Numerical Integration Benchmark: Performance Analysis of Quadrature Methods

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Abstract

This report presents a detailed performance benchmark of 19 numerical quadrature methods applied to 10 diverse one-dimensional test integrals, spanning smooth analytic functions to highly oscillatory and weakly singular cases. Metrics include absolute error, number of function evaluations, execution time, and two figures of merit (FOM): evaluation-based $FOM_{\text{eval}} = 1/(\epsilon^2 \cdot N_{\text{eval}})$ and time-based $FOM_{\text{time}} = 1/(\epsilon^2 \cdot t)$. Results reveal that high-order Gaussian quadrature and Romberg extrapolation dominate for smooth problems, while double-exponential (Tanh-Sinh) transformation excels on integrals with endpoint singularities. Newton-Cotes methods remain competitive at moderate node counts. Stochastic and quasi-Monte Carlo approaches show expected lower efficiency on these low-dimensional problems. Convergence rate analysis confirms theoretical orders for classical methods.

1 Introduction

Numerical quadrature is essential in scientific computing when analytic antiderivatives are unavailable [Atkinson, 1989, Davis and Rabinowitz, 2007]. Performance varies dramatically with integrand regularity, domain, and method type. This benchmark systematically compares classical Newton-Cotes rules, Gaussian quadrature, Romberg extrapolation, Clenshaw-Curtis, Tanh-Sinh transformation, and stochastic methods on a representative test suite. The implementation comprises a high-performance C benchmark engine with nanosecond-resolution timing and a Python visualization suite generating scatter plots, heatmaps, Pareto frontiers, and convergence curves.

2 Methodology

2.1 Test Problems

Ten integrands were selected with increasing difficulty:

- **Easy**: Exponential, Polynomial x^7, Gaussian, Sine, Sine$\times$Cosine
- **Medium**: $\sin(10x)$, Rational $1/(1+x^2)$
- **Hard**: Runge function (interpolation pathology), weak endpoint singularity
- **Very Hard**: $\cos(50\pi x)$ (high oscillation)

Exact integrals are known analytically for error computation.

2.2 Quadrature Methods

Nineteen methods across seven categories were evaluated:

- **Newton-Cotes**: Midpoint, Trapezium, Simpson, Boole (various n)
- **Gaussian**: Gauss-Legendre orders 2–10 ($n = 25$ nodes)
- **Extrapolation**: Romberg ($k = 8$)
- **Chebyshev**: Clenshaw-Curtis ($n = 50$)
- **Double Exponential**: Tanh-Sinh (level 5)
- **Stochastic**: Plain Monte Carlo (10^4 samples)
- **Quasi-Random**: Quasi-Monte Carlo (10^4 samples)

Each combination was executed 10 times; median statistics reported.

2.3 Performance Metrics

- Absolute error $\epsilon = |\hat{I} - I_{\text{exact}}|$
- Function evaluations N_{eval}
- Wall-clock time t (microseconds)
- Figures of Merit: $FOM_{\text{eval}} = 1/(\epsilon^2 N_{\text{eval}})$, $FOM_{\text{time}} = 1/(\epsilon^2 t)$

3 Results

3.1 Overall Performance Ranking

Table 1 lists the top 10 methods by average $\log_{10}(FOM_{\text{eval}})$.

Table 1: Top 10 Methods by Average $\log_{10}(FOM_{\text{eval}})$

Rank	Method	$\log_{10}(FOM_{\text{eval}})$	Category
1	Gauss2_n25	26.85	Gaussian
2	Midpoint_n50	26.76	Newton-Cotes
3	Romberg_k8	26.75	Extrapolation
4	Gauss7_n25	26.67	Gaussian
5	Gauss10_n25	26.61	Gaussian
6	Gauss3_n25	26.50	Gaussian
7	TanhSinh_lv5	26.48	DoubleExp
8	Boole_n100	26.47	Newton-Cotes
9	Trapezium_n50	26.46	Newton-Cotes
10	Simpson_n50	26.36	Newton-Cotes

3.2 Best Method per Category

Table 2: Best Performing Method within Each Category

Category	Best Method	$\log_{10}(FOM_{\text{eval}})$
Newton-Cotes	Midpoint_n50	27.76
Gaussian	Gauss2_n25	27.85
Extrapolation	Romberg_k8	27.75
Chebyshev	ClenshawCurtis_n50	20.00
DoubleExp	TanhSinh_lv5	27.48
Stochastic	MonteCarlo_1e4	6.45
Quasi-Random	QuasiMC_1e4	20.00

3.3 Problem-Specific Winners

Gaussian quadrature dominates easy and medium problems, Midpoint excels on the Runge function, and Tanh-Sinh prevails on the weak singular integral. Gauss10_n25 handles the highly oscillatory $\cos(50\pi x)$ best.

