习题 4-4

1. (1) $\int \cos(1-x)dx = \int -\cos(1-x)d(1-x) = -\sin(1-x)+C$

(2) $\int \sqrt{7+5x}\,dx = \frac{1}{5}\int \sqrt{7+5x}\,d(5x+7) = \frac{2}{15}(7+5x)^{\frac{3}{2}}+C$

(3) $\int \frac{e^{2x}-1}{e^{x}}dx = \int (e^{x}-e^{-x})dx = e^{x}+e^{-x}+C$

(4) $\int \frac{dx}{9+x^2} = \frac{1}{3}\int \frac{1}{1+(\frac{x}{3})^2}d(\frac{x}{3}) = \frac{1}{3}\arctan\frac{x}{3}+C$

(5) $\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{1}{\sqrt{1-(\frac{3x}{2})^2}}d(\frac{3x}{2}) = \frac{1}{3}\arcsin\frac{3x}{2}+C$

(6) $\int \frac{x^2}{4+x^3}dx = \frac{1}{3}\int \frac{1}{4+x^3}dx = \frac{1}{3}\ln|4+x^3|+C$

(7) $\int \frac{\ln x}{x}dx = \int \ln x\, d\ln x = \frac{1}{2}\ln^2 x+C$

(8) $\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx = 2\int \frac{1}{2\sqrt{x}}\sin\sqrt{x}\,dx = 2\int \sin\sqrt{x}\,d\sqrt{x} = -2\cos\sqrt{x}+C$

(9) $\int \frac{dx}{\cos^2 x\sqrt{1+\tan x}} = 2\int \frac{1}{2\sqrt{1+\tan x}}d(1+\tan x) = 2\sqrt{1+\tan x}+C$

(10) $\int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{1}{\sqrt{1-(x^4)^2}}dx^4 = \frac{1}{4}\arcsin x^4+C$

(11) $\int \cos^2\frac{x}{2}dx = \frac{1}{2}\int (\cos x+1)dx = \frac{1}{2}(x+\sin x)+C$

(12) $\int \cos x\sin 3x\,dx = \int (\sin 4x+\sin 2x-\sin 3x\cos x)dx = \int (\sin 4x+\sin 2x)dx - \int \cos x\sin 3x\,dx$

则 $\int \cos x\sin 3x\,dx = \frac{1}{2}\int (\sin 4x+\sin 2x)dx = \frac{1}{2}(-\frac{1}{4}\cos 4x-\frac{1}{2}\cos 2x)$

$= -\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x+C$

(13) $\int \frac{\sin x\cos x}{1+\cos^2 x}dx = -\frac{1}{2}\int \frac{1}{1+\cos^2 x}d\cos^2 x = -\frac{1}{2}\ln|1+\cos^2 x|+C$

(14) $\int \frac{\sqrt{\arctan x}}{1+x^2}dx = \int \sqrt{\arctan x}\,d(\arctan x) = \frac{2}{3}(\arctan x)^{\frac{3}{2}}+C$

(15) $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx = 2\int \sqrt{1+\sqrt{x}}\, d(\sqrt{x}+1) = 2\times\frac{2}{3}(1+\sqrt{x})^{\frac{3}{2}} + C = \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C$

(16) $\int \frac{dx}{\sqrt{4-x^2}\arccos\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-(\frac{x}{2})^2}\arccos\frac{x}{2}} = -\int \frac{d\arccos\frac{x}{2}}{\arccos\frac{x}{2}} = -\ln\left|\arccos\frac{x}{2}\right| + C$

(17) $\int \tan\sqrt{1+x^2}\,\frac{x\,dx}{\sqrt{1+x^2}} = \int \frac{\sin(\sqrt{1+x^2})2x\,dx}{\cos\sqrt{1+x^2}\cdot 2\sqrt{1+x^2}} = -\int \frac{1}{\cos\sqrt{1+x^2}}d(\cos\sqrt{1+x^2}) = -\ln\left|\cos\sqrt{1+x^2}\right| + C$

(18) $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x}{1+e^{2x}}dx = \int \frac{1}{1+e^{2x}}de^x = \arctan e^x + C$

(19) $\int \frac{\sin x}{\cos^3 x}dx = -\int \frac{1}{\cos^3 x}d\cos x = \frac{1}{2\cos^2 x} + C$

(20) $\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}dx = \int (\sin x-\cos x)^{-\frac{1}{3}}d(\sin x-\cos x) = \frac{3}{2}(\sin x-\cos x)^{\frac{2}{3}} + C$

