Politecnico di Milano
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Master of Science in Mathematical Engineering

An Advanced Signature Scheme: Schnorr Algorithm and its Benefits to the Bitcoin Ecosystem

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Abstract

Nowadays in the Bitcoin protocol it is used ECDSA as underlying digital signature scheme: its drawbacks are well known, but a solution exists. The Schnorr signature algorithm, although lacking a standardization, is superior in every aspect to ECDSA. Starting from the mathematical and cryptographic foundations, the aim of this work is to present the two schemes, delve into the technicalities of their comparison and study the benefits to higher level constructions allowed by Schnorr (multi-signatures, threshold signatures, cross-chain atomic swaps and the Lightning Network). The discussion proceeds according to the recent BIP by Pieter Wuille about Schnorr's Bitcoin standardization.

Acknowledgements

Da scrivere.

Chapter 1

Introduction

1.1 Problem under analysis

Bitcoin ([23]) is a decentralized network whose original aim was to provide a peer-to-peer electronic payment system. In recent years it has astonished people who had the opportunity to study it, with an increasing number of supporters and detractors worldwide: it has been called a bubble, a Ponzi scheme, it has been labelled as sound money and store of value. But labels can be dangerous, and excessive labeling is usually unuseful: for this reason the present work does not aim at settling the dispute. Bitcoin is hard to understand, being at the crossroad of various disciplines, going from networking theory to cryptography, from economics to game theory. Among this multitude, mathematics and cryptography are perhaps the easiest way to approach Bitcoin, being difficult to misinterpret these two disciplines: of course there are engineering tradeoffs and debates around them, but the technical foundations are sound.
For this reason, the present work wants to give a brief but satisfactory introduction to the mathematical structures and the cryptographic primitives behind Bitcoin, focusing in particular on the role played by the digital signature scheme. We will outline all the problems presented by ECDSA, the scheme actually implemented, and show how they are solved by Schnorr, a scheme that probably will be introduced in the protocol in a couple of years. On top of that, we will delve into the technicalities of higher level constructions enabled by Schnorr, that will help in solving some of the key problems of Bitcoin, namely privacy and fungibility[1]. Bitcoin was intended to be anonymous, but it is not: it is pseudonymous, in the sense that it is

[1]Fungibility is the property of a good or a commodity whose individual units are interchangeable.

possible to link transactions to a unique identity through blockchain analysis. If this digital identity is unveiled, connecting to a real identity, that person could be persecuted for his involvement in Bitcoin (e.g. in authoritarian regime). The lack of privacy contributes to the lack of fungibility, allowing to distinguish between different bitcoins[2]. These problems affects heavily the monetary use case and in general have to be properly addressed: their improvement will be the *leitmotiv* of the entire present work.

The present thesis has been written during the author's involvement in the Digital Gold Institute, a research and development center focused on teaching, consulting, and advising about scarcity in digital domain (bitcoin and crypto-assets) and the underlying blockchain technology. The author has actively contributed to the development of the Python library that can be found at `https://github.com/dginst/BitcoinBlockchainTechnology`: for this reason, some optimization procedures are presented throughout the thesis.

1.2 Thesis structure

In Chapter 2 we will start presenting the mathematical foundations of the cryptography underpinning the Bitcoin protocol: in particular we will recall the concepts of group and field (Section 2.1), we will introduce the general idea of elliptic curve (EC in Section 2.2) and show how to induce a group structure on an EC through a proper definition of the addition operation (Section 2.2.1).

In Chapter 3 we will give an overview of the public key cryptography based on elliptic curves: thanks to the mathematical introduction of Chapter 2 we will be able to understand what an elliptic curve over a finite field is (Section 3.1); then we will discuss elliptic curves' parameters selection and validation (Section 3.1.1), we will present the basic idea behind asymmetric cryptography based on EC (Section 3.1.2) and we will show how to improve greatly the core operation of asymmetric elliptic curve cryptography (ECC), scalar multiplication, through a proper change of coordinates (Section 3.1.3). As a practical use case the section will be concluded with the presentation of the Bitcoin curve (secp256k1 in Section 3.1.4). The chapter will be concluded by the description of the hard problem at the basis of elliptic curves' widespread

[2]Typically Bitcoin identifies the protocol, while bitcoins are the coins that is possible to transfer through the protocol.

adoption in recent years, the so called discrete logarithm problem (DLP in Section 3.2).

Chapter 4 will deal with ECDSA (Section 4.1) and Schnorr (Section 4.2). Throughout the chapter many topics will be touched, among which the issues of ECDSA, Schnorr's solutions and multi-signature schemes.

Finally in Chapter 5 we will delve in the technicalities of Schnorr's applications: we will study a multi-signature scheme that recovers key aggregation (MuSig in Section 5.1), a threshold signature scheme (Section 5.2) and the benefits arising from the use of adaptor signatures (Section 5.3) for cross-chain atomic swaps (Section 5.3.1) and for the Lightning Network (Section 5.3.2).

Chapter 6 will conclude the thesis with a summary in which we will draw interesting conclusions.

1.3 Notation

- Prime numbers: the lowercase letter p is used to represent an odd prime number;
- Fields: in Chapter 2 for a general field the letter K is used, while the finite field of order $q = p^k$, where p is an odd prime and k an integer, are represented as $\mathbb{F}_q$;
- Elliptic curves: in general an elliptic curve over a field K is denoted by $E(K)$, which represents the set of points satisfying the generalized Weierstrass equation with coefficients in K plus the point at infinity. From Chapter 3 onward, we will deal with EC over finite fields: in this case lowercase letters are used to denote scalars, while the uppercase equivalent denotes the linked EC point, e.g. $qG = Q = (x_Q, y_Q)$ (G is reserved to the generator of the group). Whenever a second generator is needed we use the capital letter H: this does not constitute a conflict with the cofactor of the group since typically the two generators are NUMS (*nothing up my sleeves*), meaning that we do not know the discrete logarithm of one with respect to the other and viceversa;
- Elliptic curves' key pair: the pair of private and public key is denoted as $\{q, Q\}$, where $Q = qG$. Whenever a second point is needed we use

the couple $\{r, R\}$; if more key pairs are needed subscripts are used, e.g. $q_1G = Q_1 = (x_1, y_1)$, $q_2G = Q_2 = (x_2, y_2)$ and $q_3G = Q_3 = (x_3, y_3)$. Notice that, for the sake of clarity, also the coordinate representation is adapted.

- Algorithms:
  - $||$ refers to byte array concatenation;
  - $a \leftarrow b$ refers to the operation of assignment;
  - $z \overset{\$}{\leftarrow} Z$ denotes uniform sampling from the set Z and assignment to z;
  - The function $\text{bytes}(x)$, where x is an integer, returns the byte encoding of x;
  - The function $\text{bytes}(Q)$, where Q is an EC point, returns $bytes(\text{0x02}+(y_Q\&1))\ ||\ bytes(x_Q)$[3];
  - The function $\text{int}(x)$, where x is a byte array, returns the unsigned integer whose byte encoding is x;
  - The function $\text{hash}(x)$, where x is a byte array, returns the hash of x. In particular, when dealing with the Schnorr signature, it returns the 32 byte SHA-256 of x;
  - The function $\text{jacobi}(x)$, where x is an integer, returns the Jacobi symbol $\left(\frac{x}{p}\right)$. In general we have (gcd here stands for greatest common divisor):

$$\left(\frac{x}{p}\right) = \begin{cases} 0, & \text{if } \gcd(x, p) \neq 1 \\ \pm 1, & \text{if } \gcd(x, p) = 1 \end{cases}$$

    Moreover we have that if $\left(\frac{x}{p}\right) = -1$ then x is not a quadratic residue modulo p (i.e. it has not a square root modulo p) and that if x is a quadratic residue modulo p and $\gcd(x, p) = 1$, then $\left(\frac{x}{p}\right) = 1$. However, unlike the Legendre symbol, if $\left(\frac{x}{p}\right) = 1$ then x may or may not be a quadratic residue modulo p. Fortunately enough, since p is an odd prime we have that the Jacobi symbol is equal to the Legendre symbol. Thus we can check only whether

[3]An EC point can be expressed in compressed or uncompressed form, as suggested in [1]: the compressed form requires the x coordinate and one additional byte to make explicit the y coordinate, while the uncompressed form requires both the coordinates.

$\left(\frac{x}{p}\right) = 1$ to assess if x is or is not a quadratic residue modulo p. Moreover, by Euler's criterion we have that $\left(\frac{x}{p}\right) = x^{\frac{p-1}{2}} \pmod{p}$, so that we have an efficient way to compute the Jacobi symbol.

Chapter 2

Mathematical background

The present study starts with an introduction to the mathematical foundations required for its understanding. The reader that is familiar with abstract algebra and elliptic curve cryptography may skip the first two chapters to deal directly with the digital signature schemes presented in Chapter 4.
We start with a brief presentation of the group and field's structures, deferring the introduction to elliptic curves mathematics to the second section.

2.1 Groups and fields

Definition 2.1.1. (Group)*: A group is a non empty set G together with a binary operation $\bullet : G \times G \to G$ (called the group law of G) that satisfies:*

- *Closure:* $\forall a, b \in G \Longrightarrow a \bullet b \in G$;
- *Associativity:* $\forall a, b, c \in G \Longrightarrow (a \bullet b) \bullet c = a \bullet (b \bullet c)$;
- *Identity:* $\exists e \in G \mid \forall a \in G,\ e \bullet a = a \bullet e = a$. *Such an element e is unique;*
- *Invertibility:* $\forall a \in G,\ \exists b \in G \mid a \bullet b = b \bullet a = e$. *This element is unique and it is commonly denoted either as a^{-1} or $-a$, depending on the notation (multiplicative or additive).*

As a careful reader may have noticed, it is not required the property of commutativity. This means that, in general, the result of the operation may depend on the order of the operands. If this is not the case, meaning that the group operation is also commutative, the group is called abelian.
Consider now a group $(G, +)$: hereinafter we will stick with the additive

notation, unless otherwise specified. If G is finite, i.e. it has a finite number of elements, we define the order of G as the number of elements in G. We can also define the order of an element $g \in G$ to be the smallest integer $k > 0$ such that $kg = 0$[1], where with 0 we denote the additive identity. If k is the order of g, then:

$$ig = jg \Longleftrightarrow i \equiv j \pmod{k}.$$

Indeed:

$$ig = jg \iff (i-j)g = 0 \iff i-j = nk,\ n \in \mathbb{Z}\backslash\{0\},$$

from which the thesis.
Next we state a basic result from group theory, that will turn out to be very useful in the following:

Theorem 2.1.1. (Lagrange's theorem): *Let G be a finite group.*

1. *Let H be a subgroup*[2] *of G. Then the order of H divides the order of G;*

2. *Let $g \in G$. Then the order of g divides the order of G.*

For the cryptographic applications that we will study, a particular family of groups turns out to be very important: cyclic groups. A cyclic group is a group that is generated by a single element: this means that it contains (at least) an element g such that every other element of the group can be obtained by repeatedly applying the group operation to g, i.e. $\forall h \in G$, $\exists n \in \{1, ..., k\} \mid h = ng$, where k as before denotes the order of g. All the group's elements satisfying this property are called generator of the group.
Two interesting properties of cyclic groups are the following:

1. Every cyclic group is abelian;

2. Every finite group has at least a cyclic subgroup.

Now let's get to the mathematical concept of field. A field is a non empty set on which are defined two binary operations, usually called addition and multiplication. It has to be an abelian group under addition (with 0 denoting

[1]In an additive group it is natural to define the scalar multiplication: given $g \in G$ and $k \in \mathbb{N}$, $k \neq 0$, we have that $kg = g + g + ... + g$, where the summation is repeated $k-1$ times. In a similar fashion, it is natural to define exponentiation in the multiplicative case.

[2]Given a group $(G, +)$, a subset H is a subgroup of G if H endowed with the restriction of the group operation to $H \times H$ is a group.

the additive identity) and the non-zero elements have to form an abelian group under multiplication. Finally, the multiplication has to be distributive over the addition.
More precisely we can give the following definition.

Definition 2.1.2. (Field)*: A field is a set $K \neq \emptyset$ endowed with two binary operations $+$ and $* : K \times K \to K$, such that:*

- *$(K, +)$ is an abelian group, whose identity element is denoted by 0. This group is called the additive group of the field;*
- *$(K\backslash\{0\}, *)$ is an abelian group. It is called the multiplicative group of the field, sometimes denoted by $K^\times$ or by K^*;*
- *$\forall a, b, c \in K : \; a * (b + c) = a * b + a * c.$*

Another key role in the following will be played by finite fields: a finite field is simply a field with a finite number of elements. It can be shown that a finite field of order q, denoted as $\mathbb{F}_q$, exists if and only if $q = p^k$, where p is a prime number and k is a positive integer. The field of a given order is unique, up to an isomorphism (this means that there are different representations of a unique mathematical object). In a field of order p^k, the characteristic of the field (as defined below) is p: thus adding p copies of any element results in the additive identity.
In particular we will deal with prime finite fields, i.e. the finite fields whose order is a prime number. Finite fields of order a prime p may be represented by the set $\mathbb{Z}_p = \{0, 1, ..., p - 1\}$ with addition and multiplication defined through modular arithmetic, that is: given $a, b \in \mathbb{F}_p$ then $r = a + b$ and $s = a * b$ are integers in $[0, p - 1]$, defined as the remainder of the divisions $(a+b)/p$ and $a*b/p$, respectively. This is usually written as: $a+b \equiv r \pmod p$ and $a*b \equiv s \pmod p$ or as $a+b \equiv r$ in $\mathbb{F}_p$ and $a*b \equiv s$ in $\mathbb{F}_p$. It is easy to see that in this case the additive identity is the integer 0, while the multiplicative identity is the integer 1.
Subtraction and division are defined via the additive and multiplicative inverses:

- Subtraction: given $a \in \mathbb{F}_p$, we know by definition that there is a unique element $-a \in \mathbb{F}_p$ such that $a + (-a) = 0 \pmod p$. Then we define subtraction as: $a - b = a + (-b) \pmod p$, $\forall a, b \in \mathbb{F}_p$;
- Division: $\forall a \in \mathbb{F}_p\backslash\{0\}$, $\exists a^{-1} \mid a * a^{-1} = 1 \pmod p$, again by definition. Thus we can defined division as: $a/b = a * (b)^{-1}$, $\forall a, b \in \mathbb{F}_p$, $b \neq 0$.

Notice that not all the finite fields $\mathbb{F}_q$ are isomorphic to $\mathbb{Z}_q$: consider for example the case $q = 2^2 = 4$. Here $p = 2$ and $k = 2$. Since two is a prime number, we have that there exists a field with four elements. But since four is not a prime number, we cannot represent this field through the set $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Consider the element 2: $2 * 0 = 0 \pmod 4$, $2 * 1 = 2 \pmod 4$, $2 * 2 = 0 \pmod 4$, $2 * 3 = 2 \pmod 4$. As we can see it does not admit a multiplicative inverse in $\mathbb{Z}_4$: for this reason $\mathbb{Z}_4$ is not a field and cannot be used as representation of $\mathbb{F}_4$.

We end this section giving the definition of field characteristic, that will be useful when defining the equation of an elliptic curve.

Definition 2.1.3. (Field characteristic): *Given a field* $(K, +, *)$, *we denote by 0 and 1 the additive and multiplicative identities, respectively.*
The characteristic of K, *denoted as* $char(K)$, *is defined to be the smallest* $n \in \mathbb{Z}\backslash\{0\}$ *for which the following relation holds:*

$$\underbrace{1 + ... + 1}_{n\ times} = 0$$

In case this relation never holds, the field is said to have characteristic zero.

2.2 Elliptic curves

In the most general case, an elliptic curve over a field K is defined through the equation:

$$y^2 + a_1xy + a_2y = x^3 + a_3x^2 + a_4x + a_5,$$

where $a_1, ..., a_5 \in K$. This form is called generalized Weierstrass equation, and can be simplified according to the characteristic of the field. In particular, if it is different from two and three we can write:

$$y^2 = x^3 + ax + b,\ \ a, b \in K.$$

This latter equation is called Weierstrass equation, and it is the most widely used.
We will denote an elliptic curve over a field K through the notation $E(K)$, that represents the set of points satisfying the defining equation. By construction, this set also contains a special point, called the point at infinity (its role will be clarified in the next section):

$$E(K) = \{\infty\} \cup \{(x, y) \in K \times K \mid y^2 = x^3 + ax + b\}.$$

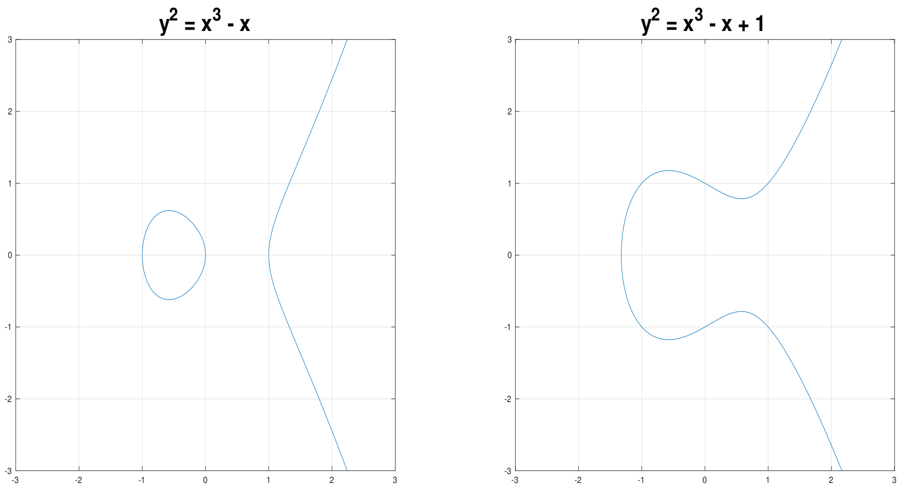


Figure 2.1: Two examples of elliptic curves over the real numbers.

Source: `https://en.wikipedia.org/wiki/Elliptic_curve#Elliptic_curves_over_the_real_numbers`.

To start familiarizing with elliptic curves, it could be useful to consider the case in which $K = \mathbb{R}$, the set of real numbers. In Figure 2.1 we can look at a couple of elliptic curves' real graph. As we can see, the elliptic curves are symmetric with respect to the x-axis, so that for every feasible x there are two y values satisfying the Weierstrass equation.
Typically the Weierstrass equation is coupled with the constraint that the discriminant of the cubic $x^3 + ax + b$ is different from zero. This is equivalent to ask that:

$$4a^3 + 27b^2 \neq 0,$$

in order to avoid multiple roots. Geometrically, this requirement means that there are no cusps, self intersections or isolated points[3]. We can also link the sign of the discriminant to the number of components of the curve: the real graph of a curve has two components if its discriminant is positive, and one component if it is negative, as exemplified in Figure 2.1 (the discriminant is $\Delta = -16(4a^3 + 27b^2)$).

[3]This requirement has also other implications: it can be shown that, without it, the elliptic curve $E_{ns}(K)$ defined over the non-singular points has an isomorphism with simpler algebraic structures, such as K itself or $K^\times$. This problem typically arises when dealing with elliptic curves modulo composites.

