Algorithmic Game Theory, Spring 2022
Homework 1

Zhengyang Liu

Instructions:

1. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
2. Please type your solutions if possible in LaTeX or word whatever is suitable.
3. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.

Problem 1. (2pt) What's your favourite meal/dish in our canteen? *Please answer in Chinese.*

Problem 2. (5pt) Show that the two definition of Nash Equilibrium mentioned in class are equivalent. For convenience, we list these two definition.

Definition 1. A pair of strategies $(\mathbf{x}, \mathbf{y})$ is NE iff

$$\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}'^T R\mathbf{y}, \forall \mathbf{x} \in \Delta_m;$$
$$\mathbf{x}^T C\mathbf{y} \geq \mathbf{x}^T C\mathbf{y}', \forall \mathbf{y}' \in \Delta_n$$

Definition 2. A pair of strategies $(\mathbf{x}, \mathbf{y})$ is NE iff

$$x_i > 0 \Rightarrow \mathbf{e}_i^T R\mathbf{y} \geq \mathbf{e}_k^T R\mathbf{y}, \forall k \in [m];$$
$$y_j > 0 \Rightarrow \mathbf{x}^T C\mathbf{e}_j \geq \mathbf{x}^T C\mathbf{e}_l, \forall l \in [n]$$

Problem 3. (5pt) Show that any symmetric game (R, C) where $R = C^T$ has a symmetric Nash Equilibrium $(\mathbf{x}, \mathbf{x})$. *Hint: modify the proof of Nash's Theorem.*

Problem 4. This problem is to prove the Sperner's Lemma, a combinatorial version of Brouwer's Fixed Point Theorem. Given a grid as Figure 1, we first color the boundary using three colors in a legal way as the figure says, and then color the internal nodes arbitrarily. Prove that there exists one tri-chromatic triangle, i.e., a small unit triangle whose nodes are colored by all the three colors. You should prove this lemma using two methods as follows.

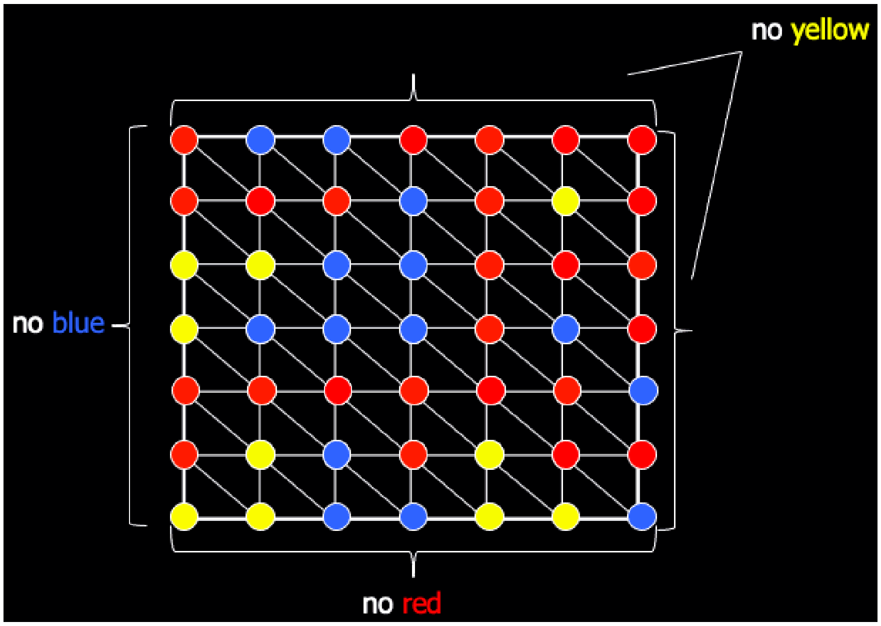

Figure 1: An example of Sperner's Lemma

1. (3pt) The first method is using *double-counting*, that is, we count the number of some object from two different views. In this problem, we can prove the lemma by counting the number of yellow-blue edges of all the unit triangles.

2. (5pt) The second method is using *path-following*. Actually, PPAD is inspired by this lemma! (Recall the problem End-of-A-Line) One can define each triangle as a node in the graph. How to define directed edges is the crucial part. Another issue is the initial source node (0^n in the problem EoAL).