Successive Over-Relaxation

Vasily Arzhanov

Reactor Physics, KTH

Overview

- Convergence condition
- Stopping criterion
- Optimal SOR parameter
- Red-Black ordering
- Hexagonal FD meshes
- Block modifications

Iteration Matrix

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \mathbf{A} = \mathbf{P} - \mathbf{N}$$

$$\mathbf{P}\mathbf{x} = \mathbf{N}\mathbf{x} + \mathbf{b}$$

$$\mathbf{P}\mathbf{x}^{(k+1)} = \mathbf{N}\mathbf{x}^{(k)} + \mathbf{b} \qquad \mathbf{x}^{(k+1)} = \mathbf{P}^{-1}\mathbf{N}\mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{b}$$

$$\mathbf{x}^{(k+1)} = \mathbf{B}\mathbf{x}^{(k)} + \mathbf{f} \qquad \mathbf{B} \equiv \mathbf{P}^{-1}\mathbf{N} = \mathbf{I} - \mathbf{P}^{-1}\mathbf{A}$$

Convergence Condition

$$\mathbf{e}^{(k)} \equiv \mathbf{x}^{(k)} - \mathbf{x} \longrightarrow \mathbf{e}^{(k)} = \mathbf{B}\mathbf{e}^{(k-1)} = \mathbf{B}^k \mathbf{e}^{(0)}$$

$$\mathbf{B}\mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \mathbf{e}^{(0)} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

$$\mathbf{B}^k \mathbf{e}^{(0)} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n$$

$$\rho(\mathbf{B}) < 1 \longrightarrow \forall i \quad |\lambda_i| < 1 \longrightarrow \mathbf{B}^k \mathbf{e}^{(0)} \xrightarrow[k\to\infty]{} 0$$

$$\rho(\mathbf{B}) > 1 \longrightarrow \exists l \quad |\lambda_l| > 1 \longrightarrow c_l\lambda_l^k\mathbf{v}_l \xrightarrow[k\to\infty]{} \infty$$

Definitions

Convergence Factor after k: $\left\|\mathbf{B}^k\right\|$

Average Convergence Factor: $\left\|\mathbf{B}^k\right\|^{1/k} \xrightarrow[k\to\infty]{} \rho(\mathbf{B})$

Average Convergence Rate: $R_k(\mathbf{B}) = -\frac{1}{k}\log\left(\left\|\mathbf{B}^k\right\|\right)$

Assymptotic Convergence Rate: $R(\mathbf{B}) = \lim_{k\to\infty} R_k(\mathbf{B}) = -\log\rho(\mathbf{B})$

Stoping Criterion

$$\frac{\left\|\mathbf{e}^{(k)}\right\|}{\left\|\mathbf{e}^{(0)}\right\|} \leq \varepsilon \sim 10^{-6} \qquad \mathbf{e}^{(k)} = \mathbf{B}^k \mathbf{e}^{(0)}$$

$$\frac{\left\|\mathbf{e}^{(k)}\right\|}{\left\|\mathbf{e}^{(0)}\right\|} \leq \left\|\mathbf{B}^k\right\| \approx \rho^k\left(\mathbf{B}\right) \leq \varepsilon$$

$$k \geq \frac{\log \varepsilon}{\log \rho\left(\mathbf{B}\right)}$$

SOR Method

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) +$$

$$+ \left(1-\omega\right) x_i^{(k)}$$

2D NDE

$$-\phi_{xx}(x,y)-\phi_{yy}(x,y)+B^2\phi(x,y)=S(x,y) \qquad a\le x,y\le b$$

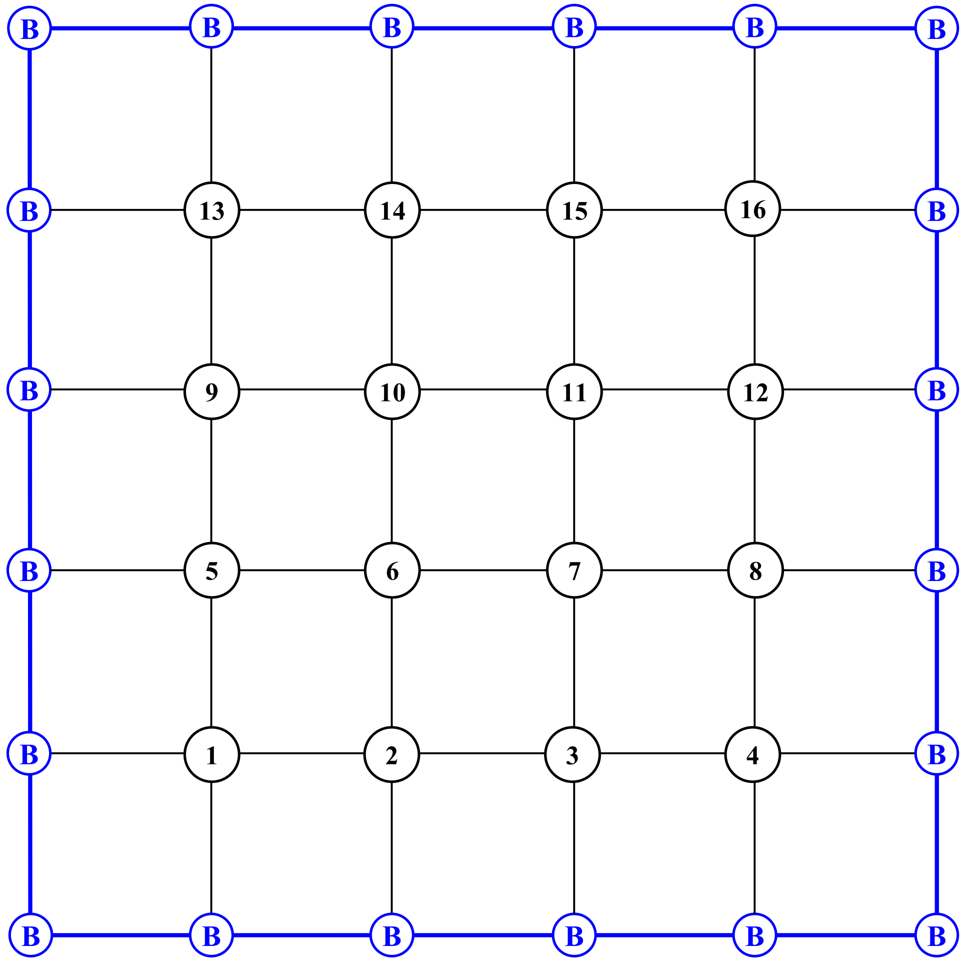
$$x_i=a+i*h; \qquad y_j=a+j*h; \qquad i,j=0,1,\ldots,N+1.$$

$$\phi_{i,j}\equiv\phi(x_i,y_j); \qquad S_{i,j}\equiv S(x_i,y_j).$$

$$\phi_{xx}(x_i,y_j)\approx\frac{\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}}{h^2}; \quad \phi_{yy}(x_i,y_j)\approx\frac{\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}}{h^2};$$

$$-\phi_{i,j-1}-\phi_{i-1,j}+\left(4+B^2h^2\right)\phi_{i,j}-\phi_{i+1,j}-\phi_{i,j+1}=S_{i,j}h^2 \qquad i,j=1,\ldots,N$$

