University of Strathclyde

Maths Skills Support Centre

Foundations - Equations

1. Linear Equations
2. Quadratic Equations
3. Polynomial Equations
4. Simultaneous Equations
5. Logarithmic and Exponential Equations

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1 Linear Equations

An equation is **linear** if it is in the form

$$ax + b = 0$$

for some real numbers $a(\neq 0)$ and b. The solution of the equation is obtained by making x the subject:

$$\begin{aligned} ax + b = 0 &\Rightarrow ax = -b \\ &\Rightarrow x = -\frac{b}{a}. \end{aligned}$$

Example 1.1 Solve the following equations for x.

1) $-3x - 4 = 0$

2) $\dfrac{x}{2} + 11 = 0$

3) $-6x + 7 = 3x - 2$

4) $8 + \dfrac{2}{x} = -1$

$$\begin{aligned} 1)\ -3x - 4 = 0 &\Rightarrow -3x = 4 \\ &\Rightarrow x = -\frac{4}{3} \end{aligned}$$

$$\begin{aligned} 2)\ \frac{x}{2} + 11 = 0 &\Rightarrow \frac{x}{2} = -11 \\ &\Rightarrow x = -22. \end{aligned}$$

$$\begin{aligned} 3)\ -6x + 7 = 3x - 2 &\Rightarrow -6x - 3x = -2 - 7 \quad \text{(combine } x\text{-terms)} \\ &\Rightarrow -9x = -9 \\ &\Rightarrow x = 1. \end{aligned}$$

$$\begin{aligned} 4)\ 8 + \frac{2}{x} = -1 &\Rightarrow 8x + 2 = -x \quad \text{(multiply both sides by } x\text{)} \\ &\Rightarrow 9x = -2 \quad \text{(combine } x\text{-terms)} \\ &\Rightarrow x = -\frac{2}{9}. \end{aligned}$$

Exercise 1.1 Solve the following equations.

1) $7x + 49 = 0$

2) $-2x + \dfrac{1}{3} = 0$

3) $\dfrac{y}{2} + 4 = y - 2$

4) $\dfrac{y}{4} - 3 = -\dfrac{5y}{4} + 5$

5) $5 = \dfrac{2}{z} + 1$

6) $\dfrac{8}{z} + 3 = \dfrac{4}{z} - 2$

2 Quadratic Equations

A **quadratic equation** is one in the form

$$ax^2 + bx + c = 0,$$

where $a(\neq 0)$, b and c are real numbers. These equations can be solved by either *factorising* the quadratic expression or *completing the square* (see Foundations - Algebra for details on these techniques).

Example 2.1 Solve the following quadratic equations.

1) $4x^2 - 6x = 0$

2) $x^2 - 7x - 16 = 0$

3) $x^2 + 8x - 2 = 0$

4) $-2x^2 - 6x + 1 = 0$

1) The first step is to factorise the left-hand side of the equation:

$$4x^2 - 6x = 0 \Rightarrow 2x(2x - 3) = 0.$$

If the product of two numbers equals zero then at least one of them must be zero, so:

$$2x = 0 \text{ or } 2x - 3 = 0.$$

It follows that

$$x = 0 \quad \text{or} \quad x = \frac{3}{2}.$$

2) Factorise the left-hand side:

$$x^2 - 7x - 18 = 0 \Rightarrow (x - 9)(x + 2) = 0.$$

Therefore

$$x - 9 = 0 \text{ or } x + 2 = 0.$$

Hence

$$x = 9 \text{ or } x = -2.$$

3) It is not so obvious how to factorise $x^2 + 8x - 2$ so we instead complete the square:

$$x^2 + 8x - 2 = (x + 4)^2 - 16 - 2 = (x + 4)^2 - 18.$$

Therefore

$$x^2 + 8x - 2 = 0 \Rightarrow (x + 4)^2 - 18 = 0 \Rightarrow (x + 4)^2 = 18.$$

We deduce that

$$x + 4 = \sqrt{18} \text{ or } x + 4 = -\sqrt{18}.$$

Hence

$$x = -4 + \sqrt{18} \text{ or } x = -4 - \sqrt{18}.$$

4) Complete the square:

$$-2x^2 - 6x + 1 = -2(x^2 + 3x) + 1 = -2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4}\right] + 1 = -2\left(x + \frac{3}{2}\right)^2 + \frac{11}{2}.$$

