Sustainable Energy Transformation Technologies, SH2706

Lecture No 11

Title:

Heat Conduction in ETS

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Outline

- Basic definitions in heat conduction theory
- Steady-state heat conduction
  - plane wall
  - hollow cylinder
  - composite wall
  - critical insulation thickness
  - infinite cylinder with uniform heat sources
  - infinite cylinder with heat sources from nuclear fission
- Transient heat conduction
  - lumped heat capacity model
  - general one-dimensional transient model

Heat Conduction – Fourier Law

- Heat conduction refers to the transfer of heat by means of molecular interactions without any accompanying macroscopic displacement of matter.
- The flow of heat by conduction in isotropic media is governed by the **Fourier law**:

$$\mathbf{q}''(\mathbf{r},t) = -\lambda \nabla T(\mathbf{r},t)$$

q” – heat flux vector [W m^{-2}]

λ - thermal conductivity [W m^{-1} K^{-1}]

T - temperature, [K]

r - location vector [m]

t – time [s]

Heat Conduction Equation

- A non-stationary temperature distribution in an arbitrary volume is described by the following equation, resulting from the energy balance in the conducting material:

$$\frac{\partial\left(\rho c_p T\right)}{\partial t}+\nabla\cdot\left(\rho c_p T\mathbf{v}\right)=-\nabla\cdot\mathbf{q}''-\boldsymbol{\tau}:\nabla\mathbf{v}+\left(\frac{\partial\ln\upsilon}{\partial\ln T}\right)_p\frac{Dp}{Dt}+\rho T\frac{Dc_p}{Dt}+q'''$$

- Assuming heat transfer in solid with heat sources:

$$\frac{\partial}{\partial t}\left[\rho\cdot c_p\cdot T(\mathbf{r},t)\right]=q'''\left(\mathbf{r},t\right)-\nabla\cdot\mathbf{q}''\left(\mathbf{r},t\right)$$

ρ – density of conducting matter [kg m^{-3}] c_p – specific heat [J kg^{-1} K^{-1}]

q''' – volumetric heat source [W m^{-3}]

Heat Conduction Equation

- Using the Fourier law, the conductivity equation becomes,

$$\frac{\partial}{\partial t}\left[\rho \cdot c_p \cdot T(\mathbf{r},t)\right] - \nabla \cdot \lambda \nabla T(\mathbf{r},t) = q'''(\mathbf{r},t)$$

- In general material properties can be functions of location, temperature and time.

- In some cases it can be assumed that they are constant

Heat Conduction Equation

- For constant properties, the heat conduction equation becomes,

$$\rho \cdot c_p \frac{\partial T(\mathbf{r},t)}{\partial t} - \lambda \nabla^2 T(\mathbf{r},t) = q'''(\mathbf{r},t)$$

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - a \nabla^2 T(\mathbf{r},t) = \frac{q'''(\mathbf{r},t)}{\rho \cdot c_p}$$

$$a = \frac{\lambda}{\rho \cdot c_p}$$

a – thermal diffusivity [m^2 s^{-1}]

Fourier and Poisson Equations

- If there are no volumetric heat sources, the heat conduction equation becomes (**Fourier equation**),

$$\frac{\partial T(\mathbf{r},t)}{\partial t} = a\nabla^2 T(\mathbf{r},t)$$

- For steady-state conduction in a material with constant thermal conductivity and given volumetric heat sources the so-called **Poisson equation** is obtained,

$$\nabla^2 T(\mathbf{r}) = -\frac{q'''(\mathbf{r})}{\lambda}$$

Laplace Equation

- Finally, Laplace equation is obtained when no heat sources are present

$$\nabla^2 T(\mathbf{r}) = 0$$

  - in Cartesian coordinates

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

  - in cylindrical coordinates

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right)$$

  - in spherical coordinates

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\cot\theta \frac{\partial}{\partial \theta} \right)$$

Laplace Equation

- Further simplifications are obtained for the axisymmetric heat conduction,

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)$$

- and for the radius-dependent only heat conduction in a sphere,

$$\nabla^2 \equiv \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right)$$

Initial and Boundary Conditions

- To solve the differential equation of heat conduction, additional conditions are required:
  - boundary conditions, describing how the body interacts with the environment
  - in case of transient solution, initial conditions giving the temperature distribution at time $t = 0$
- Four types of boundary conditions are applied:
  - with given boundary temperature (Dirichlet's b.c.)
  - with given boundary heat flux (Neumann's b.c.)
  - with given fluid temperature and heat transfer coefficient (Robin - using Newton's law of cooling)
  - solid-solid contact boundary condition

Boundary Conditions

- The boundary conditions have the following form:
  - 1st kind – Dirichlet:

$$T\big|_{boundary} = G\big|_{boundary}$$

G – given function (temperature)

  - 2nd kind – Neumann:

$$-\lambda \frac{dT_F}{dr}\bigg|_{boundary} = F\big|_{boundary}$$

F – given function (heat flux)

  - 3rd kind – Robin:

$$-\lambda \frac{dT}{dr}\bigg|_{boundary} = h\left(T_{boundary} - T_{fluid}\right)$$

h – given heat flux coefficient

  - 4th kind:

$$\lambda_1 \frac{dT}{dr}\bigg|_{boundary-1} = \lambda_2 \frac{dT}{dr}\bigg|_{boundary-2}$$

$$T\big|_{boundary-1} = T\big|_{boundary-2}$$

Plane Infinite Wall (1)

- Assume steady-state heat conduction through a plane infinite wall

Conduction equation

$$\nabla^2 T(\mathbf{r}) = 0 \Rightarrow \frac{d^2 T(x)}{dx^2} = 0$$

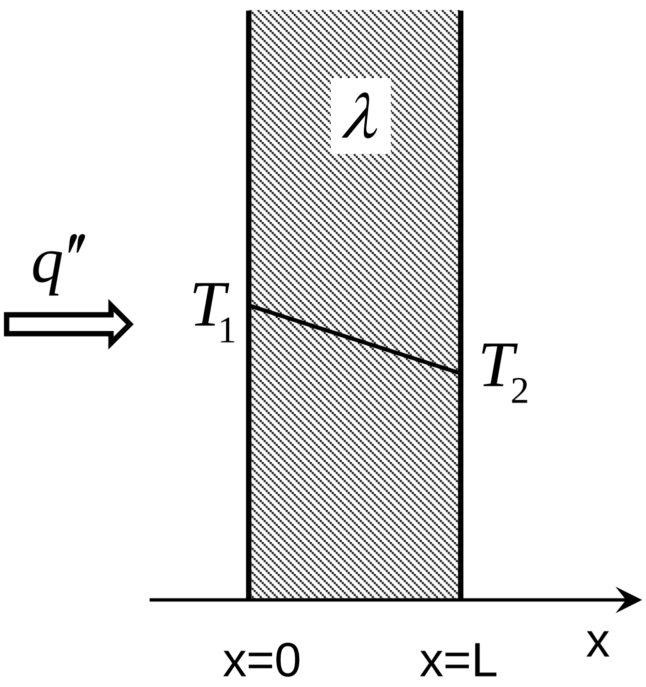
Boundary conditions

$$T(x)\big|_{x=0} = T_1$$

$$T(x)\big|_{x=L} = T_2$$

Solution

$$T(x) = T_1 + \frac{x}{L}\left(T_2 - T_1\right)$$

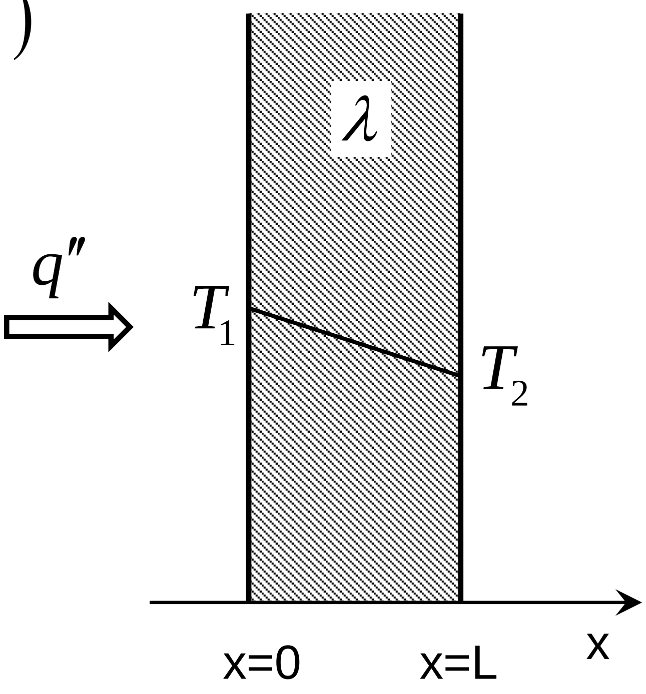


Plane Infinite Wall (2)

- Heat flux can now be found as:

$$q'' = -\lambda \frac{dT(x)}{dx} = -\lambda \frac{T_2 - T_1}{L} = \frac{\lambda}{L}\left(T_1 - T_2\right)$$

