Contents

1 Introdution

Let F be a finite extension of p-adic number field $\mathbb{Q}_p$ and K a quadratic extension of F. Let $z \mapsto \bar{z}$ be the nontrivial F-automorphism in K. Define $K^1 = \{z \in K^\times | z\bar{z} = 1\}$. Fix an element $\kappa \in K$ that satisfies $\bar{\kappa} = -\kappa$. Define a symplectic form $\langle\ ,\ \rangle$ on K by formula $\langle x, y\rangle = Tr_{K/F}(\kappa\bar{x}y)$. Because now K is a symplectic space over F, we can consider the Heisenberg group H associated to K, which is the set $K \times F$ with group law $(x, s)(y, t) = (x+y, s+t+\frac{1}{2}\langle x, y\rangle)$ for $x, y \in K$, $t \in F$. The symplectic group $Sp(K)$ could act on the Heisenberg group H through its action on K, i.e. $g \bullet (x, t) = (gx, t)$ for $g \in Sp(K)$, $(x, t) \in H$. Stone-Von Neumann theorem asserts that, if we fix a smooth irreducible central representation (ρ, H, V), then for each $g \in Sp(K)$, there exits a $M \in GL(V)$ such that

$$M\rho(h) = \rho(g \bullet h)M \qquad \forall h \in H \tag{1}$$

Moreover for a fixed g, the choice of M is unique upto a scalar.

Denote by $\widetilde{Sp}(K)$ the subgroup of $Sp(K) \times GL(V)$ formed by pairs of (g, M) that satisfy (1). Projecting $\widetilde{Sp}(K)$ into the second coordinate gives us a representation $\widetilde{Sp}(K) \to GL(V)$. Projecting $\widetilde{Sp}(K)$ into the first coordinate gives us the exact sequence below.

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \widetilde{Sp}(K) \longrightarrow Sp(K) \longrightarrow 1$$

In [2], Murase and Sugano gives a splitting map from $K^\times$ to $\widetilde{Sp}(K)$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \widetilde{Sp}(K) & \longrightarrow & Sp(K) & \longrightarrow & 1 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & K^\times & & \end{array}$$

In above diagram, the map from $K^\times$ to $Sp(K)$ is a composition of two maps as follows

$$K^\times \longrightarrow K^1 \hookrightarrow Sp(K)$$

where the first map is defined by $z \mapsto \bar{z}/z$, and the sencond map is defined by $u \mapsto$multiplying by u. If we compose the splitting map $K^\times \to \widetilde{Sp}(K)$ with $\widetilde{Sp}(K) \to GL(V)$, we got a representation $\mathcal{M} : K^\times \to GL(V)$, which we denote as $(\mathcal{M}, K^\times, V)$. In [2], $\mathcal{M}$ is decomposed into a direct sum of eigenspaces, and $\mathcal{M}$ is shown to be multiplicity free which means each eigenspace has dimension one. Further, [2] gives a critirion for which character of $K^\times$ appears as an eigencharacter.

In a different but equivalent setting, under the assumption 2 is a unit in F, [6] provides another splitting, and gives a different criterion of which character appears as an eigencharacter. When a character appears, [6] writes down an explicit formula for the eigenvector. The goal of our paper is to write down explicit formulas of eigenvectors without the assumption 2 is a unit in F.

Let us go into some details. Denote the integer ring of F,K by O_F, O_K individually and their unit groups by $O_F^\times$ and $O_K^\times$. Fix a nontrivial additive character ψ of F. If L is a lattice in K, we define $L_* = \{x \in K|\psi(\langle x, y\rangle) = 1\ \forall y \in L\}$. Fix a lattice $\mathcal{L}$ in K that satisfies $\mathcal{L} = \mathcal{L}_*$ and $\frac{1}{2}(l+\bar{l}) \in \mathcal{L}$ for $l \in \mathcal{L}$. We further assume $\mathcal{L}$ is an ideal in K. Denote by $\mathcal{S}(K)$ the space of locally constant compactly supported functions on K. The space V mentioned above is realized as follows

$$V = \{\Phi \in \mathcal{S}(K)|\Phi(x+l) = \psi(\frac{1}{2}\langle x, l\rangle + \frac{1}{4}\langle l, \bar{l}\rangle)\Phi(x) \ \ \forall x \in K,\ \forall l \in \mathcal{L}\}$$

It can be seen that V has a basis $\{\Phi_{x,\mathcal{L}}|x \in K/\mathcal{L}\}$, where $\Phi_{x,\mathcal{L}}$ is the unique function in V that is supported on $x + \mathcal{L}$ and has value 1 at x.

Let $\mathfrak{l}$ be the ideal in K defined in section 8.4 and r the element in K defined in 5.4. For now we just need to know $\mathfrak{l}$ is some ideal and r is some element in K. Define $V(\mathfrak{l}) = \{\Phi \in V|\Phi$ is supported on $r + \mathfrak{l}_*\}$. If $\mathfrak{a}$ is an ideal of K contained in $\mathfrak{l}$ and $\mathfrak{a}_* = \alpha O_K$ for some $\alpha \in K$, define $V_{\text{prim}}(\mathfrak{a}) = \{\Phi \in V|\Phi$ is supported on $r + \alpha O_K^\times\}$. Let's skip the detailed reailization of $\mathcal{M}$ for now, but the representation $(\mathcal{M}, K^\times, V)$ can be decomposed into a direct sum of subrepresentations $V = V(\mathfrak{l}) \oplus (\oplus_{\mathfrak{a}\subsetneq\mathfrak{l}} V_{\text{prim}}(\mathfrak{a}))$, where $\mathfrak{a}$ runs throught all ideals of K strictly contained in $\mathfrak{l}$.

Let $U_\mathfrak{a}$, $G_\mathfrak{a}$ be the subgroup of $K^\times$ defined in section 6.2 and lemma 23 individually. Now the main result of this paper is the following.

1) The set $\{\Phi_{s,\mathcal{L}}|s \in r + \mathfrak{l}_*/\mathcal{L}\}$ is a basis of $V(\mathfrak{l})$ consisting of eigenfunctions.

2) A character χ appears in $V_{\text{prim}}(\mathfrak{a})$ if and only if $\chi|_{F^\times} = \omega$, $\chi(U_\mathfrak{a}) = 1$ and $\chi|_{G_\mathfrak{a}} = \chi_{r,\mathcal{L}} C_x$ for some $x \in \alpha O_K^\times$, where $\chi_{r,\mathcal{L}}$ is the eigencharacter associated to $\Phi_{r,\mathcal{L}}$ and

$$C_x(z) = \psi(\frac{1}{2}\langle ux, x\rangle + \frac{1}{2}\langle r, (1-u)x\rangle + \frac{1}{4}\langle (1-u)x, (1-\bar{u})\bar{x}\rangle)$$

with $u = \bar{z}/z$.

If χ appears, the eigenfunction associated to χ is given by

$$\sum_{z \in K^\times/G_\mathfrak{a}} \chi(z^{-1})\chi_{r,\mathcal{L}}(z)\psi(\frac{1}{2}\langle ux, r\rangle)\Phi_{r+ux,\mathcal{L}}$$

Remark. In this introduction, for the purpose of both being simple and presenting the best result of the paper, we have made one assumption which does not always hold, that is $\mathcal{L}$ is an ideal in K. Our paper presents how one can write down eigenfunctions without assuming $\mathcal{L}$ is an ideal in K, but only when $\mathcal{L}$ is an ideal in K, our solution is completely explicit.

2 Background

Denote by p a prime number. We are mostly interested in the case where $p = 2$. Let F be a finite extension of $\mathbb{Q}_p$. Let K be a finite dimensional F-vector space equipped with a symplectic form $\langle \ , \ \rangle$. Hence K will also be mentioned by the name symplectic space. Recall that a **symplectic form** $\langle \ , \ \rangle$ on K is a non-degenerate bilinear form on K such that $\langle a, b\rangle = -\langle b, a\rangle$ for any vectors a, b in K. Start from section 3, K will actually be a qudratic extension of F.

Let ψ be a non-trivial continous group homomorphism from F to $\mathbb{C}^1$, the multiplicative group of complex numbers with absolute value 1.

2.1 Definition of Heisenberg group

The **Heisenberg group** associated with the symplectic vector space K is the set $K \times F$, equipped with the product topology and the group law

$$(x, s)(y, t) = (x + y, s + t + \frac{1}{2}\langle x, y\rangle) \qquad x, y \in K \quad s, t \in F$$

We write H for the Heisenberg group we just defined. Two facts come in handy. First $t \mapsto (0, t)$ is a monomorphism from F to H and its image is the center of H; Second multiplication in H commutes in the following fashion $(x, s)(y, t) = (0, \langle x, y\rangle)(y, t)(x, s)$.

2.2 Schur's Lemma

Let G be a Hausdorff, locally compact toplogical group. Suppose G is a countable union of compact subsets. Suppose the unit element e has a neighborhood basis consisting of open compact sets.

A representation of G in a complex vector space E is a group homomorphism $\pi : G \to GL(E)$, where $GL(E)$ is the set of linear automorphism of E. We say that π is **smooth,** if for any $\xi \in E$, the set $\{g \in G | \pi(g)\xi = \xi\}$ is open in G. We remark that E here is treated without any topology.

(Schur's lemma) Let π be a smooth representation of G in a complex vector space E. If a linear transformation $T : E \to E$ satisfies $T\pi(g) = \pi(g)T$ for $\forall g \in G$, then T is a scalar.

For a proof of the lemma, see page 18 in [1].

2.3 Stone-Von Neumann Theorem

Theorem (Stone, Von Neumann) Up to isomorphism, there is a unique smooth, irreducible representation π of H in a complex vector space E such that

$$\pi(0, t)\xi = \psi(t)\xi \quad \text{for all } t \in F, \ \ \xi \in E$$

For a proof of the theorem, see page 28 in [3].

2.4 Projective Representation

A **projective representation** of a group G in a complex vector space E is a map $\rho : G \to GL(E)$ such that

$$\rho(x)\rho(y) = c(x,y)\rho(xy) \quad \forall x, y \in G$$

for some function $c : G \times G \to \mathbb{C}^\times$. A projective representation is not a representation in general. However if there is a function $\gamma : G \to \mathbb{C}^\times$ such that

$$c(x,y) = \gamma(xy)\gamma(x)^{-1}\gamma(y)^{-1} \quad \forall x, y \in G$$

then we see the map $\gamma\rho$ is representation. In this section we will see a projective representation of symplectic group.

The **symplectic group** of K is the group of F-linear automorphism in K that preserves the symplectic form $\langle\ ,\ \rangle$. The symplectic group of K is denoted by $Sp(K)$. Let $G = Sp(K)$. We see G acts on H throught its action on K. We use a big dot to denote this action

$$g \bullet (x,t) = (gx,t)$$

where $g \in G$ and $(x,t) \in H$.

Let ρ be a representation of H in a complex vector space E that satisfies all the conditions in Stone-Von Neumann theorem. Let $g \in G$, then Stone-Von Neumann theorem asserts that there exists a linear isomorphism $M(g) : E \to E$ that satisfies

$$M(g)\rho(h) = \rho(g \bullet h)M(g) \quad \forall h \in H \tag{2}$$

Moreover Schur's lemma asserts that if another linear isomorphism $T : E \to E$ satisfies $T\rho(h) = \rho(g \bullet h)T$ for all $h \in H$, then T is a scalar times $M(g)$. So $M(g)$ is unique upto to scalar in $\mathbb{C}^\times$.

Now we make a choice of $M(g)$ for all $g \in G$. Then we have a map $M : G \to GL(E)$. Let g_1, g_2 be any two elements in G. We see both $M(g_1g_2)$ and $M(g_1)M(g_2)$ can substitiute for T in the equation $T\rho(h) = \rho(g_1g_2 \bullet h)T$. This means $M(g_1g_2)$ and $M(g_1)M(g_2)$ differ by a scalar in $\mathbb{C}^\times$. This means M is a projective representation of G.

Notice if G is a group that has a homorphism image in $Sp(K)$, the above argument is still valid.

2.5 Integration

Let $S(W)$ be the space of locally constant compactly supported functions on W. A **Haar measure** on W is a positive invariant linear function $\int :\ S(W) \to \mathbb{C}$. Being positive means that if $f \in S(W)$ and $f > 0$ then $\int f > 0$. Being invariant means that

$$\int f = \int f_x \quad \forall x \in W,\ f \in S(W)$$

where f_x is defined by $f_x(y) = f(x+y)$ for any $y \in W$. Haar measure is unique up to a positive scalar.

The exisitence of Haar measure and its uniqueness holds for any locally compact hausdorff topological group. For the groups considered in this paper, a short proof for exisitence and uniqueness can be found in page 10 in [1].

Let E be $\mathbb{C}$-vector space. A Haar measure $\int : S(W) \to \mathbb{C}$ can be extended to $\int : S(W) \otimes E \to E$, where $S(W) \otimes E$ should be identified with the space of locally constant compactly supported E-value functions on W. We refer readers to page 8 in [1]. If N is an open compact subset of W and g: $W \to E$ is a locally constant funtion, then $g1_N \in S(W) \otimes E$, where 1_N is the function that has value 1 on N and 0 elsewhere. We define $\int_N g = \int g1_N$.

If g: $W \to E$ is a locally constant funtion that satisfies

$$\int_{x+N} g = 0 \quad \forall x \in K \text{ and } x \notin N$$

for some open compact subgroup N of W, we define

$$\int_K g = \int_N g$$

Let W be a finite dimentional F-vector space equipped with a nondenegerate bilinear form $[\ ,\ ]$. Let L be a compact open subgroup of W. Then

$$\int_L \psi([x,y])dx = \begin{cases} \int_L dx & x \in L_* \\ 0 & \text{otherwise} \end{cases}$$

where

$$L_* = \{y \in W | \psi([l,y]) = 1 \ \ \forall l \in L\}$$

is also a compact open subgroup of W. The lower star notation is taken from page 38 in [5].

2.6 Model

Let $G = Sp(K)$. Let ρ be a representation of H in a complex vector space E as in Stone-Von Neumann theorem. We saw in section 2.4 that there is a projective representation $M : G \to GL(E)$ associated with ρ. In this section we will see a general process of realizing M as integrals. For any $g \in G$, we will see an explicit construction of $M(g)$ such that (2) holds.