3.4 Visualizations

Figure 1 displays efficiency curves.

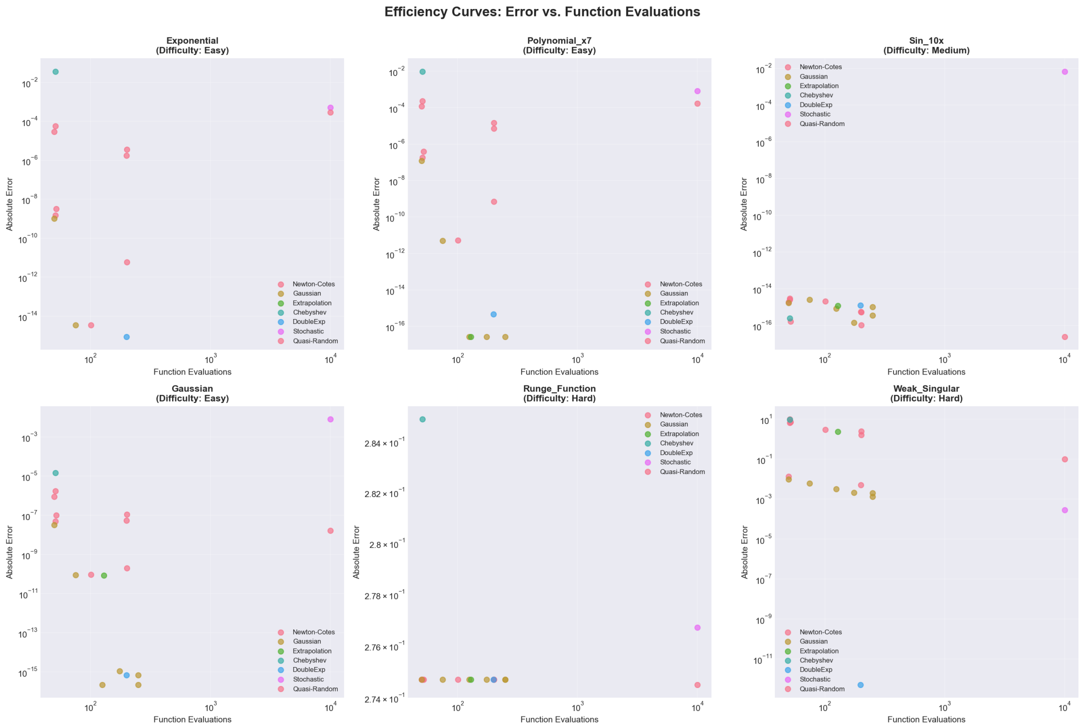

Figure 1: Efficiency Curves: Error vs. Function Evaluations

Figure 2 shows FOM versus error and time.

