Ordinary Differential Equations

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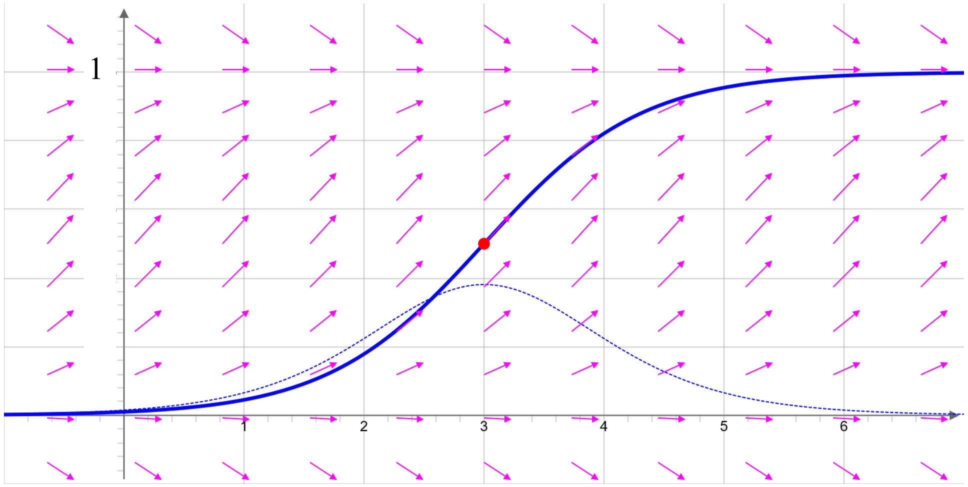
Overview

- ODE Overview
- Examples of ODE
- ODE Classification
- Euler Methods
- Local/Global Error
- Improvements
- Runge-Kutta Methods

Logistic Differential Equation

Isaac Newton, 1624 – 1727: $F = m \times a$ [1687]

Pierre-François Verhulst (1804–1849): $y' = y \times (1 - y)$ [1845]

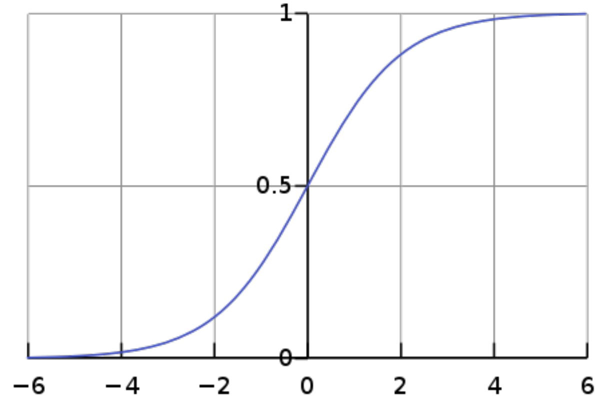


$$\begin{cases} y' = y(1-y) & \text{Equation} \\ t \in [0, T] & \text{Interval} \\ y(0) = y_0 & \text{Initial value} \end{cases}$$

Logistic Function

$$y'(x) = y(x)\left[1 - y(x)\right]$$
$$y(0) = 1/2$$

$$y(x) = \frac{e^x}{e^x + 1}$$



The logistic (sigmoid) function was introduced between 1838 and 1847 when modelling the population growth in Belgium.

The logistic function finds application in artificial neural networks, biomathematics, biology, ecology, chemistry, demography, economics, geoscience, psychology, sociology, linguistics, statistics etc.

Bernoulli Equation

In 1695, Jacob (Jacque, James) Bernoulli, 1605-1705, suggested the equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

The following year, Gottfried Leibnitz solved it.

Moving with friction $a = \dot{v} = -\mu v$ (Ideal case)

$$\dot{v} = -\mu v + \nu v^3$$ (More realistic)

1st Order ODE

$$\frac{dy}{dt} = f\left(t, y\right)$$

Notation: t for independent and y for dependent variable.

$$\begin{cases} y'(t) = f\left(t, y\right) & t \in \left[0, T\right] \\ y(0) = y_0 & \text{IVP (Initial Value Problem)} \\ y(T) = y_f & \text{FVP (Final Value Problem)} \end{cases}$$

Assumption: $f\left(t, y\right) \in \mathrm{C}(D)$

$$f_t'\left(t, y\right), f_y'\left(t, y\right) \in \mathrm{C}(D)$$

Special Case, *f*(*t*,*y*) = *f*(*t*)

$$\begin{cases} y'(t) = f(t) \\ y(0) = y_0 \end{cases} \longrightarrow y(t) = y_0 + \int_0^t f(\tau)d\tau$$

$$y'(t) = \cos(t) \longrightarrow y(t) = y_0 + \sin(t)$$

$$y'(t) = e^{-t^2} \longrightarrow y(t) = y_0 + \int_0^t e^{-\tau^2} d\tau$$

Joseph Liouville:
No solution in terms of elementary functions

Special Case, *f*(*t*,*y*) = *f*(*y*)

$$\begin{cases} \dfrac{dy}{dt} = f(y) \\ y(0) = y_0 \end{cases} \longrightarrow \frac{dy}{f(y)} = dt \longrightarrow t + C = \int \frac{dy}{f(y)}$$

$$\frac{dy}{dt} = -e^{y} \longrightarrow -e^{-y}dy = dt$$

$$e^{-y} = -\int e^{-y}dy = \int dt = t + C$$

$$y = -\ln\left(t + C\right)$$

Simple Example

$$f(t, y) = -5y$$

$$\downarrow$$

$$y'(t) = -5y(t) \quad \leftarrow \quad y(t) = e^{\lambda t}$$

Leonard Euler

$$\lambda e^{\lambda t} = -5e^{\lambda t} \quad \rightarrow \quad \lambda = -5$$

1707 – 1783.

$$y(t) = Ce^{-5t}$$

IVP $\quad y(0) = 2 \quad \rightarrow \quad y(t) = 2e^{-5t}$

Constant Coefficients IVP

$$\begin{cases} y'(t) = ay + b \\ y(t_0) = y_0 \end{cases} \qquad f(t,y) = ay + b$$

(1) $y' = a\left(y + b/a\right)$

(3) $\tilde{y} \equiv y + b/a \rightarrow \tilde{y}' = a\tilde{y} \rightarrow \tilde{y} = Ce^{at}$

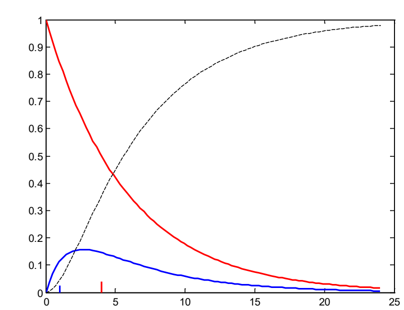
(2) $\left(y + b/a\right)' = a\left(y + b/a\right)$

(4) $\tilde{y} = y + b/a = Ce^{at} \rightarrow y = Ce^{at} - b/a$

General $\quad y(t) = \left(y_0 + b/a\right)e^{a(t-t_0)} - b/a$

Commonly $t_0 = 0$ $\quad y(t) = y_0 e^{at} - b/a\left(1 - e^{at}\right)$

Often $a = -\alpha$ $\quad y(t) = y_0 e^{-\alpha t} + b/\alpha\left(1 - e^{-\alpha t}\right)$



Special Class of ODE

$$\begin{cases} y' = f(t,y) = g(t)y + h(t) & t \in [0,T] \\ y(0) = y_0 \end{cases}$$

(1) $$G(t) = \int g(t)dt$$

(2) $$y(t) = e^{G(t)} \int e^{-G(t)} h(t) dt$$

Example

$$\begin{cases} y' = ty + t^3 \\ y(0) = y_0 \end{cases} \qquad \begin{array}{l} g(t) = t \\ h(t) = t^3 \end{array}$$

(1) $G(t) = \int g(t)dt = t^2/2$

$$t^2/2 \equiv u$$

(2) $\int e^{-G(t)} h(t)dt = \int e^{-t^2/2} t^3 dt = 2\left[-\frac{t^2}{2}e^{-t^2/2} - e^{-t^2/2} + C\right]$

