Home Assignments in Numerical Methods

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Preamble. Ordinary Differential Equations (ODEs) frequently arise in many areas of mathematics, physics, natural and social sciences. The vast majority of physical laws can be formulated in the form of differential equations. Examples are: (1) Newton's second law of motion; (2) the logistics equation used to simulate the population growth; and many others.

$$(1)\ mx''(t) = F\left(x(t)\right); \quad (2)\ y'(t) = y(1-y).$$

The canonical form of an initial value problem (IVP) is written as

$$\begin{cases} y'(t) = f(t,y) & t \in [T_0, T_1] \\ y(T_0) = y_0 \end{cases} \tag{1}$$

A wide majority of interesting physical problems has no closed-form solution thus increasing the role of approximate numerical solutions.

Lipschitz continuity, named after German Mathematician Rudolf Lipschitz, is of paramount importance in ODEs. It is a strong form of uniform continuity for functions. A function $f(t,y)$ is said to be Lipschitz continuous in the variable y on the rectangle $R = [T_0, T_1] \times [Y_0, Y_1]$ if there exists a constant L (Lipschitz constant) that, for each (t, y_1) and (t, y_2) in the rectangle R, guarantees the inequality

$$\left| f(t,y_2) - f(t,y_1) \right| \le L \left| y_2 - y_1 \right|.$$

A continuous function is not necessarily Lipschitz continuous. On contrary, if a function is Lipschitz continuous, it is continuous in the ordinary sense moreover a continuously differentiable in y function is Lipschitz continuous because

$$f(t,y_2) - f(t,y_1) = f_y'(t,\xi)\left(y_2 - y_1\right) \text{ for some } \xi \in \left[y_1, y_2\right].$$

Hence, the Lipschitz constant may be set to be $L = \max\limits_{(t,\xi)\in R} \left| f_y'(t,\xi) \right|$.

Theorem 1. If $f(t,y)$ is Lipschitz continuous in y on $[T_0, T_1] \times [Y_0, Y_1]$, and $Y_0 \le y_0 \le Y_1$ then there exists $\tau \in [T_0, T_1]$ such that the initial value problem (1) has exactly one solution $y(t)$ on the interval $T_0 \le t \le \tau$. Moreover, if f is Lipschitz continuous on $[T_0, T_1] \times [-\infty, \infty]$, then there exists exactly one solution on $T_0 \le t \le T_1$.

Exercise 1 (1p).

(a) Let a function be defined as $f(t,y) = ty + t^3$. Using the definition, find the Lipschitz constant, L, in the variable y on the rectangle $[0, T_1] \times (-\infty, \infty)$.

(b) Determine the Lipschitz constant for the function $f(t,y) = te^{ty}$ on the rectangle $[0, T_1] \times [0, Y_1]$.

Exercise 2 (1p). We are trying to solve the initial value problem

$$\begin{cases} y'(t) = 2ty^2 & t \in [0, 2] \\ y(0) = 1 \end{cases}$$

On which interval $[0, \tau]$ does this initial value problem actually have a unique solution?

In numerical analysis, the (standard) Euler method, also known as the Explicit Euler Method or the Forward Euler Method, is probably the simplest and most basic technique for numerical integration of ordinary differential equations with a given initial value. The Euler method often serves as the basis to construct more accurate methods for example predictor-corrector schemes.

If a discretization of IVP (1) is a set of discrete values $T_0 = t_0 < t_1 < \ldots < t_N = T_1$, which determines discretization steps $h_j = t_{j+1} - t_j$, the Euler method reads as

$$y_{j+1} = y_j + h_j f(t_j, y_j)$$

Very often, one uses a uniform discretization

$$t_j = T_0 + jh \text{ here } h = (T_1 - T_0)/N \text{ and } j = 0,1,\ldots,N .$$

It is very important to use an unambiguous and clear notation. To this end, the exact solution to IVP (1) will be denoted as $y(t)$. Accordingly, $y(t_j)$ is the exact solution evaluated at $t = t_j$, whereas y_j stands for an approximate number $y_j \approx y(t_j)$ as given by a numerical method.

