POLITECNICO
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An Advanced Signature Scheme:

Schnorr Algorithm and its Benefits to the Bitcoin Ecosystem

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20th December 2018

Introduction

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1. Efficiency (DER encoding, no batch validation, modular inversion);
2. Poor implementation of higher level constructions (low privacy and fungibility, scales badly);
3. Not provably secure (malleable).

Outline

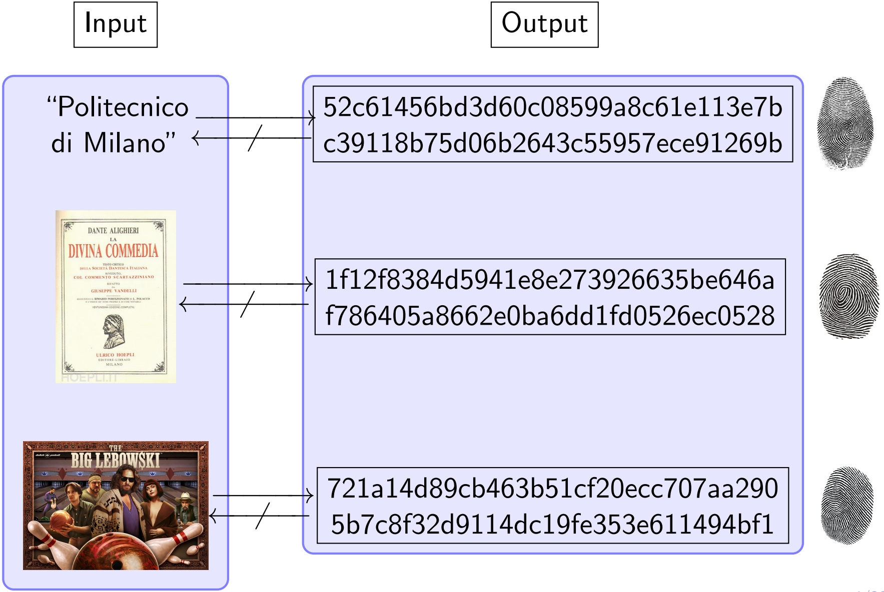
Mathematical background and cryptographic primitives

- Hash functions
- Elliptic curve cryptography

Digital signature schemes

ECSSA applications

Hash functions ($\simeq$ Random functions)



Input

Output

“Politecnico di Milano” $\rightarrow$ 52c61456bd3d60c08599a8c61e113e7b c39118b75d06b2643c55957ece91269b $\not\leftarrow$

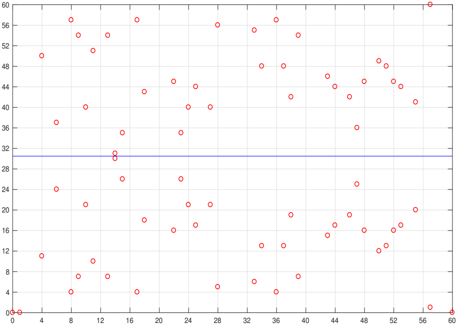
1f12f8384d5941e8e273926635be646a f786405a8662e0ba6dd1fd0526ec0528

721a14d89cb463b51cf20ecc707aa290 5b7c8f32d9114dc19fe353e611494bf1

Elliptic curve cryptography

An elliptic curve over a finite field is defined by:

$$E(\mathbb{F}_p):\ y^2 = x^3+ax+b\,(\mathrm{mod}\ p).$$



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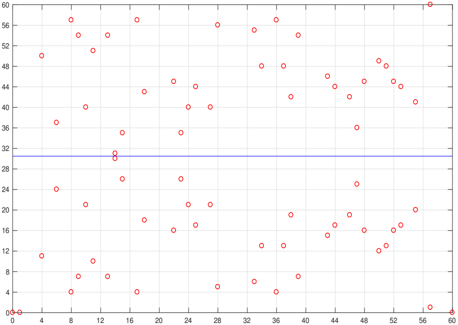
It is possible to define:

- Addition:
  $Q_3 := Q_1 + Q_2$,
  $\forall Q_1, Q_2 \in E(\mathbb{F}_p)$;



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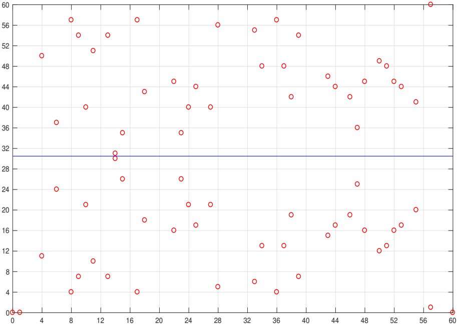

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- Addition:
  $Q_3 := Q_1 + Q_2$,
  $\forall Q_1, Q_2 \in E(\mathbb{F}_p)$;
- Scalar multiplication:
  $nG = G + ... + G$,
  $\forall G \in E(\mathbb{F}_p), \forall n \in \mathbb{N}$.

Multiplication's computational asymmetry is the core of ECC.

Discrete logarithm problem

Fixed $G \in E(\mathbb{F}_p)$, we can define $Q = qG \quad \forall q \in [1, ..., n-1]$:

- The direct operation $q \mapsto Q$ is efficient;
- The inverse operation $Q \mapsto q$ is computationally infeasible for certain groups.

Double and add algorithm: $q = 41$

$41 = 1 + 8 + 32 = 2^0 + 2^3 + 2^5$, i.e. $(41)_2 = 101001 \implies$

$$41G = G + 8G + 32G.$$

5 point doubling and 2 additions vs. 40 additions.

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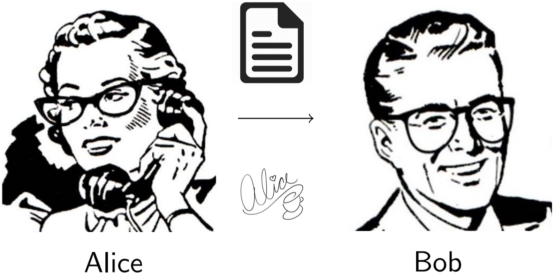
Digital signature schemes
- ECDSA
- ECSSA

ECSSA applications

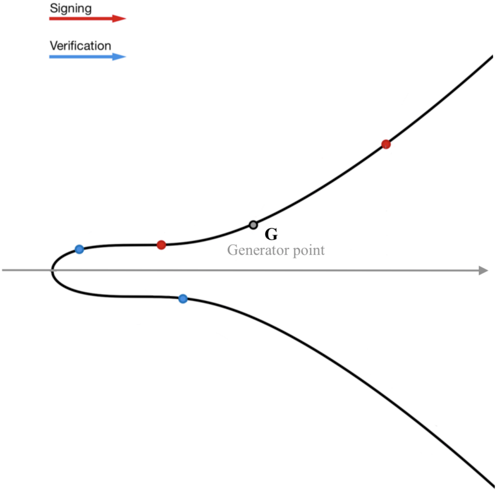
Digital signature



- Authentication: the recipient is confident that the message comes from the alleged sender;
- Non repudiation: the sender cannot deny having sent the message;
- Integrity: ensures that the message has not been altered during transmission.

