习题 2-3

1. 解：(1) $3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\frac{x^2-y}{x-y^2}$

(2) $y+x\frac{dy}{dx}=e^{x+y}\cdot(1+\frac{dy}{dx})$

$\Rightarrow \frac{dy}{dx}=\frac{y(x-1)}{x(1-y)}$

(3) $\frac{dy}{dx}=-e^{y}-xe^{y}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{-e^y}{1+xe^y}$

(4). $\cos y+x(-\sin y)\frac{dy}{dx}=\cos(x+y)(1+\frac{dy}{dx})$

$\Rightarrow \frac{dy}{dx}=\frac{\cos y-\cos(x+y)}{x\sin y+\cos(x+y)}$

(5) $1+\frac{1}{2\sqrt{xy}}(y+x\frac{dy}{dx})+\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\frac{-2\sqrt{xy}-y}{2\sqrt{xy}+x}$

(6). $-\sin(xy)\cdot(y+x\frac{dy}{dx})=1$

$\Rightarrow \frac{dy}{dx}=\frac{-1-y\sin(xy)}{x\sin(xy)}$

(7) $\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=1+\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{y}\cdot(2\sqrt{x}-1)}{\sqrt{x}(1-2\sqrt{y})}$

(8) $\frac{1}{\sqrt{1+y^2}}\frac{dy}{dx}=e^{x+y}\cdot(1+\frac{dy}{dx})$

$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-y^2}\,e^{x+y}}{1-\sqrt{1-y^2}\,e^{x+y}}$

2. 解：(1) $2x+2y+2x\frac{dy}{dx}-2y\frac{dy}{dx}=2$

$\Rightarrow \frac{dy}{dx}\Big|_{(2,4)}=\frac{5}{2}$.

(2) $\frac{dy}{dx}\cdot e^x+ye^x+\frac{1}{y}\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}\Big|_{(0,1)}=-\frac{1}{2}$

(3) $\frac{dy}{dx}=-\sin x+\frac{1}{2}\cos y\frac{dy}{dx}$

则 $\frac{dy}{dx}\Big|_{(2,4)}=-2$

(4) $\cos(xy)\cdot(y+x\frac{dy}{dx})+\frac{1}{y-x}(\frac{dy}{dx}-1)=1$，$x=0$ 时，$y=1$

则 $\frac{dy}{dx}\Big|_{(0,1)}=1$

(5). $x=0$ 时，$y=1$，又 $e^{y}\cdot\frac{dy}{dx}+x\frac{dy}{dx}+y=0$

则 $\frac{dy}{dx}\Big|_{(0,1)}=-\frac{1}{e}$

3. 证明：设曲线上任一点 (x_0, y_0)　$(x_0, y_0\geq 0)$

则 $\sqrt{x_0}+\sqrt{y_0}=\sqrt{a}$.

又 $\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$.

则 $\frac{dy}{dx}\Big|_{(x_0,y_0)}=-\frac{\sqrt{y_0}}{\sqrt{x_0}}$.

则切线方程为 $y-y_0=\frac{-\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)$.

令 $y=0$. 得 $x=\frac{y_0\sqrt{x_0}}{\sqrt{y_0}}+x_0=\sqrt{x_0y_0}+x_0$

令 $x=0$　得 $y=\sqrt{x_0y_0}+y_0$.

则 $\sqrt{x_0y_0}+x_0+\sqrt{x_0y_0}+y_0=(\sqrt{x_0}+\sqrt{y_0})^2=a$.

即截距之和为 a

证毕.

4. 解：设圆为：$(x-a)^2+(y-b)^2=r^2$.

则两边对 x 求导：$2(x-a)+2(y-b)\frac{dy}{dx}=0$.

因切点为 $(4,2)$. 又直线的斜率为 $-\frac{3}{4}$.

则 $\frac{dy}{dx}\Big|_{(4,2)}=\frac{a-x}{y-b}\Big|_{(4,2)}=\frac{a-4}{2-b}=-\frac{3}{4}$ ①

又圆过 $(-5,5)$. 则 $(-5-a)^2+(5-b)^2=r^2$ ②

且过 $(4,2)$　则 $(4-a)^2+(2-b)^2=r^2$ ③

由①~③得：$a=-5$. $b=-10$. $r=15$

则圆方程为 $(x+5)^2+(y+10)^2=15^2$

5. 解：对于 $y=x^2+ax+b$，有 $y'=2x+a$

对于 $2y=-1+xy^3$，有 $2y'=y^3+3xy^2\cdot y' \Rightarrow y'=\frac{y^3}{2-3xy^2}$

在 $(1,-1)$ 处相切，则：$2+a=\frac{-1}{2-3}=1 \Rightarrow a=-1$

又 $y=x^2+ax+b$ 过 $(1,-1)$，则

$-1=1+a+b=b$

$\Rightarrow \begin{cases} a=-1 \\ b=-1 \end{cases}$.