2D FD Mesh



GS for 2D NDE

$$-\phi_{i,j-1} - \phi_{i-1,j} + \left(4 + B^2h^2\right)\phi_{i,j} - \phi_{i+1,j} - \phi_{i,j+1} = S_{i,j}h^2$$

$$-\phi_{i,j-1}^{(k+1)} - \phi_{i-1,j}^{(k+1)} + \left(4 + B^2h^2\right)\phi_{i,j}^{(k+1)} - \phi_{i+1,j}^{(k)} - \phi_{i,j+1}^{(k)} = S_{i,j}h^2 \qquad i,j = 1,\ldots,N$$

$$\phi_{i,j}^{(k+1)} = \frac{1}{4 + B^2h^2}\left(S_{i,j}h^2 + \phi_{i,j-1}^{(k+1)} + \phi_{i-1,j}^{(k+1)} + \phi_{i+1,j}^{(k)} + \phi_{i,j+1}^{(k)}\right)$$

SOR for 2D NDE

$$-\phi_{i,j-1} - \phi_{i-1,j} + \left(4 + B^2h^2\right)\phi_{i,j} - \phi_{i+1,j} - \phi_{i,j+1} = S_{i,j}h^2$$

$$-\phi_{i,j-1}^{(k+1)} - \phi_{i-1,j}^{(k+1)} + \left(4 + B^2h^2\right)\phi_{i,j}^{(k+1)} - \phi_{i+1,j}^{(k)} - \phi_{i,j+1}^{(k)} = S_{i,j}h^2 \qquad i,j = 1,\ldots,N$$

$$\phi_{i,j}^{(k+1)} = \frac{\omega}{4 + B^2h^2}\left(S_{i,j}h^2 + \phi_{i,j-1}^{(k+1)} + \phi_{i-1,j}^{(k+1)} + \phi_{i+1,j}^{(k)} + \phi_{i,j+1}^{(k)}\right) +$$

$$+\left(1 - \omega\right)\phi_{i,j}^{(k)}$$

SOR for General FD

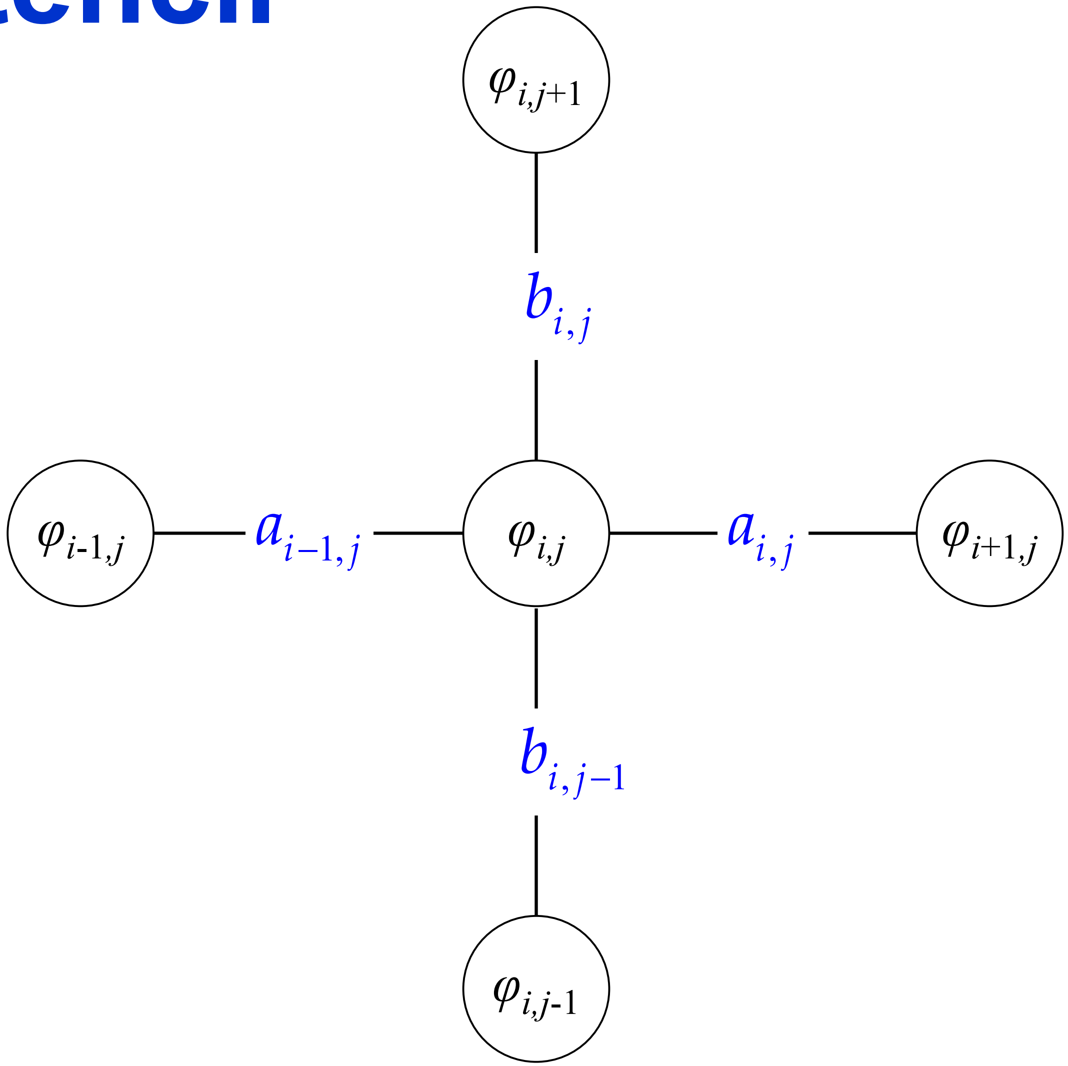
$$-b_{i,j-1}\phi_{i,j-1} - a_{i-1,j}\phi_{i-1,j} + c_{i,j}\phi_{i,j} - a_{i,j}\phi_{i+1,j} - b_{i,j}\phi_{i,j+1} = q_{i,j}$$

$$\phi_{i,j} = \frac{1}{c_{i,j}}\left[q_{i,j} + b_{i,j-1}\phi_{i,j-1} + a_{i-1,j}\phi_{i-1,j} + a_{i,j}\phi_{i+1,j} + b_{i,j}\phi_{i,j+1}\right]$$

$$\hat{\phi}_{i,j}^{(k+1)} = \frac{1}{c_{i,j}}\left[q_{i,j} + b_{i,j-1}\phi_{i,j-1}^{(k+1)} + a_{i-1,j}\phi_{i-1,j}^{(k+1)} + a_{i,j}\phi_{i+1,j}^{(k)} + b_{i,j}\phi_{i,j+1}^{(k)}\right]$$

$$\phi_{i,j}^{(k+1)} = \omega\hat{\phi}_{i,j}^{(k+1)} + \left(1-\omega\right)\phi_{i,j}^{(k)}$$

5-Point Stencil



Convergence Conditions for SOR Method

For any ω it holds $\rho(\mathbf{B}_\omega) \geq |\omega - 1|$ therefore SOR fails to converge if $\omega \leq 0$ or $\omega \geq 2$.