Let $g \in G$, $\Phi \in E$. Denote by W the F-vector space $K/\ker(1-g)$. Define

$$M(g)\Phi = \int_W \psi(\frac{1}{2}\langle w, gw\rangle)\rho((1-g)w, 0)\Phi \ \mathrm{d}w$$

We next check that this integral is well defined. The integrand is a well defined E-valuded function on W. Next let $x \in W$, and L an open compact subgroup of W sufficientely small that $\psi(\frac{1}{2}\langle l, gl\rangle) = 1$ and $\rho((1-g)l, 0)\Phi = \Phi$

for all $l \in L$. To shorten notation, denote $R(w) = \psi(\frac{1}{2}\langle w, gw\rangle)\rho((1-g)w, 0)$. It can be checked that

$$\int_{x+L} R(w)\Phi \, \mathrm{d}w = \Phi' \int_L \psi(\langle l, (1-g^{-1})x\rangle) \, \mathrm{d}l$$

where $\Phi' = \psi(\frac{1}{2}\langle x, gx\rangle)\rho((1-g)x, 0)\Phi$. The bilinear form $[x, y] := \langle x, (1-g^{-1})y\rangle$ on W is nondegenerate. If we define L_* with respect to $[\ ,\ ]$ as in section 2.5, we have $\int_{x+L} R(w)\Phi \, \mathrm{d}w = 0$ for $x \notin L_*$. Now if we choose L that also satifies $L \subset L_*$, we have $\int_{x+L_*} R(w)\Phi \, \mathrm{d}w = 0$ if $x \notin L_*$. So the integal for $M(g)\Phi$ is defined in the sense in section 2.5.

To check (2) holds, we take $\Phi \in E$ and $(x, t) \in H$ and write out

$$\rho(g \bullet (x, t))M(g)\Phi = \int_W \psi(\frac{1}{2}\langle w, gw\rangle)\rho(gx, t)\rho((1-g)w, 0)\Phi \, \mathrm{d}w$$

One just need to make a change of variable $w \mapsto w + x$ to see the right side is also $M(g)\rho((x, t))\Phi$.

Above computation is taken from page 37-38 in [3]. For a proof that $M(g) \in GL(E)$ we refer the reader to page 39 in [3] or page 278 in [2].

2.7 Induced representation

Let h be a closed subgroup of H, and let $\rho : h \to \mathbb{C}^\times$ be a smooth representation of h in $\mathbb{C}$. Notice that that requiring the representation ρ to be smooth is the same as requiring ρ to be continous. Let $\mathrm{Ind}_h^H \mathbb{C}$ be the space of functions $f : H \to \mathbb{C}$ satisfying the following two conditions:

1) $f(ab) = \rho(a)f(b)$ for all $a \in h,\ b \in H$.

2) There is an open subgroup $h_f \subset H$ such that $f(ba) = f(b)$ for all $a \in h_f,\ b \in H$.

A smooth representation π of H in $\mathrm{Ind}_h^H \mathbb{C}$ can be defined by formula $(\pi(a)f)(b) = f(ba)$. The representation π is said to be induced by ρ. We denote this representation often by $\mathrm{Ind}_h^H \rho$. See page 21 in [1].

If f is a function on H and $(x, t) \in H$, in this paper we write $f(x, t)$ instead of $f((x, t))$.

2.8 Models

The representation in Stone-Von Neumann theorem can be realized as an induced representation.

Let A be a closed subgroup of K. We define A_*, the subgroup associated to A by duality, as follows.

$$A_* = \{z \in K | \psi(\langle a, z\rangle) = 1 \text{ for all } a \in A\}$$

Then A_* is a closed subgroup of K and $A_{**} = A$. If A_1, A_2 are two closed subgroup of K, we have $(A_1 + A_2)_* = (A_1)_* \cap (A_2)_*$. When $(A_1)_* + (A_2)_*$ is closed, we have $(A_1 \cap A_2)_* = (A_1)_* + (A_2)_*$. For example when A_1 is open, hence $(A_1)_*$ is compact, then $(A_1)_* + (A_2)_*$ is closed.

Let A be a closed subgroup of K such that $A = A_*$. Suppose $\widetilde{\psi} : A \times F \to \mathbb{C}^1$ is a continous homomorhism such that $\widetilde{\psi}(0,t) = \psi(t)$ for any $t \in F$. It is a fact that $\mathrm{Ind}_{A\times F}^{H}\widetilde{\psi}$ is a realization of the representation of H that satisfies the condition in Stone-Von Neumann theorem. For a proof of this result, see page 28-29 in [3].

2.9 Lattice model

Let $S(K)$ be the space of locally constant compactly supported functions on K. If A is compact, we claim that $\mathrm{Ind}_{A\times F}^{H}\widetilde{\psi}$ can be identified with the representation ρ of H in the space

$$\{\Phi \in S(K) | \Phi(x+a) = \widetilde{\psi}(a, \frac{1}{2}\langle x,a\rangle)\Phi(x) \quad \text{for all } x \in K,\ a \in A\}$$

with the action

$$(\rho(y,t)\Phi)(x) = \psi(\frac{1}{2}\langle x,y\rangle + t)\Phi(x+y) \qquad \text{for all } x \in K,\quad (y,t) \in H \tag{3}$$

To see this, let $f \in \mathrm{Ind}_{A\times F}^{H}\mathbb{C}$. Notice $f(x,t) = \psi(t)f(x,0)$ for all $x \in K,\ t \in F$. If we define a function $\Phi : K \to \mathbb{C}$ by setting $\Phi(x) = f(x,0)$, we see f is completely decided by Φ. The identification is done by passing the properties of f and the action of ρ on f to those of Φ. It is not hard to see that Φ is locally constant, satisfies the expected transformation property and (3) holds. To see Φ is has a compact support we start with the following identity.

$$f((x,0)(a,0)) = \psi(\langle x,a\rangle)f((a,0)(x,0)) \qquad \text{for all } x \in K,\quad a \in A$$

When a is small, we have $f((x,0)(a,0)) = \Phi(x) = f((a,0)(x,0))$. So for a small enough open subgroup N of K, we have

$$\Phi(x) = \psi(\langle x,a\rangle)\varphi(x) \qquad \text{for all } x \in K,\quad a \in N \cap A$$

If $\psi([x,a]) \neq 1$ for some $a \in N \cap A$, then $\Phi(x) = 0$. In another word if $x \notin (N \cap A)_* = N_* + A$, then $\varphi(x) = 0$. Because N_* and A are compact, Φ is compactly supported.

An F-**lattice** in K is an open compact subgroup of K which is closed under multiplication by elements in O_F, the integer ring of F. Let L be an F-lattice in K, we call the identification of $\mathrm{Ind}_{L\times F}^{H}\widetilde{\psi}$ as above the **lattice model** of the representation in Stone-Von Neumann theorem.

2.10 character of second degree

If G_1 and G_2 are two abelian groups, a **bicharacter** of $G_1 \times G_2$ is a function $f : G_1 \times G_2 \to \mathbb{C}^1$ that satisfies

$$f(a+b,c) = f(a,c)f(b,c) \qquad f(a,b+c) = f(a,b)f(a,c)$$

A continous funtion $f : K \to \mathbb{C}^1$ is called a **character of second degree** if the function $(a,b) \mapsto f(a+b)f(a)^{-1}f(b)^{-1}$ is a bicharacter of $K \times K$.

Suppose f is character of second degree of K. There exists a unique F-linear transformation T on K such that we have for all a,$b \in K$

$$f(a+b)f(a)^{-1}f(b)^{-1} = \psi(\langle a, T(b)\rangle)$$

We say f and T are associated with each other.

2.11 Absolute value

Let T be a F-linear automorphism in K. The **absolute value** of T, denoted as $|T|$, is defined by the following formula

$$\int_K \Phi(x)dx = |T| \int_K \Phi(T(x)) \, \mathrm{d}x \quad \forall \Phi \in S(K)$$

If K is a finite extention of F. The absolute value of the automorphism in K given by $y \mapsto xy$ for some $x \in K^\times$, is denoted as $|x|_K$ and called the absolute value of x. Notice we have defined $|\ |_F$ if we take $K = F$. It is a fact that if K is a finite extension of F, the following holds

$$|x|_K = |N_{K/F}x|_F \quad \forall x \in K$$

For reference of this section, see page 3 and 139 in [5].

2.12 Fourier transformation

Fourier transfomation of a function $\Phi \in S(K)$ is defined by

$$\hat{\Phi}(y) = \int_K \Phi(x)\psi(\langle x, y\rangle) \, \mathrm{d}x$$

In this paper we choose the Haar measure such that$\hat{\hat{\Phi}}(x) = \Phi(-x)$.

Let L be an F-lattice in K. If we denote $|L| = \int_L \, \mathrm{d}x$, a consequence of our choice is that

$$|L||L_*| = 1$$

This is because $\hat{1}_L = |L|1_{L_*}$.

2.13 Weil constant

If $\Phi \in S(K)$ and f is a continous function on K then the convolution of Φ with f is given by

$$\Phi * f(x) = \int_K \Phi(y) f(x-y) \, \mathrm{d}y$$

Notice if f is a character of second degree on K, the Fourier transformation of f is not well defined.

(Theorem Weil) Let f be a character of second degree of K, associated to the nondegenerate F-linear transformation T on K. There exists a $\lambda(f) \in \mathbb{C}^1$ such that

$$\widehat{\Phi * f} = \lambda(f) \hat{\Phi} \hat{f} \quad \forall \Phi \in S(K) \tag{4}$$

where $\hat{}$ is the Fourier transformation operator except $\hat{f}$ is defined by $\hat{f}(x) = |T|^{-\frac{1}{2}} f(T^{-1}(x))^{-1}$. $\square$

If we take Fourier transformation on both sides of (4) and evaluate at 0, we get

(Corollary Weil) Hypothesis being the same as in the Weil Theorem, we have

$$\int_K \Phi(x) f(-x) dx = \lambda(f) \int_K \hat{\Phi}(x) \hat{f}(x) dx \quad \forall \Phi \in S(K)$$

For the original statement of above theorem, see page 161 in [4]

2.14 Eigenspace decompostion

When we say (π, G, E) is a representation, we mean π is a representation of a group G in a complex vector space E. We also say E is an G-module.

(Lemma-Definition) Let E be an G-module. Then the following are equivalent:

a) E is the direct sum of irreducible submodules.

b) E is generated by its irreducible submodules.

c) Each submodule $E' \subset E$ has a complement, a submodule E'' such that $E = E' \oplus E''$.

If E satisfied these conditions, we say E is **completely reducible**. The submodule of a completely reducible module is also completely reducible.

(Lemma) Let (π, G, E) be a representation, and N a normal subgroup of G of finite index. If N acts on E by scalar multiplication then π is completely reducible.

proof. Let $g_1, \cdots, g_n$ be coset representatives of G/N, where $n = [G : N]$. If E_1 is a G-submodule of E, then there is a subvector space E_2 such that $E = E_1 \oplus E_2$ as vector spaces. Let P_0 be the projection of E onto E_1 along E_2. If we set $P = \frac{1}{n} \sum_{i=1}^{n} \pi(g_i) P_0 \pi(g_i^{-1})$, then P does not depend on the choice of g_is, and P is G-module homomorphism. Therefore, its kernel is a complement of E_1. $\square$

These two lemmas are taken from pages 16-17 in [1].

(Lemma) Let (π, G, E) be a smooth irreducible representation, where G is an abelian group that satisfies the conditions in Schur's lemma in section 2.2. Then E is one dimensional.

Proof. If $g \in G$, then $\pi(g)$ is G-module automorphism on E. Schur's lemma implies G acts on E by scalar multiplication. However E is irreducible so E is one dimensional. □

(Theorem) Let (π, G, E) be a smooth representation, where G is an abelian group that satisfies the conditions in Schur's lemma in section 2.2. Let N be a subgroup of G of finite index. If N acts on E by scalar multiplication then π is the direct sum of eigenspaces, one-dimensional subrepresentations.

3 Representation $(\mathcal{M}, K^{\times}, V)$

This section introduces the setup in [2], which is also the beginning point of our paper. From now on, F is a finite extension of $\mathbb{Q}_p$, and K is a quadratic extension of F. We fix a non-trivial continous group homomorphism $\psi : F \to \mathbb{C}^1$. Let κ be an element in $K^{\times}$ with $\bar{\kappa} = -\kappa$, where $x \mapsto \bar{x}$ is the unique nontrivial field automorphism of K/F. One defines a symplectic form $\langle\ ,\ \rangle$ on K by setting

$$\langle x, y\rangle = Tr_{K/F}(\kappa \bar{x} y) \qquad x, y \in K \tag{5}$$

It is immediate that $\langle x, y\rangle = \langle \bar{y}, \bar{x}\rangle$ and $\langle zx, y\rangle = \langle x, \bar{z}y\rangle$ for $x, y, z \in K$.

Let H be the Heisenberg group associated with the symplectic space $(K, \langle\ ,\ \rangle)$. Let $\mathcal{L}$ be an F-lattice in K that satisfies the following two conditions

$$\text{(i) } \mathcal{L} = \mathcal{L}_* \qquad\qquad \text{(ii) } l \in \mathcal{L} \Rightarrow \frac{1}{2}(l + \bar{l}) \in \mathcal{L}$$

Later we will construct lattice $\mathcal{L}$ explicitly.

Remark. In [2], the authors fix κ from the beginning. However our main result in this paper depends in a crucial way on the choice of κ when 2 is not a unit in F.

3.1 Representation (ρ, H, V)

We define a continous group homomorphism $\widetilde{\psi} : \mathcal{L} \times F \to \mathbb{C}^1$ that satisfies $\widetilde{\psi}(0, t) = \psi(t)$ for all $t \in F$, by setting

$$\widetilde{\psi}(l, t) = \psi(\frac{1}{4}\langle l, \bar{l}\rangle + t) \quad \text{ for all } l \in \mathcal{L},\ t \in F$$

We remark that if 2 is a unit in F , then $\psi(\frac{1}{4}\langle l, \bar{l}\rangle) = 1$, hence $\widetilde{\psi}(l, t) = \psi(t)$.