6. 解：$y=\frac{a}{x}$，则 $y'=-\frac{a}{x^2}$，对两曲线上任意一点 $N(x_0,\frac{a}{x_0})$

其切线为 $y-\frac{a}{x_0}=-\frac{a}{x_0^2}(x-x_0)$ 即：$y=-\frac{a}{x_0^2}x+\frac{2a}{x_0}$.

令 $x=0$，得 $y=\frac{2a}{x_0}$. 令 $y=0$，得 $x=2x_0$.

则与坐标轴两交点为 $(0,\frac{2a}{x_0})$ 和 $(2x_0,0)$

因 $\frac{0+2x_0}{2}=x_0$，$\frac{\frac{2a}{x_0}+0}{2}=\frac{a}{x_0}$

已 pN 位于两交点中点，

即原命题得证.

7. 解：(1) $\ln y=\cos x\ln\sin x$，两边对 x 求导.

$\frac{1}{y}\frac{dy}{dx}=-\sin x\ln\sin x+\cos x\frac{\cos x}{\sin x}$

$\Rightarrow \frac{dy}{dx}=\sin x^{\cos x}(\cos x\cot x-\sin x\ln\sin x)$

(2) $\ln|y|=\ln\left|\frac{(2x+3)^4\sqrt{x-6}}{\sqrt[3]{x+1}}\right|=\ln|(2x+3)^4|+\ln|\sqrt{x-6}|-\ln|\sqrt[3]{x+1}|$

$=4\ln|2x+3|+\frac{1}{2}\ln|x-6|-\frac{1}{3}\ln|x+1|$

两边对 x 求导，得 $\frac{1}{y}y'=\frac{4\times 2}{2x+3}+\frac{1}{2}\times\frac{1}{x-6}-\frac{1}{3}\frac{1}{x+1}$

$\Rightarrow y'=\frac{(2x+3)^4\sqrt{x-6}}{\sqrt[3]{x+1}}\left(\frac{8}{2x+3}+\frac{1}{2x-12}-\frac{1}{3x+3}\right)$

(3) $\ln|y|=x^2\ln|x|+x^x\ln(2)$，两边对 x 求导.

$\frac{1}{y}y'=2x\ln|x|+x+(e^{x\ln|x|}\ln 2)'=2x\ln|x|+x+(\ln|x|+1)e^{x\ln|x|}\ln 2$

$\Rightarrow y'=x^{x^2+1}(1+2\ln x)+2^{x^x}\ln 2\cdot x^x(\ln x+1)$

(4) $\ln|y| = \sqrt{x}\ln|x| + \ln 2$，两边对$x$求导，得：

$$\frac{1}{y}y' = \frac{1}{2\sqrt{x}}\ln|x| + \frac{1}{\sqrt{x}},$$

$$\Rightarrow y' = x^{\sqrt{x}-\frac{1}{2}}(2+\ln x)$$

(5) $\ln|y| = x\ln|\ln x|$，两边对x求导，得：

$$\frac{1}{y}\cdot y' = \ln|\ln x| + x\frac{1}{\ln x}\cdot\frac{1}{x}.$$

$$\Rightarrow y' = (\ln x)^x\left(\frac{1}{\ln x} + \ln\ln x\right)$$

(6) $\ln|y| = \ln\left|\sqrt[3]{\frac{x(x^2+1)}{(x^2-1)^2}}\right| = \frac{1}{3}\ln\left|\frac{x(x^2+1)}{(x^2-1)^2}\right| = \frac{1}{3}\ln|x| + \frac{1}{3}\ln|x^2+1| - \frac{2}{3}\ln|x^2-1|$，两边求导

$$\frac{1}{y}y' = \frac{1}{3}\cdot\frac{2x}{x^2+1} + \frac{1}{3}\frac{1}{x} - \frac{2}{3}\frac{2x}{x^2-1}$$

$$\Rightarrow y' = \frac{x^4+6x^2+1}{3x(1-x^4)}\sqrt[3]{\frac{x(x^2+1)}{(x^2-1)^2}}$$

(7) $\ln|x^y| = \ln|y^x| \Leftrightarrow y\ln|x| = x\ln|y|$，两边求导.

