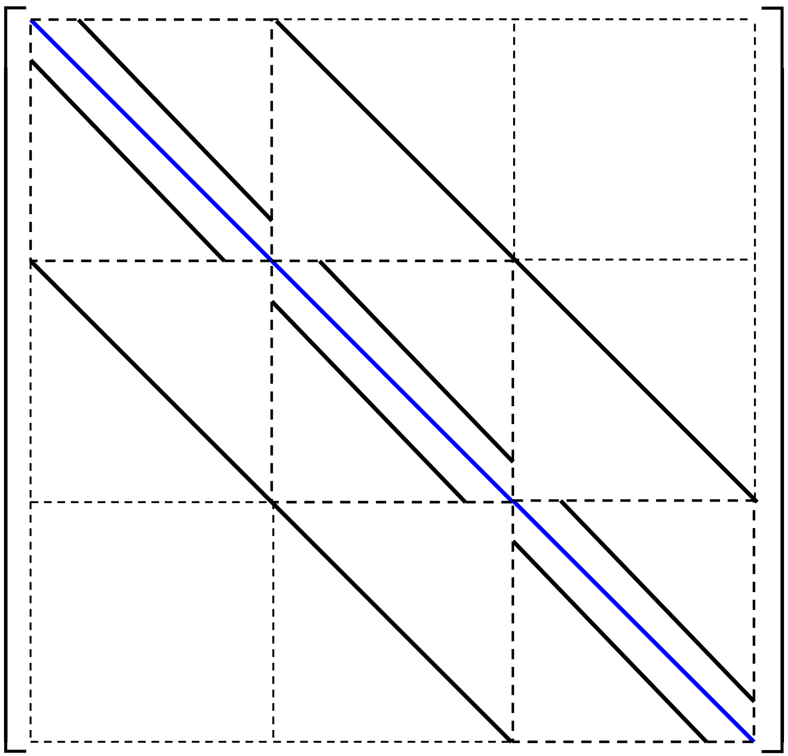
2D: Number of unknowns = N^2

3D: Number of unknowns = N^3

FD Equations in Matrix Form

$$\mathbf{x} \equiv \begin{bmatrix} \phi_{1,1}, & \cdots, & \phi_{N,N} \end{bmatrix}^T; \quad \mathbf{b} \equiv \begin{bmatrix} S_{1,1}h^2, & \cdots, & S_{N,N}h^2 \end{bmatrix}^T; \longrightarrow \mathbf{Ax} = \mathbf{b}$$

$\mathbf{A} =$



2D: Size($\mathbf{A}$) $= N^2 \times N^2$

nnz($\mathbf{A}$) $\approx 5N^2$

3D: Size($\mathbf{A}$) $= N^3 \times N^3$

nnz($\mathbf{A}$) $\approx 7N^3$

Convergent Series

$$\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)} \xrightarrow[k\to\infty]{} \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Error vector $$\mathbf{e}^{(k)} \equiv \mathbf{x}^{(k)} - \mathbf{x} \xrightarrow[k\to\infty]{} \mathbf{0}$$

$$\left\|\mathbf{e}^{(k)}\right\| \xrightarrow[k\to\infty]{} 0 \qquad \frac{\left\|\mathbf{e}^{(k)}\right\|}{\left\|\mathbf{x}\right\|} = \frac{\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\|}{\left\|\mathbf{x}\right\|} \xrightarrow[k\to\infty]{} 0$$

Multi-Order Methods

$$\mathbf{x}^{(0)} = f_0\left(\mathbf{A},\mathbf{b}\right)$$

$$\mathbf{x}^{(1)} = f_1\left(\mathbf{x}^{(0)},\mathbf{A},\mathbf{b}\right)$$

$$\mathbf{x}^{(2)} = f_2\left(\mathbf{x}^{(1)},\mathbf{x}^{(0)},\mathbf{A},\mathbf{b}\right)$$

$$\mathbf{x}^{(3)} = f_3\left(\mathbf{x}^{(2)},\mathbf{x}^{(1)},\mathbf{x}^{(0)},\mathbf{A},\mathbf{b}\right)$$

$$\vdots$$

$$\mathbf{x}^{(n+1)} = f_{n+1}(\underbrace{\mathbf{x}^{(n)},\mathbf{x}^{(n-1)},\mathbf{x}^{(n-2)}}_{m=3},\mathbf{A},\mathbf{b})$$

Nonstationary method of order $m = 3$.

