Lecture No 10

Title:
Transient Flows in Channels

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Outline of the Lecture

- Transient equations
  - Homogeneous Equilibrium Model: 3-equation model
- Example applications:
  - Channel response to step-change of inlet velocity
  - Channel response to step change of power
    - time variation of void fraction, mixture density, enthalpy and velocity

Channel dynamics using Drift-Flux Model

Governing Equations - HEM (1)

- 3-equation HEM model, in which both liquid and vapor are at saturations (in thermodynamic equilibrium) and both phases move with the same velocity
  - The conservation equations are written for the two-phase mixture

P_H – heated perimeter

D_h - diameter

G – mass flux
i – enthalpy
p – pressure
A- area
q" – heat flux
z – distance
t - time

mass $$\frac{\partial \rho_m}{\partial t} + \frac{1}{A}\frac{\partial (GA)}{\partial z} = 0$$

energy $$\frac{\partial \left[(\rho_m i_M - p) A \right]}{\partial t} + \frac{\partial (G i_m A)}{\partial z} = q'' P_H$$

momentum $$\frac{\partial G}{\partial t} + \frac{1}{A}\frac{\partial}{\partial z}\left(\frac{G^2 A}{\rho_m}\right) + \frac{\partial p}{\partial z} + \left[\phi_{lo}^2 \frac{4C_{f,lo}}{D_h} + \sum_{i=1}^{n} \xi_i \phi_{lo,i}^2 \delta(z - z_i) \right] \frac{G^2}{2\rho_f} + \rho_m g \sin\varphi = 0$$

Governing Equations - HEM (2)

- Closure relationships for the HEM model
  - Note that subscripts are now changed to indicate the saturation conditions: *l → f and v → g*

Mixture enthalpy

$$i_m = i_f(1-x) + i_g x$$

Mass-weighted enthalpy

$$i_M = \frac{\rho_f i_f (1-\alpha) + \rho_g i_g \alpha}{\rho_m}$$

Mixture density

$$\rho_m = \sum_k \rho_k \alpha_k = (1-\alpha)\rho_f + \alpha\rho_g$$

Volumetric Continuity Equation

- We can combine mass conservation equations of both phases as $\frac{\partial(\alpha\rho_v)}{\partial t}+\frac{\partial(\alpha\rho_v U_v)}{\partial z}=\Gamma \Rightarrow \rho_v\frac{\partial(\alpha U_v)}{\partial z}=\Gamma-\alpha U_v\frac{\partial(\rho_v)}{\partial z}-\frac{\partial(\alpha\rho_v)}{\partial t}$

$$\rho_l\frac{\partial(1-\alpha)}{\partial t}+\frac{\partial\left[(1-\alpha)\rho_l U_l\right]}{\partial z}=-\Gamma \Rightarrow \rho_l\frac{\partial\left[(1-\alpha)U_l\right]}{\partial z}=-\Gamma-(1-\alpha)U_l\frac{\partial\rho_l}{\partial z}-\frac{\partial\left[(1-\alpha)\rho_l\right]}{\partial t}$$

- But $\alpha U_v=J_v$ and $(1-\alpha)U_l=J_l$
- Thus $\frac{\partial J_v}{\partial z}=\frac{\Gamma}{\rho_v}-\frac{J_v}{\rho_v}\frac{\partial\rho_v}{\partial z}-\frac{1}{\rho_v}\frac{\partial(\alpha\rho_v)}{\partial t}$ $\qquad$ $\frac{\partial J_l}{\partial z}=-\frac{\Gamma}{\rho_l}-\frac{J_l}{\rho_l}\frac{\partial\rho_l}{\partial z}-\frac{1}{\rho_l}\frac{\partial\left[(1-\alpha)\rho_l\right]}{\partial t}$
- Summing up the equations, and since $J=J_v+J_l$, we get

$$\frac{\partial J}{\partial z}=\Gamma v_{fg}-\left(\frac{1-\alpha}{\rho_f}\frac{d\rho_f}{dp}+\frac{\alpha}{\rho_g}\frac{d\rho_g}{dp}\right)\frac{\partial p}{\partial t}-\left(\frac{J_f}{\rho_f}\frac{d\rho_f}{dp}+\frac{J_g}{\rho_g}\frac{d\rho_g}{dp}\right)\frac{\partial p}{\partial z}$$

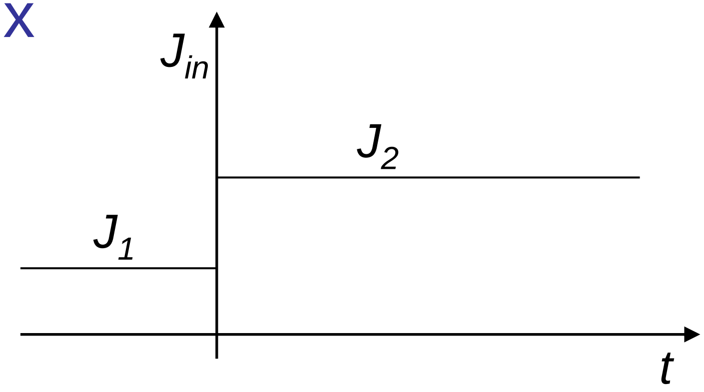
$$v_{fg}=\frac{1}{\rho_g}-\frac{1}{\rho_f}=v_g-v_f$$

This is volumetric continuity equation. We take *l->f* and *v->g*

Example Application – Step Change of Inlet Velocity – HEM (1)

- Consider a case with a step change of inlet velocity to a boiling channel with a constant cross-section
- Assumptions
  - Saturated inlet conditions
  - Uniform distribution of heat flux
  - Homogeneous flow model
  - Incompressible flow

$$J_{in}(t) = \begin{cases} J_1 & for \quad t \le 0 \\ J_2 & for \quad t > 0 \end{cases}$$



Step Change of Inlet Velocity – HEM (2)

- Taking into account the assumptions, the volumetric continuity equation becomes:

$$\frac{\partial J}{\partial z} = \Gamma v_{fg} - \underbrace{\left(\frac{1-\alpha}{\rho_f} \overbrace{\cancel{\frac{d\rho_f}{dp}}}^{0} + \frac{\alpha}{\rho_g} \overbrace{\cancel{\frac{d\rho_g}{dp}}}^{0} \right) \frac{\partial p}{\partial t} - \left(\frac{J_f}{\rho_f} \overbrace{\cancel{\frac{d\rho_f}{dp}}}^{0} + \frac{J_g}{\rho_g} \overbrace{\cancel{\frac{d\rho_g}{dp}}}^{0} \right)}_{\text{Incompressible flow}} \frac{\partial p}{\partial z} = \Gamma v_{fg}$$

$$\Gamma = \frac{q'' P_H}{A i_{fg}}$$

Evaporation rate

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial G}{\partial z} = 0$$

$$\Omega \equiv \Gamma v_{fg}$$

- From the mass conservation equation we get

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial G}{\partial z} = \frac{\partial \rho_m}{\partial t} + \frac{\partial (\rho_m J)}{\partial z} = \frac{\partial \rho_m}{\partial t} + \rho_m \frac{\partial J}{\partial z} + J \frac{\partial \rho_m}{\partial z} = 0 \Rightarrow \frac{\partial \rho_m}{\partial t} + J \frac{\partial \rho_m}{\partial z} = -\rho_m \frac{\partial J}{\partial z} = -\rho_m \Omega$$

Step Change of Inlet Velocity – HEM (3)

- Integrating the volumetric continuity equation yields:

$$\int \frac{\partial J}{\partial z} dz = \int \Gamma v_{fg} dz + f(t) \Longrightarrow J(z,t) = \Gamma v_{fg} \cdot z + f(t) = \Omega z + f(t)$$

- Substituting $z = 0$ we get

$$J(0,t) = \Omega \cdot 0 + f(t) = J_{in}(t)$$

$$\Omega \equiv \Gamma v_{fg} = \frac{q'' P_H}{A i_{fg}} v_{fg}$$

- Thus $$J(z,t) = \Omega z + J_{in}(t)$$

Step Change of Inlet Velocity – HEM (4)

- We have obtained an equation for superficial velocity distribution in the channel for any time *t* and location *z*

$$J(z,t) = \Omega \cdot z + J_{in}(t)$$

  - Note that this velocity is linearly increasing with distance z and that the step change of inlet velocity instantaneously propagates along the channel whole length