Therefore

$$-2x - 6x + 1 = 0 \Rightarrow -2\left(x + \frac{3}{2}\right)^2 + \frac{11}{2} = 0.$$

It follows that

$$\left(x + \frac{3}{2}\right)^2 = \frac{11}{4}$$

and so

$$x + \frac{3}{2} = \frac{\sqrt{11}}{2} \text{ or } x + \frac{3}{2} = -\frac{\sqrt{11}}{2}.$$

Hence

$$x = \frac{\sqrt{11} - 3}{2} \text{ or } x = \frac{-\sqrt{11} - 3}{2}.$$

WARNING! Not every quadratic equation has *real* solutions. For example,

$$x^2 + 4 = 0 \Rightarrow x^2 = -4.$$

Since $x^2 \geq 0$ for any real x, no real solutions exist.

The *general solution* of any quadratic equation can be found via the **quadratic formula**:

If $ax^2 + bx + c = 0$ then
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 2.2 Solve equation (4) from Example 2.1.
In the equation $-2x^2 - 6x + 1 = 0$, $a = -2$, $b = -6$ and $c = +1$. Therefore

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 4 \times (-2) \times 1}}{2 \times (-2)} \\ &= \frac{6 \pm \sqrt{44}}{-4} \\ &= \frac{6 \pm 2\sqrt{11}}{-4} \\ &= \frac{3 \pm \sqrt{11}}{-2} \end{aligned}$$

In other words
$$x = \frac{-3 + \sqrt{11}}{2} \quad \text{or} \quad x = \frac{-3 - \sqrt{11}}{2}$$
In the quadratic formula the term $b^2 - 4ac$ is called the *discriminant* and its value determines the possible solutions of the corresponding quadratic equation:

- if $b^2 - 4ac > 0$ then the quadratic has 2 distinct solutions;
- if $b^2 - 4ac = 0$ then the quadratic has 1 distinct solutions;
- if $b^2 - 4ac < 0$ then the quadratic has no real solutions.

Example 2.3 Solve $49x^2 + 28x + 4 = 0$.
The discriminant equals zero, $b^2 - 4ac = 28^2 - 4 \times 49 \times 4 = 0$, so the equation has one real solution:
$$x = \frac{-28 \pm 0}{2 \times 49} = -\frac{2}{7}.$$
We can see this directly by factorising:

$$49x^2 + 28x + 4 = 0 \Leftrightarrow (7x + 2)^2 = 0 \Leftrightarrow x = -\frac{2}{7}.$$

Exercise 2.1 Solve the following quadratic equations via factorisation.

1) $x^2 - 5x + 6 = 0$	2) $4x^2 - 25 = 0$	3) $2y^2 + 5y = 0$
4) $x^2 - 3x + 2 = 0$	5) $2z^2 - 11z + 5 = 0$	6) $x^2 + 7x = 44$
7) $y^2 + 2y - 8 = 0$	8) $6z^2 + 13z - 5 = 0$	9) $63y = 49 + 18y^2$

Exercise 2.2 Solve the following quadratic equations by completing the square. Leave you answers as a surd, where appropriate.

1) $x^2 - 3x - 6 = 0$	2) $2x^2 - 3x - 25 = 0$	3) $2y^2 + 5y - 1 = 0$
4) $z^2 - 3z - 2 = 0$	5) $3x^2 + 11x + 1 = 0$	6) $4y^2 + 7y + 4 = 0$

Exercise 2.3 Solve the following quadratic equations by any method, giving your answers to 3 decimal places where appropriate. Note that you may have to first simplify the equation.

