Spline Interpolation

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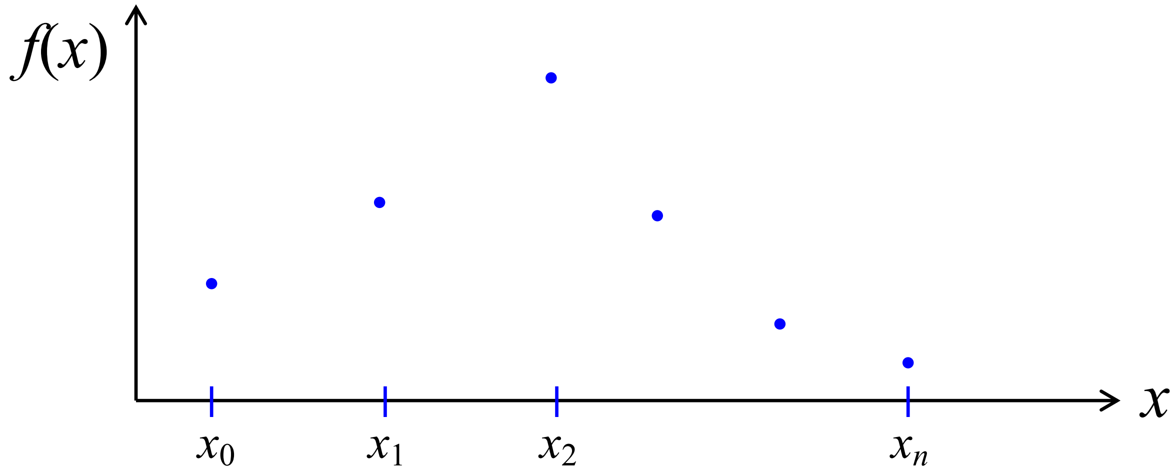
Historical Perspective

- Polynomial interpolation is an ancient practice, intensive use in 20th century.
- Shipbuilding, aircraft and car industry.
- Paul de Casteljau (Citroen) and Pierre Bézier (Renault): cubic splines.
- Cubic splines were used in computer typesetting.
- Revolution in printing: two Xerox engineers formed Adobe, PostScript.
- Steve Jobs was looking for a way to control newly invented laser printers.
- Bézier splines turned out to be a simple way to adapt the same mathematical curves to fonts with multiple printer resolutions.
- Based on PostScript, Adobe issued a more flexible format, PDF.

Interpolation Problem

x	x_0	x_1	x_2	$\cdots$	x_n
y	y_0	y_1	y_2	$\cdots$	y_n



Knots $x_0 < x_1 < \cdots < x_n$

Find a polynomial $p(x)$ such that $p(x_i) = y_i$

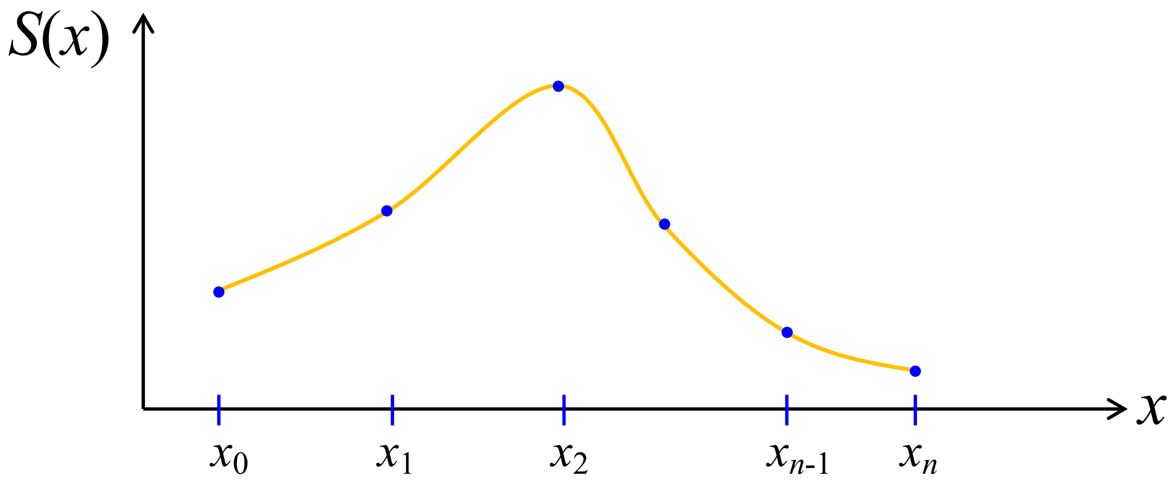
Etymology

- Schoenberg [1946] first uses the word *spline* in connection with smooth, piecewise polynomial approximations.
- The word *spline* as a thin strip of wood used by a draftsman dates back to the 1890s at least.
- Many of the ideas used in spline theory have their roots in work done in various industries such as the building of aircraft, automobiles, and ships in which splines are used extensively.

Spline in Practice

A flexible strip of metal or other material, that may be bent into a curve and used in a similar manner to a ruler to draw smooth curves between points.

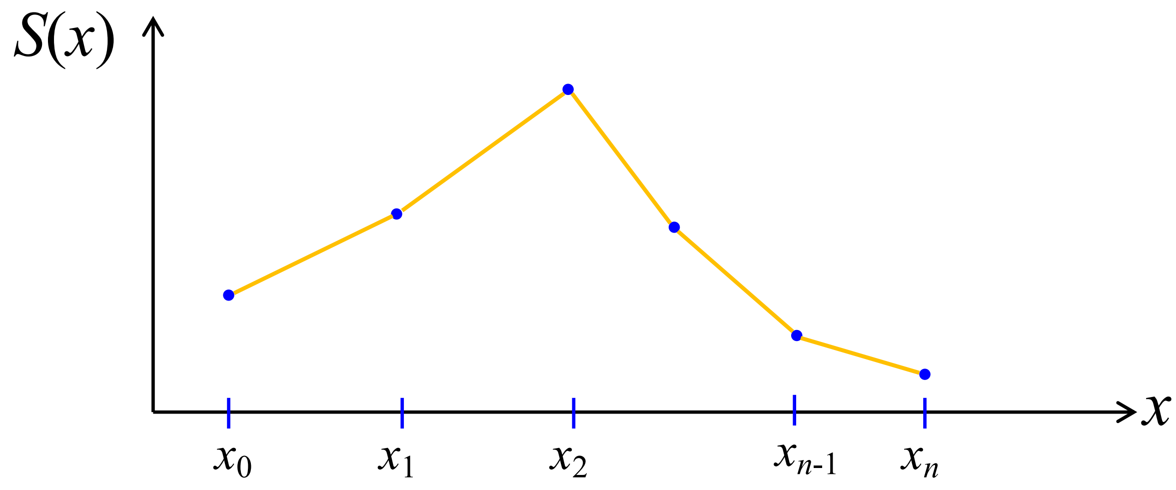
Spline in Mathematics



Definition. A spline function of degree k having knots x_0, x_1, …,x_n is a function S such that