We get back for a moment to the generalized Weierstrass equation, and show how to go from there to the more appealing Weierstrass form:

$$y^2 + a_1xy + a_2y = x^3 + a_3x^2 + a_4x + a_5,$$

with $a_1, ..., a_5$ constants in K. We have said that this is the most generic form for an elliptic curve, but what does it exactly mean? It means that it is always possible to reduce to the Weierstrass equation, unless the characteristic of the field over which the curve is defined is two or three. In these two cases, we need to resort to the general formula.
However, if the characteristic of the field is not two we can divide by two[4] and complete the square:

$$\left(y + \frac{a_1x}{2} + \frac{a_2}{2}\right)^2 = x^3 + \left(a_3 + \frac{a_1^2}{4}\right)x^2 + \left(a_4 + \frac{a_1a_2}{2}\right)x + \left(\frac{a_2^2}{4} + a_5\right),$$

which can be written as

$$y_1^2 = x^3 + {a_1}'x^2 + {a_2}'x + {a_3}',$$

with $y_1 = y + \frac{a_1x}{2} + \frac{a_2}{2}$ and new constants ${a_1}', {a_2}', {a_3}' \in K$. Moreover, if the characteristic is neither three, then we can consider $x_1 = x + \frac{{a_1}'}{3}$:

$$y_1^2 = x^3 + {a_1}'x^2 + {a_2}'x + {a_3}' = \left(x_1 - \frac{{a_1}'}{3}\right)^3 + {a_1}'\left(x_1 - \frac{{a_1}'}{3}\right)^2 + {a_2}'\left(x_1 - \frac{{a_1}'}{3}\right) + {a_3}' =$$

$$= \left(x_1 - \frac{{a_1}'}{3}\right)\left(x_1^2 + \frac{{a_1}'^2}{9} - \frac{2}{3}{a_1}'x_1 + {a_1}'x_1 - \frac{{a_1}'^2}{3} + {a_2}'\right) + {a_3}' =$$

$$= \left(x_1 - \frac{{a_1}'}{3}\right)\left(x_1^2 + \frac{1}{3}{a_1}'x_1 - \frac{2}{9}{a_1}'^2 + {a_2}'\right) + {a_3}' =$$

$$= x_1^3 + \cancel{\frac{1}{3}{a_1}'x_1^2} - \frac{2}{9}{a_1}'^2x_1 + {a_2}'x_1 - \cancel{\frac{1}{3}{a_1}'x_1^2} - \frac{1}{9}{a_1}'^2x_1 + \frac{2}{27}{a_1}'^3 - \frac{1}{3}{a_1}'{a_2}' + {a_3}' =$$

$$= x_1^3 + \left({a_2}' - \frac{1}{3}{a_1}'^2\right)x_1 + \left(\frac{2}{27}{a_1}'^3 - \frac{1}{3}{a_1}'{a_2}' + {a_3}'\right).$$

Through a proper renaming of the constants we get finally to the well known Weierstrass form: $y_1^2 = x_1^3 + ax_1 + b$ for some constants a and b in K.
Finally, for the sake of completion, we present the case of the equation $cy^2 = dx^3 + ax + b$ and how to transform it in Weierstrass form. We can multiply both sides by c^3d^2 and write $(c^2dy)^2 = (cdx)^3 + (ac^2d)(cdx) + bc^3d^2$. The change of variables $y_1 = c^2dy$ and $x_1 = cdx$ yields to the desired formulation of the equation.

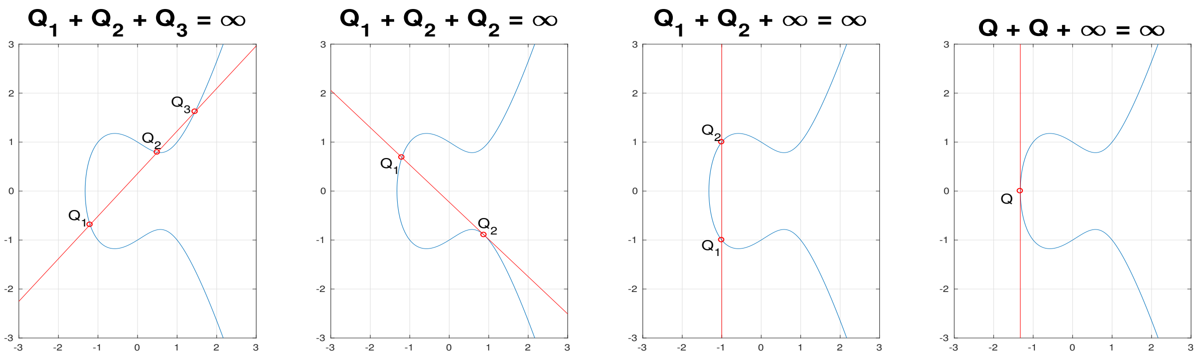
[4]Remember that in a field of characteristic two the number two acts as zero, the additive element that has not a multiplicative inverse.

2.2.1 The group law

This section is devoted to the definition of the addition operation between elements in $E(K)$: this operation turns out to be fundamental since it induces the structure of abelian group on the elliptic curve's point. Moreover, we will see why, in the definition of the curve, we had to consider the point at infinity. In particular we will consider K to be the set of real numbers, in order to give an easy graphical representation of the operation. This can be done without loss of generality, since the formulas that we will derive can be shown to be valid for every K with $\text{char}(K) \neq 2, 3$.

Loosely speaking we can say that:

- The points of the elliptic curve are the group's elements;
- The identity element is the point at infinity, denoted by ∞;
- The inverse of a point Q is the one symmetric about the x-axis: $Q = (x, y) \implies -Q = (x, -y)$;
- Addition is given by the following rule: given three aligned points on the curve Q_1, Q_2 and Q_3 their sum is $Q_1 + Q_2 + Q_3 = \infty$.



$Q_1 + Q_2 + Q_3 = \infty$
$Q_1 + Q_2 + Q_2 = \infty$
$Q_1 + Q_2 + \infty = \infty$
$Q + Q + \infty = \infty$
Q_1
Q_2
Q_3
Q

Figure 2.2: Geometric interpretation of the addition of EC points.

Source: `https://en.wikipedia.org/wiki/Elliptic_curve#The_group_law`.

In order to define properly the addition of EC points we need to consider five different cases. We start taking an elliptic curve E defined by the usual equation in Weierstrass form $y^2 = x^3 + ax + b$, $a, b \in \mathbb{R}$ and two points $Q_1 = (x_1, y_1), Q_2 = (x_2, y_2) \in E(\mathbb{R})$. Unless otherwise specified, we assume also that $Q_1, Q_2 \neq \infty$.

1. $x_1 \neq x_2$: consider the line passing through Q_1 and Q_2. In most of the cases (i.e. when neither the points are tangent points) this line will intersect the curve in a third point, that we denote as $-Q_3$. Then take the reflection with respect to the x-axis (i.e. change the sign of the y coordinate): denote this final point as Q_3.
   The algorithm specified allows us to give meaning to the formula:
   $$Q_1 + Q_2 = Q_3.$$
   The one outlined above is the geometrical intuition, but we can obtain algebraic formulas for the operation. The line passing through Q_1 and Q_2 has equation $y = m(x - x_1) + y_1$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$ is its slope. Notice that if $x_1 = x_2$ the line is vertical, but we will treat this case later on. We can find the intersections between the line and the curve through a substitution: $(m(x - x_1) + y_1)^2 = x^3 + ax + b$. We have $m^2(x^2 - 2x_1x + x_1^2) + 2my_1(x - x_1) + y_1^2 = x^3 + ax + b$; this formula can be rearranged in the form:
   $$x^3 - m^2x^2 + (a + 2m^2x_1 - 2my_1)x + (b + 2my_1x_1 - m^2x_1^2 - y_1^2) = 0.$$
   In general solving a cubic equation is non trivial, but we have already two roots, namely x_1 and x_2 since Q_1 and Q_2 are points on both the line and the curve. To find the third root we notice that, given a cubic polynomial $x^3 + ax^2 + bx + c$ with roots r, s and t, we can write:
   $$x^3 + ax^2 + bx + c = (x - r)(x - s)(x - t) =$$
   $$= x^3 - (r + s + t)x^2 + (rs + rt + st)x - rst.$$
   Therefore $r + s + t = -a$, so that, if we know two roots r and s, we can find the third as $t = -a - r - s$.
   In our case we have $x_1 + x_2 + x_3 = m^2 \implies x_3 = m^2 - x_1 - x_2$. Remembering to change the y coordinate of the intersection point due to the reflection, we get Q_3 as: $x_3 = m^2 - x_1 - x_2$ and $y_3 = m(x_1 - x_3) - y_1$.

2. $x_1 = x_2 \wedge y_1 \neq y_2$: the line through Q_1 and Q_2 is vertical, therefore it intersects E at ∞. Reflecting the point at infinity across the x-axis yields to the same point. Therefore, in this case $Q_1 + Q_2 = \infty$.
   This is in agreement with what we stated before: if $x_1 = x_2 \wedge y_1 \neq y_2$ it means that $Q_2 = -Q_1$, and since the point at infinity acts as the identity element we have exactly that $Q_1 + Q_2 = \infty$.

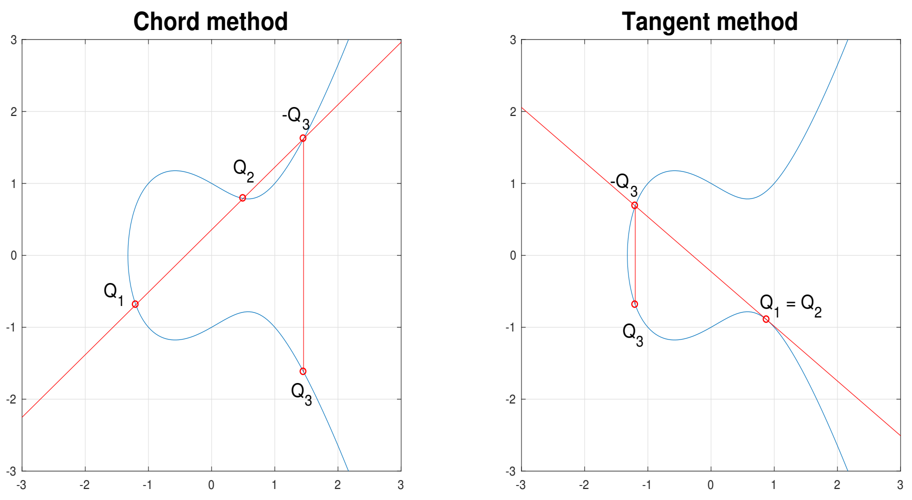


Figure 2.3: Point addition: graphical representation.

3. $Q_1 = Q_2 = (x_1, y_1) \wedge y_1 \neq 0$: when two points on a curve are very close one another, the line through them approximates a tangent line. Therefore, when the two points coincide, we take the line through them to be the tangent line. Implicit differentiation allows us to find its slope m:

$$y^2 = x^3 + ax + b \implies \frac{d(y(x)^2)}{dx} = \frac{d}{dx}(x^3 + ax + b) \implies$$

$$\implies \frac{d(y^2)}{dy}\frac{dy}{dx} = 2y\frac{dy}{dx} = 3x^2 + a \implies$$

$$\implies m = \left(\frac{dy}{dx}\right)_{x_1} = \frac{3x_1^2 + a}{2y_1}.$$

   Again, the equation of the line is $y = m(x - x_1) + y_1$. At this point we can proceed as before: the line intersects the curve in a second point. Substituting the equation of the straight line would result in a cubic equation. But we already know two roots, so that we get: $x_3 = m^2 - 2x_1$ and $y_3 = m(x_1 - x_3) - y_1$.

4. $Q_1 = Q_2 = (x_1, y_1) \wedge y_1 = 0$: as we can see from the previous point in this case the tangent line is vertical and we set $Q_1 + Q_2 = \infty$.

5. The last case we have to deal with is when one of the two points is ∞. Suppose, without loss of generality, that $Q_2 = \infty$; the line through

Q_1 and ∞ is a vertical line that intersects E in the point $-Q_1$, which is the reflection of Q_1 across the x-axis. When we reflect $-Q_1$ to get $Q_3 = Q_1 + Q_2$, we get back to Q_1. Therefore:

$$Q_1 + \infty = Q_1$$

for all points Q_1 on E. Of course, we extend this to include $\infty + \infty = \infty$.

Thanks to these algebraic formulas we can notice that when Q_1 and Q_2 have coordinates in a field K that contains a and b, then $Q_1 + Q_2$ also has coordinates in K. This means that $E(K)$ is closed under the defined addition operation.

Now we state a theorem showing formally that the couple $(E, +)$ is indeed an abelian group. The theorem with the proof can be found in [28].

Theorem 2.2.1. *The operation + as defined above on an elliptic curve E satisfies the following properties:*

1. *Commutativity:* $Q_1 + Q_2 = Q_2 + Q_1,\ \forall Q_1, Q_2 \in E$;
2. *Identity:* $Q + \infty = Q,\ \forall Q \in E$;
3. *Invertibility:* $\forall Q \in E,\ \exists Q' \in E \mid Q + Q' = \infty$. *This point will be denoted by* $-Q$;
4. *Associativity:* $(Q_1 + Q_2) + Q_3 = Q_1 + (Q_2 + Q_3),\ \forall Q_1,\ Q_2, Q_3 \in E$.

Although we won't prove the theorem, it can be useful to give a sketch of its proof. The commutativity is obvious, either from the formulas or from the fact that the line passing through Q_1 and Q_2 is the same as the line passing through Q_2 and Q_1. The identity property of the point at infinity holds by definition, as the existence of the inverse element.
The tricky part of the proof is related with associativity: either it can be proved working out the annoying computations or through a much more elegant and complex method based on projective coordinates, for which we again refer to the bibliography.

Now that we have clear in mind what does addition between EC points mean, we can define subtraction and scalar multiplication:

- Subtraction: given $Q, R \in E$, we set $Q - R = Q + (-R)$. We can do this since everything is perfectly defined in the right hand side of the equation;

- Scalar multiplication: $\forall k \in \mathbb{N}\backslash\{0\},\ kQ = Q + Q + ... + Q$, where the addition is repeated $k - 1$ times.

Notice that to compute the scalar multiplication for a large k it is inefficient to add Q to itself repeatedly. There are several faster approaches, the simplest called the **Double and Add Algorithm**. We stress this fact since, as we will see later on, scalar multiplication and its computational asymmetry are the key ingredients for elliptic curve cryptography: the fact is that it is easily computable[5] but its inversion, the so called discrete logarithm (DL), altough not provable to be hard, is believed to be so: this means that in general we have no efficient algorithm to compute it.

Double and Add Algorithm: To compute kQ, start with the binary representation of k: $k = k_0 + 2k_1 + 2^2k_2 + ... + 2^nk_n$, where $k_0, ..., k_n \in \{0, 1\}$.

Algorithm 2.1 Double and Add algorithm

```
1: procedure DOUBLE_AND_ADD($k$, $Q$)
2:     $R \leftarrow \infty$
3:     for $i \leftarrow 1, n$ do
4:         if $k_i = 1$ then
5:             $R \leftarrow Q + R$
6:         end if
7:         $Q \leftarrow Q + Q$
8:     end for
9:     return $R$
10: end procedure
```

The algorithm can be further sped up through the pre-computation of a vector of multiples of Q: this approach is not always feasible, since the point Q is not always fixed[6].

[5]The double and add algorithm requires polynomial time in the number of bits n representing the numbers involved, compared to the exponential time required by the naive approach; that is, assuming that the point doubling and the point addition require a constant time, we need $O(n^\alpha),\ \alpha > 1$ elementary operations, compared to $O(2^n)$. In particular on average it requires n doublings and $0.5 * n$ additions, since the binary expansion of a random integer has an equal number of 1's and 0's.

[6]The point Q is fixed, for example, when computing the public key from a private key. For a thorough exposition of these concepts we refer to Chapter 3.

Chapter 3

Elliptic curve cryptography

In this chapter we will analyse the cryptographic primitives and assumptions that will allow in the next chapter to introduce the digital signature schemes that are the core of the present work. The first argument will be the specialization of elliptic curves to finite fields, the mathematical structure underpinning the whole elliptic curve cryptography.

3.1 Elliptic curves over finite fields

Up to now we considered the field over which the curve is defined to be the set of real numbers: this is not at all the unique possibility. The typical choice in cryptography is $K = \mathbb{F}_p$, where $\mathbb{F}_p$ is the finite field with p elements. Since there are only finitely many pairs (x, y) with $x, y \in \mathbb{F}_p$, the group $E(\mathbb{F}_p)$ is finite. In practice we consider:

$$\{(x, y) \in \mathbb{F}_p^2 \mid y^2 \equiv x^3 + ax + b \pmod{p},\ 4a^3 + 27b^2 \not\equiv 0 \pmod{p}\} \cup \{\infty\}.$$

It can be shown that the formulas for point addition are the same derived in Section 2.2.1, but where the calculations are done modulo p[1].

[1]Notice that reduction modulo p can be executed much faster if the prime p is a Mersenne or a pseudo-Mersenne prime, i.e. if $p = 2^d - 1$ for some d in the first case or if $p \simeq 2^d$ in the second one.

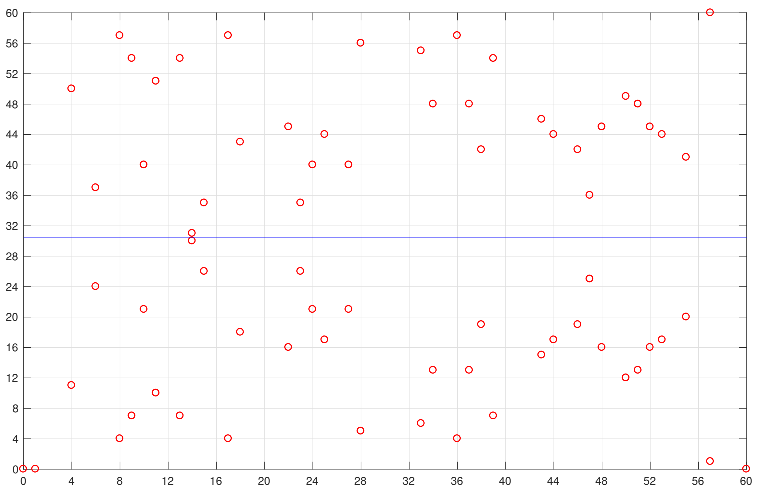


Figure 3.1: The curve $y^2 = x^3 - x$ over $\mathbb{F}_{61}$.

Source: `https://en.wikipedia.org/wiki/Elliptic_curve#Elliptic_curves_over_finite_fields`.

In Figure 3.1 it is represented the typical shape of an elliptic curve over a finite field: we can see that we are not anymore dealing with a smooth curve, but with a finite set of points scattered along the plane. We can notice a certain degree of symmetry with respect to the line $y = \frac{p}{2}$, with the exception of the points with y coordinate equal to zero, that correspond to the roots of the cubic $x^3 + ax + b$ in $\mathbb{F}_p$.
It is still possible to define a geometric method for point addition: the general idea is that we draw the line passing through the points considered. Since we are working in modular arithmetic, this line "repeats" itself along the plane. Once the line intersects a third point, we take the opposite one to be the result of the addition. The approach is presented in Figure 3.2, however the method can be somewhat counterintuitive, especially when dealing with point doubling, so we stick with the algebraic notation.

The number of points on an EC over $\mathbb{F}_q$ plays a central role in the cryptographic applications, so we state here a useful theorem that allows to give an estimate:

Theorem 3.1.1 (Hasse's theorem). *Let $\mathbb{F}_q$ be a finite field and let E be an elliptic curve defined over $\mathbb{F}_q$. Then the order of $E(\mathbb{F}_q)$ (i.e. the number of points) satisfies:*

$$|q + 1 - \#E(\mathbb{F}_q)| \leq 2\sqrt{q}$$

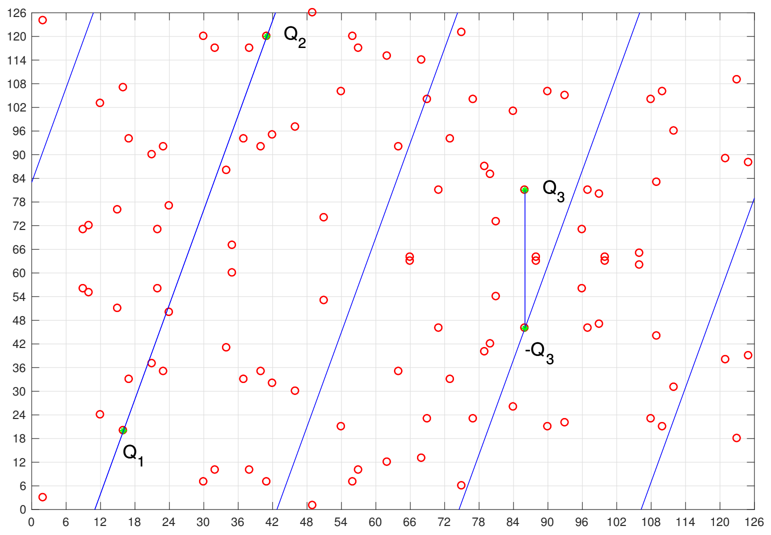


Figure 3.2: Geometric representation for the addition of points on the elliptic curve $y^2 = x^3 - x + 3$ over $\mathbb{F}_{127}$.

Source: [8].

