# FD METHOD FOR NEUTRON DIFFUSION EQUATION

Vasily Arzhanov

Reactor Physics, KTH

# Overview

- Stationary NDE in 1D
- Finite-Difference mesh in 1D
- Integro-Interpolation Method
- Three point FD equations in 1D
- Finite-Difference mesh in 2D
- Five point FD equations in 2D

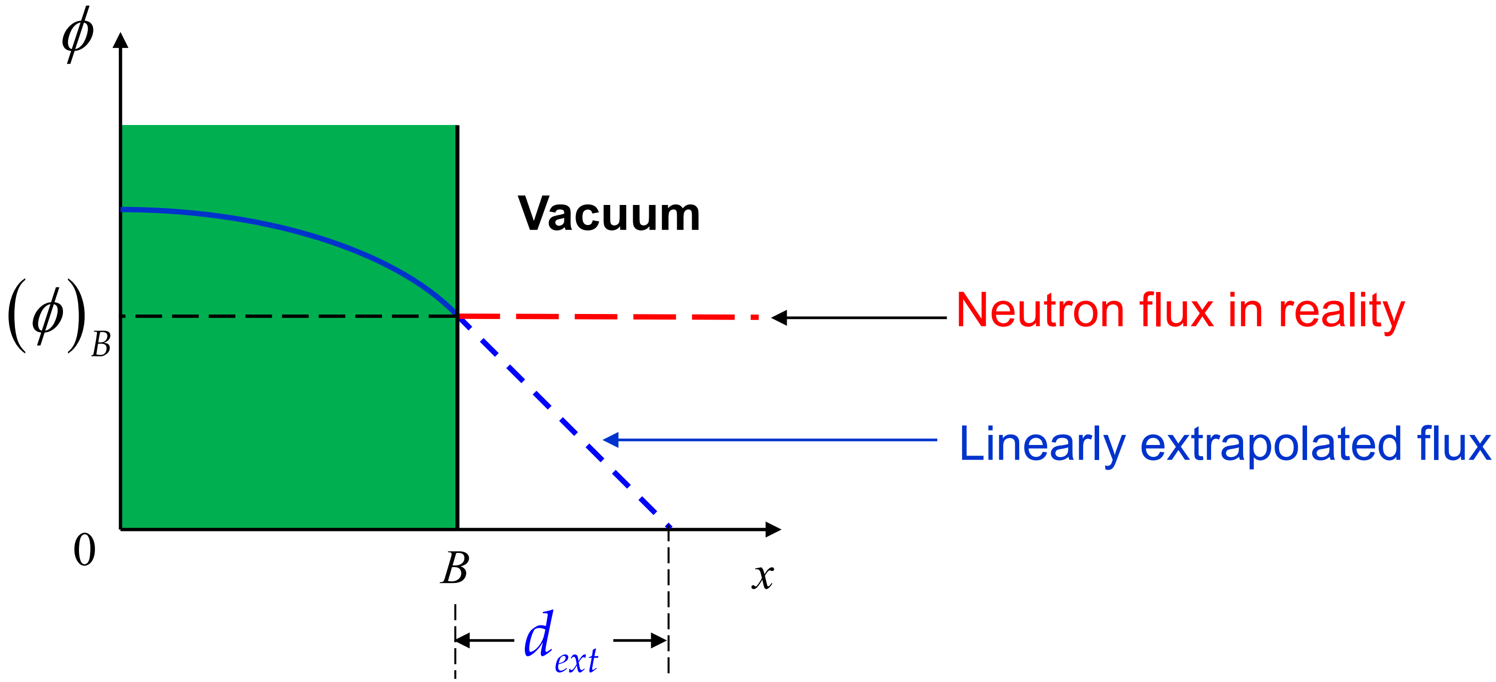
# Stationary NDE in 1D

$$-\nabla\left(D\nabla\phi\right)+\Sigma_a\phi=S+\nu\Sigma_f\phi \qquad J\equiv -D\nabla\phi=-D\frac{d\phi}{dx}$$

$$\frac{d}{dx}J+\Sigma_r\phi=S \qquad \Sigma_r\equiv\Sigma_a-\nu\Sigma_f \qquad dV=x^\alpha dx$$

$$\frac{1}{x^\alpha}\frac{d}{dx}\left(x^\alpha J\right)+\Sigma_r\phi=S \qquad \alpha=\begin{cases}0 & \text{Cartesian}\\ 1 & \text{Cylindrical}\\ 2 & \text{Spherical}\end{cases}$$

# Extrapolated Length




BC: $\left(\dfrac{\partial \phi}{\partial x}\right)_B = -\dfrac{(\phi)_B}{d_{ext}}$ $\quad d_{ext} = \begin{cases} 2/3\,\lambda_{tr} & \text{Diffusion theory} \\ 0.71\lambda_{tr} & \text{Transport theory} \end{cases}$

# Boundary Conditions in 1D

1) Zero $\qquad \phi\big|_B = 0$

2) Reflection $\qquad \left.\dfrac{\partial \phi}{\partial n}\right|_B = 0$

3) Extrapolation $\qquad \dfrac{\partial \phi}{\partial n} + \dfrac{1}{d_{ext}}\phi = 0 \longrightarrow \phi_{lin}(d_{ext}) = 0$

3.1) Diffusion $\qquad d_{ext} = \dfrac{2}{3}\lambda_{tr} = 2D$

3.2) Transport $\qquad d_{ext} = 0.71\lambda_{tr} = 2.13D$

3.3) Albedo $\qquad d_{ext} = \dfrac{1+\alpha}{1-\alpha}2D \longrightarrow \left(J_- = \alpha J_+\right)\Big|_{\partial V} = 0$

# Unified Boundary Conditions

$$\tilde{D}\frac{\partial\phi}{\partial n}+\gamma\phi=0 \qquad \tilde{D}=\begin{cases}0 & \text{ZeroBC}\\ D & \text{Otherwise}\end{cases}$$

1) Zero $\gamma=1$

2) Reflection $\gamma=0$

3) Extrapolation $\gamma=D/d_{ext}$

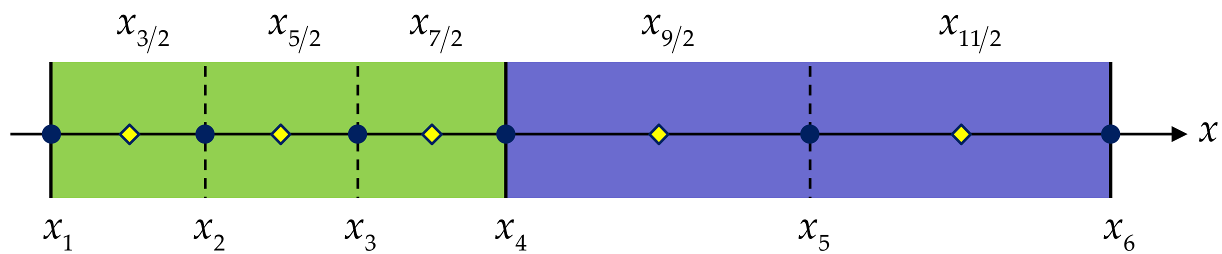
   3.1) Diffusion $\gamma=1/2$

   3.2) Transport $\gamma=0.469$

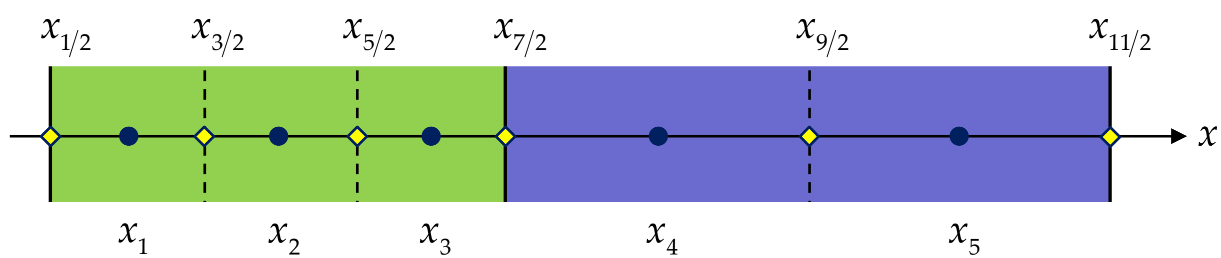
   3.3) Albedo $\gamma=\frac{1}{2}\cdot\frac{1-\alpha}{1+\alpha}$

