Games & Nash Equilibrium

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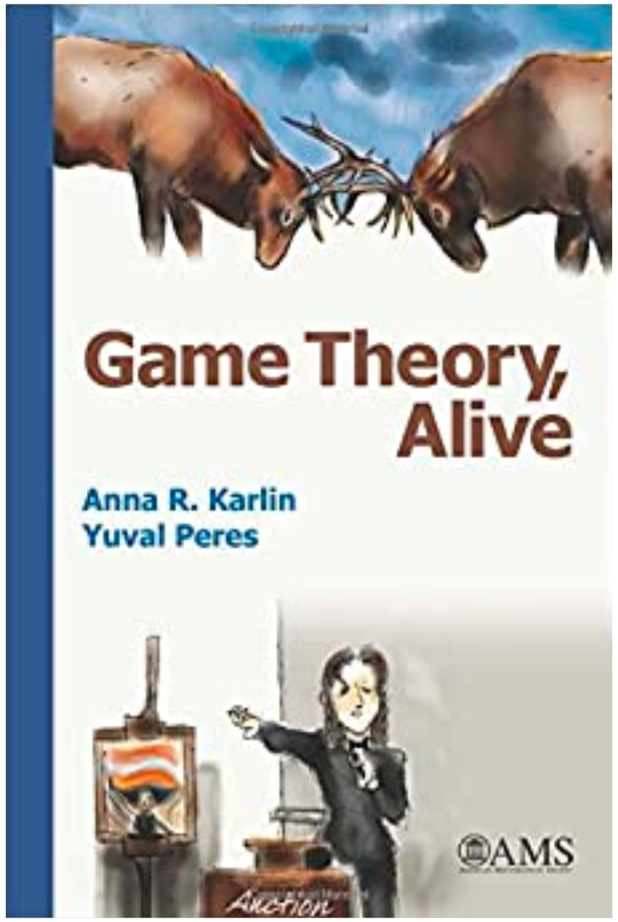
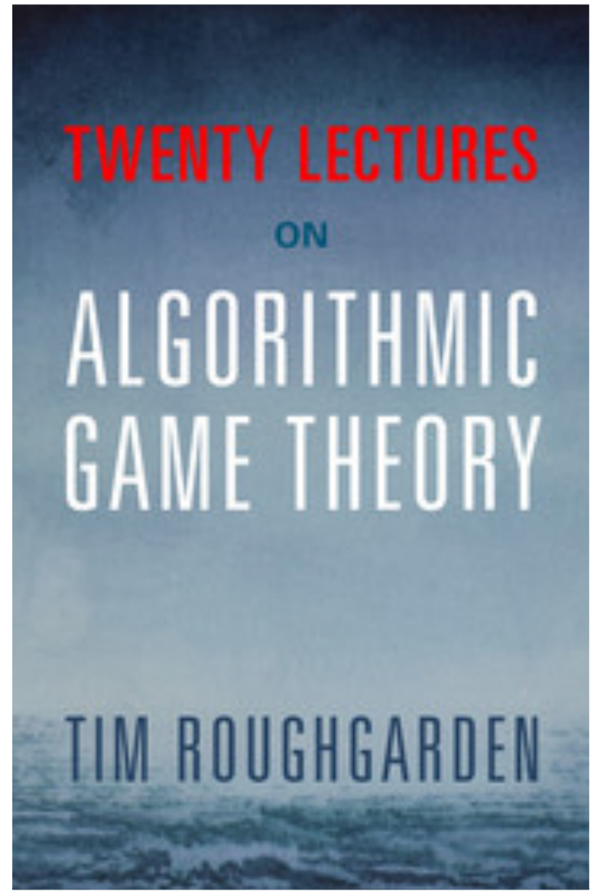
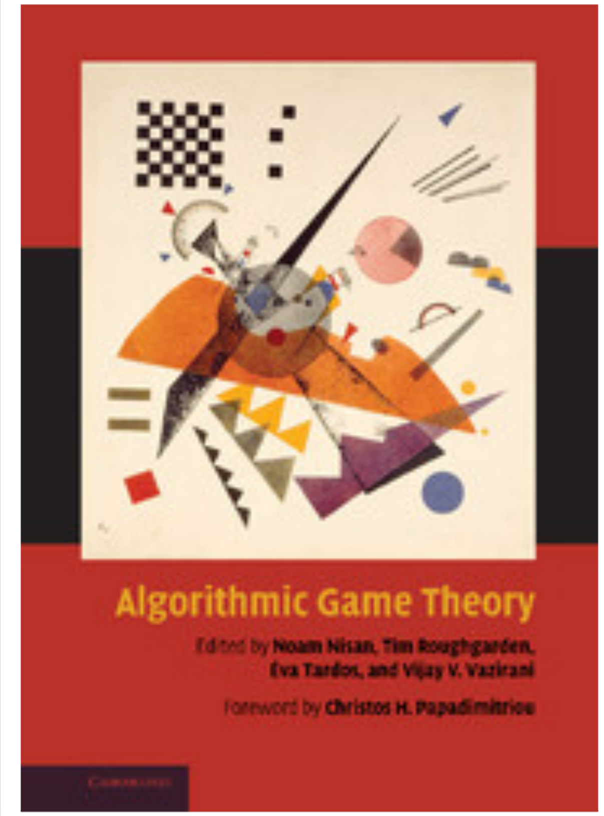
Reference







