Algorithmic Game Theory, Spring 2022 Homework 1

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Problem 1.

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Problem 2.

Proof. To prove that the two definition is equivalent, we prove it from two side.

(i) Definition 1 $\Rightarrow$ Definition 2:

Because row player and column player's proof is similar, we only prove row player's side.

KNOWN: A pair of strategies $(\mathbf{x}, \mathbf{y})$ is NE and $\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}'^T R\mathbf{y}, \forall \mathbf{x}' \in \Delta_m$.

(a) $\mathbf{x}$ is a pure strategy, assume that $x_s = 1$, so $\mathbf{x} = \mathbf{e}_s$.

$$\because \mathbf{e}_k \in \Delta_m, \forall k \in [m]$$
$$\therefore \mathbf{e}_s^T R\mathbf{y} = \mathbf{x}^T R\mathbf{y} \geq \mathbf{e}_k^T R\mathbf{y}, \forall k \in [m]$$

(b) $\mathbf{x}$ is a mixed strategy.

We assume that

$$x_s > 0 \Rightarrow \mathbf{e}_s^T R\mathbf{y} < \mathbf{e}_k^T R\mathbf{y}, \forall k \in [m]$$
$$\sum_{j=1}^{n} R_{s,j} y_j < \sum_{j=1}^{n} R_{k,j} y_j, \forall k \in [m]$$

Then, we let

$$\mathbf{x}' = (x_i') = \begin{cases} x_i & i \neq k, s \\ 0 & i = s \\ x_k + x_s & i = k \end{cases}$$

$$\begin{aligned}
\mathbf{x}'^T R\mathbf{y} &= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i' R_{i,j} y_j \\
&= \sum_{i=1, i\neq s,k}^{m} \sum_{j=1}^{n} x_i' R_{i,j} y_j + \sum_{j=1}^{n} x_s' R_{s,j} y_j + \sum_{j=1}^{n} x_k' R_{k,j} y_j \\
&= \sum_{i=1, i\neq s,k}^{m} \sum_{j=1}^{n} x_i' R_{i,j} y_j + 0 + \sum_{j=1}^{n} (x_k + x_s) R_{k,j} y_j \\
&> \sum_{i=1, i\neq s,k}^{m} \sum_{j=1}^{n} x_i' R_{i,j} y_j + \sum_{j=1}^{n} x_s R_{s,j} y_j + \sum_{j=1}^{n} x_k R_{k,j} y_j \\
&= \mathbf{x}^T R\mathbf{y}
\end{aligned}$$

However, it is known that $\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}'^T R\mathbf{y}, \forall \mathbf{x}' \in \Delta_m$.

Therefore, our assumption is incorrect. So,

$$x_s > 0 \Rightarrow \mathbf{e}_s^T R\mathbf{y} \geq \mathbf{e}_k^T R\mathbf{y}, \forall k \in [m]$$

(ii) Definition 2 $\Rightarrow$ Definition 1:

Because row player and column player's proof is similar, we only prove row player's side.

KNOWN: A pair of strategies $(\mathbf{x}, \mathbf{y})$ is NE and $x_s > 0 \Rightarrow \mathbf{e}_s^T R\mathbf{y} \geq \mathbf{e}_k^T R\mathbf{y}, \forall k \in [m]$.

Let set P contains subscript s that $x_s > 0$.

For all subscripts in set P, we can know from definition 2 that $\mathbf{e}_s^T R\mathbf{y} = \mathbf{e}_t^T R\mathbf{y}, \forall s, t \in P$, and $\mathbf{e}_s^T R\mathbf{y} \neq \mathbf{e}_t^T R\mathbf{y}, \forall s \in P, t \notin P$.Because

$$\sum_{i \in P} x_i = 1$$

$$\mathbf{x}^T R\mathbf{y} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i R_{i,j} y_j = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} R_{i,j} y_j = 1 * \sum_{j=1}^{n} R_{s,j} y_j = \sum_{j=1}^{n} R_{s,j} y_j, s \in P$$