We note that q''>0 when $T_1 > T_2$



- Total rate of heat transfer through wall area A is thus:

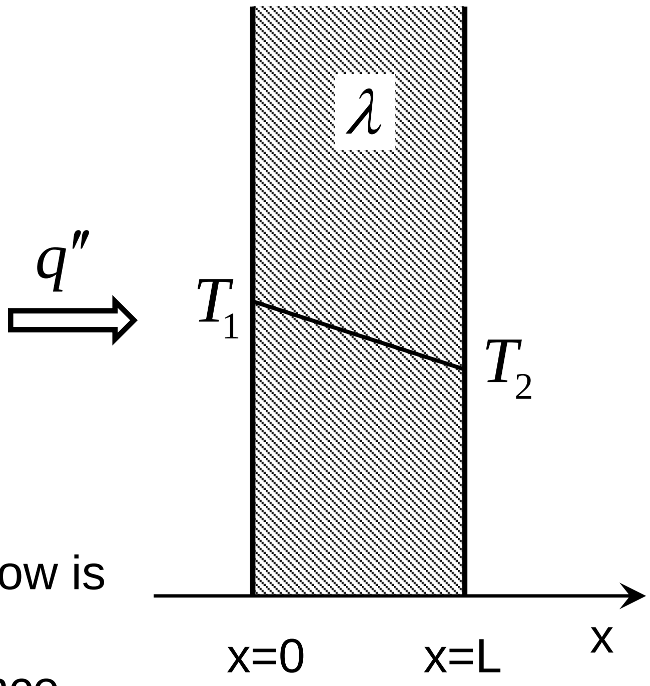
$$q = q''A = \frac{\lambda A}{L}\left(T_1 - T_2\right)$$

Plane Infinite Wall (3)

- The ratio of the temperature difference to the associated rate of heat transfer is called thermal resistance:

$$R_t \equiv \frac{T_1 - T_2}{q} = \frac{L}{\lambda A}$$

- Thus if the resistance of a wall is known, the related rate of heat transfer is found as:



$$q = \frac{T_1 - T_2}{R_t}$$

This is in analogy to Ohm's law according to which the current flow is equal to the ratio of the voltage difference to the electric resistance

Plane Infinite Wall (4)

- If the wall is separating two fluids with temperatures $T_{1\infty}$ and $T_{2\infty}$, with known heat transfer coefficients h_1 and h_2 we have:

Conduction equation

$$\nabla^2 T(\mathbf{r}) = 0 \Rightarrow \frac{d^2T(x)}{dx^2} = 0$$

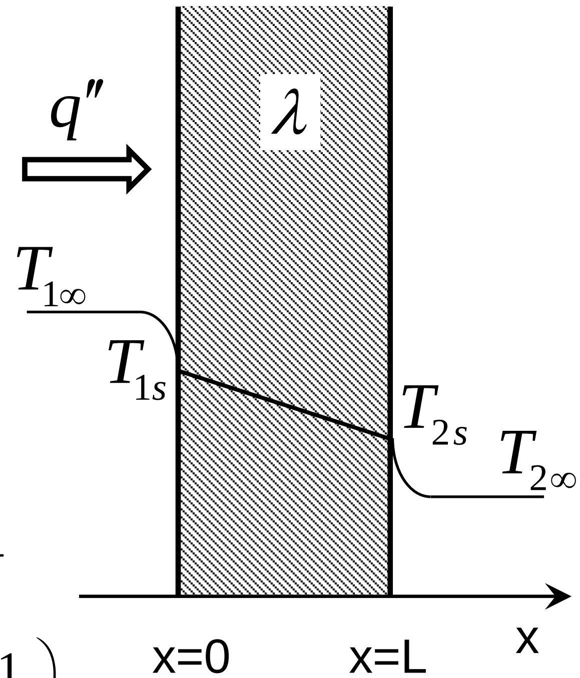
Boundary conditions

$$-\lambda \left.\frac{dT}{dx}\right|_{x=0} = h_1\left(T_{1\infty} - T_{1s}\right) = q'' \quad \Rightarrow \quad T_{1\infty} - T_{1s} = \frac{q''}{h_1}$$

$$\lambda \left.\frac{dT}{dx}\right|_{x=L} = h_2\left(T_{2\infty} - T_{2s}\right) = q'' \quad \Rightarrow \quad T_{2\infty} - T_{2s} = \frac{q''}{h_2}$$

Solution in wall

$$q'' = \lambda \frac{T_{1s} - T_{2s}}{L} \quad \Rightarrow \quad T_{1s} - T_{2s} = \frac{q''L}{\lambda}$$



Adding the three equations yields

$$T_{1\infty} - T_{2\infty} = q''\left(\frac{1}{h_1} + \frac{L}{\lambda} + \frac{1}{h_2}\right)$$

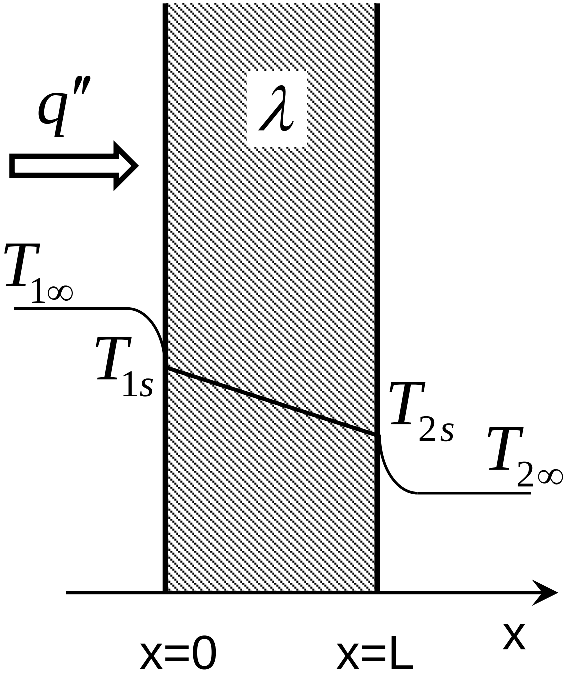
Plane Infinite Wall (5)

- Thus the heat flux through the wall is

$$q'' = \frac{T_{1\infty} - T_{2\infty}}{\left(\frac{1}{h_1} + \frac{L}{\lambda} + \frac{1}{h_2}\right)} = \frac{T_{1\infty} - T_{2\infty}}{AR_t}$$

- where the thermal resistance of the wall with convection on both sides and area A is

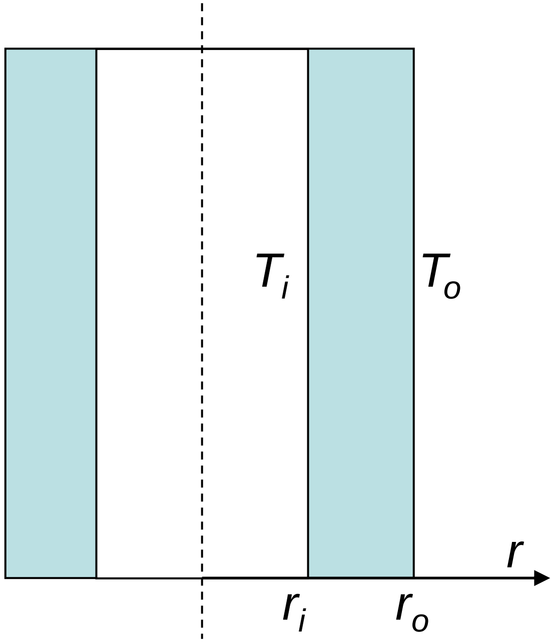
$$R_t = \frac{1}{Ah_1} + \frac{L}{A\lambda} + \frac{1}{Ah_2}$$



- The unit of thermal resistance is K/W

Infinite Hollow Cylinder (1)

- Problem description:
  - Find steady-state temperature distribution in an infinite hollow cylinder, which has constant inner temperature T_i and constant outer temperature T_o.