Necessary condition: $0 < \omega < 2$

If $\mathbf{A}$ is symmetric and positive definite, then SOR is convergent iff $0 < \omega < 2$

If $\mathbf{A}$ enjoys the A-property and if $\mathbf{B_J}$ has real eigenvalues, then SOR converges iff $\rho(\mathbf{B_J}) < 1$ and $0 < \omega < 2$ moreover

Optimal SOR Parameter

$$\mu \equiv \rho\left(\mathbf{B_J}\right)$$

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\mu^2}} = 1 + \left(\frac{\mu}{1+\sqrt{1-\mu^2}}\right)^2$$

Convergence Rate

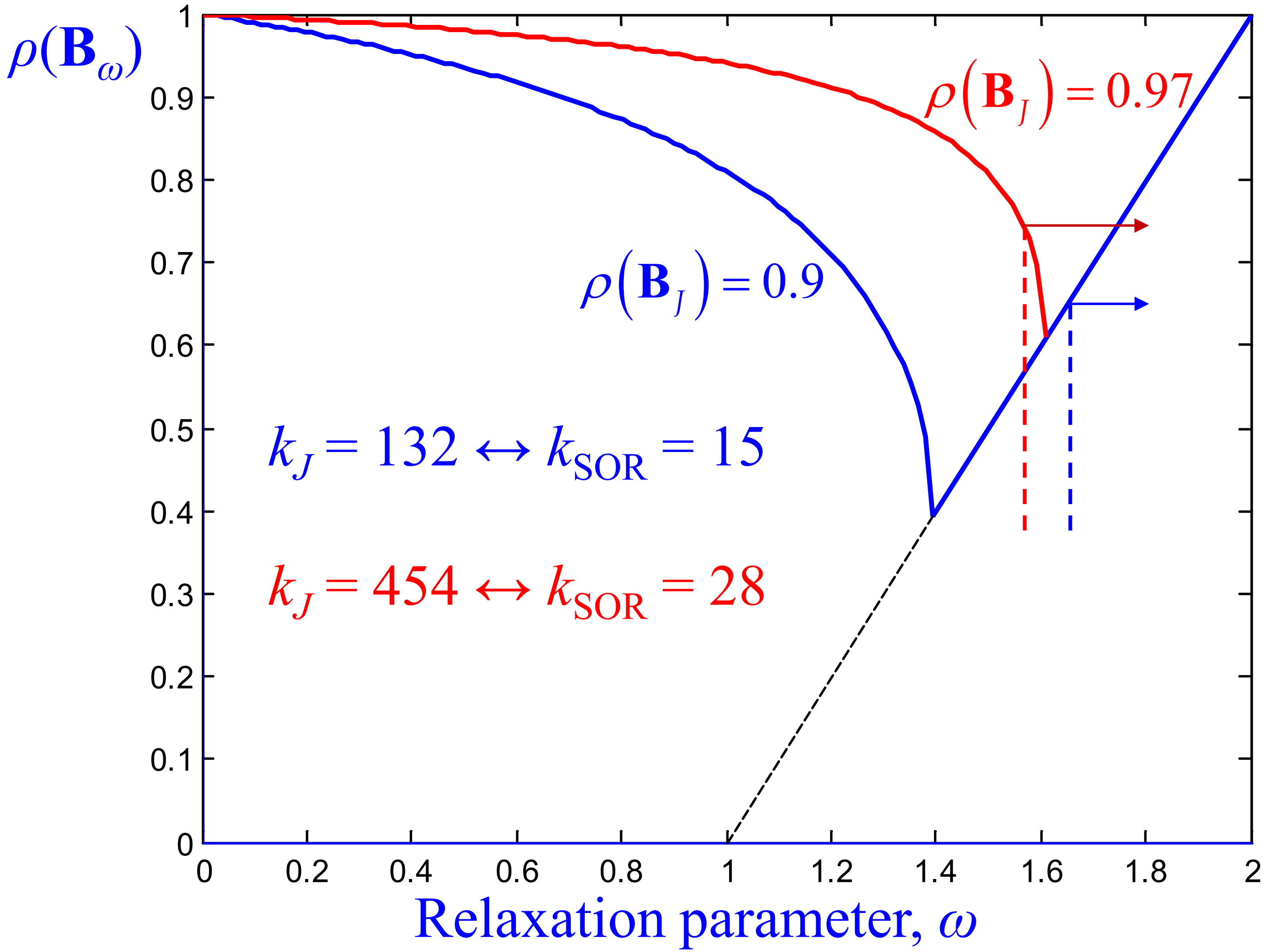
$$\rho\left(\mathbf{B}_{opt}\right) = \frac{1-\sqrt{1-\mu^2}}{1+\sqrt{1-\mu^2}}$$

SOR Spectral Radius

$$\mu \equiv \rho\left(\mathbf{B}_J\right)$$

$$\rho\left(\mathbf{B}_\omega\right) = \begin{cases} \left[\omega\mu/2 + \sqrt{\left(\omega\mu/2\right)^2 + 1 - \omega}\right]^2 & 0 < \omega \le \omega_{opt} \\ \omega - 1 & \omega_{opt} \le \omega < 2 \end{cases}$$

Spectral Radius Plot



Consistently Ordered

$$\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$$

$$\mathbf{B} = \mathbf{D}^{-1}\left(\mathbf{L} + \mathbf{U}\right)$$

$$\mathbf{B}(\alpha) \equiv \mathbf{D}^{-1}\left(\alpha\mathbf{L} + \frac{1}{\alpha}\mathbf{U}\right)$$