Loosely speaking, Hasse's theorem tells us that the number of points over $E(\mathbb{F}_q)$ increases linearly with the order of the field.
Another observation we want to do is that if $\#E(\mathbb{F}_p)$ is a prime number, then $E(\mathbb{F}_p)$ is a cyclic group: this is a direct consequences of Lagrange's theorem. Since the order of each subgroup is a divisor of the order of the group, we have that, if the order of the group is a prime number, there is no subgroup (except the trivial one, comprising only the point at infinity). This means that it had to exists a generator of the whole group. Suppose this is not the case, i.e. $\nexists G \in E(\mathbb{F}_p) \mid \forall Q \in E(\mathbb{F}_p)\ \exists k \in [1, ..., \#E(\mathbb{F}_p)]$ s.t. $Q = kG$. This means that, $\forall G \in E(\mathbb{F}_p)$, there exists at least a Q for which the relation $Q = kG$ does not hold. Fix G and start adding it repeatedly to itself. Of course we would reach different points in $E(\mathbb{F}_p)$ due to the closure of the addition. However we won't be able to reach Q, by definition. But eventually we would reach the point at infinity (when k gets equal to the order of G): then we would start again, meaning that we would have found a cyclic subgroup. But this is in contrast with Lagrange's theorem, so that we can conclude that the whole set of elliptic curve's points form a cyclic group if $\#E(\mathbb{F}_q)$ is a prime number.

3.1.1 Elliptic curve domain parameters

In order to rely on ECC, the parties involved in a scheme need to agree on a set of parameters called elliptic curve domain parameters. A typical choice is to rely on standardized curves (e.g. those defined in [2]); however, it is still possible to choose other parameters, but this has to be done carefully, as we will explain in this section. We start defining EC parameters, following closely [1].

Elliptic curve parameters are defined as a sextuple $T = (p, a, b, G, n, h)$, where:

- The integer p specifies the prime finite field $\mathbb{F}_p$;
- The two elements $a, b \in \mathbb{F}_p$ specifies the elliptic curve $E(\mathbb{F}_p)$ through the Weierstrass equation;
- G is a point on $E(\mathbb{F}_p)$;
- n is the order of G, i.e. the number of elements of the cyclic subgroup generated by G (that can coincide with the whole $E(\mathbb{F}_p)$);
- $h = \frac{\#E(\mathbb{F}_p)}{n}$ is an integer[2] called the cofactor.

There are different ways in which we can choose these parameters. For example, they can be generated at random or not. Random generation is a conservative choice, since it offers a guarantee against future special purpose attacks. In this case it could be better to use verifiable random parameters, meaning that (a, b) and/or G are obtained as output from a secure hash function[3], applied to some seed S. When choosing verifiable random parameters, the sextuple T should be coupled with the seed value S to allow parameters' validation. However, non random curves are typically built in particular ways to ensure efficient computations. Standardized curves, that comprises both types of EC, have the advantage of underpin interoperability, plus an high degree of reliability having been heavily tested.
Now that we explained what they are and how they can be chosen, we will show some constraints and the validation process for the sextuple T.

Parameters' selection:

[2]We can deduce that h is an integer directly from Lagrange's theorem.

[3]An hash function in general is a function that takes as input strings of arbitrary length and outputs in a fixed length sequence of bytes. For a discussion on the security properties of an hash function we refer to the appendix of [1].

1. Select the approximate security level in bits, t[4];
2. Choose a prime p such that $\lceil log_2 p \rceil = 2t$ if $80 < t < 256$, such that $\lceil log_2 p \rceil = 521$ if $t = 256$ and such that $\lceil log_2 p \rceil = 192$ if $t = 80$;
3. Select a and b in $\mathbb{F}_p$ such that:
   - $4a^3 + 27b^2 \neq 0 \pmod{p}$;
   - $\#E(\mathbb{F}_p) \neq p$;
4. Select $G \in E(\mathbb{F}_p)$ such that:
   - $p^B \neq 1 \pmod{n}, \ \forall 1 \leq B < 100$;
   - $h \leq 2^{\frac{t}{8}}$;
   - $n - 1$ and $n + 1$ should each have a large prime factor r, which is large in the sense that $log_n r > \frac{19}{20}$.

These requirements may look strange, but are all needed in order to avoid special attacks. In particular, the two requirements at point 3 are needed in order for the curve to be non singular and to avoid Smart's attack[5], respectively.
The three requirements at point 4 instead are needed to ensure that the curve is robust against MOV and Cheon's attack, and to ensure that the order of G is sufficiently high.
Notice however that the choice of the parameters does not secure against all possible attacks. Indeed other kind of attacks may be possible, such as side channel attacks: timing and power analysis are an example. They exploit the differences in the point addition and doubling operations, that lead to different timings or power consumptions. Possible solutions are a change of coordinates or the use of Edwards curves, a special family of elliptic curves for which addition and doubling can be done with the same operation.

Parameters' validation: Given $T = (p, a, b, G, n, h)$ and t, the parameters are deemed valid if:

1. p is a prime such that $\lceil log_2 p \rceil = 2t$ if $80 < t < 256$, or such that $\lceil log_2 p \rceil = 521$ if $t = 256$, or such that $\lceil log_2 p \rceil = 192$ if $t = 80$;

[4]The security level should be chosen such that solving the ECDLP requires 2^t curve's operations. The SEC standard that we are following restricts to $t \in \{80, 112, 128, 192, 256\}$.

[5]The points on the curve are mapped to the elements of the additive group of $\mathbb{F}_p$, so that the discrete logarithm problem is solvable in polynomial time through the **Extended Euclidean Algorithm**.

2. a, b, x_G and y_G are integers in $[0, ..., p-1]$;
3. $4a^3 + 27b^2 \neq 0 \pmod{p}$;
4. $y_G^2 = x_G^3 + ax_G + b \pmod{p}$;
5. n is a prime number;
6. $h \leq 2^{\frac{t}{8}}$ and $h = \lfloor (\sqrt{p}+1)^2/n \rfloor$;
7. $nG = \infty$;
8. $p^B \neq 1 \pmod{n}$, $\forall 1 \leq B < 100$ and $n \neq p$.

If the parameters are verifiably random, it should also be checked that (a, b) and/or G have been correctly derived from the seed S.
These checks are needed to verify that: the curve has the required difficulty level, it is defined over $\mathbb{F}_p$, it is non singular, G is a point of the curve, the DL is difficult, h is effectively the cofactor of G, G has order n and neither MOV nor Smart's attacks are possible.

3.1.2 Elliptic curve key pairs

In this section we would like to briefly analyse the core of the public key cryptography based on elliptic curves: the concept of elliptic curve key pair. Shortly, public key cryptography is a cryptographic system that relies on pairs of keys, a public and a private key, with different roles. The asymmetry of the keys is the reason behind the fact that public key cryptography is sometimes called also asymmetric cryptography. We talk about asymmetry since public keys can be published, while private keys must be kept secret, as the names suggest. This accomplishes two functions: authentication, since everybody with the public key can verify that the holder of the paired private key sent a signed message, and encryption, where the public key is used for encryption, but only the owner of the private counterpart can decrypt.

Given some elliptic curve domain parameters $T = (p, a, b, G, n, h)$, an elliptic curve key pair $\{q, Q\}$ associated with T consists of an elliptic curve secret key q which is an integer in $[1, n-1]$ and an elliptic curve public key $Q = (x_Q, y_Q)$, which is the point $Q = qG$.
The choice of the colours is intended to make clear what is secret and what can be made public.
To prevent some attacks (e.g. small subgroup attacks) it is desirable to validate the public key of the counterparties. This can be achieved following these steps:

1. Check that $Q \neq \infty$;
2. Check that x_Q and y_Q are integers in $[0, p-1]$ and that $y_Q^2 = x_Q^3 + ax_Q + b \pmod{p}$;
3. Check that $nQ = \infty$.

Through this algorithm we are checking that Q is effectively a point on the curve different from the point at infinity and that it belongs to the cyclic subgroups generated by G. Indeed, if the relation $Q = qG$ holds we have that $nQ = n(qG) = q(nG) = q\infty = \infty$. If this would not be the case, the third check would fail.

3.1.3 Jacobian coordinates

We have told that for EC over finite fields the same formulas previously seen hold, with the calculations done modulo p. We have not told that this comes with a downside: in particular we have now to deal with modular inversion, an operation that, although can be done efficiently, is up to two orders of magnitude slower than field's multiplication.
In the next chapter we will deal with digital signatures and we will see that ECDSA requires modular inversion. This traduces in poor performances when it comes to a system like Bitcoin in which signatures are (potentially) verified by each participant in the network to validate transactions. Therefore, it is sometimes advantageous to avoid modular inversion, either in the formulas for point addition or in the signature scheme. The first case can be solved through a change of coordinates: the approach consists in writing all the points as points in a projective space, the key idea being to defer the divisions by multiplying them into a denominator, represented then as a new coordinate, instead of performing every division immediately. Only at the very end we perform a single division to convert from projective coordinates back to affine coordinates.

Jacobian coordinates are a modification of projective coordinates, typically used since they lead to a faster doubling procedure. In affine form, each elliptic curve point has two coordinates (x, y). In projective form, each point will have three coordinates (X, Y, Z), with the restriction that Z is never zero for points different from the point at infinity. The forward mapping is given by $(x, y) \mapsto (xz^2, yz^3, z)$, for any non zero z (usually chosen to be 1 for convenience). The reverse mapping is given by $(X, Y, Z) \mapsto (X/Z^2, Y/Z^3)$,

as long as Z is non zero.
The elliptic curve $y^2 = x^3 + ax + b$ becomes:

$$Y^2 = X^3 + aXZ^4 + bZ^6.$$

The point at infinity now has the coordinates $\infty = (1 : 1 : 0)$: indeed, substituting $Z = 0$ in the equation defining the curve we get $Y^2 = X^3$; then we simply normalize.
Let $Q_i = (X_i : Y_i : Z_i)$, $i = 1, 2$ be points on the elliptic curve $Y^2 = X^3 + aXZ^4 + bZ^6$. Then:

$$(X_3 : Y_3 : Z_3) = (X_1 : Y_1 : Z_1) + (X_2 : Y_2 : Z_2),$$

where X_3, Y_3 and Z_3 are computed as follows:

- $Q_1 \neq \pm Q_2$: we have the affine test $x_1 = x_2$, that in jacobian coordinates correspond to the check $X_1/Z_1^2 = X_2/Z_2^2 \implies X_1 Z_2^2 = X_2 Z_1^2$.
  We start again from the slope of the line passing through Q_1 and Q_2.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{Y_2/Z_2^3 - Y_1/Z_1^3}{X_2/Z_2^2 - X_1/Z_1^2} = \frac{Y_2/Z_2^3 - Y_1/Z_1^3}{X_2/Z_2^2 - X_1/Z_1^2} \cdot \frac{Z_1^3 Z_2^3}{Z_1^3 Z_2^3} =$$

$$= \frac{Y_2 Z_1^3 - Y_1 Z_2^3}{X_2 Z_1^3 Z_2 - X_1 Z_1 Z_2^3}.$$

  Defining $T = Y_1 Z_2^3$, $U = Y_2 Z_1^3$, $W = U - T$, $R = X_1 Z_2^2$, $S = X_2 Z_1^2$ and $V = S - R$ we can write:

$$m = \frac{U - T}{SZ_1Z_2 - RZ_1Z_2} = \frac{W}{VZ_1Z_2}.$$

  Now consider:

$$x_3 = m^2 - x_1 - x_2 = \frac{W^2}{V^2 Z_1^2 Z_2^2} - \frac{X_1}{Z_1^2} - \frac{X_2}{Z_2^2} = \frac{W^2}{V^2 Z_1^2 Z_2^2} - \frac{X_1}{Z_1^2} \cdot \frac{V^2 Z_2^2}{V^2 Z_2^2} - \frac{X_2}{Z_2^2} \cdot \frac{V^2 Z_1^2}{V^2 Z_1^2} =$$

$$= \frac{W^2 - V^2 X_1 Z_2^2 - V^2 X_2 Z_1^2}{V^2 Z_1^2 Z_2^2} = \frac{W^2 - V^2 R - V^2 S}{V^2 Z_1^2 Z_2^2} = \frac{W^2 - V^2(S + R)}{V^2 Z_1^2 Z_2^2} =$$

$$= \frac{W^2 - V^2(S - R + 2R)}{V^2 Z_1^2 Z_2^2} = \frac{W^2 - V^3 - 2RV^2}{V^2 Z_1^2 Z_2^2}.$$

$$y_3 = m(x_1 - x_3) - y_1 = \frac{W}{VZ_1Z_2} \left(\frac{X_1}{Z_1^2} - \frac{W^2 - V^3 - 2RV^2}{V^2 Z_1^2 Z_2^2} \right) - \frac{Y_1}{Z_1^3} =$$

$$= \frac{W}{VZ_1Z_2} \left(\frac{V^2 X_1 Z_2^2 - W^2 + V^3 + 2RV^2}{V^2 Z_1^2 Z_2^2} \right) - \frac{Y_1}{Z_1^3} =$$

$$= \frac{W(RV^2 - [W^2 - V^3 - 2RV^2]) - V^3Y_1Z_2^3}{V^3Z_1^3Z_2^3} =$$
$$\frac{W(RV^2 - [W^2 - V^3 - 2RV^2]) - TV^3}{V^3Z_1^3Z_2^3}.$$

Thus, we can write:

$$X_3 = W^2 - V^3 - 2RV^2,\ Y_3 = W(RV^2 - X_3) - TV^3,\ Z_3 = VZ_1Z_2,$$

where

$$T = Y_1Z_2^3,\ U = Y_2Z_1^3,\ R = X_1Z_2^2,\ S = X_2Z_1^2,\ V = S-R,\ W = U-T.$$

- $Q_1 = Q_2$: now we have to check if $y = 0$. This is equivalent to check if $Y/Z^3 = 0 \implies Y = 0$, since $Z \neq 0$.

  $$m = \frac{3x^2 + a}{2y} = \frac{3(X/Z^2)^2 + a}{2Y/Z^3} = \frac{3X^2/Z^4 + a}{2Y/Z^3} \cdot \frac{Z^4}{Z^4} = \frac{3X^2 + aZ^4}{2YZ} = \frac{W}{2YZ},$$

  if we define $W = 3X^2 + aZ^4$.

  $$x_3 = m^2 - 2x = \frac{W^2}{4Y^2Z^2} - 2\frac{X}{Z^2} = \frac{W^2 - 8XY^2}{4Y^2Z^2} = \frac{W^2 - 2V}{4Y^2Z^2},$$

  once we define $V = 4XY^2$.

  $$y_3 = m(x - x_3) - y = \frac{W}{2YZ}\left(\frac{X}{Z^2} - \frac{W^2 - 2V}{4Y^2Z^2}\right) - \frac{Y}{Z^3} =$$
  $$= \frac{W(4XY^2 - [W^2 - 2V])}{8Y^3Z^3} - \frac{Y}{Z^3} = \frac{W(4XY^2 - [W^2 - 2V]) - 8Y^4}{8Y^3Z^3} =$$
  $$= \frac{W(V - [W^2 - 2V]) - 8Y^4}{8Y^3Z^3}.$$

  Finally we can write:

  $$X_3 = W^2 - 2V,\ Y_3 = W(V - X_3) - 8Y^4,\ Z_3 = 2YZ,$$

  where

  $$W = 3X^2 + aZ^4,\ V = 4XY^2.$$

- Once again we can consider jointly the separate cases $Q_1 = Q_2 \wedge y = 0$ and $Q_1 = -Q_2$: indeed, by definition, we have $Q_1 + Q_2 = \infty$.

Looking at the formulas we can see that no inversion is involved in the calculations. Moreover, a further speed up is possible when doubling if we set $a = -3$[6]: we have $W = 3X^2 - 3Z^4 = 3(X^2 - Z^4) = 3(X + Z^2)(X - Z^2)$, which can be computed via one squaring and one multiplication rather than via three squarings.

We implemented this approach in the library that can be found at `https://github.com/dginst/BitcoinBlockchainTechnology` to improve the efficiency of the curve's operations. The speed up checks have been performed considering the curves secp192k1, secp192r1, secp224k1, secp224r1, secp256k1, secp256r1, secp384r1 and secp521r1 as described in [2], resulting in a scalar multiplication that is from six to seven times faster in all the cases.

3.1.4 The Bitcoin curve: secp256k1

Since this work is mainly focused on applications that can be deployed in Bitcoin, we would like to present the elliptic curve over a finite field used there. This could also suggest the shape of the curves used in practice.

The curve is named secp256k1: the naming is not casual, and its analysis could help a better understanding of the curve's properties. It begins with **sec** to denote "Standards for Efficient Cryptography", the documentation from which it is taken; then it follows a **p**, denoting the use of parameters over a prime field $\mathbb{F}_p$, in contrast with the so called binary fields $\mathbb{F}_{2^m}$, denoted by the letter **t**; the **p** is followed by a number, 256, denoting the length in bits of the field size p, that suggests the difficulty of solving the DL on the curve; then in comes a **k** to denote parameters associated with a Koblitz curve[7], to be distinguished from an **r**, that would denote the use of verifiably random parameters; at last we find the sequence number **1**, meaning that this curve is the first, actually the unique, with all these characteristics.
We have already discussed the benefits of using a random curve; however, secp256k1 was constructed in a special non random way to ensure efficient computations.
Here follows the sextuple T defining the secp256k1 curve:

- The finite field $\mathbb{F}_p$ is defined by the pseudo-Mersenne prime number:

  $p =$ FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF

[6]This is the reason why some standardized curves make this choice for the parameter a.

[7]The name Koblitz curve used here has the same meaning as in [2].

FFFFFFFFF FFFFFFFFE FFFFFFC2F =

$= 2^{256} - 2^{32} - 2^9 - 2^8 - 2^7 - 2^6 - 2^4 - 1.$

- The defining equation E: $y^2 = x^3 + ax + b$ is determined by:

  $a =$ 00000000 00000000 00000000 00000000 00000000 00000000 00000000 00000000 =

  $= 0$;

  $b =$ 00000000 00000000 00000000 00000000 00000000 00000000 00000000 00000007 =

  $= 7$.

  Hence E: $y^2 = x^3 + 7$.

- The point G in compressed form is[8]:

  $G =$ 02 79BE667E F9DCBBAC 55A06295 CE870B07 029BFCDB 2DCE28D9 59F2815B 16F81798,

  and in uncompressed form:

  $G =$ 04 79BE667E F9DCBBAC 55A06295 CE870B07 029BFCDB 2DCE28D9 59F2815B 16F81798 483ADA77 26A3C465 5DA4FBFC

  0E1108A8 FD17B448 A6855419 9C47D08F FB10D4B8.

- Finally, the order n of G and the cofactor are:

  $n =$ FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFE BAAEDCE6 AF48A03B BFD25E8C D0364141,

  $h = 01$

[8]The starting bytes 0x02 and 0x04 mean exactly that the first expression of G is in compressed form, while the second one is uncompressed. For what concerns the compressed form we need to have informations about the y coordinate. 0x02 is used when it is even, while 0x03 is used when it is odd: if y is a square root in $\mathbb{F}_p$, we know that also $-y \pmod{p} = p - y$ is a square root. Since p is an odd prime, we are sure that one of the roots is odd while the other is even.

3.2 The discrete logarithm problem

In this section we will deal more in depth with the obscure concept of discrete logarithm problem.
In general the DLP is defined over a group $\mathbb{G}$. For the moment write $\mathbb{G}$ in multiplicative notation and consider $x, y \in \mathbb{G}$ such that y is in the subgroup generated by x. The DLP is the problem of determining an integer $k \geq 1$ such that $x^k = y$. This explain the name, since it would mean to find $k = log_x y$.
Typically, the cryptosystems whose security is based on ECC, depend on the difficulty of solving the DLP defined over the EC (ECDLP). One way of attacking the DLP is simple brute force: try all possible values of k until one works. This is impractical when the answer k is an integer of several hundred digits, which is a typical size used in cryptography (indeed this approach is fully exponential in the number of bits representing k[9]). Therefore, better techniques are needed.
More formally, the DLP can be defined as follows.

