3/6/20

Maths Assignment

1. Find the difference Equation satisfying by

(i) $y_n = A3^n + B5^n$ (ii) $y_x = (A+Bx)2^x$.

Sol: (i) let the difference Equation be

$$y_{n+2} + C_1 y_{n+1} + C_2 y_n = 0$$

let $y_n = \alpha^n$ be the trial solution.

$\therefore$ The Equation becomes

$$\alpha^{n+2} + C_1 \alpha^{n+1} + C_2 \alpha^n = 0$$

$$\alpha^n(\alpha^2 + C_1\alpha + C_2) = 0$$

$$\alpha^n \neq 0.$$

$$\Rightarrow \alpha^2 + C_1\alpha + C_2 = 0.$$

From the given solution form, the roots of the Equation are 3 and 5 substituting $\alpha_1 = 3$ and $\alpha_2 = 5$ in the Equation.

$$9 + 3C_1 + C_2 = 0 \quad -① $$

$$25 + 5C_1 + C_2 = 0 \quad - ②$$

① - ②

$$-16 - 2C_1 = 0 \Rightarrow \boxed{C_1 = -8}$$

$$9 - 24 + C_2 = 0 \Rightarrow \boxed{C_2 = 15}$$

$\therefore$ The difference Equation is,

$$y_{n+2} - 8y_{n+1} + 15y_n = 0$$

(ii) let the differential Equation be

$$y_{x+2} + c_1 y_{x+1} + c_2 y_x = 0$$

let $y_x = r^x$ be the trial solutions.

Now, the Equation becomes

$$r^{x+2} + c_1 r^{x+1} + c_2 r^x = 0$$

$$r^x(r^2 + c_1 r + c_2) = 0$$

r^x cannot be zero.

So, $r^2 + c_1 r + c_2 = 0.$

From the given solution form, the Equation has Equal root and that root is 2.

The roots of Equation are $r_1 = 2$ and $r_2 = 2$.

So, by sum and product of root,

$$c_1 = -(r_1 + r_2) = -4$$

$$c_2 = r_1 \times r_2 = 4$$

$\therefore$ The difference Equation is

$$y_{x+2} - 4y_{x+1} + 4y_x = 0.$$

2. Solve the following difference equation.

(i) $(\Delta^2 - 3\Delta + 2) y_n = 0.$

(ii) $f(x+3) - 3f(x+1) - 2f(x) = 0$

Sol) (i) The differential Equation can be written as

$$y_{n+2} - 3y_{n+1} + 2y_n = 0.$$

Then, let $y_n = \alpha^n$ be the trial solution.

On substituting,

$$\alpha^{n+2} - 3\alpha^{n+1} + 2\alpha^n = 0$$

$$\alpha^n(\alpha^2 - 3\alpha + 2) = 0$$

α^n cannot be zero.

So, $\alpha^2 - 3\alpha + 2 = 0$

$$\alpha^2 - 2\alpha - \alpha + 2 = 0$$

$$(\alpha - 1)(\alpha - 2) = 0$$

$$\alpha = 1, 2.$$

∴ The general solution for the given differencial Equation is of the form,

$$y_n = C_1\alpha_1^n + C_2\alpha_2^n$$

where $\alpha_1 = 1,\ \alpha_2 = 2$

$$\therefore y_n = C_1(1)^n + C_2(2)^n$$

$$y_n = C_1 + C_2 2^n$$

(ii) The given differential Equation can be written as

$$y_{x+3} - 3y_{x+1} - 2y_x = 0.$$

let $y_x = \alpha^x$ be the trial solution.

Then, Equation can be

$$\alpha^{x+3} - 3\alpha^{x+1} - 2\alpha^x = 0$$

$$\alpha^x(\alpha^3 - 3\alpha - 2) = 0$$

$$\alpha^2 \neq 0.$$

$$\alpha^3 - 3\alpha - 2 = 0$$

Clearly -1 is a root of the Equation by trial and Error. So, reducing the equation to second degree

$$\begin{array}{r|rrrr} & 1 & 0 & -3 & 2 \\ -1 & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$\alpha^2 - \alpha - 2 = 0$$

$$\alpha^2 - 2\alpha + \alpha - 2 = 0$$

$$\alpha(\alpha - 2) + 1(\alpha - 2) = 0$$

$$(\alpha + 1)(\alpha - 2) = 0$$

$\therefore$ Roots are $\alpha_1 = -1$, $\alpha_2 = -1$ and $\alpha_3 = 2$.

$\therefore$ The general solution is

$$y_x = (C_1 + C_2 x)(-1)^x + C_3(2)^x.$$

3. Solve the following difference Equations

(i) $u_{p+2} - 2\cos\frac{1}{2}\, u_{p+1} + u_p = \sin\frac{p}{2}$

(ii) $(E^2 - 5E + 6)y_n = 4^k(k^2 - k + 5)$

Sol: (i) The Equation in symbolic form is

$$(E^2 - 2\cos\tfrac{1}{2}\cdot E + 1)u_p = \sin\tfrac{p}{2}.$$

The auxiliary Equation is

$$E^2 - 2\cos\tfrac{1}{2}\, E + 1 = 0.$$

$$E = \frac{2\cos\frac{1}{2} \pm \sqrt{4\cos^2\frac{1}{2} - 4}}{2} = \cos\tfrac{1}{2} \pm i\sin\tfrac{1}{2}.$$

The solution for the Homogenious part is

$$u_h = C_1\cos\tfrac{p}{2} + C_2\sin\tfrac{p}{2}.$$

For non-homogenious part,

$$u_{nh} = \frac{1}{E^2 - 2E\cos\frac{1}{2} + 1}\sin\tfrac{p}{2}$$

$$= \frac{1}{E^2 - 2E\cos\frac{1}{2} + 1}\left(\frac{e^{ip/2} - e^{ip/2}}{2}\right)$$

$$= \frac{1}{E^2 - E(e^{ip/2} + e^{-ip/2}) + 1}\left(\frac{e^{ip/2} - e^{-ip/2}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{e^{ip/2}}{(E - e^{i/2})(E - e^{-i/2})} - \frac{e^{-ip/2}}{(E - e^{i/2})(E - e^{-i/2})}\right]$$

$$= \frac{1}{2}\left[\frac{e^{ip/2}}{(E - e^{i/2})(E - e^{-i/2})} - \frac{e^{-ip/2}}{(E - e^{i/2})(E - e^{-i/2})}\right]$$