We use A to substitute $\sum_{j=1}^{n} R_{s,j} y_j, s \in P$

If there exists $\mathbf{x}' \in \Delta_m$ which satisfies $\mathbf{x}^T R\mathbf{y} < \mathbf{x}'^T R\mathbf{y}$, then

$$\begin{aligned}
\mathbf{x}^T R\mathbf{y} &= A \\
&= \sum_{i \in P} x_i * A \\
&< \mathbf{x}'^T R\mathbf{y} \\
&= \sum_{i \in P} x'_i * A + \sum_{i \notin P} x'_i * (\sum_{j=1}^{n} R_{i,j} y_j) \\
&\leq \sum_{i \in P} x'_i * A + \sum_{i \notin P} x'_i * A \\
&= \sum_{i=1}^{m} x'_i * A \\
&= A
\end{aligned}$$

Obviously, $A < A$ is incorrect, so for all $\mathbf{x}' \in \Delta_m$, $\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}'^T R\mathbf{y}$.

In conclusion, the two definition of Nash Equilibrium mentioned in class are equivalent.

Problem 3.

Proof. Because the game is symmetric, so matrix R and C are both m*m matrix.

We define a domain

$$\mathbf{D} = \{(\mathbf{x}, ..., \mathbf{x}) \in \Delta_m \times ... \times \Delta_m : \mathbf{x} \in \Delta_m\}$$

Then, we define a function f as follows. For all $\underset{\sim}{x} \in \mathbf{D}, \underset{\sim}{x} \xmapsto{f} \underset{\sim}{y}$ for all $p \in [2]$ and $s_p \in S_p$:

$$y_p(s_p) := \frac{x_p(s_p) + Gain_{p;s_p}(\underset{\sim}{x})}{1 + \sum_{s'_p \in S_p} Gain_{p;s'_p}(\underset{\sim}{x})}$$

$$Gain_{p;s_p}(\underset{\sim}{x}) = max\{u_p(s_p;\underset{\sim}{x}_{-p}) - u_p(\underset{\sim}{x}), 0\}$$

For row player and column player,

$$x_1(s_p) = x_2(s_p)$$

$$\begin{aligned}
Gain_{1;s_p}(\underset{\sim}{x}) &= max\{\mathbf{e}_{s_p}^T R\mathbf{x} - \mathbf{x}^T R\mathbf{x}, 0\} \\
&= max\{(\mathbf{e}_{s_p}^T R\mathbf{x} - \mathbf{x}^T R\mathbf{x})^T, 0\} \\
&= max\{\mathbf{x}^T R^T \mathbf{e}_{s_p} - \mathbf{x}^T R^T \mathbf{x}, 0\} \\
&= max\{\mathbf{x}^T C\mathbf{e}_{s_p} - \mathbf{x}^T C\mathbf{x}, 0\} \\
&= Gain_{2;s_p}(\underset{\sim}{x})
\end{aligned}$$

$$\begin{aligned}
y_1(s_p) &= \frac{x_1(s_p) + Gain_{1;s_p}(\underset{\sim}{x})}{1 + \sum_{s'_p \in S_p} Gain_{1;s'_p}(\underset{\sim}{x})} \\
&= \frac{x_2(s_p) + Gain_{2;s_p}(\underset{\sim}{x})}{1 + \sum_{s'_p \in S_p} Gain_{2;s'_p}(\underset{\sim}{x})} \\
&= y_2(s_p)
\end{aligned}$$

$$\therefore \underset{\sim}{y} \in \mathbf{D}$$

So, f: $\mathbf{D} \to \mathbf{D}$. It is obviously that f is continuous and **D** is convex. Meanwhile, **D** is closed and bounded, so it is compact. So, Brouwer' s fixed point theorem ensures the existence of a fixed point of f, which satisfies $\underset{\sim}{x} = f(\underset{\sim}{x})$.