It is a fact that, upto isomorphism, $\mathrm{Ind}_{\mathcal{L}\times F}^{H}\widetilde{\psi}$ is the unique smooth irreducible representation of H describe in Stone-Von Neumann theorem. We saw, in section 2.9 that it could be identified with a representation of ρ in the complex space V, where V is given by

$$V = \{\Phi \in S(K) | \Phi(x+l) = \psi(\frac{1}{2}\langle x, l\rangle + \frac{1}{4}\langle l, \bar{l}\rangle)\Phi(x) \quad \text{for all } x \in K,\ l \in \mathcal{L}\}$$

and the ρ is given by

$$(\rho(y,t)\Phi)(x) = \psi(\frac{1}{2}\langle x, y\rangle + t)\Phi(x+y) \qquad \text{for all } x \in K, \quad (y,t) \in H$$

3.2 Projective representation $(M, K^\times, V)$

We denote $u = \bar{z}/z$ for any $z \in K^\times$. We define an action of $K^\times$ on H as $z \bullet (x,t) = (ux, t)$ for all $z \in K^\times$, $(x,t) \in H$. Define a projective representation $M : K^\times \to GL(V)$. Let $\Phi \in V$, we put

$$M(z)\Phi = \begin{cases} \Phi & z \in F^\times \\ |1-u|_K^{1/2} \int_K \psi(\frac{1}{2}\langle x, ux\rangle)\rho((1-u)x, 0)\Phi & z \in K^\times - F^\times \end{cases}$$

We saw in section 2.6 that M is a projective representation of $K^\times$ in V. So there exist a function $c : K^\times \times K^\times \to \mathbb{C}^\times$ such that

$$M(z_1)M(z_2) = c(z_1, z_2)M(z_1 z_2) \quad \forall z_1, z_2 \in K^\times \tag{6}$$

We record a property of M, which we saw in section 2.6, below for later use.

$$M(z)\rho(x,t) = \rho(ux,t)M(z) \quad \text{for all } z \in K^\times,\ (x,t) \in H \tag{7}$$

3.3 Weil Constant

Let $a \in F^\times$. We define a characher of second degree $f_a : K \to \mathbb{C}^1$ by putting

$$f_a(x) = \psi(ax\bar{x}) \quad \forall x \in K$$

The linear isomorphism T_a associated with f_a is given by

$$T_a(x) = ax/\kappa \quad \forall x \in K$$

Then we saw in section 2.13 there is a constant $\lambda(a) \in \mathbb{C}^1$ such that

$$\int_K \Phi(x)\psi(ax\bar{x})\ \mathrm{d}x = \lambda(a)|\kappa/a|_K^{1/2} \int_K \widehat{\Phi}(x)\psi(\kappa^2 x\bar{x}/a)\ \mathrm{d}x \tag{8}$$

where the Haar measure is chosen as in section 2.13. In [2], $\lambda(a)$ is asserted to have the form

$$\lambda(a) = \lambda_K(\psi)\omega(a) \quad \forall a \in F^\times$$

where $\lambda_K(\psi) = \lambda(1)$, and ω is the group homomorphism on $F^\times$ defined by

$$\omega(a) = \begin{cases} 1 & a \in N_{K/F}K^\times \\ -1 & \text{otherwise} \end{cases}$$

3.4 Murase-Sugano Splitting

One defines a function $\gamma : K^\times \to \mathbb{C}^\times$ as follows

$$\gamma(z) = \begin{cases} \omega(z) & z \in F^\times \\ \omega(\frac{z-\bar{z}}{\kappa})/\lambda_K(\psi) & \text{otherwise} \end{cases}$$

and a map $\mathcal{M} : K^\times \to GL(V)$ as follows

$$\mathcal{M}(z) = \gamma(z)M(z)$$

It is shown in [2] that $c(z_1, z_2)$ as in equation (6) satisfies $c(z_1, z_2) = \gamma(z_1 z_2)\gamma(z_1)^{-1}\gamma(z_2)^{-1}$ for any $z_1, z_2 \in K^\times$.

(Theorem Murase-Sugano) The map $\mathcal{M}$ is a smooth representation of $K^\times$ in the complex vector space V.

4 Subgroups of $K^\times$

Let O_F, O_K be the ring of integers of F, K, and $O_F^\times$, $O_K^\times$ be the group of units of O_F, O_K. Let π be a prime element in F and Π a prime element in K. Define $K^1 = \{u \in K | u\bar{u} = 1\}$. Any $u \in K^1$ can be written as $\bar{z}/z$ for some $z \in K^\times$ according to Hilber 90.

4.1 Notation $\theta, \delta_{K/F}$ and U_l

We know

$$O_K = xO_F \oplus yO_F$$

for some $x, y \in O_K$. If K/F is unramified, then one must have $x, y \in O_K^\times$ and one can divide both sides by x. If K/F is ramified one has $O_K = O_F \oplus \Pi O_F$. Either way one has $O_K = O_F \oplus \theta O_F$ for some $\theta \in O_K$. We fix such a θ. If K/F is unramified we know θ is unit. If K/F is ramified we take θ that is a prime element of K .

Define

$$\delta_{K/F} = \text{ord}_F N_{K/F}(\theta - \bar{\theta})$$

It follows from the definition that if K/F is ramified then $(\bar{\theta} - \theta) \in \Pi^{\delta_{K/F}} O_K^\times$.

Some other useful facts are that if K/F is unramified, then $\delta_{K/F} = 0$ and $O_F^\times \subset N_{K/F}K^\times$. If K/F is ramified, then $\delta_{K/F} > 0$ and $\delta_{K/F}$ is the smallest integer n such that $1 + \pi^n O_F \subset N_{K/F}K^\times$.

Define $U_0 = O_K^\times$. For integer $l \geq 1$, define

$$U_l = 1 + \Pi^l O_K$$

Suppose $l \geq 1$. Notice that if K/F is unramified, then $U_l = 1 + \pi^l O_K$; if K/F is ramified, then $U_{2l} = 1 + \pi^l O_K$.

4.2 Correspondence

Proposition 1 *Suppose K/F is unramified. Let $z \in K^\times$, $u = \bar{z}/z$, and integer $l \geq 0$. Then*

$$z \in F^\times U_l \Longleftrightarrow u \in U_l$$

Proof. If $l = 0$, the proof is trivial once we realize $F^\times O_F^\times = K^\times$. Now assume $l \geq 1$.

($\Rightarrow$) Suppose $z \in U_l$, we need to show $u \in U_l$. Write $z = 1 + \pi^l x$ for some $x \in O_K$. Write $x = a + b\theta$ for some $a, b \in O_F$. Put these together we have $u - 1 = \pi^l b(\bar{\theta} - \theta)/z$. Both z and $\bar{\theta} - \theta$ are units, and $b \in O_F$, so $u - 1 \in \pi^l O_K$. Hence $u \in U_l$.

($\Leftarrow$) Suppose $u \in U_l$, we need to show $z \in F^\times U_l$. Write $z = cx$ for some $c \in F^\times$, $x \in O_K^\times$. Write $x = a + b\theta$ for some $a, b \in O_F$. Put these together we have $u - 1 = b(\bar{\theta} - \theta)/x$. However $u \in U_l$, hence $b(\bar{\theta} - \theta)/x \in \pi^l O_K$. Both $\bar{\theta} - \theta$ and x are units, so $b \in \pi^l O_K$. Now $a = x - b\theta$ is a unit since x and θ are units and $b \in \pi^l O_K$. Moreover $a \in O_F$, so $a \in O_F^\times$. Finally $z = ca^{-1}(1 + ba^{-1}\theta)$, where $ca^{-1} \in F^\times$ and $1 + ba^{-1}\theta \in U_l$. Hence $z \in F^\times U_l$. $\square$

Lemma 1 *Suppose K/F is ramfied. Let $z \in \Pi F^\times O_K^\times$ and $u = \bar{z}/z$. Then $u - 1 \in \Pi^{\delta_{K/F}-1} O_K^\times$.*

Proof. Write $z = cx$ for some $c \in F^\times$, $x \in \Pi O_K^\times$. Write $x = a + b\theta$ for some $a, b \in O_F$. Notice a and $b\theta$ have different absolute values since θ is a prime element of K. The fact x is a prime element forces that $a \in \pi O_F$, $b \in O_F^\times$. Now $u - 1 = b(\bar{\theta} - \theta)/x$ with $b \in O_F^\times$, $\bar{\theta} - \theta \in \Pi^{\delta_{K/F}} O_K^\times$ and $x \in \Pi O_K^\times$. Hence $u - 1 \in \Pi^{\delta_{K/F}-1} O_K^\times$. $\square$

Proposition 2 *Suppose K/F is ramified. Then $K^1 \subset U_{\delta_{K/F}-1}$ and for integer $m \geq 1$*

$$K^1 \cap U_{2m+\delta_{K/F}} = K^1 \cap U_{2m+\delta_{K/F}-1}$$

Proof. We first show that $K^1 \subset U_{\delta_{K/F}-1}$. If $\delta_{K/F} = 1$, this is obvious. So we assume $\delta_{K/F} > 1$. Suppose $u \in K^1$, we need to show $u \in U_{\delta_{K/F}-1}$. Write $u = \bar{z}/z$ for some $z \in K^\times$. Notice $K^\times$ is the disjoint union of $F^\times O_K^\times$ and $\Pi F^\times O_K^\times$. We make a choice of z such that either $z \in O_K^\times$ or $z \in \Pi O_K^\times$. Write $z = a + b\theta$ for some $a, b \in O_F$. Then $u - 1 = b(\bar{\theta} - \theta)/z$, where $b \in O_F$, $(\bar{\theta} - \theta) \in \Pi^{\delta_{K/F}} O_K^\times$ and $z^{-1} \in \Pi^{-1} O_K$. So $u \in U_{\delta_{K/F}-1}$.

Next we show the equality. Suppose $u \in K^1 \cap U_{2m+\delta_{K/F}-1}$, we need to show $u \in U_{2m+\delta_{K/F}}$. Write $u = \bar{z}/z$ for some $z \in O_K^\times$ or $z \in \Pi O_K^\times$. However if $z \in \Pi O_K^\times$, then $u - 1 \in \Pi^{\delta_{K/F}-1} O_K^\times$ by lemma 1. But this is impossible since we assumed $u \in U_{2m+\delta_{K/F}-1}$ and $m \geq 1$. So z is a unit. Write $z = a + b\theta$ for some $a, b \in O_F$. Then $u - 1 = b(\bar{\theta} - \theta)/z$. Hence $u \in U_{2m+\delta_{K/F}-1}$ implies $b \in \pi^m \Pi^{-1} O_K$. However the fact $b \in F^\times$ forces $b \in \pi^m O_K$. Now we look at $u - 1 = b(\bar{\theta} - \theta)/z$ again and conclude $u \in U_{2m+\delta_{K/F}}$. $\square$

Proposition 3 *Suppose K/F is ramified. Let $z \in K^\times$, $u = \bar{z}/z$, and integer $l \geq 0$. We have $F^\times U_{2l+1} = F^\times U_{2l}$ and*

$$z \in F^\times U_{2l+1} \Longleftrightarrow u \in U_{2l+\delta_{K/F}}$$

Proof. We first show $F^\times U_{2l} = F^\times U_{2l+1}$.

(Case $l = 0$) Suppose $z \in U_0$, we need to show $z \in F^\times U_1$. Write $z = a + b\theta$ for some $a, b \in O_F$. Then notice $a = z - b\theta$ is a unit since z is a unit, $b \in O_F$ and θ is a prime element. As a result, $z \in F^\times U_1$ since $z = a(1 + a^{-1}b\theta)$, where $a \in F^\times$ and $(1 + a^{-1}b\theta) \in U_1$.

(Case $l \geq 1$) Suppose $z \in U_{2l}$, we need to show $z \in F^\times U_{2l+1}$. Write $z = 1 + \pi^l x$ for some $x \in O_K$. Write $x = a + b\theta$ for some $a, b \in O_F$. Put these together we have $z = 1 + \pi^l a + \pi^l b\theta$. Notice $1 + \pi^l a$ is a unit in O_F and we denote it by c. Then $z = c(1 + c^{-1}b\pi^l\theta)$, where $c \in F^\times$ and $1 + c^{-1}b\pi^l\theta \in U_{2l+1}$. Hence $z \in U_{2l+1}$.

Next we show $z \in F^\times U_{2l} \Longleftrightarrow u \in U_{2l+\delta_{K/F}}$.

($\Rightarrow$) Suppose $z \in U_{2l}$, we need to show $u \in U_{2l+\delta_{K/F}}$. As we saw above we write $z = 1 + \pi^l a + \pi^l b\theta$ for some $a, b \in O_F$. Then $u - 1 = \pi^l b(\bar{\theta} - \theta)/z$. We see $b \in O_F$, $(\bar{\theta} - \theta) \in \Pi^{\delta_{K/F}} O_K^\times$, and z is a unit. Hence $u - 1 \in \pi^l \Pi^{\delta_{K/F}} O_K^\times$, i.e. $u \in U_{2l+\delta_{K/F}}$.

($\Leftarrow$) Suppose $u \in U_{2l+\delta_{K/F}}$, we will show $z \in F^\times U_{2l+1}$. First notice that z can not be in $\Pi F^\times O_K^\times$. If otherwise, by lemma 1, $u - 1 \in \Pi^{\delta_{K/F}-1} O_K$ which contradicts the assumption $u \in U_{2l+\delta_{K/F}}$ and $l \geq 0$. So z has to be in $F^\times O_K^\times$. Write $z = cx$ for some $c \in F^\times$, $x \in O_K^\times$. Write $x = a + b\theta$ for some $a, b \in O_F$. The situation forces $a \in O_F^\times$. Meanwhile, $u - 1 = b(\bar{\theta} - \theta)/z$ and $u \in U_{2l+\delta_{K/F}}$ implies $b \in \pi^l O_K$. Now $z = ca(1 + a^{-1}b\theta)$ with $ca \in F^\times$ and $1 + a^{-1}b\theta \in U_{2l+1}$. So $z \in F^\times U_{2l+1}$. □

5 Space $V(\mathfrak{a})$

5.1 Notations v and t_x

Define

$$v = -\frac{\bar{\theta}}{\theta - \bar{\theta}}$$

Following properties are handy for later computation

(i) $v + \bar{v} = 1$

(ii) If $\delta_{K/F} = 0$, then $v \in O_K^\times$.

(iii) If $\delta_{K/F} > 0$, then $v \in \Pi^{1-\delta_{K/F}} O_K^\times$.

