Convergent adaptive Finite Element Methods for the solution of the EEG forward problem with the help of the subtraction method

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1. Tasks
2. The subtraction approach and general setting
3. Treatment of hanging nodes
4. Error estimator and AFEM
5. Validation and tests with DUNE-FEM
6. Results and outlook

Outline

Tasks

- Use of adaptive Finite Element Methods (AFEM) to solve the subtraction forward problem on hexahedral meshes
- Introduction and derivation of a residual based error estimator to enable reasonable local mesh-refinement for efficient reduction of numerical errors
- Treatment of occuring problems with hanging nodes in the FEM-approach in locally refined hexahedral meshes
- Implementation and tests of the introduced AFEM with the help of the "**D**istributed and **U**nified **N**umerics **E**nvironment" (**DUNE**)

Outline

The subtraction forward problem

The **subtraction forward problem** (see [1]) is to find $\phi^{corr} \in H^1(\Omega)$ for a domain $\Omega \subset \mathbb{R}^3$ such that

$$\int_\Omega \langle \sigma(x)\nabla\phi^{corr}, \nabla v(x)\rangle \mathrm{d}x = \int_\Omega \langle \underbrace{(\sigma^\infty - \sigma(x))\nabla\phi^\infty(x)}_{=:I}, \nabla v(x)\rangle \mathrm{d}x - \int_{\partial\Omega} \underbrace{\langle \sigma^\infty \nabla\phi^\infty(x), \mathbf{n}(x)\rangle}_{=:g} v(x)\mathrm{d}x, \qquad (1)$$

$$\int_\Omega \phi^{corr}(x)\mathrm{d}x = -\int_\Omega \phi^\infty(x)\mathrm{d}x \qquad (2)$$

hold for all $v \in H^1(\Omega)$, where $\mathbf{n}$ denotes the surface unit-outer normal and $\sigma^\infty \in \mathbb{R}$ the isotropic conductivity tensor at source position $y \in \Omega$.

General discretization setting

- The domain $\Omega \subset \mathbb{R}^3$ is defined by $\overline{\Omega} = \bigcup_{i=0}^{n} \overline{\Omega}_i$ and an isotropic conductivity tensor $\sigma_i \in \mathbb{R}$ is assigned to each domain $\Omega_i \in \mathbb{R}^3$
- A conform, shape regular mesh T_h for mesh-size h as a decomposition of Ω into hexahedrons is used, i.e. $\overline{\Omega} = \bigcup_{j=0}^{m} \overline{K}_j$ for $K_j \in T_h$
- The space of linear finite elements on hexahedral meshes is defined by $Q_h^1 := \{v_h \in \mathrm{C}^0(\Omega) |\ v_h|_K \in \mathbb{Q}^1(K), K \in T_h\}$
- FEM then yields: $u_h \in Q_h^1$ is called solution of the linear finite element method if $B(u_h, v_h) = f(v_h)$ holds $\forall v_h \in Q_h^1$ for a $H^1(\Omega)$-elliptic bilinear form B and $f \in H^{-1}(\Omega)$ as the right-hand side functional.

Outline

The local refinement of a hexahedral mesh T leads to an irregular mesh T' and to the occurence of so called **hanging nodes**. Hence $u_h \in Q_h^1$ on T' is not in $H^1(\Omega)$ due to missing continuity across element faces with hanging nodes.

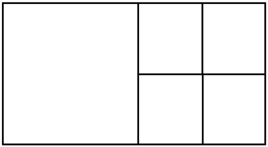


Figure : An 1-irregular mesh T'

Therefore it is not suitable as a solution of the standard FEM-approach. To solve this problem the following space is introduced:

$$D_h := \left\{ u \in L^2(\Omega) |\ u|_K \in \mathbb{Q}^1(K), K \in T_h \right\}, \tag{3}$$

with $D_h \not\subset C^0(\Omega)$.