- To find trajectories of particles moving in the channel, we have to solve the following differential equation

$$\frac{dz}{dt} \equiv J(z,t) = \Omega \cdot z + J_{in}(t)$$

Step Change of Inlet Velocity – HEM (5)

- Consider the equation for $t < 0$, then the inlet velocity to the channel is J_1 and we are solving the following equation

$$\frac{dz}{dt} = \Omega \cdot z + J_1 \Rightarrow \frac{dz}{\Omega \cdot z + J_1} = dt$$

- Now let t_0 be the entrance time of a particle to the channel and let $t_0 < 0$; we integrate the equation from t_0 to time $t < 0$ (particle will move then from $z = 0$ to z)

$$\int_0^z \frac{d\zeta}{\Omega \cdot \zeta + J_1} = \int_{t_0}^t d\tau \Longrightarrow \frac{1}{\Omega} \ln\left(\Omega \cdot \zeta + J_1\right)\Big|_0^z = t - t_0 \Longrightarrow z = \frac{J_1}{\Omega}\left[e^{\Omega(t-t_0)} - 1\right]$$

Step Change of Inlet Velocity– HEM (6)

- Thus we found a trajectory of a particle that enters the channel at time $t_0 < 0$ (point A in figure)

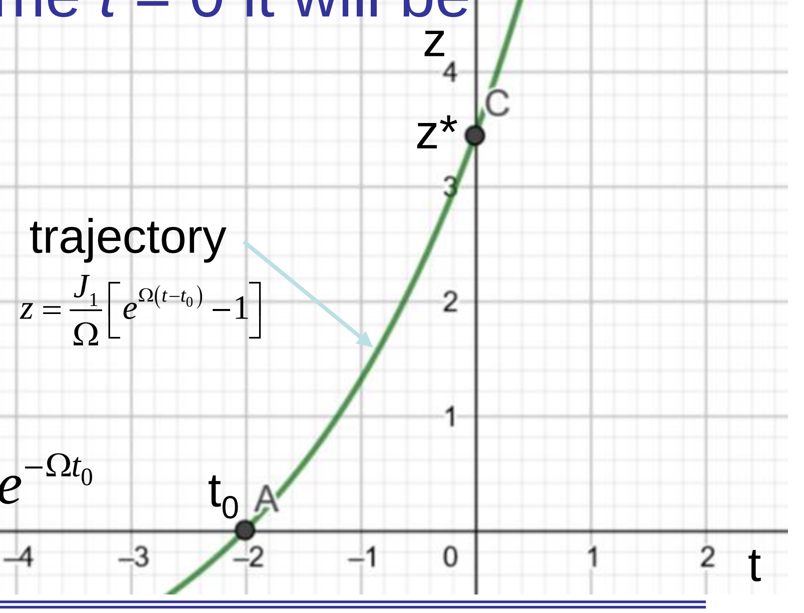
$$z = \frac{J_1}{\Omega}\left[e^{\Omega(t-t_0)} - 1\right]$$

- The particle accelerates and at time $t = 0$ it will be located at (point C in figure)

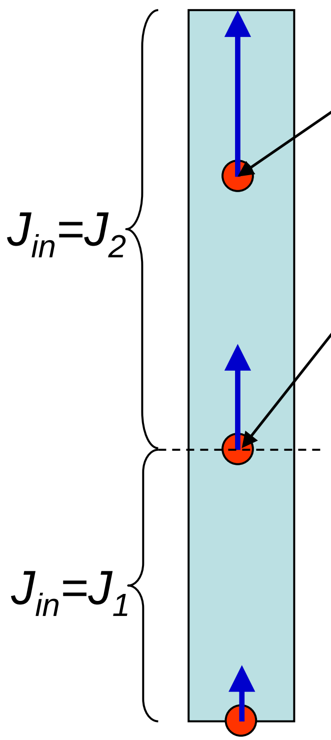
$$z^* = \frac{J_1}{\Omega}\left[e^{-\Omega t_0} - 1\right]$$

- And will have velocity

$$J\left(z^*, 0^-\right) = \Omega \cdot z^* + J_1 = J_1 e^{-\Omega t_0}$$



Step Change of Inlet Velocity – HEM (7)



sudden velocity change

z(t), J(t), t>0

$$J\left(z^*,0^+\right)=\Omega z^* + J_2 = J_1 e^{-\Omega t_0} + J_2 - J_1$$

$$J\left(z^*,0^-\right)=\Omega z^* + J_1 = J_1 e^{-\Omega t_0}$$

*t=0, z=z**

$$z^* = \frac{J_1}{\Omega}\left[e^{-\Omega t_0} - 1\right]$$

$t=t_0<0$

Step Change of Inlet Velocity – HEM (8)

- For time t > 0, the differential equation for particle motion becomes

$$\frac{dz}{dt} = \Omega z + J_2 \Rightarrow \frac{dz}{\Omega z + J_2} = dt$$

- Now we integrate the equation from time 0^+ to time $t > 0$ (and correspondingly from location z^* to location z)

$$\int_{z^*}^{z} \frac{d\zeta}{\Omega\zeta + J_2} = \int_0^t d\tau \qquad \frac{1}{\Omega}\ln\left(\frac{\Omega z + J_2}{\Omega z^* + J_2}\right) = t \qquad z = \frac{1}{\Omega}\left[\left(J_2 + \Omega z^*\right)e^{\Omega t} - J_2\right]$$

and finally, for $t_0 < 0$, $t > 0$

$$z\left(t;t_0\right) = \frac{J_2 - J_1}{\Omega} e^{\Omega t} + \frac{J_1}{\Omega} e^{\Omega\left(t - t_0\right)} - \frac{J_2}{\Omega}$$

Step Change of Inlet Velocity – HEM (9)

- When $t_0 > 0$ then particle enters the channel after step change of the inlet velocity and the trajectory equation is as follows

$$\frac{dz}{dt} = \Omega z + J_2 \Rightarrow \frac{dz}{\Omega z + J_2} = dt$$

- And we integrate from inlet (z = 0, $t = t_0$) to a location z (time t)

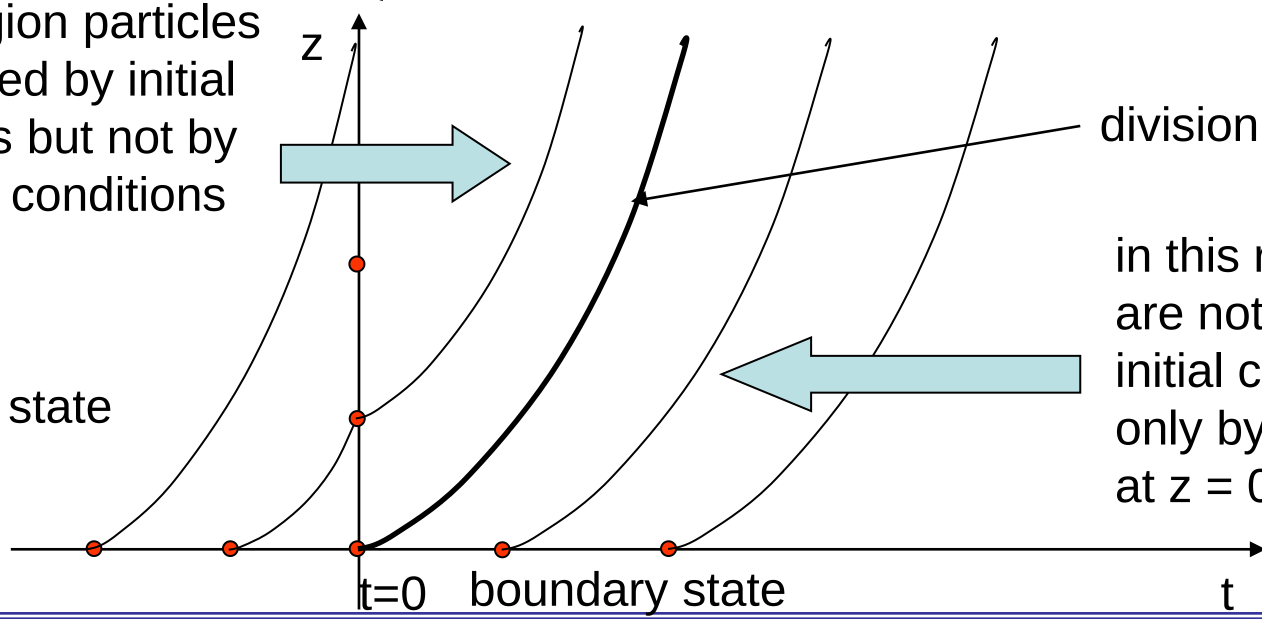
$$\int_0^z \frac{d\zeta}{\Omega\zeta + J_2} = \int_{t_0}^t d\tau \Longrightarrow \ln\left(\Omega\zeta + J_2\right)\Big|_0^z = \Omega\left(t - t_0\right) \Longrightarrow z = \frac{J_2}{\Omega}\left[e^{\Omega\left(t-t_0\right)} - 1\right]$$

