#### **Combinatorial Auctions**

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#### Recap



- Regular distributions can be complex ..
- The prophet's inequality テルミラスギュ
- $\Rightarrow \frac{1}{2}$ -approx auction with bidder-specific reserve prices.
- BK Theorem. By i.i.d. regular distributions, we can show that the second-price auction maxs the expected revenue over all DSIC auctions that always allocate the item.

Much more difficult for Multi-Parameter problems.

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# General Mechanism Design Environme



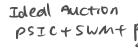
- n strategic participants, or "agents";
- a finite set  $\Omega$  of outcomes;  $\varphi_{\text{phy}}$
- each agent i has a private non-negative valuation  $v_i(\omega)$  for each outcome  $\omega \in \Omega$ .

The outcome set  $\Omega$  is abstract and could be very large. The social welfare of an outcome  $\omega \in \Omega$  is defined as  $\sum_{i=1}^{n} v_i(\omega)$ .

- Single-Item Auction へっして、で、で、は) マルが動
- Combinatorial Auctions: there are multiple indivisible items for sale, and bidders can have complex preferences between different subsets of items (called bundle). With n bidders and a set M of m items, the outcomes of  $\Omega$  correspond to n-vector  $(S_1, S_2, \ldots, S_n)$ , with  $S_i \subseteq M$ .

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# Theorem (The VCG Theorem)

In every general mechanism design environment, there is a <u>DSIC</u> welfare-maximizing mechanism, which is VCG mechanism.

- DSIC mechanism is tricky, we need affocation and payment rules to be coupled carefully.
- The first step is to assume, with justification, that agents truthfully report their private information, and then figure out which outcome to pick. We define the allocation rule x by

$$\mathbf{x}(\mathbf{b}) = \operatorname{argmax}_{\omega \in \Omega} \sum_{i=1}^{n} b_i(\omega).$$
 (1)

• The second step is to define a payment rule. Myerson 's Lemma doesn't hold beyond single-parameter environments. Even not clear how to define "monotonicity" of an allocation rule.

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## Go back to Vickrey Auctions



- How much does player i hurt everyone else by participating in the auctions?
- If i does not get the item, then it doesn't hurt anyone. The only difference is that the winner might be charged more money.
- if *i* does get the item, then it was the highest bid. So everyone else has social welfare 0. But if *i* didn't join, the item would have gone to the bidder with the highest bidder other than *i*. So the "total harm" caused by player *i* is the second highest price, which is the price defined in the auction.

"Charging an agent her externality" remains well defined in general mechanism design environments, and corresponds to the payment rule

$$p_{i}(\mathbf{b}) = \left(\max_{\omega \in \Omega} \sum_{j \neq i} b_{j}(\omega) - \sum_{j \neq i} b_{j}(\underline{\omega}^{*}), \right)$$

$$\text{where } \omega^{*} = \mathbf{x}(\mathbf{b}) \text{ is the outcome chosen from the equation in Eq (1).}$$

 $p_i(\mathbf{b})$  is non-negative. Why?

#### Definition (VCG Mechanism)

A mechanism  $(\mathbf{x}, \mathbf{p})$  with allocation and payment rules, respectively, is a Vickrey-Clarke-Groves or VCG mechanism.

An alternative interpretation:

$$p_i(\mathbf{b}) = \underbrace{b_i(\omega^*)}_{\text{bid}} - \underbrace{\left[\sum_{j=1}^n b_j(\omega^*) - \max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)\right]}_{\text{rebate}}.$$

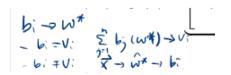
#### Proof of the VCG Theorem



Fix i and  $\mathbf{b}_{-i}$ , when the chosen outcome  $\mathbf{x}(\mathbf{b})$  is  $\omega^*$ , we write i's utility as

$$v_i(\omega^*) - p_i(\mathbf{b}) = \underbrace{\left[v_i(\omega^*) + \sum_{j \neq i} b_j(\omega^*)\right]}_{(\mathsf{A})} - \underbrace{\left[\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)\right]}_{(\mathsf{B})}.$$

Term (B) is constant, when fixing  $\mathbf{b}_{-i}$ . We only need to optimize the first term (A), which is maximized when  $b_i = v_i$ .  $\square$ 



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#### **Practical Considerations**



- Getting the reports  $\mathbf{b}_1,\dots,\mathbf{b}_n$  from the agents.  $2^m$  parameters per bidder, a thousand when m=10 and a million when m=20..
- Recall knapsack auctions, and in more complex settings even approximate welfare maximization can be computationally intractable.
- VCG can have bad revenue and incentive properties, despite being DSIC.
- three bidders and two items, A and B.
- $v_1(AB) = 1$ ,  $v_2(AB) = v_2(A) = 1$  and  $v_3(AB) = v_3(B) = 1$ .
- The maximum welfare is two, but the VCG revenue is zero ...

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# Single-Minded Bidders



• Each bidder has a (private) set  $T_i \in M$  and there is a private parameter  $v_i \in \mathbb{R}^+$  s.t.

$$v_i(S) = \begin{cases} v_i & \text{if } S \supseteq T_i \\ 0 & \text{otherwise} \end{cases}$$
   
 • A bid in this context is a pair  $(b_i, S_i)$ .

- Next we will talk about the greedy mechanism ..

#### The Greedy Mechanism



Sort the bidders s.t.

$$\frac{\text{filts}}{\text{kontitues of }\sqrt{|S_1|}} \geq \frac{b_2}{\sqrt{|S_2|}} \geq \cdots \frac{b_n}{\sqrt{|S_n|}}.$$

- $W=\emptyset$  (W is the set of agents which can get their  $S_i$ 's.)
- For i from 1 to n do: if  $S_i \cap (\bigcup_{i \in W} S_i) = \emptyset$ , add i to W.
- Return the allocation which gives  $S_i$  to player i iff  $i \in W$ .

