Model Fitting

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Overview

- Model Fitting (Curve fitting)
- Functional Approach: Non-Linear Equation
- Algebraic Approach: Normal Equations
- Efficiency Indicators
- Polynomial Models
- Non-Polynomial Models
- Examples

Interpolation Problem

- Popular choice is equidistant nodes:

$$x_j = a + jh \qquad h = (b-a)/n$$

- Y-nodes are known

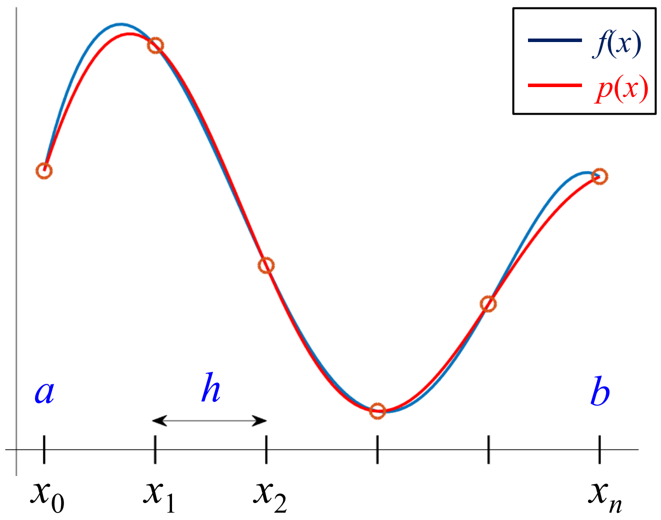
$$y_j = f(x_j)$$

- Interpolate

$$f(x) \approx p_n(x)$$

- Define error

$$E_n = \max_{a \le x \le b} \left| f(x) - p_n(x) \right|$$

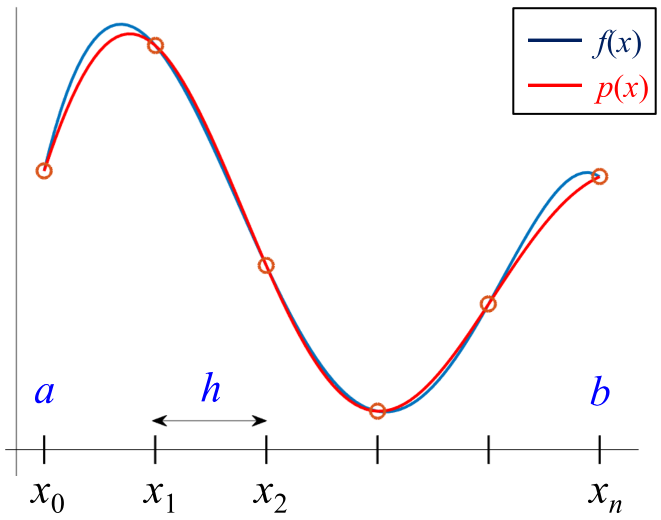


Two questions:

- Does it converge?
- How fast?

Nice Functions

Well-behaved functions such as $y = \sin(x)$ are well approximated by polynomials



Theorem

$$E_n \le \max_{a \le x \le b} \frac{\left| f^{(n+1)}(\xi) \right|}{4(n+1)} h^{n+1}$$

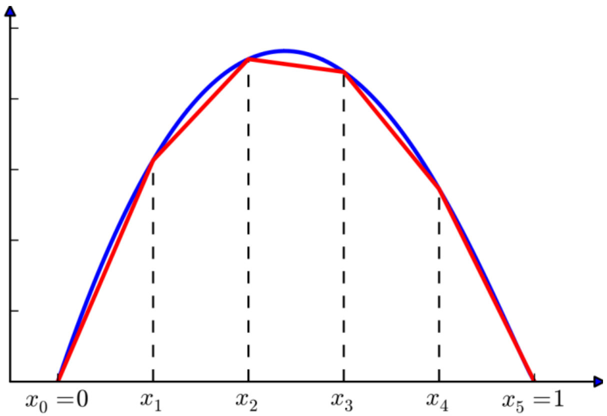
Numerical practice shows:

- Equidistant nodes are often worst
- Does not converge uniformly
- $p_n(x)$ strongly oscilates (Runge)

$$\frac{\left| f^{(n+1)}(\xi) \right|}{4(n+1)} h^{n+1} \xrightarrow[n \to \infty]{} \infty$$

Piecewise Linear

- Polynomial degree is limitted by 1.
- Does converge



Theorem

$$E_n \le Mh^2$$

$$h = \frac{b-a}{n}$$

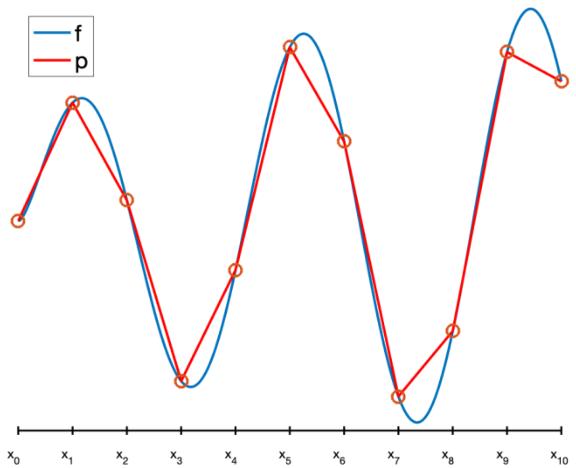
$$M = \max_{a \le \xi \le b} \frac{|f''(\xi)|}{8}$$

$$E_n \le Mh^2 \le tol \longrightarrow h \le \sqrt{tol/M}$$

$$n \ge (b-a)\sqrt{M/tol}$$

Piecewise Linear

- M does not depend on h (and n)
- Converges, $E_n \to 0$ as $O(h^2)$
- For small h, $E_n \approx Mh^2$



