Combinatorial Auctions

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组合拍卖
物品很多但不可分
人可享多个物品

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Recap

- Regular distributions can be complex ..
- The prophet's inequality 预言家不等式
- $\Rightarrow \frac{1}{2}$-approx auction with bidder-specific reserve prices.
- BK Theorem. By i.i.d. regular distributions, we can show that the second-price auction maxs the expected revenue over all DSIC auctions that always allocate the item.

Much more difficult for Multi-Parameter problems.

多参数

一般环境的机制设计

General Mechanism Design Environme

n个人

- n strategic participants, or "agents";
- a finite set Ω of outcomes; 输出
- each agent i has a private non-negative valuation $v_i(\omega)$ for each outcome $\omega \in \Omega$.

The outcome set Ω is abstract and could be very large. The social welfare of an outcome $\omega \in \Omega$ is defined as $\sum_{i=1}^{n} v_i(\omega)$.

- Single-Item Auction $\Omega = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n, \vec{0}\}$ $\vec{v}\,\omega$ → 社会福利
- Combinatorial Auctions: there are multiple indivisible items for sale, and bidders can have complex preferences between different subsets of items (called bundle). With n bidders and a set M of m items, the outcomes of Ω correspond to n-vector $(S_1, S_2, \ldots, S_n)$, with $S_i \subseteq M$.

$v_i : 2^m \to \mathbb{R}_{\geq 0}$

每人的报价

$S_i \cap S_j = \phi$

The VCG Mechanism

Vickrey Clark Groves

Ideal Auction
DSIC+SWM+P时间内
X

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非理想拍卖

Theorem (The VCG Theorem)

In every general mechanism design environment, there is a DSIC welfare-maximizing mechanism, which is VCG mechanism.

有技巧 rule 都满足

- DSIC mechanism is tricky, we need allocation and payment rules to be coupled carefully.
- 诚实 The first step is to assume, with justification, that agents truthfully report their private information, and then figure out which outcome to pick. We define the allocation rule $\mathbf{x}$ by

$$\mathbf{x}(\mathbf{b}) = \operatorname{argmax}_{\omega \in \Omega} \sum_{i=1}^{n} b_i(\omega). \tag{1}$$

- The second step is to define a payment rule. Myerson's Lemma doesn't hold beyond single-parameter environments. Even not clear how to define "monotonicity" of an allocation rule.

Go back to Vickrey Auctions

- How much does player i hurt everyone else by participating in the auctions?
- If i does not get the item, then it doesn't hurt anyone. The only difference is that the winner might be charged more money.
- if i does get the item, then it was the highest bid. So everyone else has social welfare 0. But if i didn't join, the item would have gone to the bidder with the highest bidder other than i. So the "total harm" caused by player i is the second highest price, which is the price defined in the auction.

"Charging an agent her externality" remains well defined in general mechanism design environments, and corresponds to the payment rule

payment　　i不在时的social welfare　　除了i的sw

$$p_i(\mathbf{b}) = \left(\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) \right) - \sum_{j \neq i} b_j(\omega^*),$$

间接给出也是DSIC的　　n-1个人的　　n个人但只算n-1的　　i在时的SW分配

where $\omega^* = \mathbf{x}(\mathbf{b})$ is the outcome chosen from the equation in Eq (1). $p_i(\mathbf{b})$ is non-negative. Why?

Definition (VCG Mechanism)

A mechanism $(\mathbf{x}, \mathbf{p})$ with allocation and payment rules, respectively, is a Vickrey-Clarke-Groves or VCG mechanism.

