Lectures on Thermal-Hydraulics in Nuclear Energy Engineering

Lecture No 09

Title:

Critical Flows

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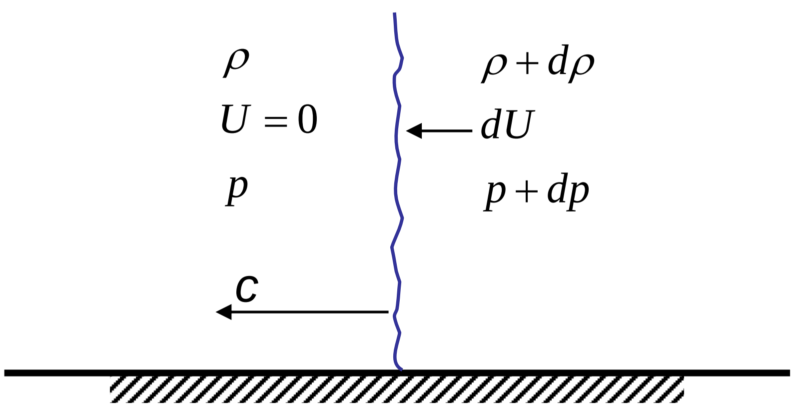
Outline of the Lecture

- Propagation of Sound Waves
  - Speed of Sound
  - Mach Number
- Stationary Gas Flow in Channels
  - Stagnation Properties
  - Critical Conditions
  - Isentropic Flow
- Discharge From a Tank – Blowdown
- Two-Phase Critical Flow
  - HEM model
  - Calculations using plots
  - Example

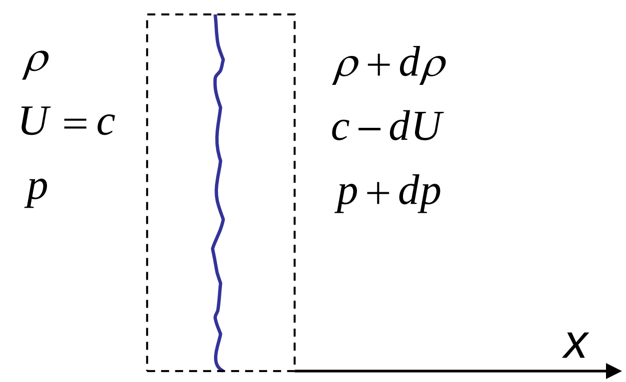
Propagation of Sound Waves

- Consider propagation of a sound wave of infinitesimal strength into an undisturbed medium

- We will determine the relation between the speed of wave propagation, *c*, to fluid property changes across the wave



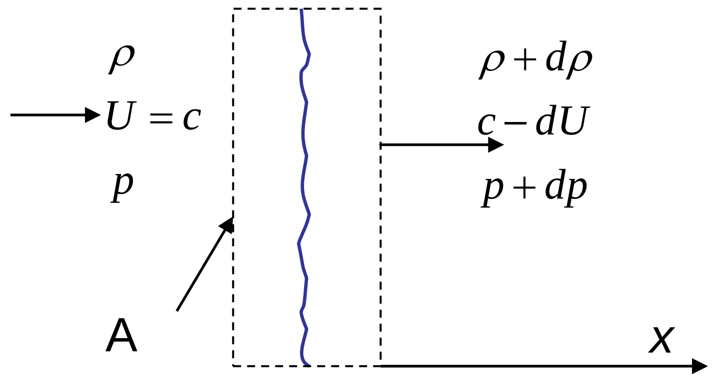
stationary system of reference



Control volume moving with wave

Speed of Sound (1)

- Speed of sound can be derived from mass and momentum conservation equations:



Control volume moving with wave

- Mass:

$$\rho c A = (\rho + d\rho)(c - dU)A$$

$$\cancel{\rho c A} = \cancel{\rho c A} - \rho dUA + d\rho cA$$

- Momentum:

$$pA + c\rho cA = (p + dp)A + (c - dU)(\rho + d\rho)(c - dU)A$$

$$\cancel{pA} + \cancel{c^2\rho A} = \cancel{pA} + dpA + \cancel{c^2\rho A} - \rho cdUA + c^2 d\rho A - \rho cdUA$$

$$dU = \frac{d\rho}{\rho}c \qquad dp = +\rho cdU - c^2 d\rho + \rho cdU \Rightarrow dp = c^2 d\rho \Rightarrow c^2 = \frac{dp}{d\rho}$$

Speed of Sound (2)

- Thus, the relationship between speed of sound as a function of property changes is as follows:

$$c^2 = \frac{dp}{d\rho}$$

- It is further assumed that pressure change is reversible and adiabatic, thus it is isentropic:

$$c = \sqrt{\left(\frac{dp}{d\rho}\right)_s}$$

For ideal gas: $\frac{p}{\rho^\kappa} = const \Rightarrow \ln p - \kappa \ln \rho = const$

$$\frac{dp}{p} - \kappa \frac{d\rho}{\rho} = 0$$

$$\left(\frac{dp}{d\rho}\right)_s = \kappa \frac{p}{\rho} = \kappa RT$$

$$c = \sqrt{\kappa RT}$$

Speed of Sound (3)

- Measurements indicate that speed of sound in gases agree with the derived equation.