The following criterion assures continuity of a function $u_h \in D_h$ (see [2]):

For $u_h \in D_h$ let the global DOF-vector be defined as

$$(u_1, u_2, \dots, u_n)^t, \tag{4}$$

where u_i is the DOF associated to the node a_i in the mesh T_h. Then u_h is globally continuous, i.e. $u_h \in C^0(\Omega)$, if and only if

$$u_i = \sum_{a_j \in \Lambda(a_i)} c(a_i) u_j \tag{5}$$

holds for all hanging nodes a_i and appropriate coefficients $c(a_i) \in \mathbb{R}$. $\Lambda(a_i)$ denotes the set of all neighboring regular nodes of a_i.

FEM on 1-irregular meshes

Let $\Omega \subset \mathbb{R}^3$ a polygonal-bounded domain, T_h a 1-irregular mesh on Ω and $R_h := D_h \cap C^0(\Omega)$. Furthermore a continuous and $H^1(\Omega)$-elliptic form $B\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and a right-hand side functional $f \in H^{-1}(\Omega)$ shall be given. Then $R_h \subset H^1(\Omega)$ and $u_h \in R_h$ is called solution of the linear finite element method on 1-irregular meshes if

$$B(u_h, v_h) = f(v_h) \text{ holds for all } v_h \in R_h. \tag{6}$$

Outline

The mesh T_h shall be refined locally with the help of a residual based error estimator η_h^2, such that the following inequality is valid for the exact solution u :

$$\|u - u_h\|_{E,\Omega}^2 \leq C\eta_h^2 \tag{7}$$

The local error estimator for $K \in T_h$ and $u_h \in R_h$ can be defined as

$$\eta_h^2(u_h, K) = h_K^2 \|\operatorname{div} \sigma \nabla u_h - \operatorname{div} I\|_{L^2(K)}^2 + h_K \sum_{\iota \in I_t} \|r(u_h)\Big|_\iota\|_{L^2(\partial K)}^2$$

with

$$r(u_h)\Big|_\iota = \begin{cases} n_t \cdot [\sigma \nabla u_h], & \text{if } \iota \subset \Omega \setminus \partial\Omega, \\ < \sigma \nabla u_h, \mathbf{n} > -g, & \text{if } \iota \subset \partial\Omega \end{cases}$$

leading to the global error estimator $\eta_h^2 := \sum_{K \in T_h} \eta_h^2(u_h, K)$ fulfilling (7).

The AFEM-algorithm for subtraction forward problem

1. Give the initial conforming hexahedral mesh T_0 and parameter $\theta \in (0,1)$, set $l = 0$;
2. Solve subtraction forward problem on T_l and obtain solution $\phi_l^{corr,y}$;
3. Compute the error estimator η_t for each element $t \in T_l$;
4. Use maximum strategy: Mark minimal elements set M_l such that for all $t \in M_l$

$$\eta_t^2 \geq \theta \cdot \max_{t' \in T_l} \eta_{t'}^2;$$

5. Refine T_l by bisection of all elements in M_l to get T_{l+1};
6. Detect and treat resulting hanging nodes appropriately;
7. Set $l := l + 1$ and go to step 2.

Convergence analysis

Using the maximum strategy the following property can be shown (analagously to [3]):

Let $\{T_l, u_l\}_{l \geq 0}$ be a sequence of meshes and solutions from the AFEM-algorithm. Let $e_l := u - u_{l+1}$ and $\epsilon_l = u_{l+1} - u_l$ denote the errors for the exact solution u. Then there exist constants $0 < \alpha_l < 1$ and $0 < \beta_l$ depending on the shape regularity of T_0, $|T_l|$, marking parameter $0 < \theta \leq 1$ and σ such that

$$\|e_{l+1}\|^2_{E,\Omega} + \beta_l \eta^2_{l+1} \leq \alpha_l (\|e_l\|^2_{E,\Omega} + \beta_l \eta^2_l). \tag{8}$$

(8) does not give convergence of the AFEM-algorithm using the maximum strategy, therefore the use of a *Dörfler* marking strategy is recommended, which assures convergence.