Put $E = e^{-i/2}$

$$= \frac{1}{2}\left(\frac{e^{iP/2}}{(E - e^{i/2})(e^{i/2} - e^{-i/2})} - \frac{e^{-iP/2}}{(E - e^{-i/2})(e^{-i/2} - e^{i/2})}\right)$$

$$= \frac{1}{4i\sin \frac{1}{2}}\left[\frac{e^{iP/2}}{E - e^{i/2}} + \frac{e^{-iP/2}}{E - e^{-i/2}}\right]$$

$$= \frac{1}{4i\sin \frac{1}{2}}\left[P e^{i/2(P-1)} + P e^{-i/2(P-1)}\right]$$

$$= \frac{P}{2\sin \frac{1}{2}}\left[\frac{e^{i/2(P-1)} + e^{-i/2(P-1)}}{2i}\right]$$

$$y_{h} = \frac{P(\cos(\frac{P-1}{2}))}{2\sin \frac{1}{2}}$$

$\therefore$ The general solution of the difference Equation is

$$u_p = u_n + u_{nh}$$

$$u_p = C_1 \cos P/_2 + C_2 \sin P/_2 + \frac{P\sin(\frac{n-1}{2})}{2\sin \frac{1}{2}}$$

(ii) The difference Equation is re-written as

$$y_{n+2} - 5y_{n+1} + 6y_n = 4^k(k^2 - k + 5)$$

let $y_n = \alpha^n$ be trial solution for the homogenious part.

$$\alpha^{n+2} - 5\alpha^{n+1} + 6\alpha^n = 0$$

$$\alpha^n(\alpha^2 - 5\alpha + 6) = 0$$

$$\alpha^n \neq 0$$

$$\therefore \alpha^2 - 5\alpha + 6 = 0.$$

$$\alpha = 2, 3.$$

The solution for homogenous part is

$$y_n : C_1 (2)^n + C_2 (3)^n.$$

for non-homogenous part,

let the solution be

$$y_n = 4^k (ak^2 + bk + c).$$

on substituting,

$$4^{k+2}(a(k+2)^2 + b(k+2) + c) - 5.4^{k+1}(a(k+1)^2 + b(k+1) + c) +$$
$$6.4^k(a(k^2) + bk + c) : 4^k(k^2 - k - 5)$$

$$16a(k+2)^2 + 16b(k+1) + 16c - 20(a(k+1)^2 + b(k+1) + c) + 6ak^2 +$$
$$6bk + 6c : k^2 - k + 5.$$

Comparing k^2 terms.

$$16ak^2 - 20ak^2 + 6ak^2 = k^2.$$

$$2a : 1$$
$$\Rightarrow \boxed{a : 1/2}$$

Comparing constant terms.

$$64a + 32b + 16c - 20a - 20b - 20c + 6c : 5.$$

$$40a + 12b + 2c : 5.$$

$$\because a : 1/2$$

$$12b + 2c = -15 - (1).$$

Comparing k terms.

$64a + 16b - 40a - 20b + 6b = -1.$

$24a + 2b = -1$ — ②

$2b + 12 = -1 \quad (\because a = 1/2)$

$2b = -13$

$\boxed{b = -13/2}$

On substituting in ①

$\boxed{c = 63/2}$

$\therefore$ The general solution of the Equation is

$$y_n = C_1(2)^n + C_2(3)^n + 4^k\left(\frac{k^2 - 13k + 63}{2}\right)$$

4) A beam of length l, supported at n points carries uniform load w per unit length. The bending moments $M_1, M_2, M_3, \ldots, M_n$ at the supports satisfy the Clapeyron's Equation:

$$M_{r+2} + 4M_{r+1} + M_r = -\tfrac{1}{2}wt^2.$$

If a beam weighing 30 kg is supported by at its ends at two other supports dividing the beam into three Equal parts of 1 metre length, show that the bending moment at each of the two middle supports is 1 kg metre.

Sol:- let $M_r = \alpha^r$ be the solution for homogenious part:

$$\alpha^{r+2} + 4\alpha^{r+1} + \alpha^r = 0$$

$$\alpha^r(\alpha^2 + 4\alpha + 1) = 0$$

$$\alpha^r \neq 0$$

$$\therefore \alpha = \frac{-4 \pm \sqrt{12}}{2}$$

$$\boxed{\alpha = -2 \pm \sqrt{3}}$$

Solution is

$$M_h = C_1(-2+\sqrt{3})^r + C_2(-2-\sqrt{3})^r$$

let the solution for non-homogenious part be

$$M_r = al^2 + bl + c.$$

So,
$$a(l+2)^2 + b(l+2) + c + 4[a(l+1)^2 + b(l+1) + c] + al^2 + blc$$
$$= -\tfrac{1}{2}\omega l^2.$$

On comparing

$\boxed{a = -\omega/12}$ $\boxed{b = \omega/6}$

Constants $\boxed{c = -\omega/18}$

$$\therefore M_{nh} = -\omega/12\, l^2 + \omega/6\, l - \omega/18.$$

The general solution is

$$M_r = M_h + M_{nh}$$

$$M_r = C_1(-2+\sqrt{3})^r + C_2(-2-\sqrt{3})^r.$$
$$-\omega/12\, l^2 + \omega/6\, l - \omega/18.$$

When $w = 30$ kgs, $l = 1m$.

The tree parts are M_1, M_2, M_3.

Middle part is neglecting constants.

$$M_2 = (-0.27)^2 + (-3.73)^2 + 5/6.$$

$$M_2 \simeq 14.4$$

Bending moments for 2 parts is

$30/14.4$ kg meters.

$\simeq 2$ kg meters.

Bending moment for Each part is 1 kg meter.

Hence proved.

(5) Solve the simultanious difference Equations.

$$u_{n+1} + v_n - 3u_n = n, \quad 3u_n + v_{n+1} - 5v_n = 4^n.$$

Sol:- The given Equation is of the form

$$(E-3)u_n + v_n = n \quad -(1).$$

$$3u_n + (E-5).v_n = 4^n \quad (2)$$

$$(1)(E-5) - (2).$$

$$[(E-3)(E-5)-3]u_n = n(E-5) - 4^n$$

$$[E^2 - 8E + 12]u_n = 1 - 4n - 4^n$$

$u_{n+2} - 8u_{n+1} + 12u_n = 1 - 4n - 4^n$.

This is of the form of normal difference Equation.

let $u_n = \alpha^n$ be solution for non homogenious part.

$\alpha^{n+2} + 12\alpha^n - 8\alpha^{n+1} = 0$

$\alpha^n(\alpha-2)(\alpha-6) = 0$

$\alpha^n \neq 0$.

$\Rightarrow y_n = C_1 2^n + C_2 6^n$

for the non homogenious part.

let the solution be

$y_{nh} = a + bn + c4^n$

$a + b(n+2) + c4^{n+2} + a + b(n+1) + c4^{n+1} + a + bn + c4^n$

$= 1 - 4n + 4^n$.