- Important feature of speed of sound in ideal gases is that it depends on temperature only.

- The speed of sound increases with the temperature.

- For air at atmospheric pressure:
  - “Summer” speed of sound (T=293 K) = $\sqrt{1.4 \cdot 287 \cdot 293} \approx 343 m/s$
  - “Winter” speed of sound (T=253 K) = $\sqrt{1.4 \cdot 287 \cdot 253} \approx 319 m/s$

Speed of Sound (4)

- In an arbitrary medium, speed of sound is given as follows:

$$c = \sqrt{\frac{C}{\rho}}$$

C – stiffness of the medium

ρ – density of the medium

- In non-dispersive media, speed of sound does not depend on wave length (air is non-dispersive)
- CO_2 is dispersive for ultrasonic frequencies (>28 kHz)
- Speed of sound in solids is given as:

$$c = \sqrt{\frac{E}{\rho}}$$

E – Young's modulus

Exercise: calculate speed of sound in steel ANSI 316: E=193 GPa, ρ=7990 kg m^{-3}

Speed of Sound (4)

- Speed of sound for fluids is given as:

$$c = \sqrt{\frac{K}{\rho}}$$

K – bulk modulus of elasticity

ρ – density of the medium

$$K = \frac{\partial p}{\partial \rho / \rho}$$

Liquid	T [K]	ρ [kgm^{-3}]	K [GPa]
Water, fresh	293	999	2.19
Water, sea	288	1025	2.27

Exercise: calculate speed of sound in fresh and sea water

Mach Number

- The ratio of the flow speed *U* to speed of sound *c* is defined as the Mach number:

$$M = \frac{U}{c}$$

- When M < 0.3, the maximum density variation in flowing gas is less than 5 %.
- Thus gas flows with M < 0.3 can be treated as incompressible.
- Compressibility effects are very important in the design of modern power plants, airplanes, fans, compressors, etc.

Stationary Gas Flow in Channels (1)

- For adiabatic flow of gas in a channel, the energy equation is reduced to

$$\frac{U^2}{2} + i = i_0$$

stagnation enthalpy

- For isentropic flow, the gas enthalpy can be expressed in terms of the temperature and pressure as

$$i = c_p T = \frac{\kappa}{\kappa - 1} RT = \frac{\kappa}{\kappa - 1} \frac{p}{\rho}$$

Stationary Gas Flow in Channels (2)

- Since

$$c = \sqrt{\kappa \frac{p}{\rho}} \qquad \frac{p}{\rho} = \frac{c^2}{\kappa}$$

- Then

$$i = \frac{\kappa}{\kappa - 1} \frac{p}{\rho} = \frac{c^2}{\kappa - 1}$$

- Energy equation becomes

$$\frac{U^2}{2} + \frac{c^2}{\kappa - 1} = \frac{c_0^2}{\kappa - 1}$$

Stationary Gas Flow in Channels (3)

- Thus, dividing by *c* yields

$$\frac{U^2}{c^2}+\frac{2}{\kappa-1}=\frac{2c_0^2}{(\kappa-1)c^2} \quad \Rightarrow \left(\frac{c_0}{c}\right)^2=1+\frac{(\kappa-1)}{2}M^2$$

- And finally

$$\boxed{\frac{c}{c_0}=\frac{1}{\sqrt{1+\frac{(\kappa-1)}{2}M^2}}}$$

Stationary Gas Flow in Channels (4)

- Flow is critical, when M = U/c = 1, thus

$$\frac{c}{c_0} = \frac{1}{\sqrt{1 + \frac{(\kappa - 1)}{2}}} = \sqrt{\frac{2}{2 + \kappa - 1}} = \sqrt{\frac{2}{\kappa + 1}}$$

- All critical parameters are marked with “*”:

$$c_* = c_0 \sqrt{\frac{2}{\kappa + 1}}$$

“critical” speed of sound, when M=1

Stationary Gas Flow in Channels (5)

- Since $c = \sqrt{\kappa RT}$ and $c_0 = \sqrt{\kappa RT_0}$

$$\frac{c}{c_0} = \frac{1}{\sqrt{1 + \frac{(\kappa - 1)}{2} M^2}}$$

$$\Downarrow$$

$$\frac{T}{T_0} = \frac{1}{1 + \frac{\kappa - 1}{2} \mathrm{M}^2} \Rightarrow \boxed{\frac{T_*}{T_0} = \frac{2}{\kappa + 1}}$$

"critical" temperature, when M=1

Stationary Gas Flow in Channels (6)

- Assuming an isentropic process, for which:

$$\frac{p^{\kappa-1}}{T^{\kappa}} = const \qquad \frac{T}{\rho^{\kappa-1}} = const$$

- The critical pressure and density can be expressed in terms of stagnation values as follows:

$$\frac{p_*}{p_0} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}} \qquad \frac{\rho_*}{\rho_0} = \left(\frac{2}{\kappa+1}\right)^{\frac{1}{\kappa-1}}$$

Exercise: find p_*/p_0 for steam taking ideal gas assumption

Stationary Gas Flow in Channels (7)