Elliptic curve digital signature algorithm

ECDSA_SIG(m, q):



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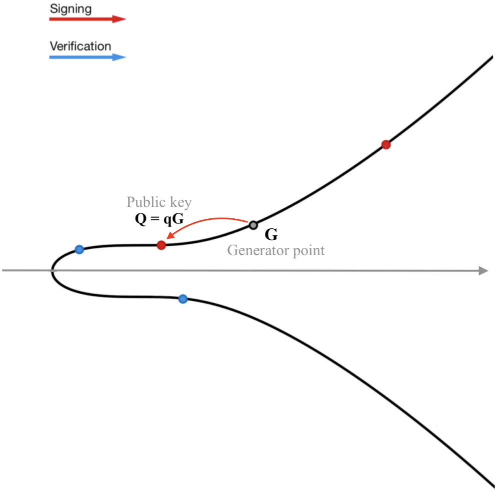

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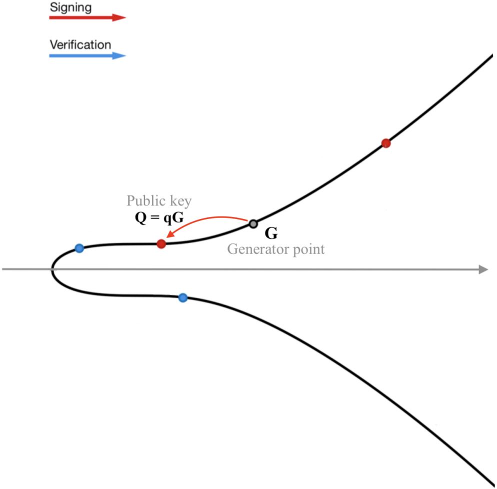

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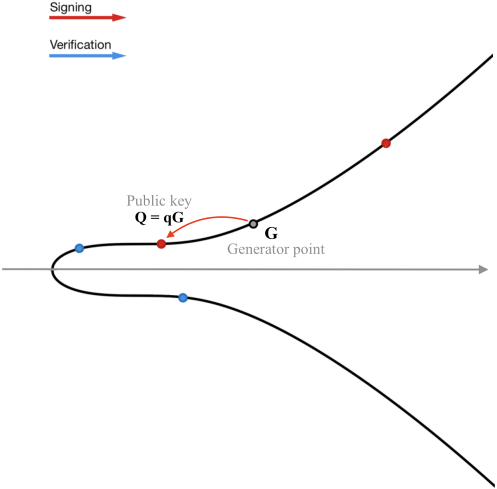

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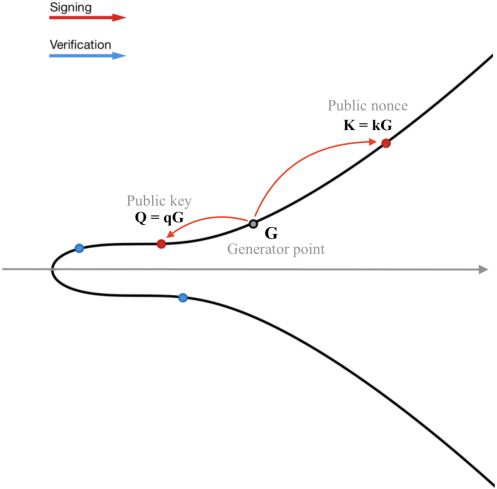

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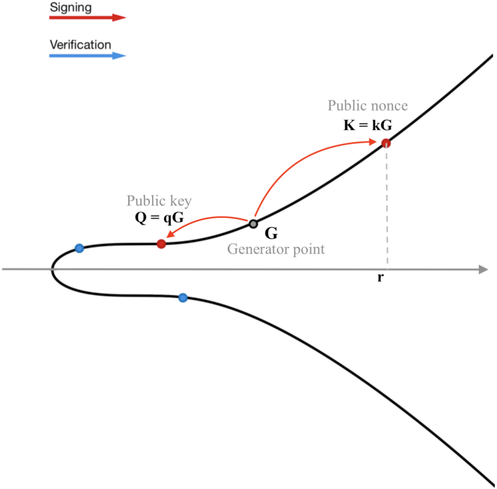

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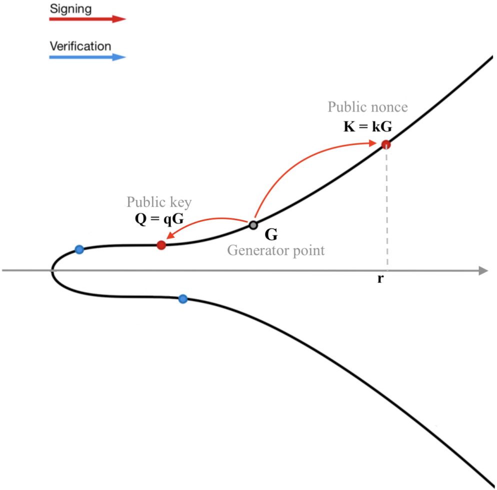

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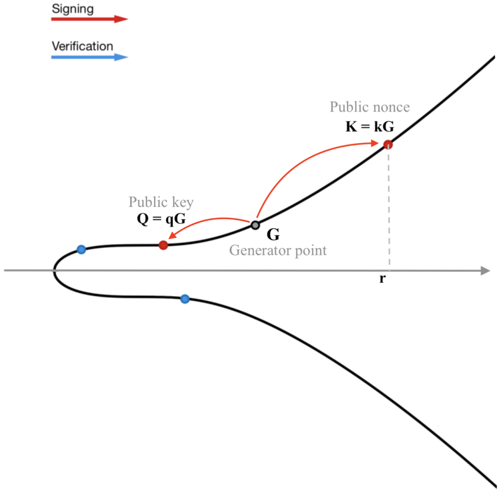

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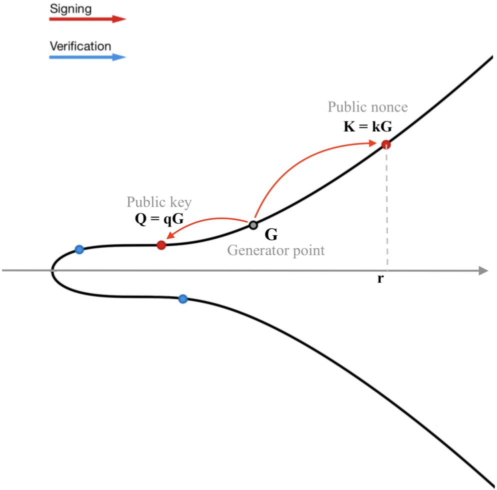