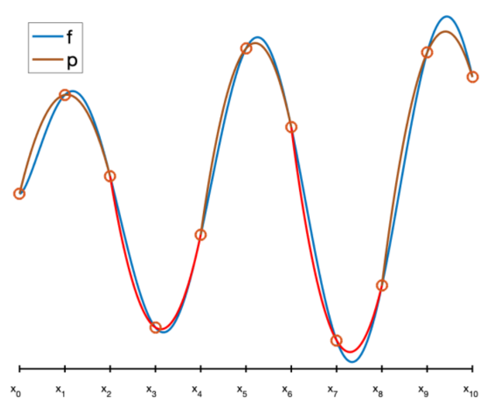
$$E_n \le Mh^2$$

$$h = \frac{b-a}{n}$$

$$M = \max_{a \le \xi \le b} \frac{|f''(\xi)|}{8}$$

Piecewise Polynomial

- Polynomial degree is limitted by k:
  - Quadratic, $k = 2$;
  - Cubic, $k = 3$ (splines)
- Converges faster with $n \to \infty$
- Beginning with $k = 4$ and higher, oscilations and instability
- Matlab, `y = interp1(xnod,ynod,x,method);`
  - method
    `'linear', 'cubic', 'spline'` etc



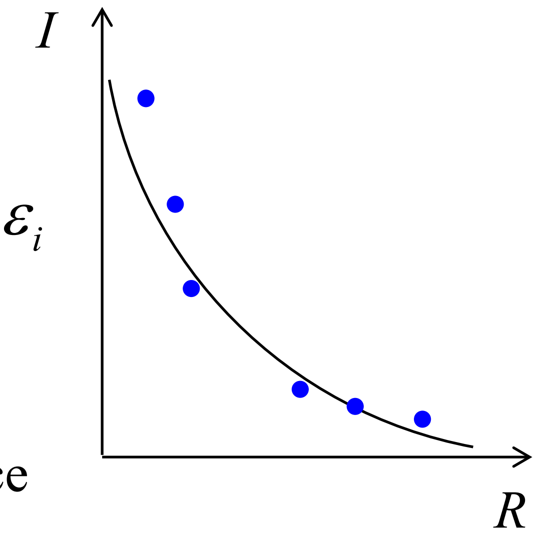
Spline

$$\|f - S\|_\infty \le \frac{5}{384} h^4 \left\|f^{(4)}\right\|_\infty$$

Curve Fitting

Constracting a curve (mathematical function) that has best fit to a series of data points.

- Ohm's law $$I = U/R$$
- Noise in each measurement $$I_i = U/R_i + \varepsilon_i$$
- Polynomyals give strong oscillations
- Least squares method can significantly reduce noise



Least Squares

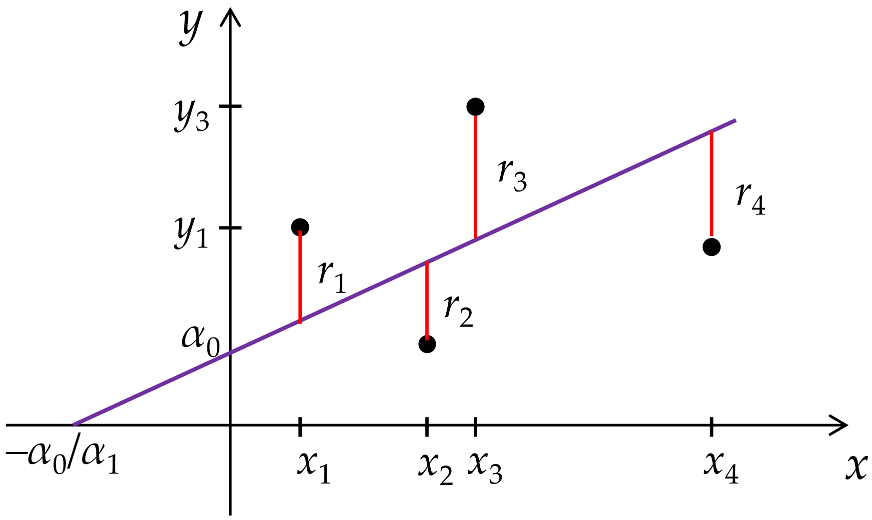
Least Squares Method, LSM, finds the straight line that minimizes the total distance from the given points.