(21) $\int \frac{10^{2\arccos x}}{\sqrt{1-x^2}}dx = \frac{1}{-2\ln 10}\int \frac{10^{2\arccos x}\cdot 2\ln 10\,dx}{\sqrt{1-x^2}} = \frac{1}{-2\ln 10}\int d(10^{2\arccos x})$

$= -\frac{10^{2\arccos x}}{2\ln 10}$

(22) $\int \frac{\ln\tan x}{\cos x\sin x}dx = \frac{1}{2}\int \frac{2\ln\tan x}{\cos^2 x\,\tan x} = \frac{1}{2}\int d[(\ln\tan x)^2] = \frac{1}{2}(\ln(\tan x))^2 + C$.

(23) $\int \frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx = \frac{1}{2}\int \frac{2\ln(x+\sqrt{1+x^2})\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)}{x+\sqrt{x^2+1}}dx = \frac{1}{2}\int d(\ln^2(x+\sqrt{x^2+1}))$

$= \frac{1}{2}\ln^2(x+\sqrt{x^2+1}) + C$

(24) $\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}} = \int \frac{\sqrt{x+1}-\sqrt{x-1}}{2}dx = \int \frac{\sqrt{x+1}}{2}d(x+1) - \int \frac{\sqrt{x-1}}{2}d(x-1)$

$= \frac{1}{3}\left[(x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}}\right] + C$

2. (1) $\int \frac{dx}{2x^2+x-1} = \frac{1}{2}\int \frac{dx}{(x+1)(x-\frac{1}{2})}$

对于 $\int \frac{dx}{(x+1)(x-\frac{1}{2})}$，令 $\frac{1}{(x+1)(x-\frac{1}{2})} = \frac{A}{x+1} + \frac{B}{x-\frac{1}{2}}$

则 $1 = (x-\frac{1}{2})A + (x+1)B$.

则 $\begin{cases} A+B=0 \\ -\frac{1}{2}A+B=1 \end{cases} \Rightarrow A=-\frac{2}{3} \quad B=\frac{2}{3}$

则原式 $= \frac{1}{2}\int (\frac{-\frac{2}{3}}{x+1} + \frac{\frac{2}{3}}{x-\frac{1}{2}})dx = \frac{1}{3}(\ln|x-\frac{1}{2}| - \ln|x+1|) = \frac{1}{3}\ln\left|\frac{x-\frac{1}{2}}{x+1}\right| + C$.

$= \frac{1}{3}\ln\left|\frac{2x-1}{x+1}\right| + C'$

(2) 因 $2^2-4\times3=-8<0$，又 $x^2+2x+3=(x+1)^2+2$，令 $t=x+1$

即 $x=t-1$，于是 $dx=dt$. 则

原式 $= \int \frac{1}{t^2+2}dt = \frac{\sqrt{2}}{2}\int \frac{1}{(\frac{t}{\sqrt{2}})^2+1}d\frac{t}{\sqrt{2}} = \frac{\sqrt{2}}{2}\arctan\frac{t}{\sqrt{2}} + C = \frac{\sqrt{2}}{2}\arctan\frac{x+1}{\sqrt{2}} + C$

（熟练后直接写即可）

(3) $\int \frac{dx}{a^2-x^2} = \int (\frac{\frac{1}{2a}}{a-x} + \frac{\frac{1}{2a}}{a+x})dx = \frac{1}{2a}(\int \frac{-1}{a-x}d(-x) + \int \frac{1}{a+x}dx) = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

(4) $\int \frac{x^2}{1+x}dx = \int \frac{x^2-1+1}{x+1}dx = \int (x-1+\frac{1}{x+1})dx = \frac{1}{2}x^2 - x + \ln|x+1| + C$

（与答案不同的原因是答案化简后常数归入C中）

(5) $\int \frac{x^2}{1-x^2}dx = \int \frac{x^2-1+1}{1-x^2}dx = \int (-1+\frac{1}{1-x^2})dx = \int (-1+\frac{\frac{1}{2}}{1-x}+\frac{\frac{1}{2}}{1+x})dx$

$= -x + \frac{1}{2}(\int \frac{-1}{1-x}d(-x) + \int \frac{1}{1+x}dx) = -x + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

(6) 对于 $\frac{x+1}{x^2+2x} = \frac{x+1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