$$y'\ln|x| + \frac{y}{x} = \ln|y| + x\frac{1}{y}y'$$

$$\Rightarrow y' = \frac{y^2-xy\ln y}{x^2-xy\ln x}$$

(8) $\ln|y| = \ln\left(\left|\left(\frac{a}{b}\right)^x\cdot\left(\frac{b}{x}\right)^a\cdot\left(\frac{x}{a}\right)^b\right|\right) = x\ln\left|\frac{a}{b}\right| + a\ln\left|\frac{b}{x}\right| + b\ln\left|\frac{x}{a}\right|$，两边求导.

$$y'\frac{1}{y} = \ln\left(\frac{a}{b}\right) + a\cdot\frac{x}{b}\cdot\left(\frac{-b}{x^2}\right) + b\frac{a}{x}\cdot\frac{1}{a}.$$

$$\Rightarrow y' = \left(\frac{a}{b}\right)^x\left(\frac{b}{x}\right)^a\left(\frac{x}{a}\right)^b\left(\ln\frac{a}{b} + \frac{b-a}{x}\right).$$

(9) $\ln|y| = \ln|x(\sin x)^{x^2}| = \ln|x| + x^2\ln|\sin x|$，两边求导.

$$y'\cdot\frac{1}{y} = \frac{1}{x} + 2x\ln|\sin x| + x^2\cdot\frac{\cos x}{\sin x}$$

$$\Rightarrow y' = x\cdot(\sin x)^{x^2}\cdot\left(\frac{1}{x} + 2x\ln|\sin x| + x^2\cot x\right).$$

(10) $\ln y = \ln\sqrt{e^{\frac{1}{x}}\sqrt{x\sqrt{\sin x}}} = \frac{1}{2}\ln\left(e^{\frac{1}{x}}\sqrt{x\sqrt{\sin x}}\right) = \frac{1}{2}\cdot\frac{1}{x} + \frac{1}{4}\ln x + \frac{1}{8}\ln\sin x$

则 $y'\frac{1}{y} = \frac{1}{2}\frac{-1}{x^2} + \frac{1}{4x} + \frac{\cos x}{8\sin x}$

$$\Rightarrow y' = \frac{1}{8}\sqrt{e^{\frac{1}{x}}\sqrt{x\sqrt{\sin x}}}\left(\frac{2}{x} - \frac{4}{x^4} + \cot x\right).$$

(11) $\ln|y| = \ln|\sqrt[5]{x}\cdot x^{\tan x}| = \frac{1}{5}\ln|x| + \tan x\ln|x|$，两边求导

$$y'\cdot\frac{1}{y} = \frac{1}{5}\cdot\frac{1}{x} + \frac{1}{\cos^2x}\ln|x| + \tan x\cdot\frac{1}{x}.$$

$$\Rightarrow y' = \sqrt[5]{x}\cdot x^{\tan x}\left(\frac{1}{5x} + \sec^2x\cdot\ln x + \frac{\tan x}{x}\right)$$

(12). $\ln|y| = \ln|x^{\frac{1}{y}}| = \frac{1}{y}\ln|x|$，两边求导.

$$y'\cdot\frac{1}{y} = -\frac{1}{y^2}y'\ln|x| + \frac{1}{xy}$$

$$\Rightarrow y' = \frac{y}{x(y+\ln x)}$$

8. (1) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{-2t}$

(2) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-1}{2\sqrt{1-t}}}{\frac{1}{2\sqrt{1+t}}} = -\sqrt{\frac{1+t}{1-t}}$

(3) $\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{a(\cos t-\cos t+t\sin t)}{a(-\sin t+t\cos t+\sin t)} = \tan t.$

(4) $\frac{dy}{dx} = \frac{y'_\theta}{x'_\theta} = \frac{\cos\theta-\theta\sin\theta}{1-\sin\theta+\theta(-\cos\theta)} = \frac{\cos\theta-\theta\sin\theta}{1-\sin\theta-\theta\cos\theta}$

(5) $\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{e^t}{e^{-t}+te^{-t}(-1)} = \frac{e^{2t}}{1-t}$

(6) $\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{1-\frac{1}{1+t^2}}{\frac{1}{1+t^2}\cdot 2t} = \frac{t}{2}$

(7) $\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{e^t\cos t-e^t\sin t}{e^t\sin t+e^t\cos t}$. 则 $\frac{dy}{dx}\Big|_{t=0} = \cancel{\frac{dy}{dx}}\ 1\cdot 0$

(8) $\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{y'_t}{\frac{1}{1+t^2}}$，又对 $2y-ty^2+e^t=5$ 两边求导得：

$$2y'_t - y^2 - 2ty\,y'_t + e^t = 0 \Rightarrow y'_t = \frac{y^2-e^t}{2(1-yt)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(y^2-e^t)(1+t^2)}{2(1-ty)}$$