- The mass flow rate can be calculated as

$$U=\sqrt{\frac{2}{\kappa-1}\left(c_0^2-c^2\right)}=\sqrt{\frac{2c_0^2}{\kappa-1}\left(1-\frac{c^2}{c_0^2}\right)}=\sqrt{\frac{2\kappa RT_0}{\kappa-1}\left(1-\frac{T}{T_0}\right)}$$

$$\Downarrow$$

$$W=\rho UA=\rho A\sqrt{\frac{2\kappa}{\kappa-1}RT_0\left(1-\frac{T}{T_0}\right)}$$

For isentropic flow, we have:

$$W=A\frac{p_0}{\sqrt{T_0}}\sqrt{\frac{2\kappa}{R(\kappa-1)}}\sqrt{\left(\frac{p}{p_0}\right)^{\frac{2}{\kappa}}-\left(\frac{p}{p_0}\right)^{\frac{\kappa+1}{\kappa}}}$$

This equation gives mass flow rate in terms of stagnation conditions and local pressure

Stationary Gas Flow in Channels (8)

- Critical mass flow rate can be obtained as follows:

$$W_* = \rho_* U_* A_*$$

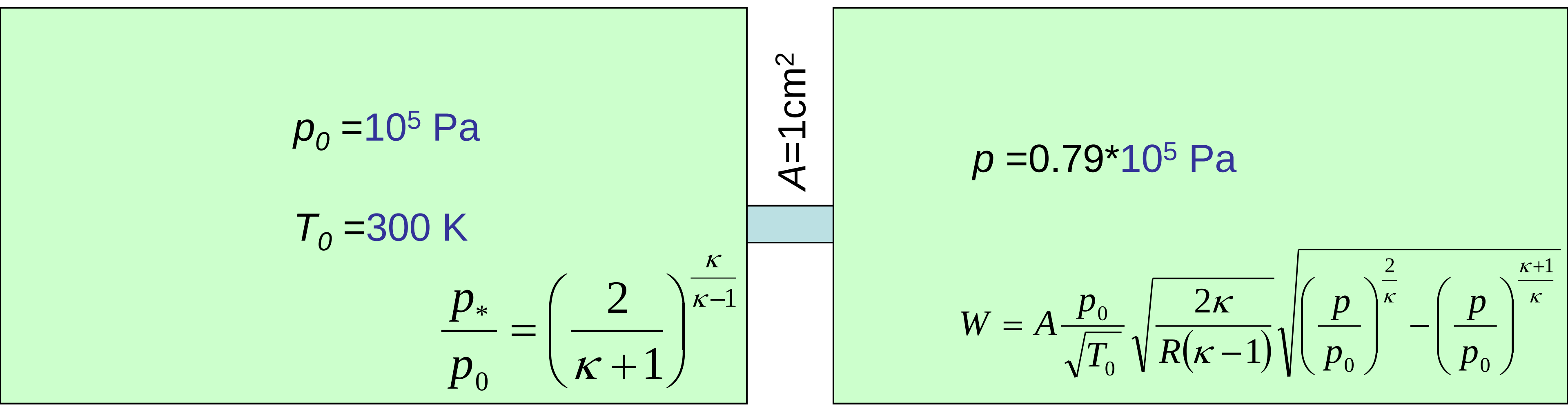
$$\frac{\rho_*}{\rho_0} = \left(\frac{2}{\kappa+1}\right)^{\frac{1}{\kappa-1}} \qquad U_* = \sqrt{\frac{2\kappa R T_0}{\kappa-1}\left(1-\frac{T_*}{T_0}\right)} \qquad \frac{T_*}{T_0} = \frac{2}{\kappa+1}$$

$$W_* = \rho_* U_* A_* = A_* \frac{p_0}{\sqrt{T_0}} \sqrt{\frac{\kappa}{R}} \sqrt{\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{\kappa-1}}}$$

Note that critical flow does not depend on the local pressure: it depends only on the stagnation pressure

Stationary Gas Flow in Channels (9)

- Exercise: Air at pressure $p_0 = 10^5$ Pa and $T_0 = 300$ K flows from one large volume to another with pressure $p = 0.79\ 10^5$ Pa through a convergent nozzle with cross-section area $A = 1\ \text{cm}^2$. Calculate the mass flow rate of the flowing air. What should be the pressure p to obtain the maximum mass flow rate?



$$\frac{p_*}{p_0} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}}$$

$$W = A\frac{p_0}{\sqrt{T_0}}\sqrt{\frac{2\kappa}{R(\kappa-1)}}\sqrt{\left(\frac{p}{p_0}\right)^{\frac{2}{\kappa}} - \left(\frac{p}{p_0}\right)^{\frac{\kappa+1}{\kappa}}}$$

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Discharge from a Tank (1)

- In many engineering applications it is necessary to analyze the discharge of compressible fluid from a tank with a finite volume through a small convergent nozzle.
- Due to the loss of fluid, the parameters in the tank are changing with time, influencing the conditions in the nozzle as well.
- Since the nozzle is small, it can be assumed that the conditions inside it are quasi steady-state.
- This assumption would not be correct for a large nozzle, in which the perturbations due to waves in the tank could not be ignored.