Related Courses

1. http://people.csail.mit.edu/costis/6853fa2011/
2. https://www.haifeng-xu.com/cs6501fa19/index.htm
3. https://www.cs.jhu.edu/~mdinitz/classes/AGT/Spring2022/
4. http://cs.brown.edu/courses/csci1440/lectures/

Reminders

- Q&A: Wed before class?
- We need math maturity (or passion?) in this class…
- Tell me anything about improving the class (slides, teaching style etc.)…
- I will post hw1 next class, but I don't know how…
- What's your purpose of enrolling this class?
- If you have any question, you can iBIT/email me. I will disband the WeChat group…

Two-Player Games

- A pair of payoff matrices (R, C) of size $m \times n$, where Row player has m actions and Column player has n actions. (action $\Longleftrightarrow$ pure strategy)

- So the meaning of $R_{i,j}$ and $C_{i,j}$?

- Mixed strategy: a distribution over pure strategies. Denote by Δ_n the set of all mixed strategies over n actions. That is,

$$\Delta_n := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = 1, x_i \geq 0\}\,.$$

- Expected payoff: given $\mathbf{x} \in \Delta_m, \mathbf{y} \in \Delta_n$, they are $\mathbf{x}^T R \mathbf{y}$ and $\mathbf{x}^T C \mathbf{y}$, just calculation…

Nash Equilibrium

Two-player version

- A pair of strategies $(\mathbf{x}, \mathbf{y})$ is NE iff neither can increase her payoff by deviating from her strategy unilaterally. That is

$$\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}'^T R\mathbf{y}, \ \forall \mathbf{x}' \in \Delta_m;$$
$$\mathbf{x}^T C\mathbf{y} \geq \mathbf{x}^T C\mathbf{y}', \ \forall \mathbf{y}' \in \Delta_n.$$

- Or an equivalent definition
  - Support of $\mathbf{x}$: $\mathrm{supp}(\mathbf{x}) := \{i \in [n] \mid x_i \neq 0\}$.
  - Each action in the support of $\mathbf{x}$ (or $\mathbf{y}$) should be the best response to the other.

Zero-Sum Games

The game with absolute conflict…

	M	T
E	3, -3	-1, 1
S	-2, 2	1, -1

- Zero-Sum iff $R + C = 0$, that is $R_{i,j} + C_{i,j} = 0$.
- Given row player using (x_1, x_2), we have
    - Column has $\mathbb{E}[M] = -3x_1 + 2x_2$, $\mathbb{E}[T] = x_1 - x_2$ and gets the better one.
    - Since zero-sum, row will choose $(x_1, x_2) \in \arg\max_{x_1, x_2} \min(3x_1 - 2x_2, -x_1 + x_2)$.

Some Observations

$$
\begin{aligned}
&\max z \\
\text{s.t.}\quad & 3x_1 - 2x_2 \quad \geq \quad z \\
& -x_1 + x_2 \quad \geq \quad z \\
& x_1 + x_2 \quad = \quad 1 \\
& x_1, x_2 \quad \geq \quad 0.
\end{aligned}
$$

$x_1 = 3/7, x_2 = 4/7, z = 1/7$

$$
\begin{aligned}
&\max w \\
\text{s.t.}\quad & -3y_1 + y_2 \quad \geq \quad w \\
& 2y_1 - y_2 \quad \geq \quad w \\
& y_1 + y_2 \quad = \quad 1 \\
& y_1, y_2 \quad \geq \quad 0,
\end{aligned}
$$

$y_1 = 2/7, y_2 = 5/7, w = -1/7$

Nash equilibrium!