(3) $y(t) = e^{G(t)} \int e^{-G(t)} h(t)dt = -t^2 - 2 + 2Ce^{t^2/2}$

Nonlinear ODE, Example

$$\frac{dy}{dt} = f(t,y) = y^2 \rightarrow \frac{dy}{y^2} = dt$$

$$-\frac{1}{y} = \int \frac{dy}{y^2} = \int dt = t + C$$

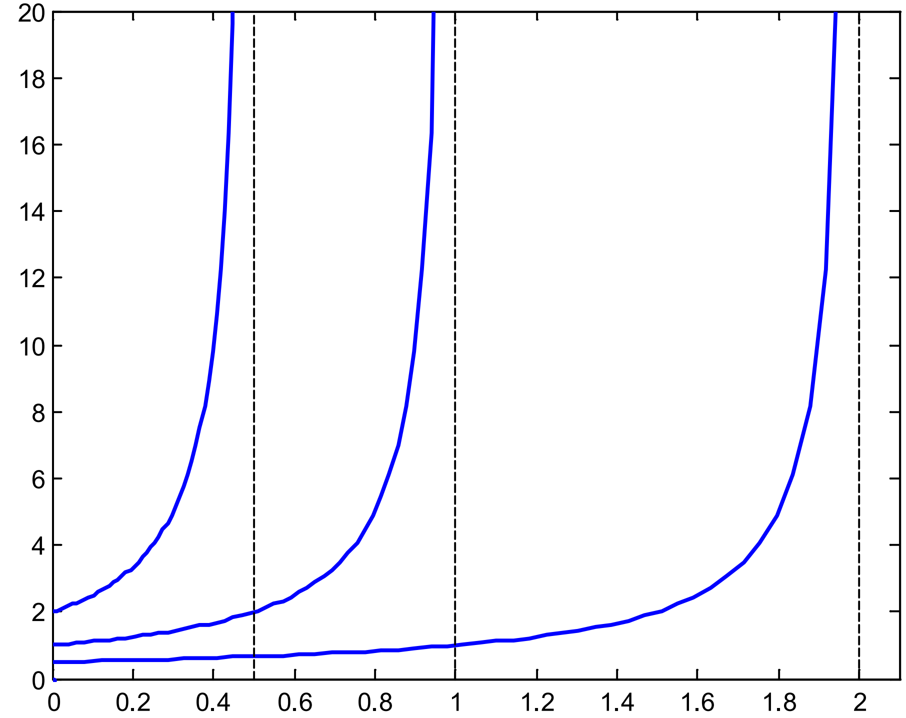
$$y(0) = y_0 \rightarrow y(t) = \frac{y_0}{1 - y_0 t}$$

Local Existence

$$f \in C(D) \longrightarrow \exists^1 y(t) \quad t \in [0, \delta]$$

$$\begin{cases} y'(t) = y^2 & t \in [0, T] \\ y(0) = y_0 \end{cases}$$

$$y(t) = \frac{y_0}{1 - y_0 t} \qquad 0 \le t < \frac{1}{y_0}$$



$y_0 = 2$ $\qquad$ $y_0 = 1$ $\qquad$ $y_0 = 0.5$

Lipschitz Functions

A function $f(t,y)$ is **Lipschitz continuous** in the variable y on the rectangle $(t,y)\in D=[a,b]\times[\alpha,\beta]$ if there exists a constant L (called the Lipschitz constant) satisfying

$$\left|f(t,y_2)-f(t,y_1)\right|\le L\left|y_2-y_1\right|$$

for each (t,y_1) and (t,y_2) in D.

$$f(t,y)=ty+t^3 \qquad 0\le t\le 1 \qquad -\infty<y<\infty$$

$$\left|f(t,y_2)-f(t,y_1)\right|=\left|ty_2-ty_1\right|\le\left|t\right|\times\left|y_2-y_1\right|$$

$$L=1$$

Some Remarks

(a) More generally, rectangle D can be any convex set;

(b) A function $f(t,y)$ that is Lipschitz continuous in y is continuous in y but not necessarily differentiable;

(c) A function $f(t,y)$ that is continuously differentiable in y is Lipschitz continuous.

MVT: $$f(t,y_2)-f(t,y_1)=\frac{\partial f(t,\xi)}{\partial y}(y_2-y_1)$$

$$L=\max_{(t,y)\in D}\left|\frac{\partial f(t,y)}{\partial y}\right|$$

Conditioning

Assume that $f(t,y)$ is Lipschitz in y on $D = [a,b]\times[\alpha,\beta]$.

If $Y(t)$ and $\tilde{Y}(t)$ are solutions in D of the differential equation

$$y' = f(t,y)$$

with initial conditions $Y(a)$ and $\tilde{Y}(a)$ respectively then

$$\left|\tilde{Y}(t) - Y(t)\right| \le e^{L(t-a)} \left|\tilde{Y}(a) - Y(a)\right|$$

$$K_{abs} = e^{L(t-a)}$$

Radioactive Decay

There are several unstable atomic species that decay to more stable forms for example, free neutrons, ^{222}Ra, ^{40}K etc.

If n is the total number of unstable particles/atoms at time t, the radiactivity law states

$$\frac{dn(t)}{dt} = -\lambda n(t) \longrightarrow n(t) = n(0)e^{-\lambda t}$$

Every radioactive process is characterized by decay constant λ.

Decay Chains

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C$$ (stable)

Parent

Daughter
1st progeny

Granddaughter
2nd progeny

Balance equation: $\boxed{\text{(Time change rate)} \;=\; \text{(Production rate)} - \text{(Destruction rate)}}$

System of equations

$$\begin{cases} \dfrac{dn_A(t)}{dt} = -\lambda_A n_A(t) \\ \dfrac{dn_B(t)}{dt} = \lambda_A n_A(t) - \lambda_B n_B(t) \\ \dfrac{dn_C(t)}{dt} = \lambda_B n_B(t) \end{cases}$$

System of ODEs

$$\mathbf{n} = \begin{bmatrix} n_A \\ n_B \\ n_C \end{bmatrix}$$

$$\begin{cases} \dfrac{dn_A(t)}{dt} = -\lambda_A n_A(t) \\ \dfrac{dn_B(t)}{dt} = \lambda_A n_A(t) - \lambda_B n_B(t) \\ \dfrac{dn_C(t)}{dt} = \lambda_B n_B(t) \end{cases}$$

$$\mathbf{F}(\mathbf{n}) = \begin{bmatrix} -\lambda_A n_A \\ \lambda_A n_A - \lambda_B n_B \\ \lambda_B n_B \end{bmatrix}$$

$$\frac{d\mathbf{n}}{dt} = \mathbf{F}(\mathbf{n})$$

System of Linear ODEs

$$\mathbf{n} = \begin{bmatrix} n_A \\ n_B \\ n_C \end{bmatrix} \qquad \mathbf{F}(\mathbf{n}) = \begin{bmatrix} -\lambda_A n_A(t) \\ \lambda_A n_A(t) - \lambda_B n_B(t) \\ \lambda_B n_B(t) \end{bmatrix} = -\underbrace{\begin{bmatrix} \lambda_A & 0 & 0 \\ -\lambda_A & \lambda_B & 0 \\ 0 & -\lambda_B & 0 \end{bmatrix}}_{\mathbf{A}} \cdot \begin{bmatrix} n_A \\ n_B \\ n_C \end{bmatrix}$$

Special but common $\quad \dfrac{d\mathbf{n}}{dt} = -\mathbf{A}\mathbf{n} \qquad \mathbf{n}(0) = \mathbf{n}_0 \in \mathrm{R}^n \qquad \mathbf{A} \in \mathrm{R}^{n\times n}$

Decay Chain Solution

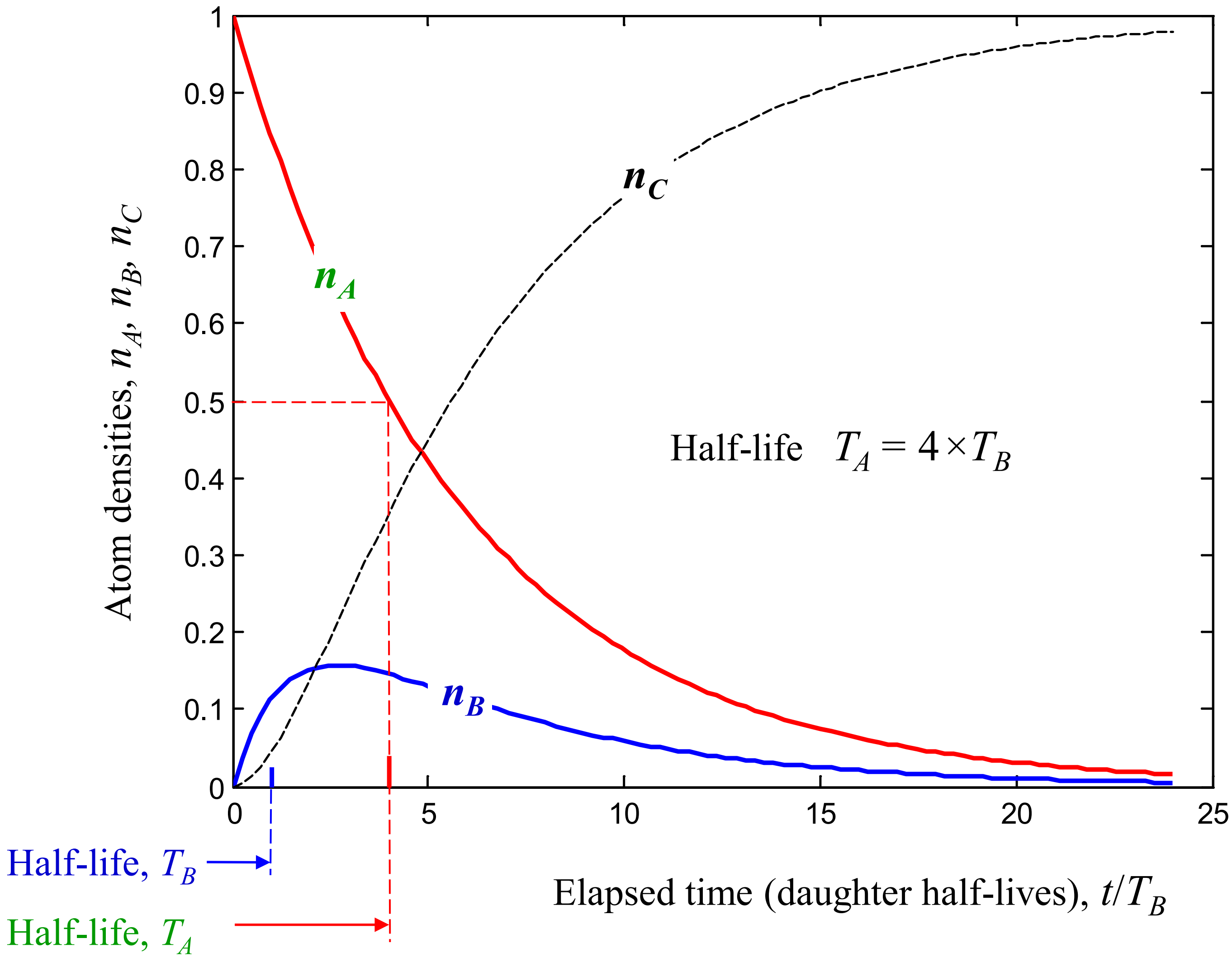
$$n_A(t) = n_A(0)e^{-\lambda_A t}$$

$$n_B(t) = n_B(0)e^{-\lambda_B t} + \frac{\lambda_A}{\lambda_B - \lambda_A} n_A(0)\left[e^{-\lambda_A t} - e^{-\lambda_B t}\right]$$

$$n_C(t) = n_C(0) + n_B(0)\left(1 - e^{-\lambda_B t}\right) + n_A(0)\left[1 - \frac{\lambda_B e^{-\lambda_A t} - \lambda_A e^{-\lambda_B t}}{\lambda_B - \lambda_A}\right]$$