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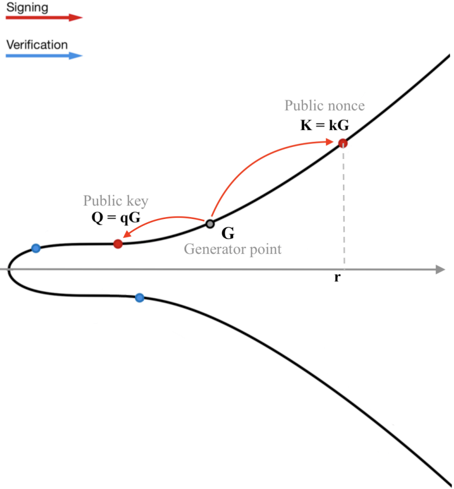

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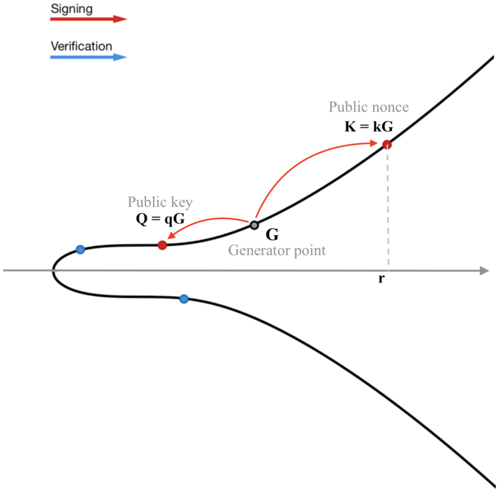

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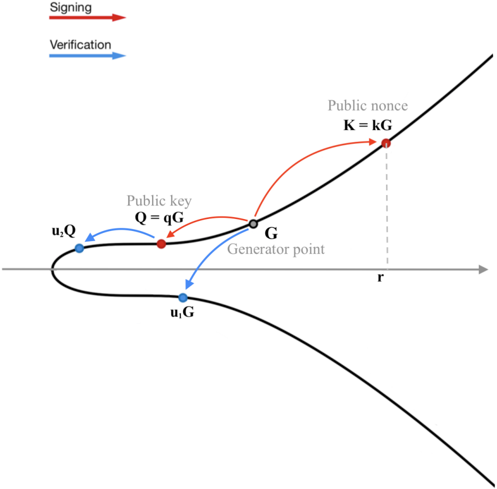

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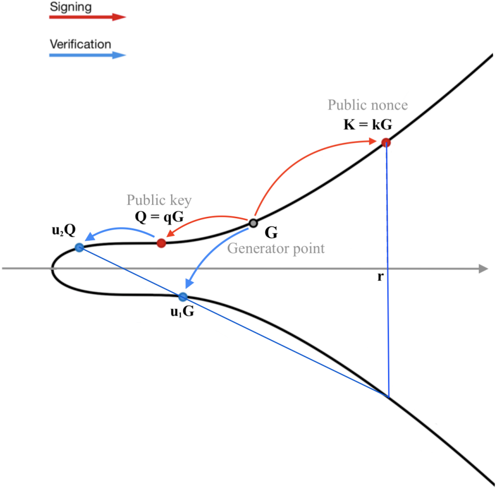

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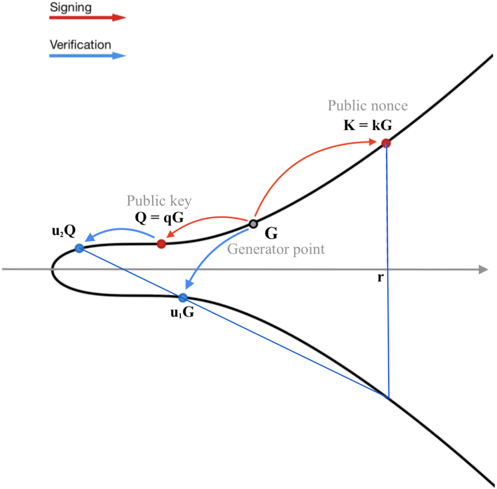

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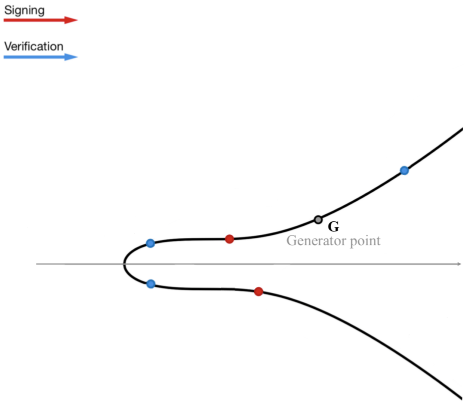

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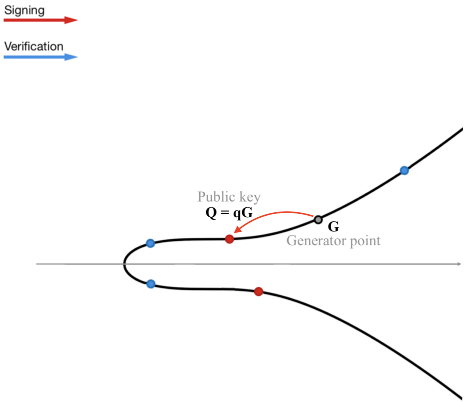

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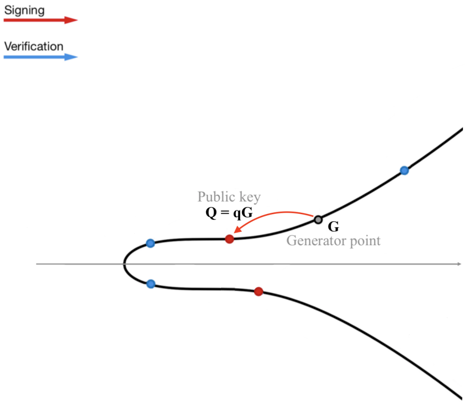

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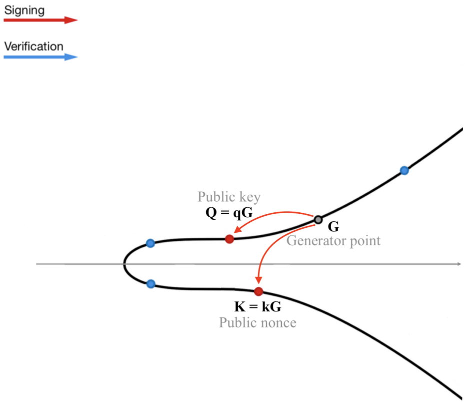

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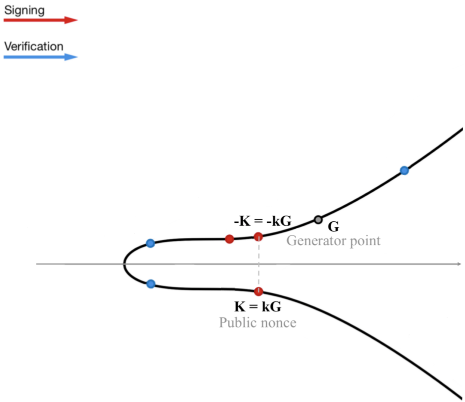

