Lemke-Howson Algorithm & PPAD

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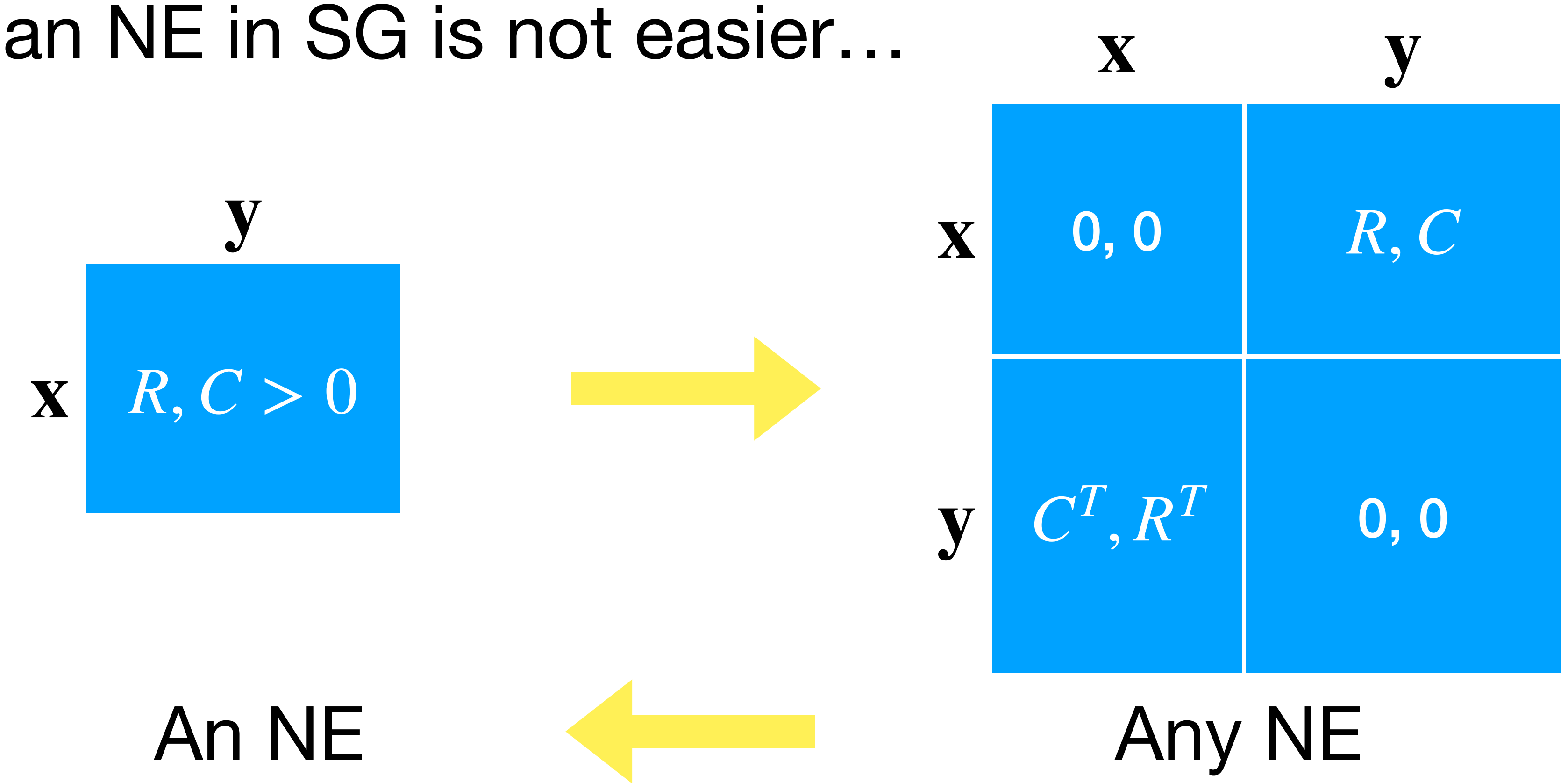
Symmetric Games (SG)