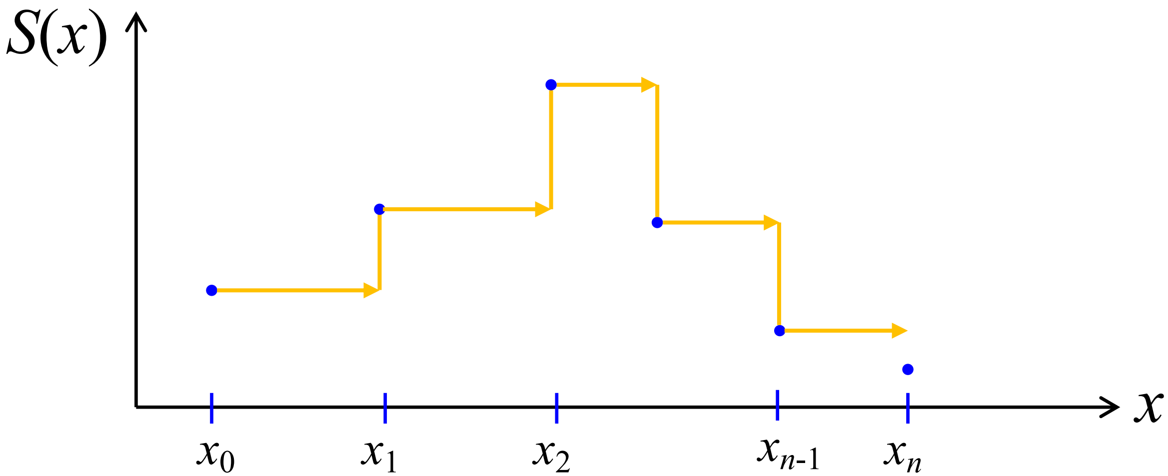
1) $S(x_i) = y_i$
2) On each $[x_{i-1}, x_i)$, S is a polynomial of degree $\leq k$
3) S has a continuous (k-1)st derivative.

Spline of Degree 1



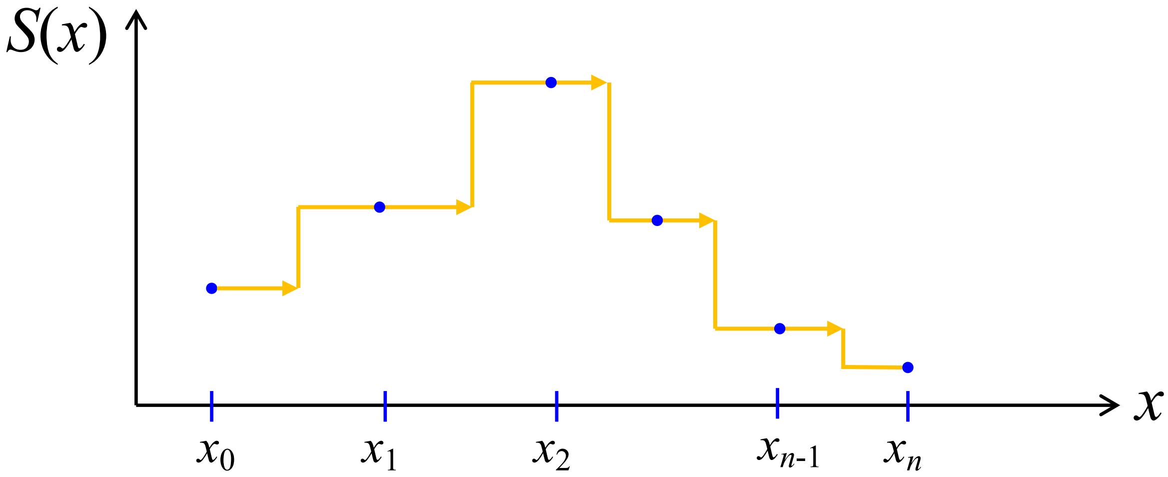
$$S(x) = \begin{cases} S_0(x) = a_0 x + b_0 & x \in [x_0, x_1) \\ S_1(x) = a_1 x + b_1 & x \in [x_1, x_2) \\ \vdots & \\ S_{n-1}(x) = a_{n-1} x + b_{n-1} & x \in [x_{n-1}, x_n) \end{cases}$$

Spline of Degree 0



$$S(x) = \begin{cases} S_0(x) = y_0 & x \in [x_0, x_1) \\ S_1(x) = y_1 & x \in [x_1, x_2) \\ \vdots & \\ S_{n-1}(x) = y_{n-1} & x \in [x_{n-1}, x_n) \end{cases}$$

Better Spline of Degree 0

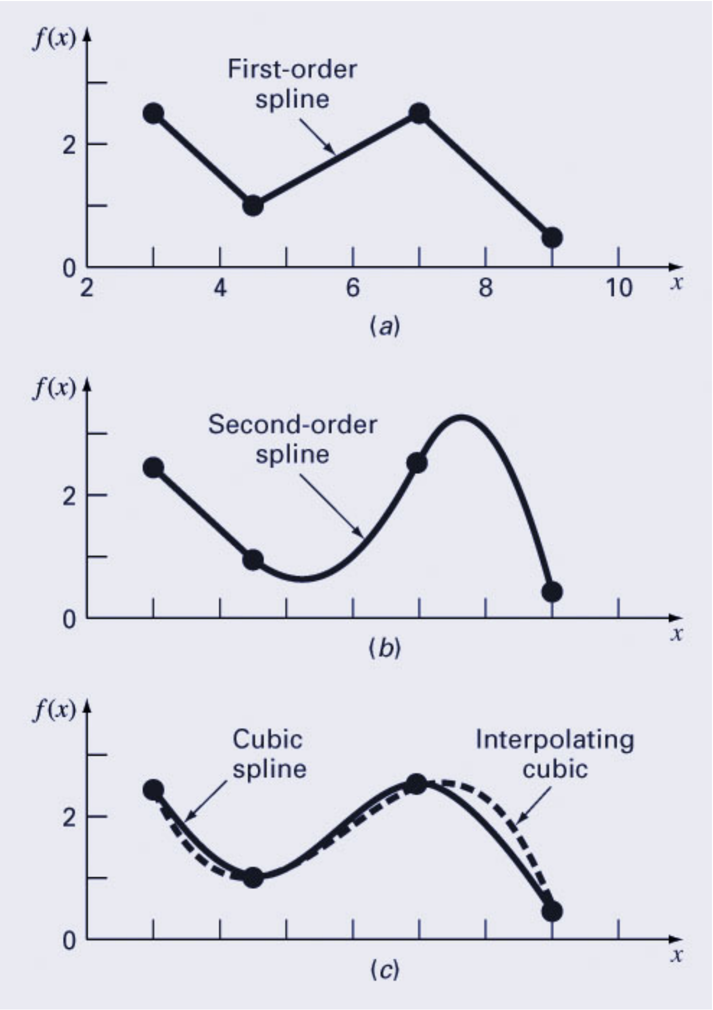


Spline Development

a) First-order splines

b) Second-order splines

c) Third-order splines



Why Cubic Splines?