Outline

Validation

The error estimator is validated with the help of the problem:

Let $\Omega := (0,1)^3$ be given, then the sinus-problem with Neumann-boundary conditions is to find $u \in H^1(\Omega)$ such that

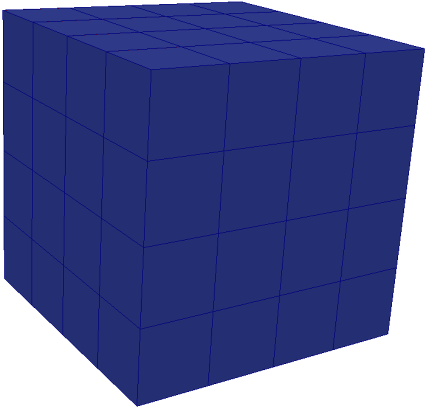
$$-\triangle u(x) = 12\pi^2 \prod_{i=1}^{3} \sin(2\pi x_i) \qquad \forall x = (x_1, x_2, x_3)^t \in \Omega \quad (9)$$

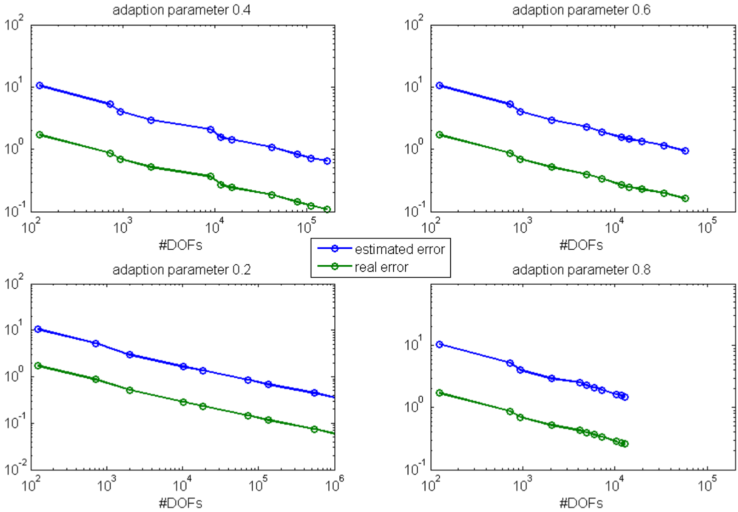
$$\langle \nabla u(x), \mathbf{n}(x) \rangle = \left\langle \nabla \left(\prod_{i=1}^{3} \sin(2\pi x_i) \right), \mathbf{n}(x) \right\rangle \quad \forall x \in \partial\Omega \quad (10)$$

and $u(0) = 0$. The exact solution is obviously given by

$$u(x) = \prod_{i=1}^{3} \sin(2\pi x_i) \ \forall x = (x_1, x_2, x_3)^t \in \Omega. \quad (11)$$

The used initial, regular mesh T_0 is defined as the uniform decomposition of Ω into 64 hexahedron as the figure below illustrates:



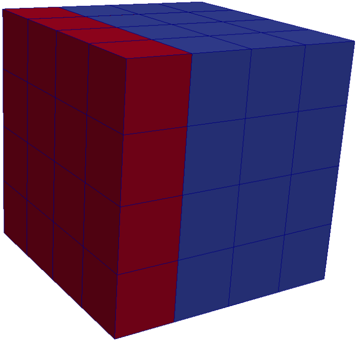

adaption parameter 0.4
adaption parameter 0.6
adaption parameter 0.2
adaption parameter 0.8
#DOFs
estimated error
real error
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}

Tests for subtraction forward problem

Let $\Omega := (0,1)^3$ and $\Omega = \Omega_0 \cup \Omega_1$ with

$$(\Omega_0, \sigma_0) \text{ with } \Omega_0 := (0,1) \times (0,0.25) \times (0,1), \qquad \sigma_0 := 0,0000042 \tag{12}$$

$$(\Omega_1, \sigma_1) \text{ with } \Omega_1 := \Omega \backslash \Omega_0, \qquad \sigma_1 := 0.00033 \tag{13}$$