LP & Duality

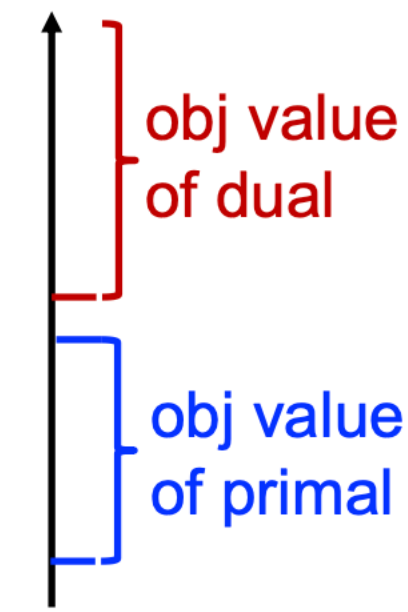
$$A = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^m \end{bmatrix} = [A_1, \ldots, A_n]$$

Primal LP

max	$c^T \cdot x$
s.t.	$Ax \leq b$
	$x \geq 0$

Dual LP

min	$b^T \cdot y$
s.t.	$A^T y \geq c$
	$y \geq 0$



- $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b, y \in \mathbb{R}^m$
- y_i is the dual variable to the primal constraint $A^i x \leq b_i$
- $A_j^T y \geq c_j$ is the dual constraint to the primal variable x_j

- Weak Duality: $c^T x \leq b^T y$
- Strong Duality: If either the primal or dual is feasible and bounded, then so is the other and OPT(primal) = OPT(dual).

LP for zero-sum games

$z'' = -z', C = -R$

$$\begin{aligned} \max\ & z \\ \text{s.t.}\ & \boldsymbol{x}^T R \geq z \mathbf{1}^T \\ & \boldsymbol{x}^T \mathbf{1} = 1 \\ & \forall i,\ x_i \geq 0. \end{aligned}$$

$$\begin{aligned} \min\ & z' \\ \text{s.t.}\ & -\boldsymbol{y}^T R^T + z' \mathbf{1}^T \geq \mathbf{0} \\ & \boldsymbol{y}^T \mathbf{1} = 1 \\ & \forall j,\ y_j \geq 0. \end{aligned}$$

$$\begin{aligned} \max\ & z'' \\ \text{s.t.}\ & C\boldsymbol{y} \geq z'' \mathbf{1} \\ & \boldsymbol{y}^T \mathbf{1} = 1 \\ & \forall j\ y_j \geq 0. \end{aligned}$$

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^T R \boldsymbol{y}.$$

$$\max_{\boldsymbol{y}} \min_{\boldsymbol{x}} \boldsymbol{x}^T C \boldsymbol{y} = -\min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^T R \boldsymbol{y}.$$

Theorem 1

If $(\mathbf{x}, z)$ is optimal for LP(1), and $(\mathbf{y}, z'')$ is optimal for LP(3), then $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium of (R, C). Moreover, the payoffs of the row/column player in this Nash equilibrium are z and $z'' = -z$ respectively.

LP $\Longleftrightarrow$ NE

$$\begin{aligned} \max\ & z \\ \text{s.t.}\quad & \boldsymbol{x}^T R \geq z \mathbf{1}^T \\ & \boldsymbol{x}^T \mathbf{1} = 1 \\ & \forall i,\ x_i \geq 0. \end{aligned} \qquad \begin{aligned} \max\ & z'' \\ \text{s.t.}\quad & C\boldsymbol{y} \geq z'' \mathbf{1} \\ & \boldsymbol{y}^T \mathbf{1} = 1 \\ & \forall j\ y_j \geq 0. \end{aligned}$$

- By def of NE, it is sufficient to show that $\mathbf{x}^T R \mathbf{y} \geq z \geq \mathbf{x}'^T R \mathbf{y}$.
- There exists a Nash equilibrium in every two-player zero-sum game.
- The Minimax Theorem: $\max_x \min_y x^T R y = \min_y \max_x x^T R y$.