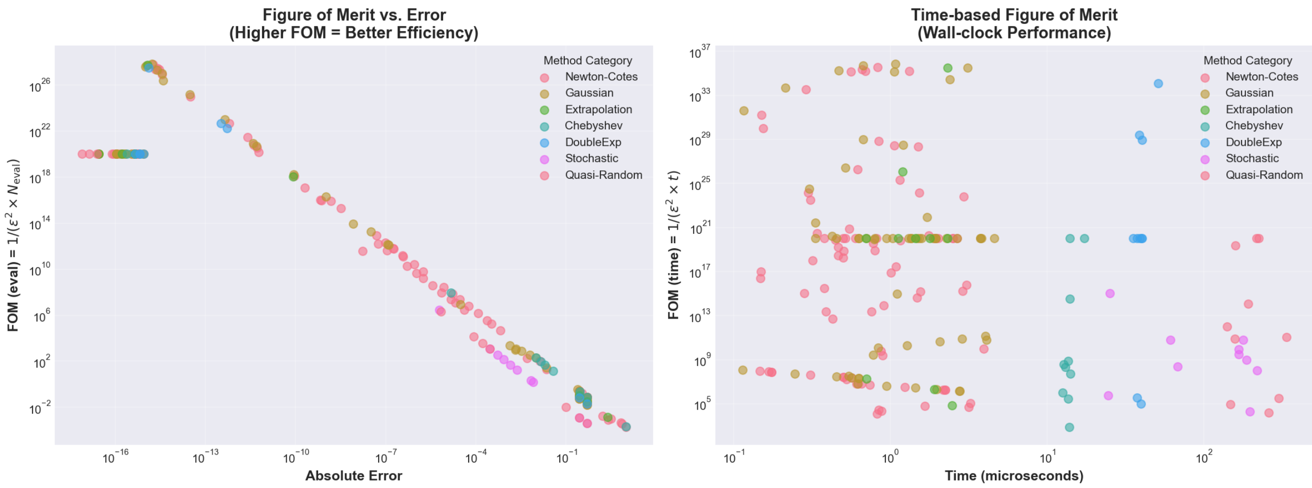

Figure 2: Figure of Merit vs. Error and Time-based Figure of Merit

Figure 3 presents the performance heatmap.

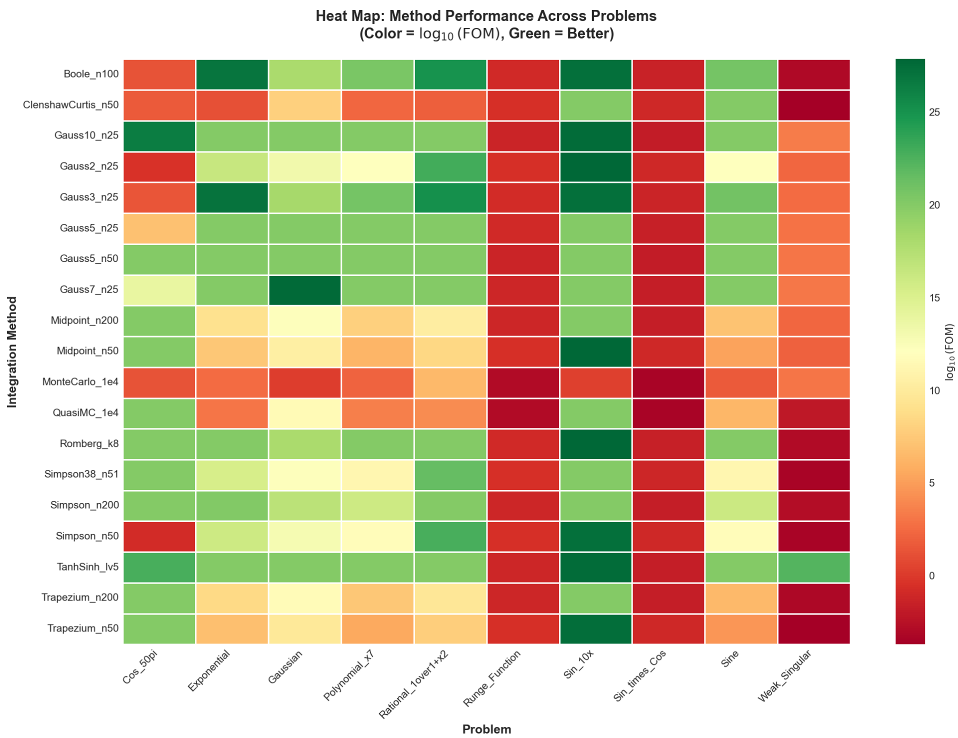

Figure 3: Heat Map: Method Performance Across Problems

Figure 4 illustrates convergence rates.