Step Change of Inlet Velocity– HEM (10)

- In summary, we have the following solutions

$$z(t,t_0)=\begin{cases}\left.\begin{array}{ll}\dfrac{J_1}{\Omega}\left[e^{\Omega(t-t_0)}-1\right] & t<0\\ \dfrac{J_1}{\Omega}e^{\Omega(t-t_0)}+\dfrac{J_2-J_1}{\Omega}e^{\Omega t}-\dfrac{J_2}{\Omega} & t>0\end{array}\right\} & \text{if} \quad t_0\le 0\\ \dfrac{J_2}{\Omega}\left[e^{\Omega(t-t_0)}-1\right] & \text{if} \quad t_0>0\end{cases}$$

in this region particles are affected by initial conditions but not by boundary conditions



division path line

in this region particles are not affected by initial conditions but only by boundary state at z = 0

initial state

Mixture Density (1)

- Now we will try to answer a question what is the mixture density at the trajectory

- For that purpose we write the mixture mass conservation equation

$$\frac{\partial \rho_m}{\partial t} + J\frac{\partial \rho_m}{\partial z} = -\rho_m \Omega$$

equation becomes ODE when integrated along trajectory

- Since $J = dz/dt$

$$\underbrace{\frac{\partial \rho_m}{\partial t} + \frac{dz}{dt}\frac{\partial \rho_m}{\partial z}}_{\text{total derivative of mixture density}} = -\rho_m \Omega \quad \Longrightarrow \quad \frac{D\rho_m}{Dt} = -\rho_m \Omega$$

density becomes a function of time only when we move with particle

Mixture Density(2)

- Thus integration of the mass conservation equation along particle trajectory gives

$$\frac{D\rho_m}{\rho_m} = -\Omega Dt \Rightarrow \int_{\rho_f}^{\rho_m} \frac{D\varsigma}{\varsigma} = -\Omega \int_{t_0}^{t} D\tau \qquad \Longrightarrow \qquad \rho_m = \rho_f e^{-\Omega(t-t_0)}$$

saturated liquid at inlet at $t = t_0$

- Now for $t_0 < 0$ and $t < 0$ we have found that

$$z = \frac{J_1}{\Omega}\left[e^{\Omega(t-t_0)} - 1\right] \quad \Longrightarrow \quad e^{\Omega(t-t_0)} = \frac{\Omega z}{J_1} + 1$$

- Substituting this to the expression for ρ_m, we get $\rho_m = \dfrac{\rho_f}{1 + \dfrac{\Omega z}{J_1}}$

Mixture Density(3)

- Similarly, for $t > 0$ and $t_0 < 0$ we obtained

$$z = \frac{J_2 - J_1}{\Omega} e^{\Omega t} + \frac{J_1}{\Omega} e^{\Omega(t-t_0)} - \frac{J_2}{\Omega}$$

$$e^{\Omega(t-t_0)} = \frac{\Omega z}{J_1} - \frac{J_2 - J_1}{J_1} e^{\Omega t} + \frac{J_2}{J_1}$$

$$e^{-\Omega(t-t_0)} = \frac{1}{\frac{\Omega z}{J_1} - \frac{J_2 - J_1}{J_1} e^{\Omega t} + \frac{J_2}{J_1}}$$

- and

$$\rho_m = \frac{\rho_f}{\frac{\Omega z}{J_1} - \frac{J_2 - J_1}{J_1} e^{\Omega t} + \frac{J_2}{J_1}}$$

Mixture Density (3)

- Consider now $t_0 > 0$ (particle enters channel after step change of the inlet velocity). We have already found that

$$z = \frac{J_2}{\Omega}\left[e^{\Omega(t-t_0)} - 1\right] \qquad e^{\Omega(t-t_0)} = \frac{\Omega z}{J_2} + 1 \qquad e^{-\Omega(t-t_0)} = \frac{1}{\frac{\Omega z}{J_2} + 1}$$

- and the density can be found as

$$\rho_m = \frac{\rho_f}{\frac{\Omega z}{J_2} + 1}$$

Mixture Density (4)

- In summary, after step change of the inlet velocity, the mixture density will change as

Steady-state in a channel before step-change

Transient due to step-change

$$\rho_m(z,t) = \frac{\rho_f}{1 + \dfrac{\Omega z}{J_1}} \qquad \text{for } t < 0$$

$$\rho_m(z,t) = \begin{cases} \dfrac{\rho_f}{\dfrac{\Omega z}{J_1} + \dfrac{J_2}{J_1} - \dfrac{J_2 - J_1}{J_1} e^{\Omega t}} & \text{for} \quad 0 \le t \le \dfrac{1}{\Omega} \ln\left(1 + \dfrac{\Omega z}{J_2}\right) \\[2ex] \dfrac{\rho_f}{1 + \dfrac{\Omega z}{J_2}} & \text{if} \quad t > \dfrac{1}{\Omega} \ln\left(1 + \dfrac{\Omega z}{J_2}\right) \end{cases}$$

Steady-state in a channel after step-change

Mass Flux

- Having mixture density, we can find mass flux $G = \rho_m J$

$$G\left(z,t\right) = \rho_f J_1 \qquad \text{for } t < 0$$

$$G(z,t) = \begin{cases} \dfrac{\left(\Omega z + J_2\right)\rho_f}{\dfrac{\Omega z}{J_1} + \dfrac{J_2}{J_1} - \dfrac{J_2 - J_1}{J_1} e^{\Omega t}} & \text{for} \quad 0 \le t \le \dfrac{1}{\Omega}\ln\left(1 + \dfrac{\Omega z}{J_2}\right) \\ \rho_f J_2 & \text{if} \quad t > \dfrac{1}{\Omega}\ln\left(1 + \dfrac{\Omega z}{J_2}\right) \end{cases}$$

Void Fraction

- Void fraction can be found as $\rho_m = \rho_f (1-\alpha) + \rho_g \alpha \Rightarrow \alpha = \dfrac{\rho_f - \rho_m}{\rho_f - \rho_g}$

$$\alpha(z,t) = \frac{\rho_f}{\rho_f - \rho_g}\left(1 - \frac{1}{1 + \dfrac{\Omega z}{J_1}}\right) \qquad \text{for } t < 0$$

$$\alpha(z,t) = \begin{cases} \dfrac{\rho_f}{\rho_f - \rho_g}\left(1 - \dfrac{1}{\dfrac{\Omega z}{J_1} + \dfrac{J_2}{J_1} - \dfrac{J_2 - J_1}{J_1} e^{\Omega t}}\right) & \text{for} \quad 0 \le t \le \dfrac{1}{\Omega}\ln\left(1 + \dfrac{\Omega z}{J_2}\right) \\ \dfrac{\rho_f}{\rho_f - \rho_g}\left(1 - \dfrac{1}{1 + \dfrac{\Omega z}{J_2}}\right) & \text{if} \quad t > \dfrac{1}{\Omega}\ln\left(1 + \dfrac{\Omega z}{J_2}\right) \end{cases}$$

Mixture Enthalpy

- Using void fraction, we can find the mixture quality as

$$\alpha = \frac{1}{1+\frac{\rho_g}{\rho_f}\frac{1-x}{x}} \Rightarrow x = \frac{1}{1+\frac{\rho_f}{\rho_g}\frac{1-\alpha}{\alpha}}$$

- And finally, we can find the mixture enthalpy as

$$i_m = i_f\left(1-x\right)+i_g x$$

Example

- Example: calculate density change in a pipe uniformly heated with $q''=6x10^5$ W/m^2

