习题3-2

1. 解：不正确，因 $\lim\limits_{x\to0}x^2+1=1$，$\lim\limits_{x\to0}x-1=-1$，则 $\lim\limits_{x\to0}\frac{x^2+1}{x-1}$ 并不能

转化为 $\frac{0}{0}$ 和 $\frac{\infty}{\infty}$ 形的不定式，不满足洛必达法则条件，

解法错误。

2. 解：(1) $\lim\limits_{x\to0}\frac{e^x-e^{-x}}{\sin x}=\lim\limits_{x\to0}\frac{e^x+e^x}{\cos x}=\frac{2}{1}=2$

(2) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\ln\sin x}{(\pi-2x)^2}=\lim\limits_{x\to\frac{\pi}{2}}\frac{\frac{1}{\sin x}\cdot\cos x}{8x-4\pi}=\lim\limits_{x\to\frac{\pi}{2}}\frac{-\frac{1}{\sin^2x}}{8}=-\frac{1}{8}$

(3) $\lim\limits_{x\to0}\frac{x-\arcsin x}{\sin x^3}=\lim\limits_{x\to0}\frac{1-\frac{1}{\sqrt{1-x^2}}}{\cos x^3\cdot3x^2}=\lim\limits_{x\to0}\frac{(1-x^2)^{-\frac{3}{2}}(-x)}{6x}=-\frac{1}{6}$

(4) $\lim\limits_{x\to0}\frac{\tan x-x}{x\ln(1+x)}=\lim\limits_{x\to0}\frac{\tan x-x}{x^3}=\lim\limits_{x\to0}\frac{\frac{1}{\cos^2x}-1}{3x^2}=\lim\limits_{x\to0}\frac{\frac{2\cos x\sin x}{\cos^4x}}{6x}$

$=\lim\limits_{x\to0}\frac{\frac{2\cos x\cos^3x+3\cos^2x\sin x2\cos x}{\cos^6x}}{6}=\frac{2}{\cancel{6}}=\frac{1}{3}$

(5) $\lim\limits_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\lim\limits_{x\to a}\frac{mx^{m-1}}{nx^{n-1}}=\frac{m}{n}a^{m-n}$

(6) $\lim\limits_{x\to0^+}\frac{\ln\tan7x}{\ln\tan2x}=\lim\limits_{x\to0^+}\frac{\frac{1}{\tan7x}\cdot\frac{7}{\cos^27x}}{\frac{1}{\tan2x}\cdot\frac{2}{\cos^22x}}=\lim\limits_{x\to0^+}\left(\frac{2}{7}\cdot\frac{7}{2}\frac{\cos^22x}{\cos^27x}\right)=1$

(7) $\lim\limits_{x\to a}\frac{a^x-x^a}{x-a}=\lim\limits_{x\to a}\frac{a^x\ln a-ax^{a-1}}{1}=a^a\ln a-a^a$

(8) $\lim\limits_{x\to1^+}\frac{\ln(x-1)-x}{\tan\frac{\pi}{2x}}=\lim\limits_{x\to1^+}\frac{\frac{1}{x-1}-1}{\frac{1}{\cos^2\frac{\pi}{2x}}\cdot\frac{-\pi}{2x^2}}=\lim\limits_{x\to1^+}\frac{(2-x)\cos^2\frac{\pi}{2}x\cdot2x^2}{(x-1)(-\pi)}$

$=\lim\limits_{x\to1^+}\frac{(8x-6x^2)\cos^2\frac{\pi}{2}x-(8x^2-4x^3)\cos\frac{\pi}{2}x\sin\frac{\pi}{2}x}{-\pi}$

$=0$

(9) $\lim\limits_{x\to+\infty}\frac{\ln(a+be^x)}{\sqrt{a+bx^2}}=\lim\limits_{x\to+\infty}\frac{\ln(be^x)}{\sqrt{b}x}=\lim\limits_{x\to+\infty}\frac{\ln b+x}{\sqrt{b}x}=\lim\limits_{x\to+\infty}\frac{1}{\sqrt{b}}=\frac{1}{\sqrt{b}}$