Balance $n_A(t) + n_B(t) + n_C(t) = n_A(0) + n_B(0) + n_C(0)$

Decay Chain Plot



Solution to Linear ODEs

$$\frac{d\mathbf{y}}{dt} = -\mathbf{A}\mathbf{y} \qquad \mathbf{y}(0) = \mathbf{y}_0 \in \mathrm{R}^n \qquad \mathbf{A} \in \mathrm{R}^{n\times n}$$

Solution is given in terms of eigenvalues, λ_j , and eigenvectors, $\mathbf{y}_j$, which is not the subject of this course.

Eigenpairs, $j = 1, \ldots, n$ $\quad \mathbf{A}\mathbf{y}_j = \lambda_j \mathbf{y}_j \qquad \mathbf{y}(t) = \sum_{j=1}^{n} c_j e^{-\lambda_j t} \mathbf{y}_j$

Matrix Exponential

$$\frac{dy}{dt} = -ay + b \qquad y(t) = e^{-at} y_0 + b/a\left(1 - e^{-at}\right)$$

$$y(t_0) = y_0 \qquad y(t) \xrightarrow[t\to\infty]{} y_{\max} = b/a$$

$$\frac{d\mathbf{y}}{dt} = -\mathbf{A}\mathbf{y} + \mathbf{b} \qquad \mathbf{y}(t) = e^{-\mathbf{A}t}\mathbf{y}_0 + \mathbf{A}^{-1}\left(\mathbf{I} - e^{-\mathbf{A}t}\right)\mathbf{b}$$

$$\mathbf{y}(t_0) = \mathbf{y}_0 \qquad e^{\mathbf{A}} \equiv \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \ldots + \frac{1}{n!}\mathbf{A}^n + \ldots$$

Matlab: expm(A)

Lotka-Voltera Equations

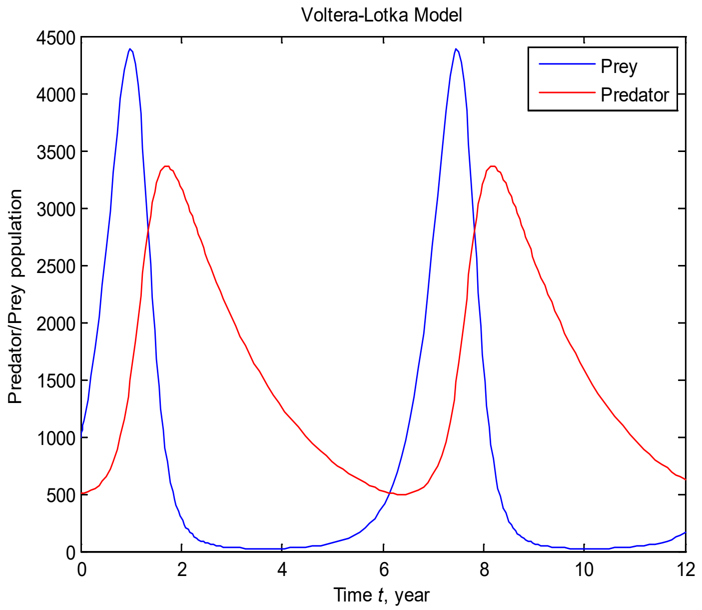
(Predator-Prey equations)

A pair of first-order nonlinear differential equations.

Let x be the number of prey (rabbits).

Let y be the number of predators (foxes).

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \lambda xy - \gamma y \end{cases} \qquad \begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$$



Lotka-Voltera in Vector Form

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \lambda xy - \gamma y \end{cases}$$

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{F}(\mathbf{u}) = \mathbf{F}(x, y) = \begin{bmatrix} \alpha x - \beta xy \\ \lambda xy - \gamma y \end{bmatrix}$$

$$\frac{d\mathbf{u}}{dt} = \mathbf{u}' = \mathbf{F}(\mathbf{u}) \qquad \mathbf{u}(0) = \mathbf{u}_0$$

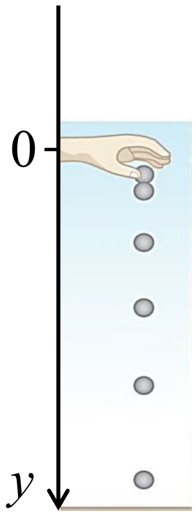
System of ODEs

$$\begin{cases} y_1' = f_1\left(t, y_1, y_2\right) \\ y_2' = f_2\left(t, y_1, y_2\right) \end{cases}$$

$$\mathbf{y} \equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \mathbf{F} \equiv \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad \begin{cases} \mathbf{y}' = \mathbf{F}\left(t, \mathbf{y}\right) \\ \mathbf{y}(0) = \mathbf{y}_0 \end{cases}$$

Second Order ODEs

Falling with friction



Newton's second law

$$ma = F_g + F_{drag}$$

$$F_g = mg$$

$$F_{drag} = -av - bv^2$$

$$y'' = g - \alpha y' - \beta (y')^2$$

$$\alpha = a/m \quad \beta = b/m$$

Derivative interpretation

$y'(t) = v$ (speed)

$y''(t) = a$ (acceleration)

Converting to First Order

$$y'' = g - \alpha y' - \beta \left(y' \right)^2$$

$$u_0 \equiv y \qquad u_0' = y' = u_1$$

$$u_1 \equiv y' \qquad u_1' = y'' = g - \alpha u_1 - \beta u_1^2$$

$$\mathbf{u} \equiv \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \qquad \mathbf{F}\left(\mathbf{u}\right) \equiv \begin{bmatrix} u_1 \\ g - \alpha u_1 - \beta u_1^2 \end{bmatrix}$$

$$\mathbf{u}' = \mathbf{F}\left(\mathbf{u}\right)$$

ODEs with Higher Derivatives

General ODE of order n

Requires n Initial Conditions (ICs)

$$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right)$$

$$\begin{aligned} u_0 &\equiv y & u_0' &= u_1 \\ u_1 &\equiv y^{(1)} & u_1' &= u_2 \\ u_2 &\equiv y^{(2)} & u_2' &= u_3 \\ &\vdots & &\vdots \\ u_{n-1} &\equiv y^{(n-1)} & u_{n-1}' &= y^{(n)} = f \end{aligned}$$

$$\begin{aligned} y(0) &= y_0 \\ y'(0) &= y_1 \\ &\vdots \\ y^{(n-1)} &= y_{n-1} \end{aligned}$$

$$\mathbf{u}' = \mathbf{F}(t, \mathbf{u}) \equiv \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ f \end{bmatrix}$$

Constant Coefficients, Hom

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y(t) = 0$$ (Homogeneous)

Try $y = e^{\lambda t}$ $\quad y^{(k)} = \lambda^k e^{\lambda t}$ (Leonard Euler, 1707 – 1783)

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0$$ (Characteristic equation)

Fundamental theorem of algebra: $\exists \lambda_i \quad i = 1, 2, \ldots, n$

Assuming $\lambda_i \neq \lambda_j \quad y_i(t) = e^{\lambda_i t}$ (Totally, *n* distinct solutions)

Multiple root, λ $\quad y(t) = t^k e^{\lambda t}$

General $\quad y_{\text{hom}}(t) = c_1 y_1(t) + \ldots + c_n y_n(t)$

Constant Coefficients, Inhom

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y(t) = g(t)$$ (Inhomogeneous)

Step 1 Find partial solution $y_{\text{part}}(t)$

Step 2 Find homogeneous solution $y_{\text{hom}}(t)$

Step 3 General solution $y(t) = y_{\text{hom}}(t) + y_{\text{part}}(t)$

ODE Classification

1) First order
2) High order
3) One equation
4) System of equations
5) IVP (FVP)
6) BVP

