Nuclear Reactor Physics

Nuclear Reactor Physics: Reactor Stability I

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Contents

Laplace transformation

Laplace transformation

Laplace transformation is useful for analysis of continuous-time dynamical systems. The one-sided Laplace transformation of a time function $f(t)$ is defined as:

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Note

Laplace transformation associates a unique image function $F(s)$ of the complex variable $s = a + jb$ with an original function $f(t)$ with t being real.

Notation

We shall denote Laplace images of original functions by capital letters, and replace the t argument with s.

Basic properties of the Laplace transformation

Laplace transformation is linear

$$L\{f_1(t) + f_2(t)\} = L\{f_1(t)\} + L\{f_2(t)\} = F_1(s) + F_2(s)$$

If c is a constant

$$L\{c\, f(t)\} = c\, L\{f(t)\} = c\, F(s)$$

Laplace transform of a derivative

$$L\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

Laplace transform of the step function $u(t)$

$$u(t) = 1 \text{ for } t > 0,$$

$$u(t) = 0 \text{ for } t \leq 0$$

The Laplace transform of $c\,u(t)$ (where c is a constant) is then

$$L\{c\,u(t)\} = c\int_0^\infty e^{-st}dt = \frac{c}{s}$$

Laplace transform of an exponential function

$$f(t) = e^{-at}$$

The Laplace transform of $f(t)$ (where a is a constant) is then

$$L\{e^{-at}\} = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = \frac{1}{s+a}$$

Basic properties of the Laplace transformation

Laplace transform of the delta function

The Laplace transformation of $\delta(t)$ is

$$\Delta(s) = \int_0^\infty e^{-st}\delta(t)dt = 1$$

Inverse Laplace transformation

Inverse Laplace transform returns the image from the Laplace domain to the original time domain:

$$L^{-1}\{F(s)\} = f(t)$$

For example:

$$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

Inverse Laplace transform of a ratio of two polynomials

If $L(s)$ and $M(s)$ are polynomials with non-zero, single valued roots and the degree of the polynomial $L(s)$ is lower than that of polynomial $M(s)$, for which it is equal to n, then:

$$L^{-1}\left\{\frac{L(s)}{M(s)}\right\} = \sum_{i=1}^{n} \frac{L(s_i)}{M'(s_i)} e^{s_i t}$$

$$L^{-1}\left\{\frac{L(s)}{sM(s)}\right\} = \frac{L(0)}{M(0)} + \sum_{i=1}^{n} \frac{L(s_i)}{s_i M'(s_i)} e^{s_i t}$$

where s_i, $i = 1, \ldots, n$ are roots of polynomial $M(s)$.

Inverse Laplace transform of a ratio of two polynomials

Example:

$$L^{-1}\left\{\frac{s+1}{s^2+5s+6}\right\}$$

Here

$$L(s) = s+1$$

$$M(s) = s^2+5s+6 \Rightarrow M'(s) = 2s+5$$

Since $s_1 = -3$ and $s_2 = -2$

$$L^{-1}\left\{\frac{s+1}{s^2+5s+6}\right\} = \frac{L(s_1)}{M'(s_1)}e^{s_1 t} + \frac{L(s_2)}{M'(s_2)}e^{s_2 t} = 2e^{-3t} - e^{-2t}$$

Transfer functions

Let's consider a system represented by a linear differential equation of order n with constant coefficients:

$$A_n \frac{d^n f_o}{dt^n} + A_{n-1} \frac{d^{n-1} f_o}{dt^{n-1}} + \ldots + A_1 \frac{df_o}{dt} + A_0 f_o = f_i(t)$$

which can be written as

$$Z f_o = f_i$$

where the Z operator is defined as

$$Z = A_n \frac{d^n}{dt^n} + A_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \ldots + A_1 \frac{d}{dt} + A_0$$

Assuming that all initial conditions are zero,

we can write the Laplace transform of

$$Zf_o = f_i$$

as

$$Z(s)F_o(s) = F_i(s)$$

The transfer function

The transfer function of the system is then

$$H(s) = \frac{F_o(s)}{F_i(s)} = \frac{1}{Z(s)}$$

Note

So, the transfer system of a linear system is not dependent on the input function $f_i(t)$.