Exercise 3 (2p). Consider the initial value problem

$$\begin{cases} y'(t) = \lambda y & t \in [0,T] \\ y(0) = y_0 \end{cases}$$

Here, T, λ and y_0 are given constants.

(a) Find the exact solution.

(b) Using a uniform discretization, derive an explicit formula for y_j as given by the forward Euler method.

(c) We expect $y_N \approx y(t_N) = y(T)$. Assuming exact arithmetic, find the limit, if it exists, $\lim\limits_{N\to\infty} y_N$. Can we state that $y_N \xrightarrow[N\to\infty]{} y(T)$?

Theorem 2. The Euler method is classified as having the first order of approximation i.e., the global error for any discrete point t_j is of the first order in h

$$e(t_j, h) \equiv e_j \equiv \left| y(t_j) - y_j \right| = O(h) \text{ with } h = \max h_j .$$

More in detail, assume that $f(t,y)$ has a Lipschitz constant L for the variable y. Let M be an upper bound for $|y''(t)|$ on $[T_0, T_1]$. Then the (global) error in the Explicit Euler Method may be estimated as

$$e(t_j, h) \equiv \left| y(t_j) - y_j \right| \le \frac{Mh}{2L}\left(e^{L(t_j - T_0)} - 1 \right)$$

A simple, but important and useful, type of separable equation is the first order non-homogeneous linear equation

$$y'(t) = f(t, y) = p(t)y + q(t) . \tag{2}$$

As seen, the right-hand side is linear in the y variable. When $p(t)$ is continuous, the Lipschitz constant can be evaluated as

$$L = \max_{(t,\xi)\in R} \left| f_y'(t,\xi) \right| = \max_t |p(t)|$$

The general solution to Eq.(2) can be found, for example, by the method of integrating factor. If $P(t)$ is any antiderivative of $p(t)$, $P(t) = \int p(t)\mathrm{d}t$, the general solution reads as

$$y(t) = e^{P(t)}\left(\int e^{-P(t)}q(t)\mathrm{d}t + C\right)$$

Here, C is an arbitrary constant of integration. This special class of equations (2) presents a handy set of illustrative examples.

Exercise 4 (4p). Find an error bound for the explicit Euler method applied to the following initial value problem

$$\begin{cases} y'(t) = ty + t^3 & t \in [0,1] \\ y(0) = 1 \end{cases}$$

To this end, do the following steps.

(a) Determine the Lipschitz constant in y on $[0,1]\times(-\infty,\infty)$.

(b) Solve the given equation analytically.

(c) Find the constant M.

(d) Estimate the global truncation error at $t = 1$.

Exercise 5 (3p). Apply the Explicit Euler Method to the following IVP.

$$\begin{cases} y'(t) = -4t^3y^2 & t \in [-10,10] \\ y(-10) = 1/10001 \end{cases} \tag{3}$$

In doing so, write a computer code that implements the Explicit Euler Method then perform the following steps.

(a) Solve the given equation analytically and plot the solution.

(b) Solve the given equation numerically with discretization steps, $h = 10^{-3}$, $h = 10^{-4}$ and $h = 10^{-5}$; and plot the numerical solutions in the same window.

(c) Using the analytic solution, directly calculate and report the global truncation errors at $t = 1$ corresponding to the suggested discretization steps.

As it was pointed out earlier, the explicit Euler method is often used as a starting point to develop more accurate numerical solvers for ODEs.