1) Propagation
2) Equilibrium
3) Eigenvalue

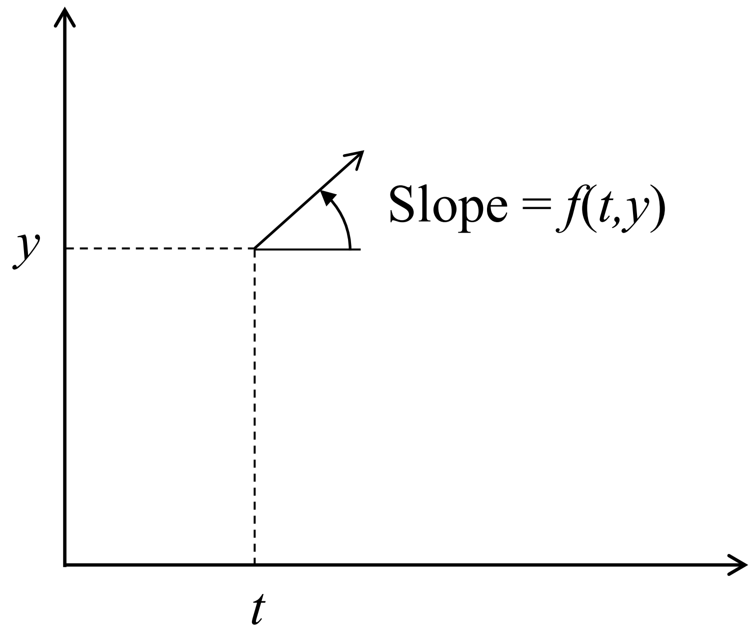
$$\begin{cases} y''(x) = \lambda y(x) \\ y(0) = y(a) = 0 \end{cases}$$

Geometric Interpretation

$$\begin{cases} y' = f\left(t, y\right) & t \in \left[0, T\right] \\ y(0) = y_0 \end{cases}$$

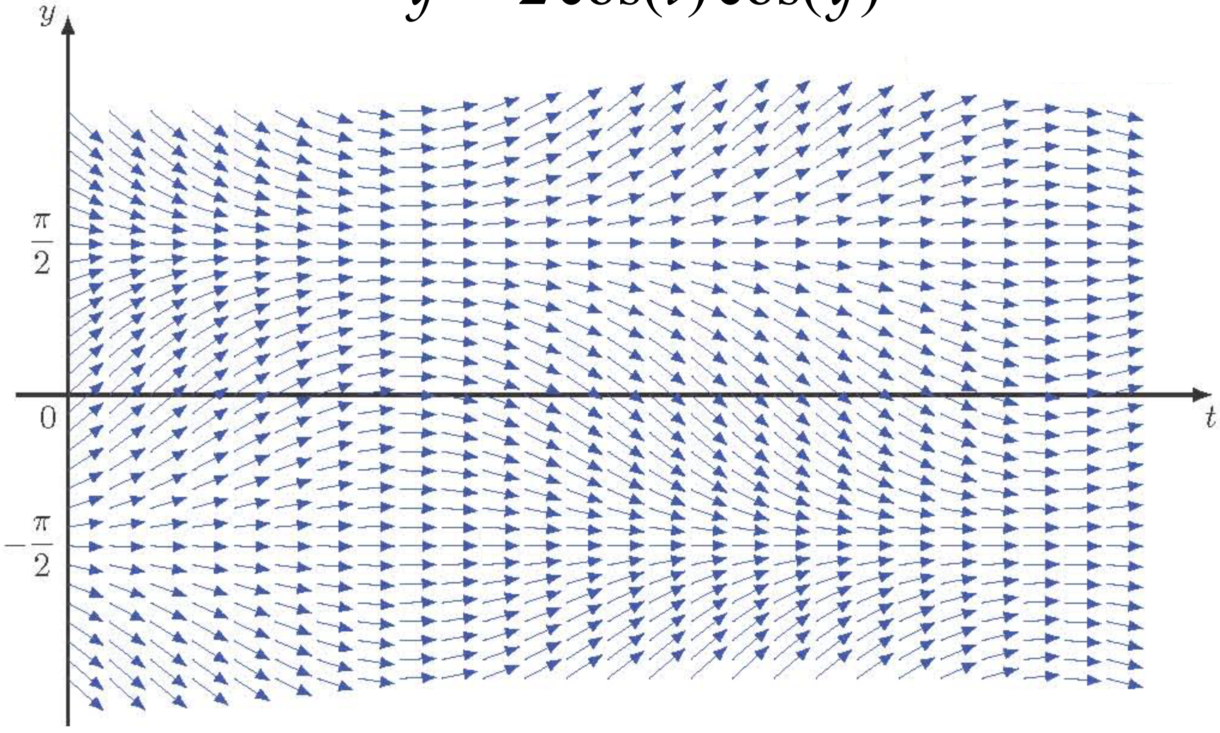
Matlab:

- `quiver(x,y,...)`
- `streamline(X,Y,...)`

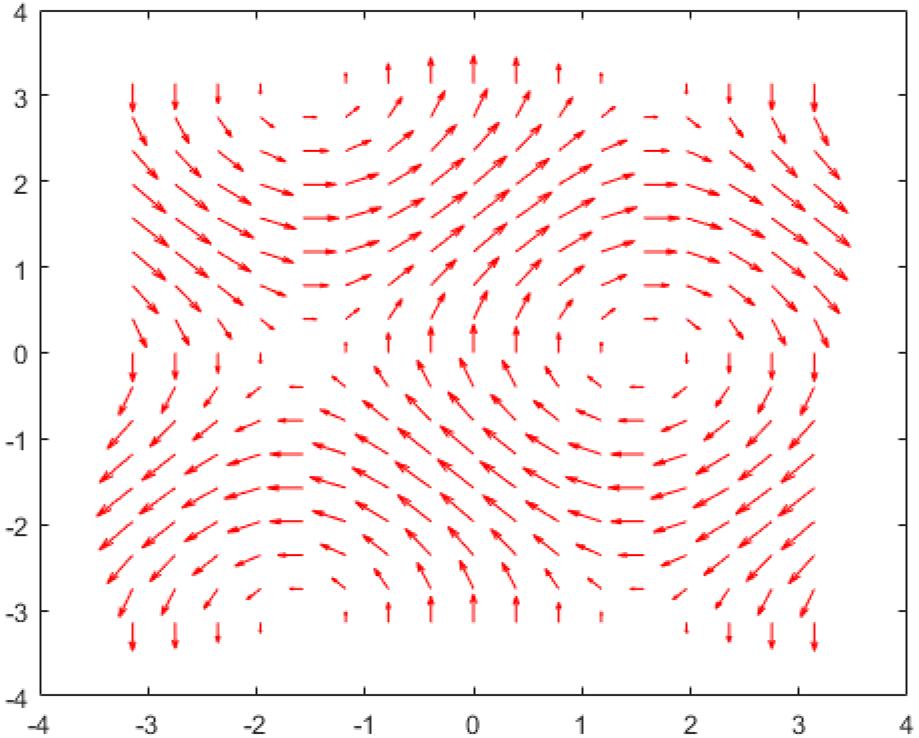


Slope Field

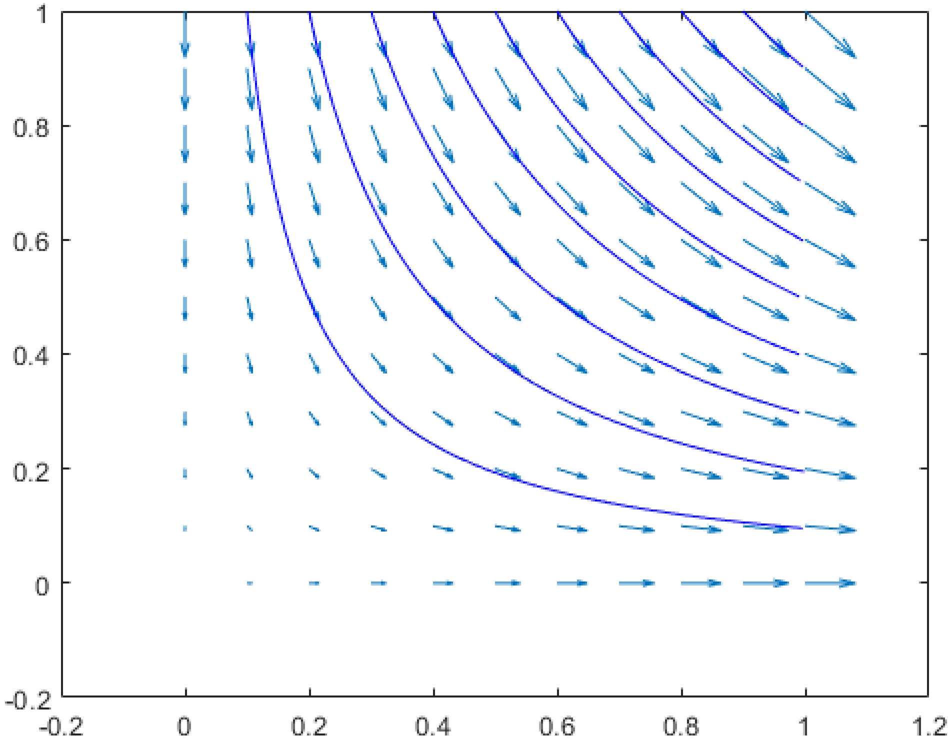
$$y' = 2\cos(t)\cos(y)$$



Arrow Plot



Streamlines



Discretization

$$\begin{cases} y' = f(t, y) & t \in [0, T] \\ y(0) = y_0 \end{cases}$$

$$0 = t_0 < t_1 < \ldots < t_N = T \qquad h_n \equiv t_{n+1} - t_n$$

$$h = \frac{T}{N} \qquad t_n = n \cdot h \qquad n = 0, 1, \ldots, N$$

Exact $y(t_n)$

Approximate $y_n \approx y(t_n)$

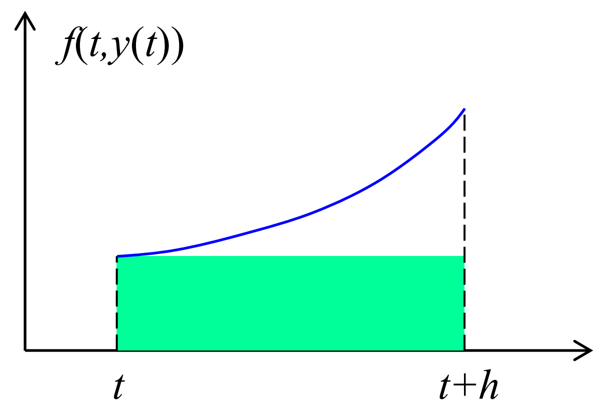
Explicit Euler Method, EEM

$$y' = f(t, y)$$

$$\frac{y(t+h) - y(t)}{h} \approx f(t, y(t)) \longrightarrow y(t+h) \approx y(t) + h \cdot f(t, y(t))$$