Lemma 2 *Let $z \in K^\times$, $u = \bar{z}/z$. Then $v(1-u) \in O_K$. Moreover if K/F is ramified and $z \in O_K^\times$ then $v(1-u) \in \Pi O_K$.*

Proof. If K/F is unramified, then $K^\times = F^\times O_K^\times$, and we can assume $z \in O_K^\times$. If K/F is ramified $K^\times$ is the disjoint union of $F^\times O_K^\times$ and $\Pi F^\times O_K^\times$ and we can assume $z \in O_K^\times$ or $z \in \Pi O_K^\times$. We make these assumptions. Write $z = a + b\theta$ for some $a, b \in O_F$. So we have

$$v(1-u) = -b\bar{\theta}/z$$

Now if K/F is unramified, then both $\bar{\theta}$ and z are units and $b \in O_F$, so $v(1-u) \in O_K$. If K/F is ramified and $z \in \Pi O_K^\times$, then $\bar{\theta}/z$ is a unit and $b \in O_F$, so $v(1-u) \in O_K$. Similarly, if K/F ramifies and $z \in O_K^\times$, then $v(1-u) \in \Pi O_K$. □

Define

$$t_x = \frac{1}{2}\langle x, vx\rangle \quad \text{for } x \in K$$

Let $u \in K^1$, then $t_{ux} = t_x$ and the following holds

$$\frac{1}{2} < x, ux > - t_{(1-u)x} = < v(1-u)x, x > \qquad \forall x \in K \tag{9}$$

5.2 Definition of $V(\mathfrak{a})$

All through this paper an **ideal** means a nonzero fractional ideal. Let $\mathfrak{a}$ be an ideal of K. One defines

$$V(\mathfrak{a}) = \{\Phi \in V | \rho(x, t_x)\Phi = \Phi \quad \forall x \in \mathfrak{a}\}$$

One sees that V is the union of all the $V(\mathfrak{a})$s and if two ideals $\mathfrak{a} \subset \mathfrak{b}$, then $V(\mathfrak{b}) \subset V(\mathfrak{a})$.

Let $\mathfrak{a}$ be an ideal of K, and $\Phi \in V(\mathfrak{a})$. On one hand $\Phi \in V$, hence the following

$$\Phi(x + l) = \psi(\frac{1}{2}\langle x, l\rangle + \frac{1}{4}\langle l, \bar{l}\rangle)\Phi(x) \qquad x \in K, l \in \mathcal{L} \tag{10}$$

On the other hand $\rho(\alpha, t_\alpha)\Phi = \Phi$ for any $\alpha \in \mathfrak{a}$, hence the following

$$\Phi(x + \alpha) = \psi(\frac{1}{2}\langle \alpha, x\rangle - t_\alpha)\Phi(x) \qquad x \in K, \alpha \in \mathfrak{a} \tag{11}$$

Equations (10) and (11) can be put into one relation

$$\Phi(x + \alpha + l) = A(x, \alpha, l)\Phi(x) \qquad x \in K, \alpha \in \mathfrak{a}, l \in \mathcal{L} \tag{12}$$

where

$$A(x, \alpha, l) = \psi(\frac{1}{2}\langle \alpha, x + l\rangle - t_\alpha + \frac{1}{2}\langle x, l\rangle + \frac{1}{4}\langle l, \bar{l}\rangle)$$

Definition 1 *If a function on K satisfies relation (12), we say the function is **almost constant** on cosets of $\mathfrak{a} + \mathcal{L}$.*

Lemma 3 *Let $\mathfrak{a}$ be an ideal of K. Let $\Phi \in V(\mathfrak{a})$. Suppose $\Phi(x) \neq 0$ for some $x \in K$, then*

$$\psi(\langle x, w\rangle + t_w + \frac{1}{4}\langle w, \bar{w}\rangle) = 1 \qquad \forall w \in \mathfrak{a} \cap \mathcal{L} \tag{13}$$

Proof. Let $w \in \mathfrak{a} \cap \mathcal{L}$. There are two expressions for $\Phi(x+w)$, one from (10) and another one from (11). Hence

$$\psi(\frac{1}{2}\langle x, w\rangle + \frac{1}{4}\langle w, \bar{w}\rangle)\Phi(x) = \psi(\frac{1}{2}\langle w, x\rangle - t_w)\Phi(x) \qquad \forall w \in \mathfrak{a} \cap \mathcal{L}$$

Then we cancel $\Phi(x)$ on both sides and rearrange the equality.

Proposition 4 *Let $\mathfrak{a}$ be an ideal of K, and $\Phi \in V(\mathfrak{a})$. Then Φ is almost constant on cosets of $\mathfrak{a} + \mathcal{L}$. If $\Phi(x) \neq 0$ for some $x \in K$, then Φ is supported on $x + \mathfrak{a}_* + \mathcal{L}$.*

Proof. We only need to show Φ is supported on $x + \mathfrak{a}_* + \mathcal{L}$. Suppose Φ is nonzero at $x, y \in K$, then x, y both satisfy (13). As a result

$$\psi(\langle y - x, w\rangle) = 1 \qquad w \in \mathfrak{a} \cap \mathcal{L}$$

In another word, $y - x \in \mathfrak{a}_* + \mathcal{L}$. So Φ is supported on $x + \mathfrak{a}_* + \mathcal{L}$.$\square$

Above proposition gives us a sense of what $V(\mathfrak{a})$ looks like. Later we will improve this proposition.

Let $\Phi \in V(\mathfrak{a})$. We take a closer look at relation (12). First notice $\Phi(x+\alpha+l)$ equals to either side of the following

$$A(x, \alpha, l)\Phi(x) = A(x, \alpha + w, l - w)\Phi(x) \tag{14}$$

where $x \in K, \alpha \in \mathfrak{a}, l \in \mathcal{L}, w \in \mathfrak{a} \cap \mathcal{L}$. Secondly notice $\Phi(x + \alpha + \beta + l + m)$ equals to either side of the following

$$A(x, \alpha + \beta, l + m)\Phi(x) = A(x + \alpha + l, \beta, m)A(x, \alpha, l)\Phi(x) \tag{15}$$

where $x \in K$ and $\alpha, \beta \in \mathfrak{a}$ and $l, m \in \mathcal{L}$.

Lemma 4 *Let $\mathfrak{a}$ be an ideal of K, and $x \in K$ such that*

$$\psi(\langle x, w\rangle + t_w + \frac{1}{4}\langle w, \bar{w}\rangle) = 1 \qquad \forall w \in \mathfrak{a} \cap \mathcal{L}$$

Then

$$A(x, \alpha, l) = A(x, \alpha + w, l - w) \qquad \forall \alpha \in \mathfrak{a}, l \in \mathcal{L}, w \in \mathfrak{a} \cap \mathcal{L} \tag{16}$$

if and only if $v\mathfrak{a} \subset \mathfrak{a}_* + \mathcal{L}$.

Proof. Direct calculation shows

$$A(x, \alpha + w, l - w) = \psi(-\langle x, w\rangle - t_w - \frac{1}{4}\langle w, \bar{w}\rangle)\psi(\langle w, \bar{v}\alpha\rangle)A(x, \alpha, l)$$

where $\alpha \in \mathfrak{a}, l \in \mathcal{L}, w \in \mathfrak{a} \cap \mathcal{L}$.

Now we see equation (16) holds if and only if

$$\psi(\langle w, \bar{v}\alpha\rangle) = 1 \qquad \forall \alpha \in \mathfrak{a}, w \in \mathfrak{a} \cap \mathcal{L}$$

This is equivalent to $v\mathfrak{a} \subset \mathfrak{a}_* + \mathcal{L}$. $\square$

Lemma 5 *Let $x \in K$, then*

$$A(x, \alpha + \beta, l + m) = A(x + \alpha + l, \beta, m)A(x, \alpha, l) \tag{17}$$

for any $\alpha, \beta \in \mathfrak{a}$, and $l, m \in \mathcal{L}$ if and only if $v\mathfrak{a} \subset \mathfrak{a}_*$.

Proof. Direct calculation shows

$$A(x, \alpha + \beta, l + m) = \psi(\langle v\alpha, \beta\rangle)A(x + \alpha + l, \beta, m)A(x, \alpha, l)$$

for any $\alpha, \beta \in \mathfrak{a}$ and $l, m \in \mathcal{L}$. So (17) holds if and only if $\psi(\langle v\alpha, \beta\rangle) = 1$ for any $\alpha, \beta \in \mathfrak{a}$. This is equivalent to $v\mathfrak{a} \subset \mathfrak{a}_*$. $\square$

Proposition 5 *Let $\mathfrak{a}$ be an ideal of K such that $v\mathfrak{a} \subset \mathfrak{a}_*$. Let $x \in K$ that satisfies*

$$\psi(\langle x, w\rangle + t_w + \frac{1}{4}\langle w, \bar{w}\rangle) = 1 \qquad \forall w \in \mathfrak{a} \cap \mathcal{L}$$

If we define

$$\Phi_{x,\mathfrak{a}+\mathcal{L}}(y) = \begin{cases} A(x, \alpha, l) & y = x + \alpha + l \textit{ for some } \alpha \in \mathfrak{a}, l \in \mathcal{L} \\ 0 & \textit{otherwise} \end{cases} \tag{18}$$

then $\Phi_{x,\mathfrak{a}+\mathcal{L}}$ is a nonzero function in $V(\mathfrak{a})$.

Proof. According to lemma 4, equation (16) holds, so $\Phi_{x,\mathfrak{a}+\mathcal{L}}$ is a well defined function. According to lemma 5, equation (17) holds, so $\Phi_{x,\mathfrak{a}+\mathcal{L}}$ satisfies (12). Lastly, $\Phi_{x,\mathfrak{a}+\mathcal{L}}(x) = 1$. So $\Phi_{x,\mathfrak{a}+\mathcal{L}}$ is a nonzero function in $V(\mathfrak{a})$. $\square$

Lemma 6 *Let $\mathfrak{a}$ be an ideal of K. Then there exists some $x \in K$ such that*

$$\psi(\langle x, w\rangle + t_w + \frac{1}{4}\langle w, \bar{w}\rangle) = 1 \qquad \forall w \in \mathfrak{a} \cap \mathcal{L}$$

if and only if $v(\mathfrak{a} \cap \mathcal{L}) \subset \mathfrak{a}_* + \mathcal{L}$.

Proof. Define $f(w) = \psi(t_w + \frac{1}{4}\langle w, \bar{w}\rangle)$ Let $w_1, w_2 \in \mathfrak{a} \cap \mathcal{L}$ then

$$f(w_1 + w_2) = f(w_1)f(w_2)\psi(\langle vw_1, w_2\rangle)$$

Now $f(w)$ is a homomorphism on $\mathfrak{a} \cap \mathcal{L}$ if and only if $\psi(\langle vw_1, w_2\rangle = 1$ for any $w_1, w_2 \in \mathfrak{a} \cap \mathcal{L}$ i.e. $v(\mathfrak{a} \cap \mathcal{L}) \subset \mathfrak{a}_* + \mathcal{L}$. On the other hand, f is a homophism on $\mathfrak{a} \cap \mathcal{L}$ if and only if there exist exists some $x \in K$ such that $f(w) = \psi(\langle w, x\rangle)$ for any $w \in \mathfrak{a} \cap \mathcal{L}$. $\square$

5.3 Relevant ideals

Definition 2 *An ideal* $\mathfrak{a}$ *of* K *is called* ***relevant*** *if* $v\mathfrak{a} \subset \mathfrak{a}_*$.

Theorem 1 *Let* $\mathfrak{a}$ *be an ideal of* K. *Then* $V(\mathfrak{a}) \neq 0$ *if and only if* $\mathfrak{a}$ *is relevant.*

Proof. ($\Rightarrow$) Suppose $V(\mathfrak{a}) \neq 0$. Then there is a $\Phi \in V(\mathfrak{a})$ and $\Phi(x) \neq 0$ for some $x \in K$. The equation (15) holds, we cancel the $\Phi(x)$ in it to get equation (17) . As a result of lemma 5, we have $v\mathfrak{a} \subset \mathfrak{a}_*$.

($\Leftarrow$) Suppose $v\mathfrak{a} \subset \mathfrak{a}_*$. According to lemma 6, there exists an $x \in K$ such that

$$\psi(\langle x, w\rangle + t_w + \frac{1}{4}\langle w, \bar{w}\rangle) = 1 \qquad \forall w \in \mathfrak{a} \cap \mathcal{L}$$

According to proposition 5, $\Phi_{x,\mathfrak{a}+\mathcal{L}}$ is a nonzero function in $V(\mathfrak{a})$. □

5.4 Notation r, n_ψ, q and $\mu_\mathfrak{a}$

Let $\mathfrak{c}$ be the largest relevant ideal in K. So $V(\mathfrak{a}) \neq 0$ if and only if $\mathfrak{a} \subset \mathfrak{c}$. According to lemma 6 there exist an $r \in K$ such that

$$\psi(\langle r, w\rangle + t_w + \frac{1}{4}\langle w, \bar{w}\rangle) = 1 \qquad \forall w \in \mathfrak{c} \cap \mathcal{L} \tag{19}$$

We fix such an $\mathbf{r}$.

Lemma 7 *Let* $\mathfrak{a}$ *be a relevant ideal of* K. *If* $x \in r + \mathfrak{a}_* + \mathcal{L}$, *and* $\Phi_{x,\mathfrak{a}+\mathcal{L}}$ *as in propsition 5, then* $\Phi_{x,\mathfrak{a}+\mathcal{L}} \in V(\mathfrak{a})$. *In particular all nonzero* $V(\mathfrak{a})$ *contains* $\Phi_{r,\mathfrak{a}+\mathcal{L}}$.

Proof. By assumption of x, we have $\psi(\langle x - r, w\rangle = 1$ for any $w \in \mathfrak{a} \cap \mathcal{L}$. This and equation (19) together imply $\psi(\langle x, w\rangle + t_w + \frac{1}{4}\langle w, \bar{w}\rangle) = 1$ for any $w \in \mathfrak{a} \cap \mathcal{L}$. Now we quote proposition 5 to conclude that $\Phi_{x,\mathfrak{a}+\mathcal{L}} \in V(\mathfrak{a})$. □

Lemma 8 *Let* $\mathfrak{a}$ *be an ideal of* K. *Then all functions in* $V(\mathfrak{a})$ *are supported on* $r + \mathfrak{a}_* + \mathcal{L}$.