Discharge from a Tank (2)

- In nuclear engineering an important process is a ***pressure vessel blowdown***, in which fluid is discharged from a tank.
- Blowdown can occur due to open-valve gas discharge from tanks or due to a ruptured pipe, which could lead to an uncontrolled fluid discharge from a pressure vessel.
- This type of accident is considered as one of the most severe in nuclear power plants, when blowdown of a reactor pressure vessel could occur during the Loss-of-Coolant Accident (LOCA), resulting from the rapture of a pipe in the reactor primary system.

Discharge from a Tank (3)

- Consider the case of **adiabatic blowdown** from a vessel containing perfect gas. The density change will be as

$$\frac{M}{V} = \rho = \rho_I \left(\frac{p}{p_I} \right)^{1/\kappa}$$

  - where subscript *I* designates the initial conditions.

- Mass conservation equation for the ideal gas in the vessel is as follows

$$W = -\frac{dM}{dt} \quad \Rightarrow \quad \left(\frac{p}{p_I} \right)^{(1-\kappa)/\kappa} \frac{d}{dt} \left(\frac{p}{p_I} \right) = -\frac{\kappa}{\rho_I} \frac{W}{V}$$

Discharge from a Tank (4)

- If critical flow takes place, then the mass flow rate depends on parameters in the vessel as follows:

$$W = A\frac{p}{\sqrt{T}}\sqrt{\frac{\kappa}{R}}\sqrt{\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{\kappa-1}}} = A\sqrt{p}\sqrt{\frac{p\kappa}{RT}}\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}} = A\sqrt{\kappa p \rho}\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}}$$

- but since for the adiabatic process:

$$\rho = \rho_I\left(\frac{p}{p_I}\right)^{1/\kappa} \Rightarrow \sqrt{\kappa p \rho} = \sqrt{\kappa p \rho_I\left(\frac{p}{p_I}\right)^{1/\kappa}} = \sqrt{\kappa\frac{p}{p_I}p_I\rho_I\left(\frac{p}{p_I}\right)^{1/\kappa}} = \sqrt{\kappa p_I \rho_I}\left(\frac{p}{p_I}\right)^{(1/\kappa+1)/2}$$

Discharge from a Tank (5)

- Thus the critical flow rate becomes:

$$W = A\sqrt{\kappa p_I \rho_I}\left(\frac{p}{p_I}\right)^{\frac{1+\kappa}{2\kappa}}\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}}$$

- and the differential equation for pressure change is:

$$\left(\frac{p}{p_I}\right)^{(1-\kappa)/\kappa}\frac{d}{dt}\left(\frac{p}{p_I}\right) = -\frac{\kappa}{\rho_I}\frac{A}{V}\left(\frac{p}{p_I}\right)^{(1+\kappa)/2\kappa}\sqrt{\kappa p_I \rho_I}\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}}$$

- or $$\left(\frac{p}{p_I}\right)^{(1-3\kappa)/2\kappa}\frac{d}{dt}\left(\frac{p}{p_I}\right) = -\frac{\kappa}{\rho_I}\frac{A}{V}\sqrt{\kappa p_I \rho_I}\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}}$$

Discharge from a Tank (6)

- Thus the differential equation to be solved is

$$\left(\frac{p}{p_I}\right)^{(1-3\kappa)/2\kappa}\frac{d}{dt}\left(\frac{p}{p_I}\right)=-\frac{\kappa}{\rho_I}\frac{A}{V}\sqrt{\kappa p_I\rho_I}\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}}$$

with initial condition at *t = 0, p/p_I = 1*

- Solution of the above differential equation is as follows:

$$\frac{p(t)}{p_I}=\left[1+\left(\frac{\kappa-1}{2}\right)\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}}\sqrt{\frac{\kappa p_I}{\rho_I}}\frac{At}{V}\right]^{\frac{-2\kappa}{\kappa-1}}$$

Characteristic time constant of the tank

Or: $$\frac{p(\tau)}{p_I}=\left[1+\sqrt{\kappa}\left(\frac{\kappa-1}{2}\right)\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}}\tau\right]^{\frac{-2\kappa}{\kappa-1}}$$

where: $$\tau=\sqrt{\frac{p_I}{\rho_I}}\frac{At}{V}=\frac{t}{\theta}$$

$$\theta=\frac{V}{A}\sqrt{\frac{\rho_I}{p_I}}$$

Discharge from a Tank (7)

- Combining the pressure-change and mass conservation equations yields the following expression for the mass change in the tank:

$$\rho = \rho_I \left(\frac{p}{p_I} \right)^{1/\kappa} \quad \Rightarrow \quad M = \rho V = V \rho_I \left(\frac{p}{p_I} \right)^{1/\kappa} = M_I \left(\frac{p}{p_I} \right)^{1/\kappa}$$

- thus:

$$\frac{M(\tau)}{M_I} = \left[1 + \sqrt{\kappa} \left(\frac{\kappa - 1}{2} \right) \left(\frac{2}{\kappa + 1} \right)^{\frac{\kappa+1}{2(\kappa-1)}} \tau \right]^{\frac{-2}{\kappa-1}}$$

here M_I is the initial mass in the tank

Discharge from a Tank (8)