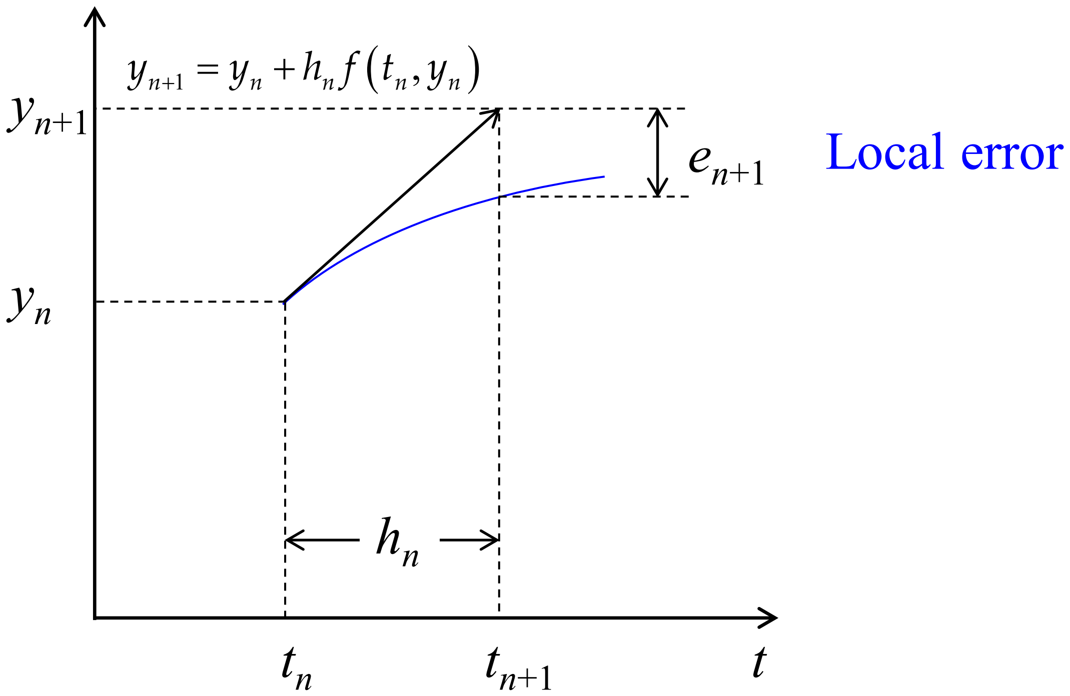
$$\frac{y_{n+1} - y_n}{h_n} = f(t_n, y_n) \longrightarrow y_{n+1} = y_n + h_n f(t_n, y_n)$$

$$y(t+h) - y(t) = \int_t^{t+h} f(\tau, y(\tau))\, d\tau \approx h \cdot f(t, y(t))$$



Left Riemann

One Step in EEM



Local Truncation Error

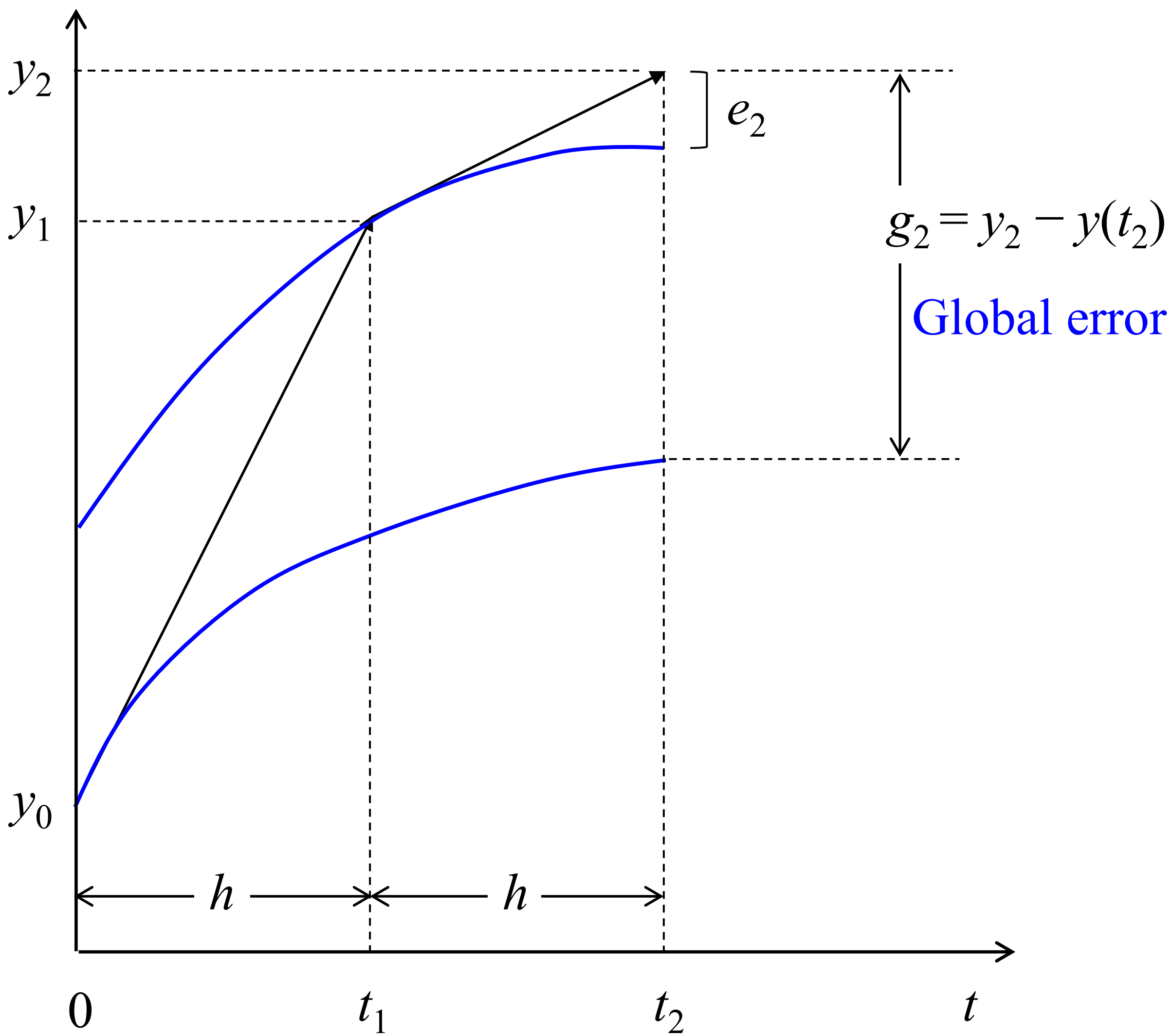
LTE is the error introduced by one step.

1) Assuming $y_n = y(t_n)$ i.e. exact

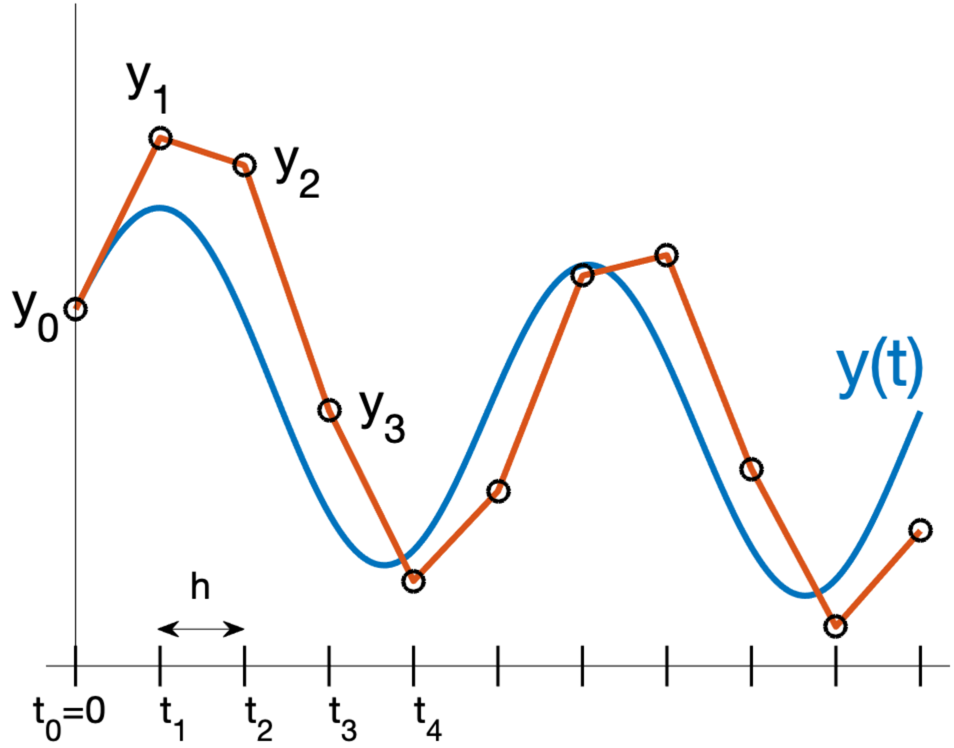
2) Calculating $e_{n+1} = y(t_{n+1}) - y_{n+1}$