# Numerical Mesh in 1D

Vertex-based

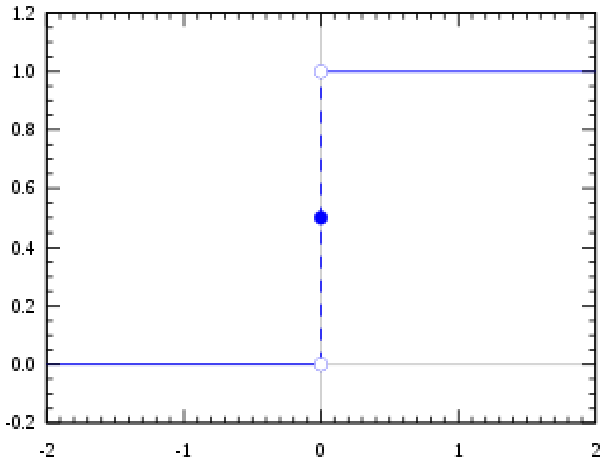


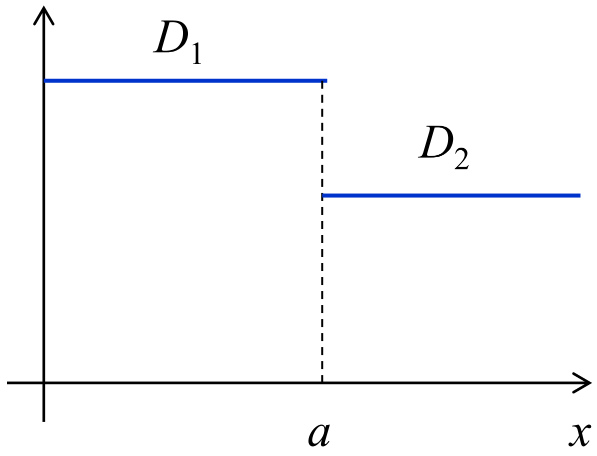

Cell-centred



# Heaviside Function

$$H(x)=\begin{cases}0 & x<0\\ 1/2 & x=0\\ 1 & x>0\end{cases}$$



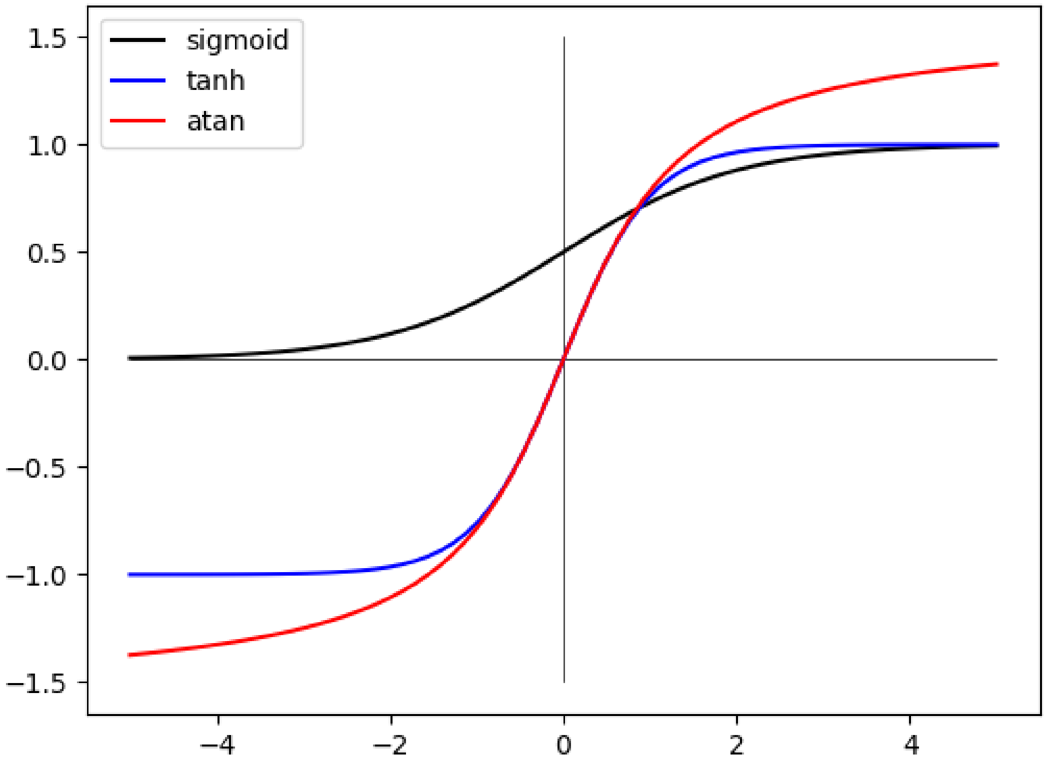

$$D(x)=D_1+\left(D_2-D_1\right)H(x-a)$$

# Approximating Heaviside

$$S(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

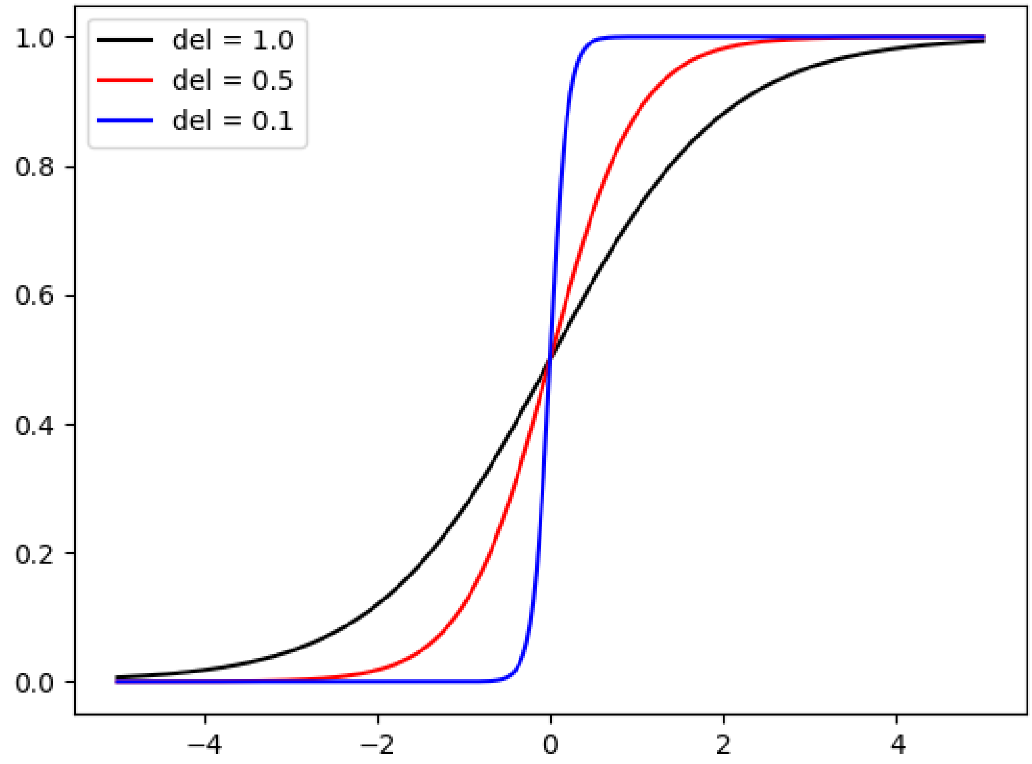
$$\arctan(x) = \tan^{-1}(x)$$



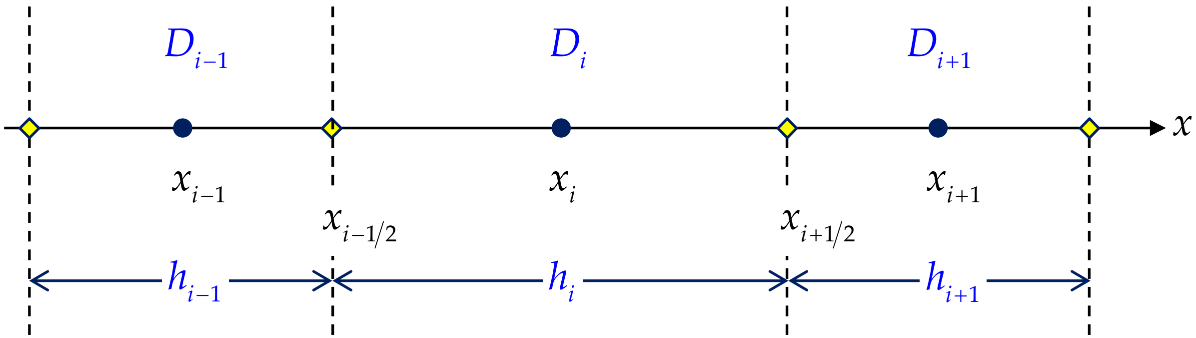
# Approximate Heaviside

$$H(x) \approx S(x/\delta)$$

$$H(x) \approx \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{\delta}$$



# Cell-Centred Mesh




$$\int_{x_{i-1/2}}^{x_{i+1/2}} x^\alpha \cdot (\quad) dx \qquad \frac{1}{x^\alpha}\frac{d}{dx}\left(x^\alpha J\right) + \Sigma_r \phi = S(x)$$

$$x^\alpha_{i+1/2} J_{i+1/2} - x^\alpha_{i-1/2} J_{i-1/2} + \int_{x_{i-1/2}}^{x_{i+1/2}} \Sigma_r \phi x^\alpha dx = \int_{x_{i-1/2}}^{x_{i+1/2}} S x^\alpha dx$$

# Approximating Integrals

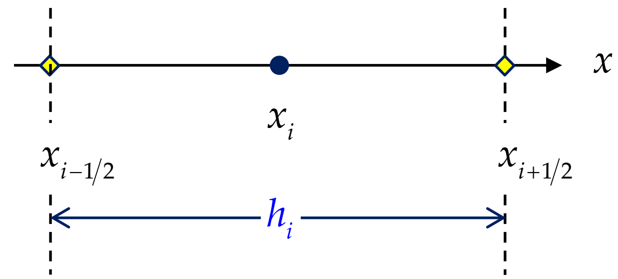
$$L_i \equiv x_{i+1/2}^{\alpha} J_{i+1/2} - x_{i-1/2}^{\alpha} J_{i-1/2} \qquad J_{i-1/2} \equiv -D \frac{d\phi}{dx}\left(x_{i-1/2}\right) \qquad J_{i+1/2} \equiv -D \frac{d\phi}{dx}\left(x_{i+1/2}\right)$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} S(x) x^{\alpha} dx \approx \frac{x_{i+1/2}^{\alpha+1} - x_{i-1/2}^{\alpha+1}}{\alpha+1} S_i = v_i S_i \qquad v_i \equiv \frac{x_{i+1/2}^{\alpha+1} - x_{i-1/2}^{\alpha+1}}{\alpha+1}$$

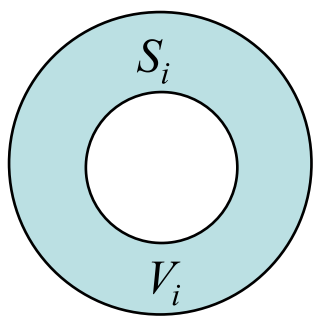
$$\int_{x_{i-1/2}}^{x_{i+1/2}} \Sigma_r \phi x^{\alpha} dx \approx v_i \Sigma_{r,i} \phi_i$$

FD eq. $L_i + v_i \Sigma_{r,i} \phi_i = v_i S_i$

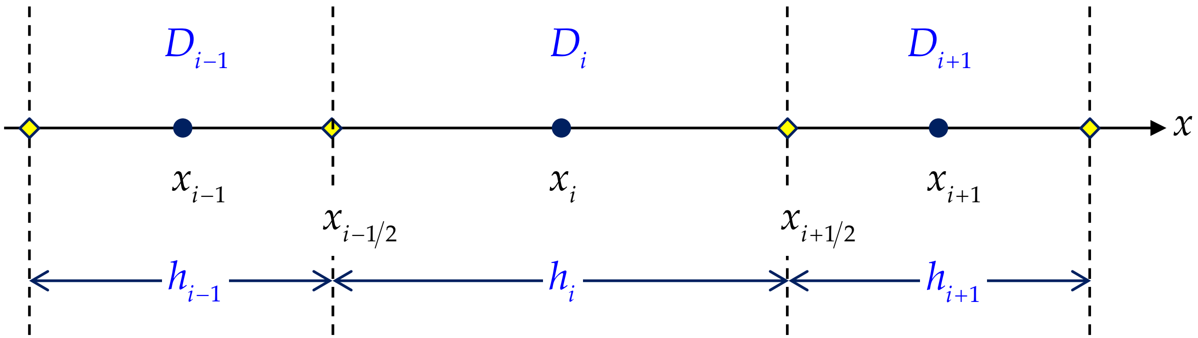
# Geometric Intepretation




$$v_i = \frac{x_{i+1/2}^{\alpha+1} - x_{i-1/2}^{\alpha+1}}{\alpha+1} = \begin{cases} x_{i+1/2} - x_{i+1/2} = h_i \\ \left(x_{i+1/2}^2 - x_{i-1/2}^2\right) \big/ 2 = S_i \big/ 2\pi \\ \left(x_{i+1/2}^3 - x_{i-1/2}^3\right) \big/ 3 = V_i \big/ 4\pi \end{cases}$$



# Approximating Currents

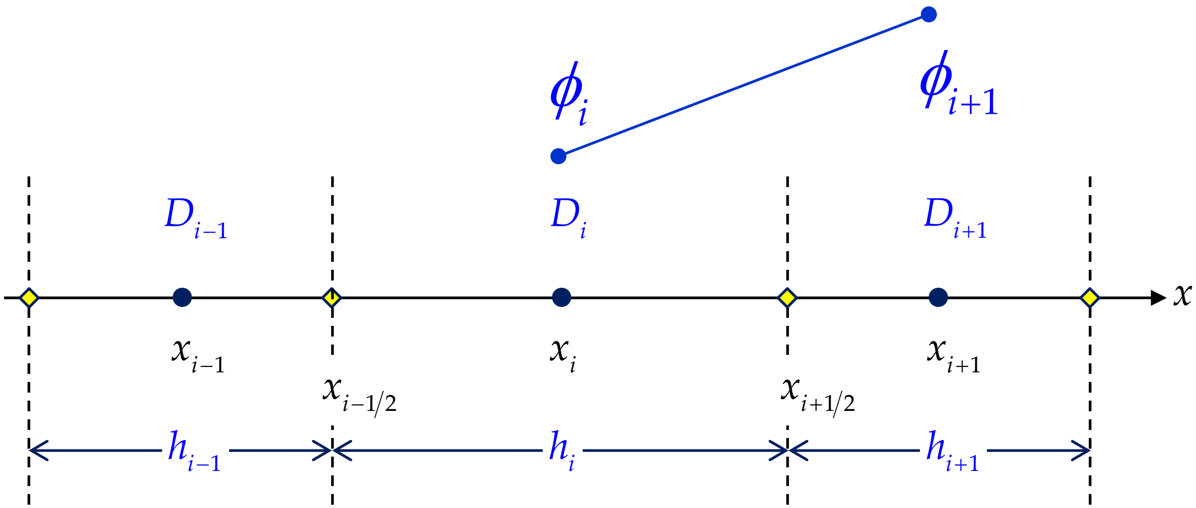



$$\overrightarrow{J_{i+1/2}} \equiv -D\frac{d\phi}{dx}\left(x_{i+1/2}\right)$$

$$D_i\phi'(x_{i+1/2}-0) = D_{i+1}\phi'(x_{i+1/2}+0)$$

$$D_i\frac{\phi_{i+1/2}-\phi_i}{h_i/2} = D_{i+1}\frac{\phi_{i+1}-\phi_{i+1/2}}{h_{i+1}/2}$$