Definition 3.2.1. *Let $\mathbb{G}$ be a cyclic group of order n and let g be a generator. An algorithm $\mathcal{A}$ is said to (t, ϵ)-solve the DLP in $\mathbb{G}$ if on input a random $h \in \mathbb{G}$, it runs in time at most t and returns $k \in \{0, ..., n-1\}$ such that $h = g^x$ with probability at least ϵ.*

The most general algorithms that works in any group has a time upper bound of $O(|\mathbb{G}|^{\frac{1}{2}})$. But it turns out that the DLP is significantly easier in some groups than it is in others. In order of difficulty:

1. The additive group of $\mathbb{F}_q$: the problem here can be stated as finding k such that $kx = y$, i.e. we only need to find the inverse of x, and we have already stated that this can be done efficiently through the extended euclidean algorithm (in particular this requires $O(log_2(q))$ elementary operations, meaning that the problem can be solved in polynomial time);

2. The multiplicative group $\mathbb{F}_q^{\times}$: the problem here is finding k such that $x^k = y$. It can be shown that there exist some algorithms that work in sub-exponential time (the so called index calculus methods);

3. Elliptic curves over finite fields: the fastest known procedure to solve the ECDLP on general curves are collision algorithms. Those algo-

[9]Assume for example to work with secp256k1, the Bitcoin curve. This means that $p \simeq 2^{256}$, and the same holds true for n. By brute force we try all possible values of k, that ranges from 1 to $n-1$. Trying all possible k would lead to a number of steps of the order $O(2^{256})$, exponential in the number of bits representing the order of the group.

rithms are fully exponential: this explains the widespread adoption of ECC in recent years

The last point is linked to the primary benefit of ECC, that is a smaller key size: since the problem is harder, in order to have the same level of security we can rely on smaller keys, reducing storage and transmission requirements.

The two most general algorithms are known as **Baby Step, Gian Step Algorithm** and as the **Pollard's ρ Method** with its variant[10]: as explained above both runs in time $O(|\mathbb{G}|^{\frac{1}{2}})$, but the first is exponential also in the memory required to run, making it absolutely impractical. But let's focus on the run time of the algorithms on the Bitcoin's curve: assuming that the scalar multiplication requires constant time on the order of milliseconds, solving the ECDLP would require time proportional to $10^{-3} * 2^{128} \simeq 10^{35}$ seconds or around 10^{28} years. It is easy to see that even relying on optimizations, super computers or parallelization, nowadays it is computationally infeasible to solve the ECDLP.

[10]For a detailed but still easily understandable presentation of the algorithms we refer to [8].

Chapter 4

Digital signature schemes

In this chapter we will present the general idea behind digital signatures: what they are and why they are used. Then we will discuss the Elliptic Curve Digital Signature Algorithm (ECDSA), the one actually used in Bitcoin. After having delved into its problems, we will present the adaptation of the Schnorr signature algorithm to elliptic curve cryptography (there is currently a discussion about its adoption in Bitcoin). The comparison will turn out to be pitiless.
For a detailed explanation of how signatures are actually used in Bitcoin we refer to Appendix A.

To understand what a digital signature scheme is, we start considering the situation in which Alice wants to sign a document that she aims sending to Bob, who wants assurance that the document comes from Alice and has not been tampered with. As for real life signatures, the aim of a proper digital signature scheme should be to prove the authenticity of a message or document, in the digital realm. Thus, we can list some properties that a signature scheme should provide:

- Authentication: the recipient is confident that the message comes from the alleged sender;
- Non repudiation: the sender cannot deny having sent the message;
- Integrity: ensures that the message has not been altered during transmission.

A naive approach to this problem would be to digitalize Alice's signature: but in this case it would be too easy for an eavesdropper to simply copy her signature and append it to any other document.

In order to achieve the listed properties, as first thing we should tie the signature to the document, so that it could not be used again: this would also solve the problem of integrity. If the message changes, the signature is not valid anymore. Moreover, it should be possible for parties other than Bob to verify that the signature is valid, and it should be possible to show that Alice must have been the person who signed the document: this would ensure the authentication and non repudiation properties.
In cryptography usually some security models are introduced for the so called unforgeability, the property that prevents adversaries from being able to forge a signature on behalf of Alice. In the following we consider the security model known as existentially unforgeability under chosen message attack (UF-CMA), where forgery means winning at the following game:

1. The signer (Alice) gives her public key to the adversary (also known as forger or attacker);

2. The adversary sends messages m_i to the challenger (Alice) and receives valid signature σ on the message m_i for the given public key. He may do this as many times as he likes;

3. The adversary produces a message $m \neq m_i$, $\forall i$ (i.e. the message has not been queried before) along with a valid signature on it.

In this model, unforgeability means that no computationally bounded adversary is able to forge a signature except with negligible probability: in this setting a forgery consists in a signature for a message/public key pair never queried. This hypothesis can be relaxed, excluding from the winning conditions only the triplets message/public key/signature obtained from a query, leading to the concept of strong unforgeability under chosen message attack (SUF-CMA).

Now that we have presented the security model, we can look more formally at the scheme: a signature scheme is a triplet of algorithms $(KeyGen, Sign, Ver)$, where the first two are randomized algorithms, while the third one is deterministic. To sign a document or message, the signer proceeds as follows:

- He runs the key generation algorithm with no inputs[1] to generate a key pair: $\{q, Q\} = KeyGen()$;

- He runs the signing algorithm on message m and private key q, resulting in a signature σ: $\sigma = Sign(m, q)$;

[1]Some authors consider as input a security parameter, but we take it as given.

- A verifier can check the validity of the signature running the verification algorithm as follows: $Ver(\sigma, m, Q) \in \{0, 1\}$. If $Ver(\sigma, m, Q) = 1$ then the signature is valid, otherwise it is not.

A proper signature scheme should satisfy the following consistency equation: $Ver(Sign(m, q), m, Q) = 1, \ \forall \{q, Q\}$ returned from the key generation algorithm.
A possibility to construct signature schemes relies on the difficulty of discrete logarithms on elliptic curves. In the next sections we will analyse deeply merits and flaws of two schemes constructed on this hypothesis.

4.1 Elliptic curve digital signature algorithm (ECDSA)

Assume that Alice wants to sign a message M and send it to Bob. The first thing to do for the parties involved is to agree on an elliptic curve determined by the set of parameters T and on a cryptographically secure hash function. This function is used to get a digest from M: usually, a digital signature can only sign small amounts of data. Applying a proper hash function to a message produces a digest which is small enough and that can act as digital fingerprint if the output space is large enough. Now Alice chooses a secret integer $q_A \in [1, ..., n-1]$ (the private key) and computes her public key $Q_A = q_A G$, that sends to Bob who has to validate it (Alice shouldn't be able to repudiate the signature due to the use of an invalid public key).
The public key can be made public thanks to the difficulty of the ECDLP; moreover, the fact that it is public allows anyone to verify Alice's signature, enforcing the non repudiation property.
Now that we have talked about the setup procedure we can give a closer look to the signing and verification algorithms, displayed by Algorithms 4.1 and 4.2 (L_n represents the length in bits of the group order n).

In the signing algorithm it is drawn at random an integer k at step 4: this value has not only to be secret, it has to be selected differently each time[2]. Otherwise the secret key would be recoverable. Indeed, let's consider two

[2]Loosely speaking, the security of the signature derives from the fact that we have one equation, the one defining s, in two unknowns, k and q. If we use the same k for another signature, we give the adversary an additional equation, allowing him to solve the system.

Algorithm 4.1 ECDSA: signing algorithm

```
procedure ECDSA_SIG($M,\ q$)
    $m \leftarrow \text{hash}(M)$
    Let $z$ be the $L_n$ leftmost bits of $m$: $z \leftarrow \text{int}(z)$
    $k \overset{\$}{\leftarrow} \{1, ..., n-1\}$
    $K \leftarrow kG$
    $r \leftarrow x_K \pmod{n}$
    if $r = 0$ then
        go to step 4
    end if
    $s \leftarrow k^{-1}(z + rq_{)} \pmod{n}$
    if $s = 0$ then
        go to step 4
    end if
    return $(\text{bytes}(r), \text{bytes}(s))$
end procedure
```

Algorithm 4.2 ECDSA: verification algorithm

```
procedure ECDSA_VER($(r, s), M, Q$ )
    $r \leftarrow \text{int}(r),\ s \leftarrow \text{int}(s)$
    if $r \notin \{1, ..., n-1\}$ or $s \notin \{1, ..., n-1\}$ then
        return False
    end if
    $m \leftarrow \text{hash}(M)$
    Let $z$ be the $L_n$ leftmost bits of $m$: $z \leftarrow \text{int}(z)$
    $u_1 \leftarrow zs^{-1} \pmod{n},\ u_2 \leftarrow rs^{-1} \pmod{n}$
    $K \leftarrow u_1G + u_2Q$
    if $K = \infty$ then
        return False
    end if
    return $r = x_K$ (mod $n$)
end procedure
```

signatures on different messages (r, s) and (r, s') derived using the same k. After having converted r, s and s' to integers, we have:

$$s = k^{-1}(z + rq_A) \pmod n,\ s' = k^{-1}(z' + rq_A) \pmod n \implies$$

$$\implies s - s' = k^{-1}(z - z') \pmod n \implies k = (z - z')(s - s')^{-1} \pmod n.$$

From just two signatures we were able to recover the common k and now we can derive the private key from the definition of s or from the definition of s':

$$q_A = r^{-1}(ks - z) \pmod n.$$

To ensure that k is unique for each message one may bypass random number generation completely, since it is one of the main sources of problems in cryptosystems, and generate deterministic signatures by deriving k from both the message and the private key as: $k = \text{hash}(q \mid\mid m)$. This approach ensures both a different k every time the message changes and unpredictability.
Moreover there are some ways to speed up the verification procedure: through the Shamir's trick, the sum of two scalar multiplication can be calculated faster than the two isolated scalar multiplication at point 9. A further speed up occurs when making K efficiently recoverable from r. In this case one may verify that $sK = zG + rQ_A$. This can be done by including the point K itself in the signature, either in compressed (x_K coordinate and a byte telling the sign of y_K, 0x02 or 0x03 if is is even or odd, respectively) or uncompressed form: the gain in speed verification in this case would come at expenses of the signature's size.

Let's give a look at the correctness' proof of the procedure:
Proof: We have to prove that $kG = K = u_1G + u_2Q_A$.

- From the definition of Q_A: $u_1G + u_2Q_A = u_1G + u_2q_AG$;
- From the definition of u_1 and u_2: $u_1G + u_2Q_A = s^{-1}(z + rq_A)G$;
- From the definition of s: $u_1G + u_2Q_A = (k^{-1}(z + rq_A))^{-1}(z + rq_A)G = k(z + rq_A)^{-1}(z + rq_A)G = kG = K$.

$\square$

At the beginning of the chapter we have said that the comparison between ECDSA and Schnorr will be pitiless, so it is clear that ECDSA is far from being perfect. Let's discuss some of its problems:

- Malleability: the signature (r, s) may be replaced with $(r, -s \pmod n)$[3], because this is an equivalent signature. This means that a third party, without access to the private key can alter an existing valid signature for a given public key and message into another signature that is valid for the same key and message. However this is not regarded as a forgery on the scheme since the message is the original one, but this possibility prevented the deployment of layer 2 solutions for the Bitcoin scalability problem (e.g. the Lightning Network described in [26] or the eltoo protocol discussed in [7]): recently this problem has been circumvented through the soft fork SegWit.
  Another solution would be to augment the verification equation with the check $s \leq (n-1)/2$;

- The signing and verification operations need the calculation of modular inverses and, as we have already pointed out, it is a slow operation. Moreover this computation can be done only if n is a prime number. Indeed we have a theorem stating that $x \in \mathbb{Z}_n$ is invertible if and only if the greatest common divisor of x and n is one[4]. This obviously holds true for every $x \in \mathbb{Z}_n$ if and only if n is a prime number. However this is not at all a problem, but just a remark: n has to be a huge prime number for the ECDLP to be hard;

- In order for ECDSA to be secure, it can be shown that we need a small cofactor h. This is due to the presence of attacks on the conversion function used in the sixth step of the signing operation, the one that entails taking the x coordinate of K and reducing it modulo n. It is told in [1] that this function has to be almost bijective for ECDSA to be secure, which means that an attacker cannot find an r for which a random K has non negligible probability of satisfying $r = x_K \pmod n$. Indeed we have that the integers $x_K = r + jn$ for $j \in \{0, 1, 2, ..., h\}$, if $h = 1$ or 2 as recommended in [2], are linked usually to one valid candidate x coordinate.

[3]With the malleated signature we would have: $u_1G+u_2Q_A = u_1G+u_2q_AG = -s^{-1}(z+rq_A)G = -(k^{-1}(z+rq_A))^{-1}(z+rq_A)G = -k(z+rq_A)^{-1}(z+rq_A)G = -kG = -K$. The verification procedure would then check whether the x coordinate of $-K$ would equal r; this would be the case since K and $-K$ share the same x coordinate.

[4]`https://www.coursera.org/learn/crypto/lecture/2YWK8/notation`, from minute 7.39.

4.1.1 ECDSA: multi-signature

Multi-signature schemes allow a group of users to cooperate (interactively or not) to sign a single message or document, usually producing a joint signature that is more compact than a collection of distinct signatures from all users. Verification usually requires the message m and the set of public keys of the signers.
Before we analyse ECDSA multi-signature, let's see formally what these schemes are.

A multi-signature algorithm is a triplet of algorithms $(KeyGen, Sign, Ver)$. To sign jointly a document or message m, each participant i of the scheme should proceed as follows:

- She generates a public key pair through the key generation algorithm: $\{q_i, Q_i\} = KeyGen()$;
- She sends her public key to all the other participants, so that every user can gather the same multiset L of public keys: $L = \{Q_1, Q_2, ..., Q_n\}$;
- She runs the signing algorithm on message m, secret key q_i and multiset of public key L: $\sigma = Sign(m, q_i, L)$;
- A verifier can check the validity of the signature through the verification algorithm: $Ver(\sigma, m, L) \in \{0, 1\}$. If it returns 1 then the signature is valid, otherwise it is not.

A proper multi-signature scheme should output to every signer the same signature σ that satisfies the following consistency equation:

$$Ver(Sign(m, q_i, L), m, L) = 1, \ \forall q_i \text{ s.t. } q_iG = Q_i \in L.$$

It is easy, given a signature scheme, to extend it to the multi-user setting in a naive way: each signer signs the message with its own private key, the final signature being the concatenation of the partial signatures. This is the approach used today in Bitcoin with ECDSA: the problem is that the signature size increase linearly in the number of participants, leading to higher fees for the users and a bloat of the blockchain size that affects every participant in the network.. Ideally, the size of the signature should be independent of the number of participant. Recently there have been proposed more efficient multi-signature protocol in [16] and [31]: however they still need review by the community and lack an actual implementation. Moreover the underlying signature scheme is the far-from-being-perfect ECDSA, so we refer the interested reader to the original papers without delving into the technicalities of these works.

ECDSA naive multi-signatures require multiple public keys in order to validate multiple signatures, i.e. despite the name, multi-signature in Bitcoin is just a tuple of distinct users' signatures.
In literature, the name multi-signature is associated to schemes that, given n participants, require the collaboration of everybody to produce a valid signature. In Bitcoin this notion is extended to the so called threshold schemes: if before we talked about n-of-n schemes, these can be though of as m-of-n, where at least $m \leq n$ participants have to collaborate to sign.

4.1.2 ECDSA: public key recovery

Given an ECDSA signature (r, s) and linked EC parameters T, it is possible to determine the public key Q associated with the private key used to generate the signature: this tool is called public key recovery and turns out to be particularly useful in bandwidth constrained environments. Potentially, many candidate public keys can be recovered from a signature, their number depending on the cofactor h. The algorithm for public key recovery is presented in Algorithm 4.3, where the function EC_point() recovers an elliptic curve point given the x coordinate (it returns **False** in case no point with that x coordinate exists).

In the set *keys* there is the correct key. Authenticity could be checked against a public key in some certificate or directory.
Through public key recovery, the verification of a signature becomes implicit: you first recover Q from (r, s). Then, the signature is deemed valid once the public key Q has been authenticated. This step is necessary since, in general, from every signature we can extract a public key.

4.2 Schnorr signature algorithm

In this section we finally pass to analyse the core of the present work, following closely the recent Bitcoin Improvement Proposal by Pieter Wuille ([29]) on the standardization of the Schnorr signature.
As stated in the BIP, ECDSA is standardized, while the Schnorr signature, due to the presence of a patent in past years, is not. As we will see this lack

Algorithm 4.3 ECDSA: public key recovery

```
 1: procedure ECDSA_RECOVERY($(r, s), M$)
 2:     $keys \leftarrow \{\}$
 3:     $r \leftarrow \text{int}(r),\ s \leftarrow \text{int}(s)$
 4:     for $j \leftarrow 0, h - 1$ do
 5:         $x \leftarrow r + jn$
 6:         $K \leftarrow \text{EC\_point}(x)$
 7:         if $K =$ False or $nK \neq \infty$ then
 8:             go to step 4
 9:         end if
10:         $m \leftarrow \text{hash}(M)$
11:         Let $z$ be the $L_n$ leftmost bits of $m$: $z \leftarrow \text{int}(z)$
12:         for $k \leftarrow 1, 2$ do
13:             $Q \leftarrow r^{-1}(sK - zG)$
14:             $keys \leftarrow keys + Q$
15:             $K \leftarrow -K$
16:         end for
17:     end for
18:     return $keys$
19: end procedure
```

hides some possibilities; on top of that, Schnorr itself is better than ECDSA in several aspects:

- Security proof: the security of Schnorr signature is proved in the random oracle model[5] assuming the ECDLP is hard. Such a proof does not exist for ECDSA, since its security proof requires the generic group model. The two proofs can be found in [12] and [6], respectively. The provably secure construction offered by Schnorr is important since it could prevent future attacks on ECDSA[6];
- Non-malleability: ECDSA, as already showed, is malleable. On the other hand, Schnorr signatures are provably non-malleable[7];
- Linearity: Schnorr verification algorithm is linear in both the terms of the signature. This property allows the construction of several higher level protocols with interesting properties, some of which will be studied in Chapter 5.

Thanks to these characteristics, the introduction of Schnorr in Bitcoin would result in privacy and efficiency improvements. Moreover, thanks to the version system introduced by SegWit, Schnorr can be deployed "easily" through a soft fork. Schnorr signature could moreover enable easier implementations of some protocols that take advantage of the Bitcoin blockchain, such as atomic swaps and payment channels.

The fact that Schnorr is not standardized, allows us to make design choices in order to implement other features in addition to the native ones:

- Batch validation: the formulation of ECDSA that is standardized cannot be validated more efficiently in batch compared to individually. Switching to Schnorr offers the opportunity to choose a formulation that allows batch validation. This can be an important feature in Bitcoin since when validating a block we need to verify multiple signatures; moreover, we are interested that all of them are valid: if the verification fails, we do not care about which signature caused the failure. We simply reject the whole block;

[5]In the random oracle model (ROM) hash functions are modelled as truly random functions whose outputs are uniformly random and can be computed by querying a public oracle.

[6]This is a major achievement since Schnorr provides better security based on the same hypothesis: we do not need to introduce stronger conditions to improve the security.

[7]Segregated witness (SegWit) solves the known malleability problem of ECDSA, but we do not know if others exist. On the other hand, Schnorr is strongly unforgeable under chosen message attacks (SUF-CMA), a property that prevents malleability.

- Fixed size: the proposed Schnorr standardization is of fixed size, 64 bytes. This is in contrast with the 70-72 bytes long ECDSA (caused by the DER encoding). This would lead to a reduction of on-chain transactions' size.

When signing with Schnorr there are two possible shape for the signature. EC Schnorr signatures for the message m and public key Q involve a point K and integers e and s which satisfy $e = \text{hash}(K \mid\mid m)$ and $sG = K + eQ$. The two different verification equations depend on whether we decide to include e or K in the signature.

- The signature is (e, s) with $e = \text{hash}(sG - eQ \mid\mid m)$. This choice avoids the difficulty of encoding an EC point in the signature;
- The signature is (K, s) with $sG = K + \text{hash}(K \mid\mid m)Q$. This formulation supports batch validation since there are no elliptic curve operations inside the hashes.