Order of approximation is p if $e_{n+1} = O(h^{p+1})$

Two Steps in EEM



Many Steps in EEM

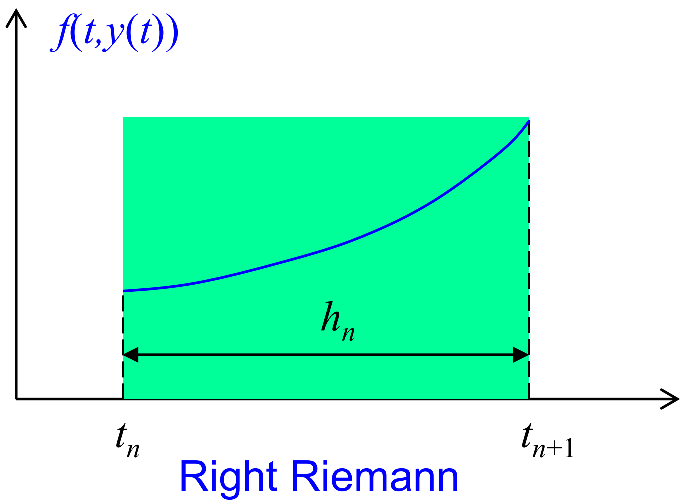


Global Error

$$t \in [0,T] \quad h = T/N \quad t_n = n \cdot h \quad n = 0,1,2,\ldots,N$$

- We define the global error at step n as $g_n \equiv y(t_n) - y_n$
- Numerical scheme (algorithm) is convergent if $g_n \to 0$
- We define order of approximation, p, as $\max\limits_{0 \le n \le N} \left| y(t_n) - y_n \right| \le C h^p$
- Typically, we may write $g_n = y(t_n) - y_n \approx C(t_n) h^p$

Implicit Euler Method, IEM



$$y' = f(t, y)$$

$$\frac{y_{n+1} - y_n}{h_n} = f(t_{n+1}, y_{n+1}) \longrightarrow y_{n+1} = y_n + h_n f(t_{n+1}, y_{n+1})$$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\, d\tau \approx h_n \cdot f(t_{n+1}, y(t_{n+1}))$$

Local/Global Error

$$y(t_n + h) = y(t_n) + y'(t_n)h + \frac{y''(\xi)}{2!}h^2$$

$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$

$$y'(t_n) = f(t_n, y_n)$$

$$e_{n+1} \equiv y(t_{n+1}) - y_{n+1} = \frac{y''(\xi)}{2!}h^2$$

Both Explicit and Implicit EM has

- Local order of approximation $p = 1$
- Global order of approximation $p = 1$

Toy Problem

$$y' = -y \qquad y(0) = 1 \longrightarrow y(t) = e^{-t}$$

$$y_{n+1} = y_n + h \cdot f\left(t_n, y_n\right) = y_n - h \cdot y_n = \left(1 - h\right) y_n$$

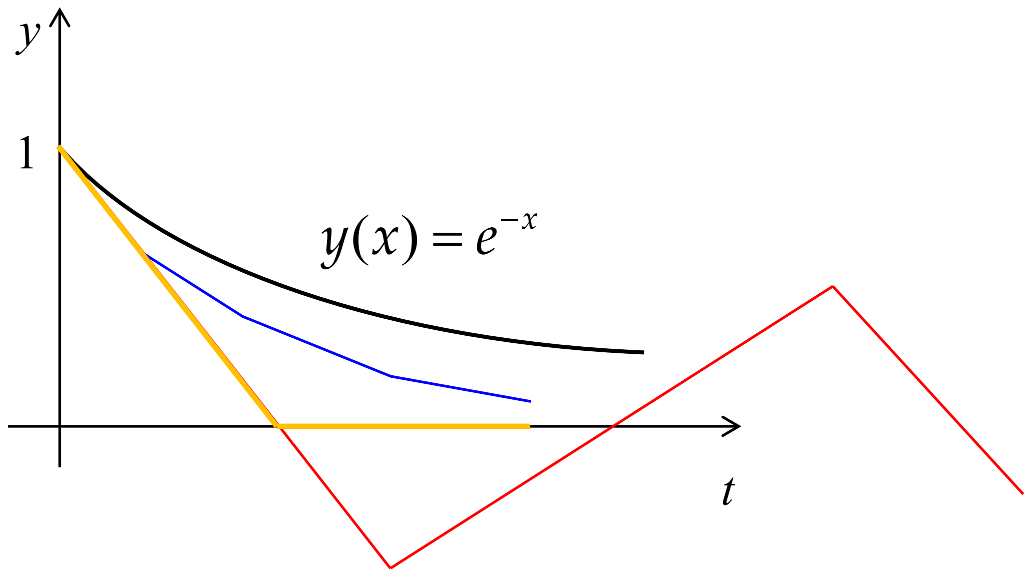
$$y_n = \left(1 - h\right)^n \cdot 1$$

$$y\left(t_n\right) = e^{-h \cdot n} = \left(e^{-h}\right)^n \approx \left(1 - h\right)^n$$

EEM Behaviour

$$y_n = (1-h)^n$$

a)	$h < 1$	$\rightarrow$ monotonic sequence y_n
b)	$h = 1$	$\rightarrow y_n = 0$
c)	$1 < h < 2$	$\rightarrow$ oscilates and approaches assymptotic solution
d)	$h = 2$	$\rightarrow y_n$ oscilates about assymptotic solution
e)	$h > 2$	$\rightarrow y_n$ diverges



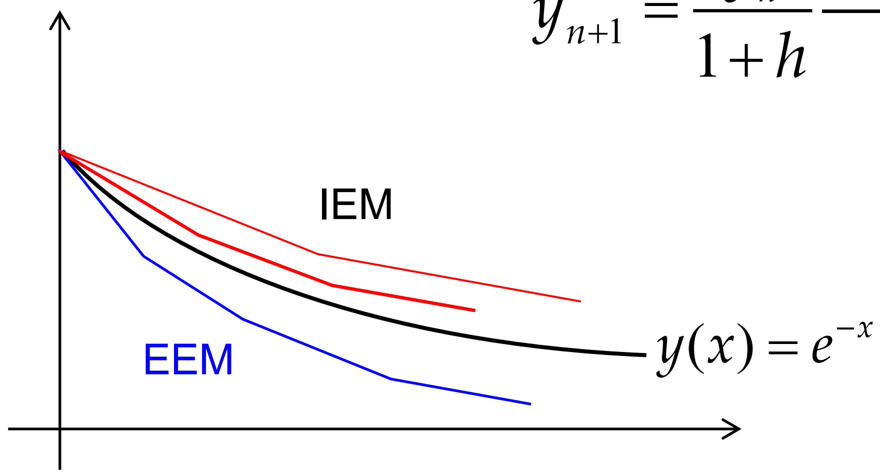
IEM Behaviour

$y' = -y \qquad y(0) = 1$

$$y_{n+1} = y_n + h \cdot f\left(t_{n+1}, y_{n+1}\right) = y_n - h \cdot y_{n+1}$$

$$y_{n+1} = \frac{y_n}{1+h} \longrightarrow y_n = \left(1+h\right)^{-n}$$ IEM

$$y_n = \left(1-h\right)^n$$ EEM

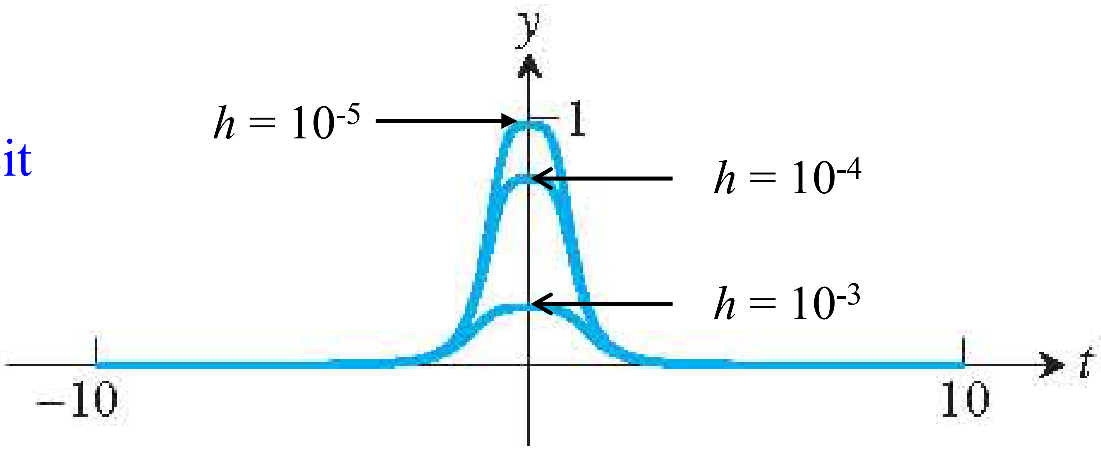


Innocent-Looking Example

$$\begin{cases} y' = -4t^3y^2 & -10 \le t \le 0 \\ y(-10) = 1/10001 \end{cases}$$

$$y(t) = \frac{1}{t^4 + 1}$$

Explicit/Implicit
Euler Method



Improving EM by Taylor

$$y' = f(t, y)$$

More terms

$$y(t+h) = y(t) + y'(t)h + y''(t)\frac{h^2}{2!} + \dots$$

$$y_{n+1} = y_n + f(t_n, y_n)h + \left.\frac{df}{dt}\right|_{t=t_n} \frac{h^2}{2!} + \dots$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \partial_t f + \partial_y f \cdot f$$