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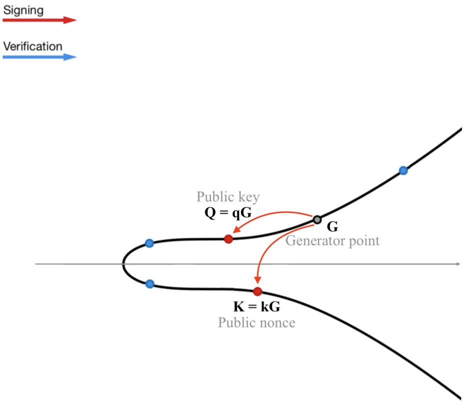

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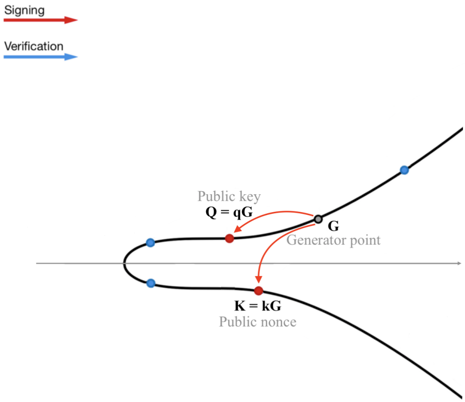

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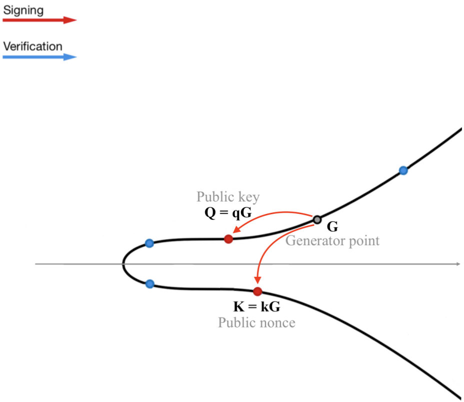

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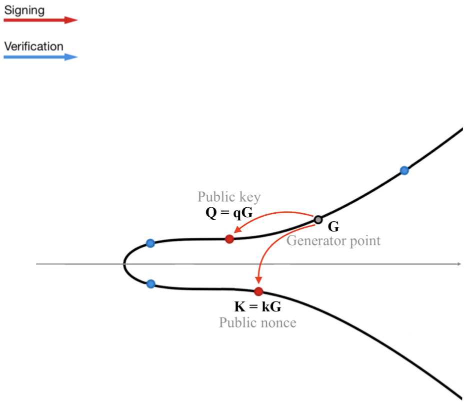

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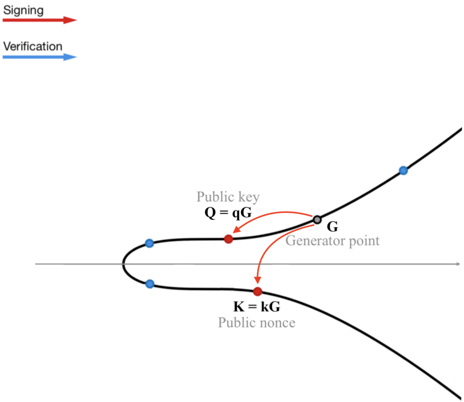

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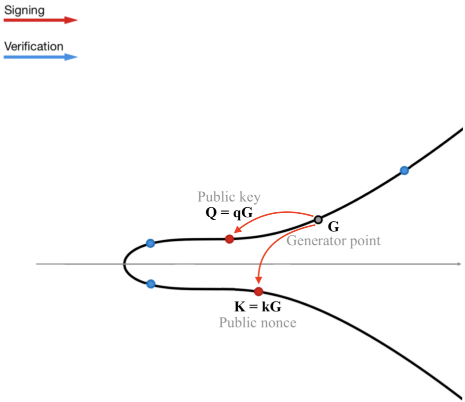

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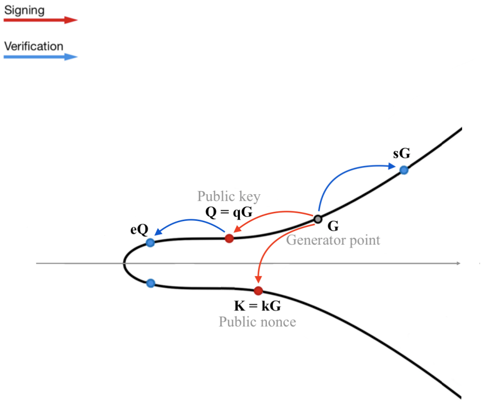

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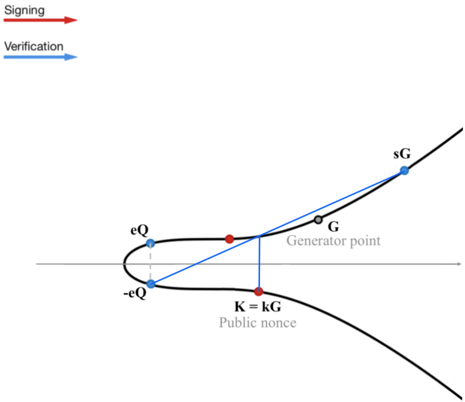

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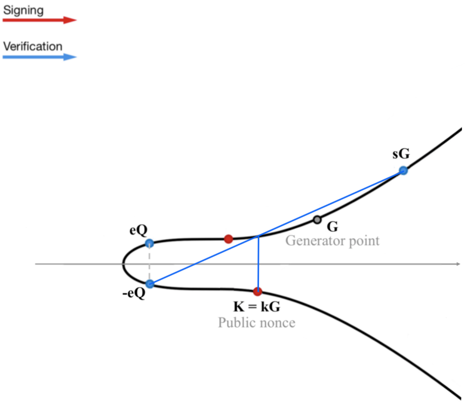

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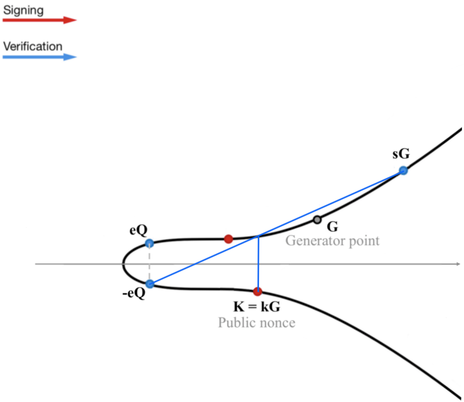
  2. $e \leftarrow \text{hash}(r||Q||m) \pmod{n}$;
  3. $K \leftarrow sG - eQ$;
  4. If $\text{jacobi}(y_K) \neq 1$ or $x_K \neq r$:
        return False;
```

Adapted from:
`https://medium.com/cryptoadvance/how-schnorr-signatures-may-improve-bitcoin-91655bcb4744`

Elliptic curve Schnorr signature algorithm