Property A

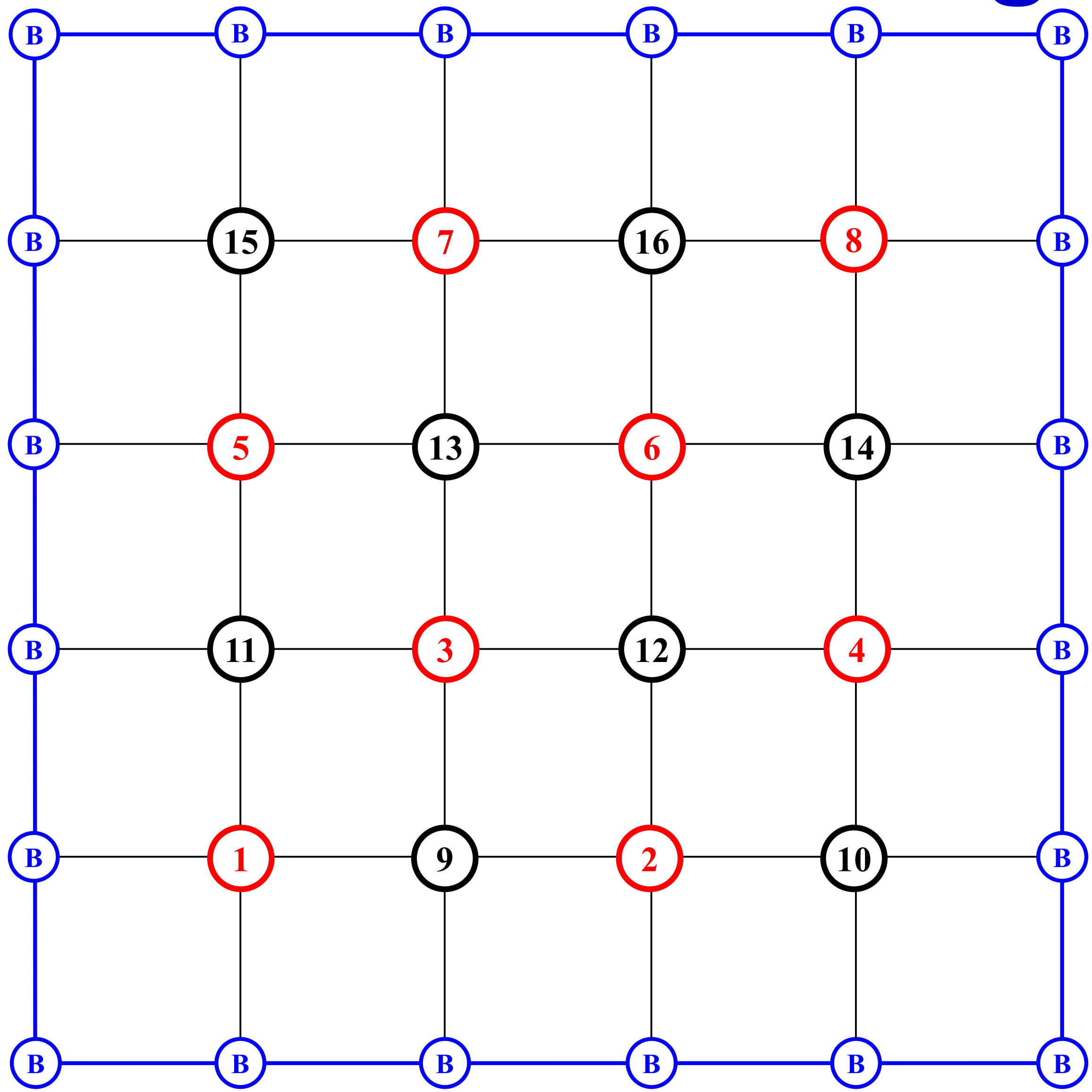
A consistently ordered matrix **A** enjoys the A-property if it can be partitioned as

$$\mathbf{PAP}^T = \left[\begin{array}{c:c} \mathbf{D}_1 & \mathbf{A}_{12} \\ \hdashline \mathbf{A}_{21} & \mathbf{D}_2 \end{array}\right]$$

$$\begin{bmatrix} \mathbf{D}_1 & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

$$\begin{cases} \mathbf{D}_1\mathbf{x}_1 = \mathbf{b}_1 - \mathbf{A}_{12}\mathbf{x}_2 \\ \mathbf{D}_2\mathbf{x}_2 = \mathbf{b}_2 - \mathbf{A}_{21}\mathbf{x}_1 \end{cases}$$