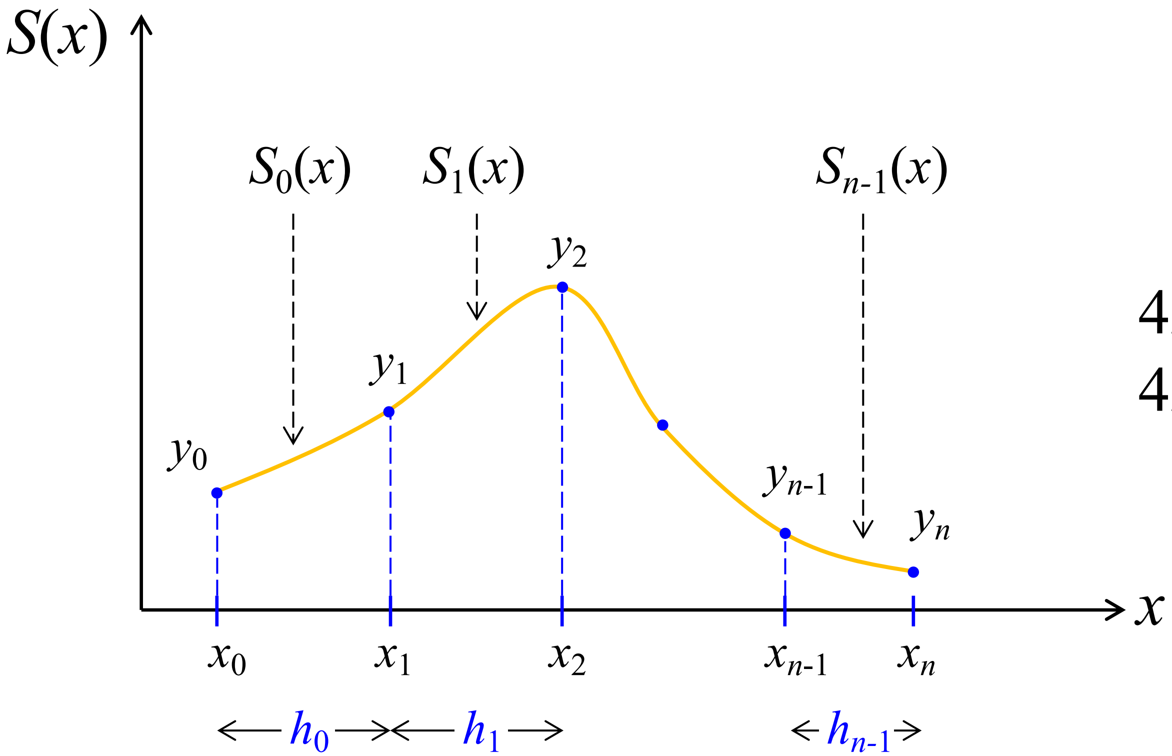
- There are many options for the order of spline
- Cubic splines are preferred due to smoothness and simplicity
  - Linear splines are discontinuous in first derivative
  - Quadric splines are discontinuous in second derivative and require setting of second derivative at some point
  - Quartic or higher splines tend to exhibit ill-conditioning or oscillations

Cubic Splines

x	x_0	x_1	x_2	$\dots$	x_n
y	y_0	y_1	y_2	$\dots$	y_n

$$S(x) = \begin{cases} S_0(x) & x \in [x_0, x_1) \\ S_1(x) & x \in [x_1, x_2) \\ \vdots & \\ S_{n-1}(x) & x \in [x_{n-1}, x_n) \end{cases}$$

Spline Pieces



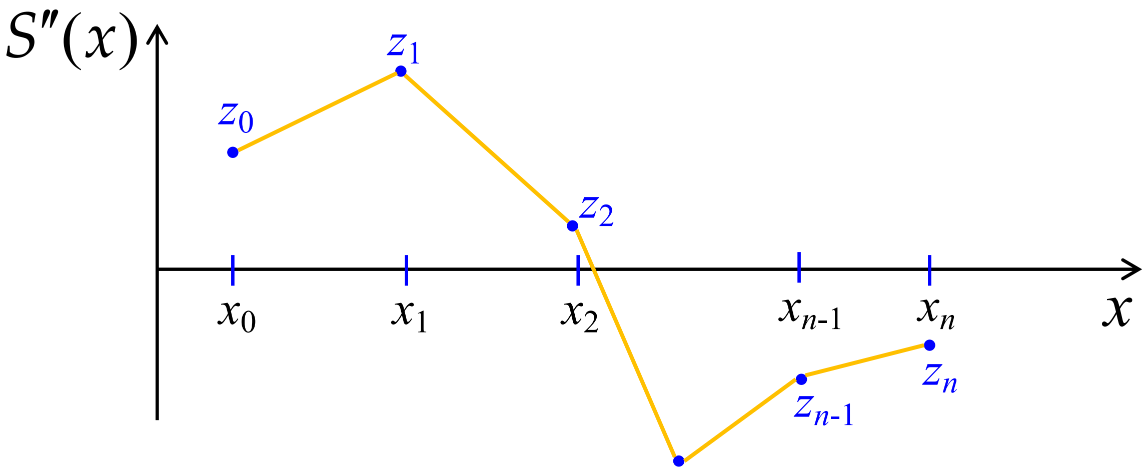
$4n$ unknowns

$4n$-2 equations

Interval $\quad S_i(x_i) = y_i \qquad S_i(x_{i+1}) = y_{i+1} \longrightarrow 2n$

Knot $\quad S'_{i-1}(x_i) = S'_i(x_i) \qquad S''_{i-1}(x_i) = S''_i(x_i) \longrightarrow 2n-2$

Second Derivative of Spline



$$\lim_{x \searrow x_i} S''(x) = z_i = \lim_{x \nearrow x_i} S''(x)$$

$$(1 \le i \le n-1)$$

$x \in [x_i, x_{i+1}]$; $\quad h_i = x_{i+1} - x_i$; $\quad S_i''(x_i) = z_i \quad S_i''(x_{i+1}) = z_{i+1}$

$$S_i''(x) = \frac{z_i}{h_i}(x_{i+1} - x) + \frac{z_{i+1}}{h_i}(x - x_i)$$

Local Piece of Spline

$$S_i(x) = \frac{z_i}{6h_i}\left(x_{i+1} - x\right)^3 + \frac{z_{i+1}}{6h_i}\left(x - x_i\right)^3 + C\left(x - x_i\right) + D\left(x_{i+1} - x\right)$$

$$S_i(x_i) = y_i \qquad\qquad S_i(x_{i+1}) = y_{i+1}$$

$$S_i(x) = \frac{z_i}{6h_i}\left(x_{i+1} - x\right)^3 + \frac{z_{i+1}}{6h_i}\left(x - x_i\right)^3 +$$

$$+\left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right)\left(x - x_i\right) + \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right)\left(x_{i+1} - x\right)$$

Equations for z_i's

$$S'_{i-1}(x_i) = S'_i(x_i) \qquad (1 \le i \le n-1)$$

$$2(h_1 + h_0)z_1 + h_1 z_2 = \frac{6}{h_1}(y_2 - y_1) - \frac{6}{h_0}(y_1 - y_0) - h_0 z_0$$

$$h_{i-1}z_{i-1} + 2(h_i + h_{i-1})z_i + h_i z_{i+1} = \frac{6}{h_i}(y_{i+1} - y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1})$$

$$h_{n-2}z_{n-2} + 2(h_{n-1} + h_{n-2})z_{n-1} = \frac{6}{h_{n-1}}(y_n - y_{n-1}) - \frac{6}{h_{n-2}}(y_{n-1} - y_{n-2}) - h_{n-1}z_n$$

Equations in Matrix Form

$$\begin{bmatrix} b_1 & h_1 & & & & \\ h_1 & b_2 & h_2 & & 0 & \\ & h_2 & b_3 & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & h_{n-3} & b_{n-2} & h_{n-2} \\ & & & & h_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{bmatrix}$$