Model function

$$f(x;\alpha_0,\alpha_1) = \alpha_0 + \alpha_1 x$$

$$r_i(\alpha_0,\alpha_1) = \alpha_0 + \alpha_1 x_i - y_i$$

$$R^2(\alpha_0,\alpha_1) \equiv \sum_{i=1}^{n} r_i^2(\alpha_0,\alpha_1)$$



Solution may be found by

- Differential calculus
- Normal equations

$$\begin{cases} \partial_0 R^2(\alpha_0,\alpha_1) = \dfrac{\partial R(\alpha_0,\alpha_1)}{\partial \alpha_0} = 0 \\ \partial_1 R^2(\alpha_0,\alpha_1) = \dfrac{\partial R(\alpha_0,\alpha_1)}{\partial \alpha_1} = 0 \end{cases}$$

Deriving System of Equations

$$R^2(\alpha_0,\alpha_1) \equiv \sum_{i=1}^{n} r_i^2(\alpha_0,\alpha_1)$$

$$\begin{cases} \dfrac{\partial R^2(\alpha_0,\alpha_1)}{\partial \alpha_0} = \displaystyle\sum_{i=1}^{N} 2r_i(\alpha_0,\alpha_1)\dfrac{\partial r_i(\alpha_0,\alpha_1)}{\partial \alpha_0} = 0 \\ \dfrac{\partial R^2(\alpha_0,\alpha_1)}{\partial \alpha_1} = \displaystyle\sum_{i=1}^{N} 2r_i(\alpha_0,\alpha_1)\dfrac{\partial r_i(\alpha_0,\alpha_1)}{\partial \alpha_1} = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial r_i(\alpha_0,\alpha_1)}{\partial \alpha_0} = \dfrac{\partial}{\partial \alpha_0}\left(\alpha_0 + \alpha_1 x_i - y_i\right) = 1 \\ \dfrac{\partial r_i(\alpha_0,\alpha_1)}{\partial \alpha_1} = \dfrac{\partial}{\partial \alpha_1}\left(\alpha_0 + \alpha_1 x_i - y_i\right) = x_i \end{cases}$$

Deriving System of Equations

$$\begin{cases} \sum_{i=1}^{N} r_i(\alpha_0,\alpha_1)\dfrac{\partial r_i(\alpha_0,\alpha_1)}{\partial \alpha_0} = \sum_{i=1}^{N} r_i(\alpha_0,\alpha_1)\times 1 = 0 \\ \sum_{i=1}^{N} r_i(\alpha_0,\alpha_1)\dfrac{\partial r_i(\alpha_0,\alpha_1)}{\partial \alpha_1} = \sum_{i=1}^{N} r_i(\alpha_0,\alpha_1)\times x_i = 0 \end{cases}$$

$$r_i(\alpha_0,\alpha_1) = \alpha_0 + \alpha_1 x_i - y_i$$

$$\begin{cases} \sum_{i=1}^{N} \left(\alpha_0 + \alpha_1 x_i - y_i\right) = 0 \\ \sum_{i=1}^{N} \left(\alpha_0 + \alpha_1 x_i - y_i\right) x_i = 0 \end{cases}$$

System of Equations

$$\begin{cases} \alpha_0 N + \alpha_1 \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i \\ \alpha_0 \sum_{i=1}^{N} x_i + \alpha_1 \sum_{i=1}^{N} x_i^2 = \sum_{i=1}^{N} x_i y_i \end{cases}$$

$$\begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i y_i \end{bmatrix}$$

$$\alpha_1 = \frac{\Delta_1}{\Delta} = \frac{N \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i \right)^2}$$

$$\alpha_0 = \frac{1}{N} \left[\sum_{i=1}^{N} y_i - \alpha_1 \sum_{i=1}^{N} x_i \right]$$

Cramer's Rule

Gabriel Cramer, 1704 – 1752. Published 1750.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} f_1 & b_1 \\ f_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{f_1 b_2 - f_2 b_1}{a_1 b_2 - a_2 b_1}; \quad y = \frac{\begin{vmatrix} a_1 & f_1 \\ a_2 & f_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1 f_2 - a_2 f_1}{a_1 b_2 - a_2 b_1}.$$

Algebraic Method

$$f(x;\alpha_0,\alpha_1) = y$$

Model function

$$\begin{cases} f(x_1;\alpha_0,\alpha_1) = \alpha_0 + \alpha_1 x_1 = y_1 \\ f(x_2;\alpha_0,\alpha_1) = \alpha_0 + \alpha_1 x_2 = y_2 \\ \vdots \\ f(x_N;\alpha_0,\alpha_1) = \alpha_0 + \alpha_1 x_N = y_N \end{cases}$$

System of equations

Matrix Form

$$\begin{cases} \alpha_0 + \alpha_1 x_1 = y_1 \\ \alpha_0 + \alpha_1 x_2 = y_2 \\ \qquad\qquad \vdots \\ \alpha_0 + \alpha_1 x_N = y_N \end{cases}$$

$$\mathbf{A}\boldsymbol{\alpha} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \mathbf{b}$$

Normal Equations

$$\mathbf{A}^T\mathbf{A}\boldsymbol{\alpha} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} \cdot \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} = \begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Sample Statistics

Two data sets can be characterised by:

$$\mathbf{x} \equiv \left\{x_1, x_2, \ldots, x_N\right\}$$

$$\mathbf{y} \equiv \left\{y_1, y_2, \ldots, y_N\right\}$$

Samples, observations, measurements etc.