```
ECSSA_VER($(r, s), m, Q$):
  1. If $r \notin \{1, ..., p-1\}$ or
     $s \notin \{1, ..., n-1\}$:
         return False;
  2. $e \leftarrow \text{hash}(r||Q||m) \pmod{n}$;
  3. $K \leftarrow sG - eQ$;
  4. If $\text{jacobi}(y_K) \neq 1$ or $x_K \neq r$:
         return False;
  5. return True.
```



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ECDSA vs. ECSSA

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- Malleable: given (r, s) also $(r, -s \pmod{n})$ is a valid signature for same message and public key;

- Provably secure (SUF-CMA) in the ROM model assuming the ECDLP is hard $\implies$ not malleable;

ECDSA vs. ECSSA

- Malleable: given (r, s) also $(r, -s \pmod{n})$ is a valid signature for same message and public key;
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- Not linear: very complex higher level constructions.

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- New encoding: fixed length, always 64 bytes;
- Batch validation scales logarithmically;
- No computational heavy operations involved;
- Linearity: easier higher level constructions.

Batch validation

A signature (K, s) is valid if $K = sG - \text{hash}(x_K \mid\mid Q \mid\mid m)Q$.
Thus, two valid signatures (K_0, s_0) and (K_1, s_1) satisfies:

$K_0 + K_1 = (s_0 + s_1)G - \text{hash}(x_{K_0} \mid\mid Q_0 \mid\mid m_0)Q_0 - \text{hash}(x_{K_1} \mid\mid Q_1 \mid\mid m_1)Q_1$.

Insecure: introduction of random factors.

$$a_0 K_0 + a_1 K_1 =$$

$= (a_0 s_0 + a_1 s_1)G - a_0 \text{hash}(x_{K_0} \mid\mid Q_0 \mid\mid m_0)Q_0 - a_1 \text{hash}(x_{K_1} \mid\mid Q_1 \mid\mid m_1)Q_1$.

Batch validation - Bos-Coster's algorithm

$$\begin{gathered}a_0 K_0 + a_1 K_1 = \\ = (a_0 - a_1) K_0 + a_1 (K_0 + K_1).\end{gathered}$$

Batch validation - Bos-Coster's algorithm

$$a_0 K_0 + a_1 K_1 =$$
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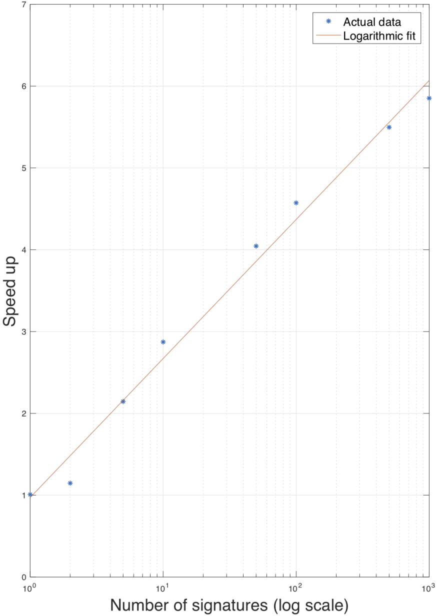
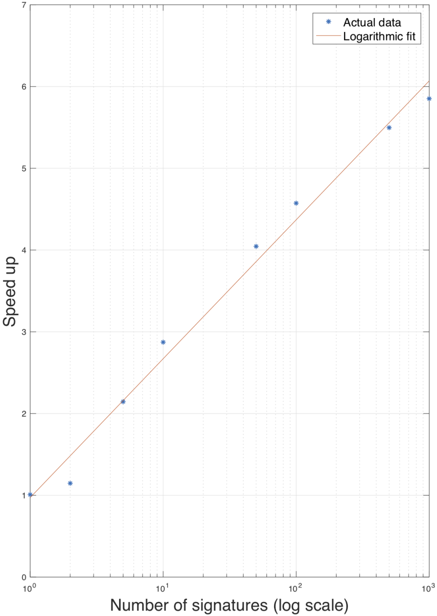
- Sort the tuples according to a_i;



Batch validation - Bos-Coster's algorithm

$$a_0 K_0 + a_1 K_1 = \\ = (a_0 - a_1) K_0 + a_1 (K_0 + K_1).$$

- Sort the tuples according to a_i;
- While the list has length larger than one:



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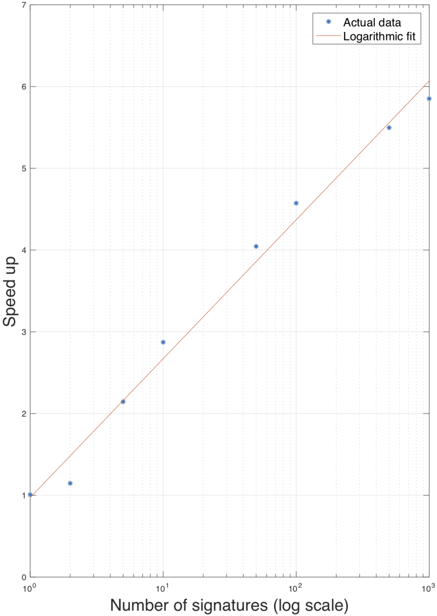

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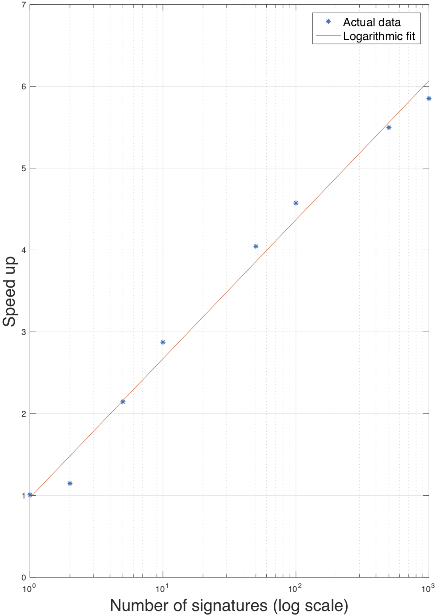

Batch validation - Bos-Coster's algorithm

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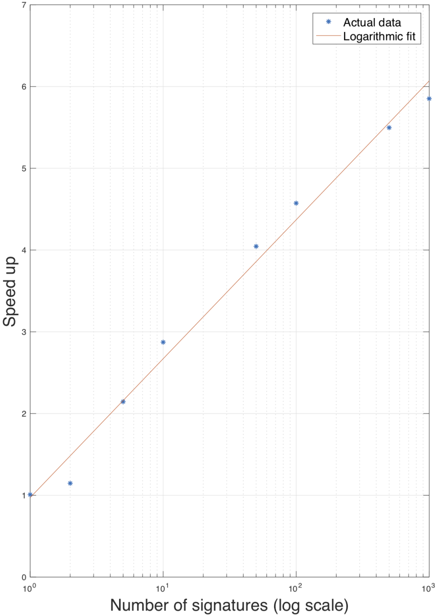
    - Sort the list again;
- When only one element remains, with very large probability it will be of the form $(1, K)$, otherwise it will be of the form (a, K).



Outline

Mathematical background and cryptographic primitives

Digital signature schemes

ECSSA applications
- MuSig
- Threshold signature scheme
- Adaptor signatures

Multi-signature schemes

Multi-signature schemes allow a group of users to cooperate to sign a single message, usually producing a joint signature that is more compact than a collection of distinct signatures. Verification usually requires the message m and the set of public keys of the signers.

Bitcoin multi-signature is implemented naively:

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- Locking script : m <pubKey1> <pubKey2> ... <pubKeyn> n OP_CHECKMULTISIG
- Unlocking script: 0 <sig1> <sig2> ... <sigm>

MuSig: compact *m*-of-*m* signature scheme

MuSig$(m, q_1, \langle L \rangle)$:

Alice

Bob

Charlotte

MuSig: compact *m*-of-*m* signature scheme