1) $x^2 - 4x - 21 = 0$	2) $2x^2 - 3x - 9 = 0$	3) $x^2 - x - 12 = 0$
4) $x^2 = 3(2x + 1) = 0$	5) $5x(x + 3) = 1$	6) $x^2 - 6x + 9 = 0$
7) $5x(x + 3) = 0$	8) $x^2 = 3x + 40$	9) $x^2 - 3x + 9 = 0$
10) $\dfrac{x+5}{3} = \dfrac{x^2-2}{4}$	11) $x = \dfrac{15}{x+5}$	12) $2x - 1 = \dfrac{8x-3}{2x+1}$
13) $\dfrac{x+2}{4} + \dfrac{3}{x-1} = 7$	14) $\dfrac{x-5}{3} = \dfrac{x^2+2}{4}$	15) $\dfrac{1}{3x} - \dfrac{1}{x+1} = 1$

3 Polynomial Equations

In this section we will look at how to solve equations of the form

$$p(x) = 0,$$

where $p(x)$ is a polynomial of degree at least 3. In general this is a hard problem but for certain equations we can solve them using the **Factor Theorem**. Before we state this theorem we need some terminology:

- If $p(a) = 0$ for some number a then we say that $x = a$ is a **root** of the polynomial $p(x)$.
- We say that $(x - a)$ is **factor** of $p(x)$ if $p(x) = (x - a)q(x)$ for some polynomial $q(x)$.

Factor Theorem Let $p(x)$ be a polynomial. Then $(x - a)$ is a factor of $p(x)$ if and only if $p(a) = 0$.

Repeated use of this theorem helps us to factorise $p(x)$ and hence solve the equation $p(x) = 0$.

Example 3.1 Show that $(x - 1)$ is a factor of $p(x) = x^3 - 2x^2 - x + 2$ and hence solve $p(x) = 0$.

By the Factor Theorem $(x-1)$ is a factor if and only if $x=1$ is a root:

$$p(1) = 1^3 - 2\times 1 - 1 + 2 = 0$$

and so $(x-1)$ is a factor of $p(x)$.
In order to solve $p(x)=0$ we must find $q(x)$ such that

$$p(x) = (x-1)q(x).$$

This is easily done using *synthetic division.*

$$\begin{array}{rr|rrrrl} \text{root} \rightarrow & 1 & 1 & -2 & -1 & 2 & \leftarrow \text{coefficients of } p(x) \\ & & 0 & 1 & -1 & -2 & \leftarrow \text{product of the root and sum of previous column values} \\ \cline{3-6} & & 1 & -1 & -2 & 0 & \leftarrow \text{sum of column values} \end{array}$$

The numbers in the last row are the coefficients of $q(x)$:

$$q(x) = x^2 - x - 2.$$

It follows that

$$\begin{aligned} x^3 - 2x^2 - x + 2 &= (x-1)(x^2 - x - 2) \\ &= (x-1)(x-2)(x+1) \end{aligned}$$

and hence

$$x^3 - 2x^2 - x + 2 = 0 \Leftrightarrow x = -1 \text{ or } x = 1 \text{ or } x = 2.$$

Example 3.2 Solve $x^3 - 8x^2 - x + 8 = 0$.
We begin by using synthetic division to test for possible factors. It is sensible to start with simple guesses for the root, e.g. $x = 1, -1, 2, -2 \ldots$.

$$x = 1 : \qquad \begin{array}{r|rrrr} 1 & 1 & 2 & -5 & -6 \\ & 0 & 1 & 3 & -2 \\ \hline & 1 & 3 & -2 & -8 \end{array}$$

Given that the final number in the last row is not zero, $x=1$ is not a root and hence $(x-1)$ is not a factor.

$$x = -1 : \qquad \begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & 0 & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