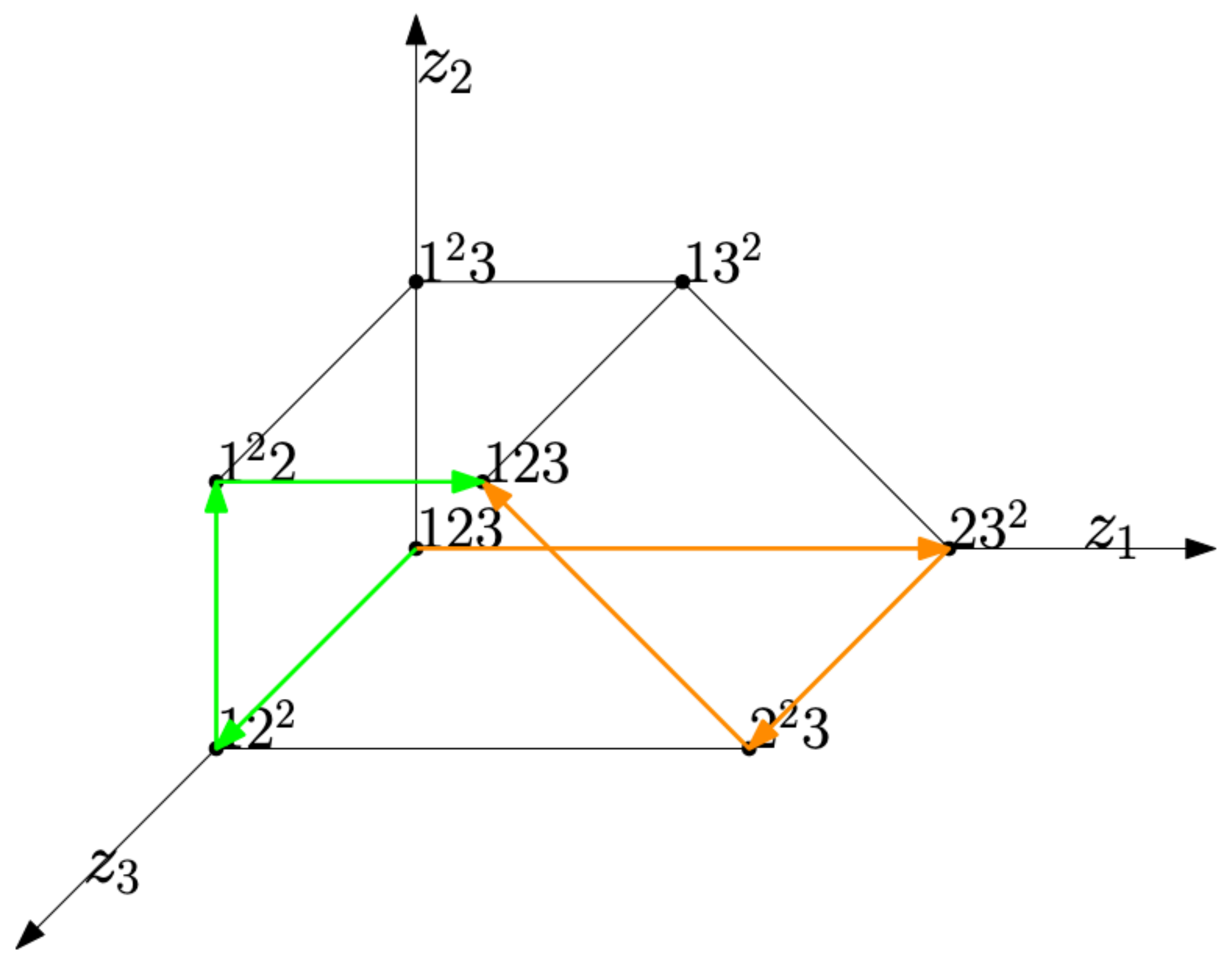
- $R = C^T$, where R, C are of size $n \times n$
- One can show that any SG has a symmetric NE $(\mathbf{x}, \mathbf{x})$
  - By the proof of Nash's Theorem
- Finding an NE in SG is not easier…



The Lemke-Howson Algo (1964)