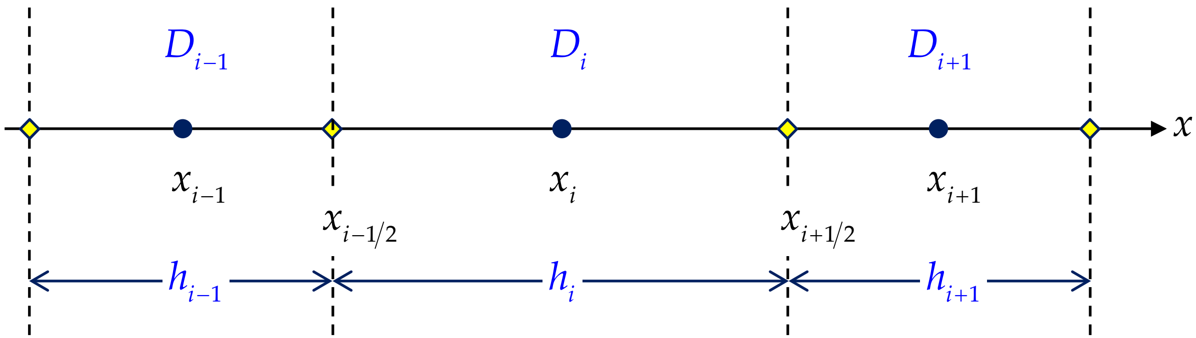
# Flux at the Interface




$$\phi_{i+1/2} = \frac{h_{i+1} D_i \phi_i + h_i D_{i+1} \phi_{i+1}}{h_{i+1} D_i + h_i D_{i+1}}$$

$$\phi_{i+1/2} = \frac{\phi_i + \phi_{i+1}}{2}$$

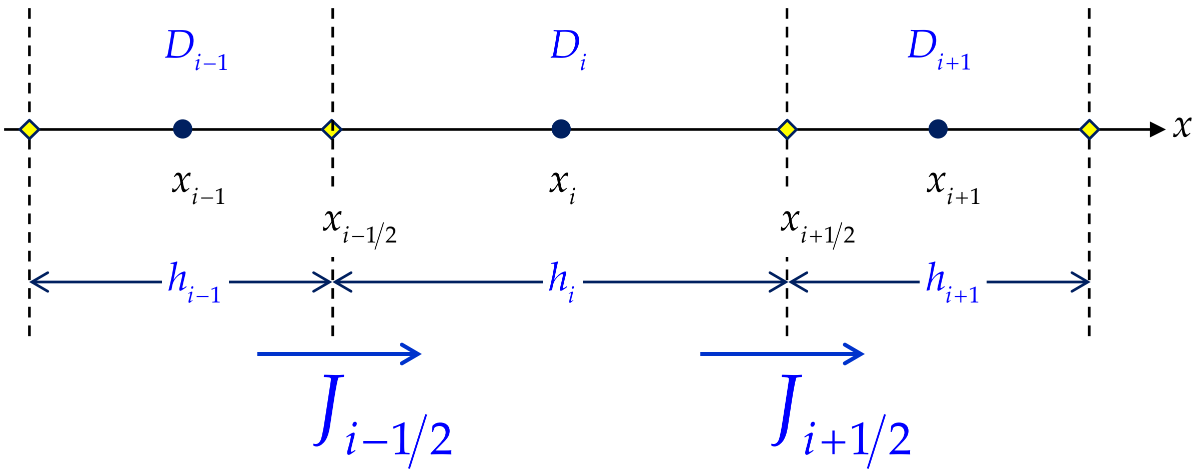
# Current through the Interface




$$J_{i+1/2} = -D\phi'_{i+1/2} \approx -D_i \frac{\phi_{i+1/2} - \phi_i}{h_i/2} = -\frac{2D_i D_{i+1}}{h_{i+1}D_i + h_i D_{i+1}}\left(\phi_{i+1} - \phi_i\right)$$

$$J_{i-1/2} = -D\phi'_{i-1/2} \approx -D_{i-1} \frac{\phi_{i-1/2} - \phi_{i-1}}{h_{i-1}/2} = -\frac{2D_{i-1} D_i}{h_i D_{i-1} + h_{i-1} D_i}\left(\phi_i - \phi_{i-1}\right)$$

# Cell Leakage




$$L_i \equiv x_{i+1/2}^{\alpha} J_{i+1/2} - x_{i-1/2}^{\alpha} J_{i-1/2}$$

# FD Equations

$$\frac{2x_{i-1/2}^{\alpha}D_{i-1}D_i}{h_iD_{i-1}+h_{i-1}D_i}\left(\phi_i-\phi_{i-1}\right)-\frac{2x_{i+1/2}^{\alpha}D_iD_{i+1}}{h_{i+1}D_i+h_iD_{i+1}}\left(\phi_{i+1}-\phi_i\right)+v_i\Sigma_{r,i}\phi_i=v_iS_i$$

$$l_{i-1/2}\equiv\frac{2x_{i-1/2}^{\alpha}D_{i-1}D_i}{h_iD_{i-1}+h_{i-1}D_i};\qquad w_i\equiv\Sigma_{r,i}v_i;\qquad q_i\equiv v_iS_i$$

$$l_{i-1/2}\left(\phi_i-\phi_{i-1}\right)-l_{i+1/2}\left(\phi_{i+1}-\phi_i\right)+w_i\phi_i=q_i\qquad i=2,3,\ldots,N-1$$

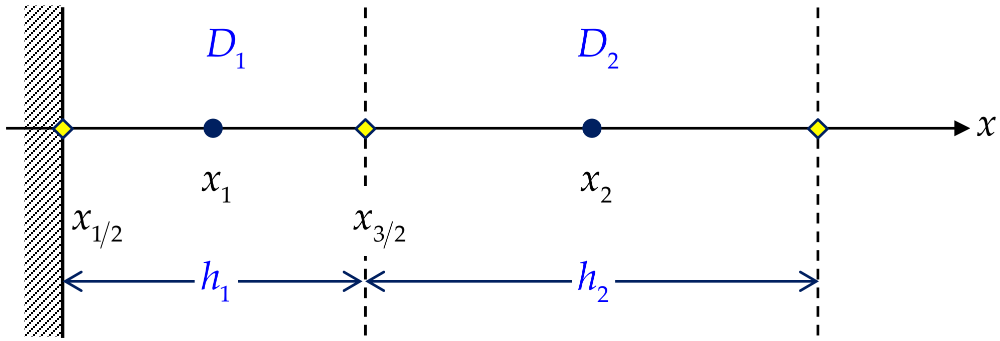
# Three Point Equations

$$a_i \equiv l_{i+1/2} = x_{i+1/2}^{\alpha} \frac{2D_i D_{i+1}}{h_{i+1} D_i + h_i D_{i+1}}$$

$$a_{i-1}\left(\phi_i - \phi_{i-1}\right) - a_i\left(\phi_{i+1} - \phi_i\right) + w_i \phi_i = q_i \qquad i = 2,3,\ldots,N-1$$

$$-a_{i-1}\phi_{i-1} + c_i\phi_i - a_i\phi_{i+1} = q_i; \qquad c_i \equiv a_{i-1} + a_i + w_i; \qquad i = 2,3,\ldots,N-1$$

# Left Boundary

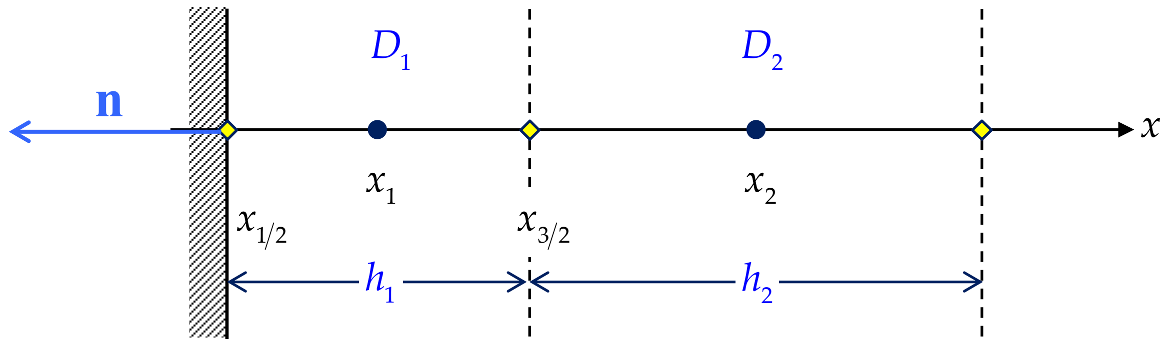



$$x_{3/2}^{\alpha} J_{3/2} - x_{1/2}^{\alpha} J_{1/2} + \int_{x_{1/2}}^{x_{3/2}} \Sigma_r \phi x^{\alpha} dx = \int_{x_{1/2}}^{x_{3/2}} S x^{\alpha} dx$$

$$J_{3/2} \equiv -D\phi'_{3/2} \approx -D_1 \frac{\phi_{3/2} - \phi_1}{h_1/2} = -\frac{2D_1 D_2}{h_2 D_1 + h_1 D_2}\left(\phi_2 - \phi_1\right)$$

# Left BC

$$\tilde{D}\frac{\partial \phi}{\partial n} + \gamma\phi = 0$$




$$-\tilde{D}\frac{\phi_1 - \phi_{1/2}}{h_1/2} + \gamma\phi_{1/2} = 0 \longrightarrow \phi_{1/2} = \frac{2\tilde{D}}{2\tilde{D} + \gamma h_1}\phi_1$$

$$J_{1/2} \equiv -D_1\phi'_{1/2} \approx -D_1\frac{\phi_1 - \phi_{1/2}}{h_1/2} = -\frac{2\gamma D_1}{2\tilde{D} + \gamma h_1}\phi_1$$

# FD Equation at the Left

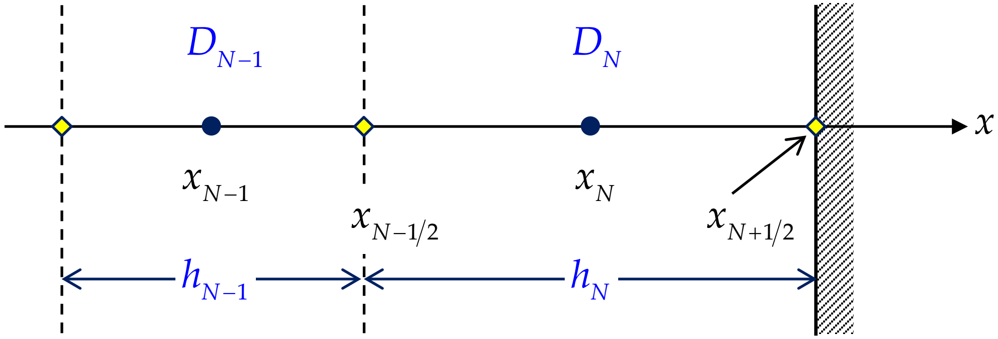
$$\left( x_{1/2}^{\alpha} \frac{2\gamma D_1}{2\tilde{D} + \gamma h_1} + a_1 + v_1 \Sigma_{r,1} \right) \phi_1 - a_1 \phi_2 = v_1 S_1$$