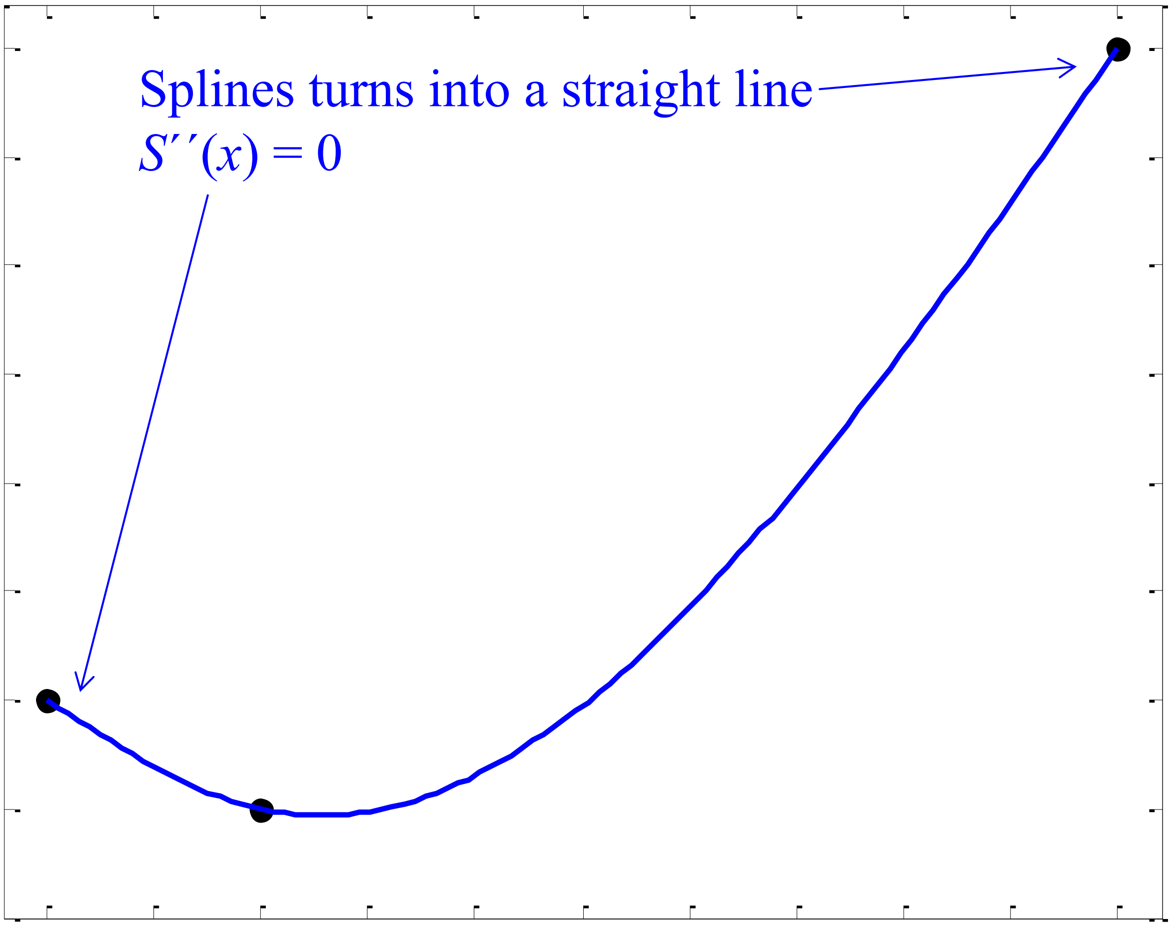
$$b_i \equiv 2\left(h_i + h_{i-1}\right); \quad v_i \equiv \frac{6}{h_i}\left(y_{i+1} - y_i\right); \quad f_i \equiv v_i - v_{i-1}.$$

$$f_1 \equiv v_1 - v_0 - h_0 z_0; \quad f_{n-1} \equiv v_{n-1} - v_{n-2} - h_{n-1} z_n;$$

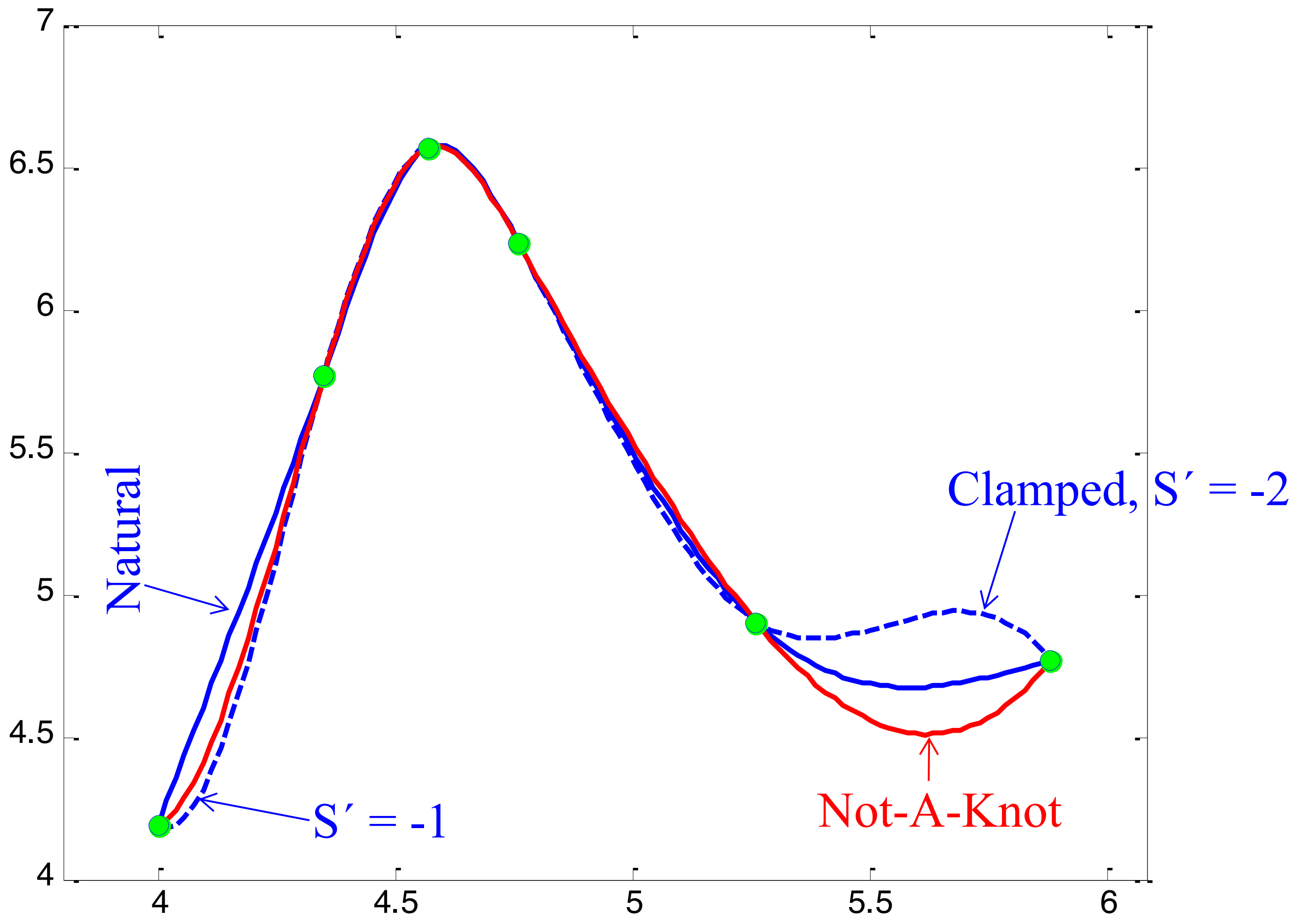
Kinds of Spline

- Natural cubic spline, $z_0 = S''(x_0) = 0 = S''(x_n) = z_n$
- Generalized natural spline, $S''(x_0) = \alpha;\ \ S''(x_n) = \beta$
- Specified third end-point derivative
- Complete cubic spline, $S'(x_0) = \alpha;\ \ S'(x_n) = \beta$ (clamped end conditions)
- Not-a-Knot condition i.e. continuous third derivative at the second and penultimate points (x_1 and x_{n-1}) results in the first two intervals having the same cubic polynomial as well as the last two ...