(9) $\frac{dy}{dx} = \frac{y'_t}{6t+2}$，对 $e^y\sin t-y+1=0$ 两边求导得：

$$e^y\cdot y'_t\sin t + e^y\cos t - y'_t = 0 \Rightarrow y'_t = \frac{e^y\cos t}{1-e^y\sin t}$$

则 $\frac{dy}{dx} = \frac{e^y\cos t}{(1-e^y\sin t)(6t+2)}$. 则 $\frac{dy}{dx}\Big|_{t=0} = \frac{e}{2}$

9. 解：$\frac{dy}{dx} = \frac{y_t'}{x_t'} = \frac{\frac{6at(1+t^2)-2t\cdot 3at^2}{(1+t^2)^2}}{\frac{3a(1+t^2)-2t\cdot 3at}{(1+t^2)^2}} = \frac{6at}{3a-3at^2}$

则在 $t=2$ 处，切线斜率为 $\frac{dy}{dx}\Big|_{t=2} = \frac{12a}{-9a} = \frac{4}{-3}$

又 $t=2$ 时，$x=\frac{6}{5}a$．$y=\frac{12}{5}a$

则切线方程为 $y-\frac{12}{5}a = \frac{4}{-3}(x-\frac{6}{5}a) \Leftrightarrow 4x+3y-12a=0$．

法线方程为 $y-\frac{12}{5}a = \frac{3}{4}(x-\frac{6}{5}a) \Leftrightarrow 3x-4y+6a=0$

10. 解：$\frac{dy}{dx} = \frac{y_t'}{x_t'} = \frac{e^t\cos t - e^t\sin t}{e^t\sin 2t + 2e^t\cos 2t} = \frac{\cos t-\sin t}{\sin 2t+2\cos 2t}$

在 $t=0$ 处．$\frac{dy}{dx}\Big|_{t=0} = \frac{1}{2}$

则法线方程为：$y-1=-2(x-0)$

$\Leftrightarrow y+2x-1=0$

11. 解：参数方程为 $\begin{cases} x=a(1+\cos\theta)\cos\theta \\ y=a(1+\cos\theta)\sin\theta \end{cases}$．

$\frac{dy}{dx} = \frac{\cos 2\theta+\cos\theta}{\sin 2\theta+\sin\theta}$．

~~因为t~~ 令 $\tan\alpha = \frac{\cos 2\theta+\cos\theta}{\sin 2\theta+\sin\theta}$

$\Rightarrow \alpha = \frac{\pi}{2}+\frac{3\theta}{2}$．

12. 解：$\rho=\sqrt{2a^2\cos 2\theta}$，则参数方程为 $\begin{cases} x=\sqrt{2a^2\cos 2\theta}\,\cos\theta \\ y=\sqrt{2a^2\cos 2\theta}\,\sin\theta \end{cases}$．

$$\frac{dy}{dx} = \frac{\frac{-2a^2\sin 2\theta\cdot 2}{2\sqrt{2a^2\cos 2\theta}}\sin\theta + \sqrt{2a^2\cos 2\theta}\cos\theta}{\frac{-2a^2\sin 2\theta\cdot 2}{2\sqrt{2a^2\cos 2\theta}}\cos\theta - \sqrt{2a^2\cos 2\theta}\sin\theta} = \frac{-2a^2\sin 2\theta\sin\theta + 2a^2\cos 2\theta\cos\theta}{-2a^2\sin 2\theta\cos\theta - 2a^2\cos 2\theta\sin\theta}$$

$$= \frac{-\sin 2\theta\sin\theta+\cos 2\theta\cos\theta}{-\sin 2\theta\cos\theta-\cos 2\theta\sin\theta} = \frac{\cos 3\theta}{-\sin 3\theta}.$$

当 $\theta=\frac{\pi}{12}$ 时．$\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{12}} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$．

13. 解：$y'=2x$，则 $\frac{dy}{dx}\Big|_{x=2} = 4$．

又 $\frac{dx}{dt}=3$．

则 $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = 4\times 3 = 12\,cm/s$．

14、解：$\tan\theta=\frac{140t}{500}$，则 $\theta=\arctan\frac{7}{25}t$.

则 $\frac{d\theta}{dt}=\frac{\frac{7}{25}}{1+(\frac{7}{25}t)^2}=\frac{175}{625+49t^2}$

当升高 500m 时. $t=\frac{500}{140}=\frac{25}{7}$ min.

则 $\frac{d\theta}{dt}\Big|_{t=\frac{25}{7}}=0.14$ 弧度/min.