$w_1 \equiv v_1 \Sigma_{r,1}$ $\quad$ $q_1 \equiv v_1 S_1$ $\quad$ $c_1 \phi_1 - a_1 \phi_2 = q_1$

$$c_1 \equiv x_{1/2}^{\alpha} \frac{2\gamma D_1}{2\tilde{D} + \gamma h_1} + a_1 + w_1$$

$$c_i \equiv a_{i-1} + a_i + w_i$$

# Right BC




$$\tilde{D}\frac{\partial\phi}{\partial n}+\gamma\phi=0$$

$$-a_{N-1}\phi_{N-1}+c_N\phi_N=q_N$$

$$-a_{N-1}\phi_{N-1}+\left(a_{N-1}+x_{N+1/2}^{\alpha}\frac{2\gamma D_N}{2\tilde{D}+\gamma h_N}+v_N\Sigma_{r,N}\right)\phi_N=v_N S_N$$

$$\left(a_1+x_{1/2}^{\alpha}\frac{2\gamma D_1}{2\tilde{D}+\gamma h_1}+v_1\Sigma_{r,1}\right)\phi_1$$

# Three Point Equations

$$
\begin{cases}
c_1\phi_1 - a_1\phi_2 = q_1 & i = 1 \\
-a_{i-1}\phi_{i-1} + c_i\phi_i - a_i\phi_{i+1} = q_i & i = 2,\ldots,N-1 \\
-a_{N-1}\phi_{N-1} + c_N\phi_N = q_N & i = N
\end{cases}
$$

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{q}$$

# Tridiagonal System

$$\mathbf{A} = \begin{bmatrix} c_1 & -a_1 & & & & 0 \\ -a_1 & c_2 & -a_2 & & & \\ & -a_2 & c_2 & -a_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -a_{N-2} & c_{N-1} & -a_{N-1} \\ 0 & & & & -a_{N-1} & c_N \end{bmatrix}$$

$$\left(\mathbf{A}^{-1}\right)_{i,j} \geq 0 \longrightarrow \boldsymbol{\phi} = \mathbf{A}^{-1}\mathbf{q} \geq 0$$

# Remark on FD Equations

$$\frac{2x_{i-1/2}^{\alpha}D_{i-1}D_i}{h_iD_{i-1}+h_{i-1}D_i}\left(\phi_i-\phi_{i-1}\right)-\frac{2x_{i+1/2}^{\alpha}D_iD_{i+1}}{h_{i+1}D_i+h_iD_{i+1}}\left(\phi_{i+1}-\phi_i\right)+v_i\Sigma_{r,i}\phi_i=v_iS_i$$

$$\frac{2x_{i-1/2}^{\alpha}D_{i-1}D_i}{v_i\left(h_iD_{i-1}+h_{i-1}D_i\right)}\left(\phi_i-\phi_{i-1}\right)-\frac{2x_{i+1/2}^{\alpha}D_iD_{i+1}}{v_i\left(h_{i+1}D_i+h_iD_{i+1}\right)}\left(\phi_{i+1}-\phi_i\right)+\Sigma_{r,i}\phi_i=S_i$$

# Special Case

$$\alpha = 0; \quad D_i = D; \quad \Sigma_{r,i} = \Sigma_r; \quad h_i = h$$

$$v_i \equiv \frac{x_{i+1/2}^{\alpha+1} - x_{i-1/2}^{\alpha+1}}{\alpha+1} = h; \quad w_i \equiv \Sigma_{r,i} v_i = h\Sigma_r; \quad q_i \equiv v_i S_i = hS_i$$

$$a_i = x_{i+1/2}^{\alpha} \frac{2D_i D_{i+1}}{h_{i+1} D_i + h_i D_{i+1}} = \frac{D}{h}$$

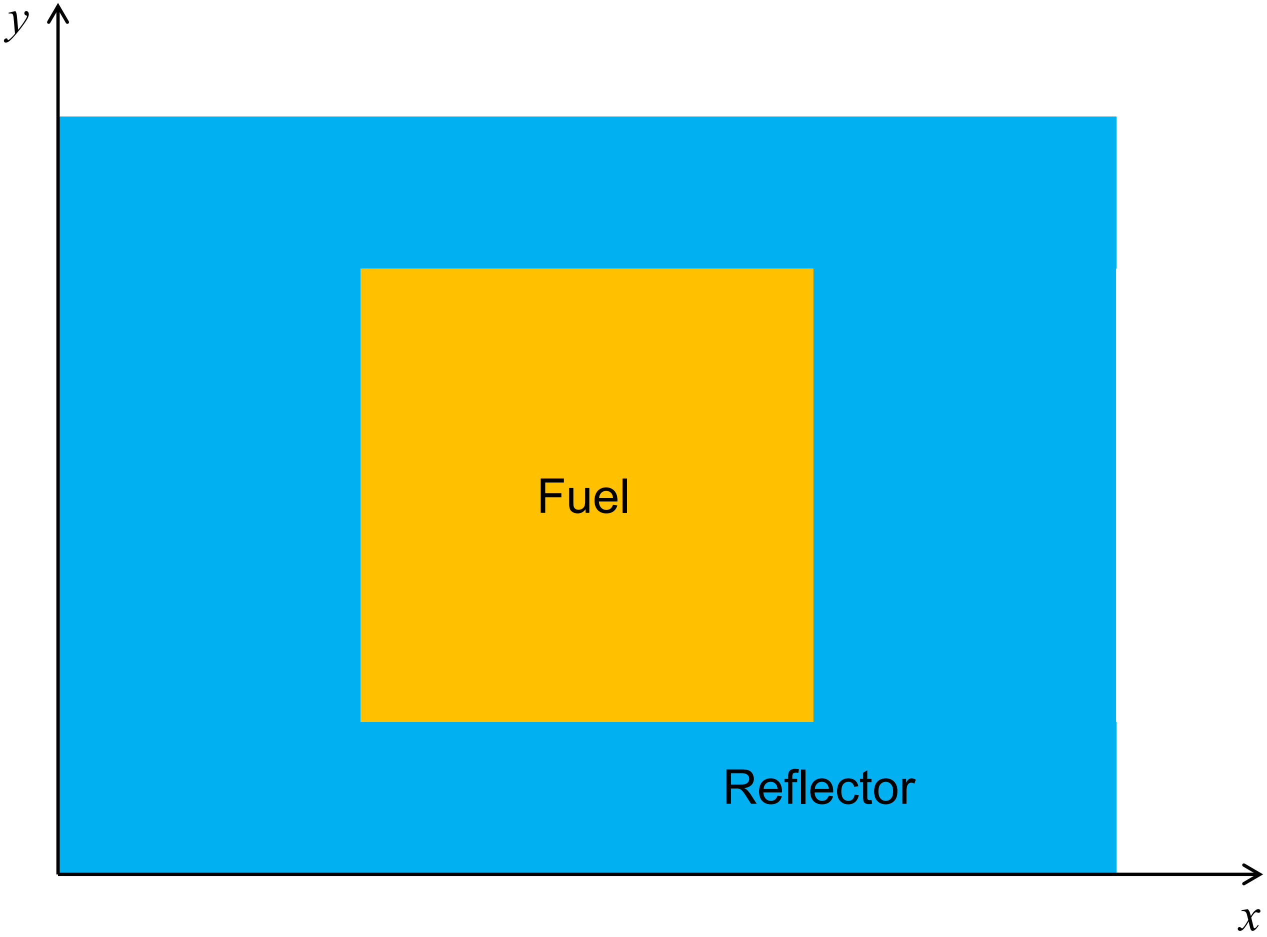
# FD Equations in Special Case

$$-a_{i-1}\phi_{i-1} + c_i\phi_i - a_i\phi_{i+1} = q_i; \qquad c_i \equiv a_{i-1} + a_i + w_i$$

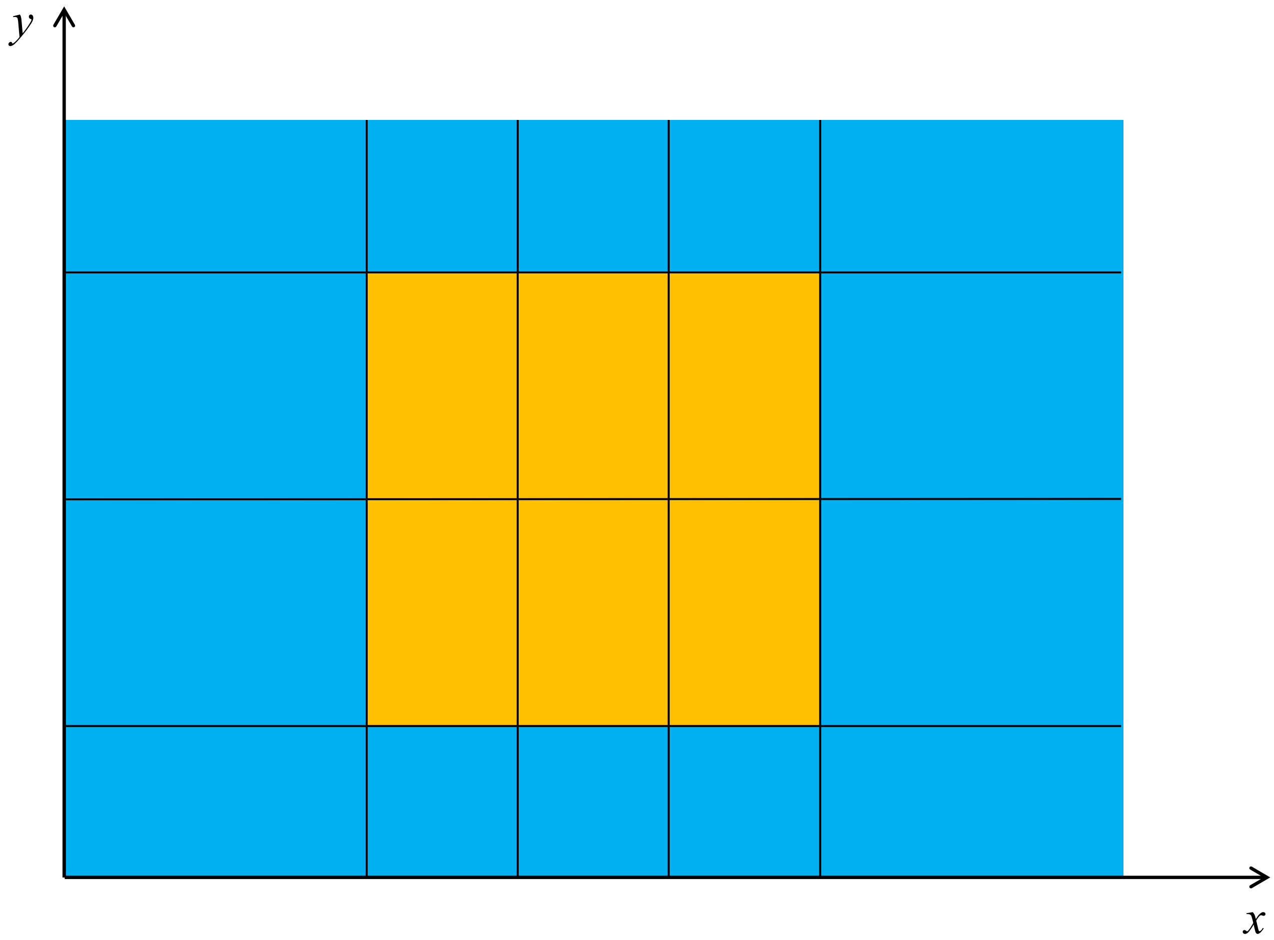
$$-\frac{D}{h}\phi_{i-1} + \left(\frac{2D}{h} + h\Sigma_r\right)\phi_i - \frac{D}{h}\phi_{i+1} = hS_i$$

$$-D\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{h^2} + \Sigma_r\phi_i = S_i$$