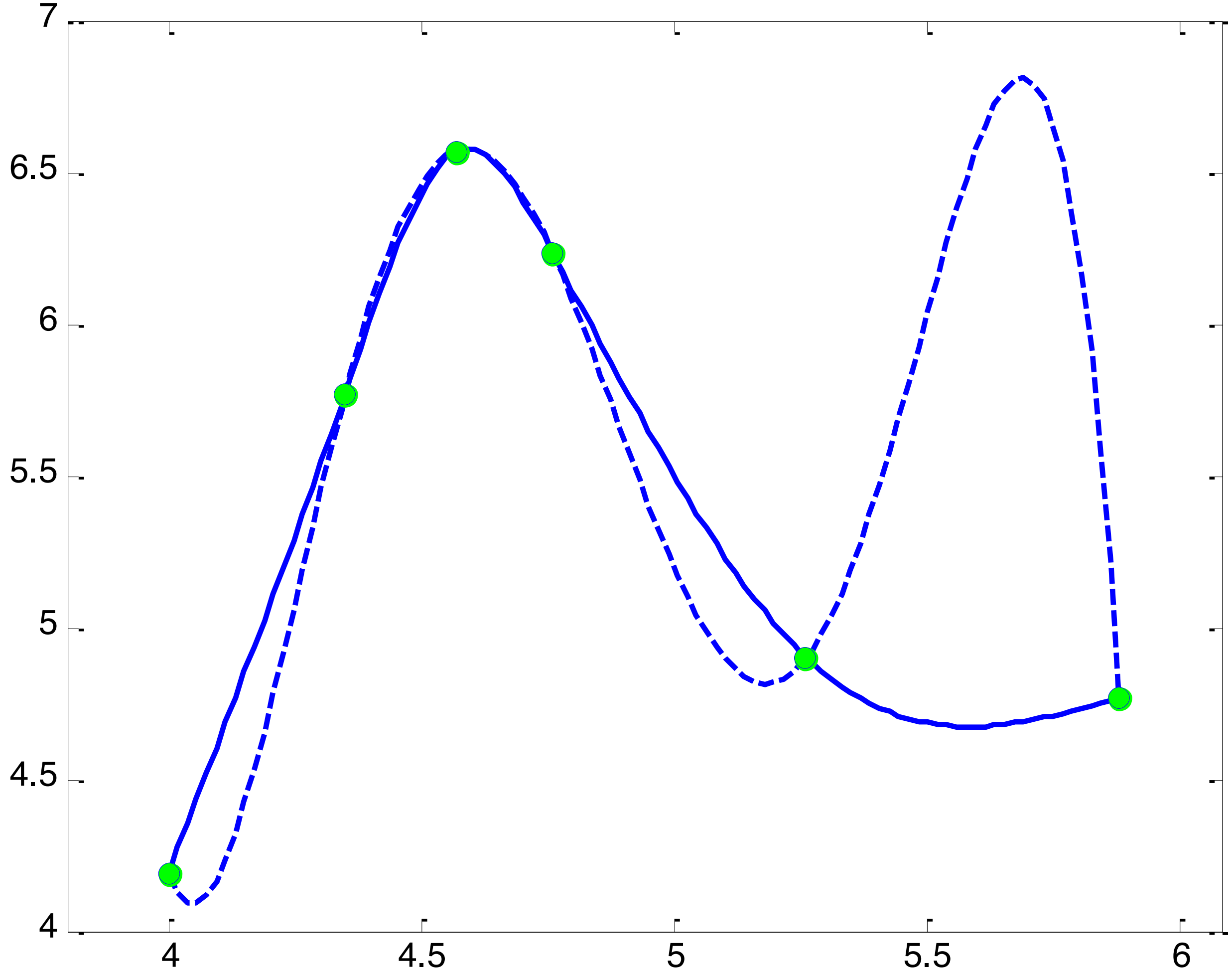
Natural Spline



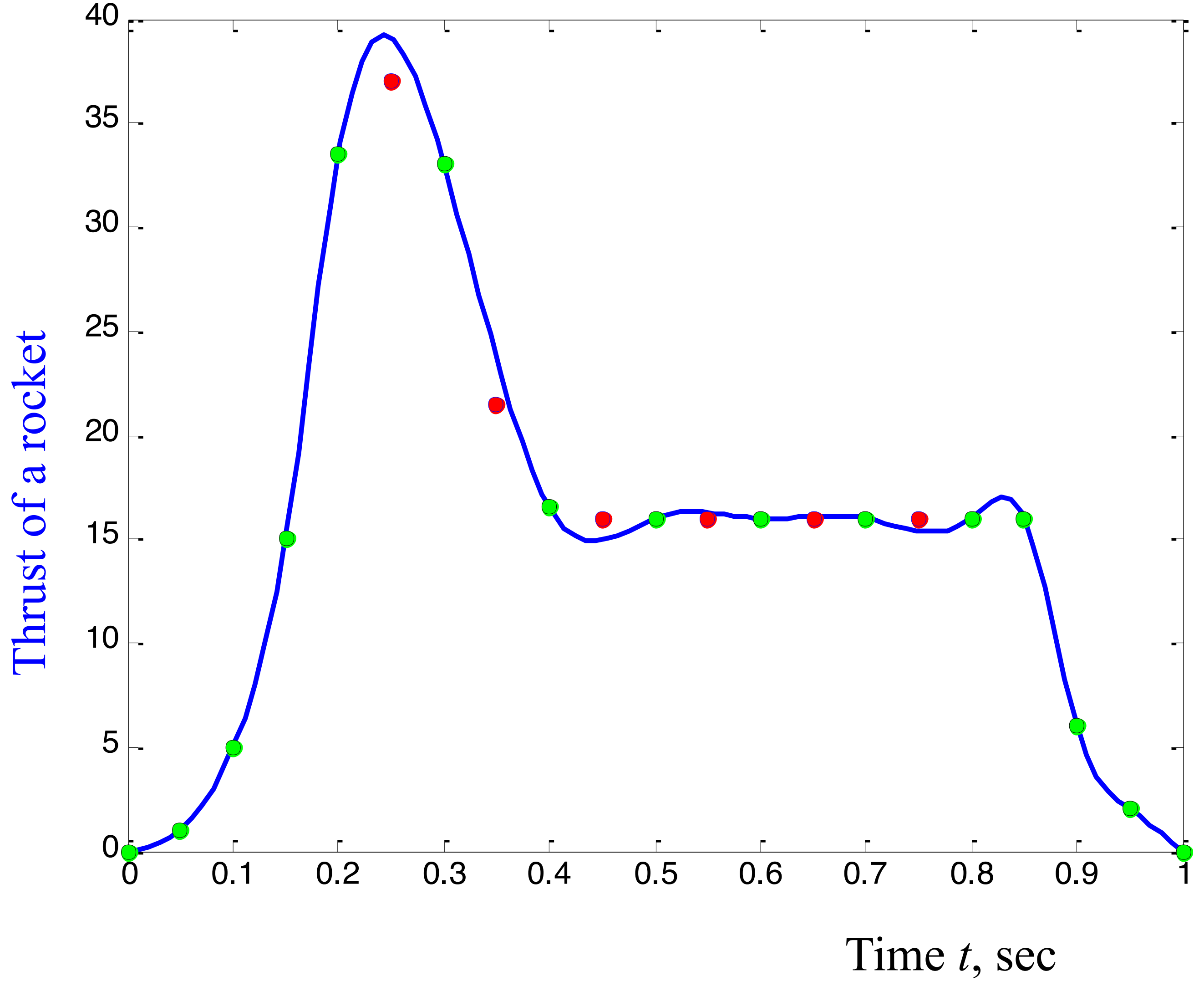
Spline Comparison



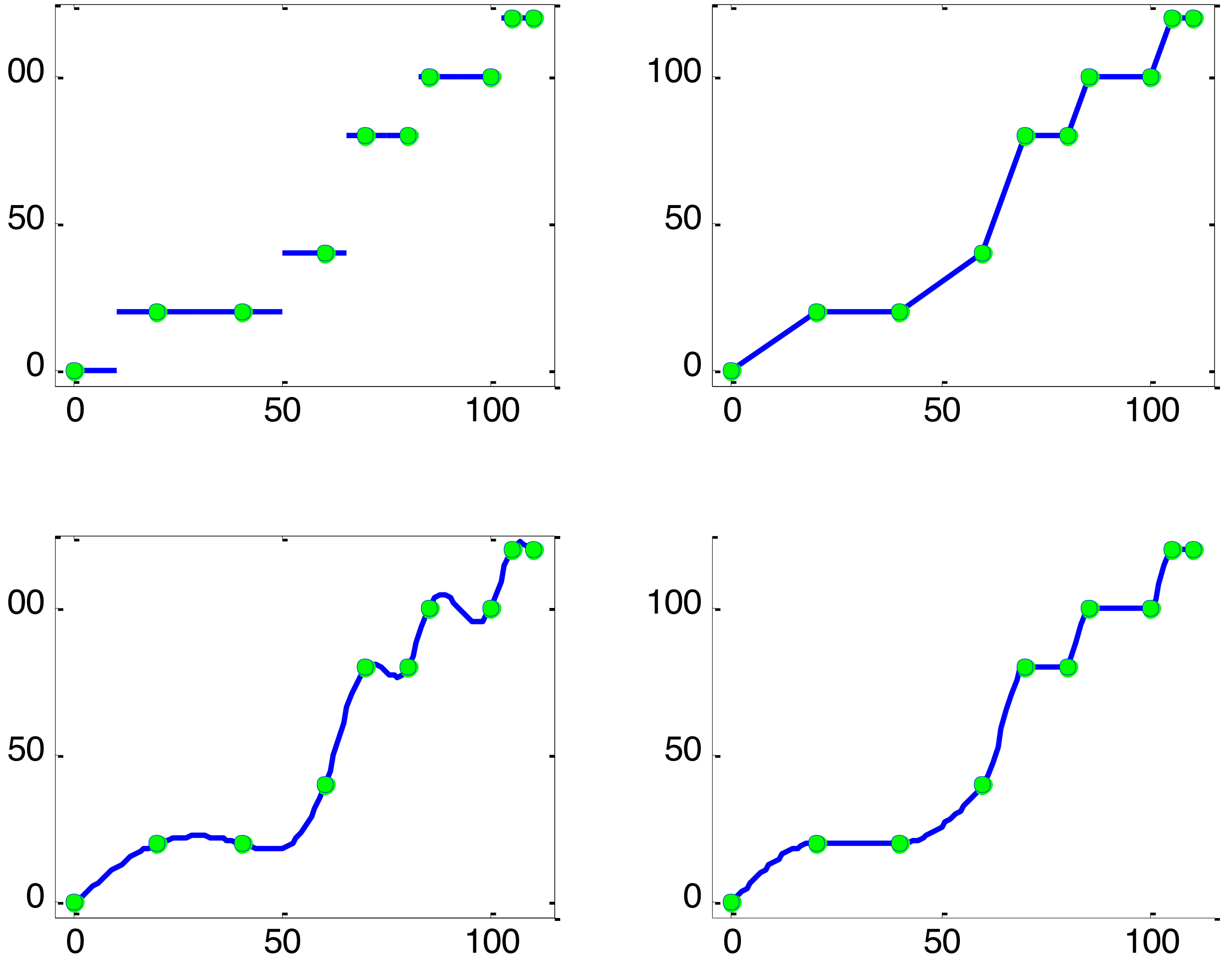
Spline vs. Polynomial



Splines are not Ideal



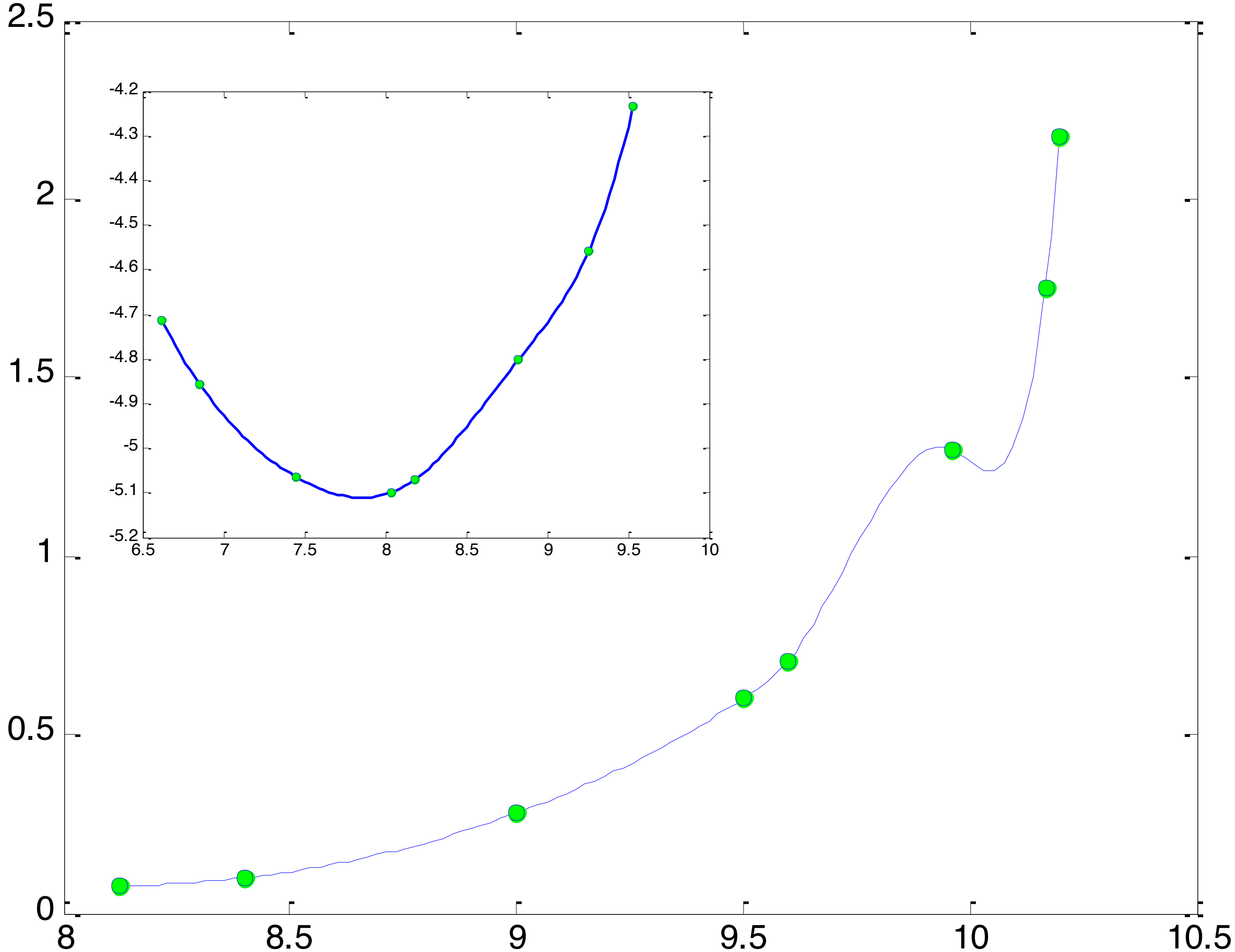
Monotonicity



PCHIP

- Piecewise Cubic Hermite Interpolating Pol.