In the cited BIP the choice falls on the second option, since it supports batch validation. Batch validation is important also because it would speed up considerably the syncing of new Bitcoin nodes that have to download the entire blockchain and validate every transaction (this would be true from the hypothetical block in which batch validation would be introduced; obviously it is not a retroactive property).
However, using the second validation rule directly comes with a downside: it is possible for a third party to convert a signature (K, s) for key Q into a signature $(K, s + a * \text{hash}(K \mid\mid m))$ for key $Q + aG$ and the same message, for any integer a. Indeed, the signature (K, s) is verified checking that:

$$sG = K + \text{hash}(K \mid\mid m)Q \implies$$

$$\implies (s + a * \text{hash}(K \mid\mid m))G = sG + a * \text{hash}(K \mid\mid m)G =$$

$$= K + \text{hash}(K \mid\mid m)Q + a * \text{hash}(K \mid\mid m)G = K + \text{hash}(K \mid\mid m)(Q + aG).$$

This is not a concern for the Bitcoin protocol itself, due to the difficulty of the ECDLP[8]. However it may be better to avoid this problem, since in some cases it could become a concern: an example is unhardened derivation in BIP32. For this reason, the BIP suggests to use key prefixed Schnorr, changing the verification equation to $sG = K + \text{hash}(K \mid\mid Q \mid\mid m)Q$. Because any

[8] An adversary could convert the signature to a public key of his choice only knowing the secret key of the signer: in this way he would be able to compute $Q_{adv} = (q + a)G$, choosing properly the integer a.

change in the public key would produce unpredictable changes in the hash, an adversary cannot use an existing signature to do anything but verify it. Notice that this choice prevents the possibility for public key derivation, a construction that is typically incompatible with batch validation[9].
Let's give a deeper look at how the formulation without key prefixing can be exploited in case of unhardened derivation using BIP32. Without entering in the details, in BIP32 it is specified how to derive key pairs from a unique master key pair, easing incredibly the backup procedure: the major achievement is that the whole tree of key pairs is recoverable from a special number called seed, that can be safely stored in efficient ways. We label the parent key pair as (q_P, Q_P) and the child key pair as (q_C, Q_C). The scheme for unhardened derivation is represented in Figure 4.1 and works as follows: we use the hash function HMAC, that outputs in the space of 512 bits, fed with the parent public key, the parent chain code (for our purposes the only thing that matters is that it is derived as half of the output of the HMAC function, i.e. it is a 256 bit string) and the child index. The result is divided in two halves, an offset and the child chaincode (that is needed for further derivations). The offset is added to the parent private key (modulo the order of the curve) to obtain the child private key. Thus, denoting the offset as f, we have the following relation: $q_C = f + q_P \pmod{n} \implies Q_C = fG + Q_P$.

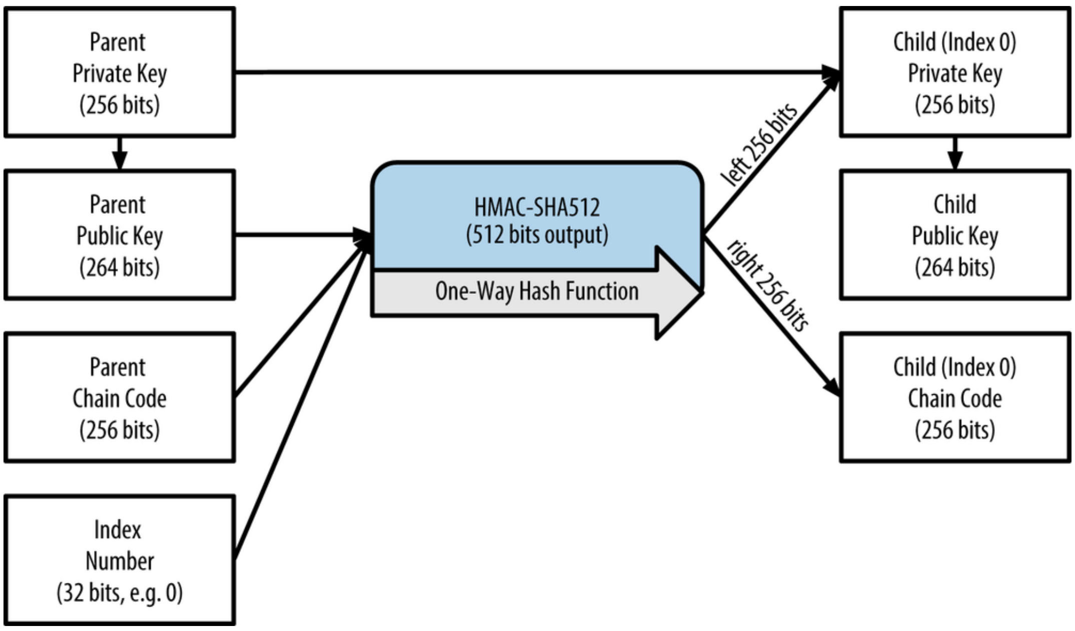

Parent Private Key (256 bits)
Parent Public Key (264 bits)
Parent Chain Code (256 bits)
Index Number (32 bits, e.g. 0)
HMAC-SHA512 (512 bits output)
One-Way Hash Function
left 256 bits
right 256 bits
Child (Index 0) Private Key (256 bits)
Child Public Key (264 bits)
Child (Index 0) Chain Code (256 bits)

Figure 4.1: BIP32's unhardened derivation.

[9]`https://bitcoin.stackexchange.com/questions/77234/schnorr-vs-ecdsa`.

Source: [3]

Now, imagine an attacker is given a valid signature (K, s) for public key Q_P and message m. Relying on the previous relation, he is able to forge a valid signature for public key Q_C on the same message m as $(K, s+\text{hash}(K \mid\mid m)f)$. Indeed we have:

$$(s + \text{hash}(K \mid\mid m)f)G = sG + \text{hash}(K \mid\mid m)fG =$$

$$= K + \text{hash}(K \mid\mid m)Q_P + \text{hash}(K \mid\mid m)fG = K + \text{hash}(K \mid\mid m)(Q_P + fG) =$$

$$= K + \text{hash}(K \mid\mid m)Q_C.$$

When substituting sG with $K + \text{hash}(K \mid\mid m)Q_P$ we relied on the fact that (K, s) is a valid signature for public key Q_P and message m. Notice that the attacker only needs a valid signature for the parent public key, the related public key and message, the parent chaincode and the child index.
This is a particular case of the previously presented weakness, that now can be exploited since the relation between the two public keys is known. Also the hardened derivation would prevent the forgery since, in place of the parent public key, the parent secret key is given as input to the hash function. To learn the relation between the two public keys, i.e. the offset, the forger would need to know also the parent private key. But this would mean that the attacker has direct access to all the funds secured by the subtree generated by the compromised parent node: the simple forgery is not anymore a concern.

Getting back to the signature specification, the first problem we have to face is how to encode efficiently the EC point in the signature. Several possibilities exist:

1. Encoding the full point through x and y coordinates, resulting in a 96 bytes signature (32 for each coordinate and another 32 for the integer s);

2. Encoding the x coordinate and use an additional byte to encode whether y is odd or even, like for compressed public keys: this would result in a 65 bytes signature;

3. Encoding only the x coordinate making the point implicit, leading to a 64 bytes signature.

In the BIP, the choice falls on the third option, since compactness is prioritized. However we cannot have ambiguity about the y coordinate, so we need break the symmetry, i.e. we need to make the whole point K implicit in its x coordinate. Again, we have multiple possibilities:

1. Select the y coordinate in the lower half of the plane;
2. Select the y coordinate that is even;
3. Select the y coordinate that is a quadratic residue[10].

As directly stated in the BIP "the third option is slower at signing time but a bit faster to verify, as the quadratic residue of the y coordinate can be computed directly for points represented in Jacobian coordinates. The two other options require a possibly expensive conversion to affine coordinates first. This would even be the case if the sign or oddness were explicitly coded". This is the statement with which the choice of the third option is justified.

Hereinafter hash will denote the SHA-256 function and the elliptic curve on which everything is worked out is intended to be secp256k1. Now we present the signing and verification algorithms[11], exemplified by Algorithms 4.4 and 4.5, where the function is_on_curve(Q) returns **True** or **False** depending on whether Q is on the curve or not. Notice that the signing algorithm can be considerably sped up giving as additional input the public key Q: as we have stated scalar multiplication (that in this case occurs at step 8) is a slow operation. Removing it results in a doubled signing speed.

We chose k in such a way that it changes each time the signature algorithm is applied to a different message; indeed even with Schnorr the predictability of this secret value can be used to recover the private key. Given (r_0, s_0) and (r_1, s_1) we have, by definition of s_i:

$$k = s_0 - e_0 q = s_1 - e_1 q \pmod{n} \implies q = (s_0 - s_1)(e_0 - e_1)^{-1} \pmod{n}.$$

[10]It can be proved that the product of two numbers is a quadratic residue when either both or none of the factors are quadratic residues. Since we have that the two y coordinates are one the negation of the other, and since -1 is not a quadratic residue when $p = 3 \pmod 4$ (as for secp256k1), we conclude that exactly one of the two roots is a quadratic residue. Notice that this choice for symmetry breaking prevents a real standardization: this formulation of Schnorr signature would not work for curves in which $p = 1 \pmod 4$.

[11]The validation procedure of the public key is incorporated in the verification algorithm to be compliant with the BIP, although in the case of ECDSA it was implicit.

Algorithm 4.4 Schnorr: signing algorithm

```
procedure SCHNORR_SIG($M,\ q$)
    $m \leftarrow \text{hash}(M)$
    $k \leftarrow \text{int}(\text{hash}(\text{bytes}(q)\ ||\ m)) \pmod n$
    $K \leftarrow kG$
    if $\text{jacobi}(y_K) \neq 1$ then
        $k \leftarrow n - k$
    end if
    $e \leftarrow \text{int}(\text{hash}(\text{bytes}(x_K)\ ||\ \text{bytes}(qG)\ ||\ m)) \pmod n$
    $s \leftarrow k + eq \pmod n$
    return $(\text{bytes}(x_K), \text{bytes}(s))$
end procedure
```

Algorithm 4.5 Schnorr: verification algorithm

```
procedure SCHNORR_VER($(r, s), M, Q$)
    if not is_on_curve($Q$) or $Q = \infty$ then
        return False
    end if
    $r \leftarrow \text{int}(r)$, $s \leftarrow \text{int}(s)$
    if $r \notin \{1, ..., p-1\}$ or $s \notin \{1, ..., n-1\}$ then
        return False
    end if
    $m \leftarrow \text{hash}(M)$
    $e \leftarrow \text{int}(\text{hash}(\text{bytes}(r)\ ||\ \text{bytes}(Q)\ ||\ m)) \pmod n$
    $K \leftarrow sG - eQ$
    if $K = \infty$ or $\text{jacobi}(y_K) \neq 1$ or $x_K \neq r$ then
        return False
    end if
    return True
end procedure
```

Again, it can be of interest to give a look to the correctness' proof.
Proof: Given the signature (r, s), we need to prove that the elliptic curve point K, recoverable from the integer r, equals $sG - eQ$. But by definition of s and by definition of public key, we have:

$$sG - eQ = (k + eq)G - eqG = kG = K.$$

$\square$

We conclude the section on Schnorr analysing the batch verification algorithm. It is pretty common for a system to verify a large number of signatures, particularly nowadays with crypto-currency widespread adoption. Thus, faster validation for a batch of signatures traduces in efficiency enhancements.
When validating a signature in the formulation that we chose, the most expensive operations are the two elliptic curve scalar multiplications involved. Using the (K, s) representation, we can verify the signature doing the hash operation first, and then perform elliptic curve operations. This is the key ingredient that allows to introduce batch validation.

We established that a signature (K, s) is valid if $K = sG - \text{hash}(K \mid\mid Q \mid\mid m)Q$. Therefore, a couple of valid signatures (K_0, s_0) and (K_1, s_1) will verify:

$$K_0 + K_1 = s_0G - \text{hash}(K_0 \mid\mid Q_0 \mid\mid m_0)Q_0 + s_1G - \text{hash}(K_1 \mid\mid Q_1 \mid\mid m_1)Q_1 \implies$$

$$K_0 + K_1 = (s_0 + s_1)G - \text{hash}(K_0 \mid\mid Q_0 \mid\mid m_0)Q_0 - \text{hash}(K_1 \mid\mid Q_1 \mid\mid m_1)Q_1.$$

We are able to factorize multiplication by G summing all the s values of the signatures: this approach reduces the number of scalar multiplication to one per signature, plus one for the aggregated s. Sadly, this is not secure at all: an attacker can produce a set of signatures that cancel each others. This could be a problem if the signatures are invalid. The attacker would convince the network that they are valid thanks to the batch validation that would succeed. There is also a simpler example that could help clarifying why this construction is not secure: imagine we have a set $(K_i, s_i),\ i \in \{1, ..., n\}$ of valid signatures. Obviously the naive batch validation would succeed. However it would be possible to switch the terms between the signatures, invalidating them all: nonetheless, the batch validation would still succeed. To work around this, we will introduce a random factor a_i per signature. Not knowing the random factor for each of his signatures, the attacker is now unable to have them cancel each others:

$$a_0K_0 + a_1K_1 =$$

$$= (a_0 s_0 + a_1 s_1)G - a_0\text{hash}(K_0 \mid\mid Q_0 \mid\mid m_0)Q_0 - a_1\text{hash}(K_1 \mid\mid Q_1 \mid\mid m_1)Q_1.$$

Usually it is possible to rely on another trick at this point, namely we can write:

$$a_0K_0 + a_1K_1 = (a_0 - a_1)K_0 + a_1(K_0 + K_1).$$

Keeping a sorted list of points and random factors, we could recursively pick the two largest a values and linked points, applying the above transformation and reinserting the new factors and points in the list: every time two factors are equal, their difference is zero and a point can be removed from the list. This traduces in less scalar multiplication and more point additions. But the more a values you have, the easier it becomes to find two that cancel each other: this approach quickly becomes profitable with the number of signatures.
However, this procedure is not feasible in Bitcoin. This is linked to the fact that batch validation is made on a block basis, meaning that we would have some thousands of signatures. But the a values are generated pseudo-randomly in $[1, ..., n-1]$, where n is the order of the curve. This number is around 2^{256}, so that it would be very unlikely to end up with two factors that cancel out.

Algorithm 4.6 presents the batch verification for a number u of signatures $(r_1, s_1), ..., (r_u, s_u)$, associated to the messages $M_1, ..., M_u$ and to public keys $Q_1, ..., Q_u$.
We highlight again the fact that the algorithm, taken from the BIP, is constructed to work on secp256k1, as clearly shown by the 14^{th} step, where we can see the defining equation. However it can be easily extended to other curves satisfying the relation $p = 3 \pmod 4$: this property is used in the 15^{th} step, that we try here to clarify. Referring to the algorithm, we have that the y coordinates are the square roots of c and can be computed as $y = \pm c^{\frac{p+1}{4}}$, if they exist, due to a lemma by Lagrange[12], applicable when $p \equiv 3 \pmod 4$. Euler's criterion tells us that, given an odd prime p and an integer d coprime to p, we have $d^{\frac{p-1}{2}} = \pm 1 \pmod p$, depending whether d is a quadratic residue ($d^{\frac{p-1}{2}} = 1 \pmod p$) or not ($d^{\frac{p-1}{2}} = -1 \pmod p$). The same criterion applied to y yields to $y^{\frac{p-1}{2}} \pmod p = \pm c^{\frac{p+1}{4}\frac{p-1}{2}} \pmod p = \pm (c^{\frac{p-1}{2}})^{\frac{p+1}{4}} \pmod p = \pm 1^{\frac{p+1}{4}} \pmod p = \pm 1 \pmod p$. Therefore, $y = c^{\frac{p+1}{4}} \pmod p$ is a quadratic residue, while $-y \pmod p$ is not: this approach is thus used to comply with the chosen symmetry breaking method.

[12]https://en.wikipedia.org/wiki/Quadratic_residue#Prime_or_prime_power_modulus.

Algorithm 4.6 Schnorr: batch verification algorithm

```
 1: procedure SCHNORR_BATCH($u, \{(r_1, s_1), ..., (r_u, s_u)\}, \{M_1, ..., M_u\}, \{Q_1, ..., Q_u\}$)
 2:     $RHS \leftarrow \infty$
 3:     $mult \leftarrow 0$
 4:     for $i \leftarrow 1, u$ do
 5:         if not is_on_curve($Q_i$) or $Q_i = \infty$ then
 6:             return False
 7:         end if
 8:         $r_i \leftarrow \text{int}(r_i)$, $s_i \leftarrow \text{int}(s_i)$
 9:         if $r_i \notin \{1, ..., p-1\}$ or $s_i \notin \{1, ..., n-1\}$ then
10:             return False
11:         end if
12:         $m_i \leftarrow \text{hash}(M_i)$
13:         $e_i \leftarrow \text{int}(\text{hash}(\text{bytes}(r_i) \mid\mid \text{bytes}(Q_i) \mid\mid m_i)) \pmod n$
14:         $c \leftarrow r^3 + 7 \pmod p$
15:         $y \leftarrow c^{\frac{p+1}{4}}$
16:         if $y^2 \neq c$ then
17:             return False
18:         end if
19:         $K_i \leftarrow (r_i, y)$
20:         if $i \neq 1$ then
21:             $a_i \overset{\$}{\leftarrow} \{1, ..., n-1\}$
22:         else
23:             $a_i \leftarrow 1$
24:         end if
25:         $mult \leftarrow mult + a_i s_i$
26:         $RHS \leftarrow a_i K_i + (a_i e_i) Q_i$
27:     end for
28:     if $multG \neq RHS$ then
29:         return False
30:     end if
31:     return True
32: end procedure
```

Chapter 5

Schnorr's applications

In this chapter we will present some of the applications that Schnorr would make deployable in Bitcoin: we will study improved multi and threshold signature schemes and we will see the construction called adaptor signature, that allows the development of the so called *scriptless scripts*, affecting considerably privacy and efficiency in cross-chain atomic swaps and in the Lightning Network.
In case the reader is not familiar with Bitcoin's inner working, we recommend the reading of Appendix A.

5.1 Multi-signature: MuSig ($\mu\Sigma$)

Usually Schnorr is presented with an implicit multi-signature scheme: given n users that want to sign a single message m, they can sign it on their own, the final signature being the sum of the so called partial signatures. This signature can then be verified against the sum of the public keys.
Let's study this scheme through an example: Alice and Bob have key pairs $\{q_A, Q_A\}$ and $\{q_B, Q_B\}$, respectively. If both participants are honest, they will proceed as follows: they exchange their public keys, computing the aggregated one $Q = Q_A + Q_B$. Then each one of them calculate as usual the public nonces K_A and K_B, defining the joint nonce as $K = K_A + K_B$. The signature would then be (x_K, s), with $s = s_A + s_B = k_A + \text{hash}(x_K \mid\mid Q \mid\mid m)q_A + k_B + \text{hash}(x_K \mid\mid Q \mid\mid m)q_B \pmod n = (k_A + k_B) + \text{hash}(x_K \mid\mid Q \mid\mid m)(q_A + q_B) \pmod n$. Looking exactly as a single user signature, the verification procedure would follow Algorithm 4.5.
This sounds great, except for the fact that it is a completely insecure scheme: we assumed that both the participants were honest, a deadly hypothesis for every cryptosystem. Imagine that it is Bob that wants to cheat. He could

simply says that his public key is $Q'_B = Q_B - Q_A$. Then, if someone sends money to the address associated to $Q = Q_A + Q'_B = Q_A + Q_B - Q_A = Q_B$, clearly Bob can control the funds by himself, being in possess of the associated private key.
This kind of attack is called rogue key attack and is a serious concern for multi-signature schemes: given n participants, a subset of $1 \leq t < n$ dishonest signers use public keys that are functions of the public keys of honest signers, allowing them to forge a signature without the aid of the honest signers for the whole set of public keys. There are certain ways to prevent such an attack: for example by ensuring that the participants own the private keys associated with the alleged public keys (now it is not possible for Bob to cheat, since it would imply breaking the ECDLP), a setting that takes the name of KOSK (knowledge of secret key).