An alternative interpretation:

其它的解释

$$p_i(\mathbf{b}) = \underbrace{b_i(\omega^*)}_{\text{bid}} - \underbrace{\left[\sum_{j=1}^{n} b_j(\omega^*) - \max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) \right]}_{\text{rebate}}.$$

优惠

Proof of the VCG Theorem

证明DSIC 真实就最大收益

Fix i and $\mathbf{b}_{-i}$, when the chosen outcome $\mathbf{x}(\mathbf{b})$ is ω^*, we write i's utility as

$$v_i(\omega^*) - p_i(\mathbf{b}) = \underbrace{\left[v_i(\omega^*) + \sum_{j \neq i} b_j(\omega^*) \right]}_{\text{(A)}} - \underbrace{\left[\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) \right]}_{\text{(B)}}.$$

Term (B) is constant, when fixing $\mathbf{b}_{-i}$. We only need to optimize the first term (A), which is maximized when $b_i = v_i$. 时 $\square$

$b_i \to \omega^*$

- $b_i = v_i$ $\quad \sum_{j=1}^{n} b_j(\omega^*) \to v_i$
- $b_i \neq v_i$ $\quad \vec{x} \to \hat{\omega}^* \to b_i$

优化目标相一致

Practical Considerations

- Getting the reports $\mathbf{b}_1, \ldots, \mathbf{b}_n$ from the agents. 2^m parameters per bidder, a thousand when $m = 10$ and a million when $m = 20$ ..
- Recall knapsack auctions, and in more complex settings even approximate welfare maximization can be computationally intractable.
- VCG can have bad revenue and incentive properties, despite being DSIC.

- three bidders and two items, A and B.
- $v_1(AB) = 1$, $v_2(AB) = v_2(A) = 1$ and $v_3(AB) = v_3(B) = 1$.
- The maximum welfare is two, but the VCG revenue is zero ..

?

Single-Minded Bidders

比一般MD简单的P时间情况

- Each bidder has a (private) set $T_i \in M$ and there is a private parameter $v_i \in \mathbb{R}^+$ s.t.

$$v_i(S) = \begin{cases} v_i & \text{if } S \supseteq T_i \\ 0 & \text{otherwise} \end{cases}$$

- A bid in this context is a pair (b_i, S_i). 估价 集合
- Next we will talk about the greedy mechanism ..

The Greedy Mechanism

- Sort the bidders s.t.

排序

相同先放size小的

$$\frac{b_1}{\sqrt{|S_1|}} \geq \frac{b_2}{\sqrt{|S_2|}} \geq \cdots \frac{b_n}{\sqrt{|S_n|}}.$$

全集

- $W = \emptyset$ (W is the set of agents which can get their S_i's.)
- For i from 1 to n do: if $S_i \cap (\cup_{j \in W} S_j) = \emptyset$, add i to W.
- Return the allocation which gives S_i to player i iff $i \in W$.

We'll set $p_i = 0$ if $i \notin W$. If $i \in W$, let $\alpha(i)$ be the minimum index such that $S_i \cap S_{\alpha(i)} \neq \emptyset$ and $S_k \cap S_{\alpha(i)} = \emptyset$ for all $k < \alpha(i)$ with $k \neq i$ and $k \in W$. That is, $\alpha(i)$ is the first player who lost due to bidder i: if i had not been participating then $\alpha(i)$ would have been in W. If no such $\alpha(i)$ exists, then we set $p_i = 0$. Otherwise, we set

$$p_i = \frac{b_{\alpha(i)}}{\sqrt{|S_{\alpha(i)}|/|S_i|}} = b_{\alpha(i)} \sqrt{\frac{|S_i|}{|S_{\alpha(i)}|}}$$

An Example

$$\frac{b_i}{\sqrt{S_i}} \geq \frac{b_{\alpha(i)}}{\sqrt{S_{\alpha(i)}}}$$

- Another way to say for the payment rule: Charge each winner the minimum b_j that they could have reported and still been a winner for S_i.
- e.g., bids are $(1, \{1\}), (0, \{1, 2, 3, 4\}), (4, \{1, 2\}), (4, \{3, 4\})$.
- $(4, \{1, 2\})$ will pay $\sqrt{2}$ and $(4, \{3, 4\})$ will pay 0.

$$1\sqrt{\frac{2}{1}} \qquad 0$$

The Analysis

P时间

- The mechanism is polynomial time and outputs a valid allocation.
- Incentive Compatibility: monotonicity and critical payment IC
- Social Welfare Approximately.