Theorem 2

If $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium of (R, C), then $(\mathbf{x}, \mathbf{x}^T R \mathbf{y})$ is an optimal solution of LP(1), and $(\mathbf{y}, -\mathbf{x}^T C \mathbf{y})$ is an optimal solution of LP (2).

Yao's Principle

How to prove an LB on randomized algorithms

- The expected cost of a randomized algorithm on the worst-case input $\geq$ the expected cost for a worst-case probability distribution on the inputs of the deterministic algorithm that performs best against that distribution.

Mixed A over $\mathscr{A}$

an algo $a \in \mathscr{A}$

Mixed X over $\mathscr{X}$

an input $x \in \mathscr{X}$

$$\max_{x \in \mathscr{X}} \mathbb{E}\left[c(A, x)\right] \geq \min_{a \in \mathscr{A}} \mathbb{E}\left[c(a, X)\right]$$

Brief History of LP

- The forefather of convex optimization problems, and the most ubiquitous.
- Developed by Kantorovich during World War II (1939) for planning the Soviet army's expenditures and returns. Kept secret.
- Discovered a few years later by George Dantzig, who in 1947 developed the simplex method for solving linear programs
- John von Neumann developed LP duality in 1947, and applied it to game theory
- Polynomial-time algorithms: Ellipsoid method (Khachiyan 1979), interior point methods (Karmarkar 1984).

Normal Form Games

- NFG: $\langle n, (S_p)_{p\in[n]}, (u_p)_{p\in[n]} \rangle$
  - # of players in the game, $[n] = \{1,\ldots,n\}$
  - A set S_p of pure strategies of player $p \in [n]$
  - A utility function $u_p : \times_{p\in[n]} S_p \to \mathbb{R}$
- Recall RSP game…

More math…

- The set Δ^{S_p} of mixed strategies to player p over S_p
- The set $S := \times_{p\in[n]} S_p$ of all the pure strategy profile.
  $\mathbf{s} = (s_1, \ldots, s_n) \sim S$
- The set $\Delta := \times_{p\in[n]} \Delta^{S_p}$ of all the mixed strategy profile.
  $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_n}) \sim \boldsymbol{\Delta}$
- Given $\mathbf{x} \in \Delta$, we define the expected payoff of player p is

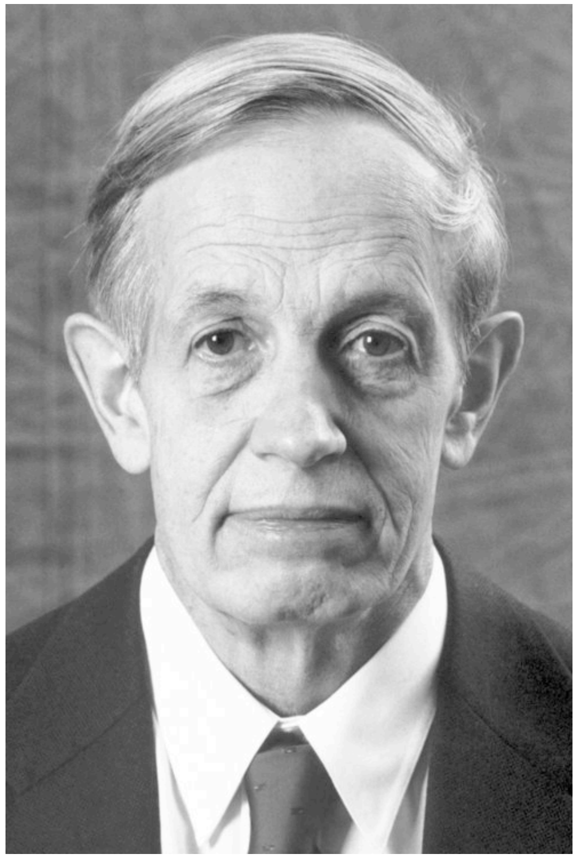
$$u_p(\mathbf{x}) = \sum_{\mathbf{s}\in S} u_p(\mathbf{s}) \prod_{q\in[n]} \mathbf{x}_q(s_q) = \mathbb{E}_{\mathbf{s}\sim\mathbf{x}}\left[u_p(\mathbf{s})\right].$$

- NE $\mathbf{x} \in \Delta$ in multi-player games iff given any $\mathbf{x}'_p \in \Delta^{S_p}$

$$u_p(\mathbf{x}) \geq u_p(\mathbf{x}'_p; \mathbf{x}_{-p})$$

“As far as I can see, there could be no theory of games … without that theorem … I thought there was nothing worth publishing until the Minimax Theorem was proved”

John von Neumann



Nash's Theorem: "Every (finte) game has a Nash equilibrium."