- We know support enumeration algo to find an exact NE.
- LH algo can be easily described in symmetric games (R, R^T).
- IDEA: fine tuning the support of the mixed strategy $\mathbf{z}$.
- Recall the constraints of an NE $R\mathbf{z} \leq 1, \mathbf{z} \geq 0$

$$R = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

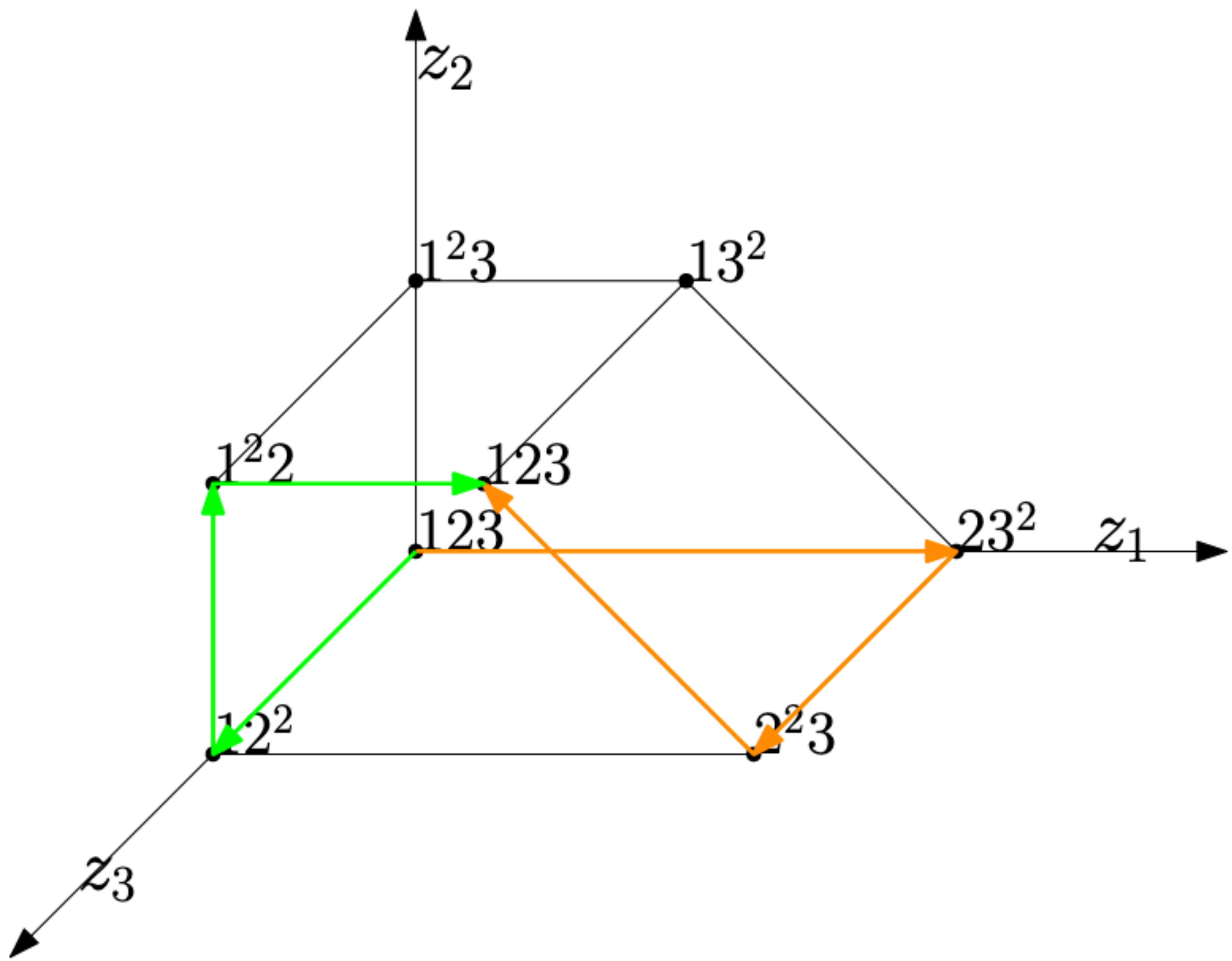


Preliminaries

- Non-degenerate Assumption! At every corner of the polytope exactly n out of the $2n$inequalities are tight.
  - Otherwise we can perturb each entry of the matrix R.
- Def: a corner $\mathbf{z}$ contains pure strategy i if $(R\mathbf{z})_i = 1$ or $z_i = 0$.
  - If $\mathbf{z} \neq \mathbf{0}$ contains all pure strategies $[n]$, we can find an NE $(\frac{\mathbf{z}}{|\mathbf{z}|_1}, \frac{\mathbf{z}}{|\mathbf{z}|_1})$. Since $z_i > 0 \Rightarrow \mathbf{e}_i^T R\mathbf{z} \geq \mathbf{e}_k^T R\mathbf{z}, \forall k \in [n]$.

The Algorithm

- Start with $(0,0,\ldots,0)$
- By non-degeneracy we have exactly n edges of the polytope adjacent to the $(0,0,\ldots,0)$ corner. Each of these edges corresponds to un-tigthening one of the $z_i > 0$ inequalities.
- Say we un-tighten $z_1 \geq 0$. Along the edge we jump to a new endpoint. Again by non-degeneracy, there exists some j s.t. $(R\mathbf{z})_j = 1$ and $z_j = 0$. (Why?)
- Check the new endpoint, is it an NE?
- If not, we have two choices now: $(R\mathbf{z})_j = 1$ and $z_j = 0$
- Hence we define a directed walk on the polytope…

