习题3-6

1. 设 $f(x)=a(x-1)^4+b(x-1)^3+c(x-1)$.

又 $f(x)=x^4-2x^3+1$

$$\Rightarrow \begin{cases} a=1 \\ b=2 \\ c=-2 \end{cases}$$

则 $f(x)=(x-1)^4+2(x-1)^3-2(x-1)$

2. (1) $f(x)=\frac{1}{1-x}=(1-x)^{-1}=1-(-1)x+\frac{(-1)(-1-1)}{2!}x^2+\cdots+(-1)^n\frac{(-1)(-1-1)\cdots(-1-n+1)}{n!}(-x)^n+R_n(x)$

$=1+x+x^2+\cdots+x^n+\frac{(-1)^n}{(\theta x-1)^{n+2}}x^{n+1} \quad (0<\theta<1)$

(2) $f(x)=\mathrm{ch}\,x=\frac{e^x+e^{-x}}{2}=\frac{1+x+\frac{x^2}{2!}+\cdots+\frac{x^{2n}}{(2n)!}+\frac{e^{\theta x}x^{2n+1}}{(2n+1)!}+1-x+\frac{x^2}{2}-\frac{x^3}{3!}+\cdots+\frac{(-1)^{2n}x^{2n}}{(2n)!}+\frac{e^{-\theta x}(-x)^{2n+1}}{(2n+1)!}}{2}$

$=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{x^{2n}}{(2n)!}+\frac{\mathrm{sh}\,\theta x}{(2n+1)!}x^{2n+1} \quad (0<\theta<1)$

(3) $f(x)=(1-2x)^{-\frac{1}{2}}=1+(-\frac{1}{2})(-2x)+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-2x)^2+\cdots+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!}(-2x)^n+R_n(-2x)$

$=1+x+\frac{3}{2!}x^2+\frac{3\times5}{3!}x^3+\cdots+\frac{(2n-1)!!}{n!}x^n+\frac{(2n+1)!!}{(n+1)!\sqrt{(1-2\theta x)^{2n+3}}} \quad (0<\theta<1)$

(4) $f(x)=\ln(1+x)-\ln(1-x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n-1}}{n}x^n+\frac{(-1)^n\cdot x^{n+1}}{(n+1)(1+\theta x)}$

$-(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots-\frac{x^n}{n}+\frac{(-1)^n(-x)^{n+1}}{(n+1)(1+\theta(-x))})$

$=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\cdots+\frac{2x^{2n-1}}{(2n-1)}+\frac{2x^{2n}}{(2n)(1+\theta x)}$

$=2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\cdots+\frac{2x^{2n-1}}{2n-1}+o(x^{2n})$

3. 引用一个结论：f在x_0处的$(n+1)$阶泰勒多项式的导数等于f'在x_0点的n阶泰勒多项式.

则 $(\arctan x)'=\frac{1}{1+x^2}=1-x^2+x^4+o(x^4)$

则 $\arctan x=0+1(x)+\frac{-1}{3}x^3+0+\frac{1}{5}x^5+o(x^5)$

$=x-\frac{1}{3}x^3+o(x^3)$.

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5) = x - \frac{x^3}{3!} + o(x^3)$

则 $\sin(\sin x) = \sin\left(x - \frac{x^3}{3!} + o(x^3)\right) = x - \frac{x^3}{3!} - \frac{\left(x - \frac{x^3}{3!}\right)^3}{3!} + o(x^9)$

$= x - \frac{1}{3}x^3 + o(x^3)$

5. 直接用泰勒公式，且 $\frac{1}{x}$ 的 n 阶导数为 $(-1)^n n!\, x^{-n-1}$

则 $f(x) = -1 - (x+1) - (x+1)^2 - \cdots - (x+1)^n + (-1)^{n+1} \frac{(x+1)^{n+1}}{[-1+\theta(x+1)]^{n+2}} \quad 0<\theta<1$

6. $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$.