  - saturated water at inlet, p = const = 70 bar

  - pipe diameter D = 10 mm

  - pipe length L = 3.65 m

  - inlet velocity: step changed at time t = 0 from 2 m/s to 1.8 m/s

- plot density of mixture as a function of time at z = 1 and 2 m from the inlet

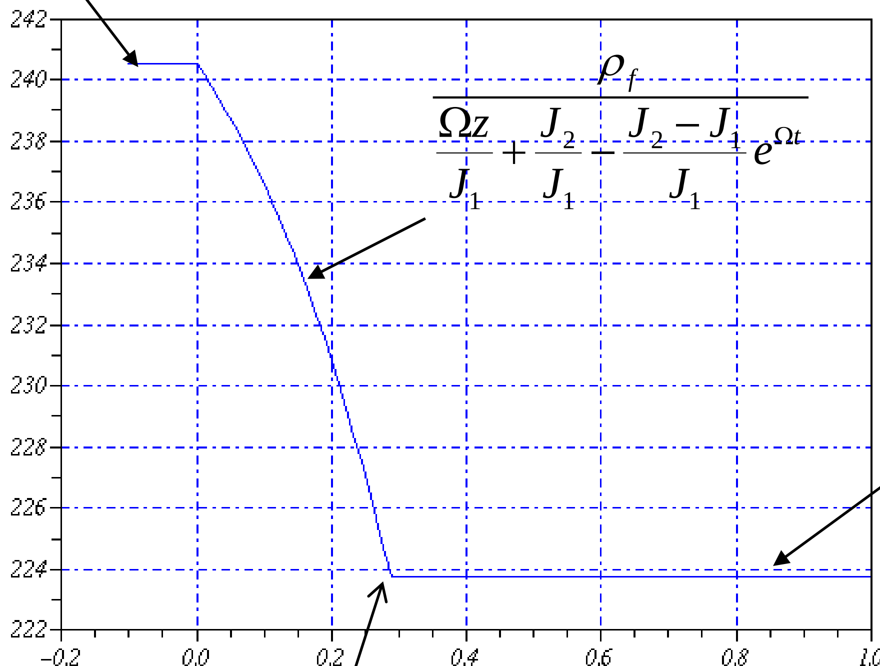
Density Versus Time (1)

$$\rho_m(z,t) = \frac{\rho_f}{1+\dfrac{\Omega z}{J_1}}$$

density change at z = 1 m from inlet

$$\frac{\rho_f}{\dfrac{\Omega z}{J_1}+\dfrac{J_2}{J_1}-\dfrac{J_2-J_1}{J_1}e^{\Omega t}}$$

