Project - 2

Applications of Differential Equations

The fluid contained in a long pipe of circular cross-section is initially rest, and is set in motion by a difference between the pressures at the two ends of the pipe suddenly imposed and maintained by external mass. This pressure difference produces immediately a uniform axial pressure gradient "$-G$" says, throughout the fluid, and so the equation to be satisfied by the axial velocity "u" is

$$\frac{\partial u}{\partial t} = \frac{G}{\rho} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

in which "G" is a constant. The boundary and initial conditions are :

$u = 0$ at $r = a$ for all "t"

$u =$ finite at $r \to 0$ for all "t"

$u = 0$ at $t = 0$ for $0 \le r \le a$.

Sol:- we have a equation for axial velocity "u" which is a second order differential equation with variable coefficients.

i.e.,

$$\frac{\partial u}{\partial t} = \frac{G}{\rho} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

we know that in a steady state motion

$$\frac{\partial u}{\partial t} = 0$$

so, we have

$$0 = \frac{G}{\rho} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

$$\Rightarrow \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right] = -\frac{G}{\rho}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{-G}{\rho\nu}$$

multiply both sides with "r"

$$\Rightarrow r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{-rG}{\rho\nu} \quad \cdots\cdots ①$$

By observing eq ① we can that the differential equation is in the form of "Cauchy - Euler equation" with forcing function as "$-\frac{rG}{\rho\nu}$"

To solve these differential equation we need to find "C.F" and "P.I"

for homogenous solution

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = 0 \quad \cdots\cdots ②$$

Let us assume that

$$\boxed{u = r^m} \quad \cdots\cdots ③$$

substitute Eq ③ in Eq ②

$$r\left[m(m-1)\, r^{m-2}\right] + m\, r^{m-1} = 0$$

$$\Rightarrow m(m-1)\, r^{m-1} + m\, r^{m-1} = 0$$

$$\Rightarrow r^{m-1}\left[m^2 - \cancel{m} + \cancel{m}\right] = 0$$

$$\boxed{m = 0, 0}$$

In cauchy - euler equation when the two roots are equal ($m_1 = m_2$), then the general solution is

$$y(x) = (C_1 + C_2 \log|r|)\, r^{m_1}$$

$$= [C_1 + C_2 \log|r|]\, r^0$$

$$= C_1 + C_2 \log|r|$$

$$C.F = C_1 + C_2 \log|r|$$

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for P.I

let us assume that

$$\boxed{u_p = Ar^2} \quad \text{------ (4)}$$

substitute eq (4) in eq (1)

$$\Rightarrow r\frac{\partial^2}{\partial r^2}[Ar^2] + \frac{\partial}{\partial r}[Ar^2] = \frac{-G}{e\nu}r$$

$$\Rightarrow 8Ar + 2Ar = \frac{-G}{e\nu}r$$

$$\Rightarrow 4Ar = \frac{-G}{e\nu}r$$

$$\boxed{A = \frac{-G}{4e\nu}}$$

$$u_p = \frac{-G}{4e\nu}r^2$$

$\therefore$ The solution of differential equation is

$$u = u_c + u_p$$

$$\boxed{u = c_1 + c_2\log|r| - \frac{G}{4e\nu}r^2}$$

i) Show that the steady state solution $u = \frac{G}{4\mu}(a^2-r^2)$, where $\mu = e\nu$ kinematic viscosity. Hence, find total volume flux.

sol:- for steady state solution we can use intial and boudary conditions.

if we observe the solution. There is a singularity at $r=0$. if $c_2 \neq 0$; we can came across the singularity. so we can assume that "$c_2 = 0$".

So, we get

$$u = c_1 - \frac{G}{4e\nu}r^2$$

$\Rightarrow \quad u = c_1 - \frac{G}{4\mu} r^2 \quad (\because \mu = 0v)$

if we use a boundary condition. when r=a then u=0 which it given.

By substituting we get

$$0 = c_1 - \frac{G}{4\mu} a^2$$

$$c_1 = \frac{G}{4\mu} a^2$$

$\therefore$ The general solution

$$u = \frac{G}{4\mu} a^2 - \frac{G}{4\mu} r^2$$

$\boxed{u = \frac{G}{4\mu}[a^2 - r^2]}$ for steady state.

$\therefore$ Hence proved

$\therefore$ The total volume flux = the rate of volume flow across a unit area.

$$= \int_0^a u \, 2\pi r \, dr$$

$$= \int_0^a \frac{G}{4\mu}[a^2 - r^2] \, 2\pi r \, dr$$

$$= \left[\frac{G}{4\mu} a^2 r^2 \pi\right]_0^a - \left[\frac{G}{8\mu} r^4 \pi\right]_0^a$$

$$= \frac{G\pi}{4\mu} a^4 - \frac{G\pi}{8\mu} a^4$$

$$= \frac{G\pi a^4}{8\mu}$$

$\boxed{\therefore \text{Total volume flux} = \frac{G\pi a^4}{8\mu}}$

ii) we wish to find the solution of equation (1) with the conditions (2). From the physical considerations it is reasonable to write the solution of (1) as

$$u(r,t) = \underbrace{\frac{G}{4\mu}(a^2-r^2)}_{\text{(steady state solution)}} + \underbrace{w(r,t)}_{\text{(transient solution)}}$$