- Sample mean $$\overline{\mathbf{x}} \equiv \frac{1}{N}\sum_{i=1}^{N} x_i$$
- Sample standard deviation $$S_{\mathbf{x}} \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{\mathbf{x}}\right)^2}$$
- Sample variance $$\mathrm{Var}(\mathbf{x}) = S_{\mathbf{x}}^2 \equiv \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{\mathbf{x}}\right)^2$$
- Sample covariance $$\mathrm{Cov}(\mathbf{x},\mathbf{y}) \equiv \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{\mathbf{x}}\right)\left(y_i - \overline{\mathbf{y}}\right)$$
- Sample correlation $$\mathrm{Corr}(\mathbf{x},\mathbf{y}) = \rho\left(\mathbf{x},\mathbf{y}\right) \equiv \frac{\mathrm{Cov}(\mathbf{x},\mathbf{y})}{S_{\mathbf{x}} \cdot S_{\mathbf{y}}}$$

Link to Sample Statistics

$$\begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i y_i \end{bmatrix}$$

$$\mathbf{y} = \{y_1, y_2, \ldots, y_N\}$$
$$\mathbf{x} = \{x_1, x_2, \ldots, x_N\}$$
$$\mathbf{x}^2 = \{x_1^2, x_2^2, \ldots, x_N^2\}$$
$$\mathbf{xy} = \{x_1 y_1, \ldots, x_N y_N\}$$

$$\begin{bmatrix} 1 & \overline{\mathbf{x}} \\ \overline{\mathbf{x}} & \overline{\mathbf{x}^2} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{y}} \\ \overline{\mathbf{xy}} \end{bmatrix}$$

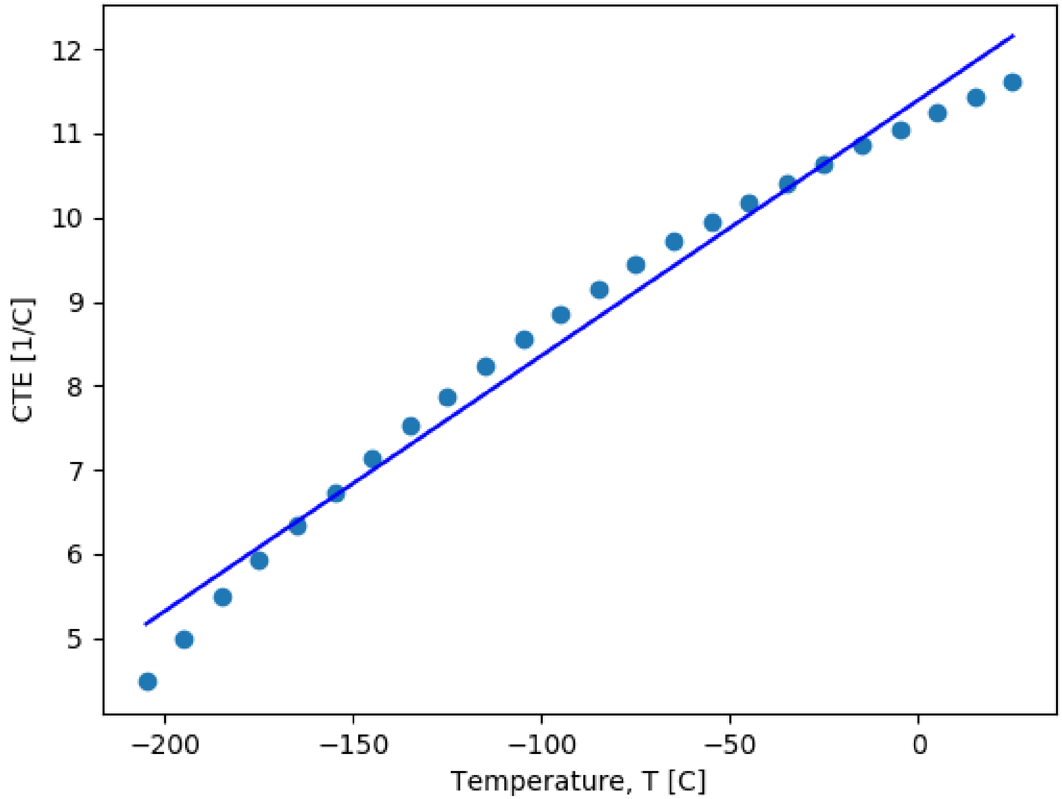
$$\alpha_1 = \frac{\Delta_1}{\Delta} = \frac{\overline{\mathbf{xy}} - \overline{\mathbf{x}} \cdot \overline{\mathbf{y}}}{\overline{\mathbf{x}^2} - \overline{\mathbf{x}}^2} = \frac{\mathrm{Cov}(\mathbf{x}, \mathbf{y})}{S_{\mathbf{x}}^2}$$

$$\alpha_0 = \overline{\mathbf{y}} - \overline{\mathbf{x}} \cdot \alpha_1$$

$$y = f(x; \alpha_0, \alpha_1) = \alpha_0 + \alpha_1 x = \alpha_1 \left(x - \overline{\mathbf{x}}\right) + \overline{\mathbf{y}} = \frac{S_{\mathbf{y}}}{S_{\mathbf{x}}} \rho(\mathbf{x}, \mathbf{y}) \left(x - \overline{\mathbf{x}}\right) + \overline{\mathbf{y}}$$