$f(\frac{1}{2}) = -\ln 2 \quad f'(\frac{1}{2}) = -2 \quad f''(\frac{1}{2}) = -4 \quad f'''(\frac{1}{2}) = -16, \quad f^{(n)}(x_0) = -2^n(n-1)!$

则 $f(x) = -\ln 2 - 2(x-\frac{1}{2}) - \frac{8}{3}(x-\frac{1}{2})^3 - \cdots - \frac{2^n}{n}(x-\frac{1}{2})^n - \frac{(x-\frac{1}{2})^{n+1}}{(n+1)\cdot[\frac{1}{2} - \theta(x-\frac{1}{2})]^{n+1}} \quad 0<\theta<1$

7. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$，则 $f'(4) = \frac{1}{4}$

$f''(x) = -(\frac{1}{2})^2 \cdot x^{-\frac{3}{2}}$ 则 $f''(4) = (-\frac{1}{2})^5$

$f'''(x) = \frac{3}{2^3}x^{-\frac{5}{2}}$ 则 $f'''(4) = \frac{3}{2^8}$

$f^{(4)}(x) = \frac{-3\times 5}{2^4}x^{-\frac{7}{2}}$

则 $\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{2^6}(x-4)^2 + \frac{1}{2^9}(x-4)^3 - \frac{5}{2^7}\frac{(x-4)^4}{(4+\theta(x-4))^{\frac{7}{2}}} \quad 0<\theta<1$

$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{15(x-4)^4}{4!\cdot 16\cdot[4+\theta(x-4)]^{\frac{7}{2}}}$

8. 解：因为 $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$

则 $2^x = e^{x\ln 2} = 1 + x\ln 2 + \frac{(x\ln 2)^2}{2!} + \frac{(x\ln 2)^3}{3!} + \cdots + \frac{(x\ln 2)^n}{n!} + o(x^n)$

则 x^n 的系数为 $\frac{(\ln 2)^n}{n!}$

9. (1) 由泰勒公式得：$\frac{x^2}{2} + 1 - \sqrt{1+x^2} = \frac{x^2}{2} + 1 - \left(1 + \frac{1}{2}x^2 + \frac{-\frac{1}{4}}{2!}x^4 + o(x^5)\right)$

$= \frac{1}{8}x^4 + o(x^5) \quad x\to 0$.

则 $\lim_{x\to 0} \frac{\frac{x^2}{2} + 1 - \sqrt{1+x^2}}{\sin x^2 \cdot x^2} = \lim_{x\to 0} \frac{\frac{1}{8}x^4 + o(x^5)}{x^4} = \frac{1}{8}$

(2) $\ln(1+x)-\sin x=\left(x+\frac{-x^2}{2}+o(x^2)\right)-\left(x+o(x^2)\right)=-\frac{1}{2}x^2+o(x^2) \quad x\to 0$

$\sqrt{1+x^2}-\cos x^2=\left(1+\frac{1}{2}x^2+o(x^3)\right)-\left(1-o(x^3)\right)=\frac{1}{2}x^2+o(x^3) \quad x\to 0.$

则原式 $=\lim\limits_{x\to 0}\dfrac{-\frac{1}{2}x^2+o(x^2)}{\frac{1}{2}x^2+o(x^2)}=-1.$

(3). 由泰勒公式得: $f(x)=x+2x^2+o(x^3) \quad x\to 0$

则 $\lim\limits_{x\to 0}\dfrac{f(x)-x}{x^2}=\lim\limits_{x\to 0}\dfrac{2x^2+o(x^3)}{x^2}=2.$

10. (1) $\sin x=x-\frac{x^3}{3!}+o(x^3)=x+o(x). \quad x\to 0$

则 $\sin x+x=2x-\frac{x^3}{3!}+o(x^3)=2x+o(x) \quad x\to 0$

则 $\sin x+x$ 是一阶无穷小, 且 $f(x)\sim 2x$.

(2). $\sin x=x-\frac{x^3}{3!}+o(x^3) \quad x\to 0$

则 $\sin x-x=-\frac{x^3}{3!}+o(x^3) \quad x\to 0$

则 $\sin x-x$ 是 3 阶无穷小, $f(x)\sim -\frac{x^3}{3!}$

(3). $\cos x=1-\frac{1}{2}x^2+\frac{x^4}{4!}+o(x^4) \quad x\to 0$

则 $\cos x-1+\frac{x^2}{2}=\frac{x^4}{4!}+o(x^4) \quad x\to 0$

则 $f(x)=\cos x-1+\frac{x^2}{2}$ 是 4 阶无穷小, $f(x)\sim\frac{x^4}{4!}$

(4). $x(e^x+1)=x\left(2+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o(x^3)\right)=2x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+o(x^4) \quad x\to 0$

