Please read carefully the assignments and address all the questions/instructions.

Exercise 1. Draw graphs of the functions

$$y = \cos(x)$$
$$y = x$$

for $0 \le x \le 2$ on the same window. Use the zoom facility to determine the point of intersection of the two curves [and, hence, the root of $x = \cos(x)$] to two significant figures.

Exercise 2. The irrational number π, also known as Archimedes' constant, is one of the fundamental mathematical constants originally defined as the ratio between a circle's circumference and its diameter, $\pi \equiv C/D$. It appears in essentially all areas of mathematics and physics. For instance, the natural numbers are related to this transcendental number through a formula discovered by Leonard Euler in 1775

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

Read how to obtain π in your computational platform and find (approximately) how many terms in the above series are needed to evaluate $\pi^2/6$ with 3 correct decimal places.

An Indian mathematician, Madhava of Sangamagrama (c. 1340 – c. 1425) found a simple representation of π, also known as Gregory's (1668) or Leibniz (1676) series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This series is notorious for its slow (logarithmic) convergence. Evaluate the number of terms in the above series to approximate π with 3 correct decimal places.

Madhava also gave a more rapidly converging series for π

$$\pi = \sqrt{12}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \cdots\right)$$

Confirm that 21 terms in the above series are enough to obtain a value correct to 11 decimal places.

Exercise 3. Read the appropriate documentation how to define vectors and access/modify their components. Note that MATLAB uses 1-based indexing whereas Python is 0-based. Define a vector **a** = (1,2,3); perform the assignment **b** = **a;** and print these vectors. Then do the following.

1) Alter the sign of the last component in **b** and print these vectors again.
2) Make sure that your computational platform supports component wise operation. To this end, perform the assignment $\mathbf{a} = -\mathbf{a}$ and print **a** again.
3) Make sure that built-in functions such as abs support component wise operations. To this end, execute, $\mathbf{a} = \text{abs}(\mathbf{a})$ and print **a** once more.
4) Read how to perform component wise multiplication of vectors. To this end, calculate and print **c** = **a*****a** (Python) or **c** = **a**.***a** (MATLAB, note dot here).
5) Explore component wise division by calculating and printing, **a** = **c**/**a** (Python) or **a** = **c**./**a** (MATLAB, note dot here).

Exercise 4. Learn how to manipulate vectors and find a built-in function to calculate various norms (magnitudes) of a vector. Define three unit vectors to represent the axes of a Cartesian coordinate system

$$\mathbf{i}=(1,0,0); \quad \mathbf{j}=(0,1,0) \quad \text{and} \quad \mathbf{k}=(0,0,1)$$

Calculate a new vector, $\mathbf{a}=3\mathbf{i}+4\mathbf{j}+12\mathbf{k}$. Print this vector and its three norms

$$\|\mathbf{a}\|_1 \quad \|\mathbf{a}\|_2 \quad \text{and} \quad \|\mathbf{a}\|_\infty$$

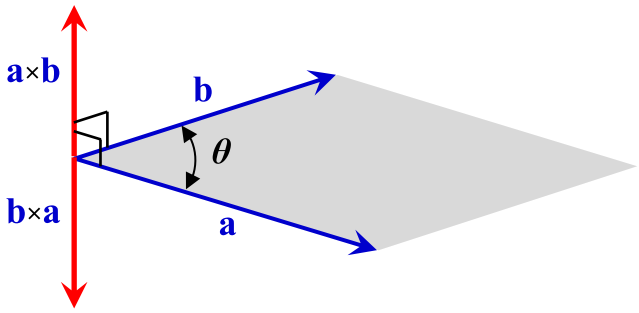
Exercise 5. The scalar product and the vector product are two ways of multiplying vectors which see the most applications in mathematics, physics and other exact sciences. The scalar product (dot product, inner product) of two geometric vectors in our physical space is defined through their magnitudes (geometric lengths), $|\mathbf{a}|$ and $|\mathbf{b}|$, as

$$\langle \mathbf{a},\mathbf{b}\rangle=(\mathbf{a},\mathbf{b})=\mathbf{a}\cdot\mathbf{b}\equiv|\mathbf{a}|\cdot|\mathbf{b}|\cdot\cos(\theta)$$

Read the appropriate documentation and find a built-in function to calculate the scalar product of two vectors. Then do the following.

1) Determine the direction cosines for the radius vector **r** = (2,3,6) using the dot product with the coordinate vectors **i**, **j** and **k** (as in Exercise 4).
2) Verify that the computed cosines satisfy a well-known identity (which one?)
3) Find the angle in degrees between **a** = (1,1,0) and **b** = (0,1,1)

Exercise 6. The vector product (outer product, cross product, directed area product) of two vectors, **a** and **b** is another vector, $\mathbf{a}\times\mathbf{b}$, that is at right angles to both and having the magnitude (length) equal to the area of a parallelogram with vectors **a** and **b** for sides (see the picture below).



