9. 证明: $x_n = \frac{1}{5+10} + \frac{1}{5^2+10} + \cdots + \frac{1}{5^n+10} = \sum_{i=1}^{n} \frac{1}{5^i+10} < \sum_{i=1}^{n} \frac{1}{5^i} = \frac{1-\frac{1}{5^n}}{1-\frac{1}{5}} = \frac{5}{4}\left(1-\frac{1}{5^n}\right) < \frac{4}{5}$

又 $x_n - x_{n-1} = \frac{1}{5^n+10} > 0$

故 $\{x_n\}$ 单调递增且有界

由单调有界定理知 $\{x_n\}$ 极限存在.

习题1-7.

1. (1) 解: $\lim\limits_{x\to 1} \frac{1-x}{1-x^3} = \frac{1}{3}$, 同阶不等价

(2) 解: $\lim\limits_{x\to 1} \frac{1-x}{\frac{1}{2}(1-x^2)} = 1$, 等价

2. (1) 解: $\lim\limits_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim\limits_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$, 同阶

(2) 解: $\lim\limits_{x\to \infty} \frac{\sqrt{x^2+2}-\sqrt{x^2+1}}{\frac{1}{x^2}} = \lim\limits_{x\to \infty} \frac{x^2}{\sqrt{x^2+2}+\sqrt{x^2+1}} = \lim\limits_{x\to \infty} \frac{x}{\sqrt{1+\frac{2}{x^2}}+\sqrt{1+\frac{1}{x^2}}} = \pm\infty$

则 $\frac{1}{x^2} = o(\sqrt{x^2+2}-\sqrt{x^2+1})$

(3) 解: $\lim\limits_{x\to 1} \frac{\frac{1-x}{1+x}}{1-\sqrt{x}} = \lim\limits_{x\to 1} \frac{1+\sqrt{x}}{1+x} = 1$. 等价

(4) 解: $\lim\limits_{x\to 0} \frac{x^2+x^3\sin\frac{1}{x}}{x^2} = \lim\limits_{x\to 0}\left(1+x\sin\frac{1}{x}\right) = 1$ 等价

(5) 解: $\lim\limits_{x\to 0} \frac{\sqrt{x+\sqrt{x}}}{\sqrt[8]{x}} = 0$. 则 $\sqrt{x+\sqrt{x}} = o(\sqrt[8]{x})$

3. (1) $\lim\limits_{x\to 0} \frac{\sqrt{x}+\sin x}{x^{\frac{1}{2}}} = \lim\limits_{x\to 0}(1+\sqrt{x}) = 1$

则 $\sqrt{x}+\sin x$ 为 x 的 $\frac{1}{2}$ 阶无穷小, 即阶为 $\frac{1}{2}$

(2) $\lim\limits_{x\to 0} \frac{x^{\frac{3}{4}}-x^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \lim\limits_{x\to 0}\left(x^{\frac{5}{12}}-1\right) = -1$.

则 $x^{\frac{3}{4}}-x^{\frac{1}{3}}$ 为 x 的 $\frac{1}{3}$ 阶无穷小.

(3) $\lim\limits_{x\to 0} \frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^{\frac{1}{8}}} = \lim\limits_{x\to 0} \sqrt{\frac{x}{\sqrt{x}}+\sqrt{\sqrt{x}+1}} = 1$

则原式为 x 的 $\frac{1}{8}$ 阶无穷小.

(4) $\lim\limits_{x\to 0} \frac{\sqrt[3]{\cos x}-1}{x^2} = \lim\limits_{x\to 0} \frac{-\frac{1}{3}(\cos x)^{-\frac{2}{3}}\sin x}{2x} = \lim\limits_{x\to 0} \frac{(\cos x)^{-\frac{2}{3}}}{-6} = -\frac{1}{6}$.

则原式为 x 的 2 阶无穷小.

(5) 解：由于 $\lim\limits_{x\to0}\frac{\tan x-\sin x}{x^3}=\lim\limits_{x\to0}\frac{\tan x(1-\cos x)}{x^3}=\lim\limits_{x\to0}\frac{\frac{1}{2}x^3}{x^3}=\frac{1}{2}$

所以原式为 x 的3阶无穷小.

(6) 解：由于 $\lim\limits_{x\to0}\frac{\sqrt{1+\tan^3x}-1}{x^3}=\lim\limits_{x\to0}\frac{\tan^3x}{x^3(\sqrt{1+\tan^3x}+1)}=\frac{1}{2}$

所以原式为 x 的3阶无穷小.

(7) 解：由于 $\lim\limits_{x\to0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}=\lim\limits_{x\to0}\frac{\tan x-\sin x}{x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}=\lim\limits_{x\to0}\frac{\frac{1}{2}x^3}{x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}$

$=\frac{1}{4}$

所以原式为 x 的3阶无穷小.

(8) 解：由于 $\lim\limits_{x\to0}\frac{\frac{x(x+1)}{1+\sqrt{x}}}{x}=1$

所以原式为 x 的1阶无穷小，即等价无穷小.

(9) 解：由于 $\lim\limits_{x\to0}\frac{\sqrt{1-\cos x}+\sqrt[3]{x\sin x}}{x^{\frac{2}{3}}}=\lim\limits_{x\to0}\left(\frac{\sqrt{1-\cos x}}{x^{\frac{2}{3}}}+\frac{(x\sin x)^{\frac{1}{3}}}{x^{\frac{2}{3}}}\right)=0+1=1$

所以原式为 x 的等价无穷小.