In this case $x = -1$ is a root and hence $(x+1)$ is a factor. It follows that

$$\begin{aligned} x^3 + 2x^2 - 5x - 6 = 0 &\Leftrightarrow (x+1)(x^2 + x - 6) = 0 \\ &\Leftrightarrow (x+1)(x-2)(x+3) = 0 \\ &\Leftrightarrow x = -3 \text{ or } x = -1 \text{ or } x = 2 \end{aligned}$$

Exercise 3.1 1) Solve the following polynomial equations.

i) $x^3 + 2x^2 - x - 2 = 0$	*ii)* $x^3 + 2x^2 - 5x + 6 = 0$
iii) $x^3 - 3x^2 - 4x + 12 = 0$	*iv)* $x^3 - 6x^2 + 15x - 14 = 0$
v) $x^3 - x^2 - 4x + 4 = 0$	*vi)* $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$
vii) $x^3 - 3x^2 - 10x + 24 = 0$	*viii)* $2x^2 + 5x^2 - 28x - 15 = 0$

2) Show that $(x+2)^2$ is a factor of $p(x) = x^3 + 3x^2 - 4$.

3) Find the value of p such that $(x-4)$ is a factor of $p(x) = 2x^3 - 5x^2 + px - 20$.

4) Find the values of a and b such that $(x^2 - 4)$ is a factor of $p(x) = x^3 + ax^2 + bx - 12$.

5) Find the values of a and b such that $(x-2)$ and $(x+3)$ are factors of $p(x) = 2x^3 + x^2 + ax + b$.

4 Simultaneous Equations

In order to solve a pair of *simultaneous equations* (or any system of equations) we are required to find the values of the unknowns that solve both equations **simultaneously**.

Example 4.1 Solve the simultaneous equations

$$\begin{aligned} 3x + 5y &= 21 \quad \text{(A)} \\ 2x + 3y &= 13 \quad \text{(B)} \end{aligned}$$

We can eliminate x from both equations by taking a combination of equations (A) and (B). First, multiply (A) by 2 and (B) by 3:

$$\begin{aligned} 6x + 10y &= 42 \quad \text{(C)} \\ 6x + 9y &= 39 \quad \text{(D)} \end{aligned}$$

Next, subtract equation (D) from equation (C) to give

$$y = 3.$$

Substituting $y = 3$ into equation (A) gives

$$3x + 15 = 21,$$

i.e. $x = 2$. It follows that the solution to the simultaneous equations is

$$x = 2 \quad \text{and} \quad y = 3.$$

Note that we could have substituted $y = 3$ into equation (B) to obtain the same solution.

WARNING! It is important to check that your solution satisfies **both** equations.

Example 4.2 Solve

$$\begin{aligned} x + y &= 1 \quad &\text{(A)} \\ x^2 + y^2 - 5 &= 0 \quad &\text{(B)} \end{aligned}$$

From (A), $y = 1 - x$. We can substitute this expression into (B) to obtain an equation in terms of x only:

$$\begin{aligned} x^2 + y^2 - 5 = 0 &\Leftrightarrow x^2 + (1-x)^2 - 5 = 0 \\ &\Leftrightarrow x^2 + 1 - 2x + x^2 - 5 = 0 \\ &\Leftrightarrow 2x^2 - 2x - 4 = 0 \\ &\Leftarrow (2x+2)(x-2) = 0 \\ &\Leftrightarrow x = -1 \quad \text{or} \quad x = 2. \end{aligned}$$

If $x = -1$ then $y = 2$, and if $x = 2$ then $y = -1$. Hence the solution of the simultaneous equations is

$$(x, y) = (-1, 2) \quad \text{or} \quad (2, -1).$$

WARNING! It is important to clearly state your solutions. For example, in the previous example, it would have been ambiguous to write $x = -1$ or $x = 2$, $y = 2$ or $y = -1$.

Exercise 4.1 Find all real solutions of the following simultaneous equations.