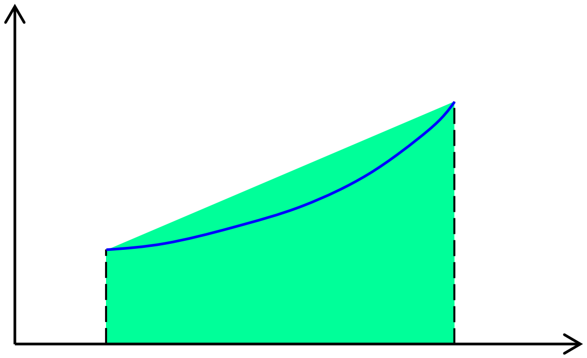
Improving EM

One-step $$y_{n+1} = \Phi\left(t_n, y_n\right)$$

Multi-step $$y_{n+1} = \Phi\left(t_n, y_n, t_{n-1}, y_{n-1}, \ldots, t_{n-k+1}, y_{n-k+1}\right)$$

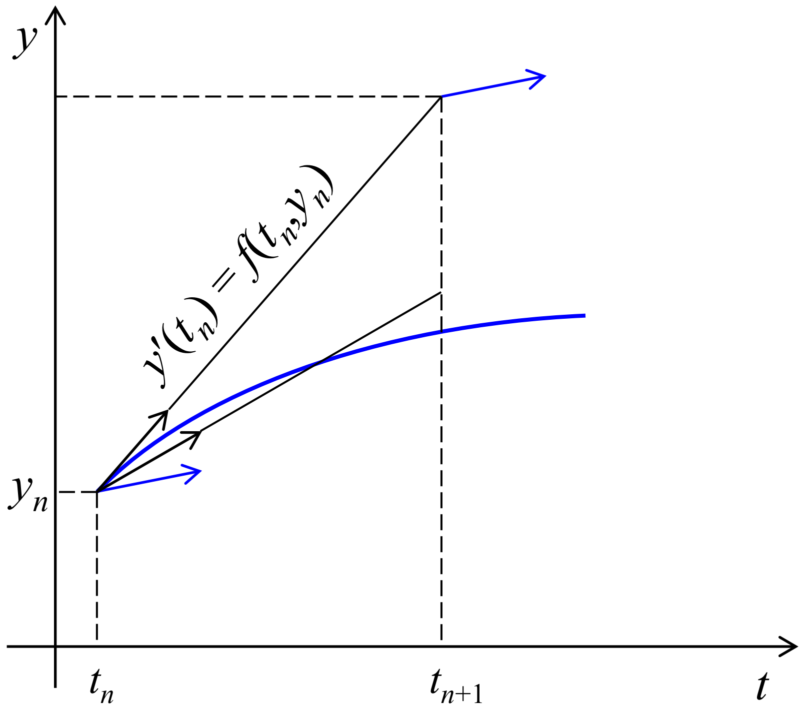
Quadrature $$y\left(t+h\right) = y\left(t\right) + \int_t^{t+h} f\left(\tau, y(\tau)\right) d\tau$$

Trapezoid $$y_{n+1} = y_n + \frac{f\left(t_n, y_n\right) + f\left(t_{n+1}, y_{n+1}\right)}{2}$$



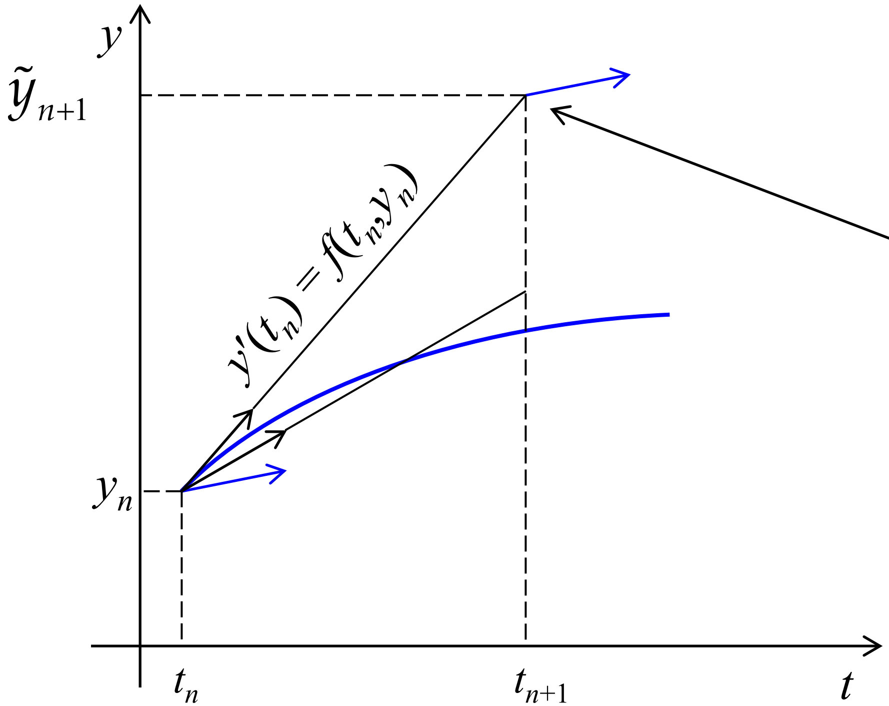
Predictor-Corrector

$$y' = f(t, y)$$



Explicit Trapezoid

$$y' = f\left(t, y\right)$$



$$y'_n = f\left(t_n, y_n\right)$$

$$\tilde{y}_{n+1} = y_n + f\left(t_n, y_n\right)h$$

$$y'_{n+1} = f\left(t_{n+1}, \tilde{y}_{n+1}\right)$$

$$\overline{y}'_n = \frac{y'_n + y'_{n+1}}{2}$$

$$y_{n+1} = y_n + \overline{y}'_n h$$

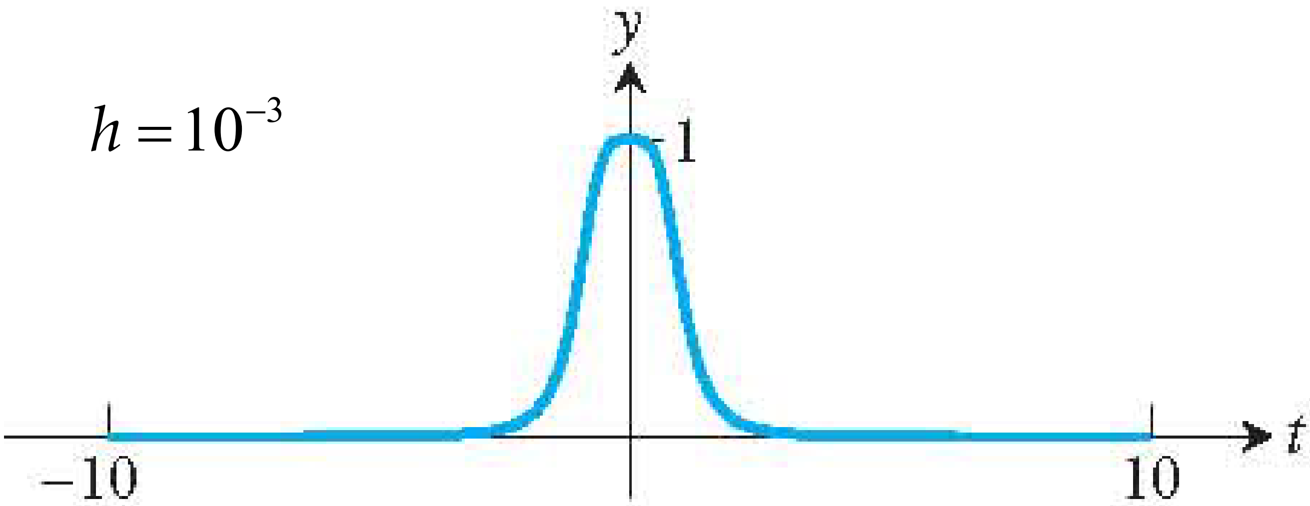
$$e_{n+1} = O\left(h^3\right) \longrightarrow p = 2$$