(10) 解: $\lim\limits_{x\to+\infty}\frac{(\ln x)^n}{x} = \lim\limits_{x\to+\infty}\frac{n(\ln x)^{n-1}\frac{1}{x}}{1} = \lim\limits_{x\to+\infty}\frac{n(n-1)\cdot(\ln x)^{n-2}}{x} = \cdots = \lim\limits_{x\to+\infty}\frac{n!}{x} = 0$

(11) $\lim\limits_{x\to0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim\limits_{x\to0}\frac{e^x-1-x}{x(e^x-1)} = \lim\limits_{x\to0}\frac{e^x-1}{e^x-1+xe^x} = \lim\limits_{x\to0}\frac{e^x}{2e^x+xe^x} = \frac{1}{2}$

(12) $\lim\limits_{x\to1^+}(x-1)\tan\frac{\pi}{2}x = \lim\limits_{x\to1^+}\frac{x-1}{\frac{1}{\tan\frac{\pi}{2}x}} = \lim\limits_{x\to1^+}\frac{1}{-\frac{1}{\sin^2\frac{\pi}{2}x}\cdot\frac{\pi}{2}} = -\frac{2}{\pi}$

(13) $\lim\limits_{x\to0}\left(\cot^2x-\frac{1}{x^2}\right) = \lim\limits_{x\to0}\frac{x^2\cos^2x-\sin^2x}{x^2\sin^2x} = \lim\limits_{x\to0}\frac{x^2\cos^2x-\sin^2x}{x^4}\cdot\frac{x^2}{\sin^2x}$

$= \lim\limits_{x\to0}\frac{x^2\cos^2x-\sin^2x}{x^4}\times\lim\limits_{x\to0}\frac{x^2}{\sin^2x} = \lim\limits_{x\to0}\frac{-2\sin x\cos x+2x\cos^2x-2x^2\cos x\sin x}{4x^3}$

$= \lim\limits_{x\to0}\frac{-\sin x+x\cos x-x^2\sin x}{2x^3} = \lim\limits_{x\to0}\frac{-\cos x+\cos x-x\sin x-2x\sin x-x^2\cos x}{6x^2}$

$= \lim\limits_{x\to0}\frac{-3\sin x-x\cos x}{6x} = \lim\limits_{x\to0}\frac{-3\cos x-\cos x+x\sin x}{6} = -\frac{2}{3}$

(14). $\lim\limits_{x\to0}\cot x\ln\frac{1+x}{1-x} = \lim\limits_{x\to0}\frac{\ln(1+x)-\ln(1-x)}{\frac{\sin x}{\cos x}} = \lim\limits_{x\to0}\frac{\frac{1}{1+x}+\frac{1}{1-x}}{\frac{1}{\cos^2x}} = \frac{1+1}{1} = 2.$

(15) $\lim\limits_{x\to\pi}(1-\tan\frac{x}{4})\sec\frac{x}{2} = \lim\limits_{x\to\pi}\frac{\frac{-1}{4\cos^2\frac{x}{4}}}{-\frac{1}{2}\sin\frac{x}{2}} = \frac{-\frac{1}{2}}{-\frac{1}{2}} = 1$

(16) $\lim\limits_{x\to1}\left(\frac{2}{x^2-1}-\frac{1}{x-1}\right) = \lim\limits_{x\to1}\left(\frac{-1}{x+1}\right) = -\frac{1}{2}$

(17) $\lim\limits_{x\to1^+}\ln x\ln(x-1) = \lim\limits_{x\to1^+}\frac{\ln(x-1)}{\frac{1}{\ln x}} = \lim\limits_{x\to1^+}\frac{\frac{1}{x-1}}{\frac{-1}{x\ln^2x}} = \lim\limits_{x\to1^+}\frac{x\ln^2x}{1-x} =$

$= \lim\limits_{x\to1^+}\frac{\ln^2x+2x\ln x\cdot\frac{1}{x}}{-1} = \lim\limits_{x\to1^+}-(\ln^2x+2\ln x) = 0$