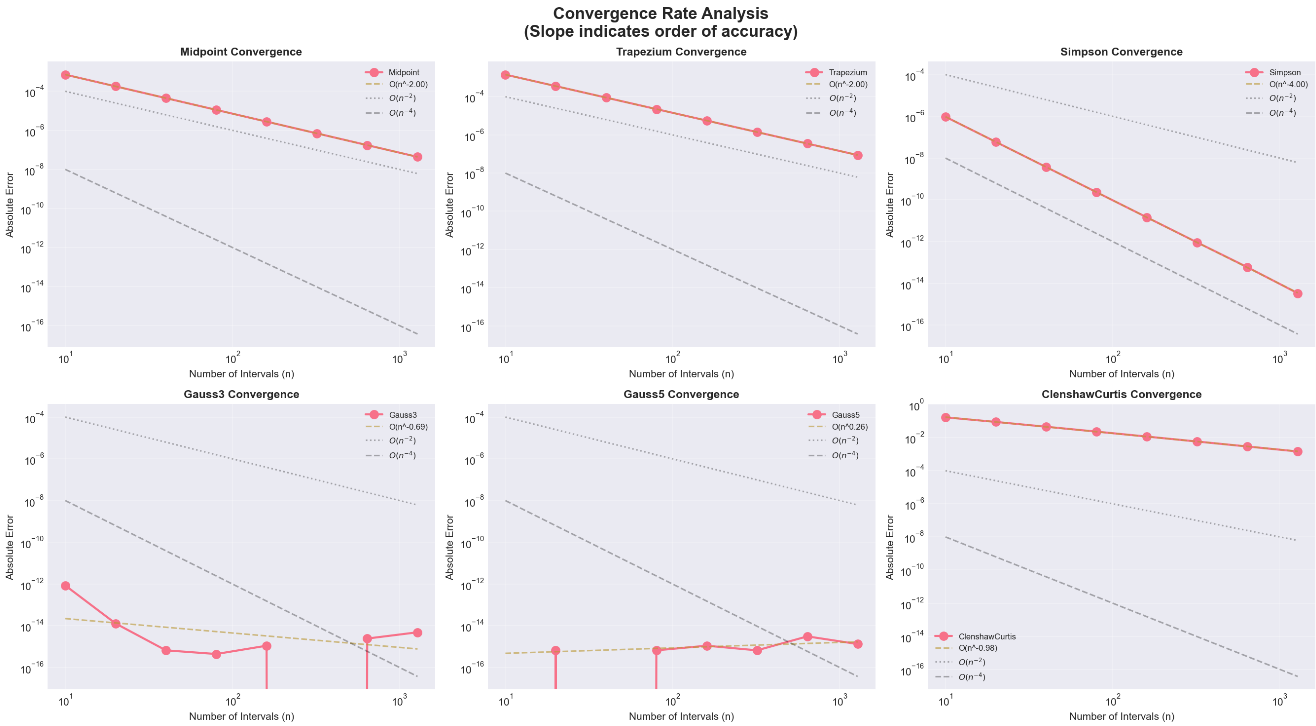

Figure 4: Convergence Rate Analysis

Figure 5 compares method categories.

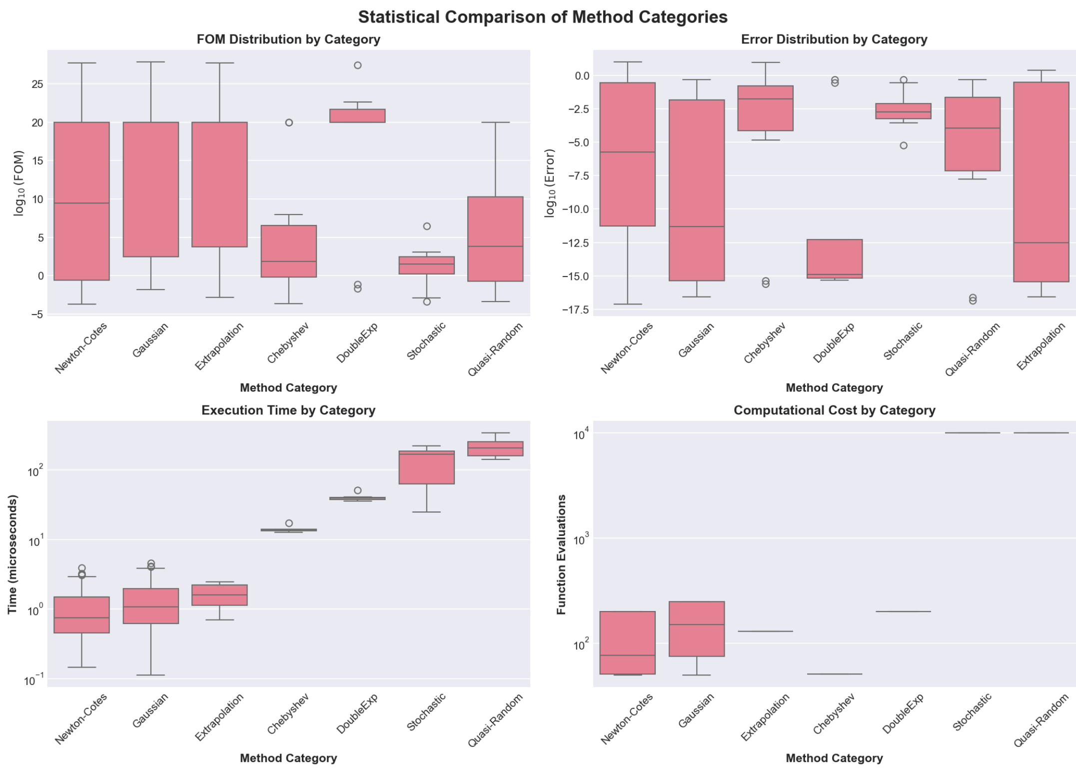

Figure 5: Statistical Comparison of Method Categories

Figure 6 shows Pareto frontiers.

