DM Assignment-1

1. i) Find a recurrence relation for the number of bit strings of length 'n' that contain the string 01.
ii) What are the inital conditions? iii) How many bit strings of length seven contain the string 01?

Sol:- (i) Let a_n represent the number of bit strings of length 'n' that contain the string 01.

First case: The bit string is a sequence beginning with 1, then the bit string of length $n-1$ has a_{n-1} possible strings that contain the string 01.

second case: The bit string is a sequence starting with k zeros allowed by a 1, there are 2^{n-k-1} bit strings of length $n-k-1$ and thus there are 2^{n-k-1} bit strings starting with k zeros followed by a 1 and then followed by a bit string of length $n-k-1$

Third case: The bit string is a sequence ending in 01. There are 2^{n-2} bit strings of length $n-2$ and thus there are 2^{n-2} bit strings of length $n-2$ followed by 01.

Adding the numbers of sequence of all three cases we obtain:

$$a_n = a_{n-1} + \sum_{k=1}^{n-1} 2^{n-k-1} = a_{n-1} + \sum_{k=0}^{n-2} 2^k = a_{n-1} + 2^{n-1} - 1$$

ii) when $n=0$, there are exactly 0 bit string containg 01.

$$\boxed{a_0 = 0}$$

when $n=1$, there are exactly 0 bit strings containing 01

(since the string contains only 1 element)

$$\boxed{a_1 = 0}$$

when $n=2$ there are exactly 1 bit string containing 01.

when we use the recurence relation in part (a). we note that $a_2 = a_1 + 2^{2-1} - 1 = 0+2-1 = 1$. Thus the recurrence relation holds for $n=2$ and thus $n=2$ does not need to be an initial condition.

iii) Let us use the recurrence relation $a_0 = 0, a_1 = 0$, $a_n = a_n = a_{n-1} + 2^{n-1} - 1$ to derive a_7:

$a_0 = 0$

$a_1 = 0$

$a_2 = a_2 + 2^{2-1} - 1 = 0+2-1 = 1$

$a_3 = a_2 + 2^{3-1} - 1 = 1+4-1 = 4$

$a_4 = a_3 + 2^{4-1} - 1 = 4+8-1 = 11$

$a_5 = a_4 + 2^{5-1} - 1 = 11+16-1 = 26$

$a_6 = a_5 + 2^{6-1} - 1 = 26+32-1 = 57$

$a_7 = a_6 + 2^{7-1} = 57+64-1 = 120$

$$\boxed{\therefore 120 \text{ bit strings}}$$

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2. A nuclear reactor has created 18 grams of a particular radioactive isotope. Every hour 1% of this radioactive isotope decays. a) set up a recurrence relation for the amount of this isotope left 'n' hours after its creation. b) What are the initial conditions for the recurrence relation. c) Solve this recurrence relation.

Sol: a) Let a_n be the amount of the isotope left 'n' hours after its creation.

At the nth hour, 1% of the a_{n-1} of isotope will have been decayed and thus the amount of the isotope a_{n-1} is decreased by 1%.

$$a_n = a_{n-1} - 1\% \times a_{n-1} = a_{n-1} - 0.01a_{n-1} = 0.99a_{n-1}$$

$$\boxed{a_n = 0.99a_{n-1}}$$

b) Initially, after 0 hours, the isotope is 18 grams.

$$\boxed{a_0 = 18}$$

(c) Repeatedly apply the recurrence relation

$$a_n = 0.99a_{n-1}$$
$$= 0.99^2 a_{n-2}$$
$$= 0.99^3 a_{n-3}$$
$$\cdots$$
$$= 0.99^{n-1} a_1$$
$$= 0.99^n a_0$$
$$= 0.99^n (18)$$
$$\boxed{a_n = 18 \cdot 0.99^n}$$

3. Find a recurrence relation for the number of ways to climb 'n' stairs if the person climbing the stairs can take one stairs or two stairs at a time. what are the initial conditions? In how many ways can this person climb a flight of eight stairs? Find an explicit expression for climbing n-th stair.

Sol:

let a_n be the number of ways to climb 'n' stairs. In order to climb 'n' stairs, a person must either start with a step of one stair and then climb $n-1$ stairs or else start with a step of two stairs and then climb $n-2$ stairs.

from this analysis we can immediately write down the recurrence relation.

Valid for all $n \geq 2$:

$$\boxed{a_n = a_{n-1} + a_{n-2}}$$

The initial conditions are $a_0 = 1$, $a_1 = 1$. since there is one way to climb no stairs and clearly only one way to climb one stair. Note that the recurrence relation is the same as that for the Fibonacci sequence, and the initial conditions are that $a_0 = f_1$ and $a_1 = f_2$, so it must be that

$a_n = f_{n+1}$ for all n.

$\therefore \boxed{a_0 = 1} \boxed{a_1 = 1}$

Each term in our sequence $\{a_n\}$ is the sum of the previous two terms, so the sequence begins $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, $a_6 = 13$, $a_7 = 21$, $a_8 = 34$. Thus a person can climb a flight of 8 stairs in 34 ways under the restriction in the problem

4. A new employee at an exciting new software company starts with a salary of \$ 50,000 and is promised that at the end of each year her salary will be double her salary of the previous year, with an extra increment of \$ 10,000 for each year she has been with company. Construct a recurrence relation for her salary for her nth year of employment. Solve this recurrence relation to find her salary for her nth year of employment.

sol:- Derivation recurrence relation

let a_n represents the salary at the nth year

First case: The salary doubles in comparison to the previous year. The salary of the previous year is a_{n-1}.

$$2a_{n-1}$$

Second case: An extra 10000 dollors is added to the salary for each of the $n-1$ years that she has been with the company.

$$10000(n-1)$$

Add all number of pairs for each case:

$$a_n = 2a_{n-1} + 10000(n-1)$$

initial conditions

At the beginning of the first year, the salary is \$ 50000

$$\boxed{a_1 = 50000}$$

The particular solution needs to satisfy the recurrence relation:

$$a_n = 2a_{n-1} + 10000(n-1)$$

$$P_1 n + P_0 = 2(P_1(n-1) + P_0) + 10000(n-1)$$

$$P_1 n + P_0 = 2P_1 n - 2P_1 + 2P_0 + 10000n - 10000$$

$$0 = (P_1 + 10000)n + (-2P_1 + P_0 - 10000)$$

All coefficients then need to be zero

$$P_1 + 10000 = 0$$

$$-2P_1 + P_0 - 10000 = 0$$

$$P_1 = -10000$$

$$P_0 = 10000 + 2P_1 = 10000 + 2(-10000) = -10000$$

The particular solution then becomes:

$$a_n^{(P)} = P_1 n + P_0 = -10000n - 10000$$

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$= \alpha \cdot 2^n - 10000n - 10000$$

Evaluate $a_n = \alpha \cdot 2^n - 10000n - 10000$ at $n = 1$.

$$50000 = a_1 = 2\alpha - 10000 - 10000$$

$$50000 = 2\alpha - 20000$$

$$70000 = 2\alpha$$

$$35000 = \alpha$$

The solution of the nonhomogeneous linear recurrence relation then becomes

$$a_n = \alpha \cdot 2^n - 10000n - 10000$$

$$a_n = 35000 \cdot 2^n - 10000n - 10000$$

$$\boxed{a_n = 70000 \cdot 2^{n-1} - 10000n - 10000}$$

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