Proof. If $\Phi \in V(\mathfrak{a})$ and $\Phi(x) \neq 0$ for some $x \in K$. Then $\psi(\langle x, w\rangle + t_w + \frac{1}{4}\langle w, \bar{w}\rangle) = 1$ for any $w \in \mathfrak{a} \cap \mathcal{L}$ by lemma 3. Notice $\mathfrak{a} \subset \mathfrak{c}$, so if we refer to (19), we see $\psi(\langle x - r, w\rangle = 1$ for any $w \in \mathfrak{a} \cap \mathcal{L}$. So $x \in r + \mathfrak{a}_* + \mathcal{L}$. □

Theorem 2 *Let* $\mathfrak{a}$ *be a relevant ideal of* K. *Then* $V(\mathfrak{a})$ *has a basis* $\{\Phi_{x,\mathfrak{a}+\mathcal{L}}\}$ *where* x *runs throught a representative set of* $r + (\mathfrak{a}_* + \mathcal{L})/(\mathfrak{a} + \mathcal{L})$. *In particular* $V(\mathfrak{a})$ *is finite dimensional.*

Proof. By lemma 7 we know $\{\Phi_{x,\mathfrak{a}+\mathcal{L}}\} \subset V(\mathfrak{a})$. By lemma 8 we know all funtions in $V(\mathfrak{a})$ are supported on $r + \mathfrak{a}_* + \mathcal{L}$; We have seen before all funtions in $V(\mathfrak{a})$ are almost constant on cosets of $\mathfrak{a} + \mathcal{L}$. Such functions can be expressed as a unique linear combination of the $\Phi_{x,\mathfrak{a}+\mathcal{L}}$s. □

Denote by n_ψ the largest integer n such that $\psi(\pi^{-n}O_F) = 1$.

Lemma 9 *Let $L = xO_F \oplus yO_F$ be an F-lattice in K, where $x, y \in K$. Then*

$$L_* = \frac{1}{\pi^{n_\psi}\langle x, y\rangle} L$$

Proof. First notice $K = xF \oplus yF$. So any element in K can be written as $ax + by$ for some $a, b \in F$. Now $ax + by \in L_*$ if and only if $\psi(a\langle x, y\rangle O_F) = \psi(b\langle x, y\rangle O_F) = 1$, which exactly means $a, b \in \pi^{-n_\psi}\langle x, y\rangle^{-1} O_F$. $\square$

We put

$$q = [O_F : \pi O_F]$$

Let $\mathrm{ord}_F : F^\times \to \mathbb{Z}$ be the group homomorphism that satisfies $\mathrm{ord}_F(\pi) = 1$. Let $a \in F^\times$, then $|a|_F = q^{-\mathrm{ord}_F(a)}$.

Let $\mathfrak{a} = \alpha O_K$ be an ideal in K, where $\alpha \in K^\times$. We define

$$\mu_{\mathfrak{a}} = \mathrm{ord}_F(\pi^{n_\psi} \frac{\kappa}{\theta - \bar{\theta}} N_{K/F}\alpha)$$

Lemma 10 *Let $\mathfrak{a}$ be an ideal of K, then $\mathfrak{a}_* = \frac{1}{\pi^{\mu_{\mathfrak{a}} + \delta_{K/F}}}\mathfrak{a}$.*

Proof. If $\mathfrak{a} = \alpha O_K$ for some $\alpha \in K^\times$, then $\mathfrak{a} = \alpha O_F \oplus \alpha\theta O_F$. Therefore $\mathfrak{a}_* = \frac{1}{\pi^{n_\psi}\langle 1, \theta\rangle N_{K/F}\alpha}\mathfrak{a} = \frac{1}{\pi^{\mu_{\mathfrak{a}} + \delta_{K/F}}}\mathfrak{a}$. $\square$

Lemma 11 *Let $\mathfrak{a}$ be an ideal of K, then $|\mathfrak{a}| = \int_{\mathfrak{a}} dx = \frac{1}{q^{\mu_{\mathfrak{a}} + \delta_{K/F}}}$.*

Proof. $|\mathfrak{a}_*| = |\mathfrak{a}\pi^{-\mu_{\mathfrak{a}} - \delta_{K/F}}| = |\mathfrak{a}||\pi^{-\mu_{\mathfrak{a}} - \delta_{K/F}}|_K = |\mathfrak{a}||\pi^{-2\mu_{\mathfrak{a}} - 2\delta_{K/F}}|_F = |\mathfrak{a}|q^{2\mu_{\mathfrak{a}} + 2\delta_{K/F}}$. On the hand $|\mathfrak{a}_*||\mathfrak{a}| = 1$. So $|\mathfrak{a}|^2 q^{2\mu_{\mathfrak{a}} + 2\delta_{K/F}} = 1$ and we proved the lemma. $\square$

Proposition 6 *Let $\mathfrak{a}$ be a ideal of K. If K/F is unramified $\mathfrak{a}$ is relevant if and only if $\mu_{\mathfrak{a}} \geq 0$. If K/F is ramified $\mathfrak{a}$ is relevant if and only if $2\mu_{\mathfrak{a}} + \delta_{K/F} + 1 \geq 0$.*

Proof. The condition $v\mathfrak{a} \subset \mathfrak{a}_*$ is equivalent to $v\pi^{\mu_{\mathfrak{a}} + \delta_{K/F}} \in O_K$. If K/F is unramified, then $\delta_{K/F} = 0$ and $v \in O_K^\times$. If K/F is ramified, then $\delta_{K/F} > 0$ and $v \in \Pi^{1 - \delta_{K/F}} O_K^\times$. The lemma follows from these facts. $\square$

As an application of theorem 2, we give a proof of a lemma in [] on the dimension of $V(\mathfrak{a})$.

Corollary 1 *If $V(\mathfrak{a}) \neq 0$ then $\dim_{\mathbb{C}} V(\mathfrak{a}) = q^{\mu_{\mathfrak{a}} + \delta_{K/F}}$*

Proof. $\dim_{\mathbb{C}} V(\mathfrak{a}) = \frac{|\mathfrak{a}_* + \mathcal{L}|}{|\mathfrak{a} + \mathcal{L}|} = \frac{|(\mathfrak{a} \cap \mathcal{L})_*|}{|\mathfrak{a} + \mathcal{L}|} = \frac{1}{|\mathfrak{a} + \mathcal{L}||\mathfrak{a} \cap \mathcal{L}|} = \frac{1}{|\mathfrak{a}||\mathcal{L}|} = \frac{1}{|\mathfrak{a}|} = q^{\mu_{\mathfrak{a}} + \delta_{K/F}}$. $\square$

Remark. Our definition of $V(\mathfrak{a})$ coincides with the definition in [2] in the case where $\mu_{\mathfrak{a}} \geq 0$. When K/F is ramified and $\mu_{\mathfrak{a}} = -1$, our definition of $V(\mathfrak{a})$ coincides with $V^1(\Pi\mathfrak{a})$ in [2].

6 Subrepresentation $(\mathcal{M}, K^\times, V(\mathfrak{a}))$

6.1 Projection $\mathcal{P}_\mathfrak{a}$

Let $\mathfrak{a}$ be an ideal of K. We define an operator $\mathcal{P}_\mathfrak{a} \in \mathrm{End}(V)$ by

$$\mathcal{P}_\mathfrak{a}\Phi = \frac{1}{|\mathfrak{a}|}\int_\mathfrak{a} \rho(x, t_x)\Phi \,\mathrm{d}x \quad \Phi \in V$$

where $|\mathfrak{a}| = \int_\mathfrak{a} \,\mathrm{d}x$.

Proposition 7 *Let $z \in K^\times$, $\mathfrak{a}$ be an ideal of K, and $\Phi \in V$. Then*

$$\mathcal{M}(z)\mathcal{P}_\mathfrak{a}\Phi = \mathcal{P}_\mathfrak{a}\mathcal{M}(z)\Phi$$

Proof. Because $M(z)$ commutes with $\rho(x,t)$ as in (7) and $t_{\bar{u}x} = t_x$, we have

$$\begin{aligned} \int_\mathfrak{a} M(z)\rho(x,t_x)\Phi \,\mathrm{d}x &= \int_\mathfrak{a} \rho(ux,t_x)M(z)\Phi \,\mathrm{d}x \\ = \int_\mathfrak{a} \rho(x,t_{\bar{u}x})M(z)\Phi \,\mathrm{d}x &= \int_\mathfrak{a} \rho(x,t_x)M(z)\Phi \,\mathrm{d}x \end{aligned}$$

The lemma follows from this and the fact $\mathcal{M}(z)$ differ $M(z)$ by a scalar. □

Proposition 8 *Let $\mathfrak{a}$ be a relevant ideal in K, then*

$$V(\mathfrak{a}) = \{\Phi \in V \mid \mathcal{P}_\mathfrak{a}\Phi = \Phi\}$$

Proof. The only nontrivial part of the proof is to show if $\Phi \in V$, then $\mathcal{P}_\mathfrak{a}\Phi \in V(\mathfrak{a})$. Let $x, y \in \mathfrak{a}$, then the following holds

$$\begin{aligned} t_x + t_y + \frac{1}{2}\langle y, x\rangle - t_{x+y} &= \frac{1}{2}\langle y,x\rangle - \frac{1}{2}\langle x, vy\rangle - \frac{1}{2}\langle y, vx\rangle \\ = \frac{1}{2}\langle x, (-1 - v + \bar{v})y\rangle &= \frac{1}{2}\langle x, -2vy\rangle \\ &= \langle -x, vy\rangle \end{aligned}$$

with this preparation, the following is easy to see

$$\begin{aligned} \rho(y,t_y)\int_\mathfrak{a} \rho(x,t_x)\Phi \,\mathrm{d}x &= \int_\mathfrak{a} \rho(x+y, t_x + t_y + \frac{1}{2}\langle y,x\rangle)\Phi \,\mathrm{d}x \\ &= \int_\mathfrak{a} \psi(\langle -x, vy\rangle)\rho(x+y, t_{x+y})\Phi \,\mathrm{d}x \\ = \int_\mathfrak{a} \rho(x+y, t_{x+y})\Phi \,\mathrm{d}x &= \int_\mathfrak{a} \rho(x,t_x)\Phi \,\mathrm{d}x \end{aligned}$$

where $\psi(\langle -x, vy\rangle) = 1$ is the place we use the condition $v\mathfrak{a} \subset \mathfrak{a}_*$. □

If we combine prosition 7, 8, we get

Proposition 9 *Let $\mathfrak{a}$ be a relevant ideal of K. Then $(\mathcal{M}, K^\times, V(\mathfrak{a}))$ is a subrepresentation $(\mathcal{M}, K^\times, V)$.* □

6.2 Notation $U_\mathfrak{a}$ and $K_\mathfrak{a}$

Let $\mathfrak{a}$ be a relevant ideal of K. If $\mu_\mathfrak{a} < 0$, define $U_\mathfrak{a} = 1 + \Pi^{2\delta_{K/F}-1}O_K$. If $\mu_\mathfrak{a} \geq 0$, Define

$$U_\mathfrak{a} = (1 + \pi^{\mu_\mathfrak{a}+\delta_{K/F}}O_K) \cap O_K^\times$$

If K/F is unramified, then $U_\mathfrak{a} = U_{\mu_\mathfrak{a}}$ for $\mu_\mathfrak{a} \geq 0$. If K/F is unramified then $U_\mathfrak{a} = U_{2\mu_\mathfrak{a}+2\delta_{K/F}}$ for $\mu_\mathfrak{a} \geq 0$.

Lemma 12 *Let $\mathfrak{a}$ be a relevant ideal of K, and K/F raimified. We have $U_\mathfrak{a} \subset F^\times U_{2\mu_\mathfrak{a}+2\delta_{K/F}}$.*

Proof. The only case that needs a proof is when $\mu_\mathfrak{a} < 0$. In this case, $F^\times U_\mathfrak{a} = F^\times U_{2\delta_{K/F}-2} \subset F^\times U_{2\mu_\mathfrak{a}+2\delta_{K/F}}$ by proposition 3. $\square$

Let $\mathfrak{a}$ be a relevant ideal in K. Define

$$K_\mathfrak{a} = F^\times U_\mathfrak{a}$$

Lemma 13 *$K_\mathfrak{a}$ is a subgroup of $K^\times$ of finite index.*

Proof. We know $U_\mathfrak{a}$ is an open subgroup of $K^\times$. $K_\mathfrak{a}$ is union of cosets of $U_\mathfrak{a}$, hence it is open. As a result $K^\times/K_\mathfrak{a}$ is discrete. On the other hand $K^\times/K_\mathfrak{a}$ is the continous image of the compact set $O_K^\times \cup \Pi O_K^\times$ under the quotient map $K^\times \to K^\times/K_\mathfrak{a}$. So $K^\times/K_\mathfrak{a}$ is compact. So $K^\times/K_\mathfrak{a}$ is finite. $\square$

6.3 Gauss sum

Let $\mathfrak{a}$ be an ideal of K and $a \in F^\times$, we define a Gauss sum

$$S_\mathfrak{a}(a) = \frac{1}{|\mathfrak{a}|}\int_\mathfrak{a} \psi(ax\bar{x})\ \mathrm{d}x$$

where $|\mathfrak{a}| = \int_\mathfrak{a}\ \mathrm{d}x$.

Lemma 14 *Let $\mathfrak{a}$ be an ideal of K and $a \in F^\times$. We have*

$$S_\mathfrak{a}(a) = S_{\mathfrak{a}_*}(\kappa^2/a)\lambda_K(\psi)\omega(a)|\kappa/a|_K^{1/2}|\mathfrak{a}|^{-1}$$

Proof. In equation (8), divide both sides by $|\mathfrak{a}|$ and replace Φ by $1_\mathfrak{a}$. $\square$

Lemma 15 *Let $\mathfrak{a}$ be an ideal of K and $a \in F^\times$. If $|a\pi^{\mu_\mathfrak{a}}\frac{\theta-\bar{\theta}}{\kappa}|_K \leq 1$, then $S_\mathfrak{a}(a) = 1$. If $|a\pi^{\mu_\mathfrak{a}+\delta_{K/F}}\frac{\theta-\bar{\theta}}{\kappa}| \geq 1$, then*

$$S_\mathfrak{a}(a) = \lambda_K(\psi)|\kappa/a|_K^{1/2}\omega(a)|\mathfrak{a}|^{-1}$$

Proof. The condition $|a\pi^{\mu_\mathfrak{a}}\frac{\theta-\bar{\theta}}{\kappa}|_K \leq 1$ is designed so that $\psi(ax\bar{x}) = 1$ for any $x \in \mathfrak{a}$. So under this condition $S_\mathfrak{a}(a) = 1$. The condition $|a\pi^{\mu_\mathfrak{a}+\delta_{K/F}}\frac{\theta-\bar{\theta}}{\kappa}| \geq 1$ is designed so that $\psi(\kappa^2 x\bar{x}/a) = 1$ for any $x \in \mathfrak{a}_*$. So under this condition $S_{\mathfrak{a}_*}(\kappa^2/a) = 1$. $\square$

Lemma 16 *Suppose that $\delta_{K/F} = 0$. Then $\lambda_K(\psi) = \omega(\pi^{n_\psi})$.*

Proof. Take $a \in F^\times$ and an ideal $\mathfrak{a}$ in K such that $|a\pi^{\mu_\mathfrak{a}} \frac{\theta-\bar{\theta}}{\kappa}|_K = 1$ then the result follows from the previous lemma . $\square$

Proposition 10 *Let $\mathfrak{a}$ be a relevant ideal of K, $z \in U_\mathfrak{a}$, $u = \bar{z}/z$. Then*

$$S_\mathfrak{a}(a) = \lambda_K(\psi)|1-u|_K^{\frac{1}{2}}\omega(\frac{\kappa}{z-\bar{z}})|\mathfrak{a}|^{-1}$$

where $a = \langle v, (1-u)^{-1}\rangle$.