Stationary Methods

$$\mathbf{x}^{(0)} = f_0\left(\mathbf{A}, \mathbf{b}\right)$$

$$\mathbf{x}^{(1)} = f_1\left(\mathbf{x}^{(0)}, \mathbf{A}, \mathbf{b}\right)$$

$$\mathbf{x}^{(2)} = f_2\left(\mathbf{x}^{(1)}, \mathbf{x}^{(0)}, \mathbf{A}, \mathbf{b}\right)$$

$$\mathbf{x}^{(3)} = f_3\left(\mathbf{x}^{(2)}, \mathbf{x}^{(1)}, \mathbf{x}^{(0)}, \mathbf{A}, \mathbf{b}\right)$$

$$\vdots$$

$$\mathbf{x}^{(n+1)} = f_3\left(\mathbf{x}^{(n)}, \mathbf{x}^{(n-1)}, \mathbf{x}^{(n-2)}, \mathbf{A}, \mathbf{b}\right)$$

Linear methods = functions f_i are linear.

Simplest Iterative Scheme

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Iteration matrix $\quad \mathbf{x}^{(k+1)} = \mathbf{B}\mathbf{x}^{(k)} + \mathbf{f}$

Consistency $\quad \mathbf{x} = \mathbf{B}\mathbf{x} + \mathbf{f} \longrightarrow \mathbf{f} = \left(\mathbf{I} - \mathbf{B}\right)\mathbf{A}^{-1}\mathbf{b}$

Error Convergence

$$\begin{cases} \mathbf{x}^{(k)} = \mathbf{B}\mathbf{x}^{(k-1)} + \mathbf{f} \\ \\ \mathbf{x} = \mathbf{B}\mathbf{x} + \mathbf{f} \end{cases} \longrightarrow \begin{cases} \mathbf{x}^{(k)} - \mathbf{x} = \mathbf{B}\left(\mathbf{x}^{(k-1)} - \mathbf{x}\right) \\ \\ \mathbf{e}^{(k)} = \mathbf{B}\mathbf{e}^{(k-1)} = \mathbf{B}^{k}\mathbf{e}^{(0)} \end{cases}$$

$$\left\|\mathbf{e}^{(k)}\right\| \le \left\|\mathbf{B}^{k}\right\| \cdot \left\|\mathbf{e}^{(0)}\right\| \le \left\|\mathbf{B}\right\|^{k} \cdot \left\|\mathbf{e}^{(0)}\right\|$$

Convergence Rate

$$\left\|\mathbf{e}^{(k)}\right\| \le \left\|\mathbf{B}^k\right\| \cdot \left\|\mathbf{e}^{(0)}\right\|$$

Error reduction factor

$$\left\|\mathbf{e}^{(k)}\right\| \le q^k \left\|\mathbf{e}^{(0)}\right\| \longrightarrow q = \left\|\mathbf{B}^k\right\|^{1/k}$$

Average factor

$$q^k \le \varepsilon \longrightarrow k \ge \frac{\log 1/\varepsilon}{\log 1/q} = \frac{\log 1/\varepsilon}{-\frac{1}{k}\log\left\|\mathbf{B}^k\right\|}$$

Average rate

Definitions

Convergence Factor after k: $\left\|\mathbf{B}^k\right\|$

Average Convergence Factor: $\left\|\mathbf{B}^k\right\|^{1/k} \xrightarrow[k\to\infty]{} \rho(\mathbf{B})$

Average Convergence Rate: $R_k(\mathbf{B}) = -\frac{1}{k}\log\left(\left\|\mathbf{B}^k\right\|\right)$

