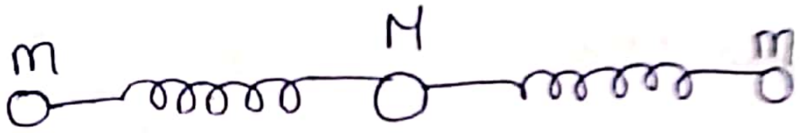
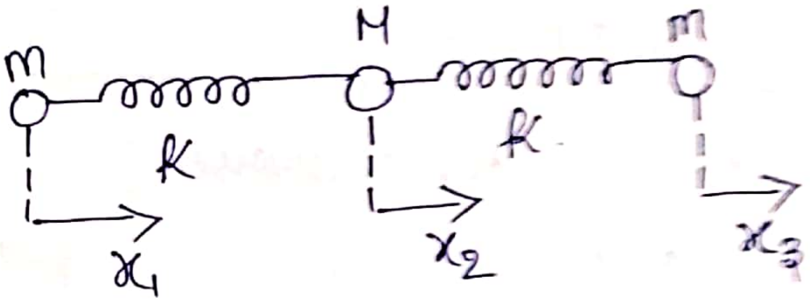
①



ⓐ Let x_1, x_2, x_3 be the displacements of the atoms from the equilibrium position at time t

From Newton's law & Hooke's law it follows



$$m x_1 = K(x_2 - x_1) \rightarrow ①$$

$$M x_2 = -K(x_2 - x_1) + K(x_3 - x_2) \rightarrow ②$$

$$m x_3 = -K(x_3 - x_2) \rightarrow ③$$

ⓑ By inserting the ansatz, into above equations

$$x_1 = a_1 \cos \omega t$$

$$x_2 = a_2 \cos \omega t$$

$$x_3 = a_3 \cos \omega t$$

hence, we get as follows

Let us consider the equation ①

$$m a_1 (\cancel{\cos \omega t}) = K(a_2 - a_1)(\cancel{\cos \omega t})$$

$$m a_1 = K a_2 - K a_1$$

$$-m a_1 + K a_2 - K a_1 = 0.$$

[∵ As we know that $\omega^2 = -1$]

$$\boxed{(m\omega^2 - K)a_1 + K a_2 = 0} \rightarrow ④$$

Let us consider equation (2)

$$Ma_2(\cancel{\cos\omega t}) = -K(a_2-a_1)\cancel{\cos\omega t} + K(a_3-a_2)\cancel{\cos\omega t}$$

$$Ma_2 = -Ka_2 + Ka_1 + Ka_3 - Ka_2$$

$$Ma_2 = Ka_1 - 2Ka_2 + Ka_3$$

$$Ka_1 - 2Ka_2 - Ma_2 + Ka_3 = 0.$$

$$\boxed{Ka_1 + (M\omega^2 - 2K)a_2 + Ka_3 = 0} \rightarrow (5).$$

Let us consider the equation (3)

$$m(a_3\cancel{\cos\omega t}) = -K(a_3-a_2)\cancel{\cos\omega t}$$

$$ma_3 = -Ka_3 + Ka_2$$

$$-Ka_3 + Ka_2 - ma_3 = 0.$$

$$\boxed{Ka_2 + (m\omega^2 - K)a_3 = 0} \rightarrow (6)$$

The Eigen frequencies of this system are obtained by setting the determinant of coefficents equal to zero.

$$\begin{vmatrix} m\omega^2 - K & K & 0 \\ K & M\omega^2 - 2K & K \\ 0 & K & m\omega^2 - K \end{vmatrix} = 0.$$

$$(m\omega^2 - K)\left[(M\omega^2 - 2K)(m\omega^2 - K) - K^2\right]$$
$$-K\left[K[m\omega^2 - K] - 0\right] + 0 = 0.$$

$$(m\omega^2-K)\left[(M\omega^2)(m\omega^2)-K(M\omega^2)-2K(m\omega^2)+2K^2-K^2\right]$$

$$-K^2\left[m\omega^2-K\right]=0.$$

$$(m\omega^2-K)\left[\omega^4 mM-KM\omega^2-2Km\omega^2+\cancel{2K^2}-\cancel{2K^2}\right]=0$$

$$(m\omega^2-K)\left[\omega^4 mM-KM\omega^2-2Km\omega^2\right]=0.$$

$$(m\omega^2-K)\left[\omega^4 Mm-\omega^2(KM+2Km)\right]=0.$$

$$\boxed{\omega^2(m\omega^2-K)\left[\omega^2 Mm-(KM+2Km)\right]=0} \rightarrow (7)$$