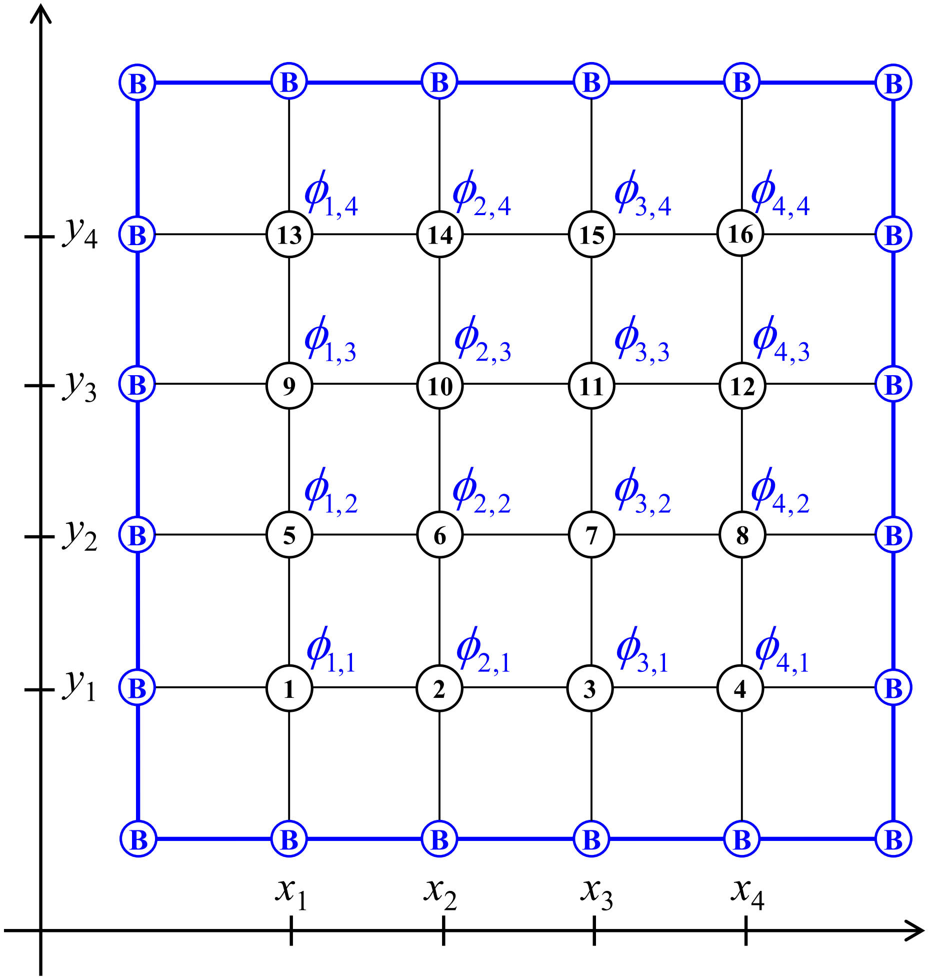
Red-Black Ordering

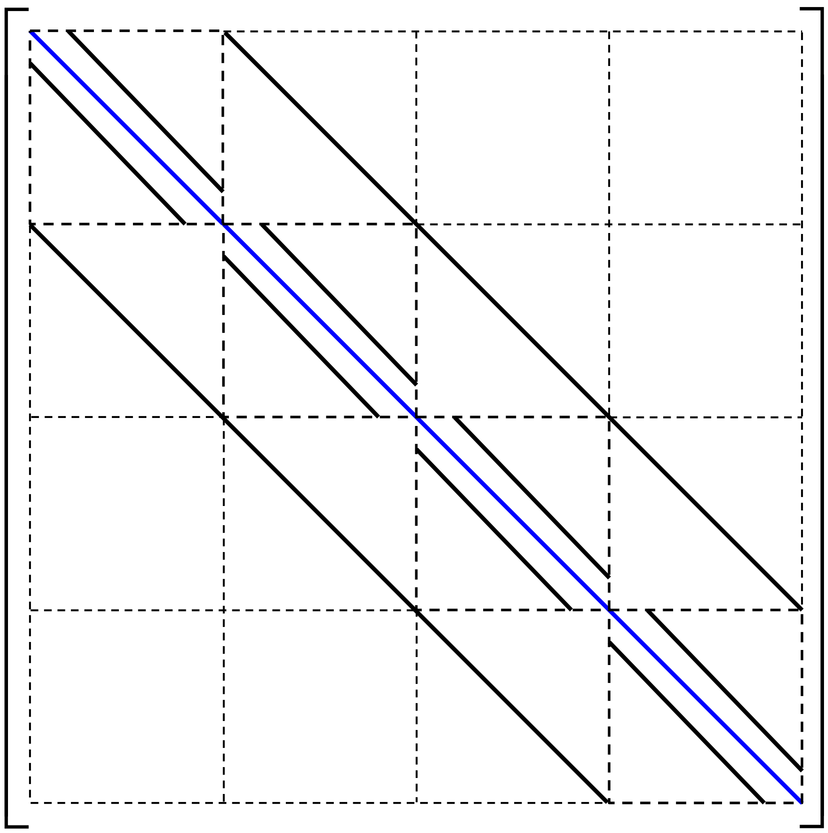


Red-Black Partioning

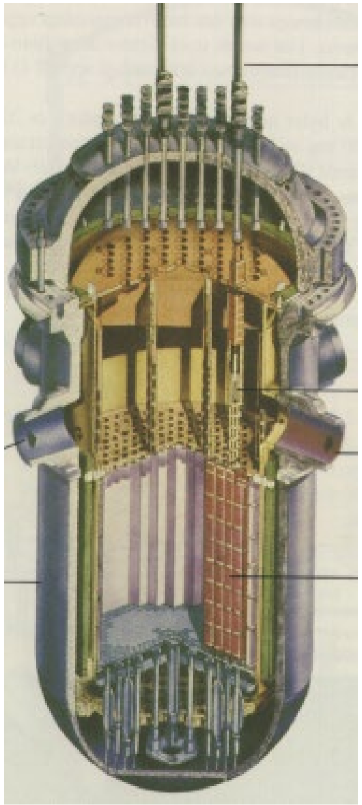
$$
\begin{array}{c}
\begin{array}{cccccccccccccccc}
1 & & & & & & & & 9 & & 11 & & & & & \\
\end{array} \\
\left[\begin{array}{cccccccc|cccccccc}
* & & & & & & & & * & & * & & & & & \\
 & * & & & & & & & * & * & & * & & & & \\
 & & * & & & & & & * & & * & * & * & & & \\
 & & & * & & & & & & * & & * & & * & & \\
 & & & & * & & & & & & * & & * & & * & \\
 & & & & & * & & & & & & * & * & * & & * \\
 & & & & & & * & & & & & & * & & * & * \\
 & & & & & & & * & & & & & & * & & * \\
\hline
* & * & * & & & & & & * & & & & & & & \\
 & * & & * & & & & & & * & & & & & & \\
* & & * & & * & & & & & & * & & & & & \\
 & * & * & * & & * & & & & & & * & & & & \\
 & & * & & * & * & * & & & & & & * & & & \\
 & & & * & & * & & * & & & & & & * & & \\
 & & & & * & & * & & & & & & & & * & \\
 & & & & & * & * & * & & & & & & & & * \\
\end{array}\right]
\end{array}
\left[\begin{array}{c}
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16
\end{array}\right]
$$