As the definition suggests.

$$\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\cdot\|\mathbf{b}\|\cdot\sin\theta$$

Hence we can use the vector product to compute the area of the parallelogram or the area of the triangle spanned by the vectors **a** and **b**.

$$A_{par}=\|\mathbf{a}\times\mathbf{b}\| \qquad A_{tri}=\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$$

The cross product can also be expressed as a formal determinant

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

Read the appropriate documentation and find a built-in function for calculating the vector product $\mathbf{a}\times\mathbf{b}$. Define three vectors

$$\mathbf{a} = (3,-1,5); \quad \mathbf{b} = (0,4,-2); \quad \mathbf{c} = (-2,3,-1).$$

Then do the following

1) Using this built-in function, compute and print the vector $\mathbf{a} \times \mathbf{b}$
2) Verify numerically the anti-commutative rule $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
3) Calculate and print the triple vector product, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
4) Verify numerically the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ also known as the BAC-CAB rule or the triple product expansion or Lagrange's formula.
5) Verify numerically the Jacobi identity,
   $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$
6) Verify numerically another identity, $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = (|\mathbf{a}| \cdot |\mathbf{b}|)^2$

Exercise 7. Hero of Alexandria (c. 10 AD – 70 AD) found a method to calculate the area of a tringle when the lengths (a, b and c) of all three sides are known. Unlike other triangle area formulae, Heron's formula (sometimes Hero's formula) does not require to calculate angles or other distances in the triangle first. It reads

$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

where $p = (a+b+c)/2$ (semiperimeter). Write a function that will accept the values a, b and c as inputs and return the value A as an output. Note that not all values of a, b and c are acceptable. They must be at least positive numbers. There is also another restriction known as the triangle inequality. Incorporate the above ideas into your function. Define also three coordinate vectors

$$\mathbf{i} = (1,0,0); \quad \mathbf{j} = (0,1,0) \quad \text{and} \quad \mathbf{k} = (0,0,1)$$

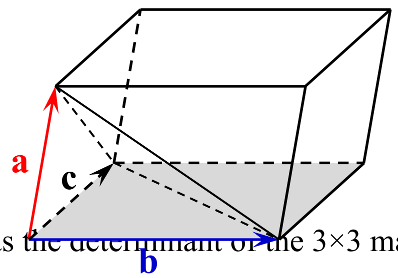
Then do the following

1) Test your function using a non-existing triangle, (1,2,4).
2) Find the area of the triangle spanned by the vectors, **a** = 2***j**-**i** and **b** = 3***k**-**i**, using your function.
3) Check the above result using the cross product.

Exercise 8. The scalar triple product (mixed product, box product, triple scalar product) is defined as

$$[\mathbf{a},\mathbf{b},\mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Geometrically, the scalar triple product is the (signed) volume of the parallelepiped, V_{par}, defined by the three vectors given, which in turn is 6 times greater than the volume of the tetrahedron (triangular pyramid), V_{tet}, formed by the vectors **a**, **b** and **c**.



The scalar triple product can also be understood as the determinant of the 3×3 matrix

$$[\mathbf{a},\mathbf{b},\mathbf{c}] \equiv \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \det(\mathbf{a},\mathbf{b},\mathbf{c}) = \pm V_{par} = \pm 6V_{tet}$$

Learn how to define and manipulate matrices and find a built-in function to calculate determinants. Then do the following.

1) Compute the volume of the tetrahedron whose vertices are the points A = (3,2,1), B = (1,2,4), C = (4,0,3) and D = (1,1,7).
2) Check your calculation numerically by evaluating the same volume with a corresponding determinant.
3) Using vectors **a** = (3,-2,5), **b** = (2,2,1) and **c** = (-4,3,2), verify the identities, $[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}]$ by means of computations.

Exercise 9. Find a built-in function for solving a system of linear algebraic equations of the form, **Ax** = **b**. Define a matrix, **A**, and a vector, **b** as

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then do the following

1) Find the solution to **Ax** = **b** using a built-in solver.
2) Verify the found solution using the built-in matrix-vector multiplication and printing the difference, **Ax** - **b**.

Exercise 10. Read and learn how to define and manipulate complex numbers. Find a built-in function for calculating the inverse of a matrix. Set then a matrix **A** and a vector **b**, that involve the imaginary unit *i*, as

$$\mathbf{A} = \begin{bmatrix} 2+2i & -1 & 0 \\ -1 & 2-2i & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1+i \\ 0 \\ 1-i \end{bmatrix}$$

Then do the following

1) Solve the system of equations, **Ax** = **b**, by first finding the inverse matrix, $\mathbf{A}^{-1}$, and then multiplying it with the right hand side **b**, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
2) Verify the found solution using the built-in matrix-vector multiplication and printing the difference, **Ax** - **b**.