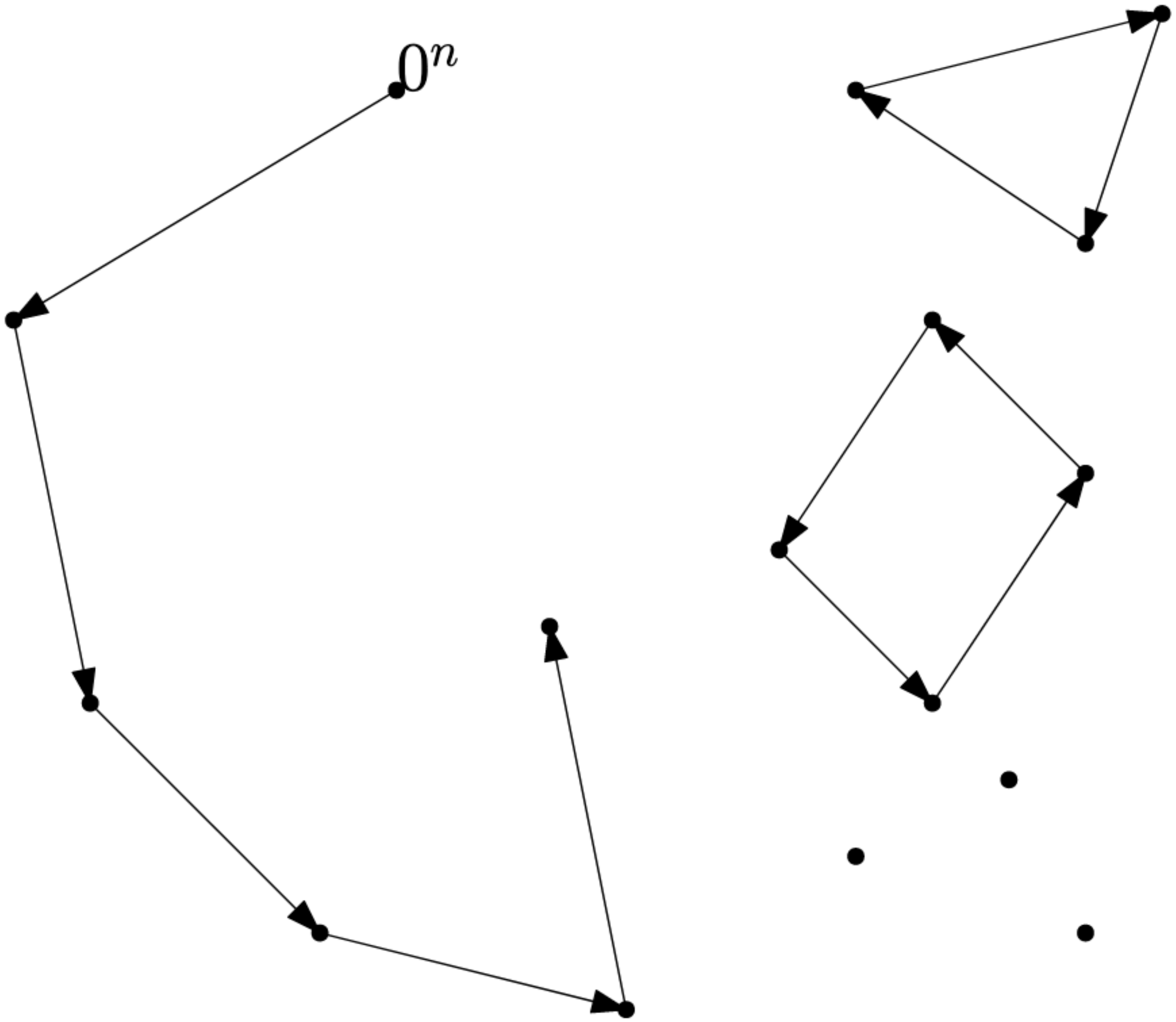


z_2
$1^2 3$
13^2
$1^2 2$
123
123
23^2
z_1
12^2
$2^2 3$
z_3

Remarks

- It needs exponential time in the worst case.
- It provides an alternative proof that a Nash equilibrium exists in 2-player games
- It also shows that there always exists a rational equilibrium in 2-player games
- Moreover, it shows that the number of NEs in any 2-player game is odd.
- It makes a fundamental contribution to the complexity of NE, fixed point and even cake cutting…

