Algorithmic Game Theory, Spring 2022 Homework 2

1120200807 黄昱欣

Problem 1. 《Harry Potter》

Problem 2. My favorite theorem is Nash's Theorem: Every (finite) game has a Nash equilibrium.

It looks simple but significant. It constructs a reduction from finding an NE to finding a fixed point by using the Brouwer's Fixed Point Theorem: Let D be a good (convex, compact) subset of $\mathbb{R}^n$. If a function $f : D \to D$ is continuous, then there exists an $x \in D$ such that $f(x) = x$. After this, we constructs a well-defined and continuous function f and prove the fixed point is an NE by contradiction. To be honest, I didn't understand the proof of this theorem in the class. Through discussion with other students, which was full of fun of thinking, I finally understood the ingenious way of proving the theorem. The process of figuring it out and its ingenious proof made me love this theorem.

Problem 3.

Proof. First, we construct $a_r(S),\ r \in [t]$, using its additivity.

$$a_r(S) = \begin{cases} 0 & S = \emptyset \\ v(S^{\pi(r)}_{<j} + \{j\}) - v(S^{\pi(r)}_{<j}) & S = \{j\},\ S^{\pi(r)}_{<j}\ is\ a\ set\ which\ contains\ elements \\ & \ before\ element\ j\ in\ arrangement\ \pi(r) \\ \sum_{i \in S} a_r(\{i\}) & otherwise \end{cases}$$

Then, we prove this equality from two side.

(i) $v(s) \le max_{r\in[t]} a_r(s)$

Let the first $|S| = n$ elements in arrangement $\pi(r)$ be the elements in S.

Assume $S = \{s_1, ..., s_n\}$, $\pi(r) = \{s_{r_1}, ..., s_{r_n}\}$.

$$\begin{aligned} a_r(S) &= \sum_{i=1}^{n} a_r(\{s_i\}) \\ &= \sum_{i=1}^{n} a_r(\{s_{r_i}\}) \\ &= v(S^{\pi(r)}_{<s_{r_1}} + \{s_{r_1}\}) - v(S^{\pi(r)}_{<s_{r_1}}) + v(S^{\pi(r)}_{<s_{r_2}} + \{s_{r_2}\}) - v(S^{\pi(r)}_{<s_{r_2}}) + ... + v(S^{\pi(r)}_{<s_{r_n}} + \{s_{r_n}\}) - v(S^{\pi(r)}_{<s_{r_n}}) \\ &= v(\{s_{r_1}\}) - v(\emptyset) + v(\{s_{r_1}, s_{r_2}\}) - v(\{s_{r_1}\}) + ... + v(\{s_{r_1}, ..., s_{r_n}\}) - v(\{s_{r_1}, ..., s_{r_{n-1}}\}) \\ &= v(\{s_{r_1}, ..., s_{r_n}\}) \\ &= v(S) \end{aligned}$$

$\therefore\ v(S) = a_r(S) \le max_{r\in[t]} a_r(S)$

(ii) $v(s) \geq max_{r\in[t]}a_r(s)$

let $\pi(c)$ be any arrangement but the first $|S|$ elements are in S, and π be any arrangement but those elements same in S must have the same relative position as those in $\pi(c)$.

It is easy to understand that

$$S^{\pi(c)}_{<i} \subseteq S^{\pi}_{<i}$$

From (i) we can know that $v(S) = a_c(S)$, then we try to prove $a_c(S) \geq max_{r\in[t]}a_r(s)$. Because $a_c(\{i\}) = v(S^{\pi(c)}_{<i} + \{i\}) - v(S^{\pi(c)}_{<i}),\ \forall i \in S$. Then Let $A = S^{\pi(c)}_{<i} + \{i\},\ B = S^{\pi(c)}_{<i} \cup (S^{\pi}_{<i} - S^{\pi(c)}_{<i})$.

Because v is submodular, so

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$$

$$v(S^{\pi}_{<i} + \{i\}) + v(S^{\pi(c)}_{<i}) \leq v(S^{\pi(c)}_{<i} + \{i\}) + v(S^{\pi}_{<i})$$

$$v(S^{\pi(c)}_{<i} + \{i\}) - v(S^{\pi(c)}_{<i}) \geq v(S^{\pi}_{<i} + \{i\}) - v(S^{\pi}_{<i})$$

$$a_c(\{i\}) \geq v(S^{\pi}_{<i} + \{i\}) - v(S^{\pi}_{<i})$$

$$a_c(\{i\}) \geq a_r(\{i\}),\ r \in [t]$$

$$\therefore v(S) = a_c(S) = \sum_{i\in S} a_c(\{i\}) \geq \sum_{i \in S} a_r(\{i\}) = a_r(S),\ r \in [t]$$

$\therefore$v(S)$\geq max_{r\in[t]}a_r(S)$.

$\therefore$v(S)= $\max_{r\in[t]}a_r(S)$. So, any monotone and normalized submodular function can be written as an XOS function.

Problem 4.