■ Show that the equation on $w(r,t)$ is

$$\frac{\partial w}{\partial t} = \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

Sol:- Given that

$$u(r,t) = \frac{G}{4\mu}(a^2-r^2) + w(r,t) \quad \cdots\cdots ⑤$$

$$\frac{\partial w}{\partial t} = \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

• Differentiate w.r.t "t" for eq ⑤ on both sides

$$\frac{\partial u}{\partial t} = 0 + \frac{\partial w}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} = \frac{G}{\rho} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]}$$

apply "$\frac{\partial}{\partial r}$" on both sides for eq ⑤

$$\frac{\partial u}{\partial r} = 0 - \frac{G(2r)}{4\mu} + \frac{\partial w}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{\partial u}{\partial r} + \frac{G r}{2\mu} \quad \cdots\cdots ⑥$$

apply "$\frac{\partial}{\partial r}$" on both sides for eq ⑥

$$\frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 u}{\partial r^2} + \frac{G}{2\mu}$$

Given that

$$\frac{\partial w}{\partial t} = \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

$$\frac{G}{\rho} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right] = \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{G}{2\mu} + \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

$$= \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{G}{2\mu} + \frac{1}{r}\left[\frac{\partial u}{\partial r} + \frac{Gr}{2\mu}\right]\right]$$

$$= \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{G}{2\mu} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{G}{2\mu}\right]$$

$$= \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{G}{\mu}\right]$$

$$= \frac{G}{\rho} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

$(\because \mu = \rho\nu)$

$$\boxed{\therefore L.H.S = R.H.S}$$

$$u(x,t) = \frac{G}{4\mu}(a^2 - r^2) + w(r,t)$$

for $w(a,t)$:-

$$u(a,t) = \frac{G}{4\mu}(a^2 - a^2) + w(a,t)$$

$$0 = 0 + w(a,t) \Rightarrow \boxed{w(a,t) = 0}$$

for $w(r,0)$:-

$$u(x,0) = \frac{G}{4\mu}(a^2 - r^2) + w(r,0)$$

$$0 = \frac{G}{4\mu}(a^2 - r^2) + w(r,0)$$

$$\boxed{w(r,0) = -\frac{G}{4\mu}(a^2 - r^2)}$$

for $w(r,t)$ as $r \to 0$

$$u(x,t) = \frac{G}{4\mu}(a^2 - 0) + w(r,t)$$

$$\text{finite} = \frac{G}{4\mu}a^2 + w(r,t)$$

$$\boxed{w(r,t) \Rightarrow \text{finite as } r \to 0}$$

6) Find the solution for $w(r,t)$

Sol: assume that

$$w(r,t) = R(r)\, T(t)$$

A particular solution of this equation which satisfies the boundary condition at $r=a$ is

By simplification

we get,

$$J_0\left(\lambda_n \frac{r}{a}\right) e^{\left(-\lambda_n^2 \frac{\nu t}{a^2}\right)}$$

where J_0 is the Bessel function of the 1^{st} kind of order zero and λ_n is one of the positive roots of $J_0(\lambda) = 0$. By using the whole set of these particular solutions, we can also satisfy the condition at $\boxed{t=0}$. Thus "w" is given by the "Fourier - Bessel" series

$$w(r,t) = \frac{G}{4\mu} \sum_{n=1}^{\infty} A_n J_0\left(\lambda_n \frac{r}{a}\right) e^{\left(-\lambda_n^2 \frac{\nu t}{a^2}\right)}$$

where the coefficients "A_n" are such as to satisfy

$$a^2 - r^2 = \sum_{n=1}^{\infty} A_n J_0\left(\lambda_n \frac{r}{a}\right)$$

i.e.,

$$A_n = \frac{2a^2}{J_1^2(\lambda_n)} \int_0^1 x(1-x^2)\, J_0(\lambda_n x)$$

$$= \frac{8a^3}{J_1(\lambda_n) \cdot \lambda_n^3}$$

$\therefore$ The velocity distribution is given by

$$u(r,t) = \frac{G}{4\mu}(a^2 - r^2) - \frac{2Ga^2}{\mu} \sum_{n=1}^{\infty} \frac{J_0\left(\lambda_n \frac{r}{a}\right)}{\lambda_n^3 J_1(\lambda_n)} \cdot e^{\left(-\lambda_n^2 \frac{\nu t}{a^2}\right)}$$

Final discussion, Initially the whole fluid has acceleration G/ρ, but as the velocity increases the restrain-ing influence of the wall spreads further into the fluid. The central portion of fluid whose velocity is increasing as Gt/ρ becomes narrower as "t" increases, until when "t" is of order "$\left(\frac{a^2}{\nu \lambda_1^2}\right)$" all parts of the fluid are subject to the effect of the wall and the velocity at "$r=0$" ceases to increase. As in previous case, the approach to the steady state is soon dominated by the 1^{st} term of the series in "$u(r,t)$".