- On each subinterval, pchip returns a cubic polynomial
- It comes through the interpolation data i.e. the spline function is continuous
- The first derivative is also continuous
- The second derivative may be discontinuous
- Instead, the spline function preserves the shape of the data and respects monotonicity

Geometric Non-Invariance

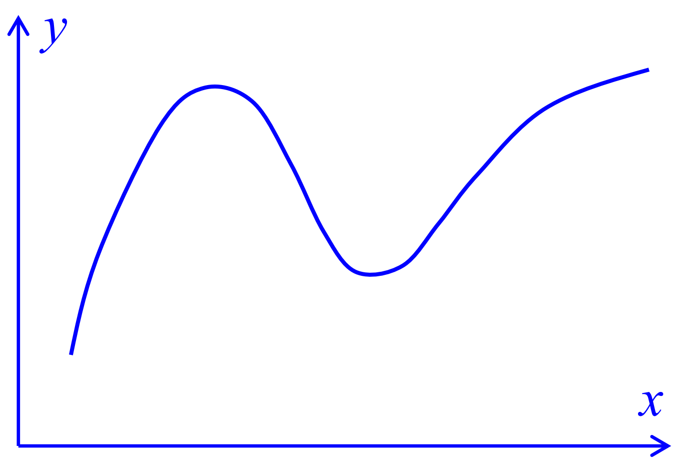