In this section we will present a provably secure multi-signature scheme of the type n-of-n. But before delving into its technicalities, it could be better to stop and talk a little about how changes are introduced in Bitcoin: deploying innovations in Bitcoin is a long procedure, due to its decentralized consensus protocol. Since it could require years to take a new feature to it, we should think about properties that would enhance Bitcoin in the long term. Today Bitcoin is missing some important properties in order to be a good method of payment: it is missing both fungibility and privacy. It is missing fungibility due to the fact that it is missing privacy: Bitcoin is pseudonymous, not anonymous, in the sense that an address is not directly linked to a physical person, but every single transaction is on the public ledger open to (possibly) every node in the network. Low privacy means that bitcoins, not being interchangeable, could be treated differently: think about the bitcoins possessed by the creator of Bitcoin, Satoshi Nakamoto, and not moved since the creation of Bitcoin. Obviously they have not the same appeal of newly minted coins.
Fortunately enough, the lack of some properties is not everlasting: for such a reason, when introducing a new feature in Bitcoin we should try to fix these problems. So, we will give now a look to some properties that, in a long term view, a new multi-signature scheme should possess:

1. Accountability: this property refers to m-of-n multi-signature schemes (also referred to as threshold schemes) and deals with the fact that for the participants of the scheme should be possible to know who signed and to show to others that they have not;

2. Usability: the ease of use is important. If an interactive scheme requires a huge number of rounds, it won't be used by anyone;

3. Privacy: third parties should learn as little about the policy of the scheme as possible (particular kind of policies could identify your transactions, leading to various problems, like censorship by miners).

After having discussed partially the reasons behind its creation, we present now the MuSig scheme ([18]). MuSig is an interactive (meaning that the scheme comprehends different rounds of communication between the participants) multi-signature scheme, based on Schnorr signature. MuSig has some very attractive properties, namely:

- The size of the signature is equal to the single user case;
- It is provably secure in the plain public key model[1];

These properties, although being appealing, are not original: MuSig shares them with others scheme, in particular with the Bellare-Neven scheme (BN scheme presented in [4]). The novelty introduced by the authors is that they recovered key aggregation, meaning that to the scheme can be associated a unique joint public key, leading to a verification algorithm that is equal to the single user case: the multi-signature can be verified with respect to a single aggregated public key, leading to greater privacy since for third parties the multi-signature policy is completely hidden.
The Bellare-Neven scheme prevents rogue key attacks relying on a particular algorithm to compute the partial signatures, avoiding a trusted setup: each participant $i \in \{1, ..., n\}$ with key pair $\{q_i, Q_i\}$ computes $s_i = k_i + c_i q_i,\ c_i = \text{hash}(\langle L \rangle \mid\mid K \mid\mid Q_i \mid\mid m)$, where $K = \sum_{i=1}^{n} K_i$ is the aggregated public nonce, m is the message to be signed and $\langle L \rangle$ is a unique encoding of the multiset of public keys $L = \{Q_1, Q_2, ..., Q_n\}$. The signature is (K, s), where $s = \sum_{i=1}^{n} s_i$, and the verification equation is: $sG = K + \sum_{i=1}^{n} c_i Q_i$. We can notice that a validator needs the whole set of public keys in order to check the signature. MuSig can be thought as a variant of BN that recovers key aggregation. The setting in which the two schemes are defined is the same: both can be proven to be secure in the plain public key model under the discrete logarithm assumption[2], modeling the hash functions involved as a public random oracle. Security is to be intended in the sense that it is infeasible for an adversary to forge multi-signatures involving at least an honest participant, that is: the adversary is not able to produce on its own a signature valid for the set of

[1]The signers are only required to have a public key: they do not have to prove ownership of it, i.e. knowledge of the associated private key.

[2]The discrete logarithm assumption requires the DL to be hard on the selected group. This means that if the DL is hard, then the scheme is secure.

public keys containing the one of the honest signer.
We stress the fact that, from the applicative point of view, key aggregation is a fundamental property: if the scheme is usable (few interaction rounds), then we get for free privacy and accountability (MuSig is an n-of-n scheme, so that it is possible to generate a valid signature only if all the participants agree). Indeed, thanks to key aggregation, verifiers will only see an aggregated public key: they wouldn't even know that it is indeed aggregate, since it is indistinguishable from a normal public key. This is also important from the point of view of efficiency: in Bitcoin every single node has the possibility of validating each transaction, meaning that verification efficiency and signature size are very important, more than the timing of the signing algorithm. This is why, although there are multi-signature schemes with fewer interaction rounds, we present here MuSig: the benefits of key aggregation are improved bandwidth (no need for communication of multiple public keys), privacy (aggregated public key indistinguishable from a normal one) and validation efficiency (as efficient as a normal Schnorr's verification).
Moreover if the aggregated public key is not given to the verifier, it is still possible to recover it just from the set of public keys of the participant, without interaction with the signers.

The plain public key setting plays a crucial role when trying to enable multi-signatures across multiple inputs of a Bitcoin transaction, resulting in a unique signature per transaction. In case the transaction spends inputs from different owners, they will obviously need to collaborate to produce the multi-signature. Such a construction would reduce further the traffic on-chain, resulting in a benefit for all the network participants. Such a change would require the introduction of new opcodes in the Bitcoin scripting language, but this can be done via a soft fork (i.e. in a backward compatible way). To see why security in the plain public key model is fundamental to enable cross-input multi-signature, think about an attacker that identifies some outputs he wants to steal, corresponding to a set $\{Q_1, Q_2, ..., Q_{n-t}\}$ of public keys. He could try to identify another set of keys $\{Q_{n-t+1}, ..., Q_n\}$ such that he can sign for the aggregated public key. He would be able to steal the coins just by sending a small amount of his own money to outputs corresponding to the keys he found and finally creating a transaction referencing the outputs he wants to steal and the newly created outputs in his possession: by construction he is able to forge a signature on his own for this transaction. But the plain public key model defends exactly against such a situation, since the game is won by the adversary if he is able to forge a signature over a set of keys that includes at least one key not in possession of the attacker.

In the setting of signature aggregation on a transaction basis, it is possible to resort to other signature schemes than MuSig (in particular the older and more reviewed BN scheme), since all the public keys involved are public.
Up to now we have talked about cross-input aggregation, that would reduce the number of signatures to one per transaction. Is it possible to go further? For example, is it possible to obtain a single signature on a block basis? Unfortunately it is not possible through MuSig, since the scheme is interactive: this obviously prevents aggregation on a block level. However the discussion would not be fair if we would not point out that changing signature scheme would allow aggregation on a block basis: for example the BLS signature scheme ([9], [10]) is non interactive when it comes to aggregation. This would enable a single signature per transaction. However, although there is interest around BLS signature, it is underpinned by a different cryptographic hypothesis than ECDSA and Schnorr. For this reason the Bitcoin community is working towards the implementation of Schnorr.

Now we can finally look at the inner working of the scheme: it is parameterized by the cyclic group $\mathbb{G}$ (a subgroup of $E(\mathbb{F}_p)$) denoted in additive notation, its order n, a generator of the group G and three hash functions hash_{com}, hash_{agg} and hash_{sign}: $\{0,1\}^* \to \{0,1\}^{L_n}$. The bit length of the order n is denoted by L_n and assumed to be a security parameter. The key generation algorithm is not presented: we assume each participant $i \in \{1, ..., m\}$ is in possession of a proper key pair $\{q_i, Q_i\}$. The signing algorithm is presented respectively in Algorithms 5.1. In the following $\langle L \rangle$ denotes a unique encoding of the multiset of public keys and the indices used in the signing algorithm are local references to the other cosigners. Moreover Algorithm 5.1 is split in interactive rounds according to the **send** and **upon reception of** commands. Verification occurs following the single signature Schnorr verification Algorithm 4.5.

As usual, here follows the proof of the correctness.
Proof: Loosely speaking, we have to prove that $sG = K + cQ$.

$$sG = \left(\sum_{i=1}^{m} s_i\right) G = \left(\sum_{i=1}^{m}(k_i + ca_iq_i)\right) G = \sum_{i=1}^{m}(k_iG + ca_iq_iG) =$$

$$= \sum_{i=1}^{m} k_iG + \sum_{i=1}^{n} ca_iq_iG = \sum_{i=1}^{m} K_i + c\sum_{i=1}^{m} a_iQ_i = K + cQ.$$

$\square$

Algorithm 5.1 MuSig: signing algorithm

```
 1: procedure MuSig_sig($M, q_1, \{Q_2, ..., Q_m\}$)
 2:     $m \leftarrow \text{hash}(M)$
 3:     $\langle L \rangle \leftarrow \{q_1 G, Q_2, ..., Q_m\}$
 4:     for $i \leftarrow 1, m$ do
 5:         $a_i \leftarrow \text{int}(\text{hash}_{agg}(\langle L \rangle \mid\mid \text{bytes}(Q_i)))$
 6:     end for
 7:     $Q \leftarrow \sum_{i=1}^{m} a_i Q_i$
 8:     $k_1 \xleftarrow{\$} \{1, ..., n-1\}$
 9:     $K_1 \leftarrow k_1 G,\ t_1 \leftarrow \text{hash}_{comm}(\text{bytes}(K_1))$
10:     send $t_1$
11:     upon reception of $t_2, ..., t_m$ send $K_1$
12:     upon reception of $K_2, ..., K_m$ do
13:         for $i \leftarrow 2, m$ do
14:             if $t_i \neq \text{hash}_{comm}(\text{bytes}(K_i))$ do
15:                 abort
16:             end if
17:         end for
18:     $K \leftarrow \sum_{i=1}^{m} K_i$
19:     if $\text{jacobi}(y_K) \neq 1$ then
20:         $k_1 \leftarrow n - k_1$
21:     end if
22:     $c \leftarrow \text{int}(\text{hash}_{sig}(\text{bytes}(x_K) \mid\mid \text{bytes}(Q) \mid\mid m))$
23:     $s_1 \leftarrow k_1 + c a_1 q_1 \pmod n$
24:     send $s_1$
25:     upon reception of $s_2, ..., s_m$ do
26:     $s \leftarrow \sum_{i=1}^{m} s_i \pmod n$
27:     return $(x_K, s)$
28: end procedure
```

For the security proof of the scheme we refer to the original paper [18].
Dealing with ECDSA and Schnorr in Chapter 4, we have seen that in the single user setting there was the possibility to derandomize the signature algorithm without loss of security, by generating the random nonce k through a deterministic function. This is done since pseudo random generation is one of the major sources of problems in cryptography. This is not anymore suggested in the multi-user setting: it is necessary to ensure to use different unpredictable values when the other signers change their K values in repeated signing attempts, otherwise secret key recovery would be possible.
Here follows an example taken from directly [18]: assume Alice and Bob have key pairs $\{q_A, Q_A\}$ and $\{q_B, Q_B\}$, respectively. They want to jointly produce a signature. Alice generates k_A and sends $K_A = k_A G$ to Bob. In a first attempt, Bob responds with K_B. Alice then computes:

$$K = K_A + K_B,$$

$$c = \text{hash}_{sig}(x_K \mid\mid Q \mid\mid m),$$
$$s_A = k_A + c a_A q_A \pmod{n},$$

and sends s_A to Bob. Bob is trying to cheat on Alice, and decides not to produce a valid s_B, and thus the protocol fails. A new signing attempt takes place, and Alice again sends the same K_A. Bob responds with $K'_B \neq K_B$. Alice then computes $c' = \text{hash}_{sig}(x_{K_A+K'_B} \mid\mid Q \mid\mid m)$ and $s'_A = k_A + c' a_A q_A \pmod{n}$ and sends s'_A to Bob. Now Bob is able to derive Alice's private key:

$$s_A - s'_A = (c - c')a_A q_A \pmod{n} \implies q_A = (c - c')^{-1} a_A^{-1}(s_A - s'_A) \pmod{n}.$$

To avoid this problem, each signer must ensure that whenever any K value sent by other cosigners or the message m changes, his k_i changes as well.

5.2 Threshold signature

With the name threshold signatures we refer to policies of the kind m-of-n, where it is necessary that at least m participants of the scheme decide to collaborate to produce a valid signature. This kind of policy is very popular in Bitcoin, since it is flexible and has many applications: for example, a single user could use such a policy to improve security, storing the keys on different machines, at the same time defeating the risk of loss of some keys.
In this section we will study a provable secure Schnorr based threshold

scheme. For the security proof we refer to [27]. The scheme is constructed on top of the Pedersen Verifiable Secret Sharing Scheme (VSS scheme) and Pedersen's multi-party protocol to generate a random shared secret, hence we start presenting these two schemes.

5.2.1 Verifiable secret sharing scheme

Hereinafter, we will refer to threshold signature schemes as t-of-m schemes in order to avoid confusion in the notation.
Typically, in a t-of-m secret sharing scheme, a trusted dealer distributes a secret s to m players $P_1, ..., P_m$ in such a way that any subgroup of at least t members can recover the secret, while any subgroup of cardinality strictly less than t learns nothing about it. A verifiable secret sharing scheme moreover prevents the dealer from cheating, since each participant can verify that his share of the secret is consistent with the others. The novelty introduced by Pedersen is that his scheme is non-interactive in the verification and does not require trust between the parties involved. Still there is a trusted dealer, a figure we would like to get rid of: this will be done through the second protocol that will be presented in Section 5.2.2. Here follows the VSS scheme proposed by Pedersen.

Fix an elliptic curve over a prime finite field $E(\mathbb{F}_p)$, characterized by the EC domain parameters $T = (a, b, p, G, n, h)$ and fix another generator H of the same cyclic group generated by G. We require that these two generators are *nothing up my sleeves* (NUMS), meaning that we do not know the discrete logarithm of one with respect to the other, and vice versa.
Assume that the dealer has a secret value $s \in \mathbb{Z}_n$ and a number $s' \in \mathbb{Z}_n$ generated at random. He commits[3] to the couple (s, s') through the so called Pedersen commitment $C_0 = sG + s'H$. The NUMS property is needed to prevent the person who commits from lying about the values he committed to. Assume indeed that the dealer knows the discrete logarithm of H with respect to G: $H = r_H G$. In this case he could write: $C_0 = sG + s'H = (s + s'r_H)G = (s \pm ar_H + s'r_H)G = (s - ar_H)G + (s'r_H + ar_H)G = (s - ar_H)G + (s' + a)H$, $\forall a \in \mathbb{Z}_n$.
After having broadcasted the commitment, the secret s can be shared among $P_1, ..., P_m$ through the following protocol:

[3]A commitment scheme is a cryptographic primitive used to commit to some secret data without revealing anything about it. Commitment schemes are designed so that it is unfeasible to change the secret: when revealed in a second moment, we have probabilistic assurance that it is the real committed value. A typical example of commitment scheme is a collision resistant hash function, e.g. SHA-256.

The dealer:

1. Chooses a couple of random polynomials of degree $t-1$:

   $$f(u) = s + f_1 u + ... + f_{t-1}u^{t-1},\ f'(u) = s' + f'_1 u + ... + f'_{t-1}u^{t-1},$$

   where s and s' are the committed values, while $f_i, f'_i \in \mathbb{Z}_n$ are randomly chosen for every $i \in \{1, ..., t-1\}$;

2. Computes $(s_i, s'_i) = (f(i) \pmod n, f'(i) \pmod n)$ for $i \in \{1, ..., m\}$;

3. Sends secretly (s_i, s'_i) to $P_i, \forall i \in \{1, ..., m\}$;

4. Broadcasts the values $C_j = f_j G + f'_j H,\ \forall j \in \{1, ..., t-1\}$.

Each participant P_i:

1. Verifies the consistency of its share of the secret as:

   $$s_i G + s'_i H = \sum_{j=0}^{t-1} i^j C_j.$$

   If this check fails, she broadcasts a compliant against the dealer;

2. For each compliant from a player i, the dealer defends himself by broadcasting the values $(s_i, s'_i) = (f(i) \pmod n, f'(i) \pmod n)$ that satisfy the checking equation at point 1;

3. Aborts the protocol if:

   - The dealer received more than t compliants in step 1;
   - He answered to a compliant in step 2 with values that violates again the checking equation.

Pedersen proved that any coalition of less than t players cannot get any information about the shared secret, provided that the discrete logarithm in $E(\mathbb{F}_p)$ is hard[4]. For the proof we refer to the original paper [24]. Although we do not look at the proof, it may still be of interest to check why the verification procedure at step 1 should succeed:

$$s_i G + s'_i H = f(i)G + f'(i)H = \sum_{j=0}^{t-1} f_j i^j G + \sum_{j=0}^{t-1} f'_j i^j H =$$

[4]This is important since we are not adding cryptographic assumptions.

$$\sum_{j=0}^{t-1} i^j (f_j G + f'_j H) = \sum_{j=0}^{t-1} i^j C_j.$$

We used the convention that $f_0 = s$ and $f'_0 = s'$. Remembering that C_0 commits to the secret and that the other C_j commits to the polynomials, we have the assurance that the dealer is not cheating. Indeed there is one and only one polynomial of degree at most $t-1$ satisfying $f(i) = s_i$, respectively $f'(i) = s'_i$, for t values of i.
This is also the key property that allows the reconstruction of the secret value from any group $\mathcal{P}$ of t participants. Indeed the members in $\mathcal{P}$ can recover the polynomial f through the Lagrange's interpolation formula, that given a set of t points $(i, s_i = f(i) \pmod n)$ returns the lowest degree polynomial (in this case a $t-1$ degree polynomial) interpolating the given points:

$$f(u) = \sum_{i \in \mathcal{P}} f(i)\omega_i(u), \text{ where } \omega_i(u) = \prod_{j \in \mathcal{P},\ j \neq i} \frac{u-j}{i-j} \pmod n.$$

Since it holds that $s = f(0)$ by definition, the group $\mathcal{P}$ can directly reconstruct the secret as:

$$s = f(0) = \sum_{i \in \mathcal{P}} f(i)\omega_i, \text{ where } \omega_i = \omega_i(0) = \prod_{j \in \mathcal{P},\ j \neq i} \frac{j}{j-i} \pmod n.$$

5.2.2 Protocol for the generation of a random shared secret

As we pointed out before, we would like to get rid of the trusted dealer, so that it is possible to generate a key pair between distrustful parties. The key generation phase of the signature scheme generates a random shared secret key in a distributed way according to the following protocol:

Each participant P_i:

1. Chooses $r_i, r'_i \in \mathbb{Z}_n$ at random and verifiably shares (r_i, r'_i) acting as the dealer according to the Pedersen's VSS scheme described above. Let the sharing polynomials of participant i be $f_i(u) = \sum_{j=0}^{t-1} a_{ij} u^j$, $f'_i(u) = \sum_{j=0}^{t-1} a'_{ij} u^j$, where $a_{i0} = r_i$ and $a'_{i0} = r'_i$. The public commitments are $C_{ik} = a_{ik} G + a'_{ik} H$ for $k \in \{0, ..., t-1\}$;

2. Sets $H_0 = \{P_j \mid P_j \text{ is not detected to be cheating at step 1}\}$. The distributed secret value r is equal to $\sum_{i \in H_0} r_i$. Each participant P_i sets his share of the secret at $s_i = \sum_{j \in H_0} f_j(i) \pmod n$ and set the value $s'_i = \sum_{j \in H_0} f'_j(i) \pmod n$;
3. Each player in H_0 broadcasts $R_i = r_i G$ via Feldman's VSS scheme:
   (a) Each player P_i in H_0 broadcasts $A_{ik} = a_{ik}G$ for $k \in \{0, ..., t-1\}$;
   (b) Each player P_j verifies the values broadcast by the other players in H_0: for each $P_i \in H_0$, P_j checks:
   $$f_i(j)G = \sum_{k=0}^{t-1} j^k A_{ik}.$$
   If the check fails for an index i, P_j complaints against P_i broadcasting the values $(f_i(j), f'_i(j))$ that satisfy the checking equation of the Pedersen's VSS scheme but not the one at point (b);
   (c) For players P_i who received at least one valid complaint, the other players run the reconstruction phase of Pedersen's VSS scheme to compute r_i, $f_i(u)$ and A_{ik} for $k \in \{0, ..., t-1\}$. All participants in H_0 set $R_i = r_i G$.