- The derived equations are valid as long as critical flow takes place, that is:

$$p(\tau) = p_I \left[1 + \sqrt{\kappa} \left(\frac{\kappa - 1}{2} \right) \left(\frac{2}{\kappa + 1} \right)^{\frac{\kappa+1}{2(\kappa-1)}} \tau \right]^{\frac{-2\kappa}{\kappa-1}} \geq \frac{p_a}{\left(\frac{2}{\kappa+1} \right)^{\frac{\kappa}{\kappa-1}}}$$

here p_a – ambient pressure

Thus the flow will remain critical for time t_{cr} given as:

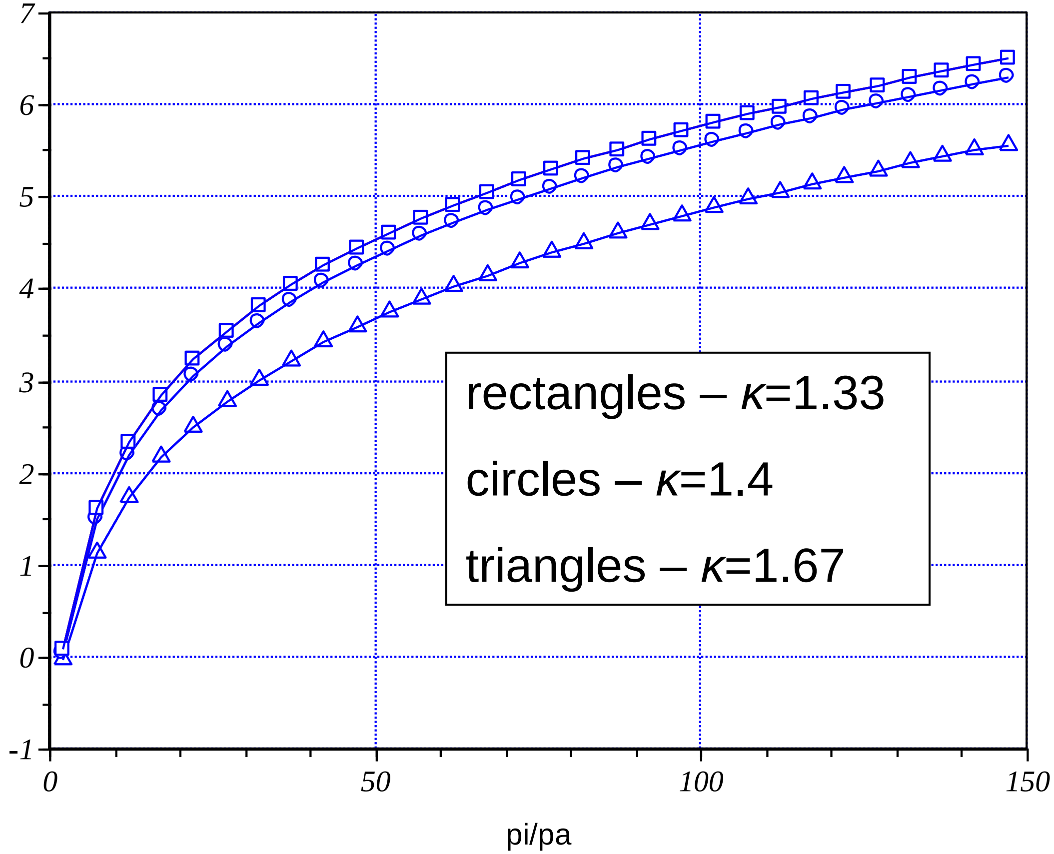
$$t_{cr} = \tau_{cr} \theta$$

$$\theta = \frac{V}{A} \sqrt{\frac{\rho_I}{p_I}}$$

$$\tau_{cr} = \frac{\left(\frac{p_I}{p_a} \right)^{\frac{\kappa-1}{2\kappa}} \left(\frac{2}{\kappa+1} \right)^{\frac{1}{2}} - 1}{\sqrt{\kappa} \left(\frac{\kappa - 1}{2} \right) \left(\frac{2}{\kappa + 1} \right)^{\frac{\kappa+1}{2(\kappa-1)}}}$$

Discharge from a Tank (9)

$$\tau_{cr} = \frac{t_{cr}}{\theta}$$



Standard curves showing dimensionless critical discharge as a function of initial pressure ratio in the tank and ambient