Transfer function

A transfer function $H(s)$ is the ratio of the output $F_o(s)$ of a system to the input $F_i(s)$ of a system in the Laplace domain, assuming its initial conditions and equilibrium point to be zero.

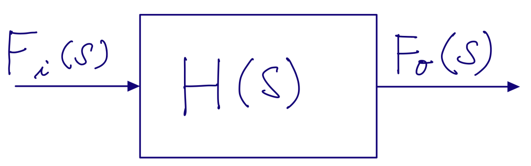


Figure 1:

- Transfer functions are used to refer to linear, time-invariant systems.
- Real systems have usually non-linear input/output characteristics.
- Nevertheless, many systems have behaviour that is close to linear when operated within nominal parameters and the transfer function is then an acceptable representation of the input/output behaviour.

Poles of the transfer function

Poles of the transfer function

$$H(s) = \frac{1}{Z(s)}$$

are the roots of

$$Z(s) = 0$$

which is called the “characteristic equation”.

If we know the transfer function H of a linear system

then its response $f_o(t)$ to an input function $f_i(t)$ can be obtained by the inverse Laplace transform of $F_o(s)$,

$$F_o(s) = H(s)F_i(s)$$

Impulse Response

Let's consider the delta function $\delta(t)$ (infinity for $t = 0$, zero for $t <> 0$) as the input function.

The Laplace transformation of $\delta(t)$ is

$$\Delta(s) = \int_0^{\infty} e^{-st}\delta(t)dt = 1$$

Therefore an impulse response (response to the delta function) is

$$F_o(s) = H(s)\Delta(s) = H(s)$$

Hence, the transfer function $H(s)$ is also the Laplace transform of the system impulse response.

Transfer functions in reactor physics

The system input and output functions can be various combinations of quantities. For instance, the input function can be:

- externally controlled reactivity,
- intensity of an external neutron source,
- coolant mass flow rate,
- coolant inlet temperature, etc.

The output function can be e.g.:

- reactor power,
- average temperature of the fuel,
- coolant outlet temperature, etc.

Hence, many various transfer functions exist for a reactor.

Frequency response

Complex numbers

Let's summarise some facts about complex numbers. A complex number z can be written in a number of forms:

- $z = x + jy$
- $z = r(\cos\varphi + j\sin\varphi)$, where:
  - $r = \sqrt{x^2 + y^2} = |z|$
  - $\varphi = \arg z = \mathrm{arctg}\frac{\mathrm{Im} z}{\mathrm{Re} z}$ is the angle from the positive real axis to the vector representing z
- $z = re^{j\varphi}$

We can write the input harmonic functions as

$$f_i(t) = \sin\omega_0 t = \mathrm{Im}(e^{j\omega_0 t})$$

Frequency Response

- Frequency response shows how the system responds to a harmonic input function.
- The system response can be e.g. reactor power, and the input function can be e.g. the externally controlled reactivity.

The input function is $f_i(t) = \sin \omega_0 t$. The Laplace transform of $f_i(t)$ is

$$F_i(s) = \frac{\omega_0}{s^2 + \omega_0^2}$$

The Laplace transform of the output $f_o(t)$ is

$$F_o(s) = H(s)\frac{\omega_0}{s^2 + \omega_0^2}$$

Frequency Response

And, the inverse Laplace transform of

$$F_o(s) = H(s)\frac{\omega_0}{s^2 + \omega_0^2}$$

is

$$f_o(t) = |H(j\omega_0)|\sin(\omega_o t + \theta) + T(t)$$

where

$$\theta = \arg H(j\omega_0)$$

and $T(t)$ decays if the real parts of all poles of H are negative.

System stability

This implies a condition for system stability: the system is stable when the real parts of all poles of the system transfer function are negative.

Frequency Response

Frequency response

So, the frequency response of a linear system is characterised by:

- the amplitude of the output function $f_o(t)$ and by
- the phase shift θ by which the response is delayed in time with respect to the input harmonic function.