```
MuSig($m, q_1, \langle L \rangle$):
  1. for $i \leftarrow 1, m$ do:
      1.1 $a_i \leftarrow \text{hash}(\langle L \rangle || Q_i)$;
```

a_1, a_2

Alice

Bob

Charlotte

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```

Q

Alice

Bob

Charlotte

MuSig: compact *m*-of-*m* signature scheme

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```

k_1

Alice

Bob

Charlotte

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  4. $K_1 \leftarrow k_1 G$,  $t_1 \leftarrow \mathrm{hash}(K_1)$;
```

K_1, t_1

Alice

Bob

Charlotte

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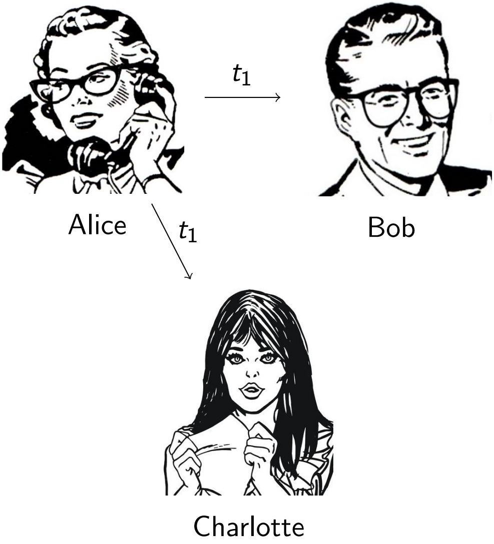
  5. send $t_1, K_1$;
```



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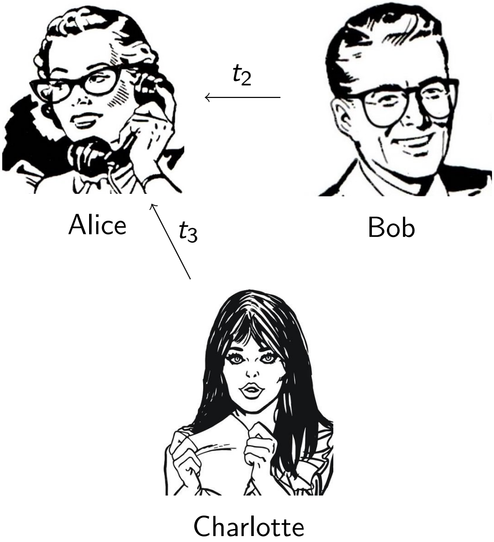

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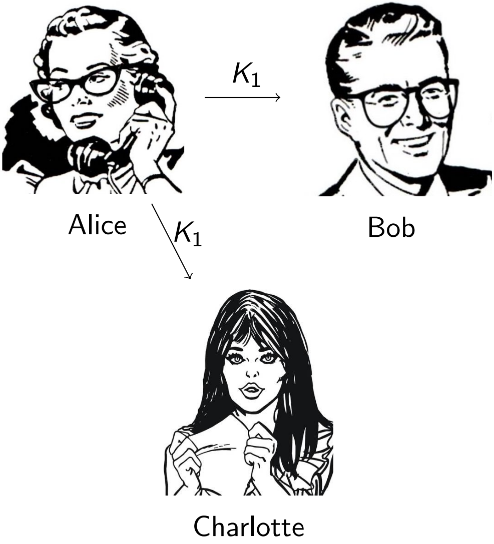

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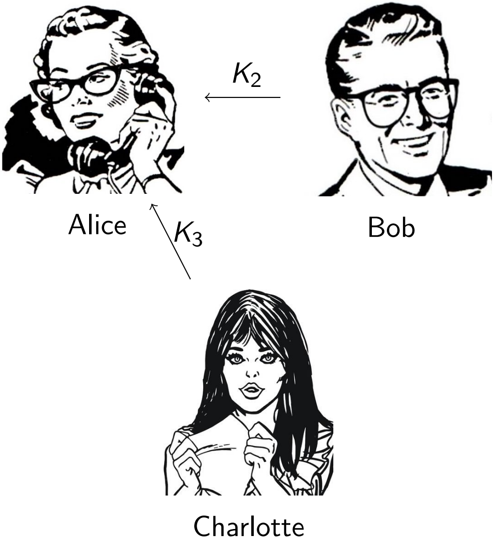

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  send $t_1, K_1$;
```

Alice

$t_2 \overset{?}{=} \text{hash}(K_2)$

$t_3 \overset{?}{=} \text{hash}(K_3)$

Bob

Charlotte

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Bob

Charlotte

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```

Alice

```
if jacobi($y_K$) $\neq$ 1:
    $k_1 \leftarrow n - k_1$
```

Bob

Charlotte

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  7. $c \leftarrow \mathrm{hash}(x_K || Q || m)$;
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Alice