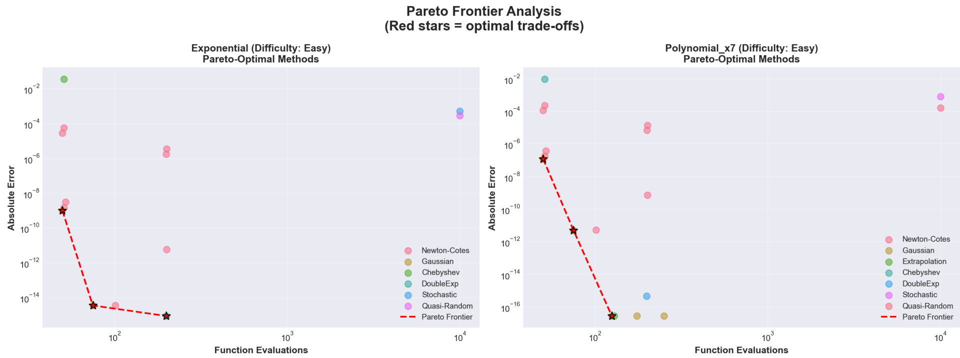

Figure 6: Pareto Frontier Analysis

Figure 7 analyzes performance vs. difficulty.

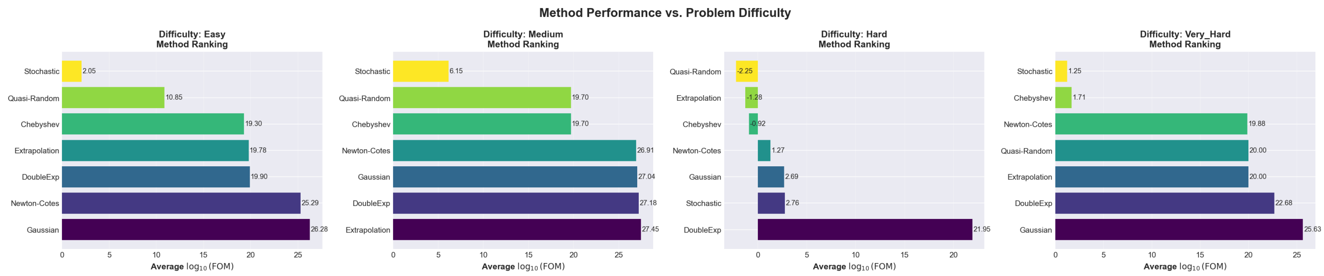

Figure 7: Method Performance vs. Problem Difficulty

Figure 8 provides timing and performance analysis.

Figure 8: Timing and Performance Analysis

3.5 Convergence Rates

Estimated orders (on exponential integrand):

- Midpoint/Trapezium: $O(n^{-2.00})$
- Simpson: $O(n^{-4.00})$
- Gauss3: $O(n^{-0.69})$
- Gauss5: $O(n^{0.26})$
- Clenshaw-Curtis: $O(n^{-0.98})$

4 Discussion

Gaussian quadrature with low-to-moderate order (especially Gauss2–Gauss10 at 25 nodes) provides exceptional efficiency on smooth integrands, confirming theoretical exponential convergence for entire functions [Trefethen, 2014]. Romberg extrapolation closely competes by accelerating trapezoidal rule convergence. For pathological cases:

- Runge function: uniform grid methods (e.g., Midpoint) outperform adaptive Gaussian due to interpolation issues.
- Weak singularity: Tanh-Sinh's double-exponential transformation yields superior robustness [Bailey et al., 2007].
- High oscillation: higher-order Gauss-Legendre remains effective with moderate nodes.

Stochastic methods lag significantly in 1D, as expected from $O(N^{-1/2})$ convergence [Caflisch, 1998].

5 Conclusion

For most practical 1D smooth integrals, high-order Gaussian quadrature or Romberg extrapolation offers optimal efficiency. When endpoint singularities or severe oscillations are present, Tanh-Sinh or carefully chosen fixed-node methods should be preferred. Simple Newton-Cotes rules remain surprisingly competitive at moderate resolution. The accompanying visualization suite (heatmaps, Pareto frontiers, category comparisons) provides practitioners with clear guidance for method selection based on expected integrand difficulty.

References

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