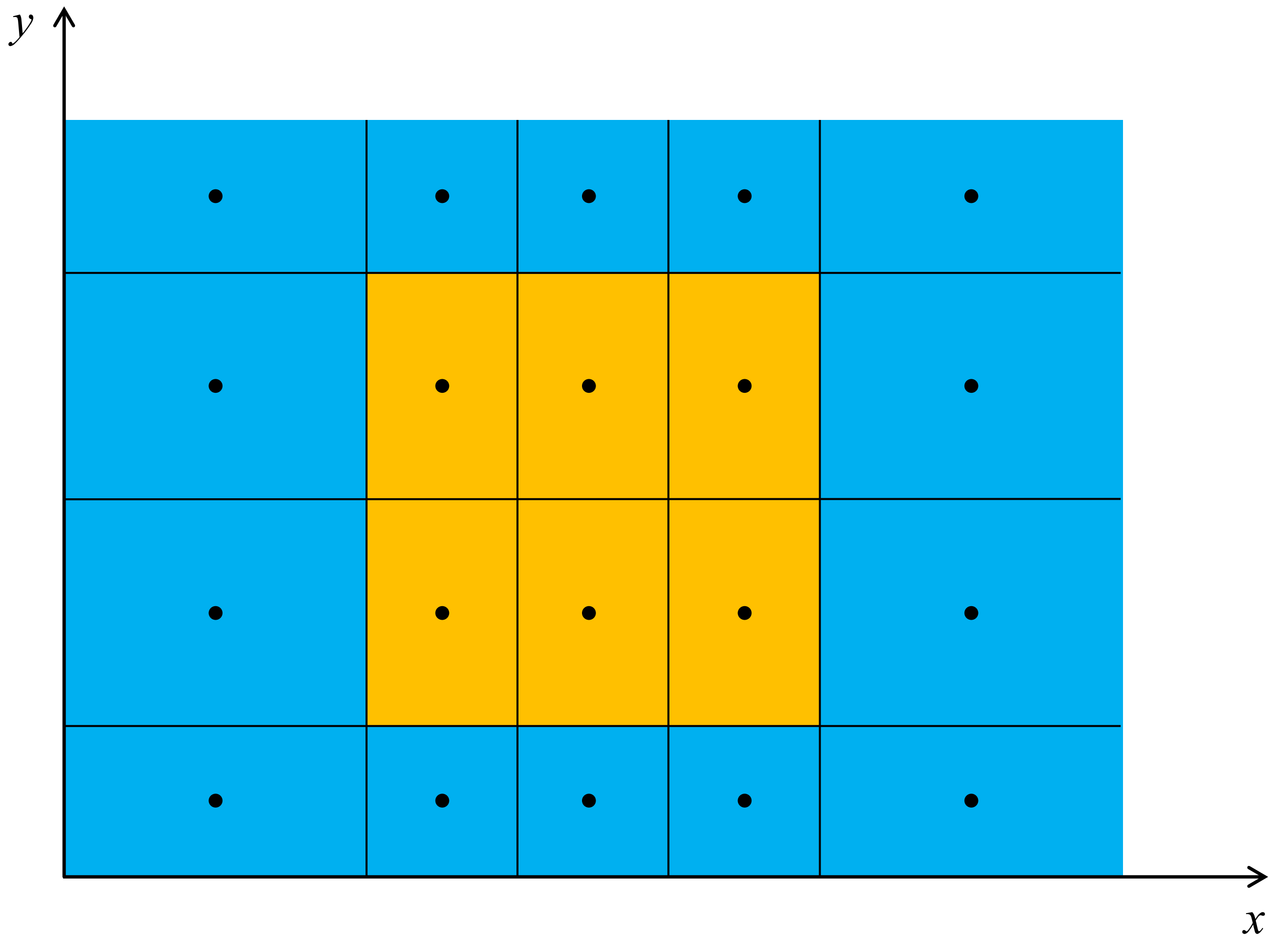
# 2D Model



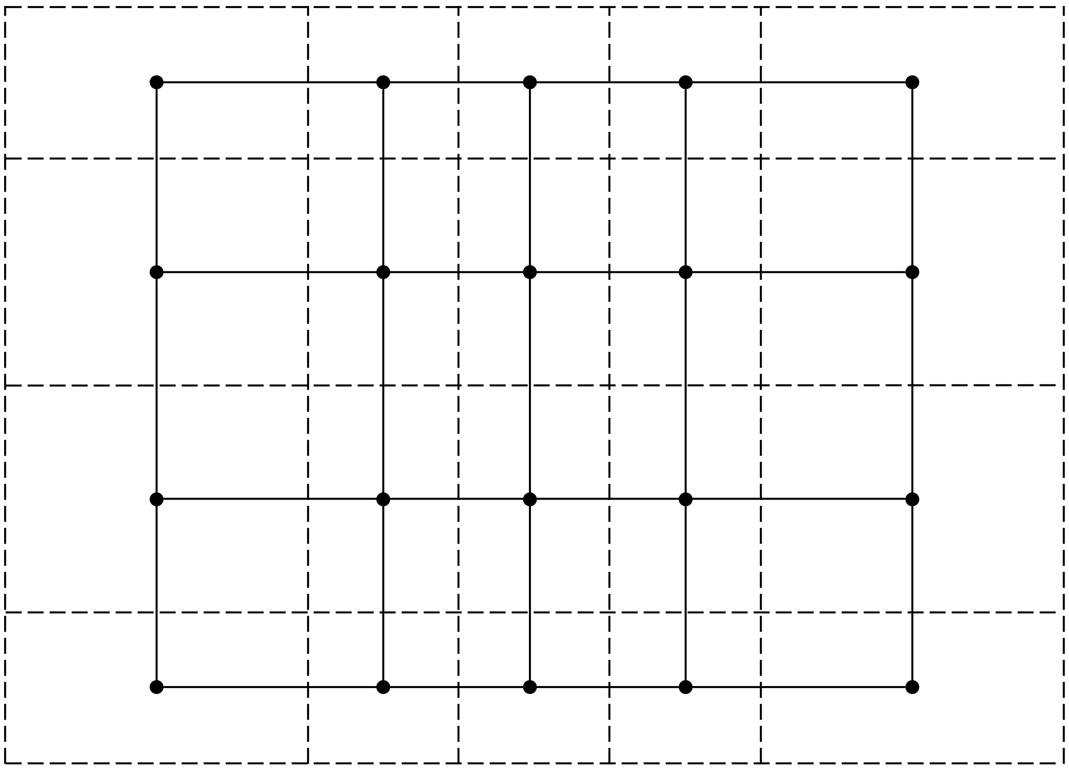
# Subintervals



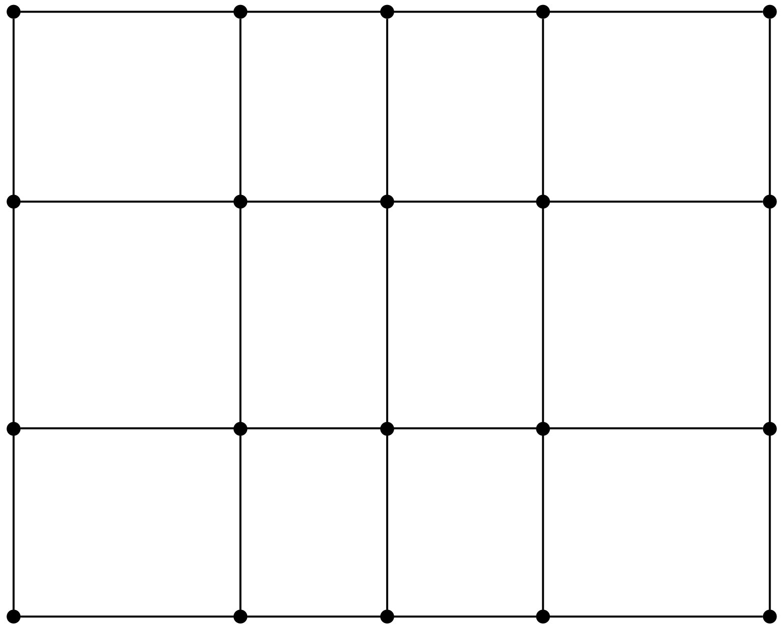
# FD Nodes



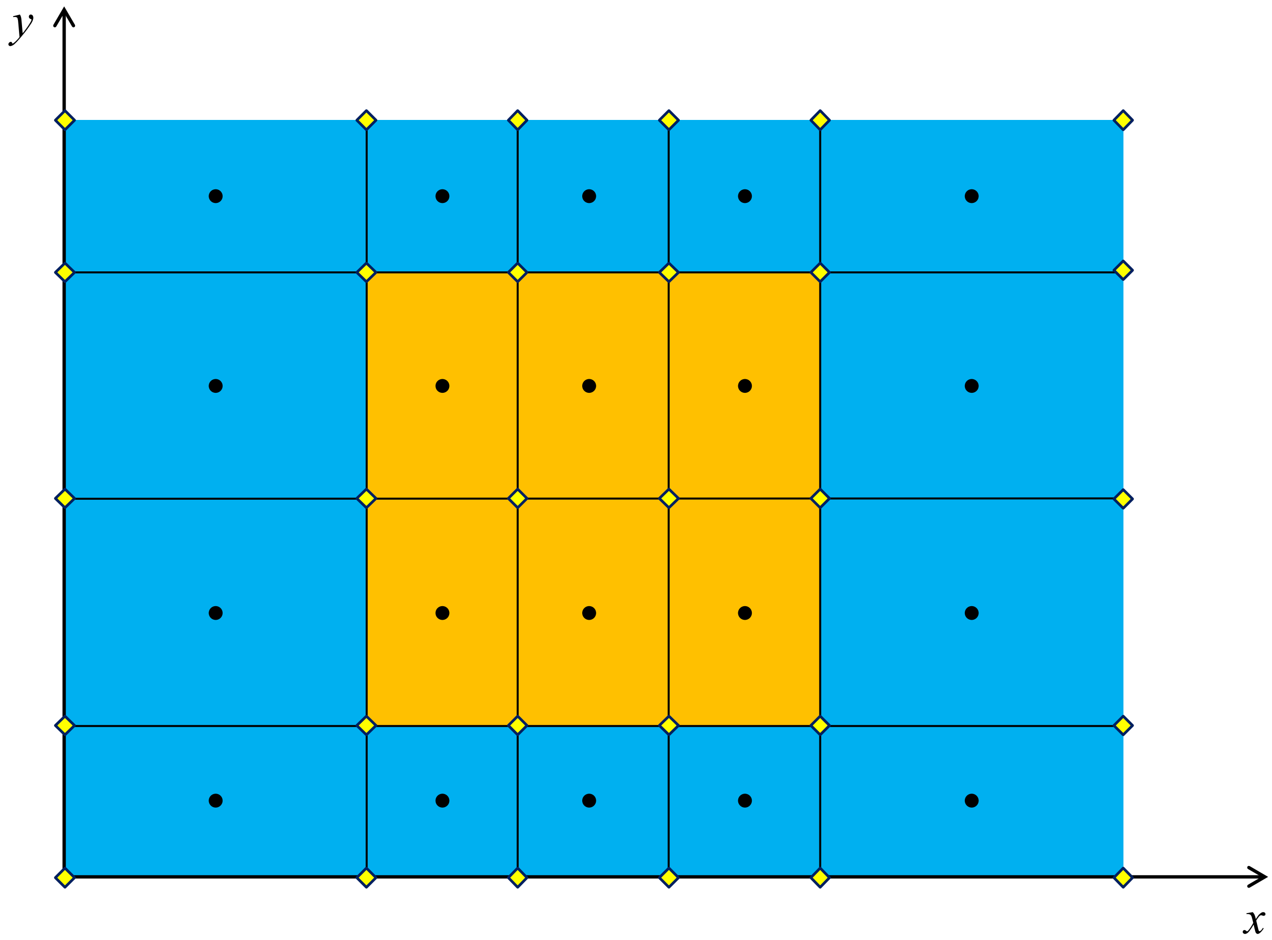
# FD Mesh and Cells



# FD Mesh



# Another Kind of FD Nodes



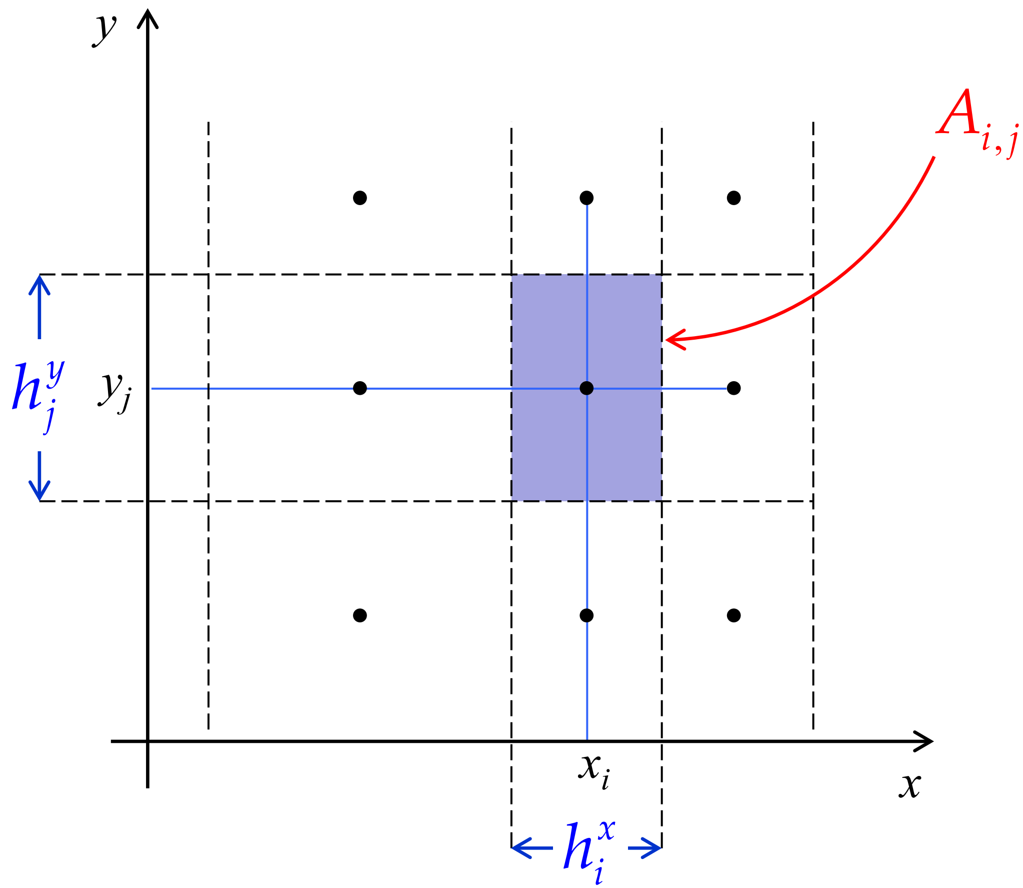
# NDE Equations in 2D

$$div\mathbf{J} + \Sigma_r \phi = S$$

$$\mathbf{J} = -D\left(\frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y\right)$$

$$div\mathbf{J} = -D\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

$$\iint_{A_{i,j}} dA$$



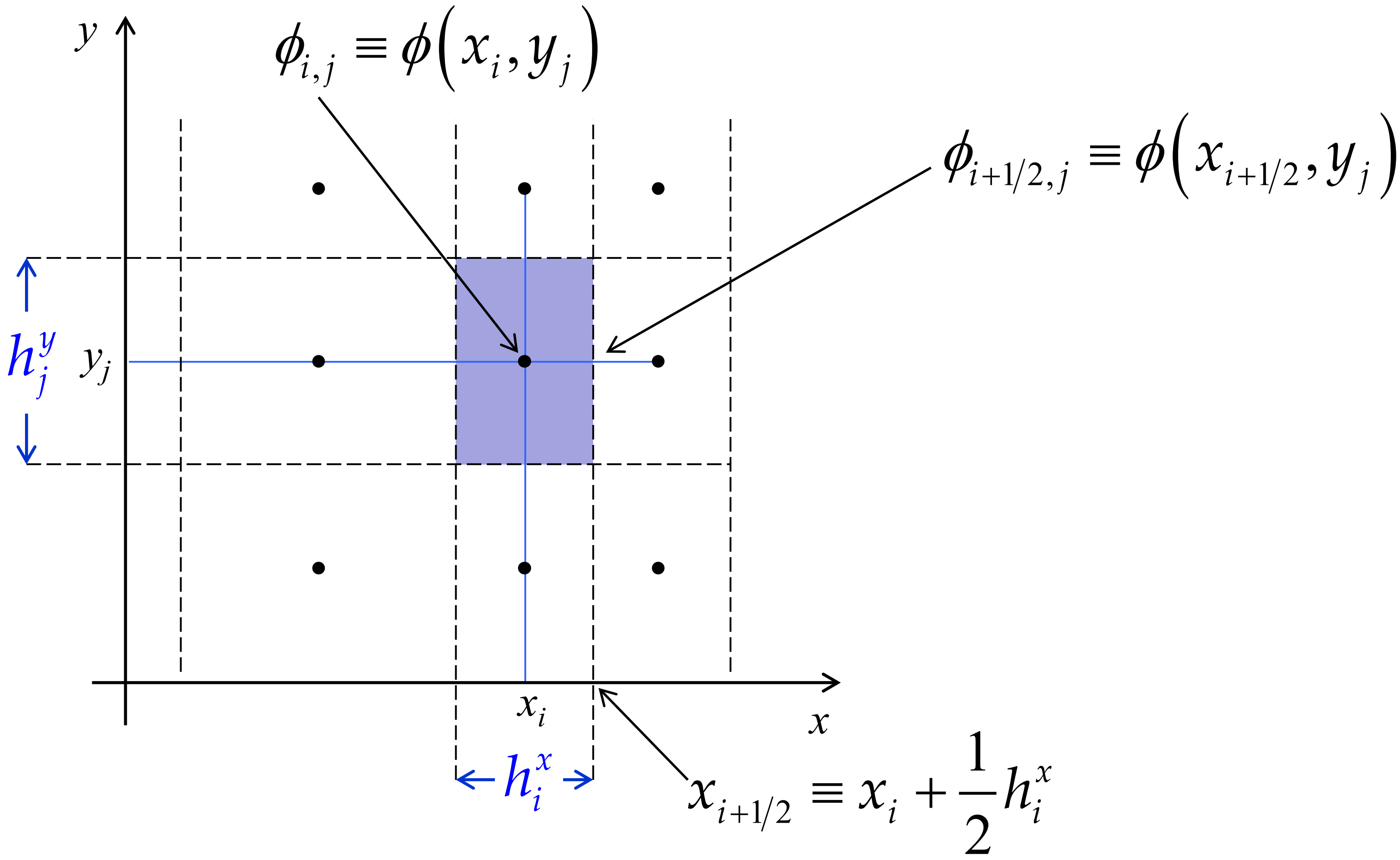
# Integrating Parts of NDE

$$\iint\limits_{A_{i,j}} S(x,y)\,dA \approx S_{i,j}A_{i,j} \qquad \iint\limits_{A_{i,j}} \Sigma_r \phi(x,y)\,dA \approx \Sigma_{r,i,j}\phi_{i,j}A_{i,j}$$

