Algorithmic Mechanism Design

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Recap

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- single-parameter environment
- Allocation and Payment
- implementable and monotone
- Myerson's Lemma
- Applications: second-price auctions and sponsored search auctions

理想
"Ideal" mechanisms for general (single-parameter) environments?

NP-hard!

Answer: Social Welfare **approximation**

近似

Definition (Knapsack Auction)

公有 个私有

参与者 物品规模 私有

Each bidder i has a publicly known *size* w_i and a private value. v_i The seller has a capacity W. The feasible set X 可行结果 is the 0-1 vectors $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} w_i x_i \leq W$, where $x_i = 1$ means that i is a winner. 拿到了

例如: 广告, 服务器

- Whenever there is a shared resource with limited capacity, you have a knapsack problem.
- ADs during the Olympic.
- Cloud storage.

单参数环境

Are Knapsack auctions the single-parameter environments? YES!

Next we first assume that the bids are truthful, and then decide the payment.

Welfare-Maximization DSIC

We define the allocation rule by

社会福利最大化

$$\mathbf{x}(\mathbf{b}) = \text{argmax}_X \sum_{i=1}^{n} b_i x_i.$$

诚实 $v_i = b_i$

单调

One can prove that this allocation rule is monotone. By Myerson's Lemma, we can provide the payment rule to make it DSIC. (Why?) But what is the breakpoint? holding other bids $\mathbf{b}_{-i}$ fixed.
The knapsack problem is NP-hard, that is, there is no polynomial-time algorithm for the allocation rule, assuming that P $\neq$ NP.[1] What should we do? SW vs. efficiency?

背包问题 断点

- Relax the efficiency? 有效性×
- Relax the welfare-optimal?

放弃社会福利最大化

[1]FPTAS (fully polynomial time approximation scheme), i.e., $1-\epsilon$-approx in time $\text{poly}(n, 1/\epsilon)$ for any given $\epsilon > 0$.

近似算法

Indeed monotone

证明x(b)

We will show that $\mathbf{x}(\mathbf{b})$ is monotone, recall that

$\mathbf{x}(\mathbf{b}) = \mathrm{argmax}_{x \in X} \sum_{i=1}^{n} b_i x_i$.

Proof.

As the proof of Myerson's Lemma, fix $i \in [n]$, other bids $\mathbf{b}_{-i}$ and $y > z > 0$. Let $\mathbf{x}^y = \mathbf{x}(\mathbf{b}_{-i}, y)$ and $\mathbf{x}^z = \mathbf{x}(\mathbf{b}_{-i}, z)$, we show that $x_i^y \geq x_i^z$. By the defs of $\mathbf{x}$ (SW maximization), we have

只有第i位改成了y

$$\mathbf{x}^y \cdot (\mathbf{b}_{-i}, y) \geq \mathbf{x}^z \cdot (\mathbf{b}_{-i}, y);$$
$$\mathbf{x}^z \cdot (\mathbf{b}_{-i}, z) \geq \mathbf{x}^y \cdot (\mathbf{b}_{-i}, z).$$

Simplifying the above, we have

$$(y - z)(x_i^y - x_i^z) \geq 0$$

□

- One of the first and most well-studied branches of AGT.
- Relax the welfare-optimal condition as little as possible (approximation).
- For single-parameter environments, we only need to design a polynomial-time and monotone allocation rule, due to Myerson's Lemma.

  ①多 ②单 的解
- Approximation algorithm to NP-hard problems.[2]
- While the search space of DSIC is smaller than that of approximation algorithms.
- One question: Does *approximate* welfare-optimal also yields a monotone allocation rule?[3]

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[2] See `https://www.ics.uci.edu/~vazirani/book.pdf` and `https://www.designofapproxalgs.com/book.pdf`.

[3] Actually, the FPTAS algorithm also yields a monotone allocation... but too hard for our lecture.

A Greedy Knapsack Heuristic

1. Sort and re-index the bidders such that
$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.$$
2. Pick winners in this order until one doesn't fit (say i^* is the max index s.t. $\sum_{j=1}^{i} w_i \leq W$), and then halt.
3. Return either the solution from the previous step ($x_i(\mathbf{b}) = 1$ if $i \leq i^*$) or the highest bidder ($x_i(\mathbf{b}) = 1$ if $i = \mathrm{argmax}_{j\in[n]} b_j$), whichever has larger social welfare.

We need the second choice...

$b_1 = 2, b_2 = W$ and $w_1 = 1, w_2 = W$ for sufficient large $W > 0$.

Our Result

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Theorem

$\geq \frac{1}{2}$ opt $\quad \frac{1}{2}$ 近似

This allocation rule is a $1/2$-approx.

Proof.

每个bi 加起后
的 ≥ 社会最大福利

Suppose that $\sum_{i=1}^{i^*} b_i < \texttt{opt}/2$. However, we have $\sum_{i=1}^{i^*+1} b_i \geq \texttt{opt}$.