(1) Because the uniform distribution,

$$f(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ 1 & otherwise \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$\therefore V(q) = F^{-1}(1-q) = 1 - q\ ,\ 0 \leq q \leq 1$$

$$\therefore R(q) = V(q) \cdot q = q(1-q)\ ,\ 0 \leq q \leq 1$$

(2)

Proof.

$$\because V(q) = F^{-1}(1-q)$$

$$\therefore F(V(q)) = 1 - q$$

$$\therefore q = 1 - F(V(q))$$

$$\therefore 1 = -F'(V(q)) \cdot V'(q)$$

$$\therefore V'(q) = -\frac{1}{F'(V(q))}$$

$$\begin{aligned} R'(q) &= (q \cdot V(q))' \\ &= V(q) + q \cdot V'(q) \\ &= V(q) - \frac{q}{F'(V(q))} \\ &= V(q) - \frac{q}{f(V(q))} \\ &= V(q) - \frac{1 - F(V(q))}{f(V(q))} \\ &= \varphi(V(q)) \end{aligned}$$

(3)

Proof. To begin with, I will transform the condition "regular" and "concave" into mathematical language.

"regular":From the definition of regular distribution, we can know that

$A\ distribution\ F\ is\ regular\ if\ its\ virtual\ valuation\ function\ v - \frac{1-F(v)}{f(v)}\ is\ non-decreasing\ in\ v.$

$\therefore$ a distribution is regular$\leftrightarrow$ $\frac{d\varphi(v)}{dv} \geq 0$

"concave": the revenue curve is concave $\leftrightarrow$ $\frac{d^2R(q)}{dq^2} \leq 0$

$$\because R'(q) = \frac{dR(q)}{dq} = \varphi(V(q))$$

$$\therefore \frac{d^2R(q)}{dq^2} = \frac{d\varphi(V(q))}{dV(q)} \cdot \frac{dV(q)}{dq}$$

$$\because F(V(q)) = 1 - q$$

$$\therefore \frac{dF(V(q))}{dq} = -1$$

$$\because \frac{dF(V(q))}{dq} = \frac{dF(V(q))}{dV(q)} \cdot \frac{dV(q)}{dq}$$

$$\therefore \frac{dF(V(q))}{dV(q)} \cdot \frac{dV(q)}{dq} = -1$$

$$\therefore f(V(q)) \cdot \frac{dV(q)}{dq} = -1$$

$$\because f(V(q)) \geq 0$$

$$\therefore \frac{dV(q)}{dq} \leq 0$$

Then, we prove it from two side:

(i) a distribution is regular $\rightarrow$ its revenue curve is concave

$$\because the\ distribution\ is\ regular$$

$$\therefore \frac{d\varphi(v)}{dv} \geq 0$$

$$\because \frac{d^2R(q)}{dq^2} = \frac{d\varphi(V(q))}{dV(q)} \cdot \frac{dV(q)}{dq}$$

$$\frac{dV(q)}{dq} \leq 0$$

$$\therefore \frac{d^2R(q)}{dq^2} \leq 0$$

Therefore, the revenue curve is concave.

(ii) a distribution is regular $\leftarrow$ its revenue curve is concave

$$\because the\ revenue\ curve\ is\ concave$$

$$\therefore \frac{d^2R(q)}{dq^2} \leq 0$$

$$\because \frac{d^2R(q)}{dq^2} = \frac{d\varphi(V(q))}{dV(q)} \cdot \frac{dV(q)}{dq}$$

$$\frac{dV(q)}{dq} \leq 0$$

$$\therefore \frac{d\varphi(V(q))}{dV(q)} \geq 0$$

$$\therefore \frac{d\varphi(v)}{dv} \geq 0$$

Therefore, the distribution is regular.

So, a distribution is regular if and only if its revenue curve is concave.

(4)

***Proof*.** From (3), we can know that the revenue curve is concave.

$$\therefore R(\lambda q_1 + (1-\lambda)q_2) \geq \lambda R(q_1) + (1-\lambda)R(q_2) \ \ \lambda \in [0,1]$$

$$q_{max} = argmax_{q\in[0,1]}R(q), \ \ q_1 = min(q_{max}, 1-q_{max}) < q_2 = max(q_{max}, 1-q_{max})$$

let $\lambda = \frac{1}{2}$, then

$$R(\frac{1}{2}q_1 + \frac{1}{2}q_2) \geq \frac{1}{2}(R(q_1) + R(q_2))$$

$$R(\frac{1}{2}(q_{max} + 1 - q_{max})) \geq \frac{1}{2}(R(q_{max}) + R(1-q_{max}))$$

$$R(\frac{1}{2}) \geq \frac{1}{2}(R(q_{max}) + R(1-q_{max}))$$

$$\because R(q) = q \cdot F^{-1}(1-q)$$

$$\because q, F^{-1}(1-q) \geq 0$$

$$\therefore R(1-q_{max}) \geq 0$$

$$\therefore R(\frac{1}{2}) \geq \frac{1}{2}R(q_{max})$$

$$R(\frac{1}{2}) \geq \frac{1}{2}max_{q\in[0,1]}R(q)$$