近似最大

Incentive Compatibility

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- Monotonicity: increasing b_i or decreasing S_i can only move i earlier in the greedy ordering.
- Critical Payment: if i wins then the price she pays is the smallest x such that i would still win if she had bid (x, S_i).
  When $\alpha(i)$ doesn't exists. Then there is no other bidder who fails to win due to bidder i. So she is charged 0, the critical payment.
  Suppose $\alpha(i)$ does exist. Then i will still win as long as it appears before $\alpha(i)$ in the ordering. Thus the critical payment is x such that $\frac{x}{\sqrt{|S_i|}} = \frac{b_{\alpha(i)}}{\sqrt{|S_{\alpha(i)}|}}$, so we have the critical payment is

$$b_{\alpha(i)}\sqrt{\frac{|S_i|}{|S_{\alpha(i)}|}} = p_i. \leq b_i$$

Both Properties are Enough

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Theorem

Any mechanism where losers pay 0 which has both the monotonicity and critical payment properties is incentive compatible.

Proof.

Fix player $i \in [n]$ and all bids other than i's. Let $u(b, S)$ be the utility that player i would get by bidding (b, S), so $u(b, S) = v_i(S) - p_i(b, S)$. By the critical payment, we know $p_i(b, S) = \inf\{x : i \text{ wins with bid } (x, S)\}$.

- non-negative utility: 0 or $v_i - p_i(b, T_i) \geq 0$.
- dominant strategy: $u(v_i, T_i) \geq u(b, T_i) \geq u(b, S)$ for all (b, S) with $T_i \subseteq S$ and (b, S) is a winning bid. (Why?)

?

□

Detailed Proof

- Easy when (b, S) is a losing bid, since non-negative utility
- Obvious if $T_i \not\subseteq S$, since receiving a bundle valued zero
- So for the first part, we just need to show $p_i(b, T_i) \leq p_i(b, S)$. Holds by the monotonicity property and $T_i \subseteq S$.
- $u(v_i, T_i) = u(b, T_i)$ when (v_i, T_i) is a winning bid, since (b, T_i) is a winning bid; $p_i(b, T_i) \geq v_i$ if (v_i, T_i) is not a winning bid.

Social Welfare

Let OPT be the winners in an optimal allocation, and W be ours.

Theorem

近似比

$$\sum_{i \in \mathrm{OPT}} v_i \le \sqrt{m} \cdot \sum_{i \in W} v_i.$$

→ 至多的与我们的策略

Let $\mathrm{OPT}_i = \{j \in \mathrm{OPT} \mid j \ge i \wedge T_i \cap T_j \ne \emptyset\}$. Note that $\mathrm{OPT} = \cup_{i \in W} \mathrm{OPT}_i$. (Why?)

柯西不等式

$$\sum_{j \in \mathrm{OPT}_i} v_j \le \frac{v_i}{\sqrt{|T_i|}} \sum_{j \in \mathrm{OPT}_i} \sqrt{|T_j|} \le \frac{v_i}{\sqrt{|T_i|}} \sqrt{|\mathrm{OPT}_i|} \sqrt{\sum_{j \in \mathrm{OPT}_i} |T_j|}.$$

We have $|\mathrm{OPT}_i| \le |T_i|$ since $T_j \cap T_{j'} = \emptyset$ (They are both in OPT!).

$$\sum_{i \in \mathrm{OPT}} v_i \le \sum_{i \in W} \sum_{j \in \mathrm{OPT}_i} v_j \le \sum_{i \in W} v_i \sqrt{m} = \sqrt{m} \sum_{i \in W} v_i.$$

Summary

- Combinatorial auctions are an important example of general MD environments.
- VCG mechanism is DSIC and social welfare maximized.
- VCG has kinds of practical issues.
- CAs with Single-minded Bidders, and its greedy mechanism.
- OPEN Improved truthful mechanisms for subadditive combinatorial auctions: Breaking the logarithmic barrier [SODA'21]

Q&A?