$\Rightarrow \boxed{a = -19/25}$ $\boxed{b = -4/5}$ $\boxed{c = 1/4}$

$\boxed{u_n = C_1 2^n + C_2 6^n - 4/5 n - 19/25 + 4^n/4}$

Substituting u_n in ①

$v_{n+1} - 3u_n + v_n = n$

~~v_n is~~ $n - 4_{n+1}\ 3 4_n$

$v_n = n - (C_1 \cdot 2^{n+1} + C_2 6^{n+1} - 4/5(n+1) - 19/25 + 4^{n+1}/4) + 3(C_1 2^n + C_2 6^n - 4n/5 - 19/25 + 4^n/4)$

$$\boxed{y_n = c_1 2^n - 3c_2 6^n - 3/5\, n - 34/25 - 4^n/4}$$

(6) Determine the z transform of the Sequence.

(i) $\{x_n\} = 3^n$ (ii) $\{x_n\} = n^2 a^n$.

(iii) $\{1,0,1,0,1,0,\ldots\}$ (iv) $4^{n+3} - 2a^n$.

(v) $\{1,1,0,0,0,1,1\}$

Sol:- (i) $z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

Here, $u_n = 3^n$

$$\therefore z(3^n) = \sum_{n=0}^{\infty} 3^n \cdot z^{-n}$$

$$= 1 + 3 \cdot z^{-1} + 3 \cdot z^{-2} + \ldots$$

$$= 1 + 3/z + (3/z)^2 + \ldots$$

$$= 1/1-3/z$$

$$\therefore z(3^n) = z/z-3.$$

(ii) $z(x_n) = z(n^2 a^n)$

$$z(n^2 a^n) = -z\, d/dz\, (z(n \cdot a^n))$$

$$= -z\frac{d}{dz}\left[\frac{-zd}{dz}(z(a^n))\right]$$

$$= \frac{-zd}{dz}\left[-z \cdot \frac{d}{dz}\left(\frac{z}{z-a}\right)\right]$$

$$= -z\, d/dz\left[-z\left(\frac{(z-a)-z}{(z-a)^2}\right)\right]$$

$$= -z\frac{d}{dz}\left(\frac{az}{(z-a)^2}\right)$$

$$= -z\left(\frac{(z-a)^2 a - az \cdot 2(z-a)}{(z-a)^2}\right)$$

$$= -z\left(\frac{-az - a^2}{(z-a)^2}\right)$$

$$z(a^n \cdot n^2) = \frac{az^2 + az}{(z-a)^3}.$$

(iii) $z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

From the given,

$u_0, u_2, u_4, \dots = 1.$

$u_1, u_3, u_5, \dots = 0$

$\therefore$ The z-transform can be written as

$$z(u_n) = u_0 z^{-0} + u_1 z^{-1} + u_2 z^{-2} + \dots$$

$$= 1 + 1/z^2 + 1/z^4 + \dots$$

$$\therefore z(u_n) = 1/1-\frac{1}{z^2} = z^2/z^2-1 = z^2/(z-1)(z+1)$$

(iv) Z transform for given is,

$$z(4^{n+3} - 2a^n) = z(64 \cdot 4^n - 2 \cdot a^n)$$

$$= 64z(4^n) - 2z(a^n)$$

$$= 64 \cdot (z/z-4) - 2 \cdot (z/z-a)$$

$$= 64z/z-4 - 2z/z-a.$$

(v) The z-transform for given sequence is,

~~$z\{4^{n+3} - 3\cdot 2^n\}$~~ =

$z(u_n) = u_0 z^{-0} + u_1 z^{-1} + \dots$

But we have $u_0, u_1, \dots$

on substituting

$1 + 1/z + 1/z^5 + 1/z^6$

$$z(u_n) = \frac{z^6 + z^5 + z + 1}{z^6}$$

⑦ Find the z-transform of

(i) $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$. (ii) $\delta(n+1)$

(iii) $a^n \sin(n\theta)$ (iv) $\{x_n\} = \cosh n\theta$.

(v) $\{x_n\} = a^n/n!$

Sol: (i) $z\left[\cos\left(\frac{n\pi}{2} + \pi/4\right)\right]$

$$= z\left(\cos\frac{n\pi}{2} \cos\frac{\pi}{4} - \sin\frac{n\pi}{4} \sin\frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{4} \cdot z\left(\cos\frac{n\pi}{2}\right) - \sin\frac{\pi}{4} \cdot z\left(\sin\frac{n\pi}{2}\right)$$

$$= 1/\sqrt{2}\left[\frac{z(z - \cos\pi/2)}{z^2 - 2z\cos\pi/2 + 1} - \frac{z\sin\pi/2}{z^2 - 2z\cos\pi/2 + 1}\right]$$

$$= 1/\sqrt{2}\left(z^2/z^2+1 - z/z^2+1\right]$$

$$= \frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

(ii) $z(\delta(n+1)] = \sum_{n=0}^{\infty} \delta(n+1) \cdot z^{-n}$

$= \delta(1) \cdot z^{-0} + \delta(2) \cdot z^{-1} + \dots$ $\left(\because \delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases} \right.$

$= 0 + 0 + \dots$ and $\left. \delta(n+1) = \begin{cases} 1 & n=-1 \\ 0 & n \neq -1 \end{cases} \right)$

$= 0$

$\therefore z(\delta(n+1)] = 0$

(iii) $z(\sin(n\theta)) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

$$z(a^n \sin(n\theta)) = z((a^{-1})^{-n} \cdot \sin(n\theta))$$

$$= \frac{(a^{-1}z)\sin\theta}{(a^{-1}z)^2 - 2(a^{-1}z)\cos\theta + 1}$$

$$\therefore Z(a^n \sin(n\theta)) = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

(iv) $z(\cosh n\theta) = z\left(\frac{e^{n\theta} + e^{-n\theta}}{2}\right)$

$$= \frac{1}{2}\left[z\{(e^{-\theta})^{-n} \cdot 1\} + z\{(e^{\theta})^{-n} \cdot 1\}\right]$$

let $u_n = 1$

$= \frac{1}{2}\left[\frac{z \cdot e^{-\theta}}{z \cdot e^{-\theta} - 1} + \frac{ze^{\theta}}{ze^{\theta} - 1}\right]$ $(\because z(1) = z/(z-1))$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{\theta} + e^{-\theta})}{z^2 - 2(e^{\theta} + e^{-\theta}) + 1}\right]$$

$$z(\cosh n\theta) = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

(vi) $z\left(a^n/n!\right) = \sum_{n=0}^{\infty} a^n/n! \cdot z^{-n}$

$= 1 + a/1!\, z^{-1} + a^2/2!\, z^{-2} + \dots$

We know that,

$e^x = 1 + x/1! + x^2/2! + \dots$

$\therefore z\left(a^n/n!\right) = e^{a/z}.$

(8) The continious-time signal $f(t) = e^{-2t}$, where ω is a real constant, is sampled when $t \geq 0$ at intervals T. Write down the general term of the sequence of samples, and calculate the z transform of the sequence.