$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \cancel{\frac{\partial^2 T}{\partial z^2}} = 0 \Rightarrow \frac{d^2 T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0$$

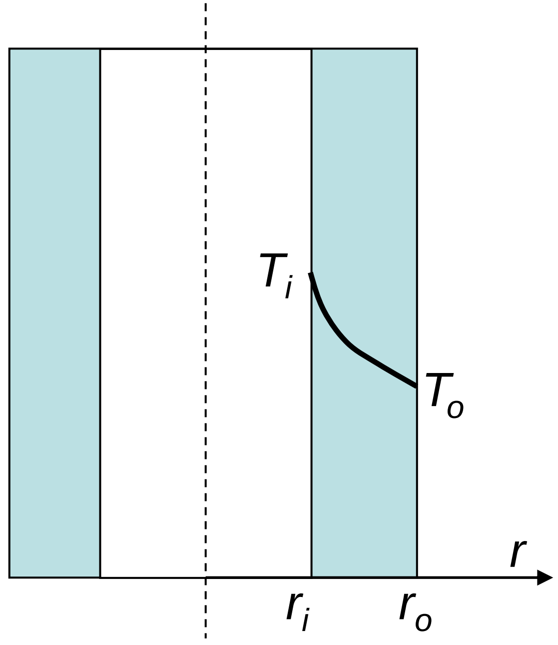
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \Rightarrow r\frac{dT}{dr} = C$$

$$T(r) = C\int\frac{dr}{r} = C\ln r + D$$

$$\left.\begin{aligned} T(r_i) &= C\ln r_i + D = T_i \\ T(r_o) &= C\ln r_o + D = T_o \end{aligned}\right\} \text{C, D}$$

Infinite Hollow Cylinder (2)

$$\left.\begin{aligned} T(r_i) &= C\ln r_i + D = T_i \\ T(r_o) &= C\ln r_o + D = T_o \end{aligned}\right\} \qquad \begin{aligned} C &= \frac{T_i - T_o}{\ln r_i - \ln r_o} \\ D &= T_o - \frac{T_i - T_o}{\ln r_i - \ln r_o}\ln r_o \end{aligned}$$



$$T(r) = C\ln r + D = \frac{T_i - T_o}{\ln r_i - \ln r_o}\ln r + T_o - \frac{T_i - T_o}{\ln r_i - \ln r_o}\ln r_o =$$

$$\frac{T_i - T_o}{\ln r_i - \ln r_o}\left(\ln r - \ln r_o\right) + T_o =$$

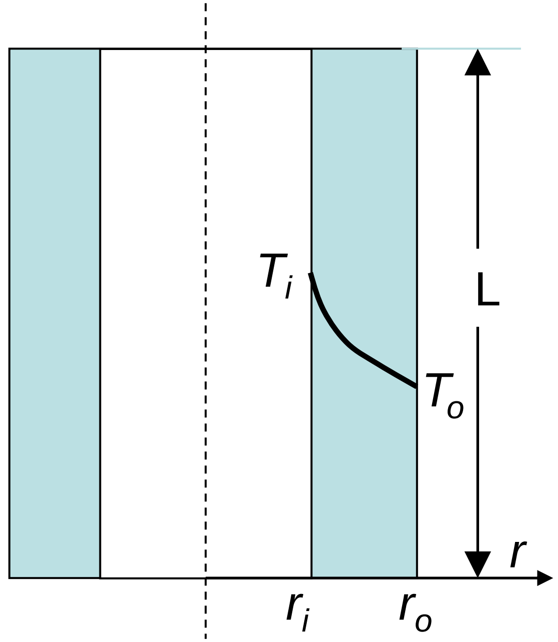
$$\left(T_i - T_o\right)\frac{\ln\left(r/r_o\right)}{\ln\left(r_i/r_o\right)} + T_o$$

Thus the temperature in a cylindrical wall has the logarithmic distribution

$$\frac{T(r) - T_o}{T_i - T_o} = \frac{\ln\left(r/r_o\right)}{\ln\left(r_i/r_o\right)}$$

Infinite Hollow Cylinder (3)

The heat flow rate *q* flowing through the wall can be calculated from the Fourier law, but now the heat transfer area is changing with the radius.



Thus taking a cylinder with length *L*, we have:

$$q = -2\pi r L \lambda \frac{dT}{dr} = const$$

Using the found temperature distribution gives:

$$q = \frac{2\pi L \lambda}{\ln\left(r_o/r_i\right)}\left(T_i - T_o\right)$$

From which we can get the linear power *q'* as:

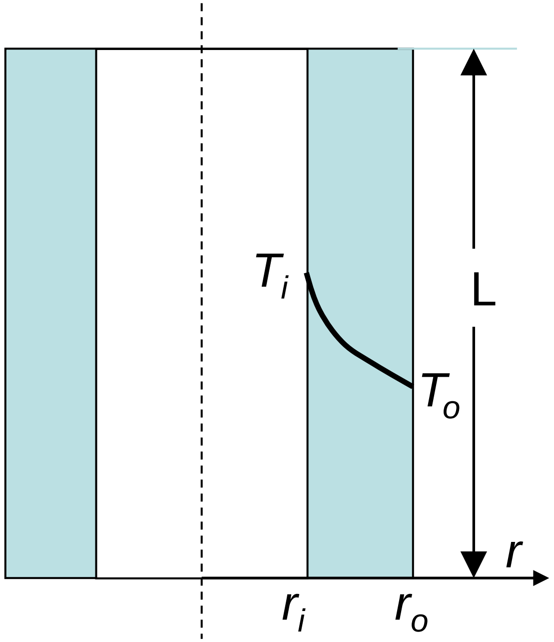
$$q' \equiv \frac{q}{L} = \frac{2\pi\lambda}{\ln\left(r_o/r_i\right)}\left(T_i - T_o\right)$$

Infinite Hollow Cylinder (4)

We can also obtain heat flux on both inner and outer surfaces as:

$$q''_o \equiv \frac{q}{2\pi r_o L} = \frac{\lambda}{r_o \ln\left(r_o/r_i\right)}\left(T_i - T_o\right)$$

$$q''_i \equiv \frac{q}{2\pi r_i L} = \frac{\lambda}{r_i \ln\left(r_o/r_i\right)}\left(T_i - T_o\right)$$



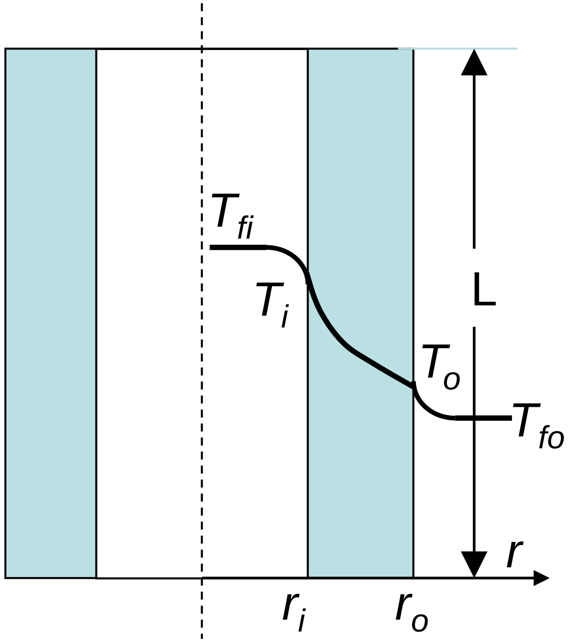
As we can see, the heat flux is different on each surface.

Similarly as for the plane wall, we can introduce the thermal resistance for heat conduction in a cylinder:

$$q' = \frac{2\pi\lambda}{\ln\left(r_o/r_i\right)}\left(T_i - T_o\right) = \frac{\left(T_i - T_o\right)}{R_t L}$$

where: $R_t = \dfrac{\ln\left(r_o/r_i\right)}{2\pi\lambda L}$

Infinite Hollow Cylinder (5)



We can now extend our result for case with convective heat transfer inside and outside the hollow cylinder, assuming known heat transfer coefficients h_i and h_o, respectively.

On the inside surface we have:

$$q' = h_i 2\pi r_i \left(T_{fi} - T_i\right) \Rightarrow T_{fi} - T_i = q' / \left(h_i 2\pi r_i\right)$$

here T_{fi} – inside fluid temperature.
On the outside surface we have:

$$q' = h_o 2\pi r_o \left(T_o - T_{fo}\right) \Rightarrow T_o - T_{fo} = q' / \left(h_o 2\pi r_o\right)$$

where T_{fo} – outside fluid temperature.
Since in solid wall we have:

$$T_i - T_o = q' \frac{\ln\left(r_o / r_i\right)}{2\pi\lambda}$$

adding yields:

$$T_{fi} - T_{fo} = q'\left(\frac{1}{h_i 2\pi r_i} + \frac{\ln\left(r_o / r_i\right)}{2\pi\lambda} + \frac{1}{h_o 2\pi r_o}\right)$$

Infinite Hollow Cylinder (6)

Thus for a hollow cylinder with convecting heat transfer on both the inner and outer surfaces, the over-all thermal resistance is:

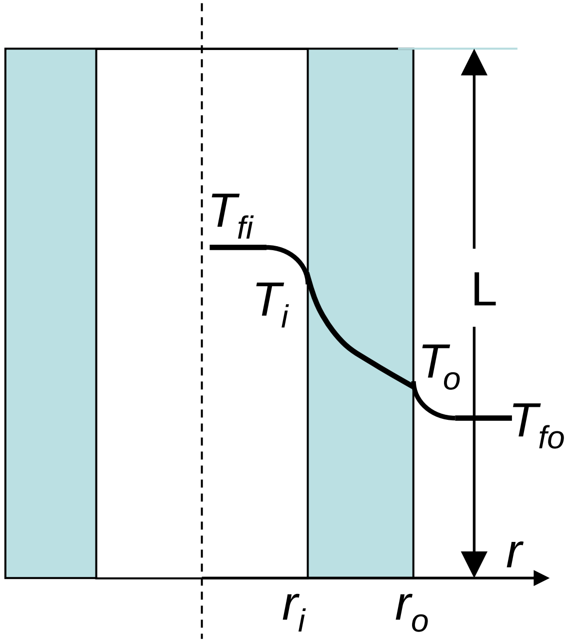
$$R_t = \frac{1}{h_i 2\pi r_i L} + \frac{\ln\left(r_o / r_i\right)}{2\pi\lambda L} + \frac{1}{h_o 2\pi r_o L}$$

where the linear heat transferred from inside to the outside of the hollow cylinder is:

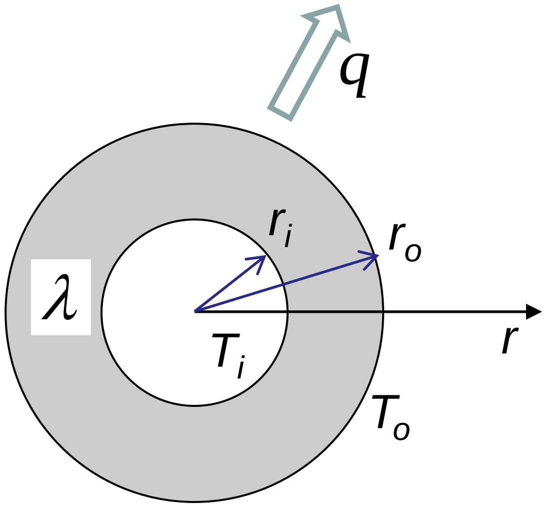
$$q' = \frac{\left(T_{fi} - T_{fo}\right)}{R_t L}$$

and the total heat is:

$$q = \frac{\left(T_{fi} - T_{fo}\right)}{R_t}$$



Hollow Sphere (1)



- The conduction equation in the spherical coordinates is

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0$$

- with the boundary conditions

$$T(r)\Big|_{r=r_i} = T_i \qquad T(r)\Big|_{r=r_o} = T_o$$

- The following temperature distribution is obtained

$$T(r) = T_i + \frac{T_i - T_o}{r_o^{-1} - r_i^{-1}}\left(r_i^{-1} - r^{-1}\right)$$

Heat transfer rate (W) is:

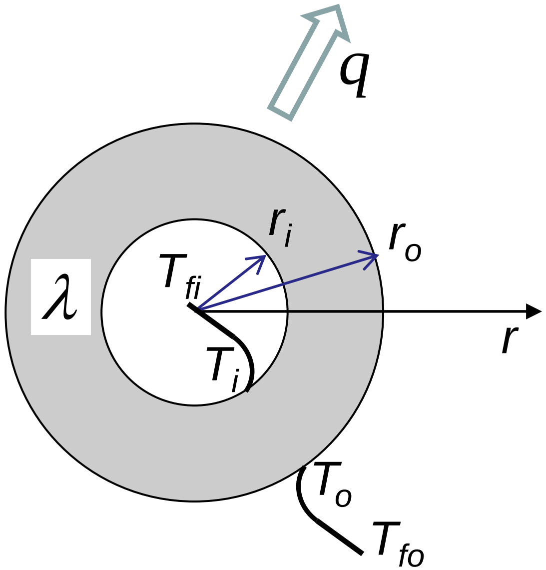
$$q = \frac{4\pi\lambda\left(T_i - T_o\right)}{r_i^{-1} - r_o^{-1}}$$

Thermal resistance:

$$R_t = \frac{r_i^{-1} - r_o^{-1}}{4\pi\lambda}$$

Hollow Sphere (2)

- Assuming the convective heat transfer on inner and outer surfaces of the hollow sphere, and employing the same procedure as for the plane wall and the cylinder, the rate of the heat transfer is as follows



$$q = \frac{T_{fi} - T_{fo}}{1/4\pi h_i r_i^2 + (r_o - r_i)/4\pi\lambda r_i r_o + 1/4\pi h_o r_o^2}$$

- Thus the thermal resistance is:

$$R_t = \frac{1}{4\pi h_i r_i^2} + \frac{r_o - r_i}{4\pi\lambda r_o r_i} + \frac{1}{4\pi h_o r_o^2}$$

Composite Plane Wall

- Our results so far can be extended to a composite walls, consisting of several layers with different properties

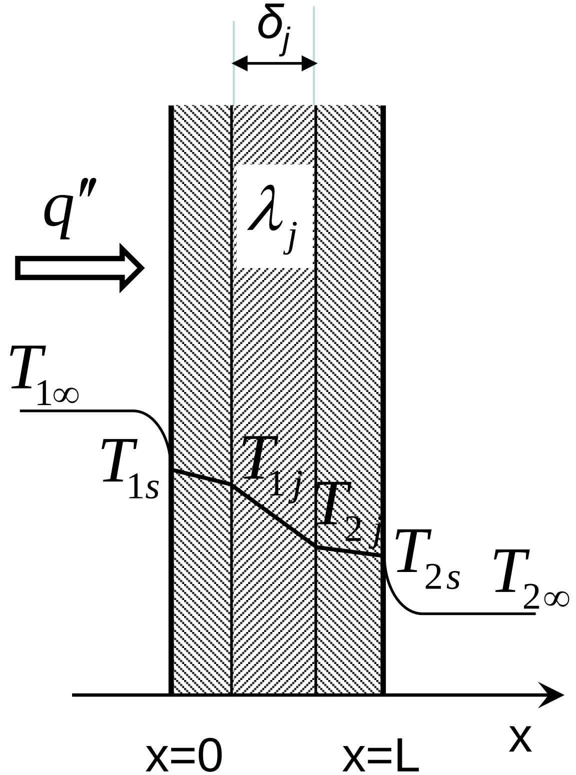
- For a plane wall we have

$$q'' = \frac{T_{1\infty} - T_{2\infty}}{\left(\frac{1}{h_1} + \sum_j \frac{\delta_j}{\lambda_j} + \frac{1}{h_2} \right)} = \frac{T_{1\infty} - T_{2\infty}}{AR_t}$$

- and

$$R_t = \frac{1}{Ah_1} + \frac{1}{A} \sum_j \frac{\delta_j}{\lambda_j} + \frac{1}{Ah_2}$$

- is the thermal resistance of the composite wall



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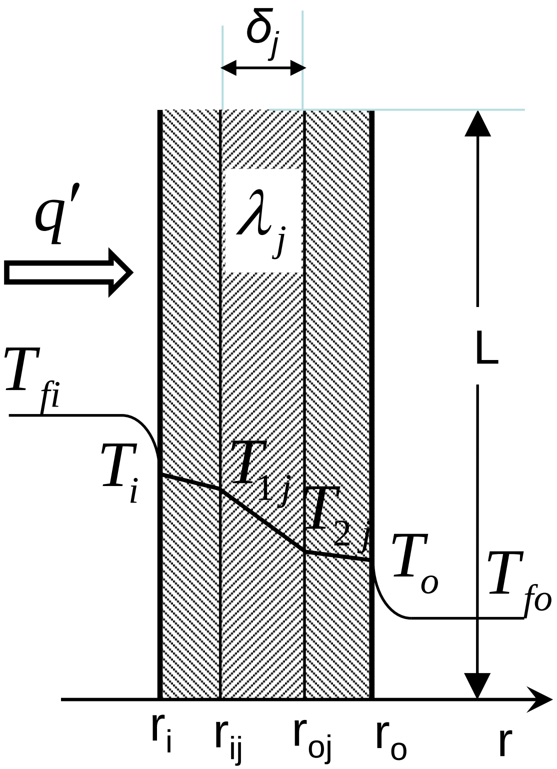
Composite Hollow Cylinder

- For a hollow cylinder with a composite wall we have

$$q' = \frac{T_{fi} - T_{fo}}{\frac{1}{h_i 2\pi r_i} + \sum_j \frac{\ln\left(r_{oj}/r_{ij}\right)}{2\pi\lambda_j} + \frac{1}{h_o 2\pi r_o}} = \frac{T_{fi} - T_{fo}}{R_t L}$$



- where

$$R_t = \frac{1}{h_i 2\pi r_i L} + \frac{1}{2\pi L}\sum_j \frac{\ln\left(r_{oj}/r_{ij}\right)}{\lambda_j} + \frac{1}{h_o 2\pi r_o L}$$

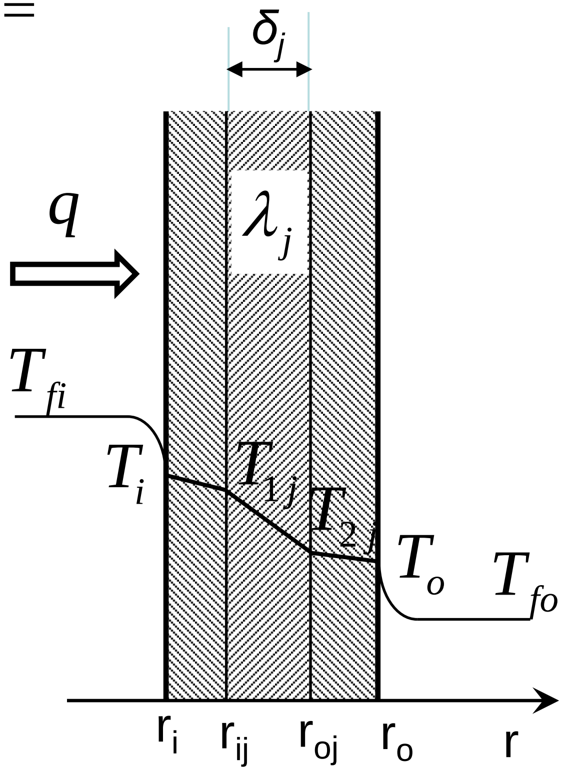
Composite Hollow Sphere

- For a hollow sphere with a composite wall we have

$$q = \frac{T_{fi} - T_{fo}}{1/4\pi h_i r_i^2 + \sum_j \left(r_{oj} - r_{ij}\right)/4\pi\lambda_j r_{ij} r_{oj} + 1/4\pi h_o r_o^2} = \frac{T_{fi} - T_{fo}}{R_t}$$