ρ_m

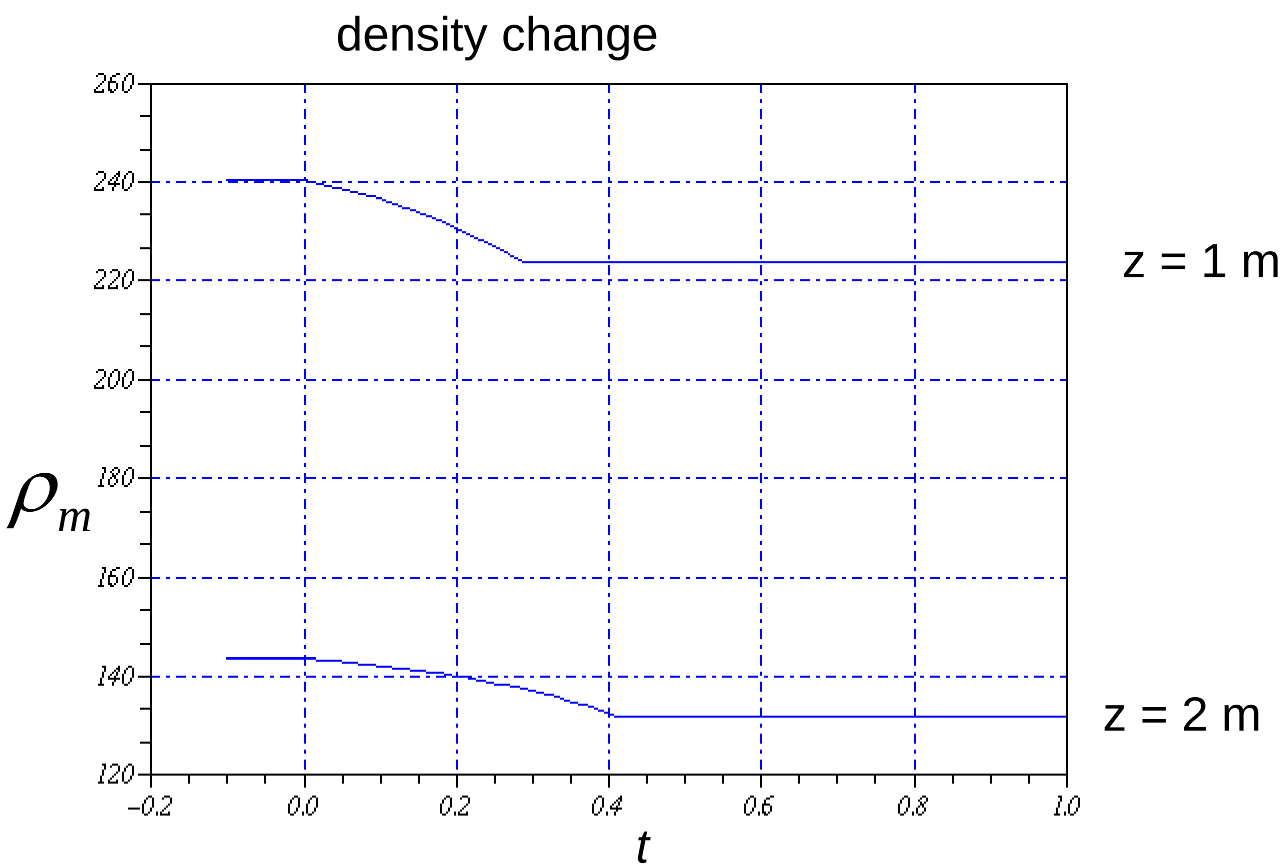


$$\frac{\rho_f}{1+\dfrac{\Omega z}{J_2}}$$

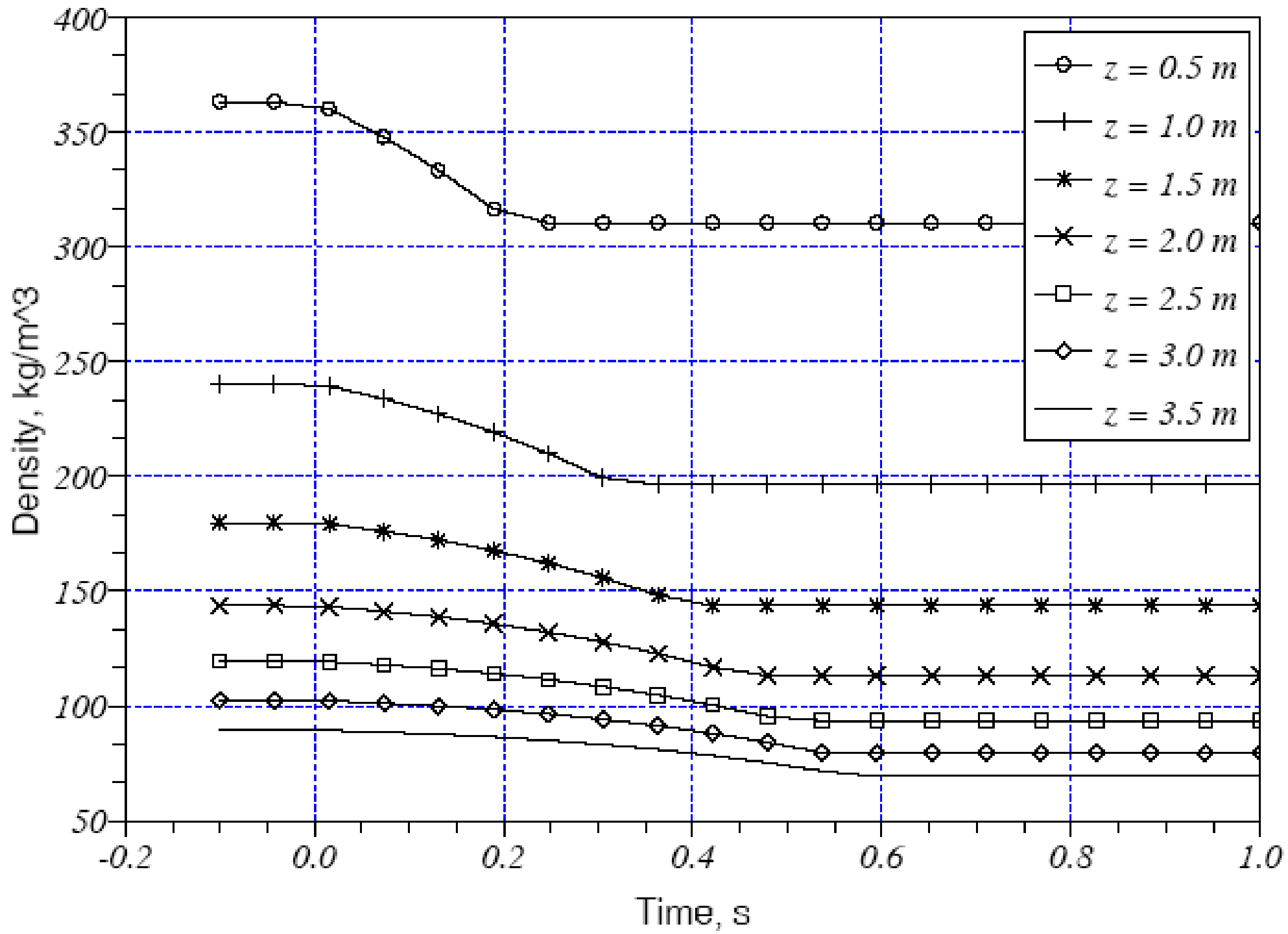
t

$$\boxed{t = \frac{1}{\Omega}\ln\left(1+\frac{\Omega z}{J_2}\right)}$$

Density Versus Time (2)

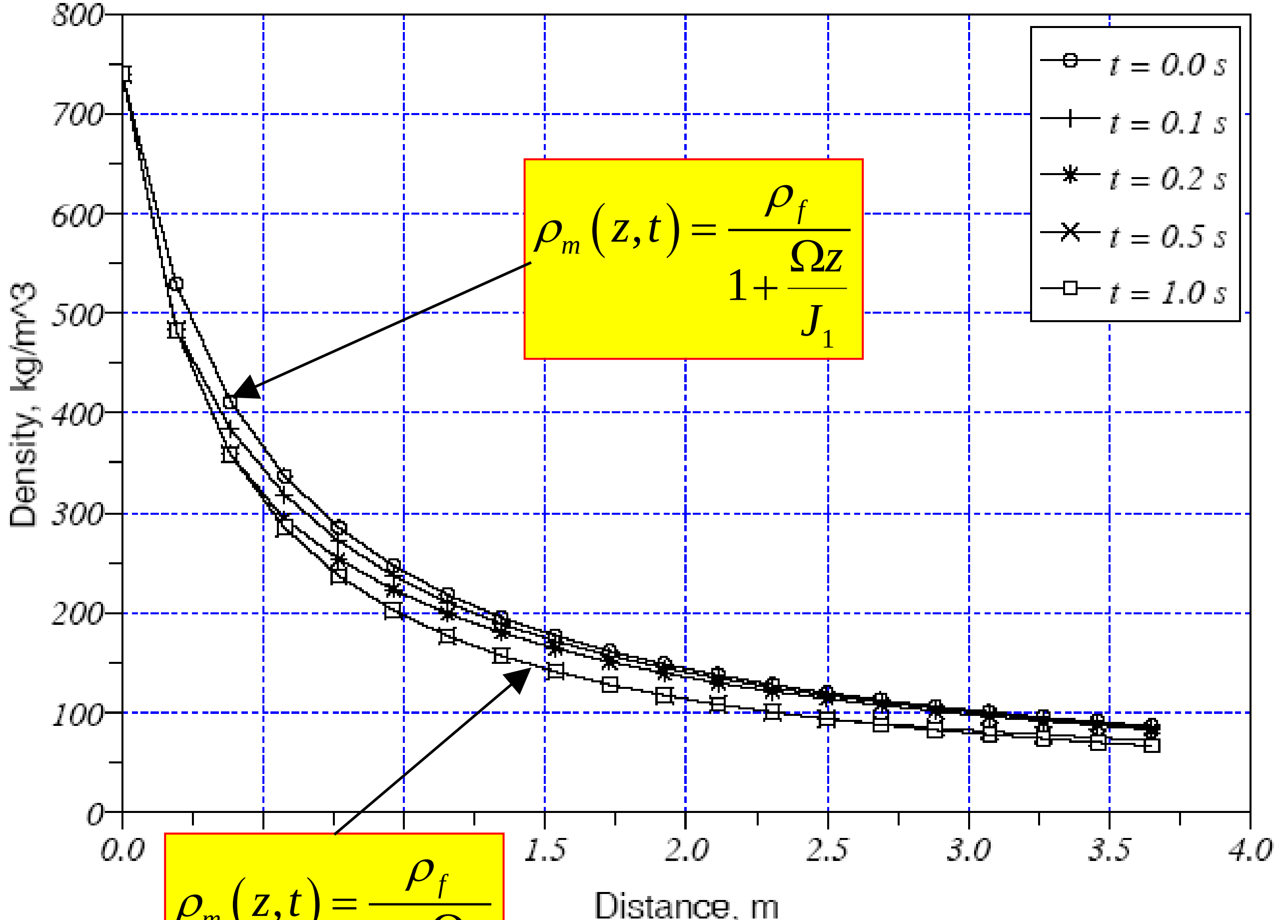


Density Versus Time (3)



Density change at different locations: at z=0.5 m ρ drops from 363 to 310 kg/m^3 within ~0.22 s. At z = 3.5 m it drops from 90 to 70 kg/m^3 within ~0.6 s.

Density Distribution Along Channel



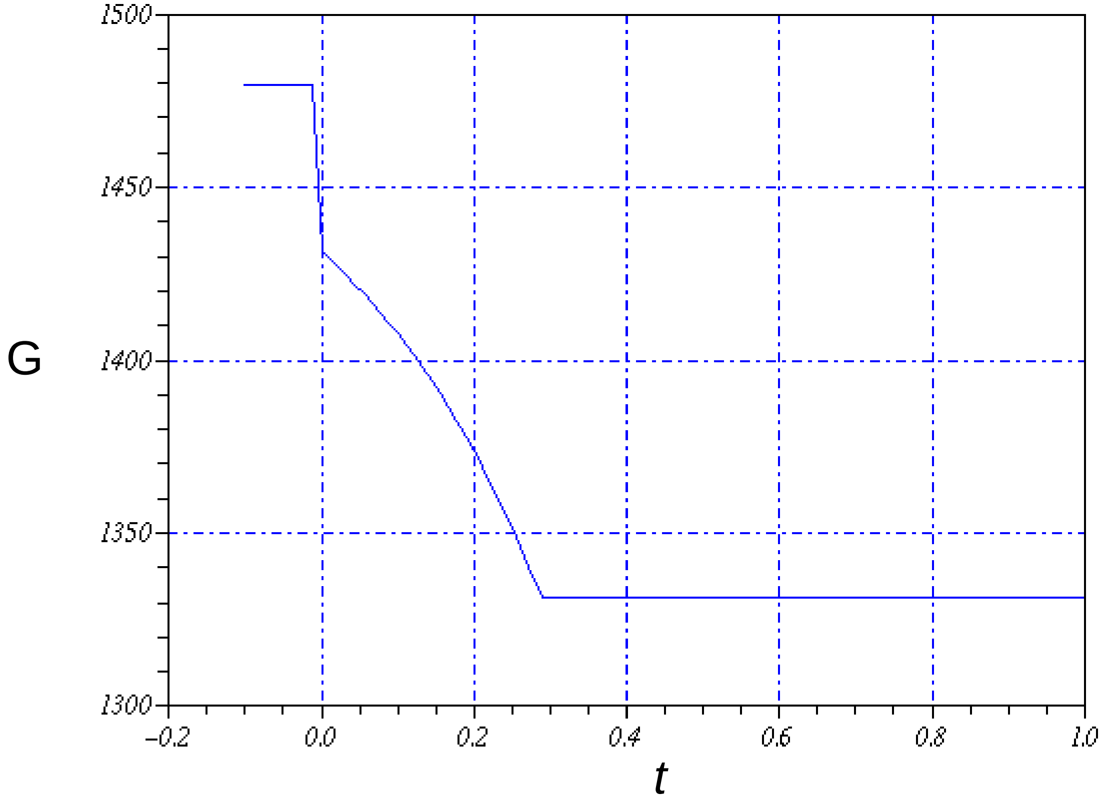
$$\rho_m(z,t) = \frac{\rho_f}{1+\frac{\Omega z}{J_1}}$$

$$\rho_m(z,t) = \frac{\rho_f}{1+\frac{\Omega z}{J_2}}$$

Mixture density along heated channel at different time instances after sudden reduction of inlet velocity