Then, to prove this fixed point $\underset{\sim}{x}$ is a Nash equilibrium, we need to prove it satisfies:

$$Gain_{p;s_p}(\underset{\sim}{x}) = 0, p \in [2], s_p \in S_p$$

If there is a player p, and

$$Gain_{p;s_p}(\underset{\sim}{x}) > 0$$

If $x_p(s_p) = 0$, the equation $\underset{\sim}{x} = f(\underset{\sim}{x})$ no longer holds. So $0 < x_p(s_p) \leq 1$. Then, because

$$x_p(s_p) > 0$$

$$Gain_{p;s_p}(\underset{\sim}{x}) = u_p(s_p : \underset{\sim}{x}_{-p}) - u_p(\underset{\sim}{x}) > 0$$

$$u_p(s_p : \underset{\sim}{x}_{-p}) > u_p(\underset{\sim}{x})$$

$$u_p(\underset{\sim}{x}) = \sum_{s'_p \in S_p} x_p(s'_p) \cdot u_p(s'_p;\underset{\sim}{x}_{-p})$$

Therefore, there must exists some other pure strategy s'_p which satisfy:

$$x_p(s'_p) > 0$$

$$u_p(s'_p : \underset{\sim}{x}_{-p}) < u_p(\underset{\sim}{x})$$

$$Gain_{p;s'_p}(\underset{\sim}{x}) = 0$$

Then,

$$y_p(s_p') = \frac{x_p(s_p') + Gain_{p;s_p'}(\underset{\sim}{x})}{1 + \sum_{s_p' \in S_p} Gain_{p;s_p'}(\underset{\sim}{x})} < x_p(s_p')$$

Hence, it shows that $\underset{\sim}{x}$ is not a fixed point, which is a contradiction to our assumption.

Therefore, $\underset{\sim}{x}$ is Nash equilibrium, and $\underset{\sim}{x} \in \mathbf{D}$. so $\underset{\sim}{x}$ is a a symmetric NE. In conclusion, any symmetric game (R, C) where $R = C^T$ has a symmetric Nash Equilibrium (x, x).

Problem 4.

1.

Proof. We let x be the sum of yellow-blue edges of all the unit triangles, and N_{ybr} be the number of triangles where the three vertices are yellow, red and blue, N_{ybb} be the number of triangles where the three vertices are yellow, blue and blue, N_{yyb} be the number of triangles where the three vertices are yellow, yellow and blue.

It is obvious that

$$x = N_{ybr} + 2N_{ybb} + 2N_{yyb}$$

On the other hand, we let x'_{yb} be the number of yellow-blue edges of triangles that lie on the boundary. And x''_{yb} be the number of yellow-blue edges of other triangles interior, because one interior edge belongs to two triangles, so

$$x = x'_{yb} + 2x''_{yb}$$

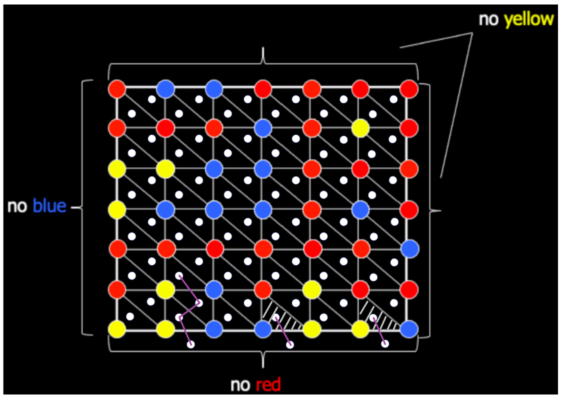
Then, we can prove that x'_{yb} is an odd number. Because only the edge without red dots can have yellow-blue edges, and the end points of this edge are blue and yellow, so it is obvious that the number of color changes is odd. And

$$x'_{yb} + 2x''_{yb} = N_{ybr} + 2N_{ybb} + 2N_{yyb}$$

Hence, N_{ybr} must be an odd number, too. It means $N_{ybr} > 0$. So there exists one tri-chromatic triangle, i.e., a small unit triangle whose nodes are colored by all the three colors.

2.

Proof. In this method, we create a graph on the grid, like:



The nodes of graph are inside each triangles and outside the yellow-blue edges. It is known from proof 1 that there is the number of nodes outside the grid is an odd number. And the edges of graph is added by: starting from the nodes outside the grid, and each time go through a yellow-blue edge, until there is no node it can reach.

As we can see, the sum degree of all nodes is an even number, and the sum degree of nodes outside the grid is an odd number, so the sum degree of nodes inside the grid is an odd number,too. Therefore, inside the grid at least exists one nodes which degree is 1, and this node is in the triangle whose nodes are colored by all the three colors.

Hence, there exists one tri-chromatic triangle, i.e., a small unit triangle whose nodes are colored by all the three colors.