- where

$$R_t = \frac{1}{4\pi h_i r_i^2} + \frac{1}{4\pi}\sum_j \frac{r_{oj} - r_{ij}}{\lambda_j r_{oj} r_{ij}} + \frac{1}{4\pi h_o r_o^2}$$

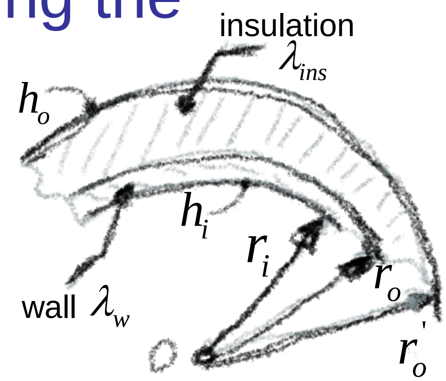


Critical Thickness of Insulation (1)

- When a plane surface is insulated then the rate of heat transfer q (W) always decreases with increasing thickness of insulation
- With increasing thickness of insulation for a cylindrical or spherical surfaces we have two contradicting effects:
  - increasing conductance resistance in the insulation layer
  - decreasing convection resistance due to increasing convection surface area
- Thus there is such thickness of the insulation when the sum of the two resistances achieve a minimum
- We call this thickness the **critical thickness of insulation**

Critical Thickness of Insulation (2)

- For a hollow cylinder with insulation (neglecting the resistance of the wall as small) we have:



$$R_t = \frac{1}{h_i 2\pi r_i L} + \underbrace{\cancel{\frac{1}{2\pi L}\frac{\ln\left(r_o/r_i\right)}{\lambda_w}}}_{\sim 0} + \frac{1}{2\pi L}\frac{\ln\left(r_o'/r_0\right)}{\lambda_{ins}} + \frac{1}{h_o 2\pi r_o' L}$$

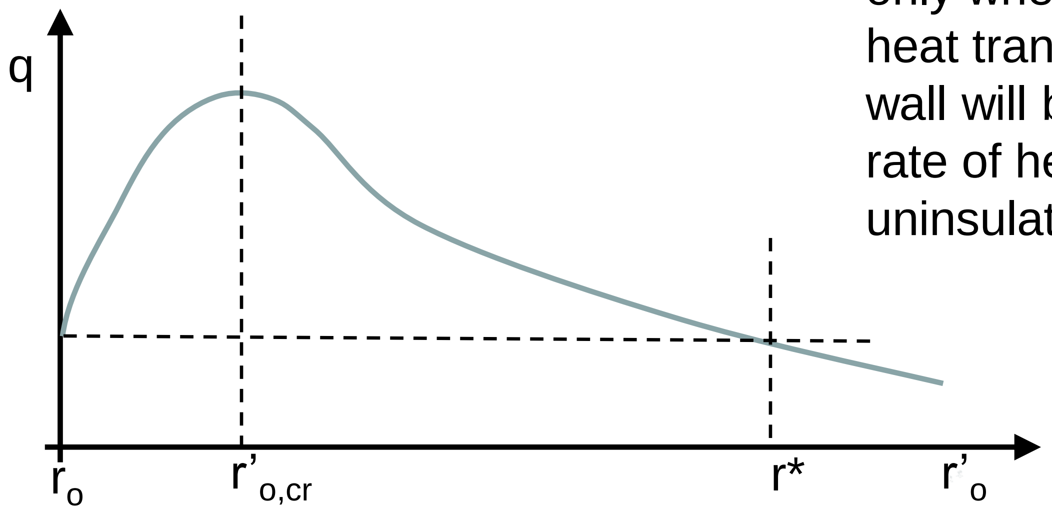
- Assuming all parameters constant but r'_o, the minimum thermal resistance can be found as

$$\frac{\partial R_t}{\partial r_o^{'}} = \frac{1}{2\pi L \lambda_{ins}}\frac{1}{r_o^{'}} - \frac{1}{h_o 2\pi \left(r_o^{'}\right)^2 L} = 0 \Rightarrow \qquad r_o^{'} = \frac{\lambda_{ins}}{h_o} = r_{o,cr}^{'} \quad \text{cylinder}$$

- For a sphere, a similar analysis gives the following critical thickness: $$r_o^{'} = \frac{2\lambda_{ins}}{h_o} = r_{o,cr}^{'} \quad \text{sphere}$$

Critical Thickness of Insulation (3)

- When insulation's outer radius is less than $r'_{o,cr}$, adding insulation will result with higher rate of heat transfer
- Only when $r'_o > r'_{o,cr}$, more insulation means decreased rate of heat transfer



only when $r'_o > r^*$, the rate of heat transfer for insulated wall will be less than the rate of heat transfer for uninsulated wall

Infinite Cylinder with Uniform Heating (1)

- The heat conduction equation is now

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{q'''}{\lambda}=0 \quad \text{or} \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{q'''}{\lambda}=0$$

- with boundary conditions

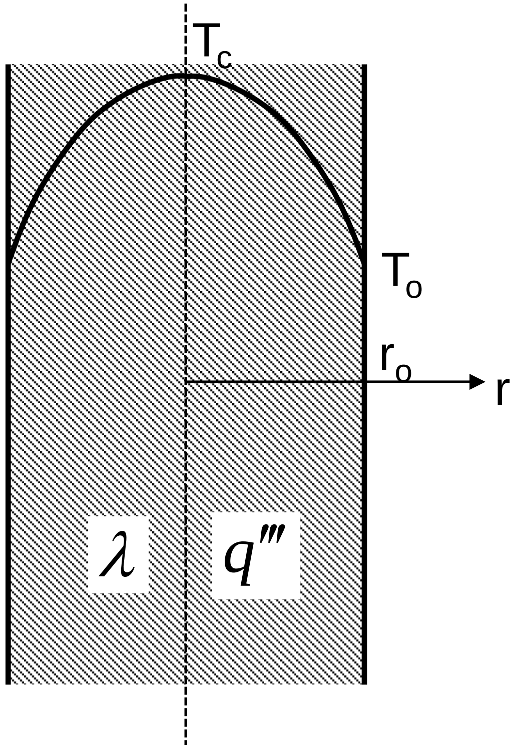
$$\left.\frac{dT}{dr}\right|_{r=0}=0, \quad \left.T(r)\right|_{r=r_o}=T_o$$

- where

q''' - heat rate per unit volume (W/m³)

λ - thermal conductivity (W/m.K)

T_o - surface temperature (K)

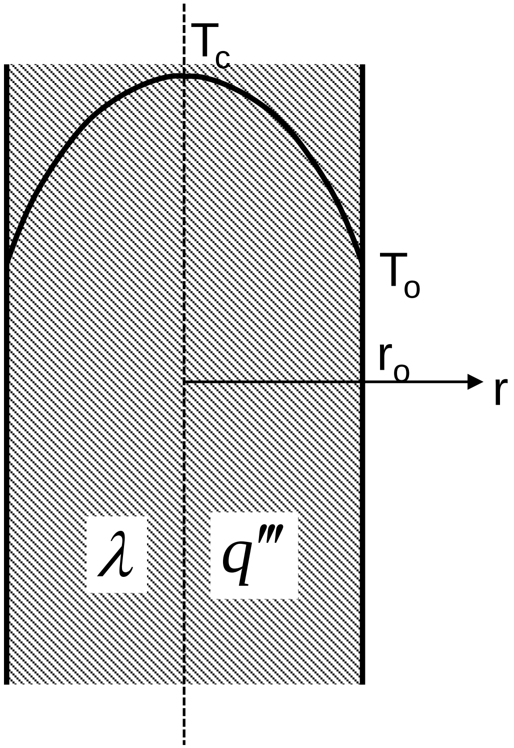


Infinite Cylinder with Uniform Heating (2)

- First integration of the equation yields

$$r\frac{dT}{dr} = -\frac{q''' r^2}{2\lambda} + C$$

- and boundary conditions $\left.\frac{dT}{dr}\right|_{r=0} = 0$ gives C = 0
- Second integration gives



$$T = -\frac{q''' r^2}{4\lambda} + D$$ and applying $\left.T(r)\right|_{r=r_o} = T_o$

yields: $D = T_o + \frac{q''' r_o^2}{4\lambda}$ thus $T = \frac{q''' r_o^2}{4\lambda}\left(1 - \frac{r^2}{r_o^2}\right) + T_o$

