Monte Carlo Methods and Simulations in Nuclear Technology

Fundamentals of probability theory and statistics

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Topics

- Discrete random variable
- Continuous random variable
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- Correlation coefficient

Discrete random variable

A discrete random variable X is a variable that takes on a finite number of values x_i, each with a certain associated probability $f_X(x_i) = P(X = x_i)$.

Continuous random variable, pdf

A continuous random variable X is a variable that takes on an infinite number of values x whose probabilities are described by a **probability density function** (pdf) $f_X(x)$. It holds that

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Cumulative distribution function (cdf) of a random variable

The probability that a continuous random variable X gives a value smaller or equal a certain x is given by the cumulative distribution function (cdf)

$$F_X(x) \equiv P(X \leq x) = \int_{-\infty}^{x} f_X(\xi) d\xi$$

The distribution function is defined for discrete random variables as

$$F_X(x) = \sum_{x_i \leq x} P(X = x_i)$$

Expectation value of a random variable

Each random variable X has an expectation value $\mathrm{E}[X]$ that is the mean of all possible values x weighted according to their probability. The expectation value of a continuous random variable is

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Similarly, the expectation value of a discrete random variable is

$$\mathrm{E}[X] = \sum_i x_i f_X(x_i)$$

Variance of a random variable

It is often important to quantify how the random values are spread about the expectation value. A common measure of the spread is the variance $\mathrm{Var}[X]$, i.e. the expected quadratic deviation from the expectation value,

$$\mathrm{Var}[X] = \mathrm{E}[(X - \mathrm{E}[X])^2]$$

It follows from the above equation that

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{E}[X^2 - 2X\mathrm{E}[X] + (\mathrm{E}[X])^2] \\ &= \mathrm{E}[X^2] - \mathrm{E}[2X\mathrm{E}[X]] + \mathrm{E}[(\mathrm{E}[X])^2] \\ &= \mathrm{E}[X^2] - 2(\mathrm{E}[X])^2 + (\mathrm{E}[X])^2 \\ &= \mathrm{E}[X^2] - (\mathrm{E}[X])^2.\end{aligned}$$

Standard deviation of a random variable

It is convenient to measure the spread with the same unit as that of the expectation value; therefore, the standard deviation σ_X has been introduced as

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

Covariance of two random variables

When working with several random variables it is useful to know how the variables relate to each other. This can be quantified by the covariance $\mathrm{Cov}[X, Y]$ of two random variables X and Y,

$$\mathrm{Cov}[X, Y] = \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])]$$

It follows from above that

$$\begin{aligned} \mathrm{Cov}[X, Y] &= \mathrm{E}\left[XY - Y\mathrm{E}[X] - X\mathrm{E}[Y] + \mathrm{E}[X]\mathrm{E}[Y]\right] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]. \end{aligned}$$

Correlation coefficient

Since the covariance is an absolute measure of the relation between two random variables it is sometimes useful to use the correlation coefficient

$$\rho_{X,Y} = \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}}$$

- The correlation coefficient is always in the interval $[-1, 1]$.
- When $\rho_{X,Y} > 0$ then X and Y are positively correlated, i.e. it is likely that both X and Y give large (or small) values during a single event observation (e.g. the relative change in neutron energy and the scattering angle during a scattering collision).
- When $\rho_{X,Y} < 0$ then X and Y are negatively correlated, i.e. it is likely that X gives a small value (relatively to its expectation value) when Y gives a large value (relatively to its expectation value) and vice versa.
- When $\rho_{X,Y} = 0$ then X and Y are uncorrelated.