Hence, we have $\max_{i \in [n]} b_i \geq b_{i^*+1} \geq \texttt{opt}/2$. □

$\sum_{i=1}^{i^*} w_i \leq W$

$\sum_{i=1}^{i^*+1} b_i = SWM \quad \sum_{i=1}^{i^*+1} w_i > W$

Theorem

Our allocation rule is also monotone.

- If $x_i(\mathbf{b}_{-i}, z) = 0$, we have $x_i(\mathbf{b}_{-i}, y) \geq 0$ where $y > z \geq 0$.
- Otherwise, i should be either the highest bidder or $i \leq i^*$. EASY!

The Revelation Principle

揭示性原理

DSIC Revisited

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Until now, we only focus on the DSIC mechanisms.
The reasons for a DSIC mechanism:

- Easy for an agent to operate (bid)
- Predict the outcome, assuming agents are rational ...

How about non-DSIC? like first-price auctions?

The DSIC Condition

估价组合

(1) For every valuation profile, the mechanism has a *dominant-strategy equilibrium* — an outcome that results from every participant playing a dominant strategy.

(2) In this dominant-strategy equilibrium, every participant truthfully reports her private information to the mechanism. (direct revelation) 直接披露

Justifying Direct Revelation

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如果非DSIC 满足(1)不满足(2)

There are mechanisms that satisfy (1) but not (2). Suppose that running the Vickrey auction on the bids ~~2b~~. 2v

The theorem below state that (1) is more important.

Theorem (Revelation Principle for DSIC Mechanisms)

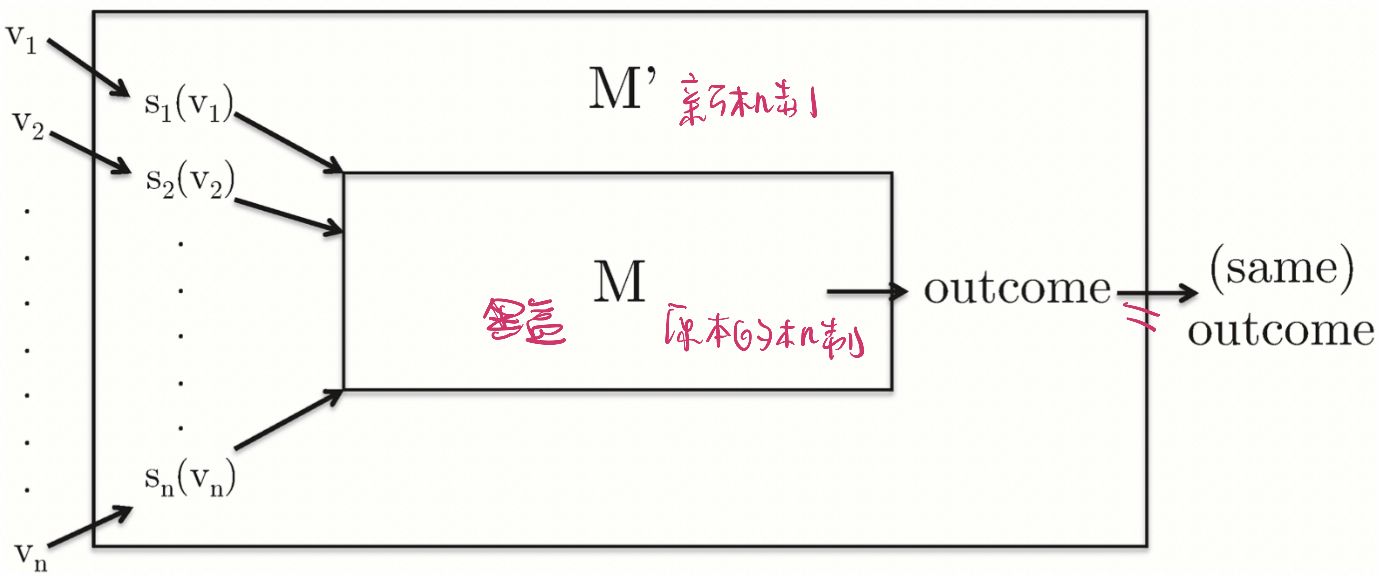
For every mechanism M in which every participant always has a dominant strategy, there is an equivalent direct-revelation *DSIC mechanism M'.*

Proof.

We know that i with the private value v_i has a dominant strategy $s_i(v_i)$ in M. We construct a new mechanism M' with the same outcome with M given the bids.

M' accepts bids $b_1, \ldots, b_n$ from the agents, and submits $s_i(b_i)$ to M.

M' is direct-revelation and DSIC. □



v_1
v_2
v_n
$s_1(v_1)$
$s_2(v_2)$
$s_n(v_n)$
M'
新机制
M
原本的机制
outcome
(same)
outcome

Summary

背包拍卖的社会福利最大化是NP难问题

- Knapsack auctions, SW optimal in this auction is NP-hard
- State-of-the-art approximation algorithms for the welfare maximization problem may or may not induce monotone allocation rules.

useful

不能直接引用

需要调整

- The revelation principle

揭示性原理

好机制→DSIC

它存在更好的

每个人报真实价格

Q&A?