15、解：$\tan\theta=\frac{2}{\alpha-200t}$ （α为角速度），$\theta=\arctan\frac{2}{\alpha-200t}$

则 $\frac{d\theta}{dt}=\frac{1}{1+(\frac{2}{\alpha-200t})^2}\cdot\frac{400}{(\alpha-200t)^2}=\frac{400}{(\alpha-200t)^2+4}=100km/h$

$\Rightarrow \alpha=\frac{5}{\pi}$ 弧度/s.

16、解：设注入水 t(min)后，水深为 h(m)，由几何知识得水面半径为 $\frac{2}{5}h$ (m).

这时水体积为 $V=\frac{1}{3}\pi\cdot(\frac{2}{5}h)^2h=\frac{4\pi}{75}h^3$，设 $h=h(t)$.

因此可得水的体积关于时间的导数为：

$V'_t=V'_h\cdot h'_t=(\frac{4\pi}{75}h^3)'\cdot h'_t=\frac{4\pi}{25}h^2\cdot h'_t$.

由假设，注水速度为 $5\ m^3/min$.

$5t=\frac{4\pi}{75}\cdot h^3$，则 $5=\frac{4\pi}{25}h^2\cdot h'_t$.

当 $h=5m$ 时，$h'_t=\frac{5}{4\pi}m/min$.

17、解：分针每分钟转 $\frac{2\pi}{60}$ 度，时针每分钟转 $\frac{2\pi}{60\times12}$ 度，假定时针不动，

则分针每分钟转 $\frac{11\pi}{30\times12}$ 度，而令逆时针方向转为正，则 2点时 ($t=0$).

$d^2=10^2+6^2-2\times10\times6\cos(\frac{\pi}{3}-\frac{11\pi t}{30\times12})$.

$\Rightarrow d'_t\big|_{t=0}=-0.0102mm/s$.

18、解：半径增大率 v，最外一圈波半径为 $r=vt$. 最外一圈波覆盖的面积为 $S=\pi r^2$

扰动水面面积的速率为：$\frac{ds}{dt}=2\pi r\cdot\frac{dr}{dt}=2\pi rv=2\pi tv^2=2\pi\times2\times6^2=144\pi\ m^2/s$

19.解：假设直升机在空中静止，汽车相对速度为V，则直升机与汽车之间的距离为$\sqrt{3^2+(4-Vt)^2}$，距离用D表示，则D对时间的导数

$$D'_t=\frac{(V^2t-4V)\sqrt{V^2t^2-8Vt+25}}{V^2t^2-8Vt+25}，（实际上D=\frac{V^2t^2-8Vt+25}{\sqrt{V^2t^2-8Vt+25}}）$$

已知直升机与汽车间距离以160km/h减小。

即$t=0$时，$D'_t=-160$，代入得

∴ $V=200km/h$。

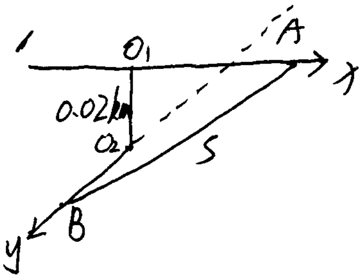
由于实际上直升机每小时120km速度前进。

则汽车速度为$200-120=80km/h$。

20.解：设$t=0$时刻人与船在x与y轴上坐标位置为O_1、O_2。

t时刻人与船分别在x与y轴上的点为A和B。

则A、B坐标为$x=4t$，$y=8t$。



此时人与船之间距离为$S=\sqrt{x^2+y^2+0.02^2}$

$$\frac{ds}{dt}=\frac{1}{2\sqrt{x^2+y^2+0.02^2}}\cdot\frac{d}{dt}(x^2+y^2+0.02^2)=\frac{2x\frac{dx}{dt}+2y\frac{dy}{dt}}{2\sqrt{x^2+y^2+0.02^2}}$$

当$t=\frac{1}{20}$小时，$x=\frac{1}{5}km$，$y=\frac{2}{5}km$。

$$\therefore \left.\frac{ds}{dt}\right|_{t=\frac{1}{20}}=2\sqrt{15}\ km/h$$

21.解：$\cos\theta=\frac{3^2+x^2-4^2}{2\cdot 3x}=\frac{x^2-7}{6x}$

$\Leftrightarrow x^2-7-6x\cos\theta=0$，两边求导

$2x\cdot x'-6(x'\cos\theta-x\sin\theta\,\theta')=0$　令$x'=\frac{dx}{dt}$，$\theta'=\frac{d\theta}{dt}$。

$\Rightarrow (x-3\cos\theta)\frac{dx}{dt}+3x\sin\theta\frac{d\theta}{dt}=0$