We'll set  $p_i=0$  if  $i\not\in W$ . If  $i\in W$ , let  $\alpha(i)$  be the minimum index such that  $S_i\cap S_{\alpha(i)}\neq\emptyset$  and  $S_k\cap S_{\alpha(i)}=\emptyset$  for all  $k<\alpha(i)$  with  $k\neq i$  and  $k\in W$ . That is,  $\alpha(i)$  is the first player who lost due to bidder i: if i had not been participating then  $\alpha(i)$  would have been in W. If no such  $\alpha(i)$  exists, then we set  $p_i=0$ . Otherwise, we set

$$p_i = \frac{b_{\alpha(i)}}{\sqrt{|S_{\alpha(i)}|/|S_i|}} = b_{\alpha(i)}\sqrt{\frac{|S_i|}{|S_{\alpha(i)}|}}$$

#### An Example



- Another way to say for the payment rule: Charge each winner the minimum  $b_j$  that they could have reported and still been a winner for  $S_i$ .
- e.g., bids are  $(1,\{1\}),(0,\{1,2,3,4\}),(4,\{1,2\}),(4,\{3,4\}).$
- $(4,\{1,2\})$  will pay  $\sqrt{2}$  and  $(4,\{3,4\})$  will pay 0.



## The Analysis



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- The mechanism is polynomial time and outputs a valid allocation.
- ullet Incentive Compatibility: monotonicity and critical payment  ${\mathcal L}$

Social Welfare Approximately.

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## **Incentive Compatibility**



- Monotonicity: increasing  $b_i$  or decreasing  $S_i$  can only move i earlier in the greedy ordering.
- Critical Payment: if i wins then the price she pays is the smallest x such that i would still win if she had bid  $(x, S_i)$ . When  $\alpha(i)$  doesn't exists. Then there is no other bidder who fails to win due to bidder i. So she is charged 0, the critical payment.

Suppose  $\alpha(i)$  does exist. Then i will still win as long as it appears before  $\alpha(i)$  in the ordering. Thus the critical payment is x such that  $\frac{x}{\sqrt{|S_i|}} = \frac{b_{\alpha(i)}}{\sqrt{|S_{\alpha(i)}|}}$ , so we have the critical payment is

$$b_{lpha(i)}\sqrt{rac{|S_i|}{|S_{lpha(i)}|}}=p_i. \le b_i$$

## **Both Properties are Enough**



#### Theorem

Any mechanism where losers pay 0 which has both the monotonicity and critical payment properties is incentive compatible.

#### Proof.

Fix player  $i \in [n]$  and all bids other than i's. Let u(b,S) be the utility that player i would get by bidding (b,S), so  $u(b,S)=v_i(S)-p_i(b,S)$ . By the critical payment, we know  $p_i(b,S)=\inf\{x: i \text{ wins with bid } (x,S)\}$ .

- non-negative utility: 0 or  $v_i p_i(b, T_i) \ge 0$ .
- dominant strategy:  $u(v_i, T_i) \ge u(b, T_i) \ge u(b, S)$  for all (b, S) with  $T_i \subseteq S$  and (b, S) is a winning bid. (Why?)

#### **Detailed Proof**



- Easy when (b, S) is a losing bid, since non-negative utility
- Obvious if  $T_i \not\subseteq S$ , since receiving a bundle valued zero
- So for the first part, we just need to show  $p_i(b, T_i) \leq p_i(b, S)$ . Holds by the monotonicity property and  $T_i \subseteq S$ .
- $u(v_i, T_i) = u(b, T_i)$  when  $(v_i, T_i)$  is a winning bid, since  $(b, T_i)$  is a winning bid;  $p_i(b, T_i) \ge v_i$  if  $(v_i, T_i)$  is not a winning bid.

#### Social Welfare



Let OPT be the winners in an optimal allocation, and W be ours.

#### Theorem

$$\sum_{i \in \mathsf{OPT}} v_i \leq \underbrace{\sqrt{m}}_{i \in W} \underbrace{\sum_{i \in W} v_i}_{v_i}.$$

Let  $\mathsf{OPT}_i = \{j \in \mathsf{OPT} \mid j \geq i \land T_i \cap T_j \neq \emptyset\}$ . Note that  $\mathsf{OPT} = \bigcup_{i \in W} \mathsf{OPT}_i$ . (Why?)

$$\sum_{j \in \mathsf{OPT}_i} v_j \leq \frac{v_i}{\sqrt{|T_i|}} \sum_{j \in \mathsf{OPT}_i} \sqrt{|T_j|} \leq \frac{v_i}{\sqrt{|T_i|}} \sqrt{|\mathsf{OPT}_i|} \sqrt{\sum_{j \in \mathsf{OPT}_i} |T_j|}.$$

We have  $|\mathsf{OPT}_i| \leq |T_i|$  since  $T_i \cap T_{i'} = \emptyset$  (They are both in OPT!).

$$\sum_{i \in \mathsf{OPT}} v_i \le \sum_{i \in W} \sum_{j \in \mathsf{OPT}_i} v_j \le \sum_{i \in W} v_i \sqrt{m} = \sqrt{m} \sum_{i \in W} v_i.$$

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# **Summary**



- Combinatorial auctions are an important example of general MD environments.
- VCG mechanism is DSIC and social welfare maximized.
- VCG has kinds of practical issues.
- CAs with Single-minded Bidders, and its greedy mechanism.
- OPEN Improved truthful mechanisms for subadditive combinatorial auctions: Breaking the logarithmic barrier [SODA'21]

# Q&A?