Coefficient of Thermal Exp.

$$\alpha_{\mathrm{L}}(T) = \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}T} \qquad (L_1 - L_0)/L_0 \ll 1 \qquad \alpha_{\mathrm{L}}(T_0) \approx \frac{1}{L_0}\frac{L_1 - L_0}{T_1 - T_0}$$

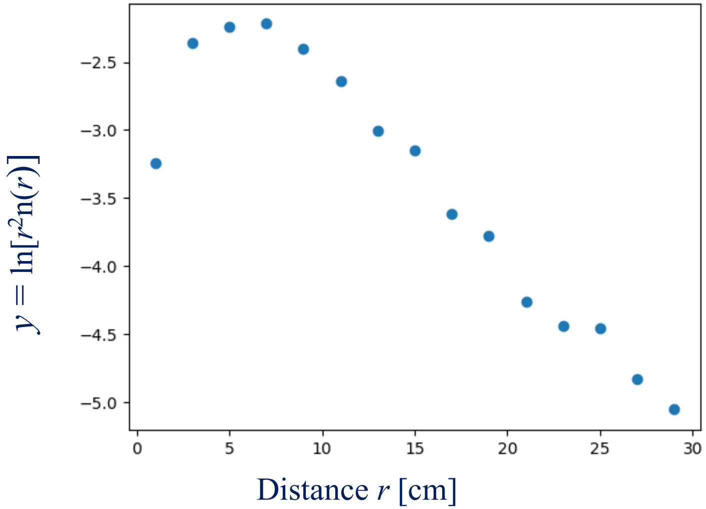


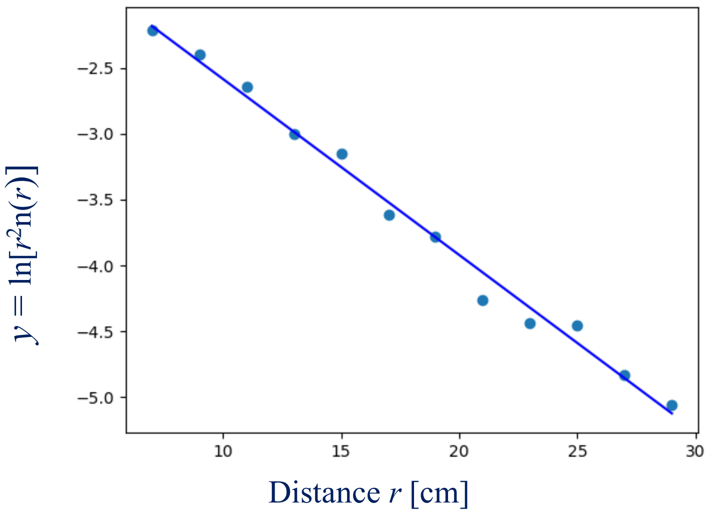
Non-Linear Problems

Reactor physics predicts that the density of fast neutrons in water from a point neutron source behaves with distance *r* from the source as