Function Parameterisation

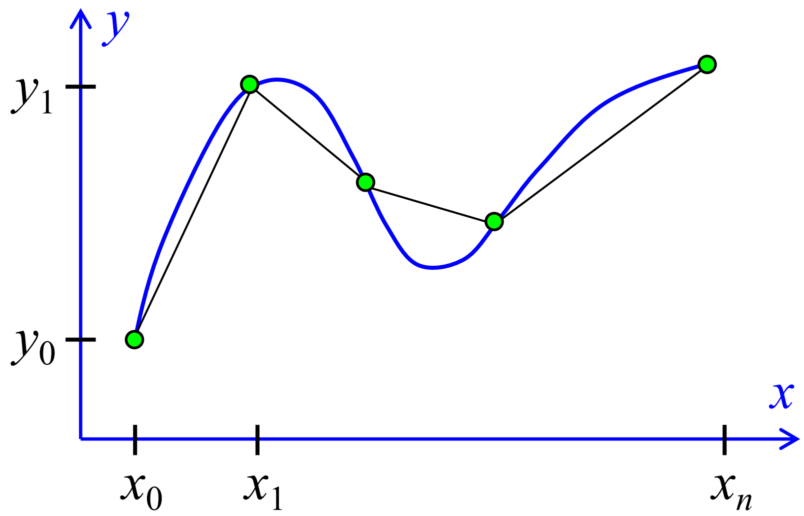
$$y = f(x) \in C^1[a,b] \longrightarrow \begin{cases} x = g(t) \in C^1[\alpha,\beta] \\ y = f(g(t)) \end{cases}$$

$$ds^2 = dx^2 + dy^2 \longrightarrow \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$s(t) = \int_a^t \sqrt{1+\left[y'(x)\right]^2}\,dx$$