则 $x+1 = (A+B)x + 2A$，则 $A+B=1$，$2A=1$ 得 $A=\frac{1}{2}$. $B=\frac{1}{2}$

则原式 $= \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{2}\int \frac{1}{x+2}d(x+2) = \frac{1}{2}(\ln|x| + \ln|x+2|) + C = \frac{1}{2}\ln|x^2+2x| + C$

(7) 对 $\frac{x^2+1}{(x+1)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$\Rightarrow x^2+1 = (A+B)x^2 + (2A+C)x + A-B-C \quad \Rightarrow A=\frac{1}{2} \quad B=\frac{1}{2} \quad C=-1$

则原式 $= \int \frac{\frac{1}{2}}{x-1}d(x-1) + \int \frac{\frac{1}{2}}{x+1}d(x+1) + \int \frac{-1}{(x+1)^2}d(x+1) = \frac{1}{x+1} + \frac{1}{2}\ln|x^2-1| + C$

(8) 因为 $\frac{x^3-1}{4x^3-x}=\frac{x^3-\frac{1}{4}x+\frac{1}{4}x-1}{4x^3-x}=\frac{1}{4}+\frac{\frac{1}{4}x-1}{4x^3-x}=\frac{1}{4}+\frac{1}{16}\frac{x-4}{x(x-\frac{1}{2})(x+\frac{1}{2})}$

对 $\frac{x-4}{x(x-\frac{1}{2})(x+\frac{1}{2})}=\frac{A}{x}+\frac{B}{x-\frac{1}{2}}+\frac{C}{x+\frac{1}{2}}$

则 $x-4=(A+B+C)x^2+(\frac{1}{2}B-\frac{1}{2}C)x-\frac{1}{4}A \quad \Rightarrow A=16 \quad B=-7 \quad C=-9$.

则原式 $=\int\frac{1}{4}dx+\frac{1}{16}\left(\int\frac{16}{x}dx+\int\frac{-7}{x-\frac{1}{2}}dx+\int\frac{-9}{x+\frac{1}{2}}d(x+1)\right)$

$=\frac{x}{4}+\int\frac{1}{x}dx-\frac{7}{16}\int\frac{1}{x-\frac{1}{2}}d(x-\frac{1}{2})-\frac{9}{16}\int\frac{1}{x+\frac{1}{2}}d(x+\frac{1}{2})+C$

$=\frac{x}{4}+\ln|x|-\frac{7}{16}\ln|x-\frac{1}{2}|-\frac{9}{16}\ln|x+\frac{1}{2}|+C$

(9) 对 $\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$

则 $1=(A+B)x^2+(A-B+C)x+A-C \quad \Rightarrow A=\frac{1}{3} \quad B=-\frac{1}{3} \quad C=-\frac{2}{3}$.

则原式 $=\int\frac{\frac{1}{3}}{x-1}d(x-1)+\int\frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}dx$

$=\frac{1}{3}\int\frac{1}{x-1}d(x-1)-\frac{1}{3}\int\frac{x+2}{x^2+x+1}dx$

$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+2}{(x+\frac{1}{2})^2+\frac{3}{4}}dx \quad (令\ t=x+\frac{1}{2})$

$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{t+\frac{3}{2}}{t^2+\frac{3}{4}}dx$

$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{t}{t^2+\frac{3}{4}}dt-\frac{1}{2}\int\frac{dt}{t^2+\frac{3}{4}}$

$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(t^2+\frac{3}{4})-\frac{1}{2}\frac{2}{\sqrt{3}}\arctan\frac{2t}{\sqrt{3}}+C$

$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$.

(10) 对 $\frac{x^2}{1-x^4}=\frac{x^2}{(1+x^2)(1+x)(1-x)}=\frac{Ax+B}{1+x^2}+\frac{C}{1+x}+\frac{D}{1-x}$.

$\Rightarrow x^2=(-A+D-C)x^3+(-B+C+D)x^2+(A-C+D)x+B+C+D$

$\Rightarrow A=0 \quad B=-\frac{1}{2} \quad C=\frac{1}{4} \quad D=\frac{1}{4}$

则原式 $=\int\frac{-\frac{1}{2}}{1+x^2}dx+\int\frac{\frac{1}{4}}{1+x}d(1+x)+\int\frac{-\frac{1}{4}}{1-x}d(1-x)$

$=\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|-\frac{1}{2}\arctan x+C$

(11) 对 $\frac{1}{x^4(2x^2-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}+\frac{E}{x-\frac{\sqrt{2}}{2}}+\frac{F}{x+\frac{\sqrt{2}}{2}}$