Infinite Cylinder with Uniform Heating (3)

- The solution $$T = \frac{q''' r_o^2}{4\lambda}\left(1 - \frac{r^2}{r_o^2}\right) + T_o$$

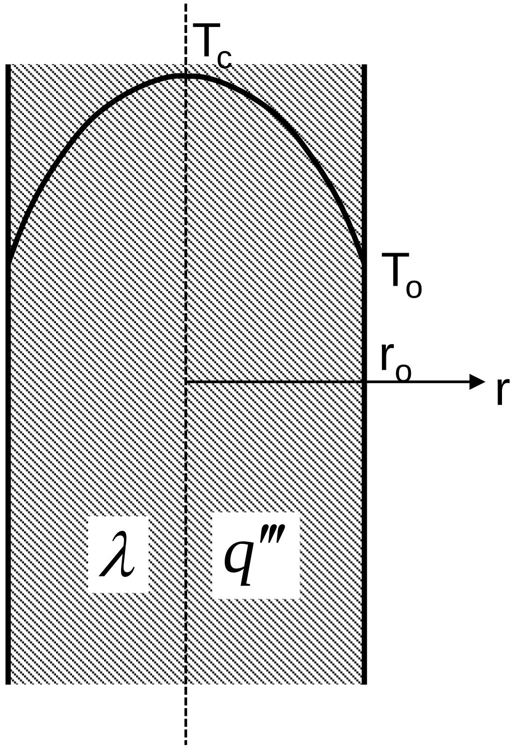
  indicates that the temperature at the centreline is:

$$T_c = \frac{q''' r_o^2}{4\lambda} + T_o$$

- We can find the linear power (power per unit length), q´(W/m) as: $q' = q''' \pi r_o^2$

thus $$T_c = \frac{q'}{4\lambda\pi} + T_o$$

We can note that the temperature at centreline depends on linear power and the temperature at the surface, but not on cylinder radius.



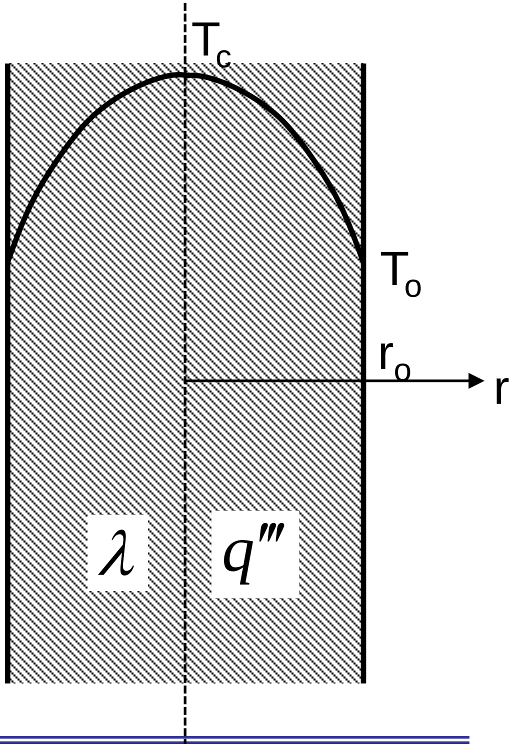
Infinite Cylinder with Nuclear Heating (1)

- The neutron flux in a fuel rod of a nuclear reactor is given as $\varphi = CI_0(kr)$, where I_0 – a modified Bessel function of the first kind, C, k – constants
- Thus, the thermal power has the following distribution: $q'''=AI_0(kr)$, and the conduction equation becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{A}{\lambda}I_0\left(kr\right)=0$$

- with boundary conditions

$$\left.\frac{dT}{dr}\right|_{r=0}=0, \quad \left.T(r)\right|_{r=r_o}=T_o$$

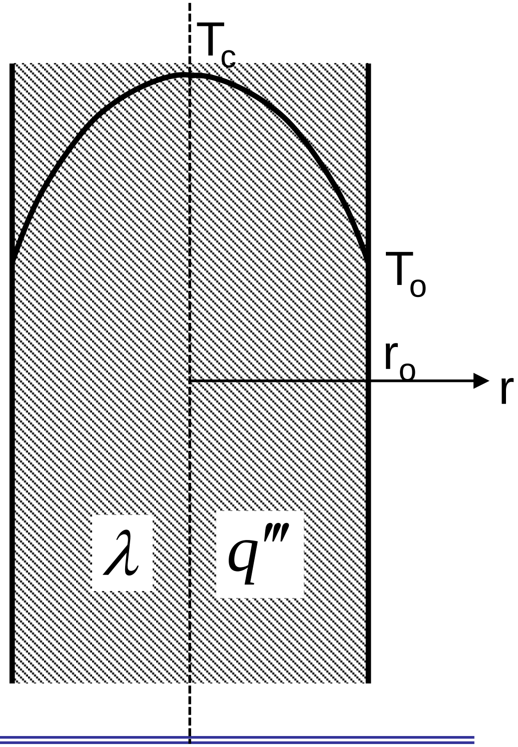


Infinite Cylinder with Nuclear Heating (2)

- The conduction equation can be transformed to the following form $$d\left(r\frac{dT}{dr}\right) = -\frac{A}{\lambda k^2} k r I_0(kr)\, d(kr)$$
- Now integration in a range from 0 to *r* gives $$r\frac{dT}{dr} = -\frac{A}{\lambda k^2} k r I_1(kr) \Rightarrow \frac{dT}{dr} = -\frac{A}{\lambda k} I_1(kr)$$
- And second integration yields

$$T = -\frac{A}{\lambda k^2} I_0(kr) + C$$ but since $$T(r)\Big|_{r=r_o} = T_o$$

we have $$T - T_o = \frac{A}{\lambda k^2}\left[I_0(kr_0) - I_0(kr)\right]$$



Infinite Cylinder with Nuclear Heating (3)

- The solution $T - T_o = \frac{A}{\lambda k^2}\left[I_0\left(kr_0\right) - I_0\left(kr\right)\right]$ contains constant *A* that can be expressed in terms of linear power q':

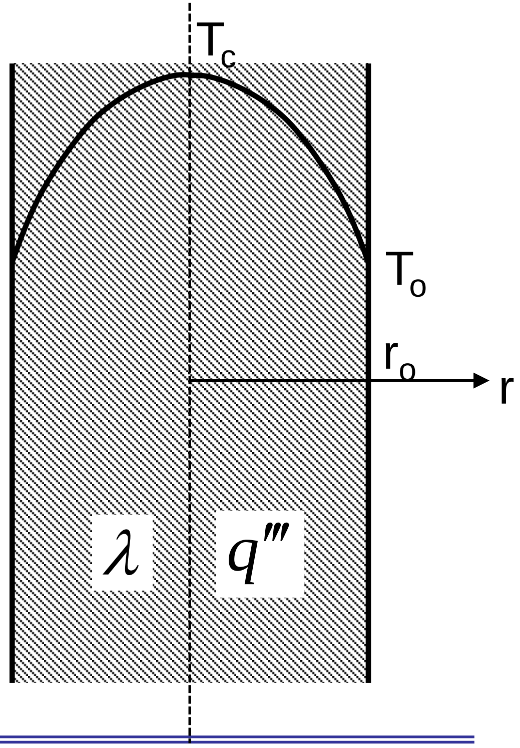
$$q' = \int_0^{r_o} q''' 2\pi r dr = 2\pi A \int_0^{r_o} r I_0\left(kr\right) dr \Rightarrow A = \frac{q' k}{2\pi r_o I_1\left(kr_o\right)}$$

- Thus the final form of the solution is:

$$T - T_o = \frac{q'}{2\pi\lambda k r_0} \frac{I_0\left(kr_0\right) - I_0\left(kr\right)}{I_1\left(kr_0\right)}$$

- and the maximum temperature T_c is:

$$T_c = \frac{q'}{2\pi\lambda k r_0} \frac{I_0\left(kr_0\right) - 1}{I_1\left(kr_0\right)} + T_o$$

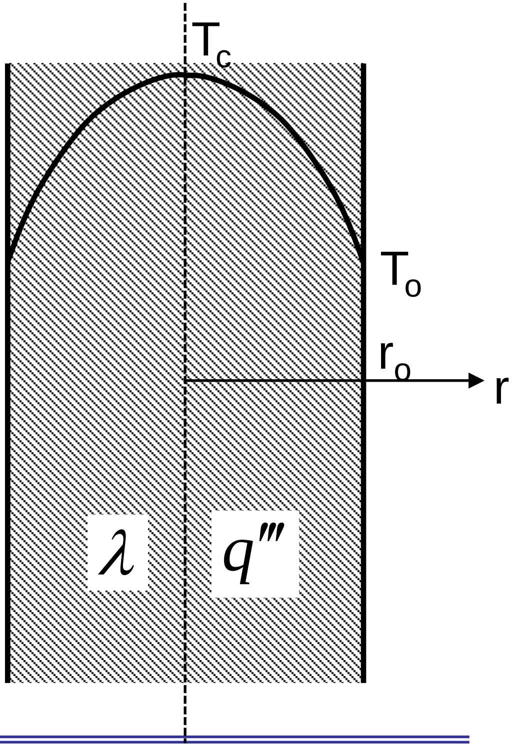


Infinite Cylinder with Nuclear Heating (4)