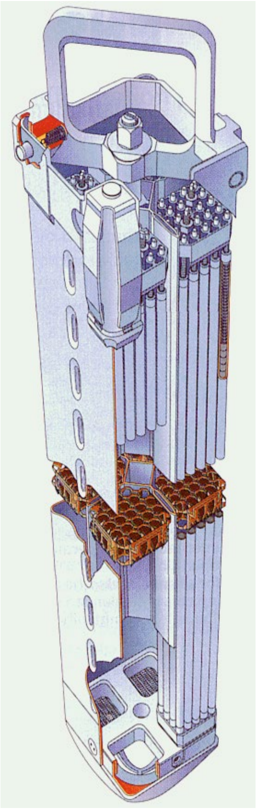
Regular FD Mesh

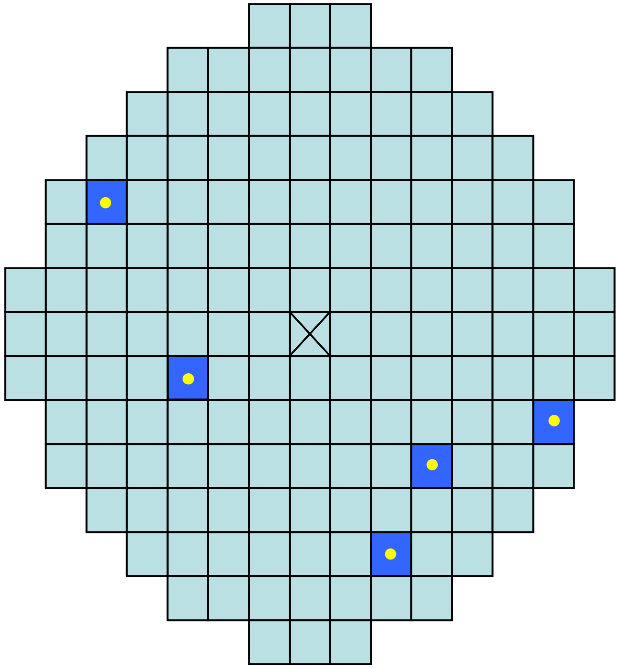




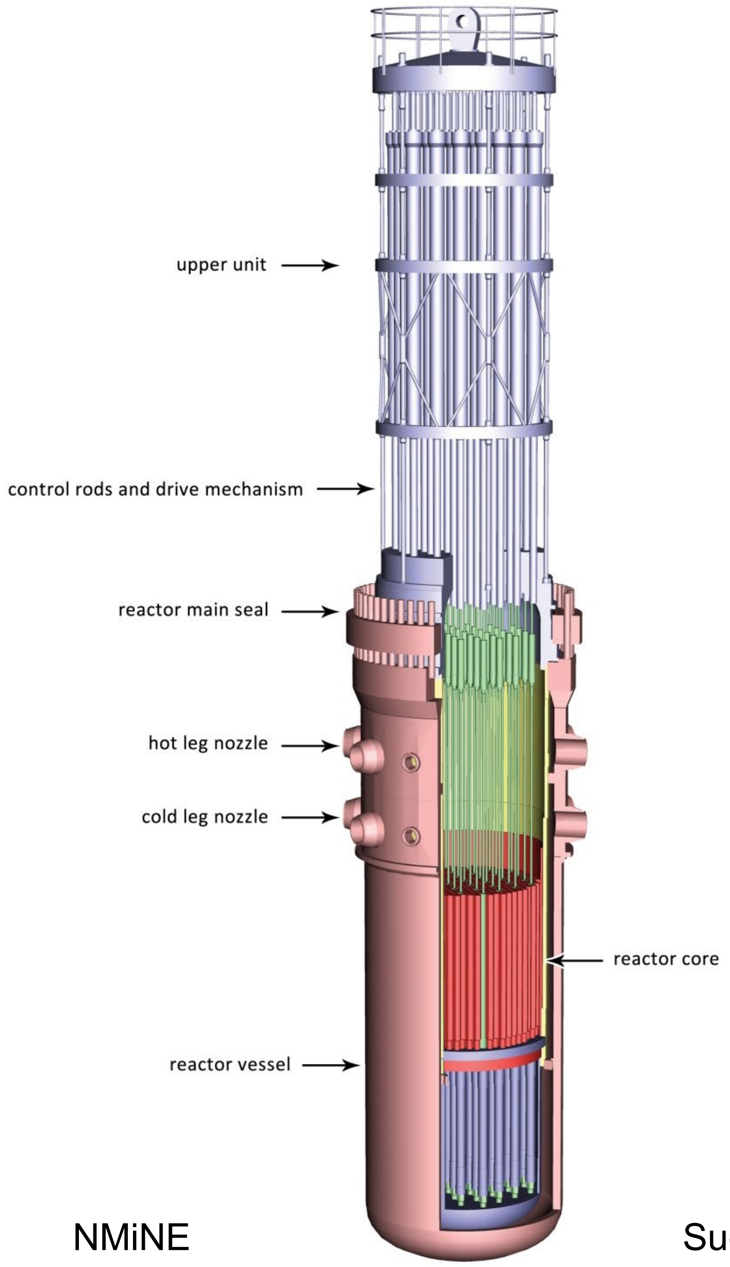
Square Reactor Lattice



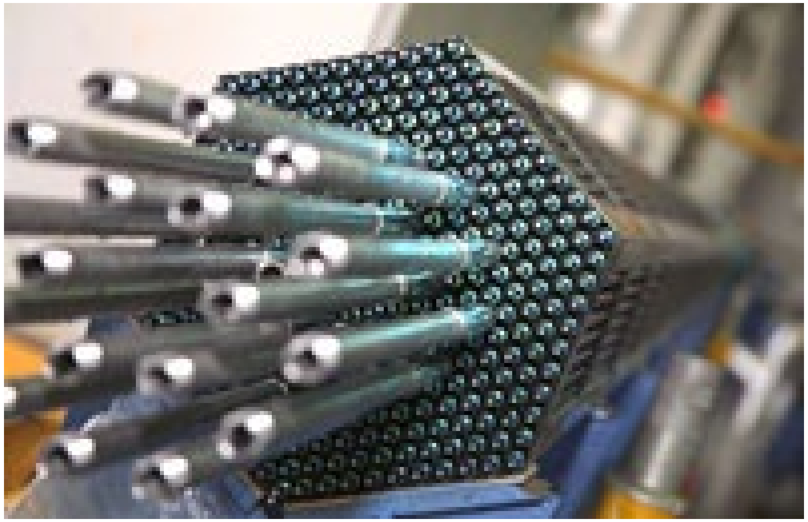




VVER-1000

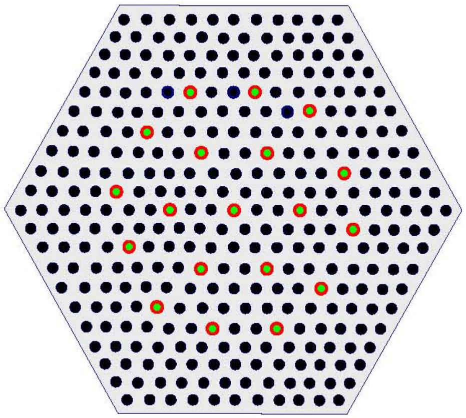


upper unit
control rods and drive mechanism
reactor main seal
hot leg nozzle
cold leg nozzle
reactor core
reactor vessel