After the execution of the protocol the following equations hold:

$$R = \sum_{i \in H_0} R_i = \sum_{i \in H_0} r_i G = \sum_{i \in H_0} A_{i0} = rG,$$

$$f(u) = \sum_{i \in H_0} f_i(u) = r + a_1 u + ... + a_{t-1}u^{t-1}, \text{ where } a_i = \sum_{j \in H_0} a_{ji},$$

$$f(i) = s_i.$$

We introduce the following notation:

$$(s_1, ..., s_n) \xleftrightarrow{\text{(t, m)}} (r|R, a_i G, H_0),\ i \in \{1, ..., t-1\}.$$

It means that s_j is the share of secret key r belonging to P_j for $j \in H_0$. The values $a_i G$ are the public commitments of the sharing polynomials $f(u)$ and (r, R) is the key pair that can be reconstructed by any subgroup of H_0 composed of at least t participants: this can be done via Lagrange's interpolation forrmula, as described in Section 5.2.1.
Before we pass to analyse the actual signature scheme, let's give a look at the checking equation at point (b) in the protocol:

$$f_i(j)G = \sum_{k=0}^{t-1} a_{ik} j^k G = \sum_{k=0}^{t-1} j^k A_{ik}.$$

5.2.3 Threshold signature scheme

Now that we have defined the primitives on which it is built, we can finally discuss the protocol that implements the t-of-m threshold signature scheme.

Key generation: All m participants have to cooperate to generate a public key Q and a share of the secret key for each participant P_j. This can be done relying on the protocol presented in Section 5.2.2. The output of the protocol is:

$$(\alpha_1, ..., \alpha_m) \xleftrightarrow{\text{(t, m)}} (q|Q, b_iG, H_0),\ i \in \{1, ..., t-1\}.$$

The α values denote the secret key share belonging to P_j. They will be used to generate a partial signature for the key pair (q, Q).

Signing algorithm: Let msg denote the message to be signed. Suppose that a subset $H_1 \subseteq H_0$ wants to issue a signature. The members of H_1 proceed as follows:

1. If $|H_1| < t$, aborts. Otherwise, the subset H_1 generates a random shared secret following again the protocol presented in Section 5.2.2. We denote the output as:

$$(\beta_1, ..., \beta_m) \xleftrightarrow{\text{(t, m)}} (k|K, c_iG, H_2),\ i \in \{1, ..., t-1\};$$

2. If jacobi$(y_K) \neq 1$, then each player i sets $\beta_i = n - \beta_i$;

3. If $|H_2| < t$, aborts. Otherwise, each $P_i \in H_2$ reveals

$$\gamma_i = \beta_i + e\alpha_i \pmod n,$$

   where $e = \text{int}(\text{hash}(\text{bytes}(x_K)\ ||\ \text{bytes}(Q)\ ||\ msg))$.

4. Each $P_i \in H_2$ verifies $\forall P_l \in H_2$:

$$\gamma_l G = \begin{cases} K + \sum_{j=1}^{t-1} c_j l^j G + e\left(Q + \sum_{j=1}^{t-1} b_j l^j G\right), & \text{if jacobi}(y_K) = 1 \\ -K - \sum_{j=1}^{t-1} c_j l^j G + e\left(Q + \sum_{j=1}^{t-1} b_j l^j G\right), & \text{if jacobi}(y_K) \neq 1 \end{cases}$$

   Let $H_3 = \{P_j \mid P_j \text{ not detected to be cheating at step } 3\}$.

5. If $|H_3| < t$, aborts. Otherwise each $P_i \in H_3$ selects an arbitrary subset $H_4 \subseteq H_3$ with $|H_4| = t$ and computes σ satisfying $\sigma = k + eq$, where:

$$\sigma = \sum_{j \in H_4} \gamma_j \omega_j,\ \text{where } \omega_j = \prod_{h \in H_4,\ h \neq j} \frac{h}{h-j}.$$

   The signature is (x_K, σ); it is verified as a simple Schnorr signature.

This concludes the presentation of the protocol. Nonetheless there are some formulas that deserve greater attention.

- Checking formula at point 4[5]:

$$\gamma_l G = (\beta_l + e\alpha_l)G = \beta_l G + e\alpha_l G =$$
$$= \left(k + \sum_{j=1}^{t-1} c_j l^j\right) G + e\left(q + \sum_{j=1}^{t-1} b_j l^j\right) G =$$
$$= K + \sum_{j=1}^{t-1} c_j l^j G + e\left(Q + \sum_{j=1}^{t-1} b_j l^j G\right).$$

- Formula used to compute σ: we defined $\gamma_i = \beta_i + e\alpha_i \pmod{n}$, $\forall i \in H_2$. In particular the equation holds for every $i \in H_4$, with $|H_4| = t$. The α and β values are defined to be the pointwise evaluation of the sharing polynomials created during the two iterations of the Pedersen protocol for the generation of a random shared secret, that we denote by $F_1(u)$ and $F_2(u)$. These polynomials have degree $t-1$, so that we can define:

$$F_3(u) = F_2(u) + eF_1(u) \implies$$
$$\implies F_3(0) = F_2(0) + eF_1(0) =$$
$$= k + eq = \sigma.$$

  At this point we can apply the Lagrange's interpolation formula, since we know that $F_3(u)$ satisfies by construction $F_3(i) = \gamma_i$:

$$F_3(u) = \sum_{j \in H_4} \gamma_j \omega_j(u), \text{ where } \omega_j(u) = \prod_{h \in H_4,\ h \neq j} \frac{u-h}{j-h}.$$

  Thus, σ can be directly computed as:

$$\sigma = F_3(0) = \sum_{j \in H_4} \gamma_j \omega_j, \text{ where } \omega_j = \prod_{h \in H_4,\ h \neq j} \frac{h}{h-j}.$$

Notice that the scheme is robust, meaning that a corrupt signer who does not follow the protocol will be detected. The validity of the partial signatures computed at step 3 is immediately tested at step 4.

[5]The formula is checked in case jacobi(y_K) = 1; the other case is proved in a similar way.

The scheme presented is complex and cumbersome, but all the computations occur off-chain: the Bitcoin blockchain would see a standard transaction, indistinguishable from others. Nobody could tell that it required a threshold signature. This means that the protocol is highly private, but it comes with the downside of not being accountable, since the participants that do not sign cannot learn who instead did it.

5.3 Adaptor signature

Adaptor signatures are the building block for the so called *scriptless script*, an innovative idea by Andrew Poelstra: the major aim is to introduce much more flexibility in systems lacking a scripting language (e.g. the Mimblewimble protocol discussed in [19] and [25]). However, it turns out that these kind of signatures have a couple of interesting applications also in Bitcoin, that we are going to explore.
Adaptor signatures are based on Schnorr and leverage its linearity property. Therefore, introducing Schnorr in Bitcoin would allow the deployment of adaptor signatures for free. As we will see in the next two sections, this tool affects significantly cross-chain atomic swaps and payment channels, where transactions can be made atomic through signatures and not through Bitcoin scipts. This traduces again in transactions that on-chain look the same as ordinary single signer transactions, improving greatly privacy and efficiency: smaller transactions result in lower fees, lower blockchain size and lower CPU requirement (since the UTXO set, i.e. the set saved in the RAM of the unspent transaction outputs, shrinks), while a unique aggregate signature hides completely the participants involved.

Adaptor signatures come from the addition of the public nonce K generated during Algorithm 4.4 with a random EC point $T = tG$: however, the secret nonce k is not updated, in the sense that it is still considered k as secret nonce instead of $k + t$. This results obviously in an invalid signature. But thanks to this construction, learning the secret integer t is equivalent to learning a valid signature for the same message and public key used for the adaptor signature.
To clarify how adaptor signatures work, let's look at an example. Consider two participants, Alice and Bob. To produce an adaptor signature Bob proceeds as shown in Algorithm 5.2 on input a message M and his private key q_B.

Algorithm 5.2 Adaptor signature

```
1: procedure ADAPTOR_SIG($M, q$)
2:     $m \leftarrow \text{hash}(M)$
3:     $k \leftarrow \text{int}(\text{hash}(\text{bytes}(q) \mid\mid m)) \pmod n$
4:     $t \overset{\$}{\leftarrow} \{1, ..., n-1\}$
5:     $K \leftarrow kG,\ T \leftarrow tG$
6:     if $\text{jacobi}(y_{K+T}) \neq 1$ then
7:         $k \leftarrow n - k,\ t \leftarrow n - t$
8:     end if
9:     $e \leftarrow \text{int}(\text{hash}(\text{bytes}(x_{K+T}) \mid\mid \text{bytes}(qG) \mid\mid m)) \pmod n$
10:    $s' \leftarrow k + eq \pmod n$
11:    return $(\text{bytes}(x_T), \text{bytes}(x_{K+T}), \text{bytes}(s'))$
12: end procedure
```

As we have already pointed out, the signature $(r', s') = (x_{K+T}, s')$ is not valid. Indeed, following the verification algorithm 4.5 Alice would compute:

$$e = \text{int}(\text{hash}(\text{bytes}(r') \mid\mid \text{bytes}(Q_B) \mid\mid m)) \pmod n \implies$$

$$\implies s'G - eQ_B = (k + eq_B)G - eQ_B = kG = K.$$

Then the algorithm would check whether or not x_K is equal to $r' = x_{K+T}$, failing. This is because the secret nonce k has not been offset by t.
Although the adaptor signature is invalid, Alice can check that it is a valid adaptor signature, meaning that it is consistent. Consistency is to be intended in the sense that learning a valid signature is equivalent to learning t and viceversa. This is done checking that:

$$s'G = K + eQ_B.$$

She can do this verification because she has s' and Q_B, while from the given x coordinates she is able to reconstruct both T and $K + T$, from which she gets K.
Assume now that Bob gives Alice a valid signature $(r', s' + t)$. It is valid by construction, indeed after the usual checks the verification algorithm would compute:

$$e = \text{int}(\text{hash}(\text{bytes}(r') \mid\mid \text{bytes}(Q_B) \mid\mid m)) \pmod n \implies$$

$$\implies (s' + t)G - eQ_B = (k + eq_B + t)G - eQ_B = (k + t)G = K + T.$$

This time, the check $x_{K+T} = r'$ would succeed.
From this valid signature and the previously received adaptor signature, Alice can immediately recover t simply by taking the difference: $s' + t - s' = t$.

Obviously also the viceversa holds true: given the adaptor signature $(r, r', s') = (x_T, x_{K+T}, s')$ and t Alice can immediately compute a valid signature as $(r', s' + t)$.

Adaptor signatures can be used jointly with the multi-signature scheme described in Section 5.1. Assuming again that it is Bob that wants to generate an adaptor signature, he would modify slightly Algorithm 5.1, while Alice follows it closely: when computing K_B at step 9, he generates at random t and computes $T = tG$. Then, he commits to $K_B + T$ and, at point 11, sends it to Alice. The final modification occurs at step 24: with s_B he sends Alice also T, so that she can verify the adaptor signature. Nothing else changes. The final joint signature is:

$$(x_{K_A+K_B+T}, s),$$

where

$$s = s_A + s_B =$$

$$= k_A + \text{hash}(\text{bytes}(x_{K_A+K_B+T}) \mid\mid \text{bytes}(Q) \mid\mid m)\text{hash}(\langle L \rangle \mid\mid \text{bytes}(Q_A))q_A +$$

$$+ k_B + \text{hash}(\text{bytes}(x_{K_A+K_B+T}) \mid\mid \text{bytes}(Q) \mid\mid m)\text{hash}(\langle L \rangle \mid\mid \text{bytes}(Q_B))q_B$$

and

$$Q = \text{hash}(\langle L \rangle \mid\mid \text{bytes}(Q_A))Q_A + \text{hash}(\langle L \rangle \mid\mid \text{bytes}(Q_B))Q_B.$$

As before this signature is not valid for Q, since Bob did not offset s_B by t. Again, learning t is equivalent to learning a valid signature.
This is the key idea behind the applications we will study in the following sections.

Adaptor signatures may not seem a big deal, but it is worth noticing that if t is some necessary data for a separate protocol, arbitrary steps of arbitrary protocols can be made equivalent to signature production. In such a case it could be useful to attach auxiliary data to the signature to ensure that the role of t is the one claimed. In particular, using the same T in multiple subsequent adaptor signatures, it is possible to make arbitrary sets of signatures atomic: once a valid signature is produced, t is revealed making all the signatures valid. Last thing worth of being pointed out is that adaptor signatures are deniable: for every signature on the blockchain one can come up with some t and construct a linked adaptor signature.

5.3.1 Atomic swap

In this section we look at one application of adaptor signatures, starting from the easier concept of cross-chain atomic swap. Notice that the Schnorr's applications discussed in Sections 5.1 and 5.2 can be extended outside the Bitcoin ecosystem. Atomic swaps instead are a concept strictly related to the crypto-currency world; for this reason we suggest to stop for a moment and give a look to Appendix A: in there we give an overview of the role that signatures play in Bitcoin and a brief introduction to Bitcoin scripting language.

A cross-chain atomic swap is the exchange of different crypto-currencies among two users in an atomic way, meaning that the swap is successful or the balances of the participants remain unchanged, in such a way that it is not possible to cheat, stealing coins from the other party. Following Bitcoin's spirit this is done in a decentralized way, without the need of resorting on a trusted third party. Atomic swaps' functionalities are important, since usually to convert a crypto-currency into another it is necessary to resort on exchanges, that charges high fees and constitutes a single point of failure.
Let's start discussing how atomic swaps are implemented nowadays: to avoid recourse on exchanges and frauds from other users, the atomicity of the swap is enforced through the so called HTLC (Hashed TimeLock Contract). It is a special kind of locking script used to lock the funds on both blockchains. It is constructed in such a way that when one party claims the funds on one chain, the other party can retrieve the coins locked in the other transaction. If something goes wrong or if too much time elapses, it is possible for both parties to get back their coins.
Considering as usual the two parties to be Alice and Bob, we imagine that they want to trade A-coins for B-coins. An HTLC constructed for this situation could have a structure similar to the following one:

```
OP_IF
  OP_HASH256 <digest> OP_EQUALVERIFY OP_DUP OP_HASH160 <Bob
    address>
OP_ELSE
  <num> OP_CHECKSEQUENCEVERIFY OP_DROP OP_DUP OP_HASH160 <Alice
    addres>
OP_ENDIF
OP_EQUALVERIFY OP_CHECKSIG
```

This HTLC would lock the funds on Alice's blockchain. The two branches of the if operator enforce two possibilities: the first one is linked to Bob, that providing the preimage of the digest[6] and a signature valid for his public key could claim the funds. This can be done through an unlocking script of the following form, where the 1 at the end is needed to force the execution of the correct if branch:

```
<Bob sig> <Bob pubkey> <preimage> 1
```

The second path corresponds to Alice's lock time refund possibility: after enough time has elapsed[7] she can claim the funds through an unlocking script that would look something like this, where again the 0 at the end enforces the execution of the else path:

```
<Alice sig> <Alice pubkey> 0
```

A similar contract, where the roles of Alice and Bob are exchanged, would lock Bob's fund on the B-coin chain. Assuming it is Bob that generates the secret preimage, he would need to send its hash (the digest) to Alice, in order to construct the transaction. The atomicity of the swap is ensured by the fact that when Bob spends the output locked by the HTLC he publishes on-chain the preimage: Alice, that needs to monitor the blockchain, could now use the same preimage to take the B-coins.
Caution has to be used when setting the time locks, in order to avoid fund loss: if Bob chooses the preimage, the time locked refund of B-coins should be greater than the one locking the A-coins. Otherwise Bob would be able to wait until his lock time expires and get his B-coins back. However Alice's lock time, being greater, would still prevents the creation of a refund transaction: Bob could publish the preimage and take all the money.

As we have just seen, Alice and Bob have to agree on some aspects of the protocol before they can proceed:

- The number of A-coins and B-coins to be exchanged (i.e. the exchange rate between the two currencies);

[6]Typically one party generates a secret and gives its hashed value (digest) to the counterparty: it is the secret value that enforces the atomicity, while the hash value acts as a commitment.

[7]The exact amount depending on <num> OP_CHECKSEQUENCEVERIFY. This is not the unique possibility to fix a time lock: the OP_CHECKSEQUENCEVERIFY operator can be substituted by OP_CHECKLOCKTIMEVERIFY.

- The addresses where they want to receive funds: each party needs an address for A-coins and one for B-coins (in case of success and failure of the atomic swap)[8];
- The hash of the locking secret: Bob generates the secret and sends the hash of it to Alice;
- The expiry time of the exchange. As we have told Alice needs enough time to redeem her funds, otherwise Bob would be able to grab them all.

Now we analyse the possibilities offered by the introduction of adaptor signatures: first of all, they would allow to bring atomic swaps also on chains which do not support a scripting language to enforce an HTLC.
For the sake of simplicity, but without loss of generality, we assume a unitary exchange rate between A-coins and B-coins. Here follows the detailed protocol to be enforced[9]:

1. Alice and Bob agree on a pair of locking scripts which are secured by an aggregated public key. Let's name these outputs O_1 and O_2: they are locked respectively by the pairs of keys (Q_1^A, Q_1^B) and (Q_2^A, Q_2^B), aggregated through the MuSig protocol described in Section 5.1. In particular:
    (a) Alice prepares a transaction TX_1 paying one A-coin into O_1, but does not sign it. Then she gives it to Bob and asks for a refund transaction that pays back the funds to Alice from TX_1, but has lock time L_1 (the two cooperates in the signature procedure). Upon reception of the refund transaction Alice can sign TX_1. If Bob broadcasts it he can do nothing since he has not Alice's partial signature; Alice can just wait until the lock time expires and broadcast the refund transaction;
    (b) Bob mirrors Alice's behaviour: he prepares TX_2 paying one B-coin to O_2 but does not sign it; he shares TX_2 and requires a signed refund transaction from Alice: it pays from TX_2 to Bob with lock time $\text{L}_2 > L_1$. Upon reception of the refund transaction (the two must interact to sign it), Bob can sign TX_2.

[8]Addresses here have to be intended in general: there are blockchain based protocols that do not have addresses at all, e.g. Mimblewimble.

[9]We present the protocol in the case both chains support Schnorr signature; actually this is not strictly required.

2. At this point Alice and Bob can broadcast TX_1 and TX_2 and both wait for confirmation on the two blockchains. If there are problems at this step, refund transactions can be used to regain control of the money after having waited for the locktime to expire;

3. Alice and Bob engage the MuSig protocol to spend from TX_1 and TX_2. Bob generates in parallel adaptor signatures for both O_1 and O_2 with the same t value and sends them to Alice. She verifies them[10]: if they are valid, she can provide her partial signature to spend TX_1,while if there are problems she aborts the protocol broadcasting the refund transaction;

4. When Bob spends the output of TX_1 thanks to Alice's signature, she learns the secret offset t (monitoring the blockchain) and becomes capable of spending from TX_2. At this point Bob cannot cheat: being L_2 greater than L_1 Alice has time to take the B-coins.

In the cooperative case (no refund transaction broadcast) the unlocking scripts of TX_1 and of TX_2 look exactly the same as an ordinary single user payment: moreover, since only Alice had the adaptor signatures allowing her to extract t, nobody but Alice and Bob can link the transactions between the two chains.
Notice that the efficiency achievement is impressive: we were able to condensate the verbose script semantics required by HTLC in a fixed size signature: this would results, once again, in greater privacy and efficiency, leading to an improvement also in fungibility.

5.3.2 Lightning Network

The Lightning Network is an example of the broader concept of payment channel, an idea originally proposed by Joseph Poon and Thaddeus Dryja in [26]: the aim of this construction is to improve the well known scalability problem of Bitcoin through the so called layer 2 solutions. The name comes from the fact that they do not scale on chain[11], but rather suggest the implementation of other protocols to handle off-chain transactions. The

[10]In particular she verifies that both are valid adaptor signatures and that the offset value used is the same.