$$\iint\limits_{A_{i,j}} div\mathbf{J}\,dA = -D_{i,j}\left(\iint\limits_{A_{i,j}} \frac{\partial^2\phi}{\partial x^2}dA + \iint\limits_{A_{i,j}} \frac{\partial^2\phi}{\partial y^2}dA\right)$$

$$I_1 \equiv D_{i,j}\iint\limits_{A_{i,j}} \frac{\partial^2\phi}{\partial x^2}dA \qquad I_2 \equiv D_{i,j}\iint\limits_{A_{i,j}} \frac{\partial^2\phi}{\partial y^2}dA$$

# Compact Notation



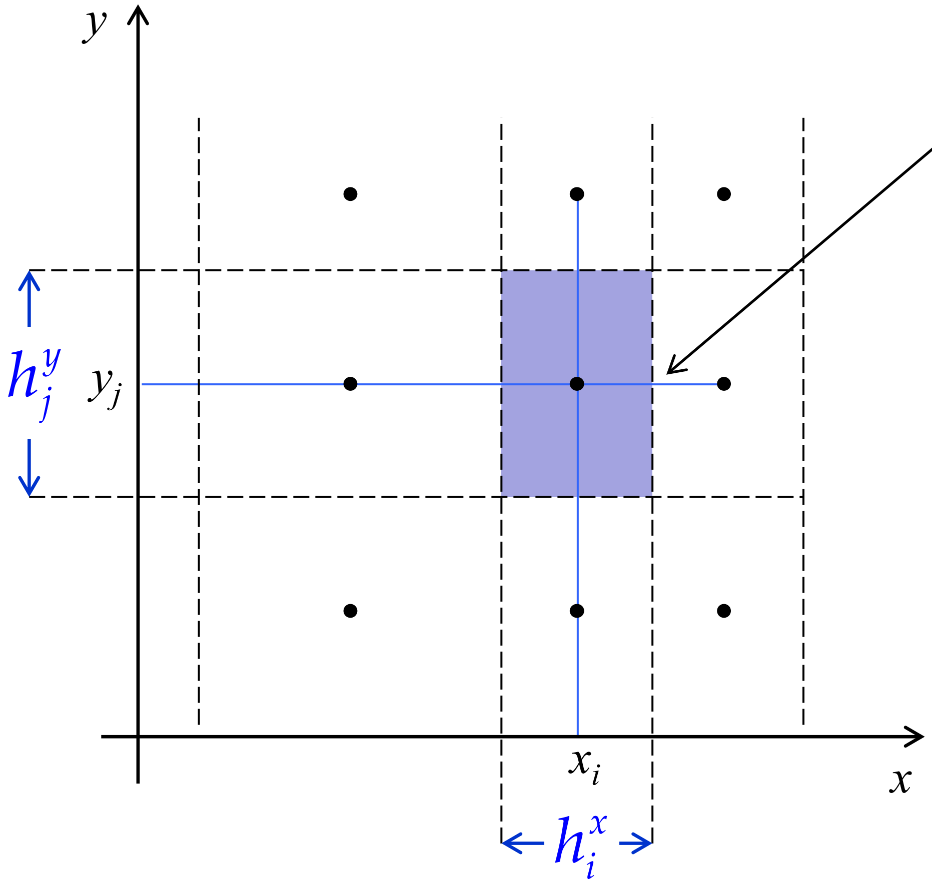
# Integral $I_1$

$$I_1 = D_{i,j} \int_{y_{j-1/2}}^{y_{j+1/2}} dy \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial^2 \phi}{\partial x^2} dx =$$

$$= D_{i,j} \int_{y_{j-1/2}}^{y_{j+1/2}} \left[ \frac{\partial \phi\left(x_{i+1/2}, y\right)}{\partial x} - \frac{\partial \phi\left(x_{i-1/2}, y\right)}{\partial x} \right] dy \approx$$

$$\approx D_{i,j} \left[ \frac{\partial \phi\left(x_{i+1/2}, y_j\right)}{\partial x} - \frac{\partial \phi\left(x_{i-1/2}, y_j\right)}{\partial x} \right] h_j^y$$