Sol:. Given $f(t) = e^{-2t}$, $t \geq 0$.

The sample values of sequence is

$1, e^{-2t}, e^{-4t}, \dots e^{-2nt}.$

$f(nt) = e^{-2nt} = (e^{-2t})^n.$

$z(a^n) = z/z-a.$

$\therefore z((e^{-2t})^n) = z/z-e^{-2t} = 1/1-e^{-2t}\cdot z^{-1}$

Scanned with CamScanner

9) The causal sequence x_k is generated by $x_k = (1/2)^k$ $(k \geq 0)$. Determine the z transform of shifted sequence x_{k-2}

Sol: Shifting x_k to right,

If $z(x_k) = U(z)$, then $z(x_{k-1}) = z^{-1}U(z)$

Z-transform of $z(x_k) = \sum_{k=0}^{\infty} (1/2)^k z^{-k}$.

$= 1 + \frac{1/2}{z} + \frac{(1/2)^2}{z^2} + \dots$

$= 1 + 1/2z + (1/2z)^2 + \dots$

$= 1/1 - 1/2z$

$z(x_k) = 2z/2z-1$

$\therefore z(x_{k-2}) = z^{-2} \cdot z(x_k)$

$= 1/z^2 \cdot 2z/2z-1$

$z(x_{k-2}) = 2/2z^2-1$.

10) Show that

$$z\{\sin(k\omega t)\} = \frac{z\sin(\omega t)}{z^2 - 2z\cos(\omega t)+1}$$

Sol: Consider $f(t) = \sin(\omega t)$

$F(z, m) = z\{\sin(\omega(kt+m))\}$

$= z\{\sin(\omega kt)\cos(\omega m) + \sin(\omega m)\cos(\omega kt)\}$

$= \cos(\omega m) = z\{\sin(k\omega t)\} + \sin(\omega m) z\{\cos(k\omega t)\}$

$$= \frac{\cos(\omega m)\, z \sin(\omega t)}{z^2 - 2z\cos(\omega t) + 1} + \frac{\sin(\omega m) \cdot z(z - \cos(\omega t))}{z^2 - 2z\cos(\omega t) + 1}$$

$$= \frac{z \sin(\omega t + m)}{z^2 - 2z\cos(\omega t) + 1}$$

let $m = 0$.

$$\therefore z(\sin(k\omega t)) = \frac{z \sin(\omega t)}{z^2 - 2z\cos(\omega t) + 1}$$

Hence proved.

11) By first resolving $Y(z)/z$ into partial fractions, find $z^{-1}[Y(z)]$ when $Y(z)$ is $z^2/(2z+1)(z-1)$

Sol: $Y(z) = z^2/(2z+1)(z-1)$

$$\frac{Y(z)}{z} = z/(2z+1)(z-1)$$

$$= A/2z+1 + B/z-1$$

for A and B,

$Az - A + 2Bz + B = z$

$A + 2B = 1$

$-A + B = 0$

$\boxed{B = 1/3}$ $\boxed{A = 1/3}$

$$\frac{Y(z)}{z} = \frac{1}{3(2z+1)} + \frac{1}{3(z-1)}$$

$$Y(z) = \frac{1 \cdot z}{6(z-(-\frac{1}{2}))} + \frac{1 \cdot z}{3(z-1)}$$

On inversion,

$$z^{-1}[Y(z)] = \frac{1}{6}\left(-\frac{1}{2}\right)^n + \frac{1}{3} \cdot (1)^n.$$

12) Using Z-transformation methods, resolve the following differential Equations.

a) $6y_{k+2} + y_{k+1} - y_k = 3$, $y_0 = y_1 = 0$

b) $2y_{n+2} - 3y_{n+1} - 2y_n = 6n+1$, $y_0 = 1, y_1 = 2$.

Sol:- a) let $y_k = \alpha^k$ be the solution part for homogenious part.

So, $6\alpha^{k+2} + \alpha^{k+1} - \alpha^k = 0$

$$\alpha^k(6\alpha^2 + \alpha - 1) = 0$$

$$\alpha^k \neq 0.$$

$$\therefore 6\alpha^2 + \alpha - 1 = 0$$

$$6\alpha^2 + 3\alpha - 2\alpha - 1 = 0$$

$$3\alpha(2\alpha+1) - 1(2\alpha+1) = 0$$

$$(3\alpha - 1)(2\alpha+1) = 0$$

$$\boxed{\alpha = \frac{1}{3}, -\frac{1}{2}}$$

$$y_n = C_1\left(\frac{1}{3}\right)^k + C_2\left(-\frac{1}{2}\right)^k.$$

for non-homogenious part,

let $y_k = k$,

$$6(k+2) + k+1 - k = 3.$$

$$6k+13 = 3$$

$$6k = -10$$

$$\boxed{k = -5/3}$$

$$y_{nh} = -5/3,$$

The general solution is

$$y_k = y_n + y_{nh}.$$

$$y_k = C_1(1/3)^k + C_2(-1/2)^k - 5/3.$$

Substituting $y_0 = 0$, $y_1 = 0$.

$$0 = C_1 + C_2 - 5/3.$$

$$C_1 + C_2 = 5/3 \quad -①.$$

$$C_1/3 - C_2/2 = 5/3 \quad -②$$

$① \times 1/2 + ②$.

$$C_1/2 + \cancel{C_2/2} + C_1/3 - \cancel{C_2/2} = 10/3.$$

$$5C_1/6 = 10/3.$$

$$C_1 = 60/15 \,,\quad \boxed{C_1 = 4}$$

$$C_2 = 5/3 - 4 \quad \boxed{C_2 = -7/3}$$

$\therefore$ The differential Equation's solution is

$$y_k = 4(3)^k + (-7/3)(-1/2)^k - 5/3$$

(b) let $y_n = \alpha^n$ be the trial solution for homogenious part.