Mass Flux Versus Time (1)

Mass flux change at z = 1 m from inlet



Note two steps in mass flux change:

(a) first rapid, connected to sudden velocity change,
(b) second, slow, connected to slow density change

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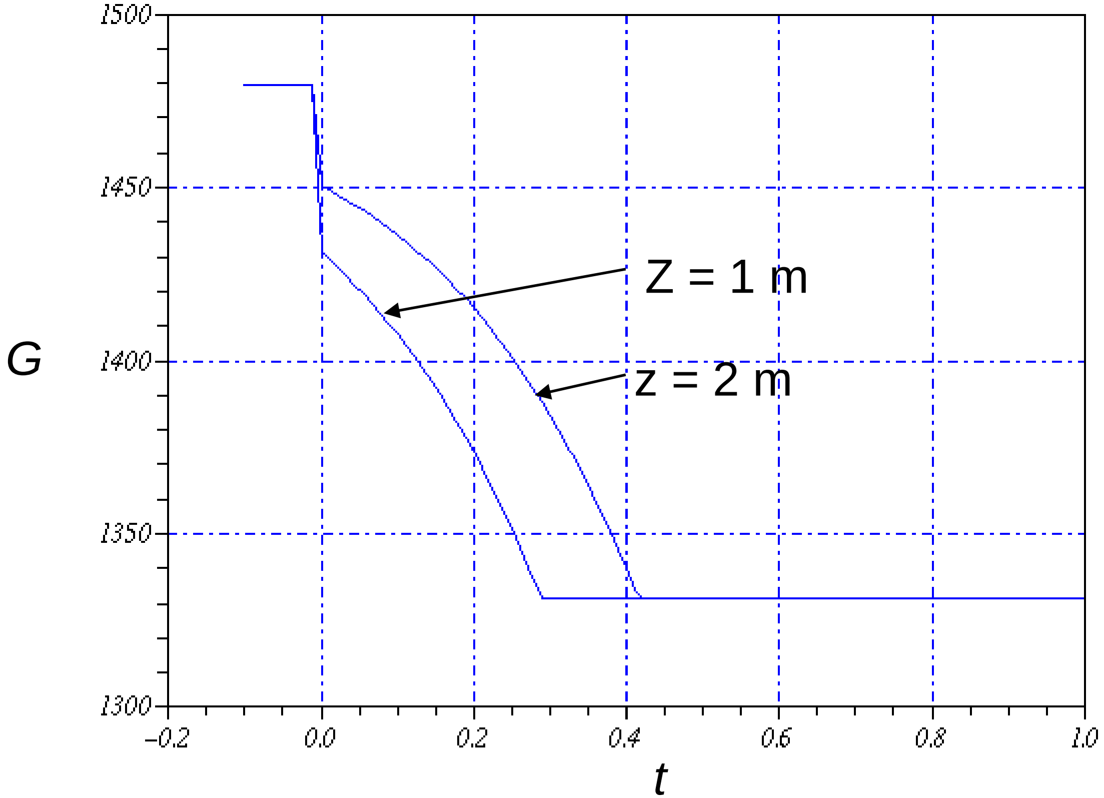
Mass Flux Versus Time (2)