$$n(r) = \frac{A}{r^2} e^{-r/\lambda}$$

$$y \equiv \ln\left[r^2 n(r)\right] = \ln A - r/\lambda$$





Regression Model

Data points

$$\mathbf{x} \equiv \{x_1, x_2, \ldots, x_N\}$$

$$\mathbf{y} \equiv \{y_1, y_2, \ldots, y_N\}$$

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_M)$$

$$y(x) = f(x; \boldsymbol{\alpha})$$

Model function with parameters

Degree of Freedom, DoF: $\mathrm{DoF} \equiv N - (M+1)$

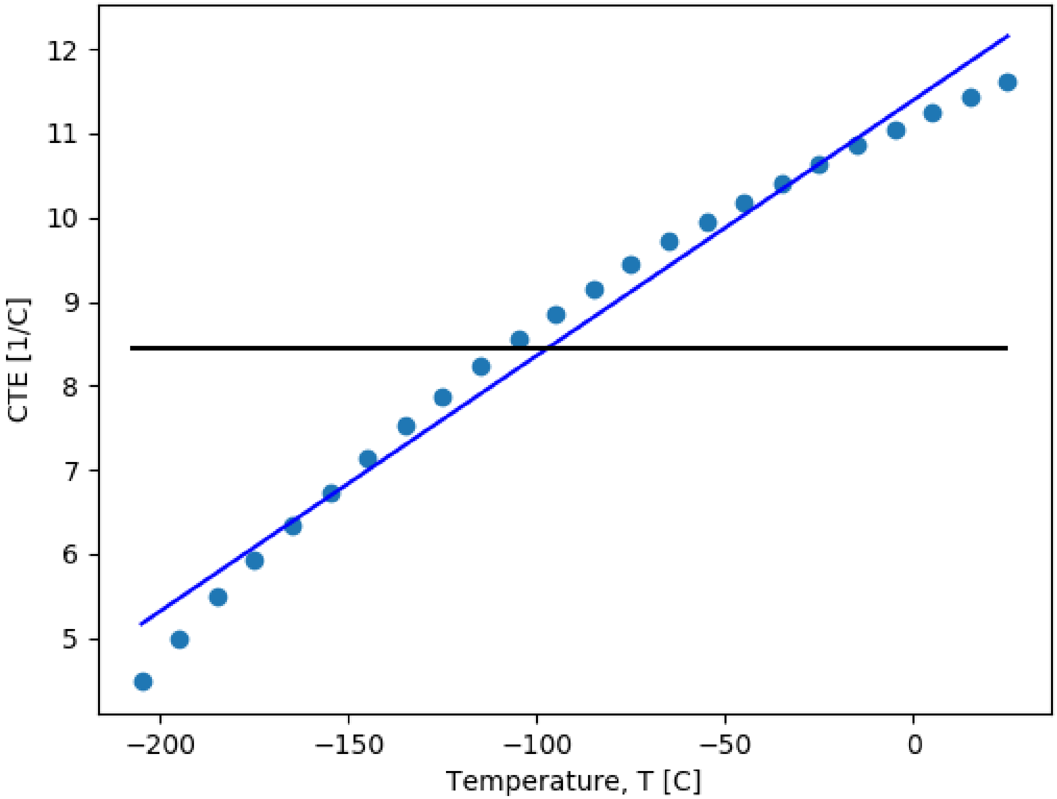
$$f(x;\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_M x^M$$

Baseline model $M = 0$:

$$y = f(x;\boldsymbol{\alpha}) = \overline{\mathbf{y}} \qquad \overline{\mathbf{y}} \equiv (y_1 + y_2 + \ldots + y_N)/N$$

Baseline for Thermal Exp.

$$\alpha_{\mathrm{L}}(T) = \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}T} \qquad \alpha_{\mathrm{L}}(T_0) \approx \frac{1}{L_0}\frac{L_1 - L_0}{T_1 - T_0}$$

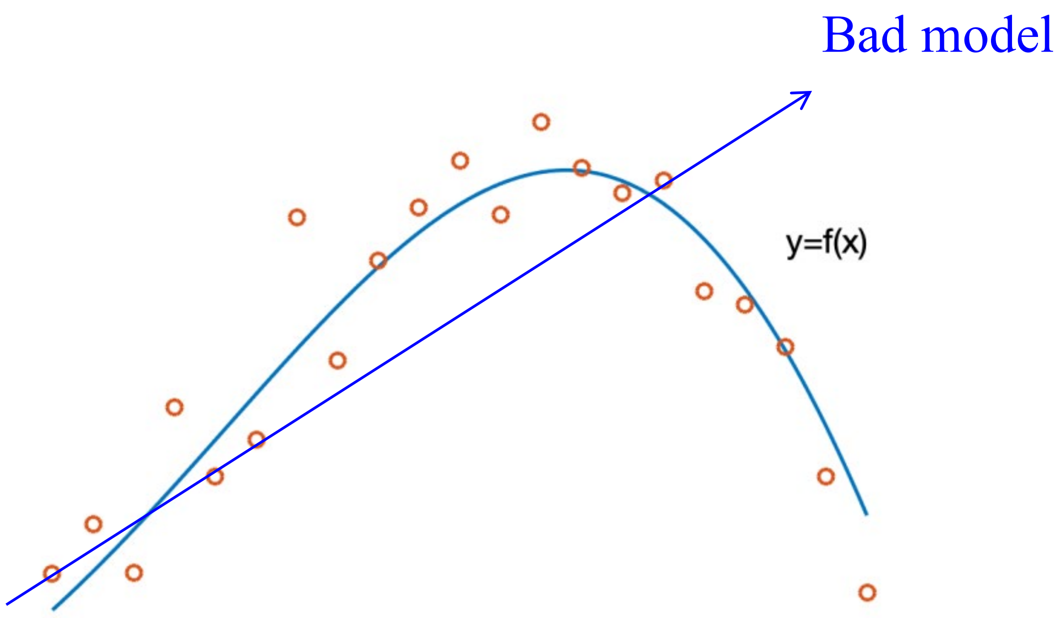


Selecting Model

A particular choice of the model may stem from:

- Physical model such as Ohm's law;
- Numerical simulation such as solving numerically Neutron Diffusion Equation, NDE;
- Previous experience e.g. the current conditions are similar to the previous ones;
- Educated guess;
- Visual inspection;
- Assumption subject to confirm or reject.