So, $2\cdot\alpha^{n+3} - 3\cdot\alpha^{n+1} - 2\alpha^n = 0.$

$$\alpha^n(2\alpha^2 - 3\alpha - 2) = 0$$

$$\alpha^n \neq 0$$

$$2\alpha^2 - 3\alpha - 2 = 0.$$

$$2\alpha^2 - 4\alpha + \alpha - 2 = 0$$

$$2\alpha(\alpha-2) + 1(\alpha-2) = 0$$

$$\boxed{\alpha = -1/2, 2}$$

$\therefore$ The solution for homogenious part is

$$y_n = C_1(-1/2)^n + C_2(2)^n$$

for non-homogenious part:

let $y_n = an + b$. So,

$$2\cdot(a(n+2)+b) - 3(a(n+1)+b) - 2(an+b) = 6n+1$$

$$2an + 4a + 2b - 3an - 3a - 3b - 2an - 2b = 6n+1$$

$$-3an + a - 3b = 6n+1$$

$$\boxed{a = 2}, \quad \boxed{b = -1}$$

$\therefore$ The general solution is

$$y_n = C_1(-1/2)^n + C_2(2)^n - 2n - 1.$$

$\because y_0 = 1, y_1 = 2,$

Scanned with CamScanner

$$1 = C_1 + C_2 - 1$$

$$C_1 + C_2 = 2 \quad (1)$$

$$2 = -\frac{C_1}{2} + 2C_2 - 3$$

$$10 = -C_1 + 4C_2 \quad (2)$$

$$(1) + (2) \quad 5C_2 = 12 \Rightarrow \boxed{C_2 = \frac{12}{5}}$$

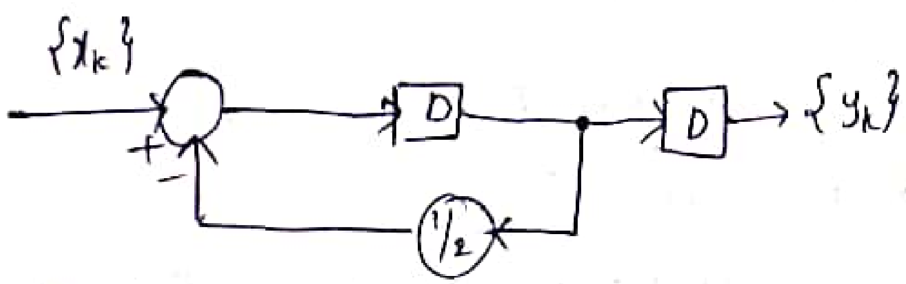
$$C_1 = 2 - \frac{12}{5} \quad \boxed{C_1 = -\frac{2}{5}}$$

∴ The solution to the difference Equation is

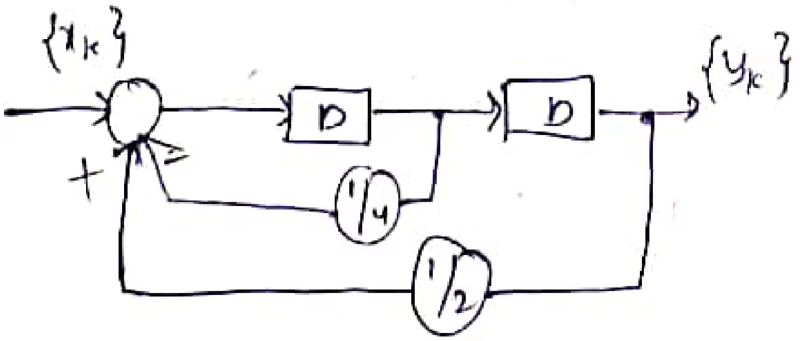
$$y_n = -\frac{2}{5}\left(-\frac{1}{2}\right)^n + \frac{12}{5}(2)^n - 2n - 1.$$

13) Find difference Equations representing discrete time systems shown below

(a)



(b)



Sol: let the signal going to left D-block is w_k and that going into right D-block is v_k.

Then,

$$y_{k+1} = v_k.$$

$$v_{k+1} = w_k = x_k - \tfrac{1}{2} v_k$$

$$y_{k+2} = y_{k+1} = x_k - \tfrac{1}{2} v_k.$$

$$= x_k - \tfrac{1}{2} v_k = x_k - \tfrac{1}{2} y_{k+1}$$

$$y_{k+2} + \tfrac{1}{2} y_{k+1} = x_k.$$

b) From a,

$$y_{k+1} = v_k..$$

and $v_{k+1} = w_k$.

$$v_{k+1} = w_k = x_k - \tfrac{1}{4} v_k - \tfrac{1}{5} y_k.$$

Then,

$$y_{k+2} = x_k - \tfrac{1}{4} y_{k+1} - \tfrac{1}{5} y_k.$$

$$x_k = y_{k+2} + \tfrac{1}{4} y_{k+1} + \tfrac{1}{5} y_k.$$

$$y_{k+2} + \tfrac{1}{4} y_{k+1} + \tfrac{1}{5} y_k = x_k.$$

14) A person's capital at the begining of, and Expenditure during, a given year k are denoted by C_k and E_k respectively, and satisfy the difference Equation.

$$C_{k+1} = 1.5 C_k - E_k \qquad E_{k+1} = 0.21 C_k + 0.5 E_k.$$

a) Show that eventually the person A's capital grows at 20% per annum.

(b) If the capital at the beginning of year 1 is £6000 and the expenditure during year 1 is £3720 then find the year in which the expenditure is minimum and the capital at the beginning of that year.

Sol: - let the transformed equation be in the form

$$\begin{bmatrix} z-3/2 & 1 \\ -0.21 & z-1/2 \end{bmatrix} \begin{bmatrix} C(z) \\ e(z) \end{bmatrix} = \begin{bmatrix} z\,C_0 \\ z\,E_0 \end{bmatrix}$$

Then

$$\begin{bmatrix} C(z) \\ e(z) \end{bmatrix} = \frac{1}{z^2-2z+0.96} \begin{bmatrix} z-1/2 & -1 \\ 0.21 & z-3/2 \end{bmatrix} \begin{bmatrix} z\,C_0 \\ z\,E_0 \end{bmatrix}$$

for $C(z)$,

$$C(z) = 1200\frac{z}{z-1.2} + 4800\frac{z}{z-0.8}$$

on inversion,

$$C_k = 1200(1.2)^k + 4800(0.8)^k.$$

It is clearly seen that there is 20% growth in C_k.

Scanned with CamScanner

b) $E_k = 1.5C_k - C_{k+1}$

$E_k = 1800(1.2)^k + 7200(0.8)^k - 1200(1.2)^{k+1} - 4800(0.8)^{k+1}$

Differentiating w.r.t and Equating to Zoore.

$0.6\log(1.2) + 5.6x\log(0.8) = 0$

$x = \left(\frac{0.8}{1.2}\right)^k$

$x = 0.0875$

$\therefore k = \dfrac{\log(0.0875)}{\log\left(\frac{0.8}{1.2}\right)} = 6.007.$

$\therefore k \simeq 6$, corresponds to 7^{th} year,

$C_0 = 4841.$

15) Find the fourier series of the function

$f(x) = \begin{cases} x - x^2, & 0 \le x \le 1 \\ 0, & -1 \le x \le 0 \end{cases}$

from the gian function,

$l = 1$ and it is full range.