[11]It is nearly impossible for Bitcoin to reach the transaction volume of centralized circuits like Visa: Bitcoin processes about ten transactions per second, while Visa has been able to handle a peak of 47000 tps. It means that Visa is around 5000 times better than Bitcoin in this regard: an on-chain scale would result in a bloat of the blockchain size.

blockchain is seen as a court to which it is necessary to resort only in controversial situations or to close the channel. Although other implementations have been suggested (e.g. the eltoo protocol[12] described in [7]), nowadays layer 2 solutions leverage the HTLC presented in Section 5.3.1. The idea is to create bidirectional channels between the participants that transact frequently: the Lightning Network can be seen as a routed path composed by these channels, in such a way that it is not necessary to open a new channel anytime two parties want to transact. We start presenting the basic working of the Lightning Network, that could be of interest on its own. Since its inner working is sort of an extension of the one presented when dealing with atomic swaps (being the atomicity the core property to be preserved), we will give only a low level overview.
First we consider the example of a single channel between Alice and Bob, then we will look at how different channels can be connected to create a multi-hop payment channel.
Alice and Bob opens a channel committing some funds in 2-of-2 multi-signature address through the so called funding transaction. To avoid fund loss, they require the other party to sign an asymmetric commitment transaction before the funding transaction is broadcast to the Bitcoin network. These transactions spend the output of the funding transaction and lock the funds in two outputs: the first spending immediately what is owed to the other party, the second paying the party holding the transaction after a time lock. For example, the outputs of Bob's commitment transaction would look something like this:

- Output 0:

```
OP_HASH160 <Alice's address> OP_EQUALVERIFY OP_CHECKSIG
```

- Output 1:

```
OP_IF
  OP_HASH160 <revocation address>
OP_ELSE
  <num> OP_CHECKSEQUENCEVERIFY OP_DROP OP_HASH160 <Bob's
  address>
OP_ENDIF
OP_EQUALVERIFY OP_CHECKSIG
```

[12]The Lightning Network enforces an honest behaviour under a threat; eltoo avoids this punitive logic enforcing the possibility to update the broadcast of an old transaction, but needs the deployment of the SIGHASH_NOINPUT soft fork proposed in [11]. Moreover three transactions have to be committed to the blockchain to open and close any channel.

In Alice's transaction the role of the two parties are exchanged. These transactions are signed by the other party, so that at any time the holder can also sign and broadcast: the revocation address is required to update the balance since it prevents the broadcast of an old state. This is a key problem for payment channels: any signed commitment transaction could be broadcast and the blockchain is not amendable, thus it is fundamental to prevent old refunding transactions from being broadcast (there is an economic incentive for the party that spent some money in the payment channel to broadcast an older version where he has more money). The revocation address is linked to a combination of Alice's public key and a revocation public key (revocation address = RIPEMD160(SHA-256($Q_A + Q_{rev}$))): through this mechanism, when Bob gives Alice the secret revocation key q_{rev}, by adding it to its private key only she is able to retrieve the funds locked in Output 1 before the lock time has elapsed. In this way, if Bob broadcast an old state Alice has some time to take action and punish Bob retrieving all the funds: this is the threat enforcing an honest behaviour we were mentioning before. Old transactions are toxic, since an erroneous broadcast would result in the funds being lost: it is better to delete intermediate states. However, revocation is not automatic: Alice has to monitor the blockchain and the same holds true for Bob.
Anytime they want to update the balance, they exchange the revocation keys and signatures for the new commitment transaction.
The closure of a channel could happen cooperatively, so that the two parties cooperate creating a transaction with the correct balance but without any time lock, or not, meaning that one of the parties broadcast a commitment transaction for which the other party has not the revocation key: the drawback in this case is that he will need to wait for the lock time to expire.

Now that we have seen the basic idea behind bidirectional channels, we can study their routing. Such a construction creates multi-hop payment channels, for which HTLC are needed: this contract ensures the atomicity of all the payments along the path.
Consider the situation in which Alice wants to pay Carol 1 BTC. There is not an open channel between them: they could open one committing to the blockchain and consequently paying the associated fees, but a better idea is to rely on the Lightning Network. It automatically finds a path going from Alice to Carol taking advantage of already opened channels. In our case we consider the channels Alice-Bob and Bob-Carol.
Alice's Lightning Network node (LN node, not to be confused with the Bitcoin node) needs to find a proper path connecting her and Carol. Notice that she is the only one knowing this path: every other participant knows

only the previous and the following nodes. Obviously Carol knows that Alice has to pay her: for this reason she generates a secret r and sends its hash $H_r = \text{hash}(r)$ to Alice. The atomicity of all the transactions is ensured by this hash value. At this point Alice constructs an HTLC paying to the hash H_r: the output of the transaction has to be slightly greater than 1 BTC (e.g. 1.002 BTC), so that the participants in the channel are rewarded with a small fee for the participation. Alice gives the HTLC to Bob, that updates the balance in the channel: Alice's balance decreases but Bob's does not increase; this is due to the fact that Alice's funds are committed in the HTLC and Bob can redeem them only knowing r. The HTLC provides a lock time refund logic, so that Bob has a limited amount of time to learn the secret. For this reason he construct another HTLC updating the state of his channel with Carol, the logic being the same as above: he locks 1.001 BTC (takes for himself 0.001 BTC as a fee) contingent to revelation of the H_r's preimage, with a smaller time lock with respect to the previous contract. The smaller time lock gives Bob enough time to take the coins from Alice's HTLC in case r is revealed by Carol. At this point Carol has secret r and can redeem the funds: she could either send the HTLC on-chain revealing r and closing the channel with Bob, or she can simply reveal the secret to Bob and update the balance of the channel. The same holds true for Bob, that has incentives to reveal r to Alice (remember that he earned a small fee). The secret acts as a sort of receipt for Alice, telling her that the payment has been successful.

In the non cooperative case, these transactions go on-chain and are easily identifiable for blockchain observers, leading to a non negligible lack of privacy. This issue can be avoided through adaptor signatures[13]. Instead of using H_r, Alice would require public keys T_B and T_C both from Bob and Carol (a unique offset could be used, but relying on one for participant allows reblinding of the offset). The funds of her transaction would be locked requiring a 2-of-2 Schnorr multi-signature: during the signature procedure she requires from Bob an adaptor signature with public offset T_B+T_C, so that Bob cannot provide a valid signature not knowing t_C. Similarly, Bob sends coins to Carol demanding for an adaptor signature with T_C as offset. At this point, Carol can produce a valid signature knowing t_C: she would grab the funds, revealing t_C (either by publishing on-chain or by telling directly to Bob) and thus allowing Bob to do the same with Alice's transaction. Notice that for this construction to work it is necessary to include a lock time refund logic.
Beyond the usual efficiency and privacy benefits,we have seen that the secret values used as offset can be reblinded between hops, allowing long chains of

[13]A much thorough discussion can be found in [17].

transactions to be made atomic while even the participants cannot identify which transactions are part of the chain. This is in sharp contrast with the HTLC based channels: although Alice does not reveal the path, all the members have the same hash and can collude to find out the nodes contributing to the channel.

Chapter 6

Conclusions

The aim of this thesis is to give an introduction to the Schnorr signature algorithm, starting from the mathematics and the cryptography behind the scheme, and present some of its amazing applications to Bitcoin, detailing the benefits and the improvements that would arise from its deployment. We started with a brief but thorough description of the mathematical structures (Chapter 2) and cryptographic primitives (Chapter 3) that underpin digital signature schemes. In Chapter 4 we have presented both ECDSA and Schnorr algorithm, respectively the one actually implemented in Bitcoin and the one that is under development. We compared the two schemes, investigating ECDSA lacks and Schnorr benefits, that ranged from security to efficiency. In particular we focused on the linearity property, that turned out to be the key for the higher level construction presented in Chapter 5.
We have seen how to traduce utilities already implemented in Bitcoin in terms of Schnorr signatures: multi-signature schemes are implemented through MuSig (Section 5.1), whose main advantage is to recover key aggregation; threshold signatures can be deployed through the protocols presented in Section 5.2, that makes them indistinguishable from a single signature; the last application we studied has been adaptor signatures and their benefits to cross-chain atomic swaps and to the Lightning Network.

The immediate benefits that Schnorr would bring to Bitcoin are improved efficiency and privacy (multi-signatures and threshold signatures would be indistinguishable from a single signature), leading also in an enhancement in fungibility. All this applications would be possible in a straightforward way after the introduction of Schnorr, that could be brought to Bitcoin through a soft-fork[1]: the fact that Schnorr is superior to ECDSA in every aspect hopefully would ease the process. We have also discussed the possibility of

[1]Improvements in the protocol have to be made without consesus split.

cross-input aggregation, a property that would allow a single signature per transaction: through a proper change in Bitcoin scripting language even this procedure would be possible through a soft fork.
The last thing we would like to point out is that, by no means, the applications presented in this thesis are the unique benefits that Schnorr could bring to Bitcoin. More complex ideas take the names of Taproot ([22]) and Graftroot ([21]), and are built on top of the concept of MAST an Pay-to-Contract: through these constructions all possible redeem scripts would look the same as a single signer transaction in the cooperative case.

Appendix A

Signatures in Bitcoin

In this appendix we will analyse the role of ECDSA in Bitcoin, starting with a general overview of the protocol, focusing our attention on transactions and blocks. Then we will give a low level description of Bitcoin scripting language, some spending policies and their relation with signatures. More complex subjects, such as SegWit, being out of the scope of the present work will not be presented. We refer the interested reader to the dedicated literature (starting from [14], [20], [13] and [15]). An easily understandable and complete introduction to Bitcoin can be found in [3].

In its basic form, a transaction is a transfer of bitcoins from a participant of the Bitcoin network to another user: it consists of some inputs and some outputs, the inputs being the funds to be spent and the outputs being the funds in possession of the new owners. Notice that bitcoins exist only in the form of unspent transaction outputs[1] (UTXO). Every transaction is broadcasted to each node in the network and each one, separately, validates it. Obviously one node propagates a transaction only if it is deemed valid: here is when signatures come in. Typically they are used to prove to the other users in the network ownership of the transferred funds. Groups of transactions are collected into a data structure called block, by special nodes in the network called miners. The action of building blocks is very expensive from a computational point of view (requiring much energy and specialized hardware this traduces in high costs): for this reason, miners are rewarded for their work with the issuance of new bitcoins. These blocks are chained one another including in the header of a block the hash of the previous one, creating the so called blockchain: an append only data structure. This means that altering a transaction requires not only the modification of the block containing it, but

[1]The so called wallets actually store the private keys needed to unlock these outputs, not bitcoins. A proper renaming would be keychain.

also of all the subsequent blocks. This traduces in the alleged immutability of the Bitcoin blockchain. In a nutshell, this is how it is possible to achieve consensus on a transactional history among a distrustful set of participants. Now let's get back to the transaction data structure and give it a deeper look. As already pointed out, a Bitcoin transaction consists of some inputs and some outputs: the inputs being references to some unspent outputs of previous transactions, and the outputs representing the new ownership of bitcoins[2]. These outputs are locked by the so called locking script (also referred to as scriptPubKey, due to historical reasons), a script written in a special stack based, Turing incomplete language: the Bitcoin scripting language. To unlock the funds (i.e. to spend them), the owner has to provide a proof of ownership in the form of an unlocking script (scriptSig), that usually comprises a digital signature. The difference in bitcoin value from inputs and outputs is collected by the miner who mines the block, and is called transaction fee (obviously the sum of the outputs' value cannot exceed the sum of the inputs' value). The presence of a scripting language, despite its simplicity, allows very flexible and complex constructions for the locking script: time lock, multi-signature, payment contingent on the unveiling of a secret are only few possibilities.

To clarify the transaction concept we present an example taken from the Bitcoin blockchain, in particular the transaction is taken from block number 286731 (determined by the block hash 0000000000000000bf3856e067ec21f4c30 a8a859cc7ed7f2de9a2b579200639[3]):

```
{
”hash”:”90b18aa54288ec610d83ff1abe90f10d8ca87fb6411a72b2e56a169f
        dc9b0219”,
”ver”:1,
”vin_sz”:1,
”vout_sz”:2,
”lock_time”:0,
”size”:226,
”in”:[
 {
  ”prev_out”:{
  ”hash”:”18798f8795ded46c3086f48d5bdabe10e1755524b43912320b81ef
          547b2f939a”,
```

[2]The inputs of a transaction are completely spent by it. This implies that if the amount is greater than the one we intend to transfer, we have to create also an output returning the difference to ourselves.

[3]https://blockexplorer.com/block/0000000000000000bf3856e067ec21f4c30a8a859cc7ed7f2de9a2b579200639.

```
   "n":0
  },
  "scriptSig":"3045022100c1efcad5cdcc0dcf7c2a79d9e1566523af9c7229
               c78ef71ee8b6300ab59aa63d02201fe27c3e6374dd3a5425a5
               77d9ca6ad8ff079800175ef9a4475bc98bcef21cf01023b027
               d54ce8b6c730e0d5833f73aec6a5bae4efe04f57d2864a6a7d
               f2af56e46"
}
],
"out":[
  {
   "value":"5.93100000",
   "scriptPubKey":"OP_DUP OP_HASH160
                   4b358739fc7984b8101278988beba0cc00867adc
                   OP_EQUALVERIFY OP_CHECKSIG"
  },
  {
   "value":"1678.06900000",
   "scriptPubKey":"OP_DUP OP_HASH160
                   55368b388ccfe22a3f837c9eee93d053460db339
                   OP_EQUALVERIFY OP_CHECKSIG"
  }
]
}
```

This is a representation of the transaction in human readable terms[4]. The first value is the transaction hash value, uniquely determining the transaction. Then we have the version number, the number of inputs, the number of outputs, the lock time[5] and the size of the transaction in bytes.
Then we have the inputs and the outputs: for what concerns the unique input we have the hash defining the transaction from which we take the unspent transaction output (UTXO) number 0, as determined from the subsequent value n; that is, we look for the transaction determined by the hash value "18798...2f939a" and take its first output. The scriptSig is the script needed to unlock the funds, that we will study in a while.
The outputs are determined by the value in bitcoins and by the scriptPubKey, that contains the mathematical puzzle that the owner has to solve to spend the coins. The scriptPubKey has this name since originally it contained the public key of the receiver: due to privacy and security reasons this is not anymore the case, and the public key has been substituted with the address[6].

[4]Source: `https://www.ccn.com/bitcoin-transaction-really-works/`.

[5]Part of a transaction that indicates the earliest time or earliest block in which the transaction may be added to the blockchain.

[6]An address is the result of the operator OP_HASH160 applied to the public key: first

Now let's analyse the locking and unlocking scripts more in detail. For the transactions of type Pay-to-PubKeyHash (P2PKH, like the ones in the example) the couple of scripts has the following form:

```
scriptPubKey: OP_DUP OP_HASH160 <pubKeyHash> OP_EQUALVERIFY
              OP_CHECKSIG

scriptSig: <sig> <pubKey>
```

Getting back to our example, we can see that the scripts in there are written exactly in this way. Thanks to the presence of the scripting operators, it is easy to recognize the locking scripts (scriptPubKey), since they have exactly the same structure, with two addresses of 20 bytes.
It is not that immediate for what concerns the unlocking script (scriptSig). But we know that it ends with a public key, that consist in 33 bytes (in compressed form), thus we can easily separate the signature from the public key:

```
sig: 3045022100c1efcad5cdcc0dcf7c2a79d9e1566523af9c7229c78ef71ee
     8b6300ab59aa63d02201fe27c3e6374dd3a5425a577d9ca6ad8ff079001
     75ef9a44475bc98bcef21cf01

pubKey: 023b027d54ce8b6c730e0d5833f73aec6a5bae4efe04f57d2864a6a7
        df2af56e46
```

Once the transaction is received by a node, it has to be validated. Table A.1 presents the validation procedure of the input[7].

After having looked at the basic construction, we can study the verification procedure for an m-of-n multi-signature transaction. But first, let's try to understand why they are so appealing.
The primary benefit is to greatly increase the difficulty of stealing the coins. With a 2-of-2 address, you can keep the two keys on separate machines, so that theft would require a double attack; it can also be used to protect against accidental loss: with a 2-of-3 address, not only does theft require obtaining 2 different keys, but you can still use the coins if you forget any single key; it can also be used for more advanced scenarios such as an address shared by multiple people, where a majority of vote is required to use the funds.

it is applied the SHA-256, then the RIPEMD160, another hash function that outputs in the space of 160 bits, or 20 bytes.

[7]The structure of the validation procedure is taken from `https://en.bitcoin.it/wiki/Transaction`.

Stack	Script	Description
Empty.	<sig> <pubKey> OP_DUP OP_HASH160 <pubKeyHash> OP_EQUALVERIFY OP_CHECKSIG	The scriptSig and the scriptPubKey are combined.
<sig> <pubKey>	OP_DUP OP_HASH160 <pubKeyHash> OP_EQUALVERIFY OP_CHECKSIG	The signature and the public key are added to the stack.
<sig> <pubKey> <pubKey>	OP_HASH160 <pubKeyHash> OP_EQUALVERIFY OP_CHECKSIG	The operator OP_DUP takes the last element on the stack and puts a duplicate of it on top.
<sig> <pubKey> <pubKeyHash>	<pubKeyHash> OP_EQUALVERIFY OP_CHECKSIG	The OP_HASH160 takes the public key as input and applies to it in sequence SHA-256 and RIPEMD160.
<sig> <pubKey> <pubKeyHash> <pubKeyHash>	OP_EQUALVERIFY OP_CHECKSIG	The address <pubKeyHash> is pushed on top of the stack.
<sig> <pubKey>	OP_CHECKSIG	OP_EQUALVERIFY checks that the last two elements on top of the stack are equal. If not, the validation fails.
True.	Empty.	OP_CHECKSIG checks the consistency of the signature with the public key associated to the address in the scriptPubKey. The validation process succeeds if at the end on the stack there is a boolean value indicating true.

Table A.1: Verification procedure for a P2PKH transaction.

Such a construction would be typically used for corporate accounts, with different layers for security, and by escrow services. For practical reasons, multi-signature transactions are fundamental for a proper custody of any digital assets.

In the m-of-n scenario, the couple of scripts would look something like this:

```
scriptPubKey: m <pubKey1> <pubKey2> ... <pubKeyn> n
OP_CHECKMULTISIG

scriptSig: 0 <sig1> <sig2> ... <sigm>
```

The scriptSig starts with a 0 due to a bug in the OP_CHECKMULTISIG operator, that requires an additional dummy element to work. It is immediate to notice that both scripts increase linearly in size: the scriptPubKey with n (the number of participants), the scriptSig with m (the number of signers). The verification procedure for a multi-signature policy is exemplified in Table A.2.

Stack	Script	Description
Empty.	0 <sig1> <sig2> ... <sigm> m <pubKey1> <pubKey2> ... <pubKeyn> n OP_CHECKMULTISIG	The scriptSig and the scriptPubKey are combined.
0 <sig1> <sig2> ... <sigm> m <pubKey1> <pubKey2> ... <pubKeyn> n	OP_CHECKMULTISIG	All the constants are added to the stack.
True.	Empty.	OP_CHECKMULTISIG checks whether the m signatures come from some of the n public keys.

Table A.2: Verification procedure for an input locked by an m-of-n policy.

As the reader may have detected another small flaw of the multi-signature construction adopted in Bitcoin is that the signatures have to be ordered as the public keys in the locking script. Since verification is made by subsequent attempts (first signature checked against first public key: if valid pass to the second signature, otherwise check the first signature against the second public key), a proper ordering traduces in a faster verification.

In the Bitcoin network, the bigger a transaction is in bytes, the more costly it is in terms of transaction fees. With the multi-signature construction, the burden had to be carried by the payer, even if it was the receiver who asked for a multi-signature locking script; on top of that, the payer knew the locking policy of the payee, with a consequent loss of privacy. This are the main reasons that led to the creation of Pay-to-Script-Hash (P2SH) transactions, that now allows many possibilities with various complex redeem paths. In a P2SH transaction the locking script is always of the following form:

```
scriptPubKey: OP_HASH160 <20 bytes hash of the redeem script>
              OP_EQUAL
```

Such a construction enables huge possibilities: in summary, it is not anymore a locking script of the type "pay to a person able to provide ownership of the private key corresponding to this address", but more generally of the kind "pay to a person validating a script whose hash is this". For a better understanding we will exemplifies the behaviour of P2SH using the previous multi-signature transaction. We have:

```
Redeem script: m <pubKey1> <pubKey2> ... <pubKeyn> n
               OP_CHECKMULTISIG

Locking script: OP_HASH160 <20 bytes hash of the redeem script>
                OP_EQUAL

Unlocking script: 0 <sig1> <sig2> ... <sigm>
```

Now the burden of the greater redeem script is on the receiver of the transaction, since the output is locked with the locking script, that is much lighter than the redeem script. When the receiver will finally try to spend this transaction, he will need to supply the serialization of the unlocking and of the redeem scripts: first the redeem script will be checked against the locking script to ensure that the hash matches, if this is the case the redeem script is validated against the unlocking script as already seen.
Once again we would like to point out that, even if P2SH was conceived in order to make multi-signature transactions more aligned with the economic interests into play, they allow a huge amount of redemption script combined with Bitcoin scripting language.

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