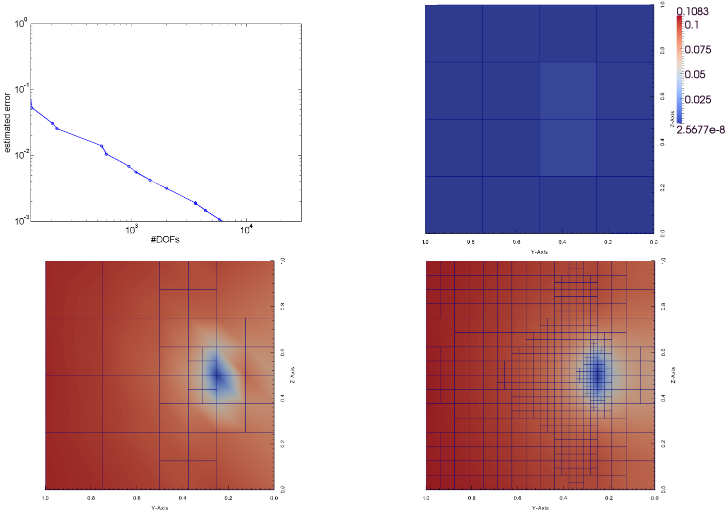


Figure : source position $(0.5, 0.4, 0.5)$, (top left) convergence history of the error estimator (top right) local estimated errors at step 0 (bottom left/right) approximated solution at step 2/14

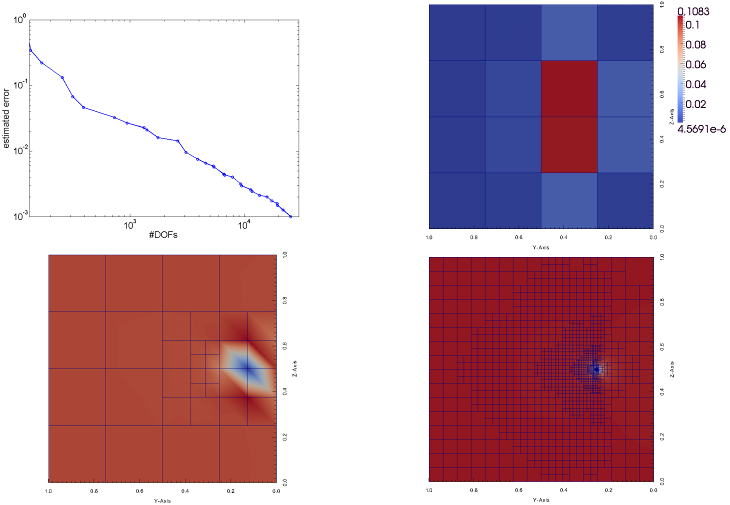


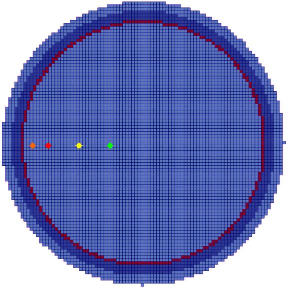
Figure : source position $(0.5, 0.26, 0.5)$, (top left) convergence history of the error estimator (top right) local estimated errors at step 0 (bottom left/right) approximated solution at step 2/35

Tests in 4-layer sphere model

Let the 4-layer sphere domain $\Omega \subset \mathbb{R}^3$ with midpoint $(0,0,0)$ be defined with radii $r_0 := 78, r_1 := 80, r_2 := 86, r_3 = 92$ for conductivities

$$\sigma_0 := 0.00033, \sigma_1 := 0.00179, \sigma_2 := 0.0000042, \sigma_3 := 0.00033. \qquad (14)$$

The sources $(-1,22,-1)$, $(-1,42,-1)$, $(-1,62,\ -1)$ and $(-1,72,-1)$ for dipole moment $M := (0,1,0)^T$ and $\theta := 0.8$ will be investigated.