$a_0 = 1/2l \int_{-l}^{l} f(x)dx.$

$a_0 = 1/2 \int_{-1}^{1} f(x)dx.$

$= 1/2 \left[\int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx\right].$

$= 1/2 \left[0 + \int_{0}^{1}(x - x^2)dx\right]$

$$= \tfrac{1}{2}\left[\left(\tfrac{x^2}{2} - \tfrac{x^3}{3}\right)_0^1\right]$$

$$a_0 = \tfrac{1}{2}\left[\tfrac{1}{2} - \tfrac{1}{3}\right] = \tfrac{1}{12}$$

$$a_n = \tfrac{1}{l}\int_{-l}^{l} f(x)\cos\left(\tfrac{n\pi}{l}x\right)dx$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$a_n = \int_{-1}^{0} f(x)\cos(n\pi x)dx + \int_0^1 f(x)\cos(n\pi x)dx$$

$$a_n = 0 + \int_0^1 (x - x^2)\cos(n\pi x)dx$$

$$a_n = \int_0^1 x\cos(n\pi x)dx - \int_0^1 x^2\cos(n\pi x)dx$$

$$a_n = \left[x\int\cos(n\pi x)dx - \int\int\cos(n\pi x)\right]_0^1$$

$$= \left[x^2\int\cos(n\pi x)dx - 2\int x\cos(n\pi x)dx\right]_0^1$$

$$a_n = \left[\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2}\right]_0^1 - \left(x^2\frac{\sin(n\pi x)}{n\pi} - 2\left[\frac{x(\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2}\right]\right]_0^1$$

$$a_n = \left(\frac{\sin(n\pi)}{n\pi} + \frac{\cos(n\pi)}{n^2\pi^2}\right) - \left(\frac{\cos 0}{n^2\pi^2}\right) - \left[\left(\frac{\sin(n\pi)}{n\pi} - 2\frac{\sin(n\pi)}{n\pi} - 2\frac{\cos(n\pi)}{n^2\pi^2}\right] - \left(-2\frac{\cos 0}{n^2\pi^2}\right)\right.$$

$$a_n = \cancel{\frac{\cos n\pi}{n^2\pi^2}} - \frac{1}{n^2\pi^2} - \frac{2}{n^2\pi^2} - \cancel{\frac{\cos n\pi}{n^2\pi^2}}$$

$$a_n = -3/n^2\pi^2$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x)\,dx$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx$$

$$b_n = \int_{-1}^{0} f(x)\sin(n\pi x)\,dx + \int_{0}^{1} f(x)\sin(n\pi x)\,dx$$

$$b_n = \int_0^1 \underset{I}{x}\ \underset{II}{\underline{\sin(n\pi x)}}\,dx - \int_0^1 \underset{I}{x^2}\ \underset{II}{\sin(n\pi x)}\,dx$$

$$b_n = \left[x\int \sin(n\pi x)\,dx - \int\int \sin(n\pi x)\,dx\right]_0^1 - \left[x^2\int \sin(n\pi x)\,dx - 2\int x\int \sin(n\pi x)\,dx\right]_0^1$$

$$b_n = \left[\frac{-x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\right]_0^1 - \left[\frac{-x^2\cos(n\pi x)}{n\pi} - 2\left[\frac{-x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\right]\right]_0^1$$

$$b_n = \left(\frac{-\cos(n\pi)}{n\pi} + 0\right) - (0+0) - \left[\left(\frac{-\cos(n\pi)}{n\pi}\right) - 2\left[\frac{-\cos(n\pi)}{n\pi} + 0\right]\right] - (0 - 2(0+0])$$

$$b_n = \cancel{\frac{-\cos(n\pi)}{n\pi}} + \cancel{\frac{\cos(n\pi)}{n\pi}} - 2\frac{\cos(n\pi)}{n\pi}$$

$$b_n = \frac{-2\cos(n\pi)}{n\pi} = \frac{-2(-1)^n}{n\pi}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\pi x) + b_n\sin(n\pi x)\right]$$

$$f(x) = 1/12 + \sum_{n=1}^{\infty}\left[\frac{-3}{n^2\pi^2}\cos(n\pi x) - \frac{2\cos(n\pi)\sin(n\pi x)}{n\pi}\right]$$

$$f(x) = 1/12 + \sum_{n=1}^{\infty}\left[\frac{-3}{n^2\pi^2}\cos(n\pi x) - \frac{2(-1)^n \sin(n\pi x)}{n\pi}\right]$$

16) Find the fourier coefficients and fourier series of the Square-wave function defined by,

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$$

Sol: $a_0 = 1/2l \int_{-l}^{l} f(x)\,dx.$

Here, $l = \pi$. and full range.

$$a_0 = 1/2\pi \int_{-\pi}^{\pi} f(x)\,dx = 1/2\pi\left(\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right)$$

$$a_0 = 1/2\pi\left(\int_{0}^{\pi} 1 \cdot dx\right) = 1/2 \qquad \boxed{a_0 = 1/2}$$

$$a_n = 1/l \int_{-l}^{l} f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$a_n = 1/\pi\left(\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right)$$

$$a_n = 1/\pi \int_{0}^{\pi} \cos nx\,dx = \frac{[\sin(nx)]_0^{\pi}}{n\pi}$$

$$\boxed{a_n = 0}$$

$$b_n = 1/\pi \int_{-\pi}^{\pi} f(x) = \sin(nx)\,dx \quad \left(\because b_n = 1/l \int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right)$$

$$b_n = 1/\pi\left(\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \int_{0}^{\pi} f(x)\sin(nx)\,dx\right)$$

$$b_n = 1/\pi \int_0^\pi \sin(nx)dx = -\frac{[\cos(nx)]_0^\pi}{n\pi}.$$

$$b_n = -\frac{(\cos(n\pi)-1)}{n\pi} = \frac{1-\cos(n\pi)}{n\pi}.$$

$$\boxed{b_n = \frac{1-(-1)^n}{n\pi}}$$

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$f(x) = 1/2 + \sum_{n=0}^{\infty} \frac{1-(-1)^n}{n\pi} \sin(nx)$$

(17.) Find Fourier Series of triangular wave function by $f(x)$

$f(x) = |x|$, for $-1 \leq x \leq 1$ and $f(x+2) = f(x)$ for all x.

for which values of is equal to the sum of its fourier Series?