- The expression for the maximum temperature

$$T_c = \frac{q'}{2\pi\lambda k r_0}\frac{I_0\left(kr_0\right)-1}{I_1\left(kr_0\right)}+T_o$$

contains Bessel function combination that can be replaced with its series expansion: $$\frac{I_0\left(x\right)-1}{xI_1\left(x\right)} \approx \frac{1}{2}\left(1-\frac{x^2}{16}\right)$$



- So we have:

$$T_c = \frac{q'}{4\pi\lambda}\left[1-\frac{\left(kr_o\right)^2}{16}\right]+T_o = \underbrace{\frac{q'}{4\pi\lambda}+T_o}_{\text{uniform power distribution}} - \underbrace{\frac{q'}{4\pi\lambda}\frac{\left(kr_o\right)^2}{16}}_{\text{nonuniformity correction}}$$

Transient Conduction

- In transient heat conduction the temperature depends on time

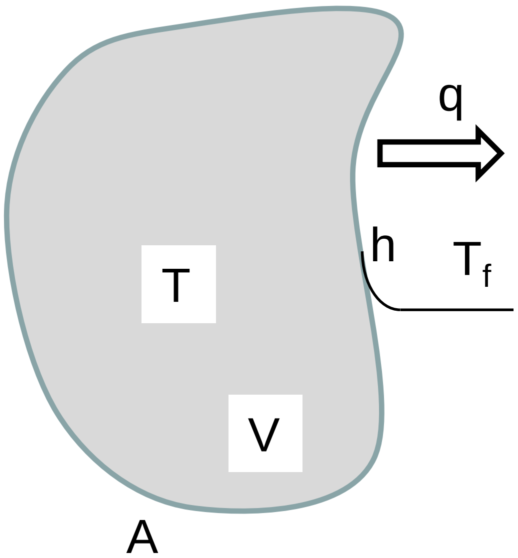
- The are three main approaches to solve such problems:
  - A lumped thermal capacity model
  - A semi-infinite region model
  - A finite-sized model

- More complicated regions are solved numerically using either finite-difference, finite-volume or finite-element method

Lumped Thermal Capacity (1)

- In this approach spatial temperature variations within the body are neglected

- For a body of arbitrary shape with volume V, surface area A, mass density ρ, specific heat c, surrounded with fluid having far-field temperature T_f, the energy equation becomes



$$\rho V c \frac{dT}{dt} = -hA\left(T - T_f\right)$$

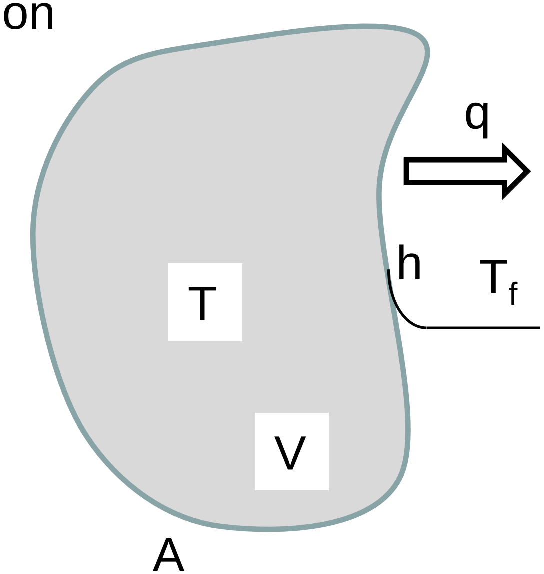
- And the initial condition is: $T\left(0\right) = T_0$

Lumped Thermal Capacity (2)

- Solution of the equation gives the following expression for the body temperature:

$$\frac{T - T_f}{T_0 - T_f} = e^{-hAt/\rho Vc}$$

here *ρVc/(hA)* has dimension (s) and is called TIME CONSTANT



- The cumulative thermal energy transferred over a period of time t is

$$q = \int_0^t \left(c\rho V \frac{dT}{dt} \right) dt = c\rho V \left(T_0 - T_f \right) \left(1 - e^{-hAt/\rho Vc} \right)$$

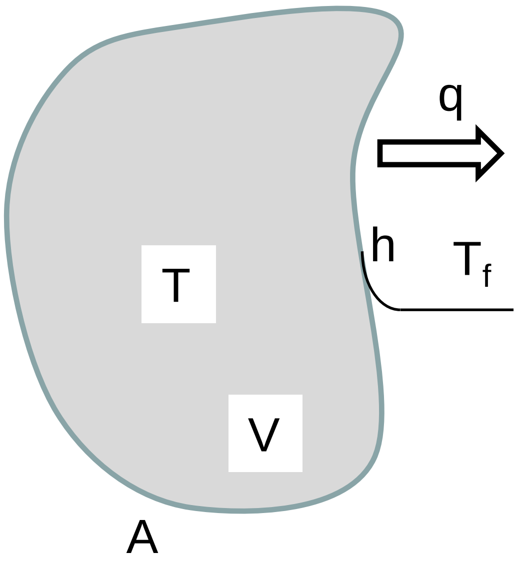
Lumped Thermal Capacity (3)

- We can express the solution equations in a non-dimensional form by introducing the following parameters:

$$L = \frac{V}{A}$$ equivalent length of the body

$$\mathrm{Bi} = \frac{hl}{\lambda}$$ Biot number, *l* – characteristic length

$$\tau = \frac{\lambda t}{\rho c l^2} = \frac{at}{l^2}$$ a=λ/(ρc) – thermal diffusivity



- Then we get

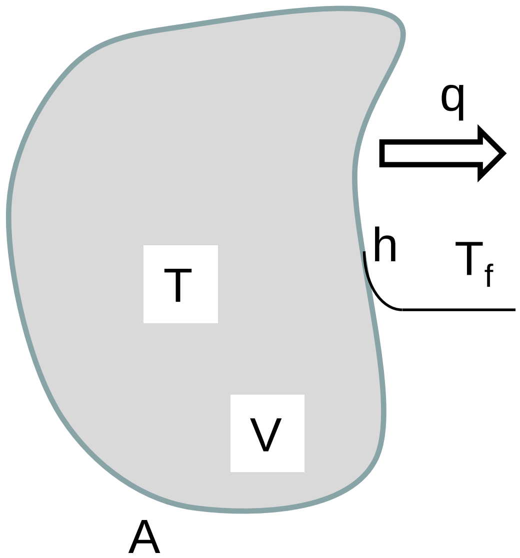
$$\frac{T - T_f}{T_0 - T_f} = e^{-\mathrm{Bi}\,\tau \frac{l}{L}}$$

The characteristic length *l* is taken as:
for infinite plate – half of its thickness
for infinite cylinder or sphere – its radius

Lumped Thermal Capacity (4)

- Lumped thermal capacity model is based on an assumption that the body's thermal conductivity is infinite

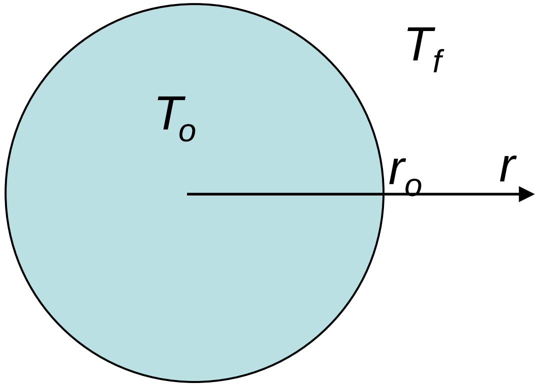
- Also, when the size of the body is not too big, and heat transfer on the body's surface is intensive, the approximation works well



- In general, when Bi < 0.1, the lumped thermal capacity model agrees with the exact model within about 5%

Transient Heat Conduction in a Sphere (1)

- A solid sphere is initially at constant temperature T_0. At time $t = 0$, the sphere is submerged into a fluid with constant far-field temperature T_f. Calculate the sphere mean temperature as a function of time.



$$a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial t}$$

$$T(r,0) = T_0$$

Initial temperature

$$-\lambda \frac{\partial T}{\partial n}\bigg|_{r=r_o} = h\left(T - T_f\right)\bigg|_{r=r_o}$$