End-of-A-Line (EoAL)



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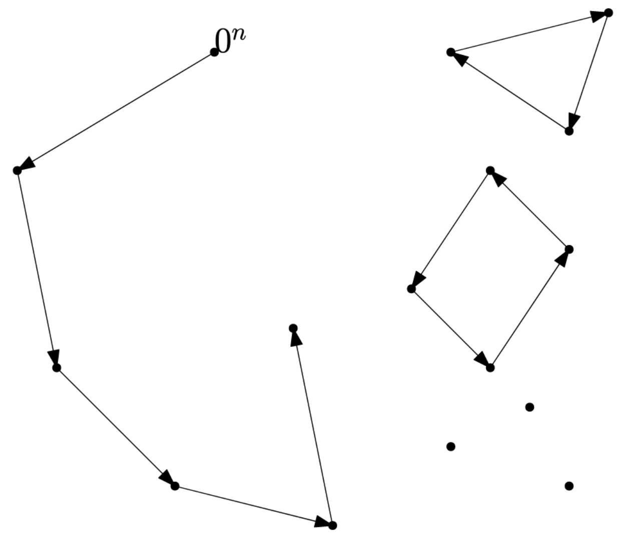


- Two Boolean circuits $S, P : \{0,1\}^n \to \{0,1\}^n$ with gates $\vee, \wedge, \neg$
- Such that $P(0^n) = 0^n \neq S(0^n)$.
- Output: a node $\mathbf{x} \in \{0,1\}^n$ s.t. $P(S(\mathbf{x})) \neq \mathbf{x}$ or $S(P(\mathbf{x})) \neq \mathbf{x} \neq 0^n$.
- Recall the LH algo, can you find the similarities?

“Open”: An explicit reduction from LH algo to EoAL?