# Continuity Condition




$$\frac{\partial \phi\left(x_{i+1/2}, y_j\right)}{\partial x}$$

$$J_x\left(x_{i+1/2} - 0\right) = J_x\left(x_{i+1/2} + 0\right)$$

$$D_{i,j} \frac{\phi_{i+1/2,j} - \phi_{i,j}}{h_i^x / 2} =$$

$$= D_{i+1,j} \frac{\phi_{i+1,j} - \phi_{i+1/2,j}}{h_{i+1}^x / 2}$$

# Flux at Cell Boundary

$$D_{i,j}\frac{\phi_{i+1/2,j}-\phi_{i,j}}{h_i^x/2}=D_{i+1,j}\frac{\phi_{i+1,j}-\phi_{i+1/2,j}}{h_{i+1}^x/2}$$

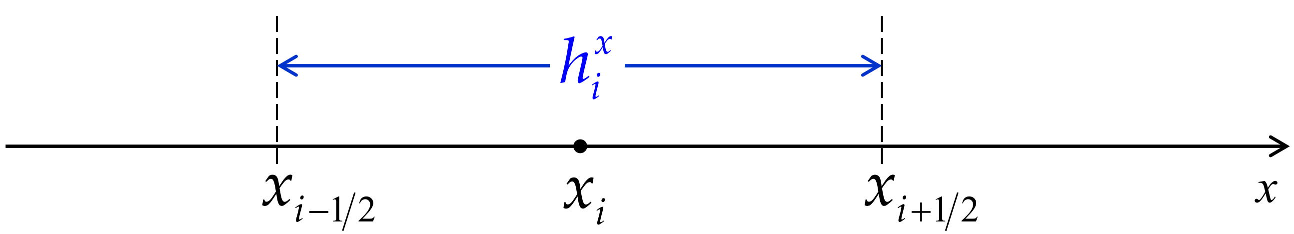
$$\phi_{i+1/2,j}=\frac{h_{i+1}^x D_{i,j}\phi_{i,j}+h_i^x D_{i+1,j}\phi_{i+1,j}}{h_{i+1}^x D_{i,j}+h_i^x D_{i+1,j}}$$

$$\phi_{i-1/2,j}=\frac{h_i^x D_{i-1,j}\phi_{i-1,j}+h_{i-1}^x D_{i,j}\phi_{i,j}}{h_i^x D_{i-1,j}+h_{i-1}^x D_{i,j}}$$

# Derivative at Cell Boundary

$$\frac{\partial \phi\left(x_{i+1/2}, y_j\right)}{\partial x} \approx \frac{\phi_{i+1/2,j} - \phi_{i,j}}{h_i^x/2} = \frac{2D_{i+1,j}}{h_{i+1}^x D_{i,j} + h_i^x D_{i+1,j}}\left(\phi_{i+1,j} - \phi_{i,j}\right)$$

$$\frac{\partial \phi\left(x_{i-1/2}, y_j\right)}{\partial x} \approx \frac{\phi_{i,j} - \phi_{i-1/2,j}}{h_i^x/2} = \frac{2D_{i-1,j}}{h_i^x D_{i-1,j} + h_{i-1}^x D_{i,j}}\left(\phi_{i,j} - \phi_{i-1,j}\right)$$



# Integral $I_1$

$$I_1 \approx D_{i,j}\left[\frac{\partial\phi\left(x_{i+1/2}, y_j\right)}{\partial x} - \frac{\partial\phi\left(x_{i-1/2}, y_j\right)}{\partial x}\right] h_j^y \approx$$

$$\approx \frac{2h_j^y D_{i,j} D_{i+1,j}}{h_{i+1}^x D_{i,j} + h_i^x D_{i+1,j}}\left(\phi_{i+1,j} - \phi_{i,j}\right) -$$

$$- \frac{2h_j^y D_{i-1,j} D_{i,j}}{h_i^x D_{i-1,j} + h_{i-1}^x D_{i,j}}\left(\phi_{i,j} - \phi_{i-1,j}\right)$$

# Integral $I_2$

$x \leftrightarrow y$

$$I_2 \approx D_{i,j}\left[\frac{\partial\phi\left(x_i, y_{j+1/2}\right)}{\partial y} - \frac{\partial\phi\left(x_i, y_{j-1/2}\right)}{\partial y}\right] h_i^x \approx$$

$i \leftrightarrow j$

$$\approx \frac{2h_i^x D_{i,j} D_{i,j+1}}{h_{j+1}^y D_{i,j} + h_j^y D_{i,j+1}}\left(\phi_{i,j+1} - \phi_{i,j}\right) -$$

$$-\frac{2h_i^x D_{i,j-1} D_{i,j}}{h_j^y D_{i,j-1} + h_{j-1}^y D_{i,j}}\left(\phi_{i,j} - \phi_{i,j-1}\right)$$

# Exact Balance in $A_{i,j}$

$$div\mathbf{J} + \Sigma_r \phi = S$$

$$-D_{i,j}\iint_{A_{i,j}} \frac{\partial^2 \phi}{\partial x^2} dA - D_{i,j}\iint_{A_{i,j}} \frac{\partial^2 \phi}{\partial y^2} dA + \iint_{A_{i,j}} \Sigma_r \phi(x,y) dA = \iint_{A_{i,j}} S(x,y) dA$$

# Approximate Balance in $A_{i,j}$

$$-D_{i,j}\iint_{A_{i,j}}\frac{\partial^2\phi}{\partial x^2}dA - D_{i,j}\iint_{A_{i,j}}\frac{\partial^2\phi}{\partial y^2}dA + \iint_{A_{i,j}}\Sigma_r\phi(x,y)dA = \iint_{A_{i,j}}S(x,y)dA$$

$$-\frac{2h_j^y D_{i,j}D_{i+1,j}}{h_{i+1}^x D_{i,j} + h_i^x D_{i+1,j}}\left(\phi_{i+1,j}-\phi_{i,j}\right) + \frac{2h_j^y D_{i-1,j}D_{i,j}}{h_i^x D_{i-1,j} + h_{i-1}^x D_{i,j}}\left(\phi_{i,j}-\phi_{i-1,j}\right) -$$

$$-\frac{2h_i^x D_{i,j}D_{i,j+1}}{h_{j+1}^y D_{i,j} + h_j^y D_{i,j+1}}\left(\phi_{i,j+1}-\phi_{i,j}\right) + \frac{2h_i^x D_{i,j-1}D_{i,j}}{h_j^y D_{i,j-1} + h_{j-1}^y D_{i,j}}\left(\phi_{i,j}-\phi_{i,j-1}\right) +$$

$$+h_i^x h_j^y \Sigma_{r,i,j}\phi_{i,j} = h_i^x h_j^y S_{i,j}$$

# Compact Notation

$$-\underbrace{\frac{2h_j^y D_{i,j} D_{i+1,j}}{h_{i+1}^x D_{i,j} + h_i^x D_{i+1,j}}}_{a_{i,j}} \left(\phi_{i+1,j} - \phi_{i,j}\right) + \underbrace{\frac{2h_j^y D_{i-1,j} D_{i,j}}{h_i^x D_{i-1,j} + h_{i-1}^x D_{i,j}}}_{a_{i-1,j}} \left(\phi_{i,j} - \phi_{i-1,j}\right) -$$

$$-\underbrace{\frac{2h_i^x D_{i,j} D_{i,j+1}}{h_{j+1}^y D_{i,j} + h_j^y D_{i,j+1}}}_{b_{i,j}} \left(\phi_{i,j+1} - \phi_{i,j}\right) + \underbrace{\frac{2h_i^x D_{i,j-1} D_{i,j}}{h_j^y D_{i,j-1} + h_{j-1}^y D_{i,j}}}_{b_{i,j-1}} \left(\phi_{i,j} - \phi_{i,j-1}\right) +$$

$$+\underbrace{h_i^x h_j^y \Sigma_{r,i,j}}_{w_{i,j}} \phi_{i,j} = \underbrace{h_i^x h_j^y S_{i,j}}_{q_{i,j}}$$

# FD in Compact Form

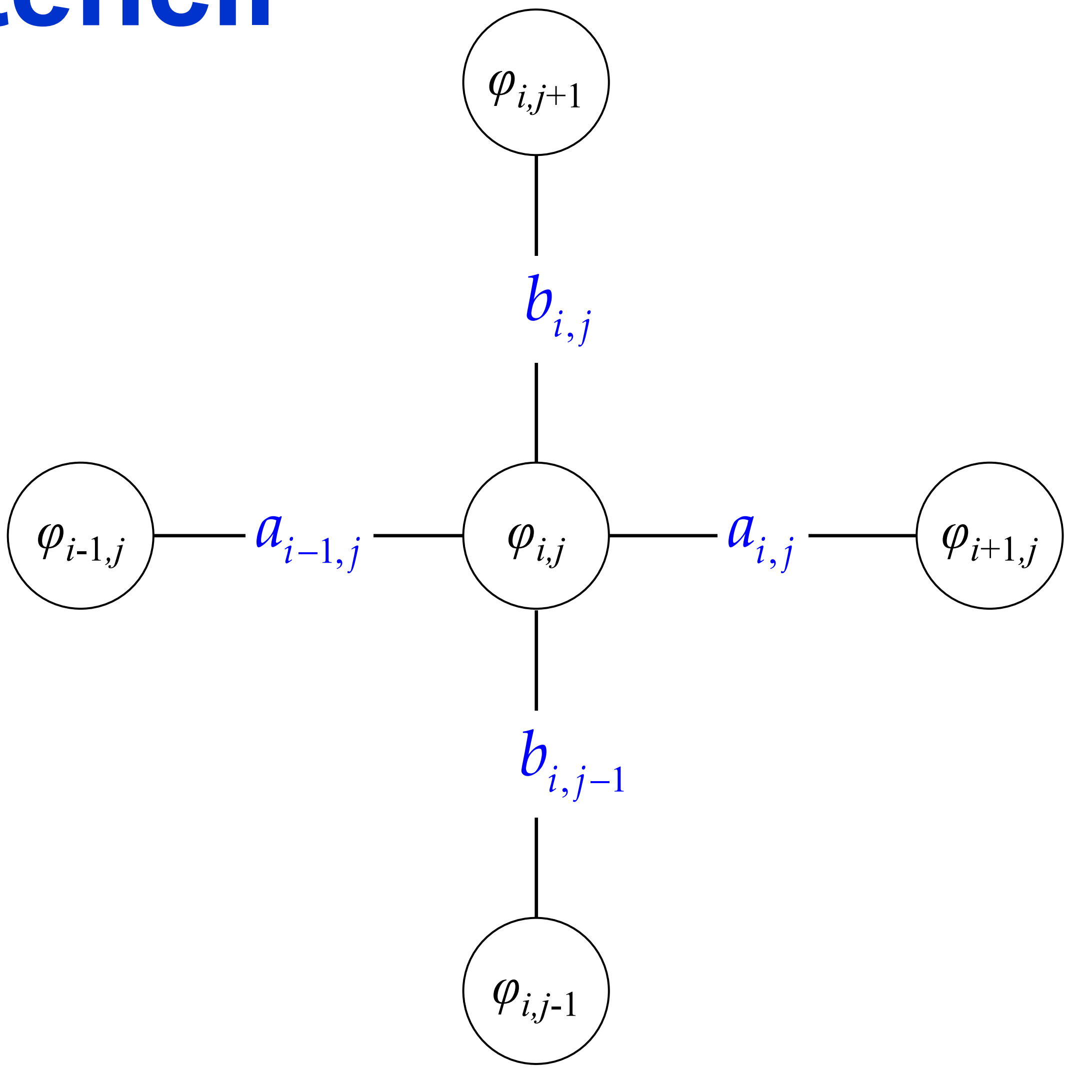
$$-a_{i,j}\left(\phi_{i+1,j}-\phi_{i,j}\right)+a_{i-1,j}\left(\phi_{i,j}-\phi_{i-1,j}\right)-$$