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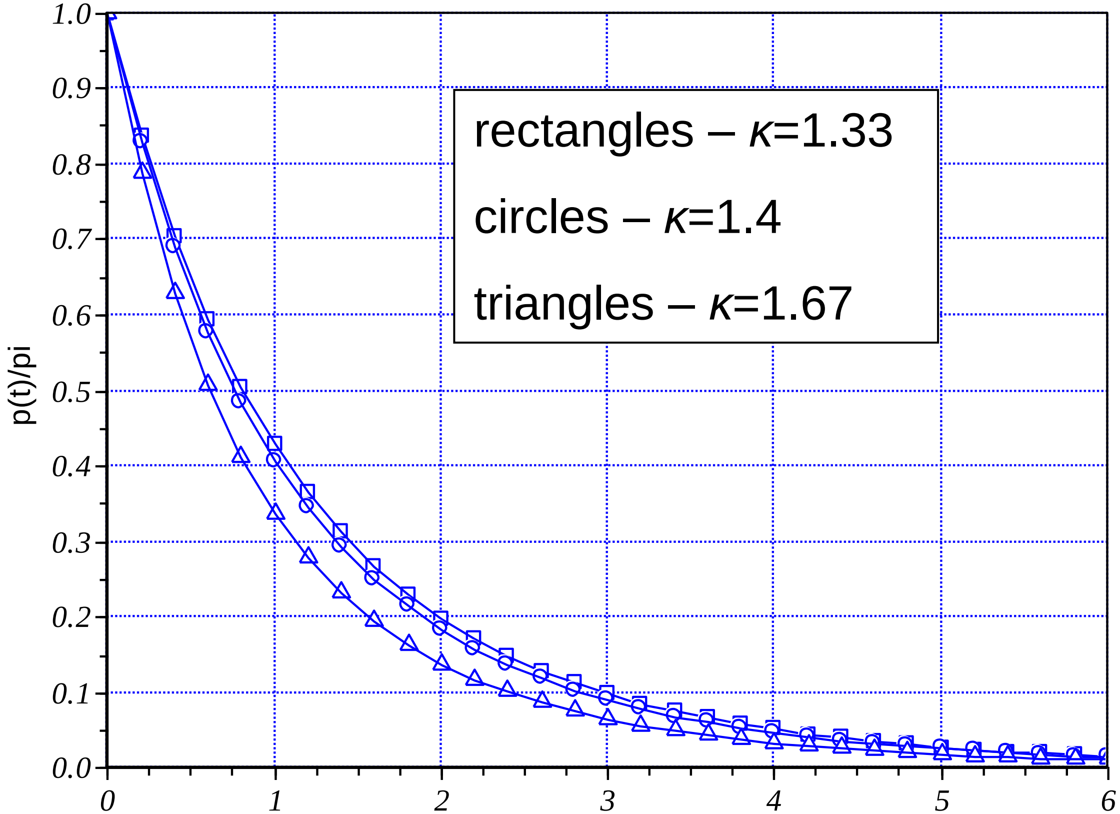
Discharge from a Tank (10)



$$\tau = \frac{t}{\theta}$$

Standard curves showing dimensionless pressure in tank as a function of dimensionless time

Discharge from a Tank (11)

- When the tank is large and is not insulated, the temperature of the gas will not change due to heat transfer with the surroundings. In that case an isothermal blowdown takes place.

- When the heat transfer with the surroundings does not take place (the tank is insulated), the adiabatic blowdown will occur.

Two-Phase Critical Flow (1)

- In single phase flows the critical flow occurs in locations where the flow velocity is equal to the local sound speed, that is where M = 1.
- In two-phase flows the situation is more complicated since the speed of sound can not be uniquely determined.
- There may be actually more than one sound speed: one for each phase.
- Clearly, some modeling assumptions are necessary to determine the critical flow in two-phase flows.

Homogeneous Equilibrium Model

- One of the first and simplest models is based on the HEM formulation.

- The model is employing two fundamental assumptions:
  - the velocities of the phases are equal and
  - the phases are in the thermodynamic equilibrium.

- With these assumptions the mixture can be treated as a single fluid and the uniqueness of the speed of sound is preserved.

Isentropic assumption

- Assuming adiabatic flow, the mixture energy equation can be written in terms of mass flux as

$$\frac{G^2}{2\rho^2} + i = i_0$$

- Further assuming isentropic flow, the mixture properties can be treated as a function of pressure only, and the equation becomes

$$G = \rho_h(p, s_0)\sqrt{2\left[i_0 - i(p, s_0)\right]}$$

Maximizing mass flux

- The critical mass flux can be found by maximizing the mass flux with respect to pressure:

$$\frac{dG}{dp} = \left(\frac{\partial \rho_h}{\partial p}\right)_s \sqrt{2(i_0 - i)} - \left(\frac{\partial i}{\partial p}\right)_s \Big/ \sqrt{2(i_0 - i)} = 0$$

- Solving this equation yields

$$\left.\sqrt{2(i_0 - i)}\right|_* = \sqrt{\left(\frac{\partial i}{\partial p}\right)_s \Big/ \left(\frac{\partial \rho_h}{\partial p}\right)_s}$$

Critical mass flux

- Finally, the critical mass flux is obtained as

$$G\big|_* = \rho_h(p_*, s_0)\sqrt{\left(\frac{\partial i}{\partial p}\right)_s \Big/ \left(\frac{\partial \rho_h}{\partial p}\right)_s}$$