4. (1) 证明：设 $\alpha=o(x^k)$. 则 $\lim\limits_{x\to0}\frac{\alpha}{x^k}=0$

则 $\lim\limits_{x\to0}\frac{2\alpha}{x^k}=\lim\limits_{x\to0}\left(\frac{\alpha}{x^k}+\frac{\alpha}{x^k}\right)=0$

$\Rightarrow \alpha+\alpha=\alpha$

即 $o(x^k)+o(x^k)=o(x^k)$

(2) 证明：因为 $k<l$. 所以 $\lim\limits_{x\to0}\frac{x^l}{x^k}=0$

所以 $x^l=o(x^k)$

$\lim\limits_{x\to0}\frac{o(x^k)+o(x^l)}{x^k}=\lim\limits_{x\to0}\frac{o(x^k)}{x^k}+\lim\limits_{x\to0}\frac{o(x^k)}{x^k}=0$

所以 $o(x^k)+o(x^l)=o(x^k)$

(3) 证明：$\lim\limits_{x\to0}\frac{o(x^k)o(x^l)}{x^{(k+l)}}=\lim\limits_{x\to0}\frac{o(x^k)}{x^k}\cdot\lim\limits_{x\to0}\frac{o(x^l)}{x^l}=0\times0=0$

所以 $o(x^k)\cdot o(x^l)=o(x^{k+l})$.

5. (1) 证明：因为 $\lim\limits_{x\to0}\dfrac{\arcsin x-x}{x}=\lim\limits_{x\to0}\dfrac{1}{\sqrt{1+x^2}}-1=0$

所以 $\arcsin x=x+o(x)\ (x\to0)$

(2) 证明：因为 $\lim\limits_{x\to0}\dfrac{\tan^2x-x^2}{x^2}=\lim\limits_{x\to0}\left(\dfrac{\tan x}{x}\right)^2-1=0$

$\therefore \tan^2x=x^2+o(x^2)\ (x\to0)$

(3) 证明：$\because \lim\limits_{x\to0}\dfrac{\sqrt[n]{1+x}-1-\frac{1}{n}x}{x}=\lim\limits_{x\to0}\dfrac{\frac{1}{n}(1+x)^{\frac{1-n}{n}}-\frac{1}{n}}{1}=0$

$\therefore \sqrt[n]{1+x}=1+\frac{1}{n}x+o(x)\ (x\to0)$

6. (1) 解：$\lim\limits_{x\to0}\dfrac{\tan2x^2}{\ln(1+3x^2)}=\lim\limits_{x\to0}\dfrac{2x^2}{3x^2}=\dfrac{2}{3}$

(2) 解：$\lim\limits_{x\to0}\dfrac{\sin(x^n)}{(\arcsin x)^m}=\lim\limits_{x\to0}\dfrac{x^n}{x^m}$. 则 $n>m$ 时，极限为 0，$n=m$ 时为 1. $n<m$ 时为 ∞.

即 $\lim\limits_{x\to0}\dfrac{\sin(x^n)}{(\arcsin x)^m}=\begin{cases}0 & n>m\\ 1 & n=m\\ \infty & n<m\end{cases}$.

(3) 解：$\lim\limits_{x\to0}\dfrac{1-\cos mx}{e^{x^2}-1}=\lim\limits_{x\to0}\dfrac{\frac{1}{2}(mx)^2}{x^2}=\dfrac{1}{2}m^2$.

(4) 解：$\lim\limits_{x\to0}\dfrac{\tan x-\sin x}{x\sin^2x}=\lim\limits_{x\to0}\dfrac{\tan x(1-\cos x)}{x\sin^2x}=\lim\limits_{x\to0}\dfrac{\frac{1}{2}x^3}{x^3}=\dfrac{1}{2}$

(5) 解：$\lim\limits_{x\to0}\dfrac{\sin x-\tan x}{(\sqrt[3]{1+x^2}-1)(\sqrt{1+\sin x}-1)}=\lim\limits_{x\to0}\dfrac{\tan x(\cos x-1)}{\frac{1}{3}x^2\cdot\frac{1}{2}x}=\lim\limits_{x\to0}\dfrac{-\frac{1}{2}x^3}{\frac{1}{6}x^3}=-3$

(6) 解：$\lim\limits_{x\to0}\dfrac{\sqrt{1+\tan^2x}-1}{(10^x-1)^2}=\lim\limits_{x\to0}\dfrac{\frac{1}{2}x^2}{x^2\ln^210}=\dfrac{1}{2\ln^210}$

(7) 解：$\lim\limits_{x\to0}\dfrac{5x^2-2(1-\cos^2x)}{6x^3+4\sin^2x}=\lim\limits_{x\to0}\dfrac{5-2\frac{(1+\cos x)(1-\cos x)}{x^2}}{6x+4\frac{\sin^2x}{x^2}}=\dfrac{5-2}{0+4}=\dfrac{3}{4}$

(8) 解：$\lim\limits_{n\to\infty}\dfrac{\tan^3\frac{1}{n}\cdot\arctan\frac{3}{n\sqrt{n}}}{\sin\frac{3}{n^3}\tan\frac{1}{\sqrt{n}}\arcsin\frac{7}{n}}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n^3}\cdot\frac{3}{n\sqrt{n}}}{\frac{3}{n^3}\cdot\frac{1}{\sqrt{n}}\cdot\frac{7}{n}}=\dfrac{1}{7}$