Visual Inspection



Sum of Squared Residuals

When parameter vector $\boldsymbol{\alpha}$ is found, the model function is defined

$$y = f(x;\boldsymbol{\alpha})$$

Predicted (computed) values: $\hat{y}_i = f(x_i;\boldsymbol{\alpha})$

$$r_i \equiv \hat{y}_i - y_i$$

$$\mathrm{SSR} \equiv R^2 \equiv \sum_{i=1}^{N} r_i^2 = \sum_{i=1}^{N} \left[\hat{y}_i - y_i\right]^2 = \sum_{i=1}^{N} \left[f(x_i;\boldsymbol{\alpha}) - y_i\right]^2$$

Also, Sum of Squared Errors, SSE; Residual Sum of Squares, RSS.

Root Mean Square Error

$$\mathrm{RMSE} \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\hat{y}_i - y_i\right]^2}$$

$$S_y \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(y_i - \overline{y}\right)^2}$$

$$\mathrm{RMSN} \equiv \mathrm{RMSE}\big/S_y$$ (normalised)

Coefficient of Determination

CoD, is a popular choice to characterize the quality of the regression model.

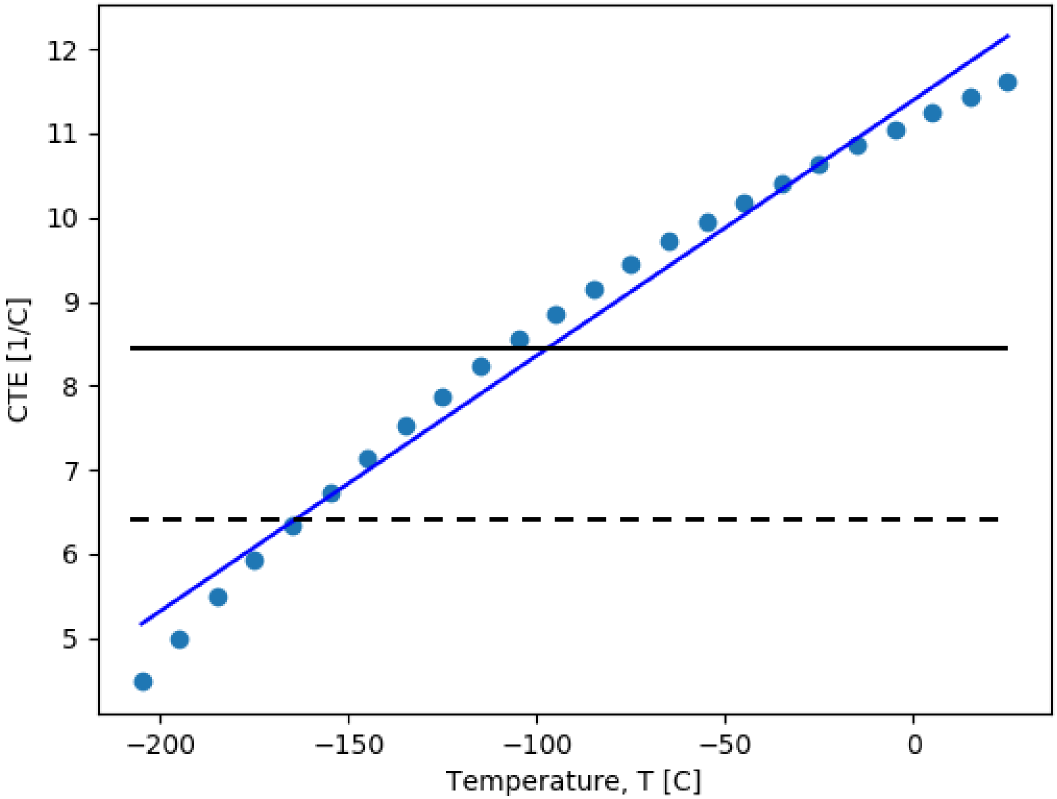
$$\mathrm{CoD} \equiv r^2 \equiv R^2 \equiv 1 - \mathrm{RMSN}^2 = 1 - \frac{\sum_{i=1}^{N} \left[\hat{y}_i - y_i \right]^2}{\sum_{i=1}^{N} \left[\bar{\mathbf{y}} - y_i \right]^2}$$

Baseline $\hat{y}_i = \bar{\mathbf{y}}$ $\quad 0 \le \mathrm{CoD} \le 1 \quad$ $\hat{y}_i = y_i$ Best

It may happen $\quad \mathrm{CoD} < 0 \quad$ Pathology !!

Unacceptable Baseline

$$\alpha_{\mathrm{L}}(T) = \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}T} \qquad \alpha_{\mathrm{L}}(T_0) \approx \frac{1}{L_0}\frac{L_1 - L_0}{T_1 - T_0}$$



CoD = 0

CoD < 0

Polynomial Model Function

$$f(x;\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots \alpha_M x^M$$

$$\begin{cases} f(x_1;\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \cdots \alpha_M x_1^M = y_1 \\ f(x_2;\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 x_2 + \alpha_2 x_2^2 + \cdots \alpha_M x_2^M = y_2 \\ \qquad\qquad \vdots \\ f(x_N;\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 x_N + \alpha_2 x_N^2 + \cdots \alpha_M x_N^M = y_N \end{cases}$$