PPAD [Papadimitriou'94]

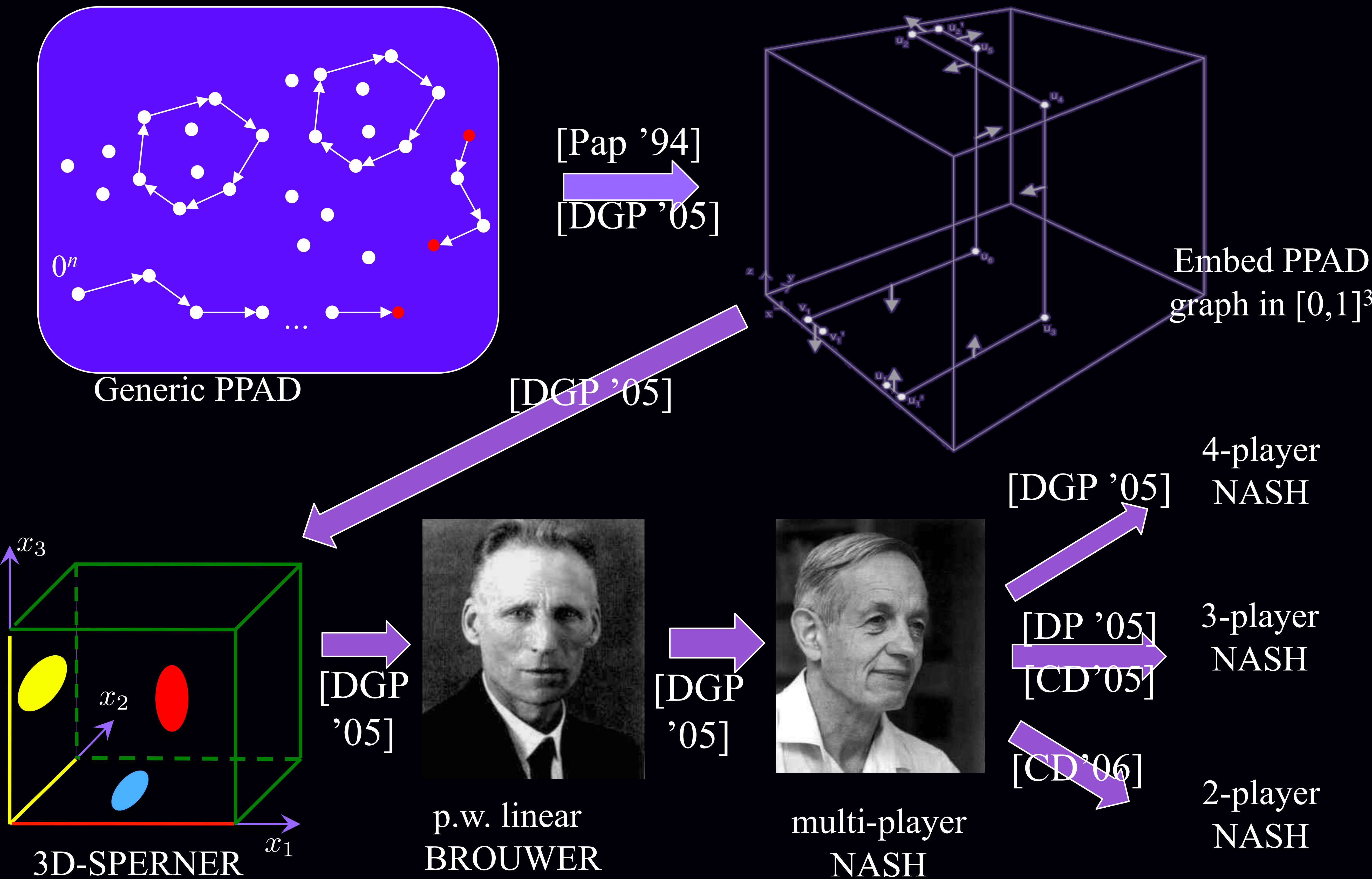
={Search problems reducible to EoAL}

- Finding an NE in games is PPAD-complete. [DGP'06a, CD'06a, DGP'06b, CD'06b, CDT'06]
- Other games: Polymatrix games, anonymous games, graphical games.
- Simpler constraints: sparse games, win-lose games.
- Simpler NE: ANE, WSNE, even "can almost people be almost happy?"
- The proofs are so involved, we will not cover in this class…

The PLAN

DGP = Daskalakis, Goldberg, Papadimitriou

CD = Chen, Deng



Q&A?

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