Convergent Sequences

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Overview

- Linear Convergence
- Rate of Convergence
- Order of Convergence
- Logarithmic Convergence
- Aitken's Δ^2-Process
- Iterated Aitken's del-Squared Process
- Other Acceleration Techniques

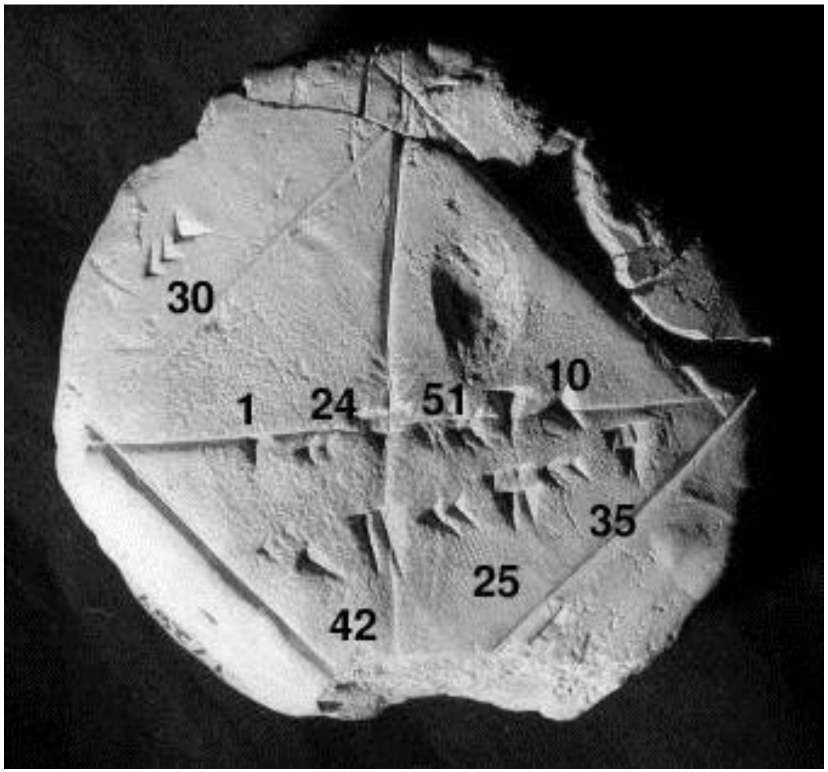
Sequences

$$F(x,d)=0; \qquad x_n \xrightarrow[n\to\infty]{} x; \qquad x \approx x_N \quad N \gg 1$$

$$\sqrt{a} \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \xrightarrow[n\to\infty]{} \sqrt{a}$$

$$\left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e; \qquad 1-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{(-1)^{n-1}}{2n-1} \xrightarrow[n\to\infty]{} \frac{\pi}{4}$$

$\sqrt{2}$ in Babylon



Babylonian clay tablet

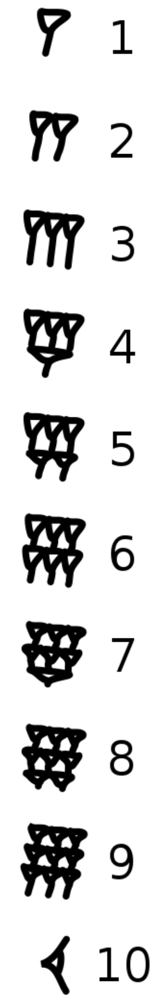
1800 – 1600 BC

$$\sqrt{2} = 1.(24)(51)(10)$$

$$= 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$$

$$= 1.4142129$$

$$= 1.4142135 \quad (ex)$$



Rate of Convergence

Let x_n be convergent i.e. $x_n \xrightarrow[n\to\infty]{} L$

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \mu \qquad \mu = \begin{cases} 0 & \text{Superlinear} \\ 0 < \mu < 1 & \text{Linear} \\ 1 & \text{Sublinear} \end{cases}$$

Rate of convergence $= \mu$

Linear Convergence

$$\frac{|x_{n+1} - L|}{|x_n - L|} \xrightarrow[n\to\infty]{} \mu \neq 0$$

$$|x_{n+1} - L| \approx \mu |x_n - L| \approx \ldots \approx \mu^n |x_1 - L|$$

$$\mu = 0.1 \longrightarrow |e_{n+1}| \approx (0.1)^n \times |e_1|$$

$$\mu = 0.9 \longrightarrow p \approx 0.05$$

Order of Convergence

In case of superlinear convergence i.e. when $\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = 0$

$$\exists p > 1 \qquad \lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|^p} = C > 0$$

Order of convergence = p $\qquad |x_{n+1} - L| \approx C|x_n - L|^p$

Logarithmic Convergence

When $$\lim_{n\to\infty}\frac{|x_{n+1}-L|}{|x_n-L|}=1 \longrightarrow |x_{n+1}-L| \approx |x_n-L|$$

True $\downarrow$ $\quad$ $\uparrow$ Not necessarily true!

$$\lim_{n\to\infty}\frac{|x_{n+2}-x_{n+1}|}{|x_{n+1}-x_n|}=1 \longrightarrow |x_{n+2}-x_{n+1}| \approx |x_{n+1}-x_n|$$

Examples

$$x_n = \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0 \qquad \frac{x_{n+1} - 0}{x_n - 0} = \frac{1}{2}$$

$$x_n = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \frac{1}{65536}, \frac{1}{4294967296}, \ldots \right\}$$

$$x_n = \frac{1}{2^{2^n}} \xrightarrow[n\to\infty]{} 0 \qquad \frac{x_{n+1} - 0}{\left(x_n - 0\right)^2} = \frac{\left(2^{2^n}\right)^2}{2^{2^{n+1}}} = 1$$

$$x_n = 1 - \frac{1}{2} + \frac{1}{3} + \ldots + \frac{\left(-1\right)^{n+1}}{n} \xrightarrow[n\to\infty]{} \ln 2$$