Heun's method

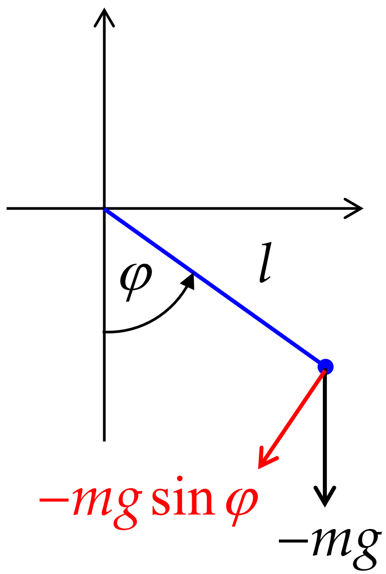
Explicit Trapezoid Example

$$\begin{cases} y' = -4t^3y^2 & -10 \le t \le 0 \\ y(-10) = 1/10001 \end{cases} \qquad y(t) = \frac{1}{t^4+1}$$

$h = 10^{-3}$



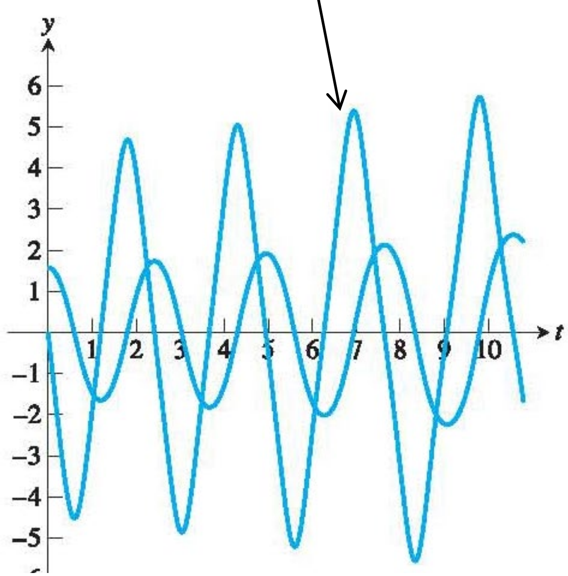
More Realistic Example



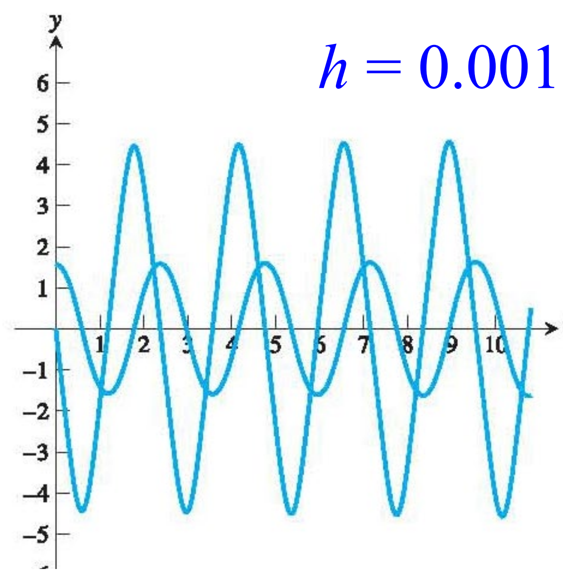
$$F = ma = ml\varphi'' = -mg\sin\varphi$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \varphi \\ \varphi' \end{bmatrix} \qquad \mathbf{y}' = \begin{bmatrix} \varphi' \\ \varphi'' \end{bmatrix} = \begin{bmatrix} y_2 \\ -l/g\sin y_1 \end{bmatrix}$$

Explicit
Trapezoid
$h = 0.01$

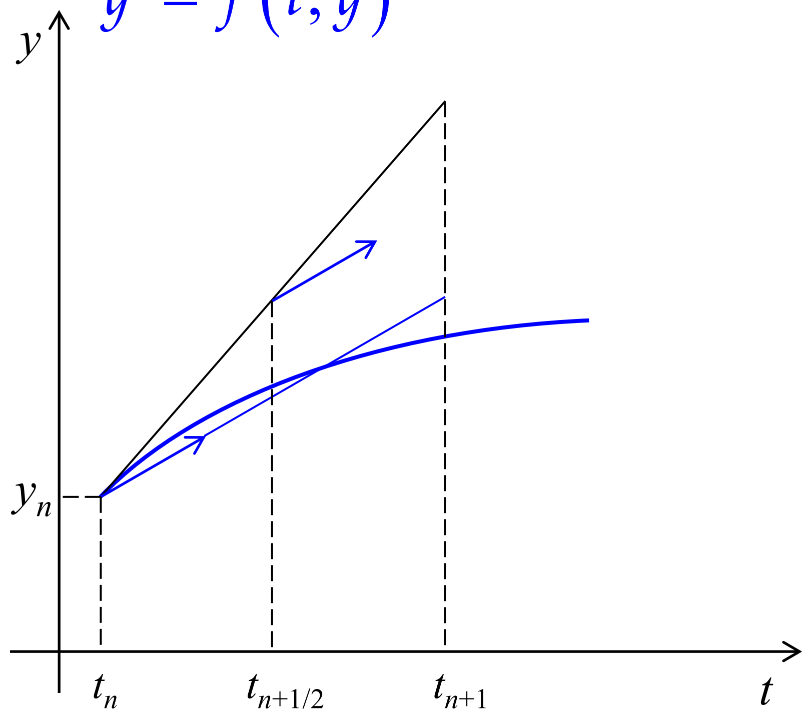


$h = 0.001$



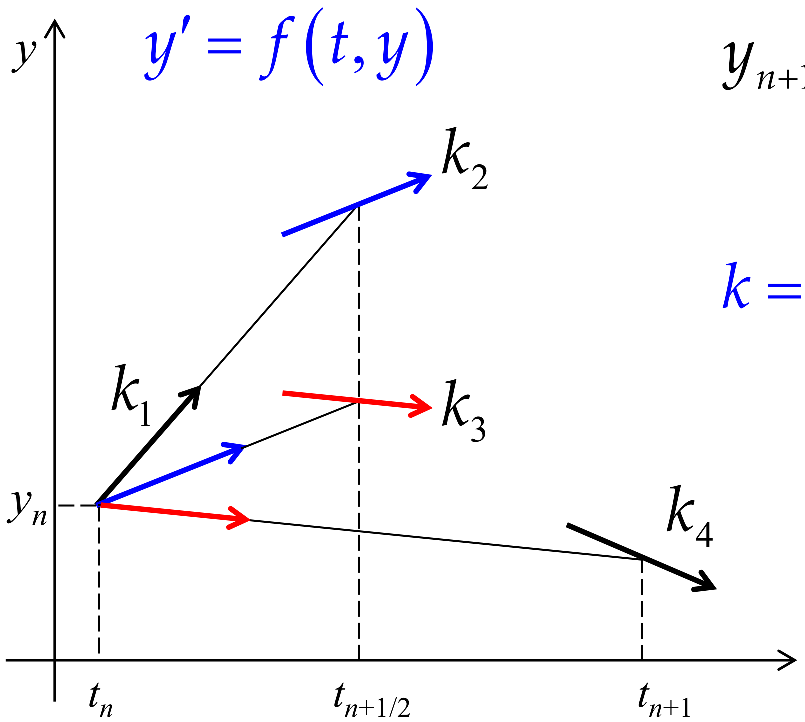
Mid-Point Method

$$y' = f\left(t, y\right)$$



$$e_{n+1} = O\left(h^3\right) \longrightarrow p = 2$$

Popular RKM



$$y_{n+1} = y_n + k \cdot h$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$\delta_{n+1} = O\left(h^5\right) \longrightarrow p = 4$$

Numerical Example

$$\begin{cases} y' = ty + t^3 \\ y(0) = 1 \end{cases} \qquad y(t) = -t^2 - 2 + 3e^{t^2/2}$$

Matlab ODE45

Steps n	Step size h	Error at $t = 1$
5	0.20000	2.38×10^{-5}
10	0.10000	1.47×10^{-6}
20	0.05000	9.03×10^{-8}
40	0.02500	5.60×10^{-9}
80	0.01250	3.48×10^{-10}
160	0.00625	2.17×10^{-11}
320	0.00312	1.35×10^{-12}
640	0.00156	7.26×10^{-14}

Runge-Kutta Methods

$$y_{n+1} = y_n + \phi\left(t_n, y_n, h\right)h$$

Increment

$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_m k_m$$

Representative slope

$$k_1 = f\left(t_n, y_n\right)$$

$$k_2 = f\left(t_n + p_1 h, y_n + q_{11} k_1 h\right)$$

$$k_3 = f\left(t_n + p_2 h, y_n + q_{21} k_1 h + q_{22} k_2 h\right)$$

$$\vdots$$

$$k_m = f\left(t_n + p_{m-1} h, y_n + q_{m-1,1} k_1 h + \ldots + q_{m-1,m-1} k_{m-1} h\right)$$

Runge-Kutta of 4th Order

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$k_1 = f\left(t_n, y_n\right)$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + k_1\frac{h}{2}\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + k_2\frac{h}{2}\right)$$

$$k_4 = f\left(t_n + h, y_n + k_3 h\right)$$

Matlab Functions

- Many ODE solvers
  `ode45, ode23, ode113, ode15s, ode23s, ode23t, ode23tb`
- They differ regarding
  - Accuracy
  - Complexity
  - Stability
  - Treating discontinuities
  - Treating stiff systems
  - Treating Diferential Algebraic Equations, DAE.

$$F(t, y, y') = 0$$

Important

- Explicit Euler Method
- Implicit Euler Method
- Local/Global Error
- Improvements
  - Quadrature
  - Predictor-Corrector (Heun, Mid-Point, RKMs)
- Runge-Kutta Method of 4^{th} Order