Functional Approach

$$r_i(\boldsymbol{\alpha}) \equiv r_i \equiv f(x_i;\boldsymbol{\alpha}) - y_i \qquad R^2(\boldsymbol{\alpha}) \equiv R^2 \equiv \sum_{i=1}^{N} r_i^2 = \sum_{i=1}^{N} \left[f(x_i;\boldsymbol{\alpha}) - y_i \right]^2$$

$$\min_{\boldsymbol{\alpha}} R^2(\boldsymbol{\alpha}) = \min_{\boldsymbol{\alpha}} \sum_{i=1}^{N} \left[f(x_i;\boldsymbol{\alpha}) - y_i \right]^2$$

$$\left\{ \frac{\partial R^2(\boldsymbol{\alpha})}{\partial \alpha_m} = 0 \qquad m = 0,1,2,\ldots,M \right.$$

$$\left\{ \sum_{i=1}^{N} 2 \left[f(x_i;\boldsymbol{\alpha}) - y_i \right] \frac{\partial f(x_i;\boldsymbol{\alpha})}{\partial \alpha_m} = 0 \qquad m = 0,1,\ldots,M \right.$$

Linear Equations

$$g_m(\boldsymbol{\alpha}) \equiv \sum_{i=1}^{N} \left[f(x_i;\boldsymbol{\alpha}) - y_i \right] \frac{\partial f(x_i;\boldsymbol{\alpha})}{\partial \alpha_m} = 0 \qquad m = 0,1,\ldots,M$$

$$\mathbf{G}(\boldsymbol{\alpha}) \equiv \begin{bmatrix} g_0(\boldsymbol{\alpha}) \\ \vdots \\ g_M(\boldsymbol{\alpha}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

Algebraic Approach

$$\begin{cases} f(x_1;\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \cdots + \alpha_M x_1^M = y_1 \\ f(x_2;\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 x_2 + \alpha_2 x_2^2 + \cdots + \alpha_M x_2^M = y_2 \\ \qquad\vdots \\ f(x_N;\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 x_N + \alpha_2 x_N^2 + \cdots + \alpha_M x_N^M = y_N \end{cases}$$

$$\begin{bmatrix} 1 & x_1 & \cdots & x_1^M \\ 1 & x_2 & \cdots & x_2^M \\ 1 & x_3 & \cdots & x_3^M \\ \vdots & \vdots & & \vdots \\ 1 & x_{N-1} & \cdots & x_{N-1}^M \\ 1 & x_N & \cdots & x_N^M \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_M \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}$$

$$\mathbf{V}\boldsymbol{\alpha} = \mathbf{y}$$

$$\mathbf{V}^T\mathbf{V}\boldsymbol{\alpha} = \mathbf{V}^T\mathbf{y}$$

Trigonometric Functions

(1) Assume model function

$$f(x;c_0,c_1) = c_0 \sin(x) + c_1 \cos(x)$$

(2) Find coefficients c_0 and c_1 that minimize residual

$$R(\mathbf{c}) = \sum_{j=1}^{n} \left[f(x_j;c_0,c_1) - y_j \right]^2 \qquad R'_{c_0}(\mathbf{c}) = 0 \quad R'_{c_1}(\mathbf{c}) = 0$$

(3) Form linear equations

$$\begin{aligned} f(x_1) &= c_0 \sin(x_1) + c_1 \cos(x_1) = y_1 \\ f(x_2) &= c_0 \sin(x_2) + c_1 \cos(x_2) = y_2 \\ &\vdots \\ f(x_N) &= c_0 \sin(x_N) + c_1 \cos(x_N) = y_N \end{aligned}$$

Matrix Form

$$\begin{bmatrix} \sin(x_1) & \cos(x_1) \\ \sin(x_2) & \cos(x_2) \\ \vdots & \vdots \\ \sin(x_N) & \cos(x_N) \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \sin(x_1) & \cos(x_1) \\ \sin(x_2) & \cos(x_2) \\ \vdots & \vdots \\ \sin(x_N) & \cos(x_N) \end{bmatrix}$$

$$\mathbf{V}^T = \begin{bmatrix} \sin(x_0) & \sin(x_1) & \cdots & \sin(x_n) \\ \cos(x_0) & \cos(x_1) & \cdots & \cos(x_n) \end{bmatrix}$$

$$\mathbf{V}^T\mathbf{V}\mathbf{c} = \begin{bmatrix} \sum \sin^2(x_i) & \sum \sin(x_i)\cos(x_i) \\ \sum \sin(x_i)\cos(x_i) & \sum \cos^2(x_i) \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \sin(x_i) \\ \sum y_i \cos(x_i) \end{bmatrix} = \mathbf{V}^T\mathbf{y}$$

Matlab

Backslash command

```
>> x = A\b
```

solves:

- $\mathbf{Ax} = \mathbf{b}$ in ordinary sense when $n = m$
- $\mathbf{Ax} = \mathbf{b}$ in least squares sense when $n \neq m$

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \times \begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

```
>> c = polyfit(x,y,m)
```

finds:

- Coefficients c of polynomial of degree m that is the best fit.

Important

- Model Fitting (Curve fitting)
- Functional Approach: Non-Linear Equation
- Algebraic Approach: Normal Equations
- Efficiency Indicators
- Polynomial Models
- Non-Polynomial Models