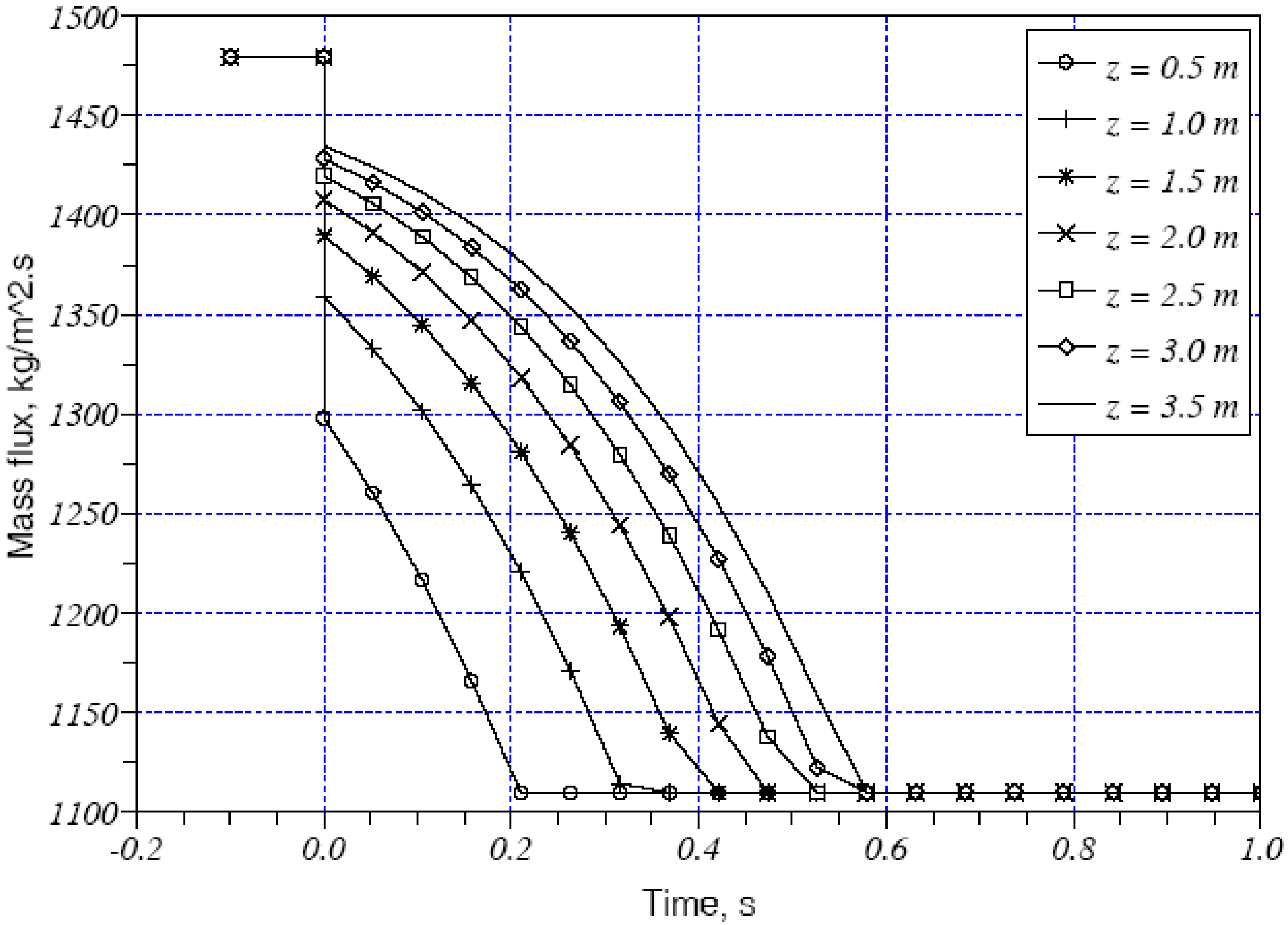
Mass flux change



Note that the rapid mass flux drop is decreasing along the channel

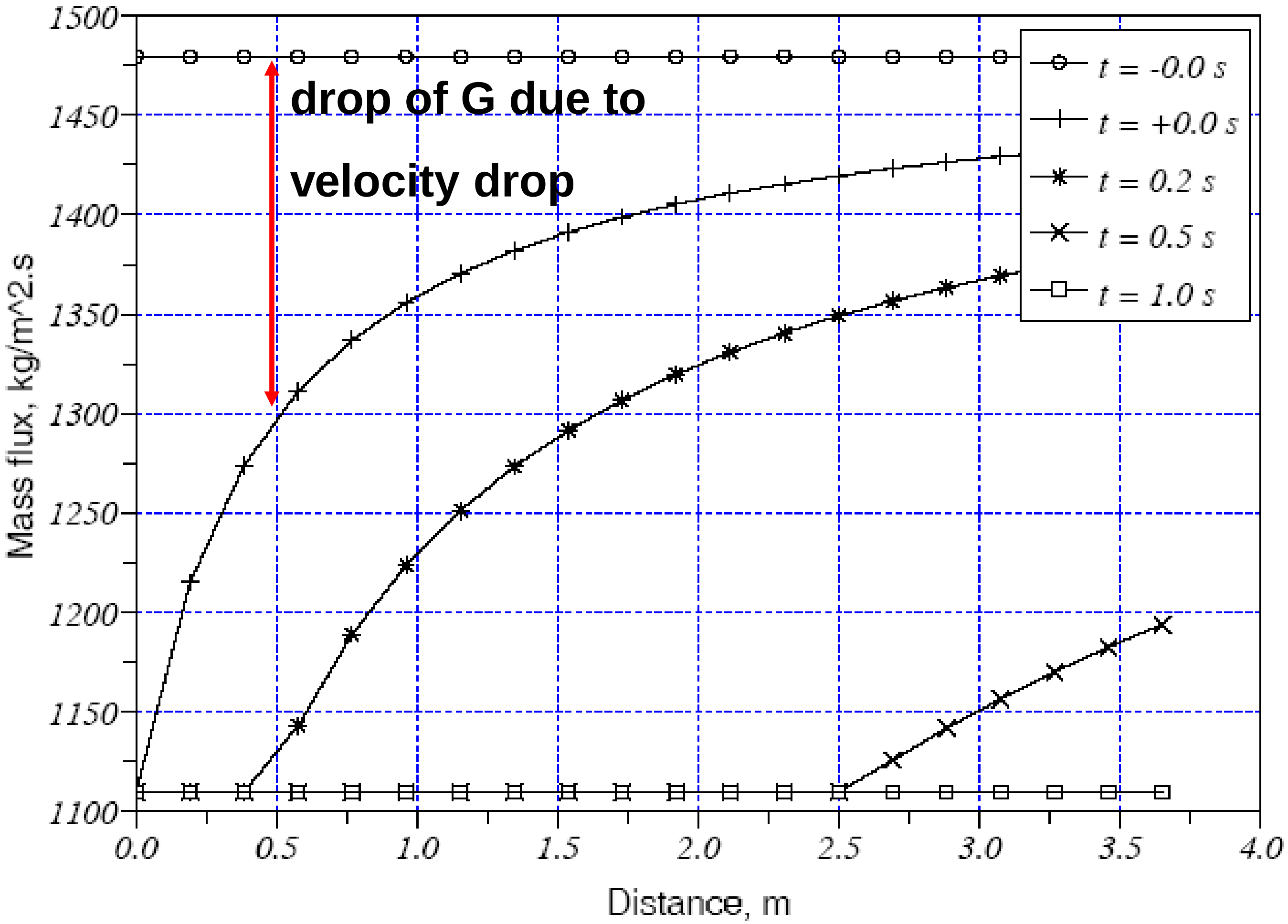
The slow drop of mass flow rate is connected to the amount of vapor that is generated and transported between inlet and current point

Mass Flux Versus Time (3)



Mass flux evolution in a boiling channel following a step change of inlet velocity

Mass Flux Versus Time (4)



Mixture mass flux distribution in a boiling channel at various time instances.

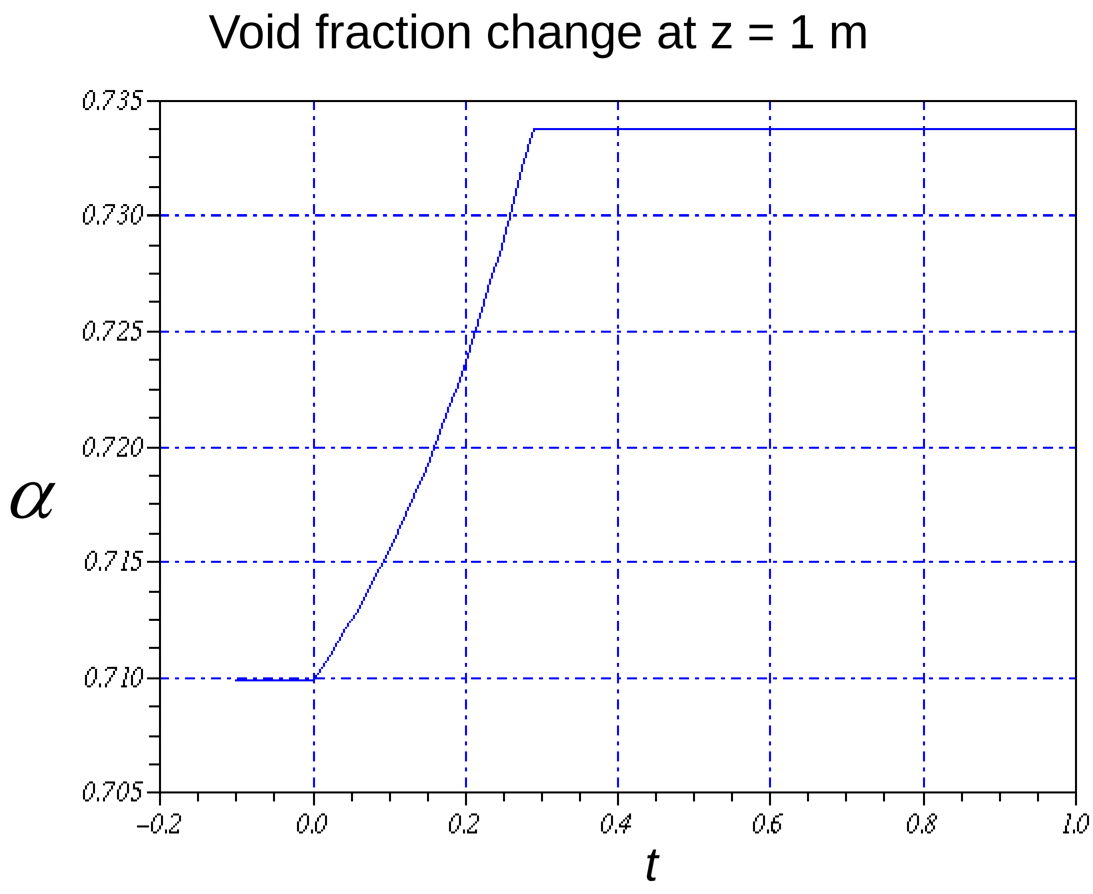
NOTE:

(a) at t=-0s mass flux is uniform everywhere

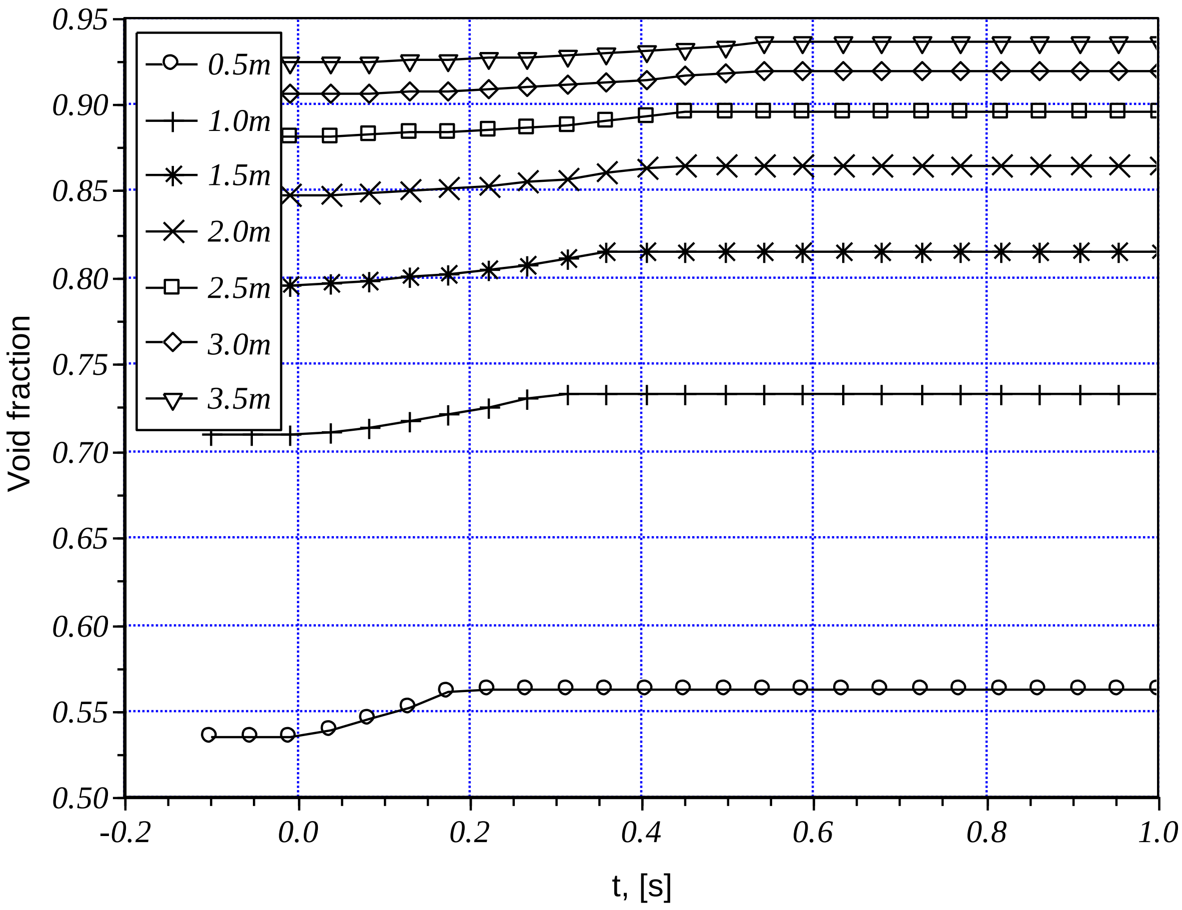
(b) at t=+0s mass flux drops at inlet to new value due to velocity drop

(c) mass flux change propagates along channel and follows the density change

Void Fraction Versus Time (1)



Void Fraction Versus Time (2)



Void transient in a boiling channel following a step change of the inlet velocity.

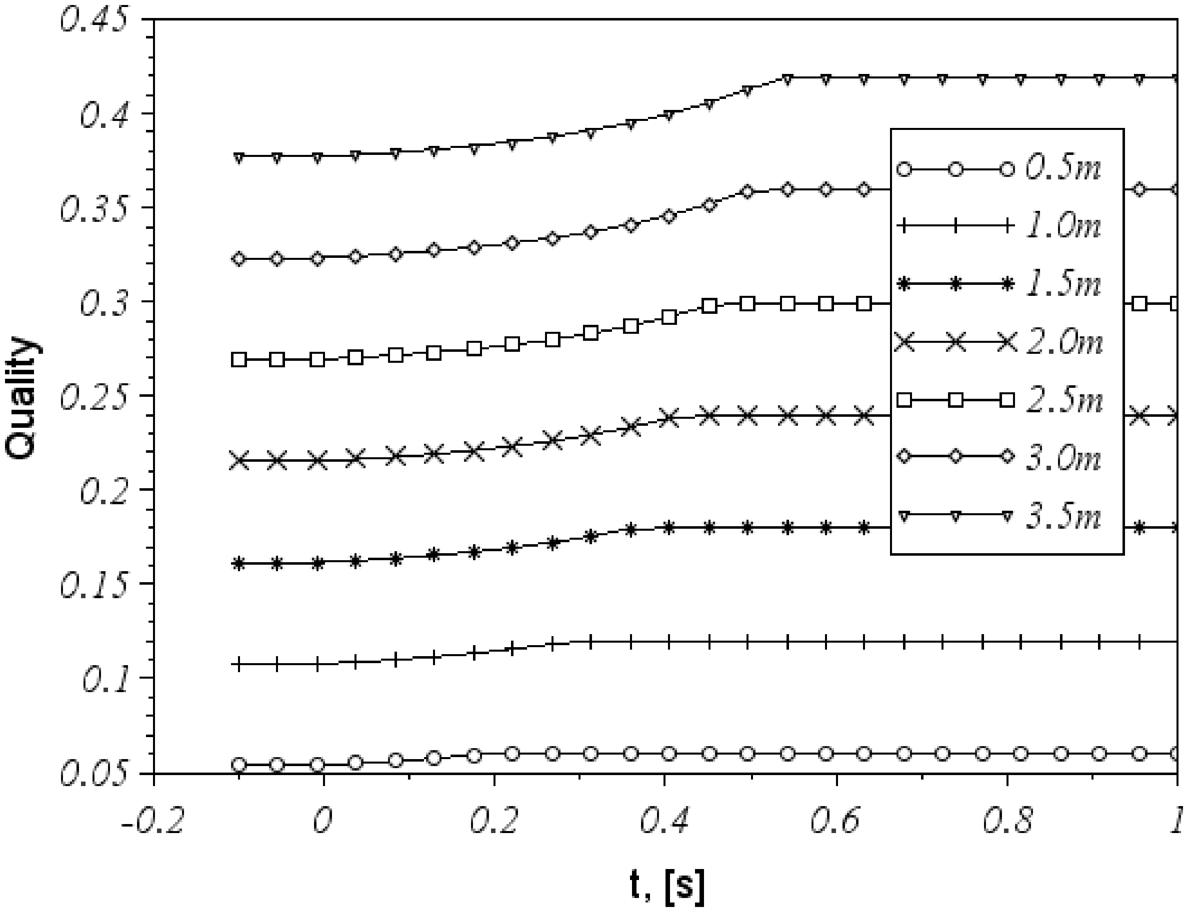
NOTE:

(a) higher amplitudes of void increase close to inlet

(b) void starts increasing in the whole channel at time t=0

(c) transient lasts ~0.2s at z=0.5m and ~0.55s at z=3.5m

Quality Versus Time



Quality transient in a boiling channel following a step change of the inlet velocity.

NOTE:

the transient follows closely the void transient

This is a direct consequence of the model we use (HEM)

Step Change of Power (1)

- Having the same assumptions as in the example of step change of the inlet velocity, we can consider a step change of heat flux:

- This can be expressed as

$$\Omega = \begin{cases} \Omega_1 & \text{for } t \le 0 \\ \Omega_2 & \text{for } t > 0 \end{cases}$$

Step Change of Power (2)

- Proceeding in a similar way as in the previous example, the trajectories are found to be as follows

$$z(t,t_0)=\begin{cases}\dfrac{J_{in}}{\Omega_1}\left[e^{\left(\Omega_2 t-\Omega_1 t_0\right)}-e^{\Omega_2 t}\right]+\dfrac{J_{in}}{\Omega_2}\left[e^{\Omega_2 t}-1\right] & \text{if} \quad t_0\le 0\\ \dfrac{J_{in}}{\Omega_2}\left[e^{\Omega_2\left(t-t_0\right)}-1\right] & \text{if} \quad t_0>0\end{cases}$$

Step Change of Power (3)

- The corresponding change in the mixture density is obtained as

$$\rho_m(z,t) = \begin{cases} \dfrac{\rho_f}{1+\dfrac{\Omega_1 z}{J_{in}}} & \text{for} \qquad t<0 \\[2ex] \dfrac{\rho_f}{\dfrac{\Omega_1 z}{J_{in}}+\dfrac{\Omega_1}{\Omega_2}+\dfrac{\Omega_2-\Omega_1}{\Omega_2}e^{\Omega_2 t}} & \text{for} \quad 0 \le t \le \dfrac{1}{\Omega_2}\ln\left(1+\dfrac{\Omega_2 z}{J_{in}}\right) \\[2ex] \dfrac{\rho_f}{1+\dfrac{\Omega_2 z}{J_{in}}} & \text{for} \qquad t > \dfrac{1}{\Omega_2}\ln\left(1+\dfrac{\Omega_2 z}{J_{in}}\right) \end{cases}$$