Getting the frequency response

To get the frequency response to a harmonic input function $\sin \omega_0 t$, simply:

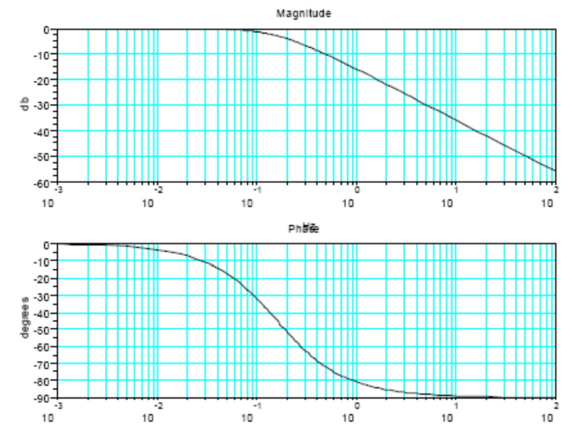
- substitute the complex number s in the transfer function $H(s)$ by $j\omega_0$, and compute
- the amplitude of the output function as $|H(j\omega_0)|$,
- the phase shift θ as $\arg H(j\omega_0)$.

Frequency Response

Bode plots

Bode plots display separately the amplitude and the phase shift as a function of ω_0.

Example of Bode plots - amplitude in dB = 20 log(amplitude)



The frequency response

- The frequency response (and the transfer function) can be used for an analysis of reactor stability.
- The transfer function depends on the actual reactor conditions. So, when the reactor power is increased the transfer function will change. Nevertheless, transfer functions measured at a low power may suggest presence of instability at a large power.
- It is a common practice to determine the frequency response of a reactor before it is operated at full power.

Determination of frequency response

Determination of frequency response

- The frequency response can be either calculated (when the system transfer function is known) or it can be measured.
- The simplest method of measuring the frequency response is the “reactor oscillator method”.

Reactor oscillator method

- In the reactor oscillator method, the reactivity is subject to small sinusoidal oscillations (e.g. by the in-and-out motion of a control rod).
- The resulting changes in power, in amplitude and phase (relative to the imposed reactivity variations), are then measured as a function of the frequency of the variations.
- The reactivity oscillations must be small so that the system may be assumed to be linear under the small variations (otherwise the measured transfer function would not be independent of the input).

Problems with the reactor oscillator method

- The measurement of power will contain small statistical fluctuations.
- In order to obtain clean reading of the power variations, the variations of reactivity must be sufficiently large.
- Large variations of the reactivity, however, will invalidate the linear analysis.
- Hence, optimal variations of the reactivity need to be found.
- Reactivity variations due to movements of control rods may not be perfectly sinusoidal, which can complicate the analysis.

Application of the transfer function

Example of measured transfer function for EBR

- Transfer function measured at several power levels for the Experimental Breeder Reactor (EBR-I) using the oscillating-rod technique.
- The reactor is stable at low and moderate powers
- A pronounced resonance peak, suggesting the approach of instability, appears at a higher power.
- Resonance appears at frequency 0.03 Hz (period of about 30s).
- An oscillation of this frequency can be readily controlled by the normal operation of the control rods.

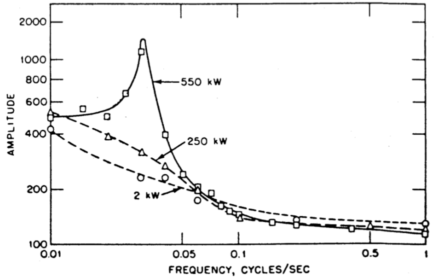


FIG. 9.17 EXPERIMENTAL TRANSFER FUNCTIONS FOR THE EBR-I (AFTER F. W. THALGOTT, *ET AL.*, REF. 57).

Another example of measured transfer function for EBWR

- Transfer function measured at several power levels for the Experimental Boiling Water Reactor (EBWR).
- Resonances appear at frequencies of about 1-2 Hz, and they increase with growing power, suggesting a possible instability.
- The oscillation period is thus 0.5 to 1 sec, which may be too short to be controlled.
- Hence, the design or operating conditions of a reactor must be adjusted to avoid the instability.