Assymptotic Convergence Rate: $R(\mathbf{B}) = \lim_{k\to\infty} R_k(\mathbf{B}) = -\log\rho(\mathbf{B})$

Convergence Theorem

$$\left\|\mathbf{e}^{(k)}\right\| \le \left\|\mathbf{B}^k\right\| \cdot \left\|\mathbf{e}^{(0)}\right\| \le \left\|\mathbf{B}\right\|^k \cdot \left\|\mathbf{e}^{(0)}\right\|$$

$$\left\|\mathbf{B}\right\| < 1 \longrightarrow \text{Iterative method is convergent}$$

$$\rho\left(\mathbf{B}\right) < 1 \longrightarrow \text{Iterative method is convergent}$$

$$\rho\left(\mathbf{B}\right) \ge 1 \longrightarrow \text{Iterative method is divergent}$$

Linear Iterative Methods

$$\mathbf{Ax} = \mathbf{b} \qquad \mathbf{A} = \mathbf{P} - \mathbf{N}$$

$\mathbf{P}$ is preconditioning matrix

$$\mathbf{Px} = \mathbf{Nx} + \mathbf{b}$$

$$\mathbf{Px}^{(k+1)} = \mathbf{Nx}^{(k)} + \mathbf{b}$$

$$\mathbf{x}^{(k+1)} = \mathbf{P}^{-1}\mathbf{Nx}^{(k)} + \mathbf{P}^{-1}\mathbf{b}$$

$$\mathbf{x}^{(k+1)} = \mathbf{Bx}^{(k)} + \mathbf{f}$$

Equivalent form

$$\mathbf{x}^{(0)} \qquad k = 0,1,2,\dots$$

$$\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{Ax}^{(k)}$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{r}^{(k)}$$

Trade-off

$$\mathbf{P}\mathbf{x}^{(k+1)} = \mathbf{N}\mathbf{x}^{(k)} + \mathbf{b} \qquad \mathbf{N} = \mathbf{P} - \mathbf{A}$$

$$\mathbf{x}^{(k+1)} = \mathbf{P}^{-1}\mathbf{N}\mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{b} = \left(\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\right)\mathbf{x}^{(k)} + \mathbf{f}$$

Fastest inversion $\mathbf{P}^{-1} \rightarrow \mathbf{P} = ?$

Fastest convergence $\rightarrow \mathbf{P} = ?$

More Matrix Norms

$$\|\mathbf{x}\|_2^2 = \mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\|\mathbf{B}\|_2 = \max_{\mathbf{x} \neq 0} \frac{\|\mathbf{Bx}\|_2}{\|\mathbf{x}\|_2}$$

$$\|\mathbf{x}\|_2^2 = \langle \mathbf{x}, \mathbf{x} \rangle = w_1 x_1^2 + w_2 x_2^2 + \cdots + w_2 x_n^2$$

$$\|\mathbf{B}\|_2 = \max_{\mathbf{x} \neq 0} \frac{\|\mathbf{Bx}\|_2}{\|\mathbf{x}\|_2}$$

$$\|\mathbf{x}\|_{\mathbf{A}}^2 = \mathbf{x} \cdot \mathbf{A}\mathbf{x} = \sum_{i,j} a_{i,j} x_i x_j$$

SPD = Symmetric Positive Definite

$$\|\mathbf{B}\|_{\mathbf{A}} = \max_{\mathbf{x} \neq 0} \frac{\|\mathbf{Bx}\|_{\mathbf{A}}}{\|\mathbf{x}\|_{\mathbf{A}}}$$