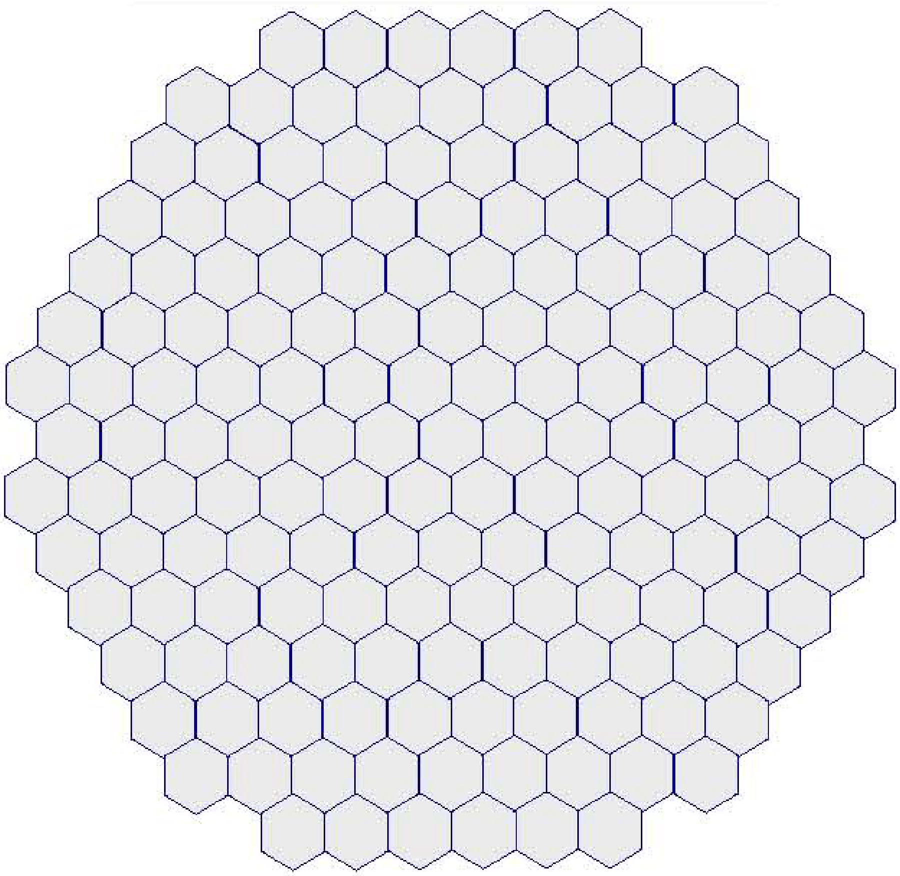


VVER-1000

Fuel assembly

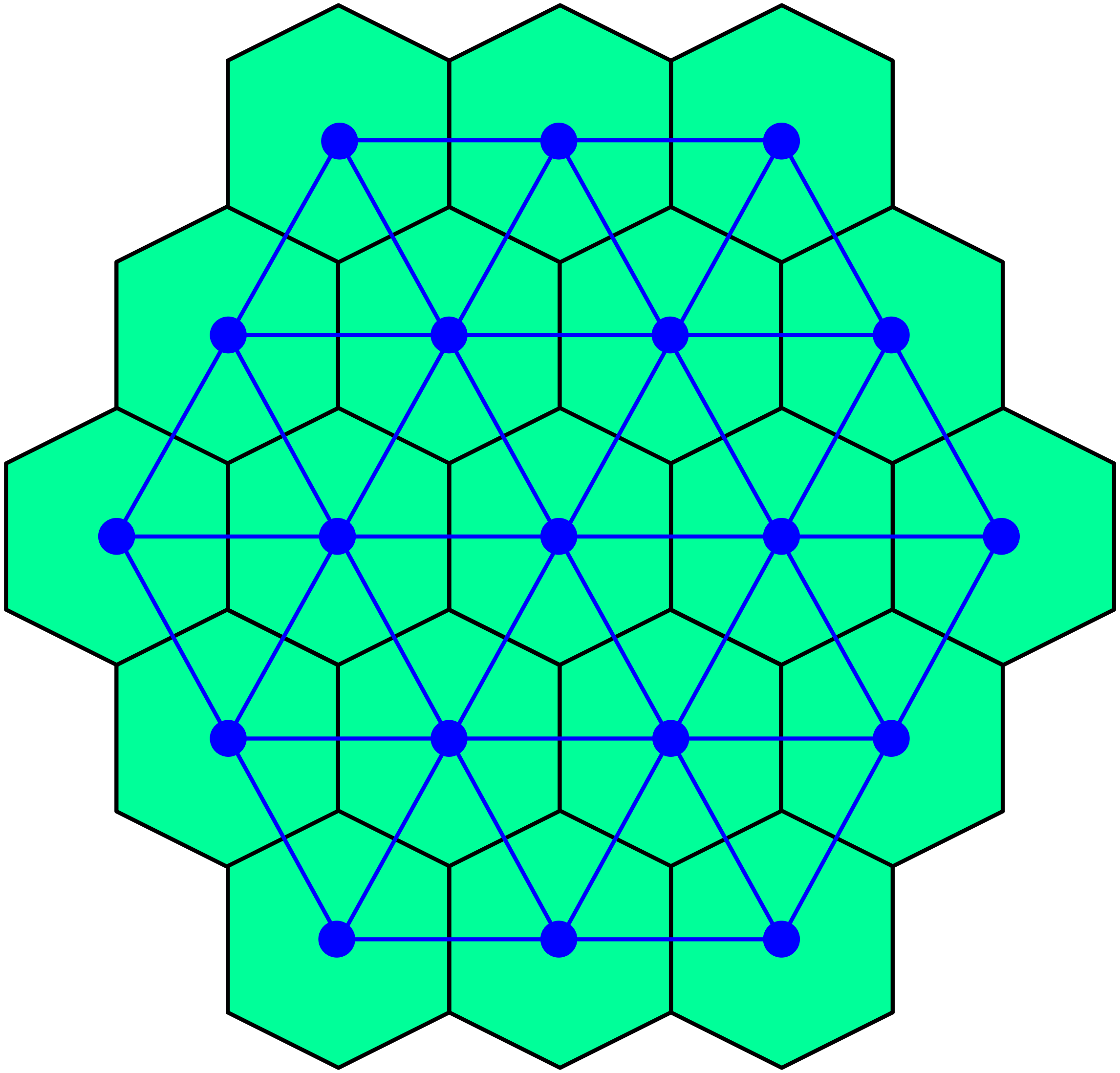


Reactor core

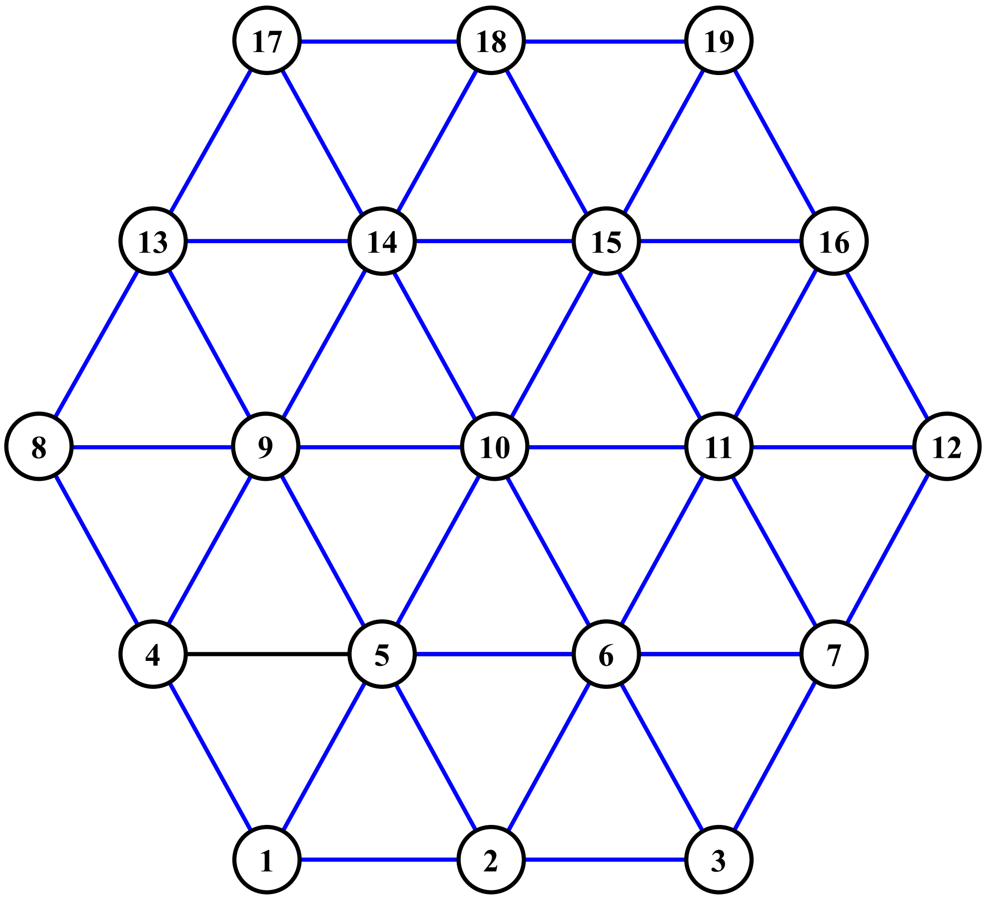


Fuel pins	- 312
Step	- 12.75 mm
Length	- 3530 mm
FA	- 163
CR	- 61
P_{th}	- 3000 MW_{th}