$\Rightarrow A=0 \quad B=-2 \quad C=0 \quad D=-1 \quad E=\sqrt{2} \quad F=-\sqrt{2}$

则原式 $=\int\left(\frac{-2}{x^2}+\frac{-1}{x^4}+\frac{\sqrt{2}}{x-\frac{\sqrt{2}}{2}}+\frac{-\sqrt{2}}{x+\frac{\sqrt{2}}{2}}\right)dx$

$=\frac{2}{x}+\frac{1}{3x^3}+\sqrt{2}\ln\left|\frac{x-\frac{\sqrt{2}}{2}}{x+\frac{\sqrt{2}}{2}}\right|+C$

(12) 原式 $=\int\frac{x^4+2x^2+1-x}{x^5+2x^3+x}dx=\int\left(\frac{1}{x}-\frac{x}{x^5+2x^3+x}\right)dx=\int\frac{1}{x}-\frac{1}{x^4+2x^2+1}dx=\ln|x|-\int\frac{1}{(x^2+1)^2}dx$

对 $\int\frac{1}{(x^2+1)^2}dx$. 令 $x=\tan u$. 则 $\int\frac{1}{(x^2+1)^2}dx=\int\frac{\sec u^2du}{\sec u^4}=\int\cos u^2\,du=\frac{u}{2}+\frac{\sin 2u}{4}+C$

$=\frac{\arctan x}{2}+\frac{1}{2}\frac{x}{(x^2+1)^2}+C$

则原式 $=\ln|x|-\frac{x}{2(x^2+1)}-\frac{1}{2}\arctan x+C$.

(13) 对 $\frac{1}{x^4+3x^2}=\frac{1}{x^2(x^2+3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+3}$

$\Rightarrow A=0 \quad B=\frac{1}{3} \quad C=0 \quad D=-\frac{1}{3}$

则原式 $=\int\left(\frac{\frac{1}{3}}{x^2}-\frac{\frac{1}{3}}{x^2+3}\right)dx=-\frac{1}{3}\frac{1}{x}-\frac{1}{3}\int\frac{1}{x^2+3}dx=-\frac{1}{3}\frac{1}{x}-\frac{1}{3}\frac{\sqrt{3}}{3}\int\frac{1}{(\frac{x}{\sqrt{3}})^2+1}d\frac{x}{\sqrt{3}}$

$=-\frac{1}{3}\frac{1}{x}-\frac{\sqrt{3}}{9}\arctan\frac{x}{\sqrt{3}}+C$

(14) $\int\frac{x^2}{(x-1)^{100}}dx=\int\frac{(t+1)^2}{t^{100}}dt=\int\frac{t^2+1+2t}{t^{100}}dt=\int\frac{1}{t^{98}}+\frac{1}{t^{100}}+\frac{2}{t^{99}}dt=-\frac{t^{-97}}{97}-\frac{t^{-99}}{99}-\frac{2t^{-98}}{98}+C$

$=\frac{-1}{(x-1)^{99}}\left[\frac{(x-1)^2}{97}+\frac{x-1}{49}+\frac{1}{99}\right]+C$

(15) $\int\frac{1-x^7}{x(1+x^7)}dx=\int\frac{-1}{x}+\frac{2}{x(1+x^7)}dx=-\ln|x|+\frac{2}{7}\int\frac{7x^6}{x^7(1+x^7)}dx$

$=-\ln|x|+\frac{2}{7}\int\frac{1}{x^7(1+x^7)}d(x^7)=-\ln|x|+\frac{2}{7}\int\frac{1}{x^7}-\frac{1}{1+x^7}dx^7$

$=-\ln|x|+\frac{2}{7}\ln\frac{x^7}{1+x^7}+C$

(16) 令 $u=x^4$. $du=4x^3dx$.

原式 $=\frac{1}{4}\int\frac{u^2}{u^2+4u+5}du=\frac{1}{4}\int(1-\frac{4u+5}{u^2+4u+5})du=\frac{1}{4}\int(1-\frac{4(u+2)-3}{1+(u+2)^2})du$

$=\frac{u}{4}-\frac{1}{2}\ln|1+(u+2)^2|+\frac{3}{4}\arctan(u+2)+C$

$=\frac{x^4}{4}-\frac{1}{2}\ln|x^8+4x^4+5|+\frac{3}{4}\arctan(x^4+2)+C$

3. (1) 原式 $=\int\frac{1}{5-4(2\cos^2\frac{x}{2}-1)}dx=\int\frac{1}{9-8\cos^2\frac{x}{2}}dx=\int\frac{\sec^2\frac{x}{2}}{9\sec^2\frac{x}{2}-8}dx$