Bob

Charlotte

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  7. $c \leftarrow \mathrm{hash}(x_K || Q || m)$;
  8. $s_1 \leftarrow k_1 + c a_1 q_1 \pmod{n}$;
```

s_1

Alice

Bob

Charlotte

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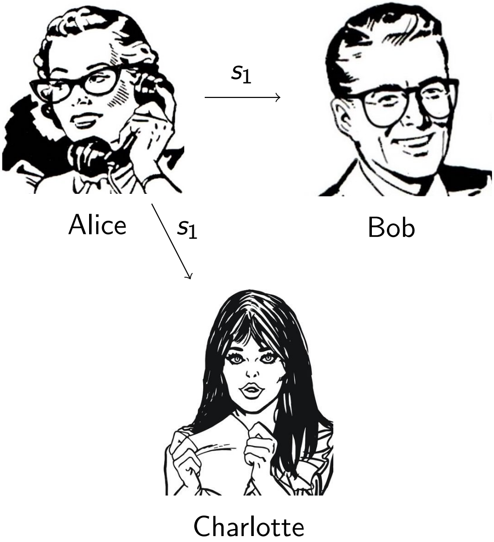

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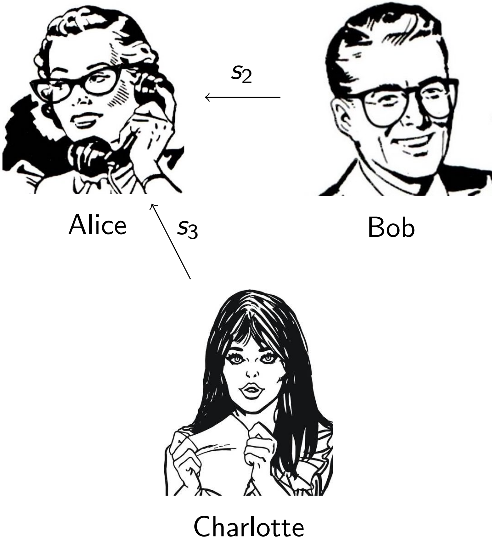

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 10. $s \leftarrow \sum_{i=1}^{m} s_i \pmod{n}$;
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Alice

Bob

Charlotte

MuSig: compact *m*-of-*m* signature scheme

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    send $s_1$;
    $s \leftarrow \sum_{i=1}^{m} s_i \pmod{n}$;
    return $(x_K, s)$.
```

(x_K, s)

Alice

Bob

Charlotte

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

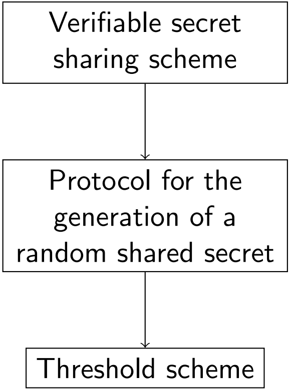
Protocol for the generation of a random shared secret

Alice

Bob

Charlotte

Threshold signature scheme (t-of-m)


Verifiable secret sharing scheme

Protocol for the generation of a random shared secret

Threshold scheme

Alice

Bob

Charlotte

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

The dealer:

Dealer: Alice

1: Bob

2: Charlotte

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

The dealer:

- generates secret s and $s' \xleftarrow{\$} \{1, ..., n-1\}$;

Dealer: Alice

s, s'

1: Bob

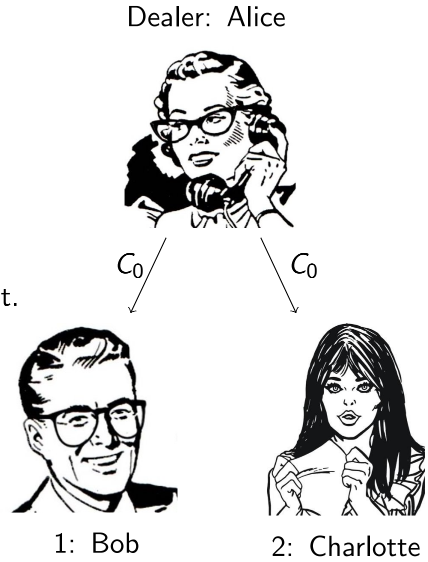
2: Charlotte

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

The dealer:

- generates secret s and $s' \overset{\$}{\leftarrow} \{1, ..., n-1\}$;
- commits to them through the Pedersen commitment $C_0 = sG + s'H$: C_0 is broadcast.



Dealer: Alice

C_0

C_0

1: Bob

2: Charlotte

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

The dealer:

- chooses random polynomials:
  $f(u) = s + f_1 u + \ldots + f_{t-1} u^{t-1}$,
  $f'(u) = s' + f'_1 u + \ldots + f'_{t-1} u^{t-1}$,
  $f_j,\ f'_j \xleftarrow{\$} \{1, \ldots, n-1\}$;

Dealer: Alice

f, f'

1: Bob

2: Charlotte

Threshold signature scheme (t-of-m)

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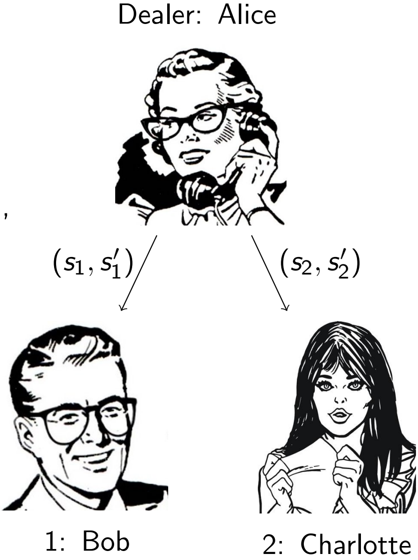
  $f_j,\ f'_j \overset{\$}{\leftarrow} \{1, \ldots, n-1\}$;
- computes $(s_i, s'_i) =$ $(f(i) \pmod{n}, f'(i) \pmod{n})$, $i \in \{1, \ldots, m\}$ and sends them secretly to P_i;



Dealer: Alice

(s_1, s'_1)

(s_2, s'_2)

1: Bob

2: Charlotte

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

The dealer:

- chooses random polynomials:
  $f(u) = s + f_1 u + \dots + f_{t-1} u^{t-1}$,
  $f'(u) = s' + f'_1 u + \dots + f'_{t-1} u^{t-1}$,
  $f_j,\ f'_j \overset{\$}{\leftarrow} \{1, \dots, n-1\}$;
- computes $(s_i, s'_i) =$
  $(f(i) \pmod{n}, f'(i) \pmod{n})$,
  $i \in \{1, \dots, m\}$ and sends them secretly to P_i;
- broadcasts the commitment to the sharing polynomials:
  $C_j = f_j G + f'_j H$,
  $j \in \{1, \dots, t-1\}$.

Dealer: Alice

C_j

C_j

1: Bob

2: Charlotte

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

The participants:

Dealer: Alice

1: Bob

2: Charlotte

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

The participants:

- verify the consistency of their shares of secret: $s_i G + s_i' H = \sum_{j=0}^{t-1} i^j C_j$;

Threshold signature scheme (t-of-m)

Verifiable secret sharing scheme

The participants:

- verify the consistency of their shares of secret: $s_i G + s'_i H = \sum_{j=0}^{t-1} i^j C_j$;
- to reconstruct the secret they rely on Lagrange's interpolation formula: $f(u) = f(i)\omega_i(u)$, where $\omega_i(u) = \prod_{j \neq i} \frac{u-j}{i-j} \pmod{n}$. $s = f(0) = s_i\omega_i$, with $\omega_i = \omega_i(0) = \prod_{j \neq i} \frac{j}{j-i} \pmod{n}$.