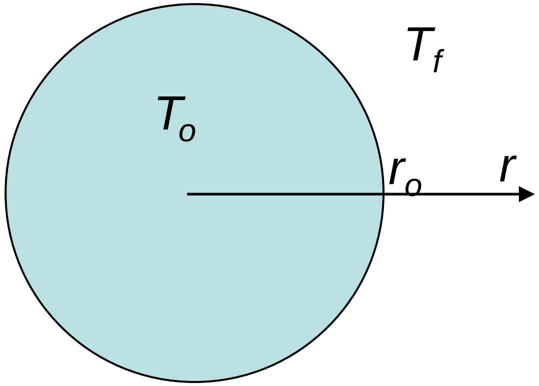
Heat convection to surrounding fluid

We can make this equation dimensionless if we use the following variables:

$$\theta = \frac{T - T_f}{T_0 - T_f} \qquad \tau = \frac{at}{r_o^2} \qquad \xi = \frac{r}{r_o} \qquad \mathrm{Bi} = \frac{hr_o}{\lambda}$$

Transient Heat Conduction in a Sphere (2)

In the dimensionless form, the set of equations is as follows:



$$\frac{\partial^2\theta}{\partial\xi^2}+\frac{2}{\xi}\frac{\partial\theta}{\partial\xi}=\frac{\partial\theta}{\partial\tau}\ \text{for}\ 0\le\xi<1\ \text{and}\ \tau>0$$

$$\frac{\partial\theta}{\partial\xi}+\mathrm{Bi}\,\theta=0\ \text{for}\ \xi=1$$

$$\theta=1\ \text{for}\ \tau=0$$

We anticipate the solution as a product of two functions:

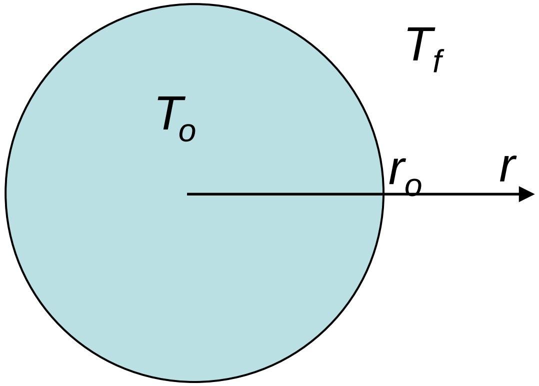
$$\theta=\mathrm{T}(\tau)\cdot\Theta(\xi)$$

And the equation becomes:

$$\frac{1}{\Theta}\left(\frac{d^2\Theta}{d\xi^2}+\frac{2}{\xi}\frac{d\Theta}{d\xi}\right)=\frac{1}{\mathrm{T}}\frac{d\mathrm{T}}{\partial\tau}=-\mu^2$$

Transient Heat Conduction in a Sphere (3)

Thus we need to solve two equations:



$$\frac{d^2\Theta}{d\xi^2} + \frac{2}{\xi}\frac{d\Theta}{d\xi} + \mu^2\Theta = 0$$

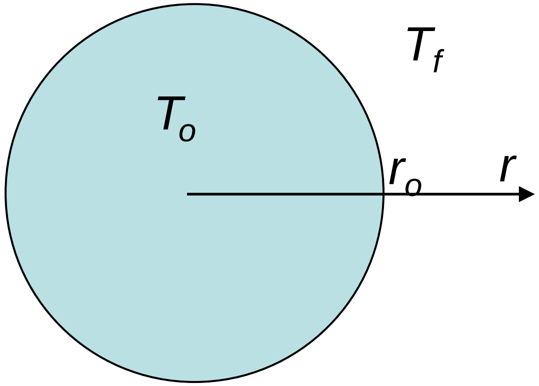
$$\frac{d\mathrm{T}}{d\tau} = -\mu^2\mathrm{T}$$

with condition at the surface:

$$\frac{\partial T\Theta}{\partial \xi} + \mathrm{Bi}T\Theta = T\left(\underbrace{\frac{d\Theta}{d\xi} + \mathrm{Bi}\Theta}_{0}\right) = 0 \text{ for } \xi = 1$$

Transient Heat Conduction in a Sphere (4)

For spatial coordinate, we need to solve:



$$\frac{d^2\Theta}{d\xi^2} + \frac{2}{\xi}\frac{d\Theta}{d\xi} + \mu^2\Theta = 0$$

$$\frac{d\Theta}{d\xi} + \mathrm{Bi}\Theta = 0 \text{ for } \xi = 1$$

This is a so-called Sturm-Liouville problem that has a solution for an infinite set of values of μ (so called eigenvalues):

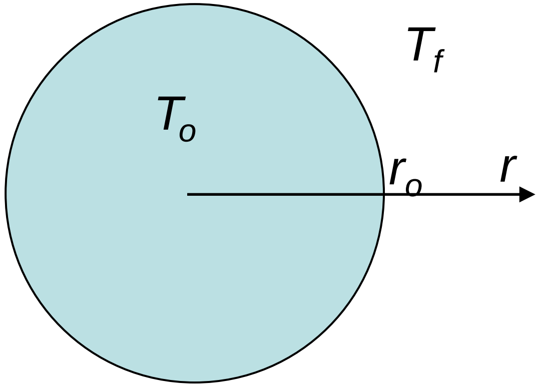
$$0 < \mu_1 < \mu_2 < \mu_3 < \mu_4 < \mu_5 < \cdots$$

With corresponding eigen functions:

$$\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \cdots$$

Transient Heat Conduction in a Sphere (5)

Equation $$\frac{d^2\Theta_k}{d\xi^2} + \frac{2}{\xi}\frac{d\Theta_k}{d\xi} + \mu_k^2\Theta_k = 0$$



has a solution:

$$\Theta_k = \frac{\sin \mu_k \xi}{\mu_k \xi}$$

Substituting the solution to the boundary condition: $$\frac{d\Theta_k}{d\xi} + \mathrm{Bi}\Theta_k = 0$$

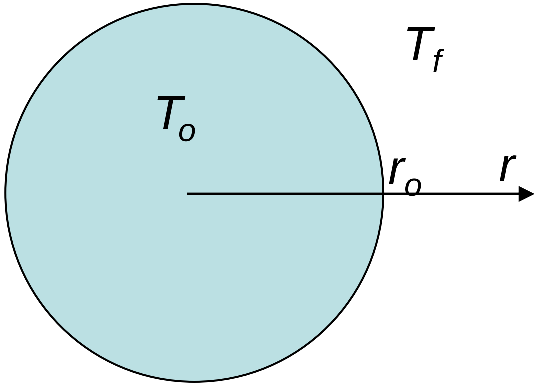
yields the following equation to determine the eigenvalues:

$$\tan \mu_k = -\frac{1}{\mathrm{Bi} - 1}\mu_k$$

Transient Heat Conduction in a Sphere (6)

The solution is then foud as:

$$\theta(\xi,\tau)=\sum_{k=1}^{\infty}A_k\frac{\sin\mu_k\xi}{\mu_k\xi}e^{-\mu_k^2\tau}$$



where:

$$A_k=\frac{2\left(\sin\mu_k-\mu_k\cos\mu_k\right)}{\mu_k-\sin\mu_k\cos\mu_k}=\left(-1\right)^{k+1}\frac{2\mathrm{Bi}\sqrt{\mu_k^2+\left(\mathrm{Bi}-1\right)^2}}{\mu_k^2+\mathrm{Bi}^2-\mathrm{Bi}}$$

General 1D Transient Model (1)

- We can write the transient one-dimensional conduction equation in a general form as

$$\frac{1}{\xi^n}\frac{\partial}{\partial \xi}\left(\xi^n \frac{\partial \theta}{\partial \xi}\right) = \frac{\partial \theta}{\partial \tau}$$

with boundary condition

$$\frac{\partial \theta}{\partial \xi} + \mathrm{Bi}\,\theta = 0 \text{ for } \xi = 1$$

and initial condition

$$\theta = 1 \text{ for } \tau = 0$$

Here n = 0 for an infinite plane, n = 1 for an infinite cylinder and n = 2 for a sphere.
In case of a plane ξ is measured from the center plane with ξ=1 at the surface.
In case of a cylinder and a sphere ξ corresponds to the radius

General 1D Transient Model (2)

- The general problem has the following solution:

$$\theta(\xi,\tau)=\sum_{k=1}^{\infty}\frac{2\mathrm{Bi}}{\mu_k^2+\mathrm{Bi}^2+2\nu\mathrm{Bi}}\frac{\xi^{\nu}J_{-\nu}(\mu_k\xi)}{J_{-\nu}(\mu_k)}e^{-\mu_k^2\tau}$$

v=(1-n)/2:
= ½ - plane
= 0 – cylinder
= -½ - sphere

where μ_k are the eigenvalues given by

$$\mu_k J_{-(\nu-1)}(\mu_k)=\mathrm{Bi}J_{-\nu}(\mu_k)$$

J_p – Bessel function of the first kind and *p*-th order

and the cumulative heat is

$$q(\tau)=c\rho V\left(T_f-T_0\right)\sum_{k=1}^{\infty}\frac{2(k+1)\mathrm{Bi}^2\left(1-e^{-\mu_k^2\tau}\right)}{\mu_k^2\left(\mu_k^2+\mathrm{Bi}^2+2\nu\mathrm{Bi}\right)}$$

General 1D Transient Model (3)

- We can note that involved fractional order Bessel functions are as follows:

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x$$

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$

$$J_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\frac{\sin x}{x} - \cos x\right)$$