Sol: $a_0 = 1/2L \int_{-L}^{L} f(x)dx$

Here, $L = 1$, $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ -x & 1 \leq x < 0 \end{cases}$

$$a_0 = 1/2\left[\int_{-1}^{0} f(x)dx + \int_0^1 f(x)dx\right]$$

$$a_0 = 1/2\left[\int_{-1}^{0} (-x)dx + \int_0^1 x\,dx\right]$$

$$a_0 = 1/2\left[-(x^2)/2 \Big|_{-1}^{0} + (x^2)/2 \Big|_0^1\right]$$

$$a_0 = \left[\frac{-(0-1)}{2} + \frac{(1-0)}{2}\right]$$

$$\boxed{a_0 = 1/2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$a_n = \int_{-1}^{0} \underset{I}{(-x)}\, \underset{II}{\cos(n\pi x)}\,dx + \int_{0}^{1}(x)\cos(n\pi x)dx$$

$$a_n = -\left[x\int\cos(n\pi x)dx - \iint\cos(n\pi x)dx\right]_{-1}^{0} +$$

$$\left[x\int\cos(n\pi x)dx - \iint\cos(n\pi x)dx\right]_0^1.$$

$$a_n = -\left(x\frac{\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2}\right]_{-1}^{0} + \left(\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2}\right]_0^1$$

$$a_n = -\left[\left(0+\frac{1}{n^2\pi^2}\right) - \left(0 + \frac{\cos(-n\pi)}{n^2\pi^2}\right)\right] + \left[\left(0+\frac{\cos(n\pi)}{n^2\pi^2}\right) - \left(0+\frac{1}{n^2\pi^2}\right)\right]$$

$$a_n = -\left[1/n^2\pi^2 - \frac{(-1)^2}{n^2\pi^2}\right] + \frac{(-1)^n}{n^2\pi^2} - 1/n^2\pi^2$$

$$a_n = -2/n^2\pi^2 + \frac{2(-1)^n}{n^2\pi^2}.$$

$$\boxed{a_n = \frac{2((-1)^n - 1)}{n^2\pi^2}}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \int_{-1}^{0} \underset{I}{(-x)}\, \underset{II}{\sin(n\pi x)}\,dx + \int_{0}^{1}(x)\sin(n\pi x)dx$$

$$b_n = -\left[x\int\sin(n\pi x)dx - \iint\sin(n\pi x)dx\right]_{-1}^{0} + \left[x\int\sin(n\pi x)dx - \right.$$

$$\left.\iint\sin(n\pi x)dx\right]_0^1.$$

$$b_n = -\left[-x\frac{\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\right]_{-1}^{0} +$$

$$\left[x\frac{\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\right]_0^1$$

$$bn = -\left[(0+0) - \left(\frac{\cos(n\pi)}{n\pi}+0\right)\right] + \left[\left(\frac{\cos(n\pi)}{n\pi}+0\right) - (0+0)\right]$$

$$bn = \frac{2\cos(n\pi)}{n\pi} \Rightarrow$$

$$\boxed{bn = \frac{2(-1)^n}{n\pi}}$$

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty}\left[\frac{2((-1)^n-1)}{n^2\pi^2}\cos(n\pi x) + \frac{2(-1)^n}{n\pi}\sin(n\pi x)\right]$$

18) Define $f: R \to R$ by $f(x) = x^2$ for $\pi \le x \le \pi$.

a) Compute fourier coefficients of f.

b) Use Parseval's theorem to deduce that

$$\sum_{n=0}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Sol: $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\cdot\left(\frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} = \pi^2/3 \qquad \boxed{a_0 = \frac{\pi^2}{3}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx.$$

$$a_n = \frac{1}{\pi}\left[x^2\int\cos(nx)\,dx - 2\int x\int\cos(nx)\,dx\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left(\frac{x^2\sin(nx)}{n} - 2\int\frac{x\sin(nx)}{n}\,dx\right]_{-\pi}^{\pi}$$

$$a_n = 1/\pi \left[\frac{x^2 \sin(nx)}{n} - 2\left[x \int \frac{\sin(nx)}{n} dx - \int\int \frac{\sin(nx)}{n} dx\right]_{-\pi}^{\pi}\right.$$

$$a_n = 1/\pi \left[0 - 2\left[-\frac{x\cos(nx)}{n^2} + \frac{\sin(nx)}{n^3}\right]_{-\pi}^{\pi}\right]$$

$$a_n = 1/\pi \left(\frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]_{-\pi}^{\pi}$$

$$a_n = 1/\pi \left[\left(\frac{2\pi\cos(n\pi)}{n^2} - \frac{2\sin(n\pi)}{n^3}\right) - \left(-\frac{2\pi\cos(nx)}{n^2} - \frac{2\sin(-nx)}{n^3}\right)\right]$$

$$a_n = 1/\pi \left[\frac{2\pi(-1)^n}{n^2} + \frac{2\pi(-1)^n}{n^2}\right]$$

$$\boxed{a_n = 4/n^2 \, (-1)^n}$$

$$\therefore \boxed{b_n = 0}.$$

$\because f(x) = f(-x)$. Even function

b) By parseval's formula,

$$1/2 \times a_0^2 + \sum_{n=0}^{\infty} (a_n^2 + b_n^2) = \left| 1/\pi \int_{-\pi}^{\pi} (f(x))^2 dx \right|$$

$$= 1/2 \left(\frac{2\pi^2}{3}\right)^2 + \sum_{n=0}^{\infty} \left[4/n^2 (-1)^n\right] = \left| 1/\pi \cdot \int_{-\pi}^{\pi} x^4 dx \right|$$

$$\frac{2\pi^4}{9} + 16 \sum_{n=0}^{\infty} 1/n^4 = \left| 1/\pi \left(x^5/5\right]_{-\pi}^{\pi} \right|$$

$$= \frac{2\pi^4}{9} + 16 \sum_{n=0}^{\infty} 1/n^4 = \left| \frac{2\pi^5}{5\pi} \right|$$

$$= 16 \sum_{n=0}^{\infty} 1/n^4 = \left| \frac{2\pi^4}{5} - \frac{2\pi^4}{9} \right|$$

$$16 \sum_{n=0}^{\infty} 1/n^4 = \frac{8\pi^4}{45}$$

$$\therefore \sum_{n=0}^{\infty} 1/n^4 = \pi^4/90$$

19) Find Eigenvalues and Eigenfunctions of two point boundary value problem.

$y'' + \lambda y = 0$, $y(-L) = y(L)$, $y'(-L) = y'(L)$ on $[-L, L]$

Sol: If $\lambda = 0$,

$y'' = 0$

$y(x) = C_1 x + C_2$

$y(L) = y(-L)$

$C_1 L + C_2 = -C_1 L + C_2$

$C_1 = 0$

$y'(-L) = y'(L)$. Trivial solution.

If $\lambda < 0$

Let $\lambda = -k^2$.