Proof. Write a as $\kappa\frac{1-(1-u)v}{1-u}$ and notice $z \in O_K^\times$.

First suppose K/F is ramified. The assumption $z \in U_\mathfrak{a}$ implies $z \in F^\times U_{2\mu_\mathfrak{a}+2\delta_{K/F}}$. Then we have fact A: $u \in U_{2\mu_\mathfrak{a}+3\delta_{K/F}}$ by lemma 3. Meanwhile $|(1-u)v|_K < 1$ by lemma 2. So we have fact B: $|a|_K = |\frac{\kappa}{1-u}|_K$. Put A and B together we know $|a\pi^{\mu_\mathfrak{a}+\delta_{K/F}}\frac{\theta-\bar{\theta}}{\kappa}| \geq 1$. So

$$\begin{aligned} S_\mathfrak{a}(a) &= \lambda_K(\psi)|\kappa/a|_K^{1/2}\omega(a)|\mathfrak{a}|^{-1} \\ &= \lambda_K(\psi)|1-u|_K^{1/2}\omega(a)|\mathfrak{a}|^{-1} \end{aligned}$$

In the ramified case it remains to show $\omega(a) = \omega(\frac{\kappa}{z-\bar{z}})$.

(Claim) If $(z-1)\bar{v} + (\bar{z}-1)v \in \pi^{\delta_{K/F}}O_F$, then $\omega(a) = \omega(\frac{\kappa}{z-\bar{z}})$.

We immediately prove the claim. From $(z-1)\bar{v}+(\bar{z}-1)v = z\bar{v}+\bar{z}v-1$, we know $z\bar{v}+\bar{z}v \in 1+\pi^{\delta_{K/F}}O_F$. So $\omega(z\bar{v}+\bar{z}v) = 1$. However $a(z-\bar{z})/\kappa = z\bar{v}+\bar{z}v$. So $\omega(a(z-\bar{z})/\kappa) = 1$.This finish the proof of the claim. Now we go check the condition in the claim holds.

If $\mu_\mathfrak{a} \geq 0$, $z \in U_\mathfrak{a}$ means we can write $z = 1+\pi^{\mu_\mathfrak{a}+\delta_{K/F}}x$ for some $x \in O_F$. Write $x = s+t\theta$ for some $s,t \in O_F$. Because $x\bar{v}+\bar{x}v = x(1-v)+\bar{x}v = x+(\bar{x}-x)v = (s+t(\theta+\bar{\theta})) \in O_F$, we have $(z-1)\bar{v}+(\bar{z}-1)v = \pi^{\mu_\mathfrak{a}+\delta_{K/F}}(x\bar{v}+\bar{x}v) \in \pi^{\delta_{K/F}}O_F$.

If $\mu_\mathfrak{a} < 0$, $z \in U_\mathfrak{a}$ means $z-1 \in \Pi^{2\delta_{K/F}-1}O_K$. Because $v \in \Pi^{1-\delta_{K/F}}O_K^\times$ and $\theta-\bar{\theta} \in \Pi^{\delta_{K/F}}O_K^\times$, we have $(z-1)\bar{v} \in (\theta-\bar{\theta})O_K$. Write $(z-1)\bar{v} = (\theta-\bar{\theta})x$ for some $x \in O_K$. Write $x = s+t\theta$ for some $s,t \in O_F$, then $(z-1)\bar{v}+(\bar{z}-1)v = (\theta-\bar{\theta})^2 t \in \pi^{\delta_{K/F}}O_F$.

Next suppose K/F is unramified. The assumption $z \in U_\mathfrak{a}$ implies $z \in U_{\mu_\mathfrak{a}}$. Then we have $u \in U_{\mu_\mathfrak{a}}$ by lemma 1. Another thing to notice is that $\delta_{K/F} = 0$.

If $|(1-u)|_K < 1$, then $|(1-u)v|_K < 1$ since $|v|_K = 1$. So we have $|a|_K = |\frac{\kappa}{1-u}|_K$. This and the condition on u results $|a\pi^{\mu_\mathfrak{a}}\frac{\theta-\bar{\theta}}{\kappa}| \geq 1$. So

$$\begin{aligned} S_\mathfrak{a}(a) &= \lambda_K(\psi)|\kappa/a|_K^{1/2}\omega(a)|\mathfrak{a}|^{-1} \\ &= \lambda_K(\psi)|1-u|_K^{1/2}\omega(a)|\mathfrak{a}|^{-1} \end{aligned}$$

We have $a(z-\bar{z})/\kappa = z\bar{v}+\bar{z}v = z(1-(1-u)v)$. Notice both z and $(1-(1-u)v)$ are units, so $a(z-\bar{z})/\kappa \in O_F^\times$. Hence $\omega(a(z-\bar{z})/\kappa) = 1$. So $\omega(a) = \omega(\frac{\kappa}{z-\bar{z}})$. Hence $S_\mathfrak{a}(a) = \lambda_K(\psi)|1-u|_K^{1/2}\omega(\frac{\kappa}{z-\bar{z}})|\mathfrak{a}|^{-1}$.

If $|(1-u)|_K = 1$, then the fact $u \in U_{\mu_\mathfrak{a}}$ forces $\mu_\mathfrak{a} = 0$. Then $|\mathfrak{a}| = 1$ by lemma 11. It can be checked $|a/\kappa|_K \leq 1$, so $|a\pi^{\mu_\mathfrak{a}} \frac{\theta-\bar{\theta}}{\kappa}| \leq 1$. Hence $S_\mathfrak{a}(a) = 1$. In order to prove the proposition, it remains to check that $\lambda_K(\psi)|1-u|_K^{\frac{1}{2}}\omega(\frac{\kappa}{z-\bar{z}})|\mathfrak{a}|^{-1} = 1$, which is simplifed to $\lambda_K(\psi)\omega(\frac{\kappa}{z-\bar{z}}) = 1$.

To prove the last equality, recall $z \in O_K^\times$, and write $z = s + t\theta$ for some $s, t \in O_F$. Then $t = z(1-u)(\theta - \bar{\theta})^{-1}$ is a product of units so $t \in O_F^\times$, hence $\omega(t) = 1$. Suppose $\mathfrak{a} = \alpha O_K$ for some $\alpha \in K^\times$, then $\pi^{n_\psi}\frac{\kappa}{\theta-\bar{\theta}}N_{K/F}\alpha = \pi^{\mu_\mathfrak{a}}$ by definition of $\mu_\mathfrak{a}$. But remember $\mu_\mathfrak{a} = 0$, so $\pi^{n_\psi}\frac{\kappa}{\theta-\bar{\theta}}N_{K/F}\alpha = 1$. Meanwhile $\lambda_K(\psi) = \omega(\pi^{n_\psi})$ by lemma 16. Put these together, we see

$$\begin{aligned} \lambda_K(\psi)\omega(\frac{\kappa}{z-\bar{z}}) &= \omega(t^{-1}\pi^{n_\psi}\frac{\kappa}{\theta-\bar{\theta}}) \\ &= \omega(t^{-1}N_{K/F}\alpha^{-1}) = 1 \quad \square \end{aligned}$$

6.4 A lemma

Lemma 17 *Let $z \in K^\times - F^\times, u = \bar{z}/z$. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of K subject to the conditions* (i) $(1-u)\mathfrak{b} \subset \mathfrak{a}$. (ii) $v(1-u)\mathfrak{b} \subset \mathfrak{b}_*$. *Then for $\Phi \in V(\mathfrak{a})$,*

$$M(z)\Phi = |1-u|_K^{\frac{1}{2}} \int_{(1-u)^{-1}\mathfrak{b}_*} \psi(\frac{1}{2}\langle x, ux\rangle)\rho((1-u)x, 0)\Phi \ dx$$

Proof. Let $x \in K$. Define $R_\Phi(x) = \psi(\frac{1}{2}\langle x, ux\rangle)\rho((1-u)x, 0)\Phi$. Put

$$I_x = \int_{x+\mathfrak{b}} R_\Phi(y) \ \mathrm{d}y$$

Because $v^{-1} \in O_K$ and $v(1-u)\mathfrak{b} \subset \mathfrak{b}_*$, one has $\mathfrak{b} \subset (1-u)^{-1}\mathfrak{b}^*$. Suppose $x \notin (1-u)^{-1}\mathfrak{b}_*$, we need to show $I_x = 0$.

$$\begin{aligned} I_x &= \int_\mathfrak{b} R_\Phi(x+b) \ \mathrm{d}b \\ &= \psi(\frac{1}{2} < x, ux >) \int_\mathfrak{b} \psi(< (1-u)b, x > + \frac{1}{2} < b, ub >) \\ &\qquad \rho((1-u)x, 0)\rho((1-u)b, 0)\Phi \ \mathrm{d}b \end{aligned}$$

Because $(1-u)\mathfrak{b} \subset \mathfrak{a}$ and $\Phi \in V(\mathfrak{a})$, the definition of $V(\mathfrak{a})$ enables us to replace $\rho((1-u)b, 0)\Phi$ by $\psi(-t_{(1-u)b})\Phi$, so

$$I_x = R_\Phi(x) \int_\mathfrak{b} \psi(< (1-u)b, x > + \frac{1}{2} < b, ub > - t_{(1-u)b}) db$$

Next use equation (9)

$$I_x = R_\Phi(x) \int_{\mathfrak{b}} \psi(< (1-u)b, x >)\psi(< v(1-u)b, b >) \, \mathrm{d}b$$

The assumption $v(1-u)\mathfrak{b} \subset \mathfrak{b}_*$ makes $\psi(< v(1-u)b, b >) = 1$. So

$$\begin{aligned} I_x &= R_\Phi(x) \int_{\mathfrak{b}} \psi(< (1-u)b, x >) \, \mathrm{d}b \\ &= R_\Phi(x) \int_{\mathfrak{b}} \psi(< b, (1-\bar{u})x >) \, \mathrm{d}b \end{aligned}$$

Now $x \notin (1-u)^{-1}\mathfrak{b}_* = (1-\bar{u})^{-1}\mathfrak{b}_*$implies

$$\int_{\mathfrak{b}} \psi(< b, (1-\bar{u})x >) \, \mathrm{d}b = 0$$

Hence $I_x = 0$. $\square$

Proposition 11 *Let $\mathfrak{a}$ be a relevant ideal of K. Then*

$$\mathcal{M}(z)\Phi = \Phi \qquad \forall z \in U_{\mathfrak{a}}, \forall \Phi \in V(\mathfrak{a})$$

Proof. Let $z \in U_{\mathfrak{a}}$, $u = \bar{z}/z$ and $\Phi \in V(\mathfrak{a})$.

Step 1. If $z \in F^\times$. Then $z \in U_{\mathfrak{a}}$ implies $z \in (1+\pi^{\delta_{K/F}} O_F) \cap O_F^\times$. So $\gamma(z) = \omega(z) = 1$ and $M(z)\Phi = \Phi$. As a result, we have $\mathcal{M}(z)\Phi = \gamma(z)M(z)\Phi = \Phi$.

For the rest of the proof we assume $z \notin F^\times$.

Step 2. In this step we show $(1-u)\mathfrak{a}_* \subset \mathfrak{a}$ and $v(1-u)\mathfrak{a}_* \subset \mathfrak{a}$. Because $\frac{1}{v} \in O_K$, it suffice to show the second inclusion. Because $\mathfrak{a}_* = \frac{1}{\pi^{\mu_{\mathfrak{a}}+\delta_{K/F}}}\mathfrak{a}$ by lemma 10. We reduce the proof to show $v(1-u) \in \pi^{\mu_{\mathfrak{a}}+\delta_{K/F}} O_K$. First suppose K/F is unramified. Then $z \in U_{\mathfrak{a}} = U_{\mu_{\mathfrak{a}}}$ implies $u \in U_{\mu_{\mathfrak{a}}}$ by proposition 1, which means $1-u \in \pi^{\mu_{\mathfrak{a}}} O_K$ when $\mu_{\mathfrak{a}} \geq 1$. However $1-u \in \pi^{\mu_{\mathfrak{a}}} O_K$ still holds when $\mu_{\mathfrak{a}} = 0$. So we have $1-u \in \pi^{\mu_{\mathfrak{a}}} O_K$. Now notice v is a unit and $\delta_{K/F} = 0$, so $v(1-u) \in \pi^{\mu_{\mathfrak{a}}+\delta_{K/F}} O_K$. Next we suppose K/F is ramified. Then $z \in U_{\mathfrak{a}}$ implies $z \in F^\times U_{2\mu_{\mathfrak{a}}+2\delta_{K/F}}$, which further implies $u \in U_{2\mu_{\mathfrak{a}}+3\delta_{K/F}}$ by proposition 3. So $1-u \in \pi^{\mu_{\mathfrak{a}}+\delta_{K/F}} \Pi^{\delta_{K/F}} O_K$. Now $v \in \Pi^{1-\delta_{K/F}} O_K^\times$, so $v(1-u) \in \pi^{\mu_{\mathfrak{a}}+\delta_{K/F}} \Pi O_K$. Therefore $v(1-u) \in \pi^{\mu_{\mathfrak{a}}+\delta_{K/F}} O_K$.

