Project - 3

i. $\frac{du}{dt} = v^2 \frac{d^2u}{dy^2}$

$u(y,t) = f(x) \cdots (1)$

$\eta = \frac{y}{\sqrt{vt}} \cdots (2)$

$\frac{du}{dt} = \frac{df}{dn} \times \frac{d\eta}{dt} \cdots (3)$

Subtitute eq (2) in eq (3)

$\frac{du}{dt} = \frac{df}{dn} \times \frac{d[y(vt)^{-1/2}]}{dt}$

$= \frac{-1}{2} y v^{-1/2} t^{-3/2} \frac{df}{dn}$

from eq (2) we get

$\frac{du}{dt} = \frac{-1}{2} t^{-1} \eta \frac{df}{dn} \cdots (4)$

$\frac{du}{dy} = \frac{df}{dn} \times \frac{dn}{dy}$

$\frac{du}{dy} = v^{-1/2} t^{-1/2} \frac{df}{dn} \cdots (5)$

from eq (5) we can write

$\frac{d^2u}{dy^2} = \frac{d}{dy}\left(\frac{du}{dy}\right)$

$= \frac{d}{dn}\left(\frac{df}{dn}\frac{dn}{dy}\right)\frac{dn}{dy}$

$= \frac{d}{dn}\left[v^{-1/2} t^{-1/2} \frac{df}{dn}\right] \times \frac{dn}{dy}$

$(\because$ from eq (2))

$$\frac{d^2u}{dy^2} = v^{-1}t^{-1}\frac{d^2f}{dn^2} \cdots (6)$$

By considering the equation, we know that

$$\frac{du}{dt} = v\frac{d^2u}{dy^2}$$

from eqn (4) & (6)

$$\Rightarrow -\frac{1}{2}t^{-1}n\frac{df}{dn} = t^{-1}\frac{d^2f}{dn^2}$$

$$\frac{d^2f}{dn^2} + \frac{1}{2}n\frac{df}{dn} = 0 \cdots (7)$$

$$f'' + \frac{1}{2}nf' = 0$$

let us assume that $f' = a$

$$a' = -\frac{1}{2}na$$

$$\Rightarrow \int\frac{a'}{a} = \int\frac{-n}{2}$$

$$\log(a) = \frac{-n^2}{4} + C_1$$

$$a = e^{-n^2/4}C_1 \quad (\because e^{C_1} \text{ is a constant})$$

$$f(n) = C_1\int_0^n e^{-n^2/4}dn$$

let $\frac{n}{2} = b$

$$\Rightarrow \frac{n^4}{4} = b^2$$

$$dn = 2db$$

$$f(n) = 2C_1\int_0^{2b} e^{-b^2}db$$

$$= C_2 \times \frac{2}{\sqrt{\pi}}\int_0^{2b} e^{-b^2}db$$

$$\boxed{f(n) = C_2 \times erf(n/2) + C_3}$$

And we all know that

$$u(y,t) = f(n)$$

$$n = \frac{y}{\sqrt{vt}}$$

$$u(y,t) = C_2 \times \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + C_3 \quad \text{-----}(8)$$

$* \; u(0,t) = U$

$$U = C_3 \; (\because \operatorname{erf}(0)=0) \quad \text{---}(9)$$

$* \; u(y,0) = 0$

$$0 = C_2 + C_3 \; (\because \operatorname{erf}(\infty) = 1) \quad \ldots(10)$$

$* \; u(\infty,t) = 0$

$$0 = C_2 + C_3 \quad \text{--}(11)$$

As we consider the equations 9, 10, 11 we get

$$\boxed{\begin{array}{l} C_3 = U \\ C_2 = -U \end{array}}$$

$$\therefore u(y,t) = U\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right] \quad \text{----}(12)$$

$* \; u(0,t) = 0$

$$\boxed{C_3 = 0} \quad \text{---}(13)$$

$* \; u(y,0) = u_0$

$$\boxed{u_0 = C_2 + C_3} \quad \text{------}(14)$$

$* \; u(\infty,t) = u_0$

$$\boxed{u_0 = C_2 + C_3} \quad \text{---}(15)$$

from the equations 13, 14, 15

$$C_3 = 0, \; C_2 = u_0$$

$$\therefore u(r,t) = u_0\left[\operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right] \quad \text{----}(16)$$

from the eq 12 & 16

The solution is

$$\boxed{u(r,t) = U\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right] + u_0\left[\operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]} \quad \text{-----}(17)$$

Sol, $u_0 = 15°C$

$u(r,t) = -5°C$

$y = 100\,cm$

$\nu = 0.0107\,cm^2/sec$

$u = 25°C$

$$\boxed{u(r,t) = u + \text{erf}\left[\frac{y}{2\sqrt{\nu t}}\right][u_0 - u]}$$

(∵ from eq 17 we write this eq as shown)

$$-5 = 25 + \text{erf}\left[\frac{50}{\sqrt{0.0107t}}\right][-10]$$

$$-30 = -10\,\text{erf}\left[\frac{50}{\sqrt{0.0107t}}\right]$$

$$3 = \text{erf}\left[\frac{50}{\sqrt{0.0107t}}\right]$$

$$\boxed{3 = \text{erf}\left[\frac{485}{\sqrt{t}}\right]}$$