Parametric Splines



$$l_k = \sqrt{\left(x_{k+1} - x_k\right)^2 + \left(y_{k+1} - y_k\right)^2}$$

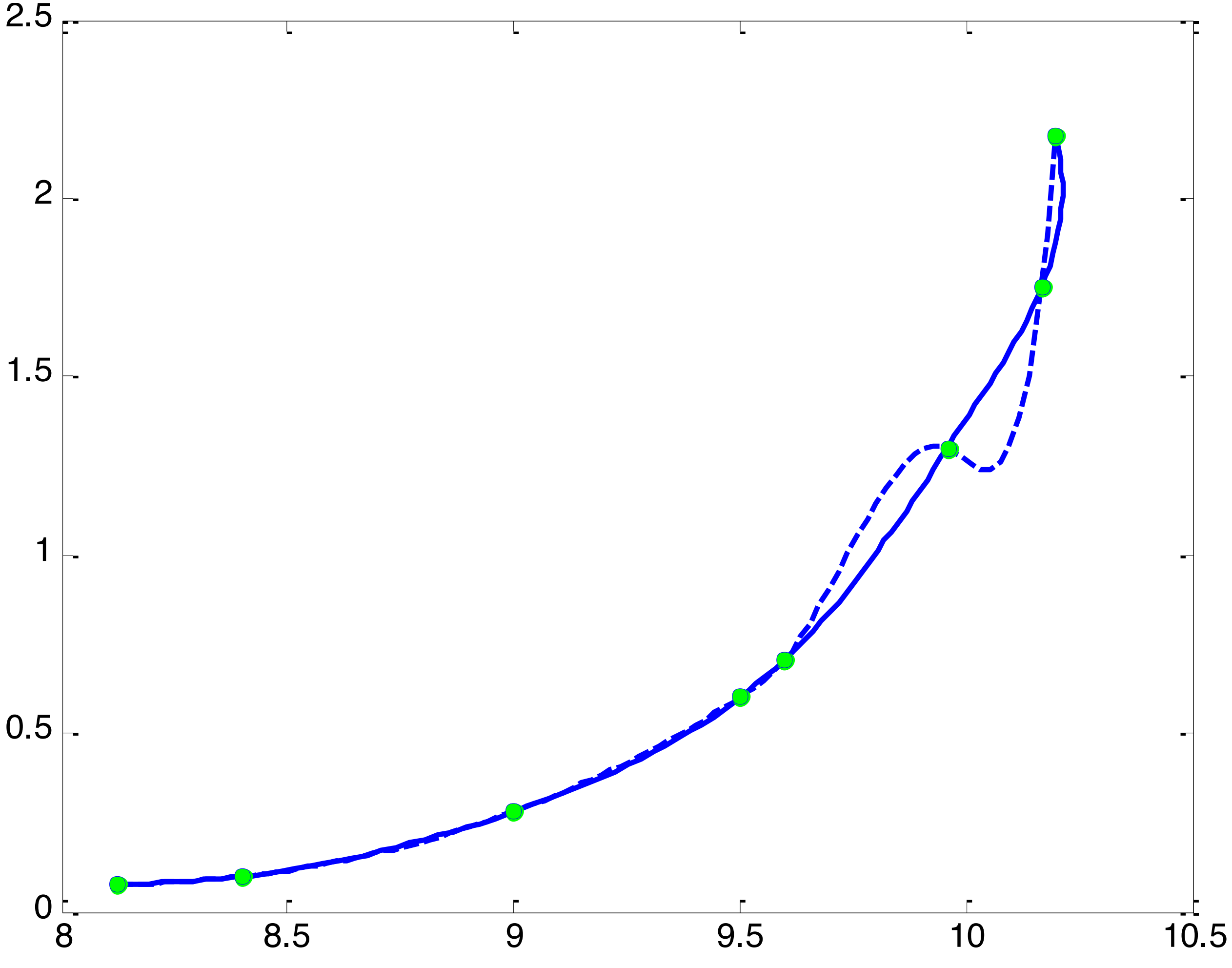
$$s_0 = 0; \quad s_{k+1} = s_k + l_k$$

$$\left(0 \le k \le n-1\right)$$

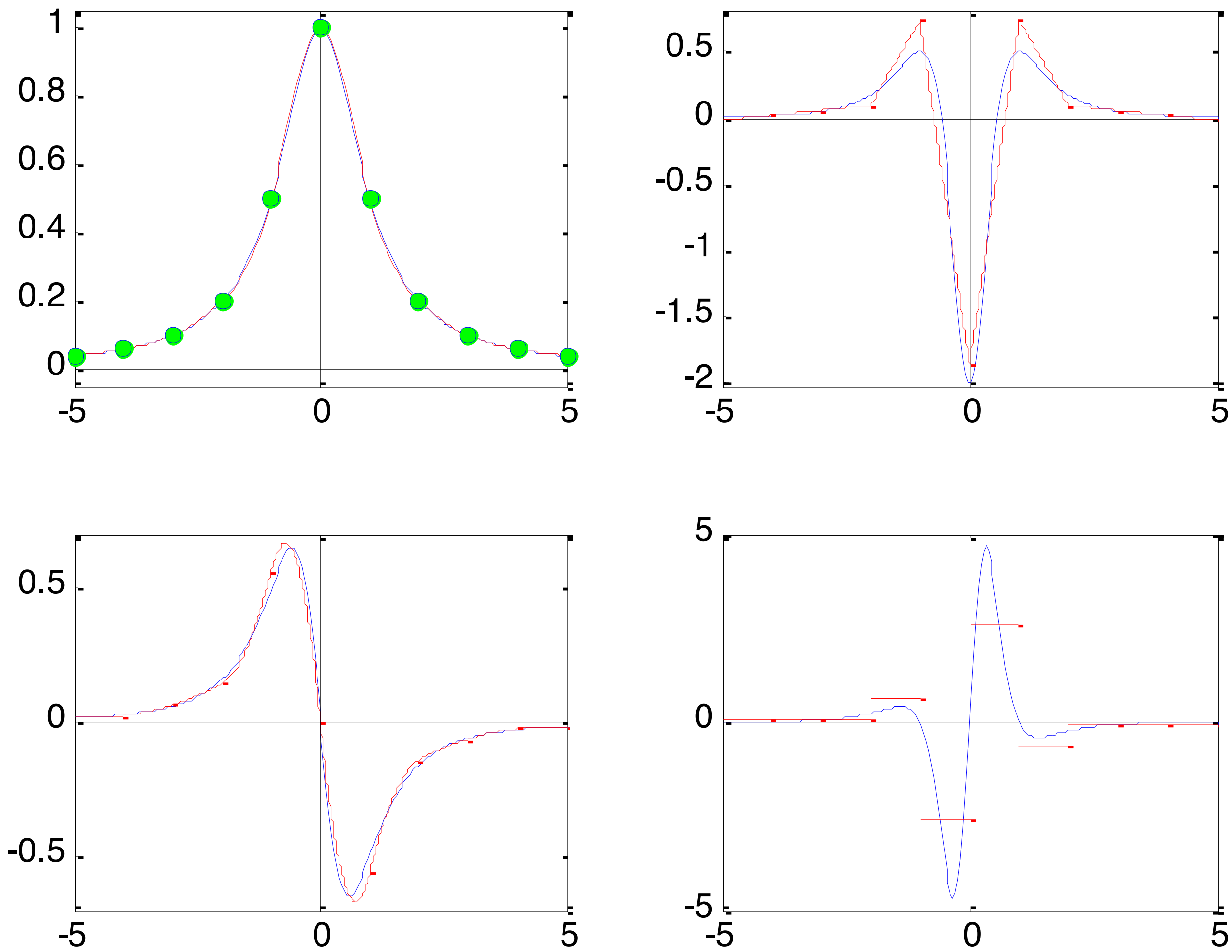
```
Input: xk, yk
x = linspace(xk(1),xk(end),N);
y = spline(xk,yk,x);
plot(x,y);
```

```
Input: xk, yk
sk = zeros(size(xk));
Loop: sk(k+1) = sk(k)+l(k);
s = linspace(0,sk(end),N);
x = spline(sk,xk,s);
y = spline(sk,yk,s);
plot(x,y);
```

Parametric Spline Plot



Derivatives of Spline



Minimum Norm Property

Theorem. Let f'' be continuous on $[a,b]$. If S is the natural cubic spline interpolating f then

$$\int_a^b \left[S''(x)\right]^2 dx \le \int_a^b \left[f''(x)\right]^2 dx \quad \sim \quad \left\|S''\right\|_2 \le \left\|f''\right\|_2$$

Here equality holds iff $f = S$.

Theorem. The curvature of a curve $y = f(x)$ is found as

$$\kappa(x) = \frac{f''(x)}{\left[1 + f'^2(x)\right]^{3/2}}$$

Accuracy

Theorem. Let $f \in C^4[a,b]$

For any partition of $[a,b]$ into subintervals of widths h_i such that $h = \max(h_i)$ and $\beta = \max(h_i)/\min(h_i)$. Let $S(x)$ be the cubic spline interpolating $f(x)$. Then

$$\left\| f - S \right\|_\infty \leq \frac{5}{384} h^4 \left\| f^{(4)} \right\|_\infty$$

$$\left\| f^{(m)} - S^{(m)} \right\|_\infty \leq C_m h^{4-m} \left\| f^{(4)} \right\|_\infty \quad (m = 0,1,2,3)$$

$$C_0 = \frac{5}{384}, \quad C_1 = \frac{1}{24}, \quad C_2 = \frac{3}{8}, \quad C_3 = \frac{\beta + \beta^{-1}}{2}.$$

Important

- Definition of Spline Function
- Cubic Spline Interpolation
- Kinds of Splines
  - Natural
  - Complete (clamped)
  - Not-A-Knot
- Cubic Hermite Spline
- Parametric Splines
- Characterisation of Natural Splines