(18) $\lim\limits_{x\to0}\left(\frac{1}{x^2}-\frac{1}{\sin^2x}\right) = \lim\limits_{x\to0}\frac{\sin^2x-x^2}{x^2\sin^2x} = \lim\limits_{x\to0}\frac{\sin^2x-x^2}{x^4} = \lim\limits_{x\to0}\frac{x+\sin x}{x}\cdot\lim\limits_{x\to0}\frac{\sin x-x}{x^3}$

$= 2\lim\limits_{x\to0}\frac{\sin x-x}{x^3} = 2\lim\limits_{x\to0}\frac{\cos x-1}{3x^2} = \frac{2}{3}\lim\limits_{x\to0}\frac{-\frac{1}{2}x^2}{x^2} = -\frac{1}{3}$

(19) $\lim\limits_{x\to\frac{\pi}{2}}(\sec x-\tan x) = \lim\limits_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right) = \lim\limits_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x} = \lim\limits_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x} = \lim\limits_{x\to\frac{\pi}{2}}\cot x = 0$

(20) $\lim\limits_{x\to0^+}\left(\frac{1}{x}\right)^{\tan x} = \lim\limits_{x\to0^+}e^{\tan x\ln\frac{1}{x}} = e^{\lim\limits_{x\to0^+}\tan x\ln\frac{1}{x}} = e^{\lim\limits_{x\to0^+}\ln\frac{1}{x}/\cot x}$

$= e^{\lim\limits_{x\to0^+}\frac{x(-\frac{1}{x^2})}{-\csc^2x}} = e^{\lim\limits_{x\to0^+}x\left(\frac{\sin x}{x}\right)^2} = e^{\lim\limits_{x\to0^+}x} = e^0 = 1$

(21). $\lim_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{\lim_{x\to\frac{\pi}{4}} \tan 2x \ln\tan x} = e^{\lim \frac{\ln\tan x}{\cot 2x}} = e^{\lim \frac{\frac{1}{\tan x}\cdot\sec^2 x}{-2\cdot\csc^2 2x}}$

$= e^{\lim \frac{\frac{1}{\sin x\cos x}}{(-1)/(2\sin^2 x\cos^2 x)}} = e^{\lim(-2\cdot\sin x\cos x)} = e^{\lim(-\sin 2x)} = e^{-1}$

(22) $\lim_{n\to\infty} (\cos\frac{1}{n})^{n^2} = \lim_{n\to\infty} e^{n^2\ln\cos\frac{1}{n}} = e^{\lim \frac{\ln\cos\frac{1}{n}}{\frac{1}{n^2}}} = e^{\lim \frac{\sin\frac{1}{n}}{-\frac{2}{n}\cos\frac{1}{n}}}$

$= e^{\lim_{n\to\infty} \frac{\cos\frac{1}{n}}{-2(\cos\frac{1}{n}-\frac{1}{n}\sin\frac{1}{n})}} = e^{-\frac{1}{2}}$

(23) $\lim_{x\to 0^+} (\cot x)^{\frac{1}{\ln x}} = \lim_{x\to 0^+} e^{\frac{1}{\ln x}\ln\cot x} = e^{\lim \frac{\ln\cot x}{\ln x}} = e^{\lim \frac{\frac{1}{\cot x}\cdot(-\csc^2 x)}{\frac{1}{x}}}$

$= e^{\lim \frac{x}{\sin x}\cdot\frac{-1}{\cos x}} = e^{1\times(-1)} = e^{-1}$

(24) $\lim_{x\to+\infty} (\frac{2}{\pi}\arctan x)^x = e^{\lim x\ln\frac{2}{\pi}\arctan x} = e^{\lim \frac{\ln\frac{2}{\pi}\arctan x}{\frac{1}{x}}} = e^{\lim \frac{\pi}{2\arctan x}\cdot\frac{2}{\pi}\cdot(\frac{1}{1+x^2})/-\frac{1}{x^2}}$

$= e^{-1\lim_{x\to+\infty} \frac{1}{\arctan x}} = e^{-1\cdot\frac{2}{\pi}} = e^{\frac{-2}{\pi}}$.