$$-b_{i,j}\left(\phi_{i,j+1}-\phi_{i,j}\right)+b_{i,j-1}\left(\phi_{i,j}-\phi_{i,j-1}\right)+w_{i,j}\phi_{i,j}=q_{i,j}$$

$$-b_{i,j-1}\phi_{i,j-1}-a_{i-1,j}\phi_{i-1,j}+c_{i,j}\phi_{i,j}-a_{i,j}\phi_{i+1,j}-b_{i,j}\phi_{i,j+1}=q_{i,j}$$

$$c_{i,j}\equiv a_{i-1,j}+a_{i,j}+b_{i,j-1}+b_{i,j}+w_{i,j}$$

# 5-Point Stencil



# FD Equations in Special Case

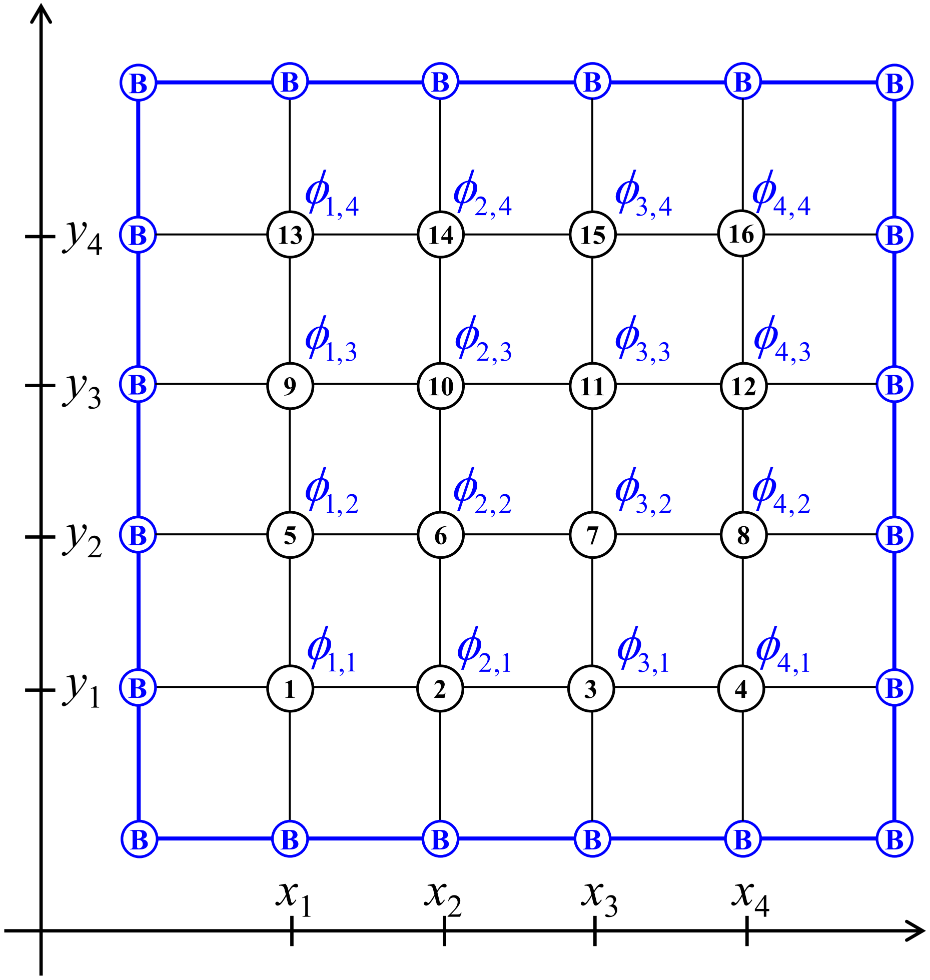
$$-\frac{2h_j^y D_{i,j} D_{i+1,j}}{h_{i+1}^x D_{i,j} + h_i^x D_{i+1,j}}\left(\phi_{i+1,j} - \phi_{i,j}\right) + \frac{2h_j^y D_{i-1,j} D_{i,j}}{h_i^x D_{i-1,j} + h_{i-1}^x D_{i,j}}\left(\phi_{i,j} - \phi_{i-1,j}\right) -$$

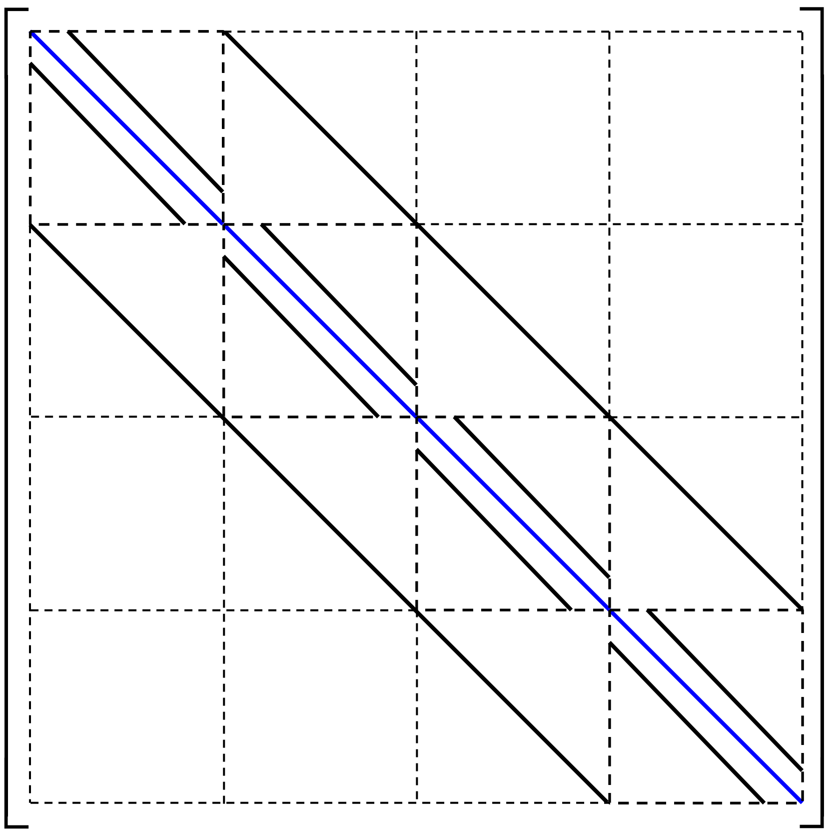
$$-\frac{2h_i^x D_{i,j} D_{i,j+1}}{h_{j+1}^y D_{i,j} + h_j^y D_{i,j+1}}\left(\phi_{i,j+1} - \phi_{i,j}\right) + \frac{2h_i^x D_{i,j-1} D_{i,j}}{h_j^y D_{i,j-1} + h_{j-1}^y D_{i,j}}\left(\phi_{i,j} - \phi_{i,j-1}\right) +$$

$$+h_i^x h_j^y \Sigma_{r,i,j} \phi_{i,j} = h_i^x h_j^y S_{i,j}$$

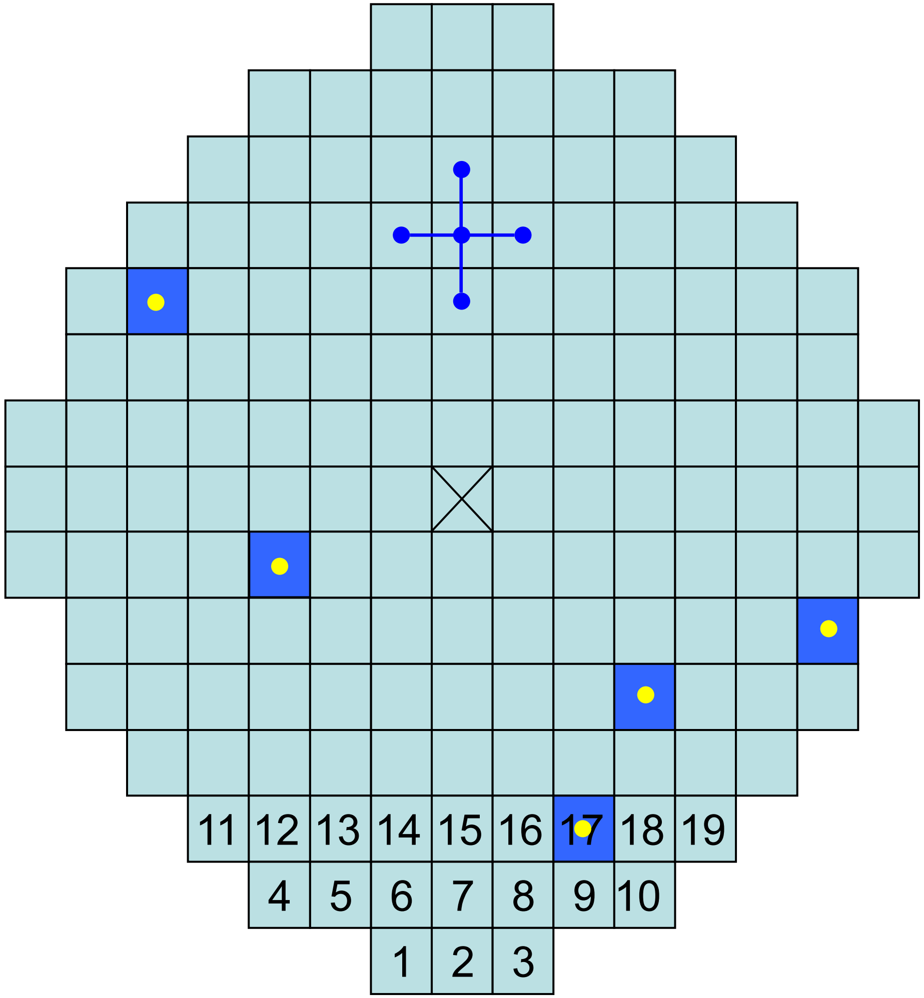
$$-D\left(\frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{h_x^2} + \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{h_y^2}\right) + \Sigma_r \phi_{i,j} = S_{i,j}$$

# Regular FD Mesh



# Square Lattice



# Matrix

$$
\mathbf{A} = \left[\begin{array}{ccc|ccccccc|ccccccccc}
* & * & & & & * & & & & & & & & & & & & & \\
* & * & * & & & & * & & & & & & & & & & & & \\
& * & * & & & & & * & & & & & & & & & & & \\
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& & & & & & & & & * & & & & & & & * & * & * \\
& & & & & & & & & & & & & & & & & * & *
\end{array}\right]
$$

# Important

- Stationary NDE in 1D
- Finite-Difference mesh in 1D
- Integro-Interpolation Method
- Three point FD equations in 1D
- Finite-Difference mesh in 2D
- Five point FD equations in 2D