Convergence Theorem 1

Let $\mathbf{A} = \mathbf{P} - \mathbf{N}$ with $\mathbf{A}$ and $\mathbf{P}$ symmetric and positive definite. If the matrix $\mathbf{2P} - \mathbf{A} = \mathbf{P} + \mathbf{N}$ is positive definite then

$$\mathbf{x}^{(k+1)} = \mathbf{P}^{-1}\mathbf{N}\mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{b} = \mathbf{B}\mathbf{x}^{(k)} + \mathbf{f}$$

is convergent for any choice of $\mathbf{x}^{(0)}$ and

$$\rho(\mathbf{B}) = \|\mathbf{B}\|_{\mathbf{A}} = \|\mathbf{B}\|_{\mathbf{P}} < 1$$

Moreover, the convergence is monotonic in the norms

$$\left\|\mathbf{e}^{(k+1)}\right\|_{\mathbf{A}} < \left\|\mathbf{e}^{(k)}\right\|_{\mathbf{A}} \quad \text{and} \quad \left\|\mathbf{e}^{(k+1)}\right\|_{\mathbf{P}} < \left\|\mathbf{e}^{(k)}\right\|_{\mathbf{P}}$$

Convergence Theorem 2

Let $\mathbf{A} = \mathbf{P} - \mathbf{N}$ with $\mathbf{A}$ symmetric and positive definite. If the matrix $\mathbf{P} + \mathbf{P}^{\mathrm{T}} - \mathbf{A} = \mathbf{P}^{\mathrm{T}} + \mathbf{N}$ is positive definite then $\mathbf{P}$ is invertible.

$$\mathbf{x}^{(k+1)} = \mathbf{P}^{-1}\mathbf{N}\mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{b} = \mathbf{B}\mathbf{x}^{(k)} + \mathbf{f}$$

is monotonically convergent in the A-norm for any choice of $\mathbf{x}^{(0)}$

$$\left\|\mathbf{e}^{(k+1)}\right\|_{\mathbf{A}} < \left\|\mathbf{e}^{(k)}\right\|_{\mathbf{A}} \quad \text{and} \quad \rho\left(\mathbf{B}\right) \le \left\|\mathbf{B}\right\|_{\mathbf{A}} < 1$$

Convergence Theorem 3

$$\mathbf{x}^{(k)} = \mathbf{B}\mathbf{x}^{(k-1)} + \mathbf{f} \qquad \delta \equiv \left\|\mathbf{B}\right\| < 1$$

$$\mathbf{x} = \mathbf{B}\mathbf{x} + \mathbf{f} \qquad \left[\rho\left(\mathbf{B}\right) < 1\right]$$

$$\mathbf{x}^{(k)} - \mathbf{x} = \mathbf{B}\left(\mathbf{x}^{(k-1)} - \mathbf{x}\right) = \ldots = \mathbf{B}^{k}\left(\mathbf{x}^{(0)} - \mathbf{x}\right)$$

$$\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\| \le \left\|\mathbf{B}^{k}\right\| \cdot \left\|\mathbf{x}^{(0)} - \mathbf{x}\right\| \le \delta^{k}\left\|\mathbf{x}^{(0)} - \mathbf{x}\right\|$$

Convergence Theorem 4

$$\mathbf{x}^{(k)} - \mathbf{x} = \mathbf{B}\left(\mathbf{x}^{(k-1)} - \mathbf{x}\right) = \mathbf{B}\left(\mathbf{x}^{(k-1)} - \mathbf{x}^{(k)} + \mathbf{x}^{(k)} - \mathbf{x}\right)$$

$$\mathbf{x}^{(k)} - \mathbf{x} = \mathbf{B}\left(\mathbf{x}^{(k)} - \mathbf{x}\right) - \mathbf{B}\left(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right)$$

$$\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\| \le \left\|\mathbf{B}\right\| \cdot \left\|\mathbf{x}^{(k)} - \mathbf{x}\right\| + \left\|\mathbf{B}\right\| \cdot \left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|$$

$$\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\| \le \frac{\left\|\mathbf{B}\right\|}{1 - \left\|\mathbf{B}\right\|} \left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\| = \frac{\delta}{1 - \delta} \left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|$$