1) $3x + y = 9$ $x + y = 5$	2) $2x - 3y = 15$ $2x + y = 3$	3) $3x + 2y = 1$ $2x - 3y = 18$
4) $3a + 2b + 3 = 0$ $5a - 3b - 16 = 0$	5) $6p = 9 + 2q$ $4q = 12 - 3p$	6) $x - y = 2$ $3x + 8y = 39$
7) $5r + 2s = 9$ $4r + 9s = 22$	8) $3d + 2e = 17$ $e - 1 = d$	9) $2x - 3y = 15$ $2x + y = 3$
10) $x(y - 1) = 0$ $x^2 + y^2 = 4$	11) $(x - 2)(y - 3) = 0$ $x^2 + y^2 = 20$	12) $2x - xy = 0$ $x^2 - y^2 = 4$
13) $y(x - 1) = 0$ $x^2 + y^2 = 1$	14) $xy - y - x + 1 = 0$ $x^2 + y^2 = 5$	15) $xy - 2y = x - 2$ $x^2 + y^2 = 2$

5 Logarithmic and Exponential Equations

In this section we will look at examples of equations featuring *logarithms* and *exponentials*. See Foundations - Functions §5 and §6 for more information about these concepts.

Example 5.1 Solve $5\log(x^2) = 10$.
Using one of the Log. Rules, we can write

$$10\log(x) = 10,$$

i.e. $\log(x) = 1$. It follows that $x = 10^1 = 10$.

Example 5.2 Solve $\ln(x) + \ln(4) = 2$.

$$\begin{aligned} \ln(x) + \ln(4) = 2 &\Rightarrow \ln(4x) = 2 \\ &\Rightarrow 4x = e^2 \\ &\Rightarrow x = \frac{1}{4}e^2. \end{aligned}$$

Example 5.3 Solve $3^{2x} - 3^{x+1} - 4 = 0$.
The trick to solving this type of equation is to spot that it is a quadratic equation for the unknown term 3^x. To see this we apply the Index Rules to the left-hand side of the equation:

$$3^{2x} - 3^{x+1} - 4 = 0 \Leftrightarrow (3^x)^2 - 3 \times 3^x - 4 = 0.$$

If we let $y = 3^x$ then we obtain

$$y^2 - 3y - 4 = 0.$$

Solving for y gives $y = -1$ or $y = 4$. It follows that

$$3^x = -1 \quad \text{or} \quad 3^x = 4.$$

The first case has no solutions since $3^x > 0$ for all real x. It follows that

$$\begin{aligned} 3^x = 4 &\Rightarrow \log(3^x) = \log(4) \\ &\Rightarrow x \log 3 = \log 4 \\ &\Rightarrow x = \frac{\log 3}{\log 4}. \end{aligned}$$

Exercise 5.1 Solve the following equations for x.

1) $\log(3y) = 2$	2) $10^x = 5$
3) $10^{3x} = 7$	4) $e^{2x} = 16$
5) $2\log x = \log 25$	6) $3\log x - 2\log x = 3\log 2$
7) $2\log x + 3\log x - 2\log(4x) = \log 4$	8) $4\log\sqrt{x} - \log(3x) = \log(x^{-2})$
9) $2\log x - \log(x+1) = \log(x+2)$	10) $\log(x+2) + \log(x-2) = \log 5 + \log x$
11) $3^{2x} + 3^x - 2 = 0$	12) $4^{2x} - 5 \times 4^x + 4 = 0$
13) $5^{2x} - 7 \times 5^x + 12 = 0$	14) $3^{2x} - 6 \times 3^x + 8 = 0$
15) $5^{2x} - 2 \times 5^x - 8 = 0$	16) $7^{2x} + 7^{x+1} + 10 = 0.$

6 Answers

Exercise 1.1

1) $x = -7$	2) $x = -\frac{1}{6}$	3) $y = 12$
4) $y = \frac{16}{3}$	5) $z = \frac{1}{2}$	6) $z = -\frac{4}{5}$