The Explicit Trapezoid Method, also known as Heun's method or the improved Euler method, is a small adjustment in the Euler's method formula. The essence of this adjustment is to evaluate the representative slope more accurately. To this end, we calculate the slope twice then do the following five simple calculations: (1) first as usual, evaluate the current slope $k_j = f(t_j, y_j)$; (2) perform one step using Euler's method, $y'_{j+1} = y_j + h_j k_j$; (3) evaluate the slope once again, $k_{j+1} = f(t_{j+1}, y'_{j+1})$; (4) evaluate the representative slope, $k_{j+1/2} = (k_j + k_{j+1})/2$; (5) finally return back to the starting point (t_j, y_j) and move to the next point using this representative slope, $y_{j+1} = y_j + h_j k_{j+1/2}$. As one formula, the Explicit Trapezoid Method may be written as

$$y_{j+1} = y_j + 0.5h_j\left[f(t_j, y_j) + f\left(t_{j+1}, y_j + h_j f(t_j, y_j)\right)\right]$$

Exercise 6 (3p). Apply the Explicit Trapezoid Method to the initial value problem (3). To this end, write a computer code that implements the Explicit Trapezoid Method then do the following steps.

(a) Solve IVP (3) numerically with the discretization step $h = 10^{-3}$, explicitly calculate the global error at $t = 1$, $e(t=1,h)$, and compare it with the global error found in Exercise 5 for the same discretization step.

(b) Solve IVP (3) numerically with the discretization step $h/2 = 0.5{\times}10^{-3}$, explicitly calculate the global error at $t = 1$, $e(t=1,h/2)$, and evaluate the ratio

$$\frac{e(t=1,h)}{e(t=1,h/2)}.$$

(c) Formulate a conjecture about the order of approximation of the Explicit Trapezoid Method i.e., suggest a value for p in $e(t,h) = O(h^p)$.

Exercise 7 (2p). The study of mathematical models for predicting the population dynamics of competing species has its origin in independent works published in the early part of the 20th century by A.J. Lotka and V. Voltera. Consider the problem of predicting the population of two species, one of which is a predator, let's say foxes, whose population at time t is $f(t)$, feeding on the other, which is the prey, let's say rabbits, whose population is $r(t)$. We will assume that the prey always has an adequate food supply and that its birth rate at any time is proportional to the number of prey alive at that time; that is the birth rate of the prey is $b_r \cdot r(t)$. The death rate of the prey depends on both the number of prey and predators alive at that time. For simplicity, we assume death rate (prey) is $d_r \cdot r(t) \cdot f(t)$. The birth rate of the predator, on the other hand, depends on its food supply, $r(t)$, as well as on the number of predators available for reproduction purposes. For this reason, we assume that the birth rate of the predator is $b_f \cdot r(t) \cdot f(t)$. The death rate of the predator will be taken as simply proportional to the number of predators alive at that time; that is, death rate (predator) is $d_f \cdot f(t)$.

Since $r'(t)$ and $f'(t)$ represent the change in the prey and predator populations respectively with respect to time, the problem is expressed by the system of non-linear differential equations

$$\begin{cases} r'(t) = b_r r(t) - d_r r(t) f(t) \\ f'(t) = b_f r(t) f(t) - d_f f(t) \end{cases}$$

a) Read about ODE solvers in MATLAB/Python based on a Runge-Kutta formulas.

b) Solve this system using the `built-in` solver for $0 \le t \le 4$, assuming that the initial population of the prey is 1000 and of the predator is 500 and that the rates are $b_r = 3;\ \ d_r = 0.002;\ \ b_f = 0.0006;\ \ d_f = 0.5$

c) Sketch a graph of the solutions to this problem, plotting both populations with time, and describe the physical phenomena represented.

d) What happens if we extend the time interval?

e) Is there a stable solution to this population model? If so, for what values r and f is the solution stable?

Exercise 8 (1p). An amount of reactivity, ρ, is promptly inserted into a critical reactor. Solve numerically the point reactor kinetics equations with one delayed neutron group using the built-in function.

$$\begin{cases} \dfrac{dn(t)}{dt} = \dfrac{\rho - \beta}{\Lambda} n(t) + \lambda C(t) \\ \dfrac{dC(t)}{dt} = \dfrac{\beta}{\Lambda} n(t) - \lambda C(t) \end{cases}$$

Consider three different cases

a) $\rho = 0.1\%$; lp = 1 ms.
b) $\rho = 0.3\%$; lp = 1 ms.
c) $\rho = 0.3\%$; lp = 60 μs.