Richardson Method

$$\mathbf{P} = \mathbf{I}$$

$$\mathbf{x}^{(k+1)} = \left(\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\right)\mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{b} = \mathbf{x}^{(k)} + \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}$$

$$\left\|\mathbf{I} - \mathbf{A}\right\| < 1 \quad \left[\rho\left(\mathbf{I} - \mathbf{A}\right) < 1\right]$$

Example

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/3 & 1 & 1/2 \\ 1/2 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/18 \\ 11/18 \\ 11/18 \end{bmatrix}$$

$$\mathbf{x}^{(1)} = \begin{bmatrix} 0.611 & 0.611 & 0.611 \end{bmatrix}^T$$

$$\mathbf{x}^{(10)} = \begin{bmatrix} 0.279 & 0.279 & 0.279 \end{bmatrix}^T$$

$$\mathbf{x}^{(40)} = \begin{bmatrix} 0.333 & 0.333 & 0.333 \end{bmatrix}^T$$

$$\mathbf{x}^{(80)} = \begin{bmatrix} 0.333333 & 0.333333 & 0.333333 \end{bmatrix}^T$$

Jacobi Method

$$\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$\mathbf{D}\mathbf{x} = \left(\mathbf{L} + \mathbf{U}\right)\mathbf{x} + \mathbf{b}$$

$$a_{ii} x_i = b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j$$

$$\mathbf{D}\mathbf{x}^{(k+1)} = \left(\mathbf{L} + \mathbf{U}\right)\mathbf{x}^{(k)} + \mathbf{b}$$

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\left(\mathbf{L} + \mathbf{U}\right)\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b}$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k)}\right)$$

$$\mathbf{B}_J = \mathbf{D}^{-1}\left(\mathbf{L} + \mathbf{U}\right)$$

Convergence of JM

$$\delta = \left\| \mathbf{D}^{-1} \left(\mathbf{L} + \mathbf{U} \right) \right\| < 1$$

$$\left\| \mathbf{D}^{-1} \left(\mathbf{L} + \mathbf{U} \right) \right\|_{\infty} = \max_{i} \frac{1}{\left| a_{ii} \right|} \sum_{j \neq i} \left| a_{ij} \right|$$

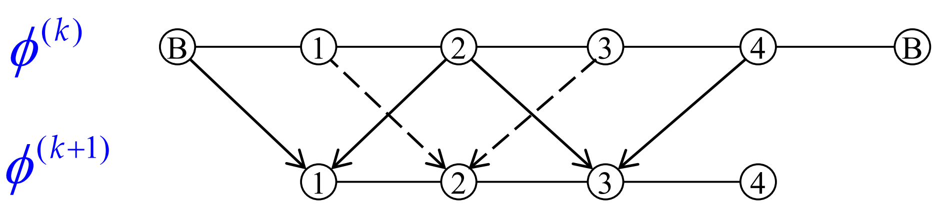
$$\left| a_{ii} \right| > \sum_{j \neq i} \left| a_{ij} \right| \longrightarrow \left\| \mathbf{D}^{-1} \left(\mathbf{L} + \mathbf{U} \right) \right\|_{\infty} < 1$$

Jacobi for 1D Diffusion

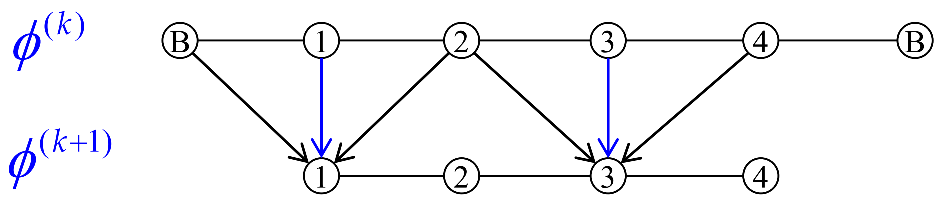
$$-\phi''(x) + B^2\phi(x) = S(x); \quad \phi''(x_i) \approx \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}$$

$$-\phi_{i-1} + \left(2 + B^2h^2\right)\phi_i - \phi_{i+1} = S_i h^2$$

$$\phi_i^{(k+1)} = \frac{1}{2 + B^2h^2}\left(\phi_{i-1}^{(k)} + \phi_{i+1}^{(k)} + S_i h^2\right)$$