Dealer: Alice

$\xleftrightarrow{s}$

1: Bob

2: Charlotte

Threshold signature scheme (t-of-m)

Protocol for the generation of a random shared secret

Each participant:

1: Alice

2: Bob

3: Charlotte

Threshold signature scheme (t-of-m)

Protocol for the generation of a random shared secret

Each participant:

- acts as the dealer in the previous protocol ($f_i(u) = \sum_{j=0}^{t-1} a_{ij} u^j$, $a_{i0} = r_i$);

1: Alice

r_1, f_1

r_2, f_2

2: Bob

r_3, f_3

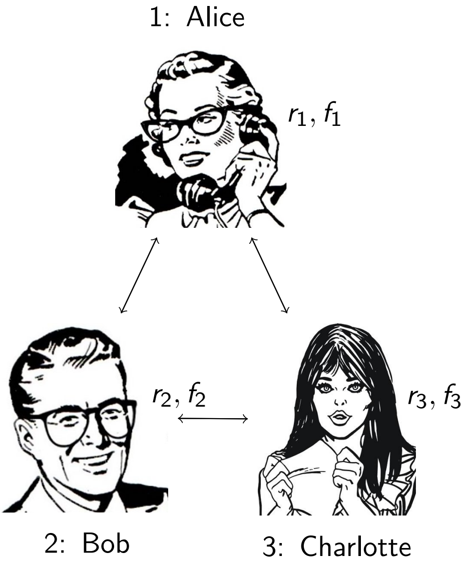
3: Charlotte

Threshold signature scheme (t-of-m)

Protocol for the generation of a random shared secret

Each participant:

- acts as the dealer in the previous protocol ($f_i(u) = \sum_{j=0}^{t-1} a_{ij} u^j$, $a_{i0} = r_i$);



1: Alice

r_1, f_1

r_2, f_2

r_3, f_3

2: Bob

3: Charlotte

Threshold signature scheme (t-of-m)

Protocol for the generation of a random shared secret

Each participant:

- acts as the dealer in the previous protocol ($f_i(u) = \sum_{j=0}^{t-1} a_{ij} u^j$, $a_{i0} = r_i$);
- at the end of the procedure the shared secret is $r = \sum_{i=1}^{m} r_i$ with shares $s_i = \sum_{j=1}^{m} f_j(i) \pmod{n}$;

1: Alice

s_1

s_2

2: Bob

s_3

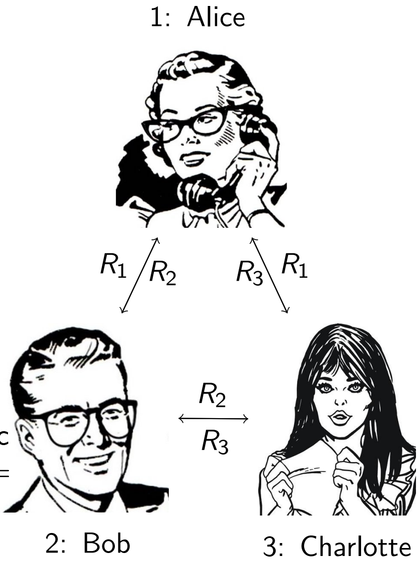
3: Charlotte

Threshold signature scheme (t-of-m)

Protocol for the generation of a random shared secret

Each participant:

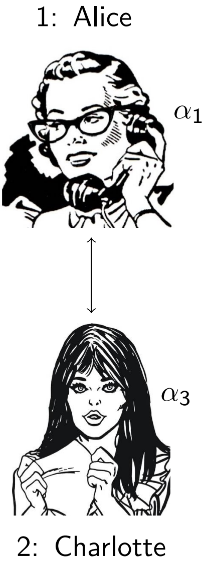
- acts as the dealer in the previous protocol ($f_i(u) = \sum_{j=0}^{t-1} a_{ij} u^j$, $a_{i0} = r_i$);
- at the end of the procedure the shared secret is $r = \sum_{i=1}^{m} r_i$ with shares $s_i = \sum_{j=1}^{m} f_j(i) \pmod{n}$;
- broadcast his share of the public key $R_j = r_j G$ ($R = \sum_{j=1} mR_j = \sum_{j=1} mr_j G = rG$).



Threshold signature scheme (t-of-m)

Threshold scheme

After having established a distributed key pair $(\alpha_1, ..., \alpha_m) \xleftrightarrow{\text{(t, m)}} (q|Q)$ through the protocol for the generation of a random shared secret (that acts as key generation protocol) the signers:



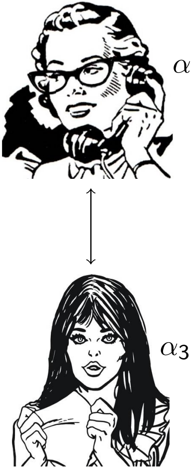
Threshold signature scheme (t-of-m)

$\boxed{\text{Threshold scheme}}$

After having established a distributed key pair $(\alpha_1, ..., \alpha_m) \xleftrightarrow{\text{(t, m)}} (q|Q)$ through the protocol for the generation of a random shared secret (that acts as key generation protocol) the signers:

- run again the same protocol to produce a nonces pair: $(\beta_1, ..., \beta_m) \xleftrightarrow{\text{(t, m)}} (k|K)$.

1: Alice



2: Charlotte

Threshold signature scheme (t-of-m)

Threshold scheme

Then each signer i:

- checks whether jacobi(y_K) $\neq 1$; if it is the case she sets $\beta_i = n - \beta_i$;

1: Alice

2: Charlotte

Threshold signature scheme (t-of-m)

Threshold scheme

Then each signer i:

- checks whether $\text{jacobi}(y_K) \neq 1$; if it is the case she sets $\beta_i = n - \beta_i$;
- reveals her partial signature: $\gamma_i = \beta_i + e\alpha_i \pmod{n}$, with $e = \text{hash}(x_K || Q || msg)$;

Threshold signature scheme (t-of-m)

Threshold scheme

Then each signer i:

- checks whether $\text{jacobi}(y_K) \neq 1$; if it is the case she sets $\beta_i = n - \beta_i$;
- reveals her partial signature: $\gamma_i = \beta_i + e\alpha_i \pmod{n}$, with $e = \text{hash}(x_K || Q || msg)$;
- computes $\sigma = \sum_{j=1}^{t} \gamma_j \omega_j$, with $\omega_j = \prod_{h \neq j} \frac{h}{h-j}$: σ is such that $\sigma = k + eq \pmod{n}$;

1: Alice

2: Charlotte

Threshold signature scheme (t-of-m)

Threshold scheme

Then each signer i:

- checks whether jacobi$(y_K) \neq 1$; if it is the case she sets $\beta_i = n - \beta_i$;
- reveals her partial signature: $\gamma_i = \beta_i + e\alpha_i \pmod{n}$, with $e = \text{hash}(x_K||Q||msg)$;
- computes $\sigma = \sum_{j=1}^{t} \gamma_j \omega_j$, with $\omega_j = \prod_{h \neq j} \frac{h}{h-j}$: σ is such that $\sigma = k + eq \pmod{n}$;
- the signature is (x_K, σ).

1: Alice

(x_K, σ)

(x_K, σ)

2: Charlotte

Adaptor signatures

Adaptor signatures