Exercise 2.1

1) $x = 2,\ 3$	2) $x = \pm\frac{5}{2}$	3) $x = 0,\ -\frac{5}{2}$
4) $x = 1,\ 2$	5) $x = 5,\ \frac{1}{2}$	6) $x = 4,\ -11$
7) $x = 2,\ -4$	8) $x = \frac{1}{3},\ -\frac{5}{2}$	9) $x = \frac{7}{6},\ \frac{7}{3}$

Exercise 2.2

1) $x = \frac{1}{2}(3 \pm \sqrt{33})$	2) $x = \frac{1}{4}(4 \pm \sqrt{216})$	3) $x = \frac{1}{4}(-5 \pm \sqrt{33})$
4) $x = \frac{1}{2}(3 \pm \sqrt{17})$	5) $x = \frac{1}{6}(-11 \pm \sqrt{109})$	6) no real solutions.

Exercise 2.3

1) $x = -3,\ 7$	2) $x = -1.5,\ 3$	3) $x = -3,\ 4$
4) $x = -0.464,\ 6.464$	5) $x = -3.065,\ 0.065$	6) $x = 3$
7) $x = -3,\ 0$	8) $x = -5,\ 8$	9) no real solutions
10) $x = -2.352,\ 3.685$	11) $x = -2.110,\ 7.110$	12) $x = 0.293,\ 1.707$
13) $x = 1.490,\ x = 25.510$	14) no real solutions	15) $x = -1.847,\ x = 0.180$

Exercise 3.1

1*i*) $x = -2,\ -1,\ 1$	*ii*) $x = -1,\ 2,\ 3$	*iii*) $x = -2,\ 2,\ 3$
iv) $x = 2$	*v*) $x = -2,\ 1,\ 2$	*vi*) $x = -4,\ -2,\ 1,\ 3$
vii) $x = -3,\ 2,\ 4$	*viii*) $x = -5,\ -\frac{1}{2},\ 3$	2) $p(x) = (x+2)^2(x-1)$
3) $p = -7$	4) $a = 3,\ b = -4$	5) $a = -13,\ b = 6$

Exercise 4.1

1) $x = 2,\ y = 3$

2) $x = 3,\ y = -3$

3) $x = 3,\ y = -4$

4) $a = \dfrac{23}{19},\ b = -\dfrac{63}{19}$

5) $p = 2,\ q = \dfrac{3}{2}$

6) $x = 5,\ y = 3$

7) $r = 1,\ s = 2$

8) $d = 3,\ e = 4$

9) $x = 3, y = -3$

10) $(x, y) = (0, 2),\ (0, -2),\ (\sqrt{3}, 1),\ (\sqrt{3}, 1)$

11) $(x, y) = (2, 4),\ (2, -4),\ (-\sqrt{11}, 3),\ (\sqrt{11}, 3)$

12) $(x, y) = (\sqrt{8}, 2),\ (-\sqrt{8}, 2)$

13 $(x, y) = (1, 0),\ (-1, 0)$

14) $(x, y) = (1, 2),\ (1, -2),\ (2, 1),\ (-2, 1)$

15) $(x, y) = (1, 1),\ (-1, 1)$

Exercise 5.1

1) $y = \dfrac{100}{3}$

2) $x = \log 15$

3) $x = \dfrac{1}{3}\log 7$

4) $x = \ln 4$

5) $x = 5$

6) $x = 8$

7) $x = 4$

8) $x = \sqrt[3]{4}$

9) $x = -2$

10) $x = -1,\ 4$

11) $x = 0$

12) $x = 0,\ 1$

13) $x = \dfrac{\log 4}{\log 5},\ \dfrac{\log 3}{\log 5}$

14) $x = \dfrac{\log 4}{\log 3},\ \dfrac{\log 2}{\log 3}$

15) $x = \dfrac{\log 4}{\log 5}$

16) No solutions