Step 3. Replace $\mathfrak{b}$ in lemma 17 by $\mathfrak{a}_*$, we get

$$M(z)\Phi = |1-u|^{\frac{1}{2}} \int_{\frac{1}{1-u}\mathfrak{a}} \psi(\frac{1}{2}\langle x, ux \rangle)\rho((1-u)x, 0)\Phi \, \mathrm{d}x$$

Because $\Phi \in V(\mathfrak{a})$ and $(1-u)x \in \mathfrak{a}$, we know $\rho((1-u)x, 0)\Phi = \psi(-t_{(1-u)x})\Phi$. So

$$M(z)\Phi = |1-u|_K^{\frac{1}{2}} \Phi \int_{\frac{1}{1-u}\mathfrak{a}} \psi(\frac{1}{2}\langle x, ux \rangle - t_{(1-u)x}) \, \mathrm{d}x$$

Now apply (9), we have

$$
\begin{aligned}
M(z)\Phi &= |1-u|_K^{\frac{1}{2}}\Phi \int_{\frac{1}{1-u}\mathfrak{a}} \psi(\langle v(1-u)x, x\rangle)\, \mathrm{d}x \\
&= |1-u|_K^{-\frac{1}{2}}\Phi \int_{\mathfrak{a}} \psi(\langle vx, \frac{x}{1-u}\rangle)\, \mathrm{d}x \\
&= |1-u|_K^{-\frac{1}{2}} |\mathfrak{a}| S_{\mathfrak{a}}(a)\Phi
\end{aligned}
$$

where $a = \langle v, \frac{1}{1-u}\rangle \in F^{\times}$.

Step 4.

$$
\begin{aligned}
\mathcal{M}(z)\Phi &= \gamma(z)M(z) \\
&= \lambda_K(\psi)^{-1}\omega(\frac{z-\bar{z}}{\kappa})|1-u|_K^{-\frac{1}{2}}|\mathfrak{a}|S_{\mathfrak{a}}(a)\Phi \\
&= \Phi
\end{aligned}
$$

where the last equality follows from propositon 10. □

Remark. The statment of above proposition is contained in [2] with a different proof. When $\mu_{\mathfrak{a}} \geq 0$, above proposition is the lemma in page 292 in [2].

Corollary 2 *Let $\mathfrak{a}$ be a relevant ideal in K, $z \in K_{\mathfrak{a}}$ and $\Phi \in V(\mathfrak{a})$. If $z = az'$ for some $a \in F^{\times}$ and $z' \in U_{\mathfrak{a}}$, then*

$$\mathcal{M}(z)\Phi = \omega(a)\Phi$$

We see that $\mathcal{M}$ is smooth, and $K_{\mathfrak{a}}$ acts on $V(\mathfrak{a})$ as scalar multiplication in the representation $(\mathcal{M}, K^{\times}, V(\mathfrak{a}))$. Hence $(\mathcal{M}, K^{\times}, V(\mathfrak{a}))$ is completely reducible. Because V is a union of all the $V(\mathfrak{a})$s, $(\mathcal{M}, K^{\times}, V)$ is completely reducible, and so is its subrepresentations. Therefore any subrepresentations of $(\mathcal{M}, K^{\times}, V)$ is a direct sum of one dimensional representations.

7 Subrepresentation $(\mathcal{M}, K^{\times}, V_{\mathbf{prim}}(\mathfrak{a}))$

7.1 Inner product on V

We define an inner product on V by

$$(\Phi, \Phi') = \int_K \Phi(z)\overline{\Phi'(z)}dz \qquad \Phi, \Phi' \in V$$

Then the representation (ρ, H, V) is unitary. More importantly the representation $(\mathcal{M}, K^{\times}, V)$ is unitary.

Proposition 12 *Let $\mathfrak{a}$ be a relevant ideal in K. Then*

$$(\mathcal{P}_{\mathfrak{a}}\Phi_1, \Phi_2) = (\Phi_1, \mathcal{P}_{\mathfrak{a}}\Phi_2) \qquad \forall \Phi, \Phi_2 \in V$$

where $\mathcal{P}_{\mathfrak{a}}$ is defined in section 6.1.

7.2 Definition of $V_{\mathbf{prim}}(\mathfrak{a})$

Let $\mathfrak{a}$ be a relevant ideal in K. Define

$$V_{\text{prim}}(\mathfrak{a}) = \{\Phi \in V(\mathfrak{a}) \mid (\Phi, \Psi) = 0 \text{ for any } \Psi \in V(\mathfrak{b}), \mathfrak{b} \supsetneqq \mathfrak{a}\}$$

It follows from the definiton that

$$V(\mathfrak{a}) = V_{\text{prim}}(\mathfrak{a}) \oplus V(\Pi^{-1}\mathfrak{a})$$

Because of proposition 12, we have

Proposition 13 *Let $\mathfrak{a}$ be a relevant ideal in K. Then $(\mathcal{M}, K^\times, V_{prim}(\mathfrak{a}))$ is a subrepsentation of $(\mathcal{M}, K^\times, V)$.*

8 Eigenfunctions

8.1 Eigencharacters

A character of $K^\times$ is a continuous group homomorphism $\chi : K^\times \to \mathbb{C}^\times$. Such an character has its image in $\mathbb{C}^1$. Let $\Phi \in V$. If

$$\mathcal{M}(z)\Phi = \chi(z)\Phi \qquad \forall z \in K^\times \tag{20}$$

for some character χ of $K^\times$ and $\Phi \neq 0$, we say Φ is an **eigenfunction** of the representation $(\mathcal{M}, K^\times, V)$ and χ is the **eigencharacter** associated to Φ. If $\Phi \in W$ for some invariant subspace W of V, we also say χ is in W. Let $\mathfrak{a}$ be a relevant ideal in K. If χ is an eigencharater in $V(\mathfrak{a})$, then $\chi|_{F^\times} = \omega$ and $\chi(U_\mathfrak{a}) = 1$ by lemma 2.

8.2 Projection operators

Let $\mathfrak{a}$ be a relevant ideal in K, and χ a charater of $K^\times$ that satisfies $\chi|_{F^\times} = \omega$ and $\chi(U_\mathfrak{a}) = 1$. Define $P_{\chi,\mathfrak{a}} : V(\mathfrak{a}) \to V(\mathfrak{a})$ by

$$P_{\chi,\mathfrak{a}}\Phi = \sum_{z \in K^\times / K_\mathfrak{a}} \chi(z^{-1})\mathcal{M}(z)\Phi \qquad \forall \Phi \in V(\mathfrak{a})$$

$P_{\chi,\mathfrak{a}}$ is a well defined map and its image satisfies equation (20). We know χ appears in $V(\mathfrak{a})$ if and only if $P_{\chi,\mathfrak{a}}\Phi \neq 0$ for some $\Phi \in V(\mathfrak{a})$.

Proposition 14 *Let $\mathfrak{a} \subset \mathcal{L}$ be a relevant ideal in K, and Φ a nonzero function in $V(\mathfrak{a})$. Then $V(\mathfrak{a})$ is generated by $\{\rho(x,0)\Phi | x \in \mathfrak{a}_*\}$.*

Proof. Under the assumption $\mathfrak{a} \subset \mathcal{L}$, a function in V is in $V(\mathfrak{a})$ if and only if it is supported on $r + \mathfrak{a}_*$. So if $x \in \mathfrak{a}_*$, then $\rho(x,0)\Phi \in V(\mathfrak{a})$.

Let $\Phi' \in V(\mathfrak{a})$. Because (ρ, H, V) is irreducible, there are $c_i \in \mathbb{C}, (x_i, t_i) \in H$ such that

$$\Phi' = \sum_i c_i \rho(x_i, t_i)\Phi$$

Notice Φ' is supported on $r + \mathfrak{a}_*$, and $\rho(x_i, t_i)\Phi$ is supported on $-x_i + r + \mathfrak{a}_*$. We can multiply both sides of the equation by $1_{r+\mathfrak{a}_*}$ to eliminate the summands with $x_i \notin \mathfrak{a}_*$. $\square$

Corollary 3 *Let $\mathfrak{a} \subset \mathcal{L}$ be a relevant ideal in K, and Φ a nonzero function in $V(\mathfrak{a})$, and χ a character of $K^\times$. Then χ appears in $V(\mathfrak{a})$ if and only if $P_{\chi,\mathfrak{a}}\rho(x,0)\Phi \neq 0$ for some $x \in \mathfrak{a}_*$.*

Corollary 4 *Let $\mathfrak{c}$ be the largest relevant ideal in K. Let $\Phi_\mathfrak{c}$ be an eigenfunction in $V(\mathfrak{c})$ with eigenfunction $\chi_\mathfrak{c}$. If χ is an eigencharacter that appears in $V(\mathfrak{a})$ for some relevant ideal $\mathfrak{a} \subset \mathcal{L}$, then there is an $x \in \mathfrak{a}_*$ such that an eigenfunction associated to χ is given by*

$$\sum_{z \in K^\times / K_\mathfrak{a}} \chi(z^{-1})\chi_\mathfrak{c}(z)\rho(-ux, 0)\Phi_\mathfrak{c}$$

where $u = \bar{z}/z$.

Proof. Notice $\Phi_\mathfrak{c} \in V(\mathfrak{c}) \subset V(\mathfrak{a})$. By corollary 3, there is an $x \in \mathfrak{a}_*$ such that the eigenfunction associated to χ is given by $\sum_{z \in K^\times/K_\mathfrak{a}} \chi(z^{-1})\Psi(z, x)$ where

$$\Psi(z, x) = \mathcal{M}(z)\rho(-x, 0)\Phi_\mathfrak{c} = \chi_\mathfrak{c}(z)\rho(-ux, 0)\Phi_\mathfrak{c} \qquad \square$$

8.3 Eigenfunctions general case

For $x \in K$, denote by $\Phi_{x,\mathcal{L}}$ the unique function in V that is supported on $x + \mathcal{L}$ and has value 1 at x. Denote by $\mathfrak{l}$ the largest relevant ideal contained in $\mathcal{L}$. Let $D_{r,\mathcal{L}}$ be an open subgroup of $K^\times$ that contains $F^\times$ and stabilizes the one dimensional space spanned by $\Phi_{r,\mathcal{L}}$. Let $\chi_{r,\mathcal{L}}$ be a character on $D_{r,\mathcal{L}}$ that satisfies

$$\mathcal{M}(z)\Phi_{r,\mathcal{L}} = \chi_{r,\mathcal{L}}(z)\Phi_{r,\mathcal{L}} \qquad \forall \in D_{r,L}$$

Lemma 18 *Let $x \in K$.*

$$\rho(-x, 0)\Phi_{r,\mathcal{L}} = \psi(\frac{1}{2}\langle x, r\rangle)\Phi_{r+x,\mathcal{L}}$$

So $\rho(-x, 0)\Phi_{r,\mathcal{L}}$ and $\Phi_{r+x,\mathcal{L}}$ differ only by a scalar.

Lemma 19 *Let $\mathfrak{a}$ be relevant ideals in K such that $\mathfrak{a} \subsetneqq \mathfrak{l}$. If $\mathfrak{a}_* = \alpha O_K$ for some $\alpha \in K$, then $V_{prim}(\mathfrak{a})$ is generated by $\{\Phi_{r+x,\mathcal{L}} | x \in \alpha O_K^\times\}$,*

Proof. By theorem 2, we know $V(\mathfrak{a})$ is generated by $\{\Phi_{r+x,\mathcal{L}} | x \in \mathfrak{a}_*\}$, and $V(\Pi^{-1}\mathfrak{a})$ is generated by $\{\Phi_{r+x,\mathcal{L}} | x \in \Pi\mathfrak{a}_*\}$. The conclusion follows from the fact that $V_{\text{prim}}(\mathfrak{a}) = \{\Phi \in V(\mathfrak{a}) | (\Phi, \Psi) = 0 \ \forall \Psi \in V(\Pi^{-1}\mathfrak{a})\}$. $\square$

Lemma 20 *Let $\mathfrak{a}$ be a relevant ideal in K such that $\mathfrak{a} \subsetneqq \mathfrak{l}$. Let χ be a character on $K^\times$ that satisfies $\chi|_{F^\times} = \omega$ and $\chi(U_\mathfrak{a}) = 1$. Write $\mathfrak{a} = \alpha O_K$ for some $\alpha \in K$. Then χ appear in $V_{prim}(\mathfrak{a})$ if and only if $P_{\chi,\mathfrak{a}}\rho(-x,0)\Phi_{r,\mathcal{L}} \neq 0$ for some $x \in \alpha O_K^\times$.*

Proof. It suffices to notice $\{\rho(-x,0)\Phi_{r,\mathcal{L}} | x \in \alpha O_K^\times\}$ generates $V_{\text{prim}}(\mathfrak{a})$ by lemma 19. $\square$

Definition 3 *Let $x \in K$. We say x is **large enough** if $\{z \in K^\times | (1-u)x \in \mathfrak{l}_*\} \subset D_{r,\mathcal{L}}$, where $u = \bar{z}/z$.*

Lemma 21 *Let $z \in K^\times$ and $u = \bar{z}/z$. Suppose $x \in K$ is large enough. The following are equivalent*

1) The support of $\mathcal{M}(z)\Phi_{r+x,\mathcal{L}}$ and $\Phi_{r+x,\mathcal{L}}$ are the same.

2) The support of $\mathcal{M}(z)\Phi_{r+x,\mathcal{L}}$ and $\Phi_{r+x,\mathcal{L}}$ overlap.

3) $(1-u)x \in \mathcal{L}$.

Proof. 1)$\Rightarrow$2) Obvious.

2)$\Rightarrow$3) We have $\mathcal{M}(z)\Phi_{r,\mathcal{L}} \in V(\mathfrak{l})$ since $\Phi_{r,\mathcal{L}} \in V(\mathfrak{l})$, therefore $\mathcal{M}(z)\Phi_{r,\mathcal{L}}$ is supported on $r + \mathfrak{l}_*$. Now by

$$\mathcal{M}(z)\rho(-x,0)\Phi_{r,\mathcal{L}} = \rho(-ux,0)\mathcal{M}(z)\Phi_{r,\mathcal{L}} \tag{21}$$

we know $\mathcal{M}(z)\Phi_{r+x,\mathcal{L}}$ is supported on $ux+r+\mathfrak{l}_*$, which overlaps with $x+r+\mathcal{L}$, the support of $\Phi_{r+x,\mathcal{L}}$. Therefore $(1-u)x \in \mathfrak{l}_*$, hence $z \in D_{r,\mathcal{L}}$. As a result, $\mathcal{M}(z)\Phi_{r,\mathcal{L}} = \chi_{r,\mathcal{L}}(z)\Phi_{r,\mathcal{L}}$. With this knowledge and (21), we see $\mathcal{M}(z)\Phi_{r+x,\mathcal{L}}$ is supported on $ux+r+\mathcal{L}$. Now this overlaps with $x+r+\mathcal{L}$ means $(1-u)x \in \mathcal{L}$.