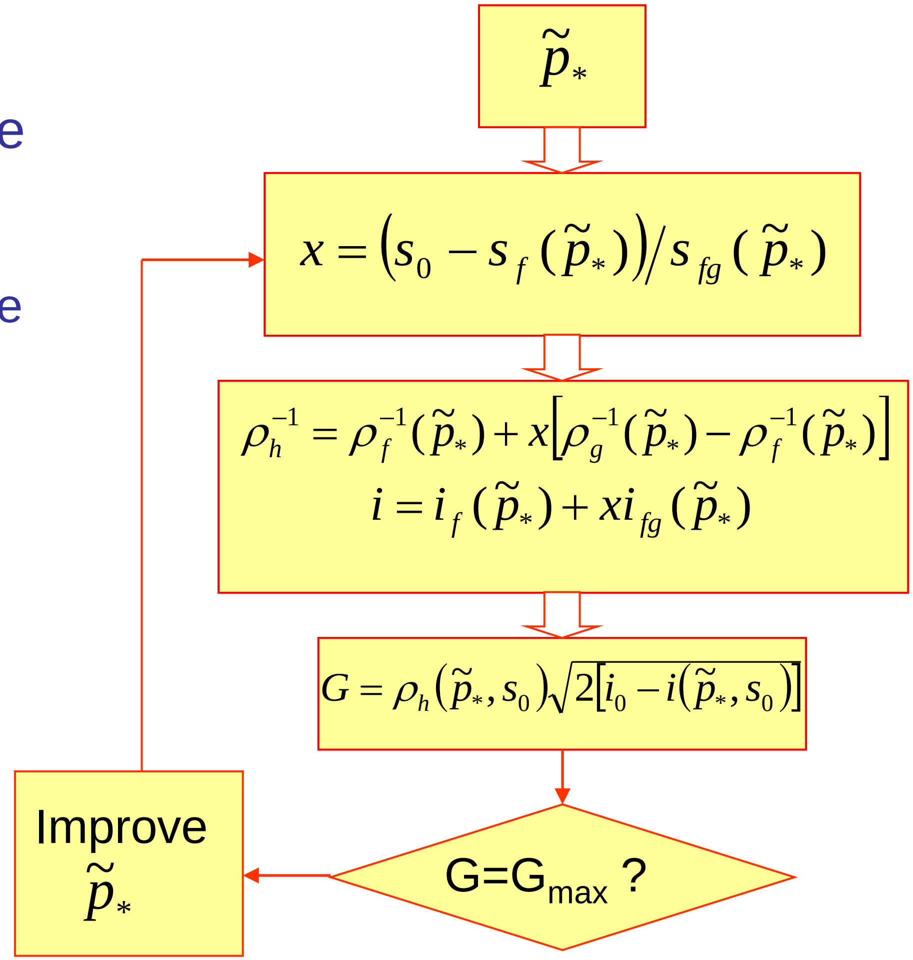
- The homogeneous density and the partial derivatives are evaluated at the critical pressure in the nozzle, which is not known yet.

- Thus, an iterative solution is necessary.

It can be complicated to find derivatives $(di/dp)_s$, etc

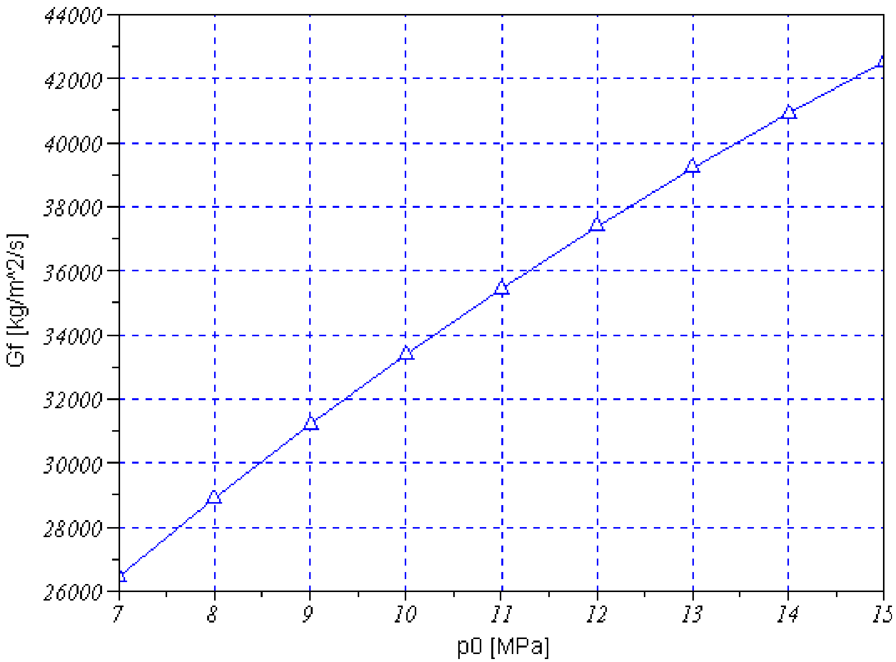
Alternative model: iterate to find maximum G