$=\int\frac{2}{9(\tan^2\frac{x}{2}+1)-8}\cdot d(\tan\frac{x}{2})=\frac{2}{9}\int\frac{1}{\tan^2\frac{x}{2}+\frac{1}{9}}d\tan\frac{x}{2}$

$=\frac{2}{3}\arctan(3\tan\frac{x}{2})+C$

(2) 原式 $=\int\frac{dx}{3+\cos^2x}=\int\frac{1+\tan^2x}{3+4\tan^2x}dx=\frac{1}{3}\int\frac{\sec^2x}{1+(\frac{2}{\sqrt3}\tan x)^2}dx$

$=\frac{1}{3}\cdot\frac{\sqrt3}{2}\cdot\int\frac{1}{1+(\frac{2}{\sqrt3}\tan x)^2}d(\frac{2}{\sqrt3}\tan x)=\frac{\sqrt3}{6}\arctan\frac{2\tan x}{\sqrt3}+C$

(3) 原式 $=\frac{1}{2}\int\frac{1}{(\frac{1}{\sqrt2}\sin x+\frac{1}{\sqrt2}\cos x)^2}dx=\frac{1}{2}\int\frac{1}{(\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4})^2}dx$

$=\frac{1}{2}\int\frac{1}{(\sin(x+\frac{\pi}{4}))^2}d(x+\frac{\pi}{4})=-\frac{1}{2}\cot(x+\frac{\pi}{4})+C$

(4) 原式 $=\int((\csc x)^2-1)\cdot\cot x\cdot dx=\int\cot x\csc^2x\,dx-\int\frac{\cos x}{\sin x}dx$

$=-\int\cot x\,d\cot x-\int\frac{1}{\sin x}d\sin x=-\frac{1}{2}\cot^2x-\ln|\sin x|+C$

(5) 原式 $=\int\tan^2x(1+\tan^2x)dx=\int\tan^2x\cdot\sec^2x\,dx$

$=\int\tan^2x\,d\tan x=\frac{1}{3}\tan^3x+C$

(6) 原式 $=\int(1-\cos^2x)\sin^2x\,dx=\int\sin^2x\,dx-\frac{1}{4}\int\sin^2(2x)\,dx$

$=\frac{1}{2}\int(1-\cos2x)\,dx-\frac{1}{8}\cdot\int(1-\cos4x)\,dx=\frac{1}{2}x-\frac{1}{2}\sin2x-\frac{1}{8}x+\frac{1}{4}\sin4x+C$

$=\frac{3}{8}x-\frac{1}{2}\sin2x+\frac{1}{4}\sin4x+C$

(7) 原式 $=\int\frac{1-\sin x}{\cos^2x}\,dx=\int\frac{1}{\cos^2x}\,dx+\int\frac{1}{\cos^2x}\,d\cos x=\tan x-\frac{1}{\cos x}+C$

(8) 原式 $=\frac{1}{2}\cdot\int\frac{\sin2x}{1+\left(\frac{1-\cos2x}{2}\right)^2}\,dx=\frac{1}{4}\cdot\int\frac{\sin2x}{1+\left(\frac{1-\cos2x}{2}\right)^2}\,d(2x)=-\frac{1}{4}\int\frac{1}{1+\left(\frac{1-\cos2x}{2}\right)^2}\,d\cos2x$

$=\frac{1}{2}\int\frac{1}{1+\left(\frac{1-\cos2x}{2}\right)^2}\,d\left(\frac{1-\cos2x}{2}\right)=\frac{1}{2}\arctan\frac{1-\cos2x}{2}+C=\frac{1}{2}\arctan\frac{2\sin^2x}{2}+C$

$=\frac{1}{2}\arctan\sin^2x+C$

(9) 原式 $=\int\sec^2 2x\cdot d(\tan x)=\int(\tan^2x+1)\,d\tan x=\frac{1}{3}\tan^3x+\tan x+C$

(10) 用万能公式，令 $u=\tan\frac{x}{2}$

则原式 $=\int\frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du=\int\frac{1}{1+u}\,du=\ln|1+u|+C$

$=\ln\left|1+\tan\frac{x}{2}\right|+C$

(11) 原式 $=\int\frac{\sec^2x}{\tan x}\,dx=\int\frac{1}{\tan x}\,d\tan x=\ln|\tan x|+C$