(25) $\lim_{x\to 0} (\frac{\arcsin x}{x})^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\frac{\ln\frac{\arcsin x}{x}}{x^2}} = e^{\lim \frac{\ln\frac{\arcsin x}{x}}{x^2}} = e^{\lim \frac{\ln\arcsin x-\ln x}{x^2}}$

$= e^{\lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}\arcsin x}-\frac{1}{x}}{2x}} = e^{\lim \frac{\frac{x-\sqrt{1-x^2}\arcsin x}{x\sqrt{1-x^2}\arcsin x}}{2x}}$

$= e^{\lim \frac{x-\sqrt{1-x^2}\arcsin x}{2x^3}} = e^{\lim \frac{1+\frac{x}{\sqrt{1-x^2}}\arcsin x-1}{6x^2}}$

$= e^{\lim \frac{\arcsin x}{6x\sqrt{1-x^2}}} = e^{\lim \frac{\arcsin x}{6x}} = e^{\frac{1}{6}}$

(26) $\lim_{x\to 0} (\cos x+x\sin x)^{\frac{1}{x^2}} = e^{\lim \frac{\ln(\cos x+x\sin x)}{x^2}} = e^{\lim \frac{\frac{1}{\cos x+x\sin x}(-\sin x+\sin x+x\cos x)}{2x}}$

$= e^{\lim \frac{\cos x}{2(\cos x+x\sin x)}} = e^{\frac{1}{2}}$

(27) $\lim_{x\to 0} (\frac{a^x+b^x}{2})^{\frac{1}{x}} = \lim_{x\to 0} \left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{2}{a^x+b^x-2}\cdot\frac{a^x+b^x-2}{2}\cdot\frac{1}{x}} = \lim_{x\to 0} e^{\frac{a^x+b^x-2}{2}\frac{1}{x}}$

$= e^{\lim_{x\to 0} \frac{1}{2}\frac{a^x+b^x-2}{x}} = e^{\frac{1}{2}\lim_{x\to 0}(a^x\ln a+b^x\ln b)} = e^{\frac{1}{2}\ln ab} = \sqrt{ab}$

(28) $\lim\limits_{x\to0}\left(\frac{a^x-x\ln a}{b^x-x\ln b}\right)^{\frac{1}{x^2}}=e^{\lim\frac{\ln\frac{a^x-x\ln a}{b^x-x\ln b}}{x^2}}=e^{\lim\frac{\ln(a^x-x\ln a)-\ln(b^x-x\ln b)}{x^2}}$

$=e^{\left[\lim\frac{\ln(a^x-x\ln a)}{x^2}-\lim\frac{\ln(b^x-\ln b)}{x^2}\right]}$

$=e^{\left[\lim\frac{\ln(1+a^x-x\ln a-1)}{x^2}-\lim\frac{\ln(1+b^x-x\ln b-1)}{x^2}\right]}$

$=e^{\left(\lim\frac{(a^x-1)\cdot\ln a}{2x}-\lim\frac{(b^x-1)\ln b}{2x}\right)}$

$=e^{\left(\lim\frac{x\ln a\ln a}{2x}-\lim\frac{x\ln b\ln b}{2x}\right)}$

$=e^{\frac{1}{2}(\ln^2a-\ln^2b)}$

(29) $\lim\limits_{x\to0}x^{\sin x}=\lim\limits_{x\to0}e^{\ln x\cdot\sin x}=\lim\limits_{x\to0}e^{\frac{\ln x}{\csc x}}=\lim\limits_{x\to0}e^{\frac{\frac{1}{x}}{-\csc^2x\cos x}}=\lim\limits_{x\to0}e^{\frac{\sin^2x}{-x\cos x}}$

$\lim\limits_{x\to0}e^{-\frac{\sin^2x}{x}\cdot\frac{1}{\cos x}}=\lim\limits_{x\to0}e^{-\sin x}=e^0=1$