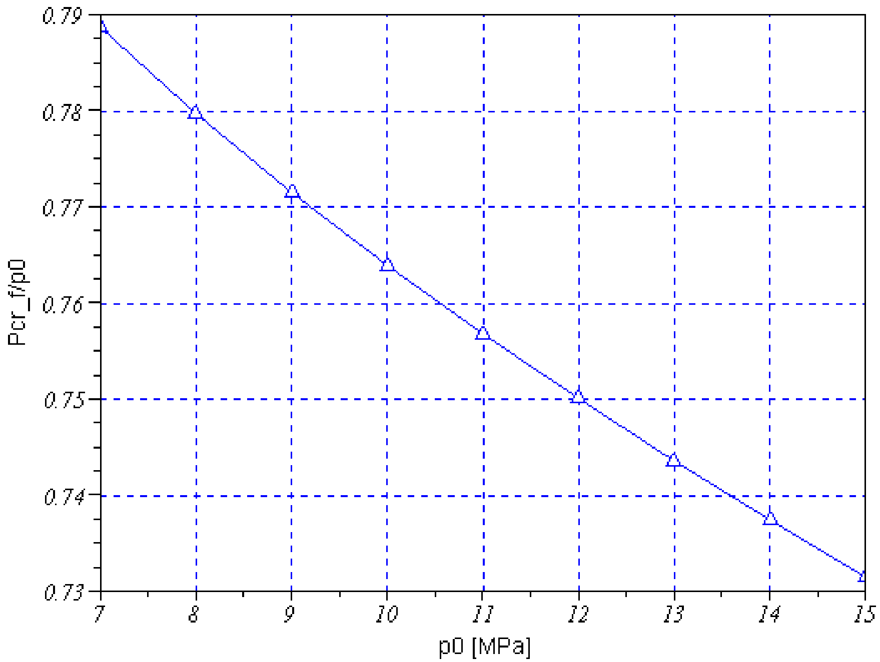
- For practical applications, the following iterative process can be employed:
  - Guess the critical pressure in the nozzle
  - Since the flow is isentropic, the entropy at the nozzle is equal to the stagnation entropy and the quality at the nozzle can be found from the change of saturation entropy
  - Find the enthalpy and homogeneous density at the nozzle
  - Calculate the mass flux and iterate until the maximum is found



Critical mass flux for sat. water

Using HEM, the following plots can be generated



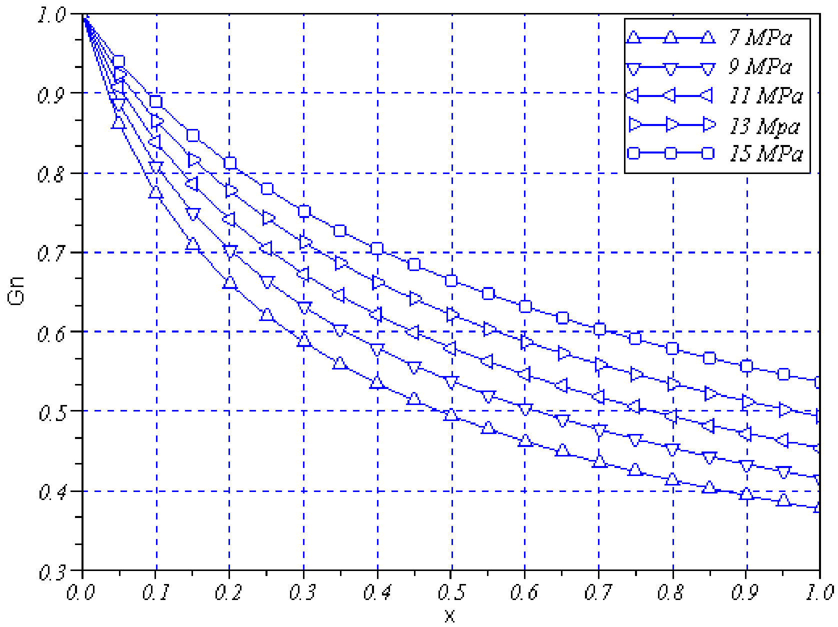


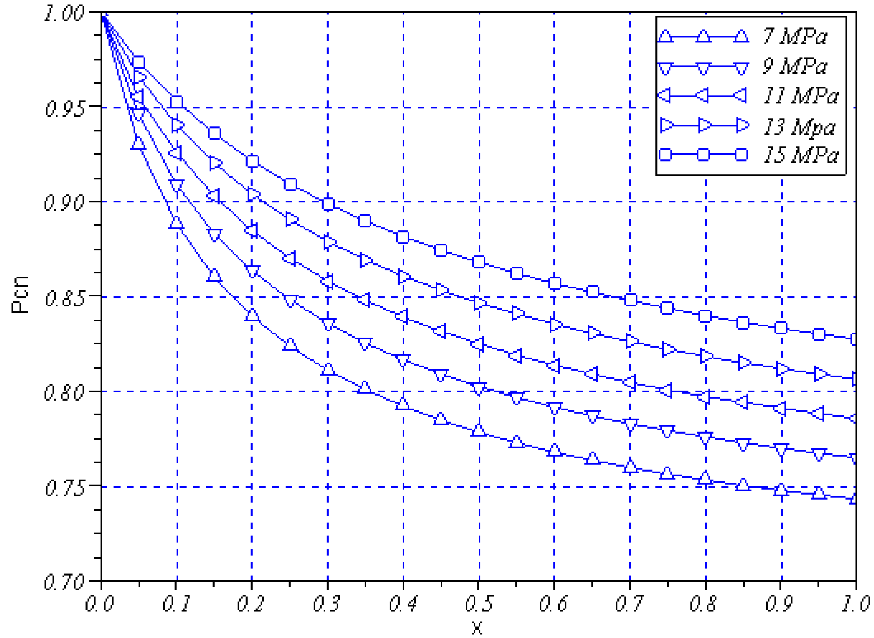
Critical flow of saturated water as a function of stagnation pressure

Critical-to-stagnation pressure ratio as a function of stagnation pressure

Critical mass flux for mixture

Using HEM, the following plots can be generated





Gn = (critical mass flux of mixture with quality x)/(critical mass flux of saturated water) = G_m/G_f

Pcn = (critical pressure of mixture with quality x)/(critical pressure of saturated water) = p_{cr_m}/p_{cr_f}