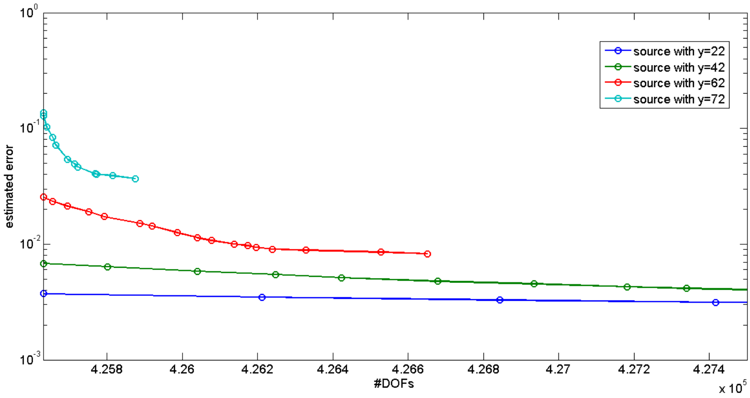


Figure : estimated errors in respect to number of DOFs for adaption step 10

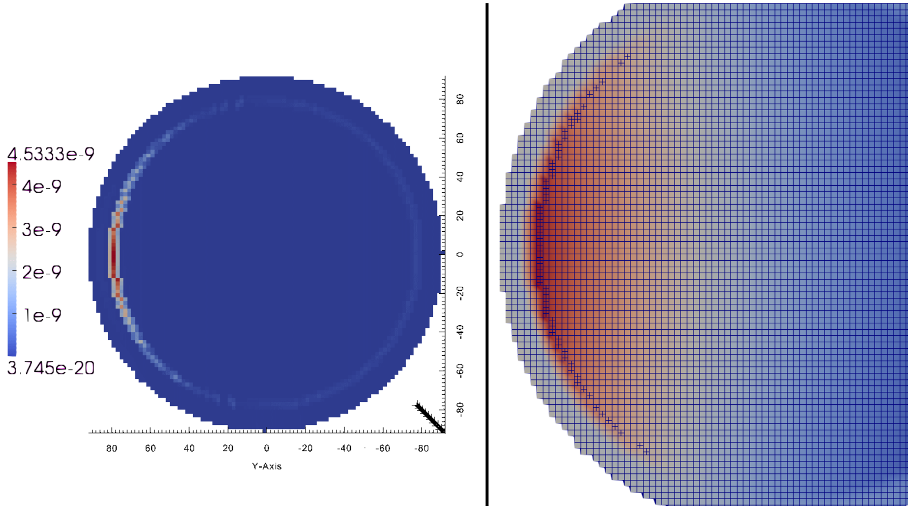


Figure : source $(-1, 22, -1)$ (left) local estimated errors at step 0 (right) approximated solution at step 10

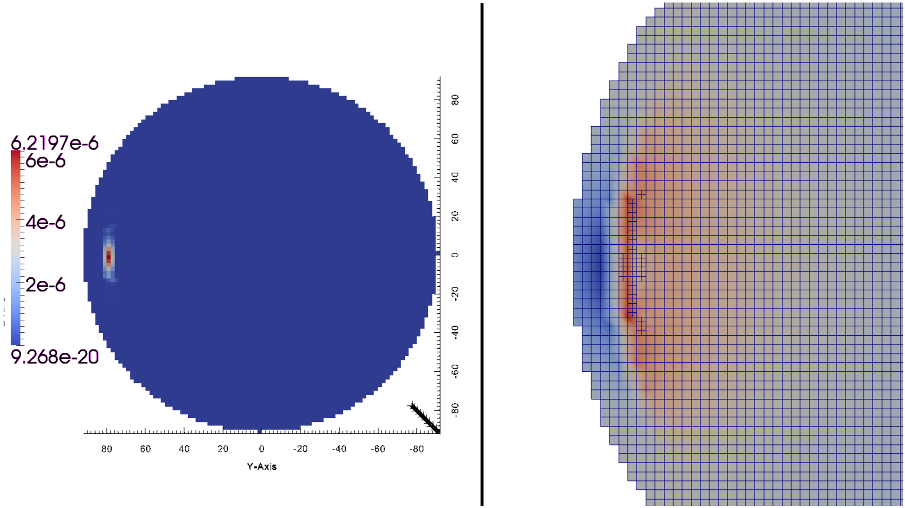


Figure : source $(-1, 62, -1)$ (left) local estimated errors at step 0 (right) approximated solution at step 14

Outline

Results

- Introduction and derivation of AFEM for special elliptic PDE and subtraction forward problem
- Convergence of the AFEM undemonstrable for chosen maximum strategy
- Implementation of AFEM with the help of DUNE shows promising results

Outlook

- Improvement (correction) of residual based error estimator possible
- Improvement of implementation desirable
- Detailed convergence study for several settings to be done
- Usage of given implementation for other applications (TMS, tDCS) interesting

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