Revenue-Maximizing Auctions

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Recap

- Virtual Valuations
- Expected revenue equals expected virtual welfare
- Regular distributions
- revenue-maximization in single-item auction: second-price auction with a reserve price $\phi^{-1}(0)$.

Pursues approximately optimal mechanisms that are simpler, more practical, and more robust than the theoretically optimal mechanism.

Optimal Auction Are Complex

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- Consider only DSIC mechanisms, that is $\mathbf{b} = \mathbf{v}$.
- The optimal single-item auction with i.i.d. bidders and a regular distributions is simple.
- How about different regular distributions? it can be very complex .. the winner may be not the highest bidder ..
- One more "intuitive" and "simpler" way is to give the item the highest bidder!
- Relax the optimal condition, can we do approximately but easily?

- Consider the following game with n stages
- In stage i, you are offered a non-negative prize π_i from a pre-defined distribution G_i.
- After seeing π_i, you can do one of the following:
  1. accept the prize and end the game
  2. discard the prize and proceed to the next stage.
- Which option will you do?

Theorem (Prophet Inequality)

For every $G_1, \ldots, G_n$ of independent distributions, there is a strategy that guarantees expected reward at least $\frac{1}{2}\mathsf{E}_{\pi\sim\mathbf{G}}\left[\max_i \pi_i\right]$. Moreover, there is such a threshold strategy, which accepts prize i iff π_i is at least some threshold t.

We will show that $\mathsf{E}_{\pi\sim\mathbf{G}}\left[\text{Payoff of the } t\text{-threshold strategy}\right] \geq \frac{1}{2}\text{OPT}$.

The Proof

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- Let z^+ denote $\max\{z, 0\}$.
- Consider a threshold strategy with threshold t. How to compare the expected payoff of this strategy with the expected payoff of a prophet?
- Let $q(t)$ denote the prob. that the threshold strategy accepts no prize at all.

We try to lower bound the t-threshold strategy ..

$$
\begin{aligned}
&(1 - q(t)) \cdot t + \\
&\sum_{i=1}^{n} \underbrace{\mathsf{E}_\pi \left[\pi_i - t \mid \pi_i \geq t, \pi_j < t \; \forall j \neq t\right] \Pr\left[\pi_i \geq t\right]}_{=\mathsf{E}[(\pi_i - t)^+]} \underbrace{\Pr\left[\pi_j < t \; \forall j \neq i\right]}_{\geq q(t)} \\
&\geq (1 - q(t) \cdot t + q(t) \sum_{i=1}^{n} \mathsf{E}_\pi \left[(\pi_i - t)^+\right].
\end{aligned}
$$

Proof II

We produce an upper bound on the prophet's expected payoff that is easy to compare to the lower bound of t-threshold strategy ..

$$
\begin{aligned}
\mathsf{E}_{\pi}\left[\max_{i\in[n]} \pi_i\right] &= \mathsf{E}_{\pi}\left[t + \max_{i\in[n]} (\pi_i - t)\right] \\
&\le t + \mathsf{E}_{\pi}\left[\max_{i\in[n]} (\pi_i - t)^+\right] \\
&\le t + \sum_{i=1}^{n} \mathsf{E}_{\pi}\left[(\pi_i - t)^+\right].
\end{aligned}
$$

The theorem holds by setting t such that $q(t) = \frac{1}{2}$. $\square$

The guarantee still holds for the strategy even if, whenever there are multiple prizes above the threshold, it somehow always picks the smallest of those.

- The regular distributions are not identical .. Let's use the prophet's inequality!
- define the i-th prize as the positive part $\varphi_i(v_i)^+$ of bidder i's virtual valuation; G_i is the corresponding distribution induced by F_i.
- We have

$$\mathsf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^{n}\varphi_i(v_i)x_i(\mathbf{v})\right]=\mathsf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\max_{i\in[n]}\varphi_i(v_i)^+\right].$$

Virtual Threshold Allocation Rule

1. Choose t such that $\Pr\left[\max_i \varphi_i(v_i)^+ \geq t\right]=\frac{1}{2}$.
2. Give the item to a bidder i with $\varphi_i(v_i)\geq t$, if any, breaking ties among multiple candidate winners arbitrarily.

Second-Price with Bidder-Specific Reserves

1. Set a reserve price $r_i = \varphi_i^{-1}(t)$ for each bidder i, with t defined as for virtual threshold allocation rules.
2. Give the item to the highest bidder that meets her reserve, if any.

Theorem (Simple Versus Optimal Auctions)

For all $n \geq 1$ and regular distributions $F_1, \ldots, F_n$, the expected revenue of a second-price auction with suitable reserve prices is at least 50% of that of the optimal auction.

Prior-Independent Mechanisms

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- When the regular distributions are unknown to the mechanism designer?
- We can continue to use distributions in the *analysis* of the mechanism, but not in their *design*..

It is better to invest your resources to recruit more serious participants than to sharpen your knowledge of their preferences .. Of course, do both if you can!

Theorem (Bulow-Klemperer Theorem)

Let F be a regular distribution and n be a positive integer. Let $\mathbf{p}$ and $\mathbf{p}^*$ denote the payment rules of the second-price auction with $n+1$ bidders and the optimal auction (for F) with n bidders, respectively. Then

$$\mathsf{E}_{\mathbf{v}\sim F^{n+1}}\left[\sum_{i=1}^{n+1} p_i(\mathbf{v})\right] \geq \mathsf{E}_{\mathbf{v}\sim F^{n}}\left[\sum_{i=1}^{n} p_i^*(\mathbf{v})\right]. \tag{1}$$

Proof of BK Theorem

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We use the following auction to facilitate the comparison.

The Fictitious Auction $\mathcal{A}$

1. Simulate an optimal n-bidder auction for F on the first n bidders $1, 2, \ldots, n$.
2. If the item was not awarded in the first step, then give the item to bidder $n + 1$ for free.

- It has the same expected revenue with that of an optimal auction with n bidders.
- It always allocates the item.

We can show

When bidders' valuations are drawn i.i.d. from a regular distribution, the second-price auction maximizes the expected revenue over all DSIC auctions that always allocate the item.

- Expected revenue equals the expected virtual welfare.
- If we are forced to sell the item, what is the best we can do?
- When F is regular, we should give the item to the highest bidder.
- The price (by Myerson) is the second highest bid.

Summary

- Regular distributions can be complex ..
- The prophet's inequality
- $\Rightarrow \frac{1}{2}$-approx auction with bidder-specific reserve prices.
- BK Theorem

Q&A?