The delayed neutron data are as follows

Group	1	2	3	4	5	6
β_j	0.000215	0.001424	0.001274	0.002568	0.000748	0.000273
λ_j	0.0124	0.0305	0.111	0.301	1.14	3.01

The average mean lifetime and the average decay constant are defined as

$$\beta = \sum_{i=1}^{6} \beta_i; \qquad \overline{\tau} = \frac{1}{\beta} \sum_{i=1}^{6} \beta_i \tau_i; \qquad \lambda = \frac{1}{\overline{\tau}}.$$

Plot the neutron density for the three cases in one window on the time interval [0, 15].

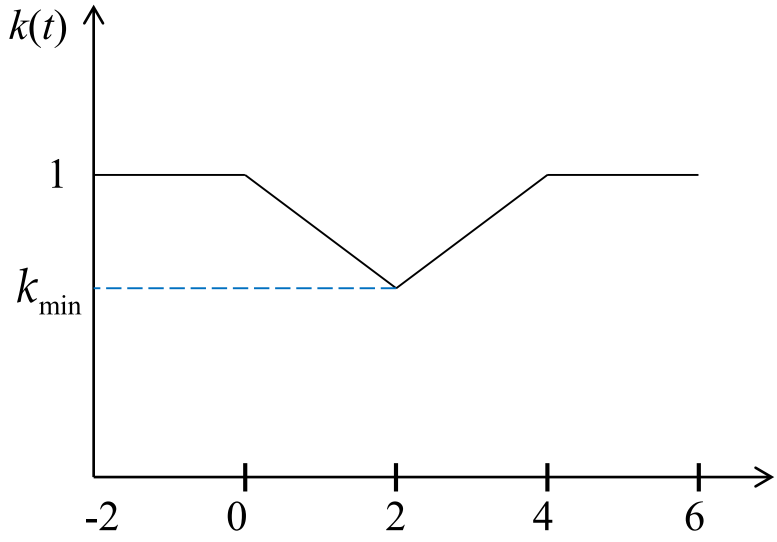
Exercise 9 (1p). (Continuation) Solve numerically the point reactor kinetics equations with one and six delayed neutron groups.

$$\begin{cases} \dfrac{dn(t)}{dt} = \dfrac{\rho - \beta}{\Lambda} n(t) + \lambda C(t) \\ \dfrac{dC_j(t)}{dt} = \dfrac{\beta_j}{\Lambda} n(t) - \lambda_j C_j(t) \end{cases}$$

Let the inserted reactivity and the prompt neutron lifetime be $\rho = \beta/2; \quad l_p = 1\text{ms}.$

Plot the neutron density as calculated by these two models in one window. What can we say about the reactor period as predicted by these two models?

Exercise 10 (2p). Let the multiplication factor, $k(t)$, change as shown in the picture.



Here the minimal value, $k_{\min}$, corresponds to the inserted amount of reactivity of $-\beta/3$ and the prompt neutron lifetime, l_p, is assumed to be 1 ms.

a) Solve numerically and then plot the solution to the point reactor kinetic equation ignoring delayed neutrons.

$$\frac{dn(t)}{dt} = \frac{k(t)-1}{l_p} n(t)$$

b) Solve numerically and then plot the solution to the point reactor kinetic equations assuming one averaged group of delayed neutrons.

$$\begin{cases} \dfrac{dn(t)}{dt} = \dfrac{\rho(t)-\beta}{\Lambda(t)} n(t) + \lambda C(t) \\ \\ \dfrac{dC(t)}{dt} = \dfrac{\beta}{\Lambda(t)} n(t) - \lambda C(t) \end{cases}$$