Then,

$y(x) = C_1 e^{kx} + C_2 e^{-kx}$

Applying boundaries,

$C_1 e^{kL} + C_2 e^{-kL} = C_1 e^{-kL} + C_2 e^{-kL}$

$C_1 (e^{kL} - e^{-kL}) = C_2 (e^{kL} - e^{-kL})$

$\boxed{C_1 = C_2}$

$$kc_1 e^{kL} - kc_2 e^{-kL} = kc_1 e^{-kL} - kc_2 e^{kL}$$

$$kc_1 (e^{\cancel{kL}} - e^{-kL}) = kc_2 (e^{\cancel{kL}} - e^{-kL})$$

$$\boxed{c_1 = c_2}$$

Solution can't be determined

If $\lambda > 0$,

$$y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

Applying bondary Conditions.

$$c_1 \cancel{\cos(-\sqrt{\lambda}L)} + c_2 \sin(-\sqrt{\lambda}L) = c_1 \cancel{\cos(\sqrt{\lambda}L)} + c_2 \sin(\sqrt{\lambda}L)$$

$$2c_2 \sin\sqrt{\lambda}L = 0$$

$$c_2 \sin\sqrt{\lambda}L = 0 \quad -(1)$$

$$y'(-L) = y'(L)$$

$$-c_1 \sin(-\sqrt{\lambda}L) + c_2 \cancel{\cos(-\sqrt{\lambda}L)} = -c_1 \sin(\sqrt{\lambda}L) + \cancel{c_2 \cos\sqrt{\lambda}L}$$

$$c_1 \sin\sqrt{\lambda}L = 0 \quad -(2).$$

From (1) and (2),

$$\sqrt{\lambda} = \frac{n\pi}{L}.$$

$$\lambda_n = \frac{n^2\pi^2}{L}$$

Each Eigenvalue has linearly independent associated Eigen functions

$$\cos\left(\frac{n\pi x}{L}\right) \text{and} \sin\left(\frac{n\pi x}{L}\right)$$

So,

$y_{1n} = \cos\left(\frac{n\pi x}{\lambda}\right)$ and

$n = 1, 2, 3, \ldots$

$y_{2n} = \sin\left(\frac{n\pi x}{\lambda}\right)$

20) Consider stress of a body at a particular point is given in the form of $Q(x_1, x_2, x_3) = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$. Find and orthogonal change of variable that eliminates the cross product terms in $Q(x_1, x_2, x_3)$.

Sol: The quadratic form can be expressed in matrix notation as

$$Q = x^T A x$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation of Matrix A is

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & -\lambda & 2 \\ 0 & 2 & -1-\lambda \end{vmatrix} = -\lambda^3 + 9\lambda$$

$$= -\lambda(\lambda+3)(\lambda-3) = 0$$

$$\lambda = 0, -3, 3$$

The Eigen values are $\lambda = 0, -3, 3$.

For $\lambda = 0$, $R_2 \to R_2 + 2R_1$.

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$x_1 = 2x_2.$$

$$x_3 = 2x_2$$

Eigen vector is

$$X = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

On Normalizing $X = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$.

For $\lambda = -3$,

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$4x_1 = 2x_2$

$\boxed{x_2 = 2x_1}$

$x_2 = -x_3$

$\boxed{x_3 = -2x_1}$

$$x = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

On normalizing, $x = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

For $\lambda = 3$.

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\boxed{x_1 = -x_2}$ $\quad 2x_2 = 4x_3$

$x_2 = 2x_3$

$x_3 = x_2/2$ $\quad \boxed{x_3 = \frac{-x_1}{2}}$

Eigen vector is, $x = \begin{bmatrix} 1 \\ -1 \\ -1/2 \end{bmatrix}$

On normalizing, $x = \begin{bmatrix} 1/3 \\ -1/3 \\ -1/6 \end{bmatrix}$

On substituting $X = Py$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 2/3 & -2/3 & -1/6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Now quadratic form is

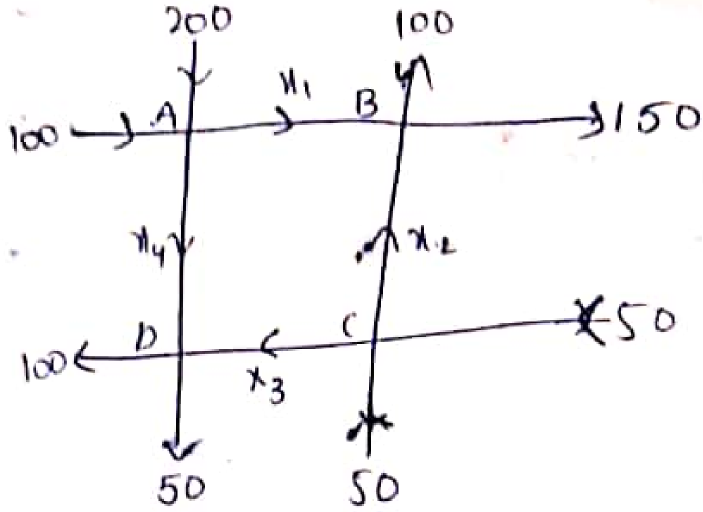
$$Q = y^T (P^T A P) y$$

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = -3y_2^2 + 3y_3^2$$

$\therefore$ No cross products.

21) Construct a system of linear Equations that describes the traffic flow in the road network of Fig. All streets are one-way streets in the directions indicated. The units are vehicles per hour. Solve the system of x_1, x_2, x_3 and x_4. What is the minimum possible flow that can be expected along branch AB?



At A,

$$200+100 = x_1 + x_4$$

$$x_1 + x_4 = 300 \quad -(1).$$

At B,

$$x_1 + x_2 = 100+150$$

$$x_1 + x_2 = 250 \quad -(2).$$

At C,

$$50+50 = x_2 + x_3$$

$$x_2 + x_3 = 100 \quad -(3).$$

At D,

$$x_3 + x_4 = 100+50$$

$$x_3 + x_4 = 150 \quad -(4).$$

Writing these 4 equations in matrix form.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 300 \\ 1 & 1 & 0 & 0 & 250 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \end{bmatrix} X = 0.$$

$R_1 \to R_1 - R_4$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 150 \\ 1 & 1 & 0 & 0 & 250 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \end{bmatrix} X = 0$$

$R_1 \to R_1 + R_3.$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 250 \\ 1 & 1 & 0 & 0 & 250 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \end{bmatrix} X = 0$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 250 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \end{bmatrix} X = 0$$

$x_3 + x_4 = 150$

$x_1 + x_2 = 250$

$x_2 + x_3 = 100$

let $x_3 = t$. Then,

$x_2 = 100 - t$

$x_4 = 150 - t$

$x_1 = 150 + t$

Hower, traffic flow at any path cannot be negative.

$\therefore 0 \le t \le 100$

The minimum flow at AB is

$x_1 = 150 + t$

Minimum value at $t = 0$

So, the minimum flow is

$\boxed{x_1 = 150}$