Transforming Sequences

$$x_n \xrightarrow[n\to\infty]{} L; \qquad y_n = T\left(x_n\right) \xrightarrow[n\to\infty]{} L$$

$$x_n = L + r_n; \qquad y_n = L + p_n$$

$$y_n = \sum_{i=1}^{\infty} a_{n,i} x_i \qquad y_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Linearly Convergent Series

$$x_n = L + c\lambda^n \longrightarrow \frac{x_{n+1} - L}{x_n - L} = \frac{c\lambda^{n+1}}{c\lambda^n} = \lambda$$

$$x_n = L + c'$$
$$x_{n+1} = L + c'\lambda$$
$$x_{n+2} = L + c'\lambda^2$$

$$x_{n+1} - x_n = c'(\lambda - 1)$$
$$x_{n+2} - x_{n+1} = c'\lambda(\lambda - 1)$$

$$\lambda = \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}$$

Uncovering Limit

$$x_{n+1} - x_n = c'(\lambda - 1) \rightarrow c' = \frac{x_{n+1} - x_n}{\lambda - 1} = \frac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n}$$

$$L = x_n - c' = x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n} = \frac{x_{n+2}x_n - x_{n+1}^2}{x_{n+2} - 2x_{n+1} + x_n}$$

Del Operator

$$L = x_n - \frac{\left(x_{n+1} - x_n\right)^2}{x_{n+2} - 2x_{n+1} + x_n} \qquad \Delta x_n \equiv x_{n+1} - x_n$$

$$\Delta^2 x_n \equiv \Delta\Delta x_n = \Delta\left(x_{n+1} - x_n\right) = \Delta x_{n+1} - \Delta x_n =$$

$$= \left(x_{n+2} - x_{n+1}\right) - \left(x_{n+1} - x_n\right) = x_{n+2} - 2x_{n+1} + x_n$$

$$L = x_n - \frac{\left(\Delta x_n\right)^2}{\Delta^2 x_n} \equiv \left(A\mathbf{x}\right)_n \qquad \mathbf{x} \equiv \left\{x_0, x_1, \ldots\right\}$$

Aitken's Δ^2 Acceleration

$$x_n \approx L + c\lambda^n$$

$$x_n = C_1 a^n + C_2 b^{n+1} + \cdots$$

$$L \approx x_n - \frac{\left(\Delta x_n\right)^2}{\Delta^2 x_n}$$

$$\left(Ax\right)_n = C_2' b^{n+1} + \cdots$$

Example Calculations

$$s_n = 1 - \frac{1}{3} + \frac{1}{5} + \ldots + \frac{(-1)^{n-1}}{2n-1} \xrightarrow[n\to\infty]{} \frac{\pi}{4}$$

n	$4s_n$	$4(As)_n$
1	4	–
2	2.666667	–
3	3.466667	3.166666
4	2.895238	3.133333
5	3.339683	3.145238
6	2.976046	3.139683
7	3.283738	3.142713
8	3.017072	3.140881
9	3.252366	3.142072

$$4s_{627} = 3.14318$$

Aitken's Iterated Δ^2 Process

$$\begin{tikzcd}
x_1 \arrow[dr] & & & \\
x_2 \arrow[r] & A_1^{(1)} \arrow[dr] & & \\
x_3 \arrow[ur] & A_2^{(1)} \arrow[r] & A_1^{(2)} \arrow[dr] & \\
x_4 & A_3^{(1)} \arrow[ur] & A_2^{(2)} \arrow[r] & A_1^{(3)} \\
x_5 & A_4^{(1)} & A_3^{(2)} \arrow[ur] & \\
x_6 & A_5^{(1)} & & \\
x_7 & & &
\end{tikzcd}$$

$$A_n^{(0)} \equiv x_n$$

$$A_n^{(k+1)} = A_n^{(k)} - \frac{\left(\Delta A_n^{(k)}\right)^2}{\Delta^2 A_n^{(k)}}$$

Example of Aitken's Iterated

x_1

4			
2.666667	3.166667		
3.466667	3.133333	3.142105	
2.895238	3.145238	3.141450	3.141599
3.339683	3.139683	3.141643	
2.976046	3.142713		
3.283738			

$x_9 = 3.25\ldots$

$$x_{1000000} = 3.141591$$

$$\pi = 3.14159265$$

Wynn's Epsilon Algorithm

$$\varepsilon_n^{(-1)} = 0; \qquad \varepsilon_n^{(0)} = x_n; \qquad \varepsilon_n^{(k+1)} = \varepsilon_{n+1}^{(k-1)} + \frac{1}{\varepsilon_{n+1}^{(k)} - \varepsilon_n^{(k)}}.$$

$$\begin{array}{cccccc}
 & x_1 & & & & \\
0 & & \varepsilon_1^{(1)} & & & \\
 & x_2 & & \varepsilon_1^{(2)} & & \\
0 & & \varepsilon_2^{(1)} & & \varepsilon_1^{(3)} & \\
 & x_3 & & \varepsilon_2^{(2)} & & \varepsilon_1^{(4)} \\
0 & & \varepsilon_3^{(1)} & & \varepsilon_2^{(3)} & \\
 & x_4 & & \varepsilon_3^{(2)} & &
\end{array}$$

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