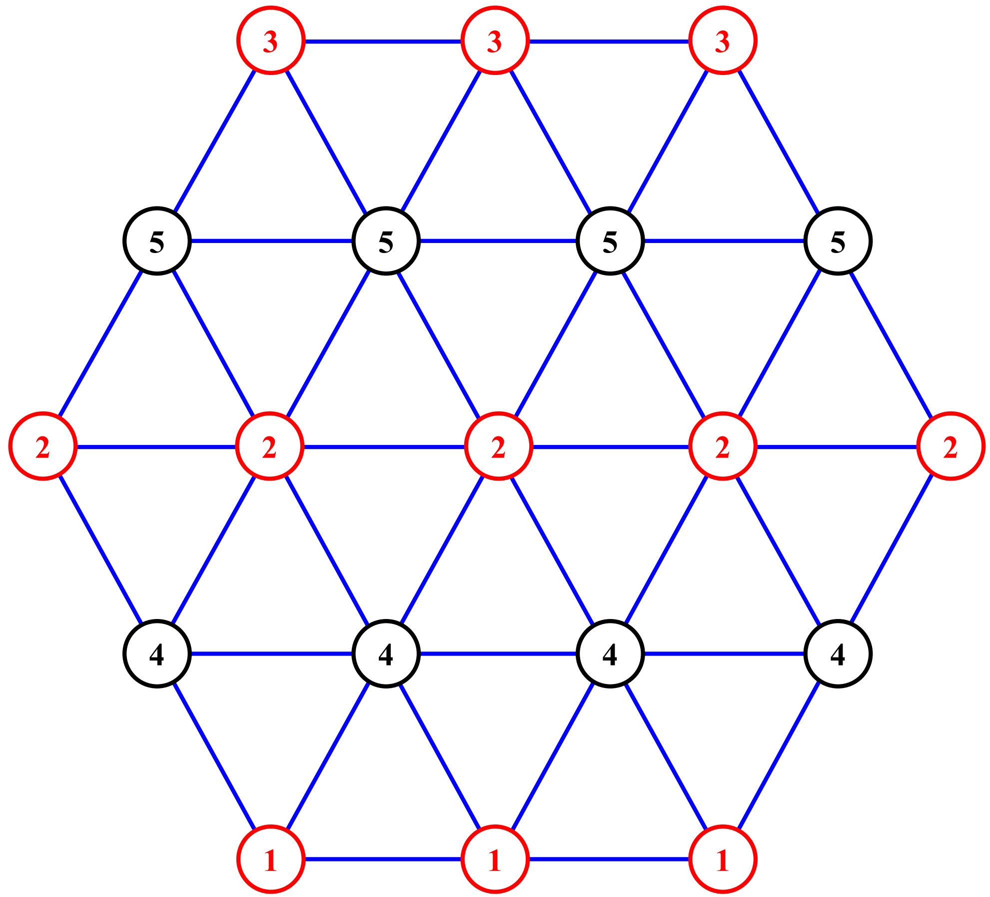
Hexagonal Lattice



Hexagonal FD Mesh



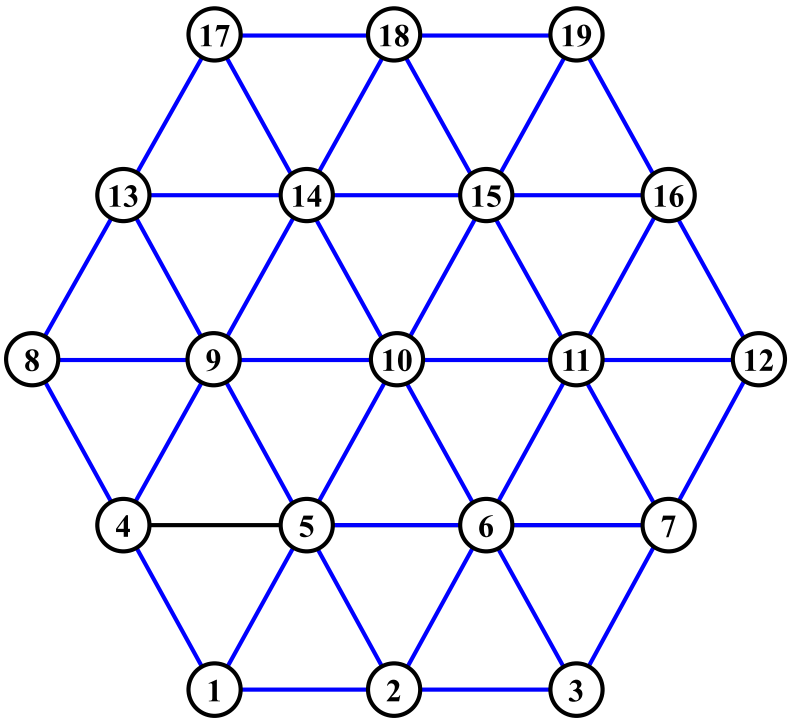
Red-Black Ordering



Block Matrices

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} & \mathbf{A}_{15} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{A}_{24} & \mathbf{A}_{25} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & \mathbf{A}_{34} & \mathbf{A}_{35} \\ \mathbf{A}_{41} & \mathbf{A}_{42} & \mathbf{A}_{43} & \mathbf{A}_{44} & \mathbf{A}_{45} \\ \mathbf{A}_{51} & \mathbf{A}_{52} & \mathbf{A}_{53} & \mathbf{A}_{54} & \mathbf{A}_{55} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \mathbf{b}_5 \end{bmatrix}$$

Hexagonal Lattices



$$
\mathbf{A} = \left[\begin{array}{ccc|cccc|ccccc|cccc|ccc}
* & * & & * & * & & & & & & & & & & & & & & \\
* & * & * & & * & * & & & & & & & & & & & & & \\
& * & * & & & * & * & & & & & & & & & & & & \\
\hline
* & & & * & * & & & * & * & & & & & & & & & & \\
* & * & & * & * & * & & & * & * & & & & & & & & & \\
& * & * & & * & * & * & & & * & * & & & & & & & & \\
& & * & & & * & * & & & & * & * & & & & & & & \\
\hline
& & & * & & & & * & * & & & & * & & & & & & \\
& & & * & * & & & * & * & * & & & * & * & & & & & \\
& & & & * & * & & & * & * & * & & & * & * & & & & \\
& & & & & * & * & & & * & * & * & & & * & * & & & \\
& & & & & & * & & & & * & * & & & & * & & & \\
\hline
& & & & & & & * & * & & & & * & * & & & * & & \\
& & & & & & & & * & * & & & * & * & * & & * & * & \\
& & & & & & & & & * & * & & & * & * & * & & * & * \\
& & & & & & & & & & * & * & & & * & * & & & * \\
\hline
& & & & & & & & & & & & * & * & & & * & * & \\
& & & & & & & & & & & & & * & * & & * & * & * \\
& & & & & & & & & & & & & & * & * & & * & *
\end{array}\right]
$$

Block Modifications

$$\mathbf{x}_i^{(k+1)} = \mathbf{A}_{ii}^{-1}\left(\mathbf{b}_i - \sum_{\substack{j=1 \\ j\neq i}}^{m} \mathbf{A}_{ij}\mathbf{x}_j^{(k)}\right)$$

$$\mathbf{x}_i^{(k+1)} = \mathbf{A}_{ii}^{-1}\left(\mathbf{b}_i - \sum_{j=1}^{i-1} \mathbf{A}_{ij}\mathbf{x}_j^{(k+1)} - \sum_{j=i+1}^{m} \mathbf{A}_{ij}\mathbf{x}_j^{(k)}\right)$$

SLOR

$$\mathbf{x}_i^{(k+1)} = \omega\mathbf{A}_{ii}^{-1}\left(\mathbf{b}_i - \sum_{j=1}^{i-1} \mathbf{A}_{ij}\mathbf{x}_j^{(k+1)} - \sum_{j=i+1}^{m} \mathbf{A}_{ij}\mathbf{x}_j^{(k)}\right) + (1-\omega)\mathbf{x}_i^{(k)}$$

Convergence Analysis

$$\rho\left(\mathbf{B}_J\right) = \cos \pi h \approx 1 - \frac{\left(\pi h\right)^2}{2}$$

$$R\left(\mathbf{B}_J\right) = -\log \rho\left(\mathbf{B}_J\right) \approx \frac{\pi^2}{2} h^2 \longrightarrow k \geq \frac{2}{\pi^2} \log 1/\varepsilon \, N^2$$

$$\rho\left(\mathbf{B}_{GS}\right) = \rho^2\left(\mathbf{B}_J\right) \approx 1 - \pi^2 h^2 \longrightarrow k \geq \frac{1}{\pi^2} \log 1/\varepsilon \, N^2$$

$$\rho\left(\mathbf{B}_{SOR}\right) = \frac{1 - \sin \pi h}{1 + \sin \pi h} \approx 1 - 2\pi h \longrightarrow k \geq \frac{1}{2\pi} \log 1/\varepsilon \, N$$

Important

- Convergence condition
- Stopping criterion
- Optimal SOR parameter
- Red-Black ordering
- Hexagonal FD meshes
- Block modifications