$\therefore$ The eigen vibrations of the system are

(1) $\boxed{\omega_1=0}$

(2) $(m\omega^2-K)=0.$

$$\omega^2=\frac{K}{m} \Rightarrow \omega=\sqrt{\frac{K}{m}}.$$

$$\boxed{\omega_2=\sqrt{\frac{K}{m}}}$$

(3) $\left[\omega^2 Mm-(KM+2Km)\right]=0.$

$$\omega^2 Mm=KM+2Km$$

$$\omega^2=\frac{K\cancel{M}}{\cancel{M}m}+\frac{2K\cancel{m}}{M\cancel{m}}$$

$$\omega=\sqrt{\frac{K}{m}+\frac{2K}{M}}$$

$$\omega=\sqrt{\frac{K}{m}\left(1+\frac{2m}{M}\right)}$$

$$\boxed{\omega_3=\sqrt{\frac{K}{m}\left(1+\frac{2m}{M}\right)}}$$

Discussion of the vibration modes:-

① Insertion of $\omega = \boxed{\omega_1 = 0}$ into the eq ①, ②, ③ we get

In eq ①.

$$(0-k)a_1 + ka_2 = 0.$$

$$ka_1 = ka_2$$

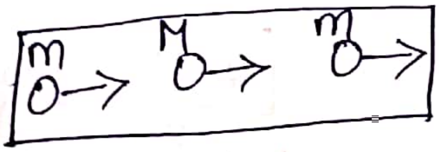
$$\boxed{a_1 = a_2}$$

In eq ③.

$$ka_2 + (0-k)a_3 = 0.$$

$$ka_2 = ka_3$$

$$\boxed{a_2 = a_3}$$



$$\therefore \quad a_1 = a_2 = a_3.$$

The eigen frequency $\omega_1 = 0$ does not correspond to a vibrational motion, but responds only a uniform translation of the entire molecule.

② Inserting $\omega = \boxed{\omega_2 = \sqrt{\frac{k}{m}}}$ into the eq ①, ②, ③ we get.

In eq ①.

$$\left(m\left(\frac{k}{m}\right) - k\right) a_1 + ka_2 = 0.$$

$$ka_2 = 0.$$

$$\therefore \boxed{a_2 = 0}$$

In EQ (2)

$$Ka_1 + \left(M\left(\frac{k}{m}\right) - 2k\right)a_2 + ka_3 = 0.$$

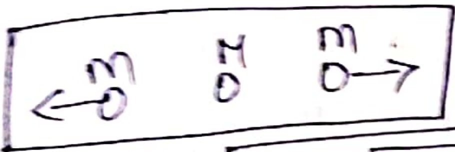
As we know $a_2 = 0$.

$$ka_1 + ka_3 = 0.$$

$$\boxed{a_1 = -a_3}$$

$\therefore a_1 = -a_3$ & $a_2 = 0$.

The central atom is at rest, while the outer atoms vibrate against each other.



(3) Inserting $\omega = \boxed{\omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}}$ into the EQ (1), (2), (3) we get

In EQ (1).

$$\left[\cancel{m}\left(\frac{k}{\cancel{m}}\left(1+\frac{2m}{M}\right)\right) - k\right]a_1 + ka_2 = 0$$

$$\left[\cancel{k} + \frac{2km}{M} - \cancel{k}\right]a_1 + ka_2 = 0.$$

$$\cancel{k}a_2 = -\frac{2\cancel{k}m}{M}a_1$$

$$\boxed{a_2 = -\left(\frac{2m}{M}\right)a_1}$$

In eq (3).

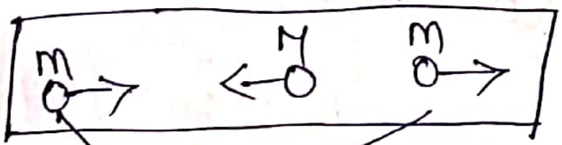
$$K\left[-\left(\frac{2m}{M}\right)a_1\right] + \left[m\left(\frac{K}{m}\left(1+\frac{2m}{M}\right)\right) - K\right]a_3 = 0.$$

$$-\frac{2Km}{M}a_1 + \left[\cancel{K} + \frac{2mK}{M} - \cancel{K}\right]a_3 = 0.$$

$$a_3\left(\frac{\cancel{2Km}}{\cancel{M}}\right) = a_1\left(\frac{\cancel{2Km}}{\cancel{M}}\right)$$

$$\boxed{a_3 = a_1}$$

The outer atoms vibrate in phase, while the central atom vibrates with the opposite phase and with another amplitude.



outer atoms are in phase.