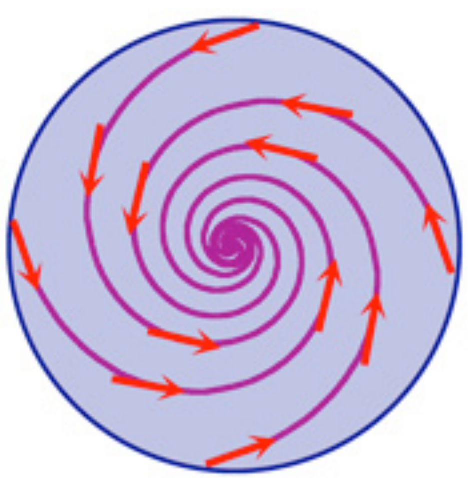
John Forbes Nash Jr.

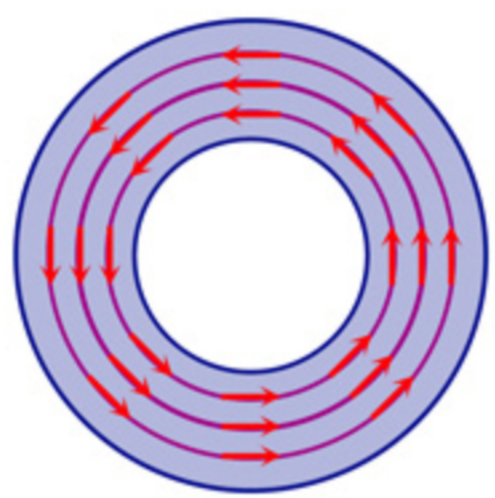
Proof of Nash's Theorem

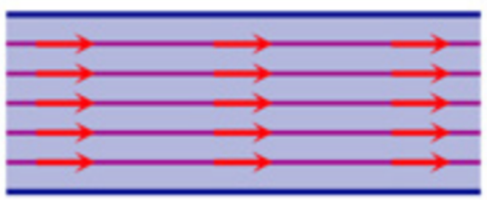
An reduction to fixed point.

- The idea is construct a reduction from the problem of finding an NE in a NFG to the problem of finding a fixed point in a well-defined domain.

- [Brouwer's Fixed Point Thm] Let D be a good (convex, compact) subset of $\mathbf{R}^n$. If a function $f : D \to D$ is continuous, then there exists an $x \in D$ such that $f(x) = x$.







- Let's make a mapping between these two problems.
- So the question is how to construct the continuous function f.
  - It's a good choice to set $f : \Delta \to \Delta$.
- We define a gain function $G_{p,s_p}(\mathbf{x}) := \max\{u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}), 0\}$.
  - Can you increase your utility when only using s_p instead of $\mathbf{x}_p$?
- We define $\mathbf{y} = f(\mathbf{x})$, where $y_{p,s_p} := \dfrac{x_{p,s_p} + G_{p,s_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} G_{p,s'_p}(\mathbf{x})}$.
- f is well-defined, continuous and Δ is good enough $\Rightarrow$ Bingo!
- Next we will show that any fixed point of f is an NE of the game.

$$y_{p,s_p} := \frac{x_{p,s_p} + G_{p,s_p}(\mathbf{x})}{1 + \sum_{s_p' \in S_p} G_{p,s_p'}(\mathbf{x})}$$

- Given $\mathbf{x} = f(\mathbf{x})$, sufficient to show that $G_{p,s_p}(\mathbf{x}) = 0$, $\forall p, s_p$
- Proof by contradiction!
  - Assume that there exists p, s_p such that $G_{p,s_p}(\mathbf{x}) > 0$
    - $x_{p,s_p} > 0$, otherwise $x_{p,s_p} = 0$ but $y_{p,s_p} > 0$
    - There exists some other pure strategy s_p' such that $x_{p,s_p'} > 0$ and $u_p(s_p'; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0$
      - By $u_p(\mathbf{x}) = \sum_{s \in S_p} x_{p,s} \cdot u_p(s; \mathbf{x}_{-p})$
    - We have $y_{p,s_p'} < x_{p,s_p'}$, so $\mathbf{x}$ is not a fixed point!

Q&A?

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