(30) $\lim\limits_{x\to\frac{\pi}{2}}(\cos x)^{\frac{\pi}{2}-x}=\lim\limits_{x\to\frac{\pi}{2}}e^{(\frac{\pi}{2}-x)\cdot\ln\cos x}=\lim\limits_{x\to\frac{\pi}{2}}e^{\frac{\ln\cos x}{\frac{1}{\frac{\pi}{2}-x}}}=\lim\limits_{x\to\frac{\pi}{2}}e^{\frac{\frac{1}{\cos x}(-\sin x)}{\frac{1}{(\frac{\pi}{2}-x)^2}}}$

$=\lim\limits_{x\to\frac{\pi}{2}}e^{-\tan x(\frac{\pi}{2}-x)^2}=\lim\limits_{x\to\frac{\pi}{2}}e^{\frac{-\sin x(\frac{\pi}{2}-x)^2}{\cos x}}=\lim\limits_{x\to\frac{\pi}{2}}e^{\frac{-\cos(\frac{\pi}{2}-x)^2-\sin x(2x-\pi)}{-\sin x}}$

$=\cancel{\lim\limits_{x\to\frac{\pi}{2}}}e^{\frac{0}{-\frac{\pi}{2}}}=1$

3. 解：$\lim\limits_{h\to0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\lim\limits_{h\to0}\frac{f'(x+h)+f'(x-h)-2f'(x)}{2h}=\lim\limits_{h\to0}\frac{f'(x+h)-f'(x)}{2h}+\lim\limits_{h\to0}\frac{f(x-h)-f(x)}{2h}$

$=\frac{1}{2}f''(x)+\frac{1}{2}f''(x)=f''(x)$

4. 解：$\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}}$，先求 $y=(1+x)^{\frac{1}{x}}$ 的导数. $\ln y=\frac{1}{x}\ln(1+x)$,

$\frac{1}{y}y'=\frac{-\ln(1+x)}{x^2}+\frac{1}{x(x+1)}$ 则 $y'=(1+x)^{\frac{1}{x}}\left(\frac{1}{x(x+1)}-\frac{\ln(1+x)}{x^2}\right)$.

则 $\lim\limits_{x\to0^+}\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}}=\lim\limits_{x\to0^+}\left[1+\frac{(1+x)^{\frac{1}{x}}}{e}-1\right]^{\frac{1}{x}}=e^{\lim\limits_{x\to0^+}\left[\frac{(1+x)^{\frac{1}{x}}}{e}-1\right]\frac{1}{x}}$.

$=e^{\lim\limits_{x\to0^+}\left[\frac{(1+x)^{\frac{1}{x}}-e}{ex}\right]}\xrightarrow{\text{洛必达}}e^{\lim\limits_{x\to0^+}\left[\frac{(1+x)^{\frac{1}{x}}\left[\frac{1}{x(x+1)}-\frac{\ln(1+x)}{x^2}\right]}{e}\right]}$

$=e^{\lim\limits_{x\to0^+}\left(\frac{1}{x(x+1)}-\frac{\ln(1+x)}{x^2}\right)}=e^{\lim\limits_{x\to0^+}\left(\frac{x-(x+1)\cdot\ln(1+x)}{x^2(x+1)}\right)}\xrightarrow{\text{洛必达}}e^{\lim\limits_{x\to0^+}\frac{1-\ln(1+x)-1}{3x^2+2x}}$

$\xrightarrow{\text{洛必达}}e^{\lim\limits_{x\to0^+}\left(\frac{\frac{-1}{x+1}}{6x+2}\right)}=e^{-\frac{1}{2}}$

下面求 $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} e^{\frac{\cos x - 1}{\ln(1+x^2)}} = e^{\lim\limits_{x\to 0^-} \frac{-\sin x}{\frac{1}{1+x^2} 2x}} = e^{\lim\limits_{x\to 0^-} \frac{-1-x^2}{2}} = e^{-\frac{1}{2}}$

则 $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = f(0) = e^{-\frac{1}{2}}$.