Jacobi Over-Relaxation



$$\phi_i^{(k+1)} = \frac{1}{2+B^2h^2}\left(\phi_{i-1}^{(k)} + \phi_{i+1}^{(k)} + S_i h^2\right)$$

JOR $$\phi_i^{(k+1)} = \frac{\omega}{2+B^2h^2}\left(\phi_{i-1}^{(k)} + \phi_{i+1}^{(k)} + S_i h^2\right) + \left(1-\omega\right)\phi_i^{(k)}$$

JOR

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \longrightarrow a_{ii} x_i = b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j$$

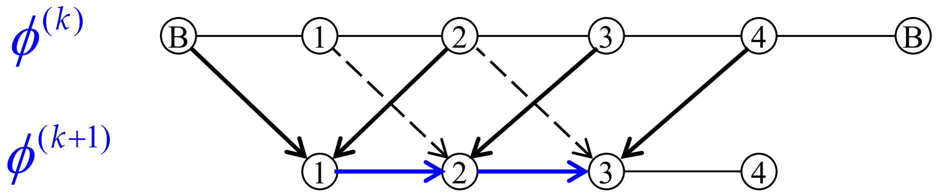
$$\hat{x}_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k)} \right)$$

$$x_i^{(k+1)} = \omega \hat{x}_i^{(k+1)} + \left(1 - \omega\right) x_i^{(k)}$$

Gauss-Seidel

$$\phi_i^{(k+1)} = \frac{1}{2 + B^2 h^2}\left(\phi_{i-1}^{(k)} + \phi_{i+1}^{(k)} + S_i h^2\right)$$

$$\phi_i^{(k+1)} = \frac{1}{2 + B^2 h^2}\left(\phi_{i-1}^{(k+1)} + \phi_{i+1}^{(k)} + S_i h^2\right)$$



GS in Matrix Form

$$\left(\mathbf{D}-\mathbf{L}-\mathbf{U}\right)\mathbf{x}=\mathbf{b}\longrightarrow\mathbf{D}\mathbf{x}=\mathbf{L}\mathbf{x}+\mathbf{U}\mathbf{x}+\mathbf{b}$$

$$\mathbf{D}\mathbf{x}^{(k+1)}=\mathbf{L}\mathbf{x}^{(k+1)}+\mathbf{U}\mathbf{x}^{(k)}+\mathbf{b}$$

$$\mathbf{x}^{(k+1)}=\left(\mathbf{D}-\mathbf{L}\right)^{-1}\mathbf{U}\mathbf{x}^{(k)}+\left(\mathbf{D}-\mathbf{L}\right)^{-1}\mathbf{b}$$

$$\mathbf{B}_J=\mathbf{D}^{-1}\left(\mathbf{L}+\mathbf{U}\right)\qquad\qquad\mathbf{B}_{GS}=\left(\mathbf{D}-\mathbf{L}\right)^{-1}\mathbf{U}$$

GS in Coordinate Form

$$\sum_{j=1}^{i-1} a_{ij} x_j + a_{ii} x_i + \sum_{j=i+1}^{n} a_{ij} x_j = b_i$$

$$a_{ii} x_i = b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j$$

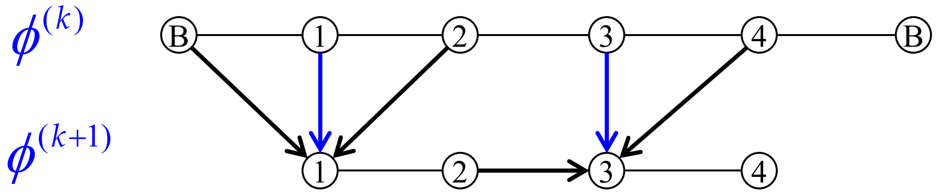
$$a_{ii} x_i^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}$$