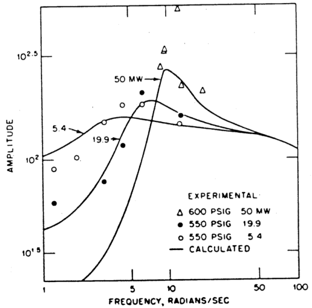


FIG. 9.18 EXPERIMENTAL AND CALCULATED TRANSFER FUNCTIONS FOR THE EBWR (AFTER T. SNYDER AND J. A. THIE, REF. 61).

Demonstration of BWR instability

Demonstration of BWR instability

2D numerical model of BWR

The oscillations are demonstrated on a simple numerical model - an array of 10 channels (assemblies), with reflective boundary conditions.

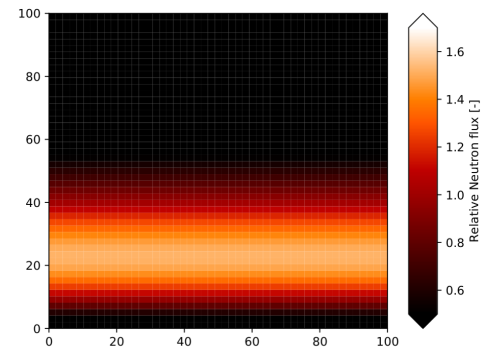
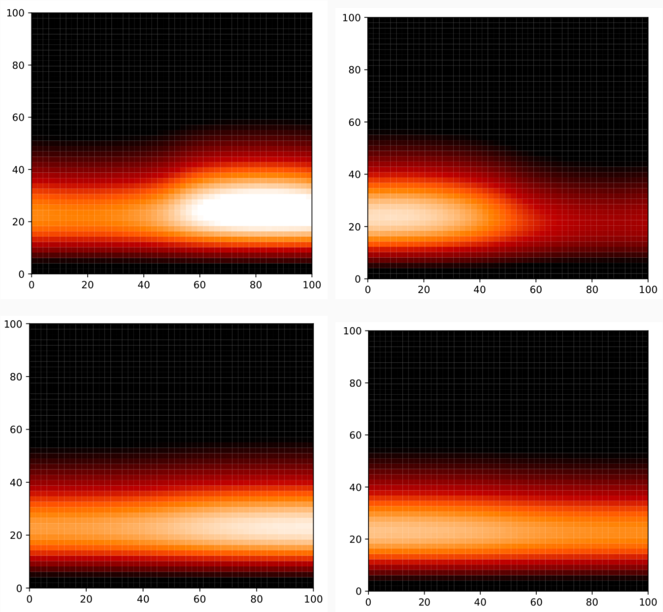
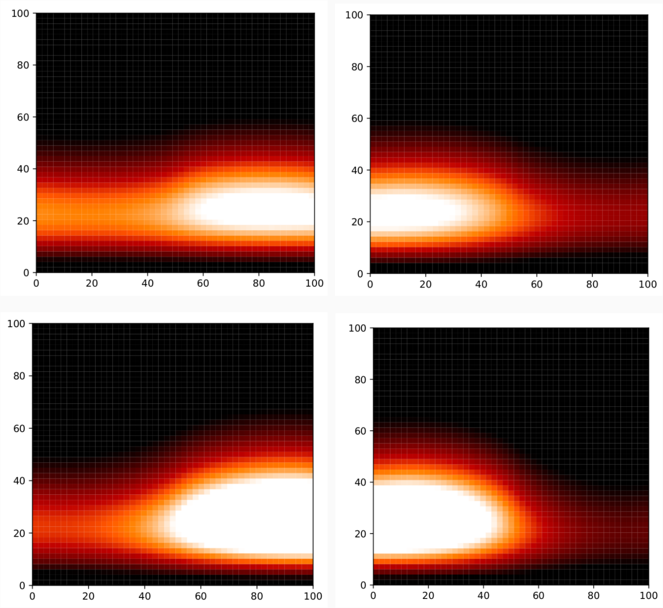


Figure 4:

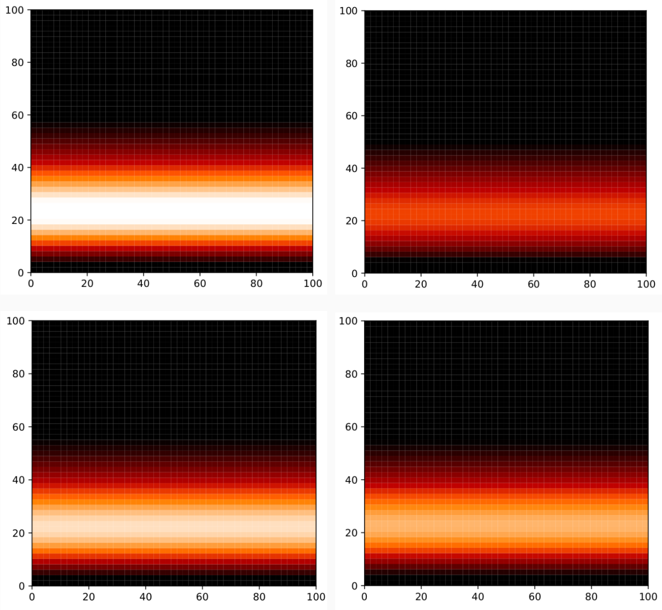
Demonstration of BWR instability - Regional stable



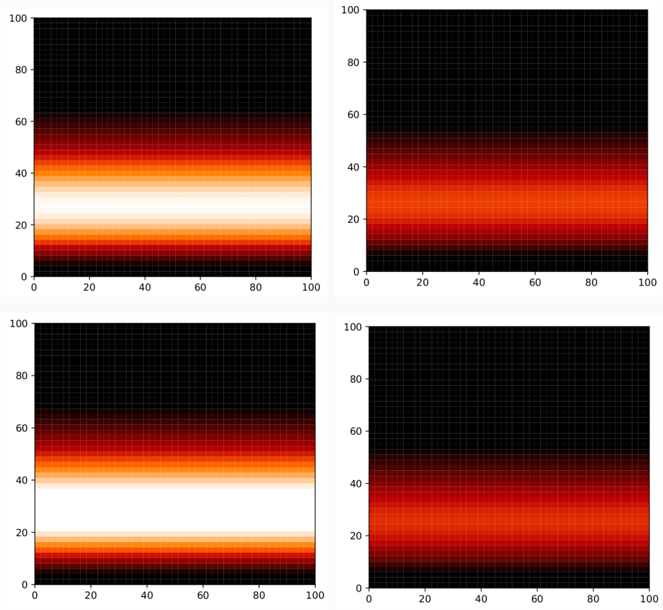
Demonstration of BWR instability - Reg. unstable



Demonstration of BWR instability - Global stable



Demonstration of BWR instability - Glob. unstable



BWR operational map

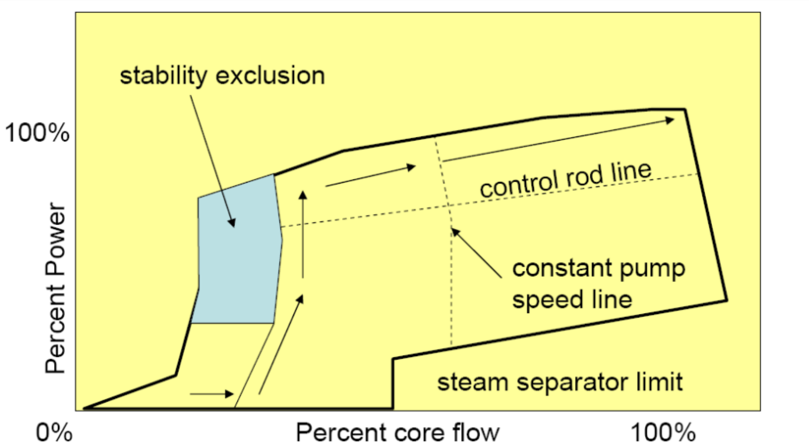


Figure 5: BWR instability region can be shown on operational map