3)$\Rightarrow$1) Since $\mathfrak{l} \subset \mathcal{L}$ hence $\mathcal{L} \subset \mathfrak{l}_*$, we know $(1-u)x \in \mathcal{L}$ implies $(1-u)x \in \mathfrak{l}_*$. Then $\mathcal{M}(z)\Phi_{r,\mathcal{L}} = t\Phi_{r,\mathcal{L}}$ for some $t \in \mathbb{C}^1$. Now (21) implies $\mathcal{M}(z)\Phi_{r+x,\mathcal{L}}$ is supported on $ux+r+\mathcal{L}$ which is the same as $x+r+\mathcal{L}$. $\square$

Lemma 22 *Suppose $x \in K$ is large enough. Define*

$$K(x,\mathcal{L}) = \{z \in K^\times | (1-u)x \in \mathcal{L}\}$$

then $z \in K(x,\mathcal{L})$ if and only if $\mathcal{M}(z)\Phi_{r+x,\mathcal{L}} = t\Phi_{r+x,\mathcal{L}}$ for some $t \in \mathbb{C}^1$. In particular $K(x,\mathcal{L})$ is a group. Further

$$\mathcal{M}(z)\Phi_{r+x,\mathcal{L}} = \chi_{r,\mathcal{L}}(z)C_x(z)\Phi_{r+x,\mathcal{L}} \qquad \forall z \in K(x,\mathcal{L})$$

where

$$C_x(z) = \psi(\frac{1}{2}\langle ux, x\rangle + \frac{1}{2}\langle r, (1-u)x\rangle + \frac{1}{4}\langle (1-u)x, (1-\bar{u})\bar{x}\rangle)$$

Proof. The first statement follows from lemma 21 and the fact that two functions in V which are supported on the same coset of $\mathcal{L}$ differ only by a constant. We next prove the second statement.

$$\begin{aligned}\rho(x,0)\mathcal{M}(z)\rho(-x,0)\Phi_{r,\mathcal{L}} &= \rho(x,0)\rho(-ux,0)\mathcal{M}(z)\Phi_{r,\mathcal{L}} \\ &= \psi(\frac{1}{2}\langle ux,x\rangle)\rho((1-u)x,0)\chi_{r,\mathcal{L}}(z)\Phi_{r,\mathcal{L}} \quad (22)\end{aligned}$$

Now $(1-u)x \in \mathcal{L}$. But if $l \in \mathcal{L}$, we have

$$\rho(l,0)\Phi_{r,\mathcal{L}} = \psi(\langle r,l\rangle + \frac{1}{4}\langle l,\bar{l}\rangle)\Phi_{r,\mathcal{L}} \tag{23}$$

Replace l by $(1-u)x$ in (23) and put the result back in (22). We get

$$\rho(x,0)\mathcal{M}(z)\rho(-x,0)\Phi_{r,\mathcal{L}} = \chi_{r,\mathcal{L}}(z)C_x(z)\Phi_{r,\mathcal{L}} \qquad \square$$

Definition 4 *A relevant ideal $\mathfrak{a}$ in K is said to be **small enough** if $\{z \in K^\times | (1-u)\mathfrak{a}_* \subset \mathfrak{l}_*\} \subset D_{r,\mathcal{L}}$.*

Theorem 3 *Let $\mathfrak{a}$ be a relevant ideal in K small enough and $\mathfrak{a} \subsetneqq \mathfrak{l}$, and χ a character on $K^\times$. Write $\mathfrak{a}_* = \alpha O_K$ for some $\alpha \in K$. Then χ appears in $V_{prim}(\mathfrak{a})$ if and only if χ satisfies*

1) $\chi|_{F^\times} = \omega$, $\chi(U_\mathfrak{a}) = 1$

2) $\chi|_{K(x,\mathcal{L})} = \chi_{r,\mathcal{L}}C_x$ for some $x \in \alpha O_K^\times$, where $K(x,\mathcal{L})$ and C_x are as defined in lemma 22.

If χ appears in $V_{prim}(\mathfrak{a})$ and x is chose as in 2), an eigenfunction associated to χ is given by

$$\sum_{z\in K^\times/K(x,\mathcal{L})} \chi(z^{-1})\mathcal{M}(z)\Phi_{r+x,\mathcal{L}}$$

Proof. Let $x \in \alpha O_K^\times$. Because

$$\{z \in K^\times | (1-u)x \subset \mathfrak{l}_*\} = \{z \in K^\times | (1-u)\mathfrak{a}_* \subset \mathfrak{l}_*\} \subset D_{r,\mathcal{L}}$$

we know x is large enough.

We consider

$$P_{\chi,\mathfrak{a}}\rho(-x,0)\Phi_{r,\mathcal{L}} = \sum_{z\in K^\times/K_\mathfrak{a}} \chi(z^{-1})\Psi(z,x)$$

where

$$\Psi(z,x) = \mathcal{M}(z)\rho(-x,0)\Phi_{r,\mathcal{L}}$$

By lemma 22 we have

$$\Psi(z,x) = \chi_{r,\mathcal{L}}(z)C_x(z)\rho(-x,0)\Phi_{r,\mathcal{L}} \qquad \forall z \in K(x,\mathcal{L})$$

Now the eigenfunction is

$$\begin{aligned}&\sum_{z\in K^\times/K_\mathfrak{a}} \chi(z^{-1})\Psi(z,x) \\ =\ & \sum_{w\in K^\times/K(x,\mathcal{L})} \chi(w^{-1})\mathcal{M}(w) \sum_{z\in K(x,\mathcal{L})/K_\mathfrak{a}} \chi(z^{-1})\Psi(z,x) \\ =\ & Q\cdot E\end{aligned}$$

where

$$Q = \sum_{z \in K(x,\mathcal{L})/K_{\mathfrak{a}}} \chi(z^{-1})\chi_{r,\mathcal{L}}(z)C_x(z)$$

and

$$E = \sum_{w \in K^\times/K(x,\mathcal{L})} \chi(w^{-1})\mathcal{M}(w)\rho(-x,0)\Phi_{r,\mathcal{L}}$$

If $\mathcal{M}(w)\rho(-x,0)\Phi_{r,\mathcal{L}}$ and $\rho(-x,0)\Phi_{r,\mathcal{L}}$ have overlapping support, then $w \in K(x,\mathcal{L})$ by lemma 21. This means, in the sum of E, the term corresponding to $w = 1$ is orthogonal to the other summands. Hence $E \neq 0$.

Now $P_{\chi,\mathfrak{a}}\rho(-x,0)\Phi_{r,\mathcal{L}} \neq 0$ if and only if $Q \neq 0$. However Q is a sum of group characters, since $z \mapsto \chi_{r,\mathcal{L}}(z)C_x(z)$ is an eigencharacter on $K(x,\mathcal{L})$ with eigenfunction $\rho(-x,0)\Phi_{r,\mathcal{L}}$. So $Q \neq 0$ if and only if

$$\chi(z) = \chi_{r,\mathcal{L}}(z)C_x(z) \qquad \forall z \in K(x,\mathcal{L})$$

The rest of theorem now follows □

8.4 Eigenfunctions when $\mathcal{L}$ is an ideal

Suppose $\mathcal{L}$ is an ideal of K, and let $\mathfrak{l} = \frac{1}{v}\mathcal{L}$. Then $v\mathfrak{l} \subset \mathfrak{l}_*$, hence $\mathfrak{l}$ is an relevant ideal. We know $V(\mathfrak{l})$ has a basis $\{\Phi_{s,\mathcal{L}} | s \in r + \mathfrak{l}_*/\mathcal{L}\}$. They are actually all eigenfunctions.

Theorem 4 *If $\mathcal{L}$ is an ideal of K. Let $\mathfrak{l} = \frac{1}{v}\mathcal{L}$, and $s \in r + \mathfrak{l}_*$. Then $\Phi_{s,\mathcal{L}}$ is an eigenfunction of $\mathcal{M}$ with eigencharacter $\chi_{s,\mathcal{L}}$ defined in the proof.*

Proof. We want to show that for any $z \in K^\times$, $\mathcal{M}(z)\Phi_{s,\mathcal{L}}$ is a scalar multiple of $\Phi_{s,\mathcal{L}}$. If $z \in F^\times$ then $\mathcal{M}(z)\Phi_{s,\mathcal{L}} = \omega(z)\Phi_{s,\mathcal{L}}$ by definition of $\mathcal{M}$. For the rest of the proof we assume $z \in K^\times - F^\times$. We know $\Phi_{s,\mathcal{L}} \in V(\mathfrak{l})$. We have $v(1-u) \in O_K$ by lemma 2, where $u = \bar{z}/z$. As a consequece we know $(1-u)\mathcal{L} \subset \mathfrak{l}$ and $v(1-u)\mathcal{L} \subset \mathcal{L}$. Now we apply Lemma 17 with $\mathfrak{a}, \mathfrak{b}$ and Φ in the lemma replaced by $\mathfrak{l}, \mathcal{L}$ and $\Phi_{s,\mathcal{L}}$. We have

$$\begin{aligned} M(z)\Phi_{s,\mathcal{L}} &= |1-u|_K^{\frac{1}{2}} \int_{\frac{1}{1-u}\mathcal{L}} \psi(\frac{1}{2}\langle x, ux\rangle)\rho((1-u)x,0)\Phi_{s,\mathcal{L}}\, \mathrm{d}x \\ &= |1-u|_K^{-\frac{1}{2}} \int_{\mathcal{L}} \psi(\frac{1}{2}\langle l, \frac{l}{1-u}\rangle)\rho(l,0)\Phi_{s,\mathcal{L}}\, \mathrm{d}l \end{aligned}$$

If we use the following formula

$$\rho(l,0)\Phi_{s,\mathcal{L}} = \psi(\langle s,l\rangle + \frac{1}{4}\langle l,\bar{l}\rangle)\Phi_{s,\mathcal{L}} \qquad \forall l \in \mathcal{L}$$

we have

$$M(z)\Phi_{s,\mathcal{L}} = B_s(z)\Phi_{s,\mathcal{L}}$$

where

$$B_s(z) = |1-u|_K^{-\frac{1}{2}} \int_{\mathcal{L}} \psi(\frac{1}{2}\langle l, \frac{l}{1-u}\rangle + \langle s, l\rangle + \frac{1}{4}\langle l, \bar{l}\rangle) \, \mathrm{d}l$$

If we define

$$\chi_{s,\mathcal{L}}(z) = \begin{cases} \omega(z) & z \in F^\times \\ \lambda_K(\psi)^{-1}\omega(\frac{z-\bar{z}}{\kappa})B_s(z) & z \in K^\times - F^\times \end{cases}$$

then we have

$$\mathcal{M}(z)\Phi_{s,\mathcal{L}} = \chi_{s,\mathcal{L}}(z)\Phi_{s,\mathcal{L}} \qquad \forall z \in K^\times$$

□

In many cases we we can decide $\chi_{s,\mathcal{L}}$ easily. If K/F is unramifed, then $\mu_{\mathfrak{l}} = 0$ and $U_{\mathfrak{l}} = O_K^\times$. If K/F is ramified, then $\mu_{\mathfrak{l}} = -1$ and $U_{\mathfrak{l}} = U_{2\delta_{K/F}-1}$. However we have $\chi_{s,\mathcal{L}}(U_{\mathfrak{l}}) = 1$. If K/F is ramified and 2 is a unit in F. We can assume $\bar{\Pi} = -\Pi$. Then $B_s(\Pi) = 1$ and $\chi_{r,\mathcal{L}}(\Pi) = \lambda_K(\psi)^{-1}\omega(\frac{2\Pi}{\kappa})$.

Lemma 23 *Let $\mathfrak{a}$ be a relevant ideal. Write $\mathfrak{a}_* = \alpha O_K$ for some $\alpha \in K$. Let $x \in \alpha O_K^\times$*

$$K(x,\mathcal{L}) = \begin{cases} K^\times & \mu_{\mathfrak{a}} \leq -1 \\ F^\times O_K^\times & \mu_{\mathfrak{a}} = 0 \\ F^\times(1+\pi^{\mu_{\mathfrak{a}}/2}O_K) & \mu_{\mathfrak{a}} > 0 \ \textit{even} \\ F^\times(1+\pi^{(\mu_{\mathfrak{a}}+1)/2}O_K) & \mu_{\mathfrak{a}} > 0 \ \textit{odd} \end{cases}$$

We denote this $K(x,\mathcal{L})$ as $G_{\mathfrak{a}}$.

Now if we take $D_{r,\mathcal{L}} = K^\times$, then any relevant ideal is small enough we have

Theorem 5 *Let $\mathfrak{a}$ be a relevant ideal in K such that $\mathfrak{a} \subsetneqq \mathfrak{l}$ and χ a character on $K^\times$. Write $\mathfrak{a}_* = \alpha O_K$ for some $\alpha \in K$. Then χ appears in $V_{prim}(\mathfrak{a})$ if and only if χ satisfies*

1) $\chi|_{F^\times} = \omega$, $\chi(U_{\mathfrak{a}}) = 1$

2) $\chi|_{G_{\mathfrak{a}}} = \chi_{r,\mathcal{L}}C_x$ for some $x \in \alpha O_K^\times$, where C_x are as defined in lemma 22.

If χ appears in $V_{prim}(\mathfrak{a})$ and x is chosen as in 2), an eigenfunction associated to χ is given by

$$\sum_{z \in K^\times/G_{\mathfrak{a}}} \chi(z^{-1})\chi_{r,\mathcal{L}}(z)\psi(\frac{1}{2}\langle ux, r\rangle)\Phi_{r+ux,\mathcal{L}}$$

where $u = \bar{z}/z$ and $\chi_{r,\mathcal{L}}$ is the eigencharacter associated to $\Phi_{r,\mathcal{L}}$.

Proof. This theorem follows from theorem 3 and the fact that $\mathcal{M}(z)\Phi_{r+x,\mathcal{L}} = \psi(\frac{1}{2}\langle r, x\rangle)\chi_{r,\mathcal{L}}(z)\psi(\langle ux, r\rangle)\Phi_{r+ux,\mathcal{L}}$. □

References

[1] I.N.Bernshtein and A.V.Zelevinskii. Representations of the group GL(n,F) where F is a non-archimedean local field. *Russian Math. Surveys 31*, 1976.

[2] Atsushi Murase and Takashi Sugano. Local theory of primitive theta function. *Compositio Mathematica 123*, 2000.

[3] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger. *Correspondances de Howe sur un corps p-adique*. Springer, 1987.

[4] André Weil. Sur certain groupes d'opérateurs unitaries. *Acta Mathematica 111*, 1964.

[5] André Weil. *Basic Number Theory*. Springer, 1973.

[6] Tonghai Yang. Eigenfunctions of the weil representation of unitary group of one variable. *Trans. Amer.Math.Soc.350*, 1998.