习题 5-1

1. (1) 2

(2) 1　　羞辱智商!!!

(3) 1

(4) 3

2. (1) 对 $x^2-xy+y^2=C$ 两边求导

$2x-y-xy'+2yy'=0$

$\Rightarrow y'=\dfrac{2x-y}{x-2y}$

满足 $(x-2y)y'=2x-y$

(2) 对 $y=\ln(xy)$ 两边求导

$y'=\dfrac{1}{xy}(y+xy')$

$\Rightarrow y'=\dfrac{y}{xy-x}$

则 $y''=\dfrac{y'(xy-x)-y(y+xy'-1)}{(xy-x)^2}=\dfrac{-xy'-y^2+y}{(xy-x)^2}$

则 $(xy-x)y''+x(y')^2+yy'-2y'$

$=\dfrac{-xy'-y^2+y}{xy-x}+\dfrac{xy^2}{(xy-x)^2}+yy'-2y'$

$=0$

(3) $x=0$ 时, $y(0)=0\cdot\int_0^0\sqrt{1+t^4}\,dt=0$

$\dfrac{dy}{dx}=\int_0^x\sqrt{1+t^4}\,dt+x\sqrt{1+x^4}$

则 $x\cdot\dfrac{dy}{dx}-y=x\int_0^x\sqrt{1+t^4}\,dt+x^2\sqrt{1+x^4}-y$

$=y+x^2\sqrt{1+x^4}-y$

$=x^2\sqrt{1+x^4}$

(4) $x=1$ 时, $y^2(1)=4-0\Rightarrow y(1)=\pm2$. 满足 $y(1)=2$

对 $y^2=4x-x\ln x$ 两边求导.

$2y\cdot y'=4-\ln x-x\cdot\frac{1}{x}\Rightarrow y'=\dfrac{4-\ln x-1}{2y}=\dfrac{3-\ln x}{2y}\Rightarrow 2y\,dy=(3-\ln x)\,dx$

则 $2xy\cdot dy=x(3-\ln x)dx=(3x-x\ln x+x-x)dx=(4x-x\ln x-x)dx$

$=(y^2-x)dx$

(5) $y(0)=\frac{1}{2}(3e^0-e^0)=1$

又 $y'=3e^{2x}+e^{-2x}$

则 $y'(0)=3e^0+e^0=4$.

$y''=6e^{2x}-2e^{-2x}=(3e^{2x}-e^{-2x})\times\frac{1}{2}\times4=4y$

即 $\frac{d^2y}{dx^2}=4y$ 则:

$\frac{d^2y}{dx^2}-4y=0$

3. (1) $y'=x^2$

(2) 设任意点斜率为 $y'(x_0)$; 则切线方程为 $y-y_0=y'(x_0)(x-x_0)$

即: $y-y_0-y'(x_0)x+y'(x_0)x_0=0$

则 $(0,0)$ 点切线距离为:

$$\frac{|-y_0+y'(x_0)x_0|}{\sqrt{1+[y'(x_0)]^2}}=x_0$$

$\Rightarrow 2x_0y_0y'(x_0)-y_0^2+x_0^2=0$

又 (x_0,y_0) 是任意的, 则:

$\Rightarrow 2xyy'-y^2+x^2=0$

(3) 设曲线方程为 $y=f(x)$

则切线在 $P(x,y)$ 处斜率为 $y'=f'(x)$.

法线斜率为 $k=-\frac{1}{y'}$

在点 (x_0,y_0) 处法线方程为 $y-y_0=-\frac{1}{y_0'}(x-x_0)$.

该法线与 x 轴交于 $(y_0y_0'+x_0,0)$

又 (x_0,y_0) 与 $(y_0y_0'+x_0,0)$ 的中点为 $\left(\frac{y_0y_0'+2x_0}{2},\frac{y_0}{2}\right)$

由题意, $\frac{y_0y_0'+2x_0}{2}=0$

即 $y_0y_0'+2x_0=0$

从而得到满足条件的曲线的微分方程:

$yy'+2x=0$

(4) 设曲线方程为$y=f(x)$

则切线在$P(x,y)$处斜率为$y'=f'(x)$

在点(x_0,y_0)处切线方程为 $y-y_0=y_0'(x-x_0)$

令$x=0$，得$y=y_0-y_0'x_0$

即PQ坐标为$(0, y_0-y_0'x_0)$

因PQ长度为2，则：

$$\sqrt{(0-x_0)^2+(y_0-y_0'x_0-y_0)^2}=2$$

$\Rightarrow x_0^2+y_0'^2 \cdot x_0^2=4$，又$\because x=2$时，$y=0$

则满足条件的曲线微分方程为：

$$\begin{cases} x^2(1+y'^2)=4 \\ y(2)=0 \end{cases}$$

(5). 设曲线方程为$y=f(x)$

则$M(x_0,y_0)$处切线为：$y-y_0=y_0'(x-x_0)$

令$x=0$，$y=y_0-y_0'x_0$ 则$Q(0, y_0-y_0'x_0)$

令$y=0$ $x=\frac{-y_0}{y_0'}+x_0$ 则$P(\frac{-y_0+y_0'x_0}{y_0'},0)$

因PM被Q平分，则：

$$\begin{cases} 2\times 0=\dfrac{-y_0+y_0'x_0}{y_0'}+x_0 \\ 2\times(y_0-y_0'x_0)=0+y_0 \end{cases}$$

$\Rightarrow 2x_0y_0'-y_0=0$

又$\because y(3)=1$

则满足条件的曲线微分方程为

$$\begin{cases} 2xy'-y=0 \\ y(3)=1 \end{cases}$$