Convergence of GS

If **A** is diagonally dominant, then the Gauss-Seidel method converges for any starting vector.

$$\left|a_{ii}\right| > \sum_{j \neq i} \left|a_{ij}\right| \longrightarrow \rho\left(\mathbf{B}_{GS}\right) = \rho\left(\left[\mathbf{D} - \mathbf{L}\right]^{-1} \mathbf{U}\right) < 1$$

Successive-Over Relaxation



$$\phi_i^{(k+1)} = \frac{\omega}{2+B^2h^2}\left(\phi_{i-1}^{(k+1)} + \phi_{i+1}^{(k)} + S_i h^2\right) + \left(1-\omega\right)\phi_i^{(k)}$$

SOR Method

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right) + (1-\omega)\, x_i^{(k)}$$

SOR in Matrix Form

$$\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$$

$$\mathbf{Dx} = \mathbf{Lx} + \mathbf{Ux} + \mathbf{b}$$

$$\mathbf{Dx}^{(k+1)} = \mathbf{Lx}^{(k+1)} + \mathbf{Ux}^{(k)} + \mathbf{b}$$

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\left(\mathbf{Lx}^{(k+1)} + \mathbf{Ux}^{(k)} + \mathbf{b}\right)$$

$$\mathbf{x}^{(k+1)} = \omega\mathbf{D}^{-1}\left(\mathbf{Lx}^{(k+1)} + \mathbf{Ux}^{(k)} + \mathbf{b}\right) + \left(1 - \omega\right)\mathbf{x}^{(k)}$$

$$\mathbf{B}_{GS} = \left(\mathbf{D} + \mathbf{L}\right)^{-1}\mathbf{U}$$

$$\mathbf{B}_{\omega} = \left(\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{L}\right)^{-1}\left[\left(1 - \omega\right)\mathbf{I} + \omega\mathbf{D}^{-1}\mathbf{U}\right]$$

Convergence Theorem

If **A** is a strictly diagonally dominant by rows
the Jacobi and Gauss-Seidel methods are convergent

If $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$ and $2\mathbf{D} - \mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{U}$ are SPD matrices
then the Jacobi method is convergent $\rho\left(\mathbf{B}_J\right) = \left\|\mathbf{B}_J\right\|_A = \left\|\mathbf{B}_J\right\|_D < 1$

If **A** is symmetric and positive definite, the Gauss-Seidel method is monotonically convergent with respect to the A-norm.

If **A** is positive definite and tridiagonal, the Gauss-Seidel method is monotonically convergent with respect to the A-norm.

$$\rho\left(\mathbf{B}_{GS}\right) = \rho^2\left(\mathbf{B}_J\right) < 1$$

General Matrices

$$\mathbf{A}_1 = \begin{bmatrix} 3 & 0 & 4 \\ 7 & 4 & 2 \\ -1 & 1 & 2 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & -6 \end{bmatrix} \qquad \mathbf{A}_4 = \begin{bmatrix} 7 & 6 & 9 \\ 4 & 5 & -4 \\ -7 & -3 & 8 \end{bmatrix} \qquad \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A}_1: \quad \rho(\mathbf{B}_J) = 1.33; \qquad \rho(\mathbf{B}_{GS}) < 1$$

$$\mathbf{A}_2: \quad \rho(\mathbf{B}_J) < 1; \qquad \rho(\mathbf{B}_{GS}) = 1.1$$

$$\mathbf{A}_3: \quad \rho(\mathbf{B}_J) = 0.44; \qquad \rho(\mathbf{B}_{GS}) = 0.018$$

$$\mathbf{A}_4: \quad \rho(\mathbf{B}_J) = 0.64; \qquad \rho(\mathbf{B}_{GS}) = 0.77$$

Important

- Classification of iterative methods
- Convergence rates
- Jacobi and Jacobi Over-Relaxation
- Gauss-Seidel
- Successive Over-Relaxation
- Convergence conditions
- Optimal parameter ω