$2(e^x-1)=2\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+o(x^3)\right)=2x+x^2+\frac{x^3}{3}+o(x^3) \quad x\to 0$

则 $f(x)=x(e^x+1)-2(e^x-1)=\frac{1}{6}x^3+\frac{1}{6}x^4+o(x^4)=\frac{1}{6}x^3+o(x^3) \quad x\to 0$

则 $f(x)$ 是 3 阶无穷小, $f(x)\sim\frac{x^3}{6}$

(5). $4\sin x=4\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^5)\right)=4x-\frac{2}{3}x^3+\frac{1}{30}x^5+o(x^5)$

$\sin x\cos x=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^5)\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)=x-\frac{2}{3}x^3+\frac{4}{30}x^5+o(x^5)$

则 $f(x)=3x-4\sin x+\sin x\cos x=\frac{1}{10}x^5+o(x^5)$

则 $f(x)$ 是 5 阶无穷小, $f(x)\sim\frac{1}{10}x^5$

11. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{e^{\theta x}}{n!} x^n$, $0<\theta<1$

当$x=\frac{1}{2}$时, $\sqrt{e} \approx 1+\frac{1}{2}+\frac{1}{8}+\cdots+\frac{x^{n-1}}{(n-1)!}$

要使误差小于等于0.001. 则只要

$$\left|R_n\left(\tfrac{1}{2}\right)\right| = \left|\frac{e^{\theta \cdot \frac{1}{2}}}{n!}\left(\tfrac{1}{2}\right)^n\right| = \left|\frac{e^{\frac{1}{2}\theta}}{2^n n!}\right| < \left|\frac{\sqrt{e}}{2^n n!}\right| < \frac{1}{1000}$$

取$n=5$即可

则 $\sqrt{e} \approx 1+\frac{1}{2}+\frac{1}{8}+\frac{1}{48}+\frac{1}{24\times16}+\frac{1}{32\times120}\approx 1.648$

12 (1) 设 $f(x)=(1+x)^{\frac{1}{3}}$

则 $$f(x) = (1+x_0)^{\frac{1}{3}} + \frac{1}{3}(1+x_0)^{-\frac{2}{3}}(x-x_0) + \frac{\left(-\frac{2}{9}\right)(1+x_0)^{-\frac{5}{3}}}{2!}(x-x_0)^2 + \frac{\frac{10}{27}(1+x_0)^{-\frac{8}{3}}}{3!}(x-x_0)^3 + o\left((x-x_0)^3\right)$$

令$x=30$. $x_0=29$.

则 $f(x) \approx 3.10724$

$$|R_3| = \left|\frac{-\frac{80}{81}(29+\theta)^{-\frac{10}{3}}}{4!}(30-29)^4\right| < \left|\frac{-\frac{80}{81}\times 29^{-\frac{10}{3}}}{4!}\times 1\right| < 1.88\times10^{-5}.$$

(2) $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + R_4(x)$, 其中 $R_4(x) = \frac{\sin\left(\frac{\pi\theta}{10}+\frac{5\pi}{2}\right)}{5!}\left(\frac{\pi}{10}\right)^5$, $0<\theta<1$.

$x=18^\circ=\frac{\pi}{10}$

$\sin 18^\circ \approx \frac{\pi}{10} - \frac{\left(\frac{\pi}{10}\right)^3}{6} + \frac{\left(\frac{\pi}{10}\right)^5}{120} \approx 0.309017$

$$\left|R_4\left(\tfrac{\pi}{10}\right)\right| = \left|\frac{\sin\left(\frac{\pi\theta}{10}+\frac{5\pi}{2}\right)}{5!}\left(\frac{\pi}{10}\right)^5\right| < \left|\frac{\sin\frac{\pi}{2}}{5!}\left(\frac{\pi}{10}\right)^5\right| < 1.3\times10^{-4}.$$

13. 解: 先用泰勒公式展开 $\ln(1+x) = x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n-1}x^{n-2}}{n-2} + o(x^{n-1})$ $x\to0$

将 $\ln(1+x)$ 展开式代入 $f(x)$ 得 $f(x) = x^3 - \frac{x^4}{2} + \cdots + \frac{(-1)^{n-1}x^n}{n-2} + o(x^n)$ $x\to0$

则 $\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n-2}$

$\Rightarrow f^{(n)}(0) = \frac{(-1)^{n-1}\cdot n!}{n-2}$.