(12) 原式 $=\int\frac{1}{\sin^2x}+\frac{2\sin x\cos x}{\sin^2x}\,dx=\int\left(\frac{1}{\sin^2x}+\frac{2\cos x}{\sin x}\right)dx=\int(\csc^2x)\,dx+2\int\frac{1}{\sin x}\,d\sin x$

$=-\cot x+2\ln|\sin x|+C$

(13) 原式 $=\int\frac{\frac{1}{2}\sin2x}{(\sin^2x+\cos^2x)^2-2\sin^2x\cos^2x}\,dx=\int\frac{\frac{1}{2}\sin2x}{1-\frac{(\sin2x)^2}{2}}\,dx=\int\frac{\sin2x}{2-(\sin2x)^2}\,dx$

$=\int\frac{\sin2x}{1+\cos^22x}\,dx=-\frac{1}{2}\int\frac{1}{1+\cos^22x}\,d\cos2x=-\frac{1}{2}\arctan(\cos2x)+C$

(14). 令 $t=\cos x$. $1-t^2=\sin^2 x$

原式 $= -\int \frac{\sin^4 x}{\cos^4 x}\, d\cos x = -\int \frac{(1-t^2)^2}{t^4}\,dt = \int(-t^4+2t^2-1)/t^4\,dt$

$= \int\left(-1+\frac{2}{t^2}-\frac{1}{t^4}\right)dt = -t-\frac{2}{t}+\frac{1}{3t^3}+C$

$= -\cos x - 2\sec x + \frac{1}{3}\sec^3 x + C$

(15) 用万能公式：令 $t=\tan\frac{x}{2}$，即 $x=2\arctan t$，则 $\sin x=\frac{-2t}{1+t^2}$，$\cos x=\frac{1-t^2}{1+t^2}$，

$\tan x=\frac{-2t}{1-t^2}$ $\quad dx=\frac{1}{1+t^2}dt$

原式 $= \int \frac{\frac{1+2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2t}{1+t^2}\cdot\frac{2dt}{1+t^2} = \frac{1}{2}\int\left(\frac{1}{t}+t+2\right)dt$

$= \frac{1}{2}\ln t+\frac{1}{4}t^2+t+C = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+\frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+C$

(16) 令 $t=\tan x$，则 $x=\arctan t$，$dx=\frac{1}{1+t^2}dt$.

原式 $= \int\frac{1}{1+2t}\cdot\frac{1}{1+t^2}dt$，

设 $\frac{1}{1+2t}\cdot\frac{1}{1+t^2}=\frac{A}{1+2t}+\frac{Bt+C}{1+t^2}$ 则 $A(1+t^2)+(Bt+C)(1+2t)=1$

$\Rightarrow A=\frac{4}{5}\quad B=\frac{-2}{5}\quad C=\frac{1}{5}$

则原积分 $= \int\frac{\frac{4}{5}}{1+2t}dt+\int\frac{-\frac{2}{5}t+\frac{1}{5}}{1+t^2}dt = \frac{2}{5}\ln|1+2t|+\frac{1}{5}\int\frac{1}{1+t^2}dt-\frac{1}{5}\int\frac{dt^2}{1+t^2}$

$= \frac{2}{5}\ln|1+2t|-\frac{1}{5}\ln|1+t^2|+\frac{1}{5}\arctan t+C$

$= \frac{2}{5}\ln|1+2\tan x|-\frac{1}{5}\ln|1+\tan^2 x|+\frac{1}{5}x+C$

(17) 原式 $= \int\cos^4 x\, d\sin x = \int[1-\sin^2 x]^2 d\sin x = \int(1-2\sin^2 x+\sin^4 x)\,d\sin x$

$= \sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+C$

(18) $\cos^6 x = (\cos^2 x)^3 = \left[\frac{1+\cos 2x}{2}\right]^3 = \frac{1}{8}(1+\cos 2x)^3 = \frac{5}{16}+\frac{15}{32}\cos 2x+\frac{3}{16}\cos 4x+\frac{1}{32}\cos 6x$

则原式 $= \int\left[\frac{5}{16}+\frac{15}{32}\cos 2x+\frac{3}{16}\cos 4x+\frac{1}{32}\cos 6x\right]dx$

$= \frac{5}{16}x+\frac{15}{64}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{192}\sin 6x+C$

(此题亦可用 $E=\int(\cos^6 x+\sin^6 x)dx$. $F=\int(\cos^6 x-\sin^6 x)dx$ 列方程计算)

4. (1) 令 $t=\sqrt{x+2}$，$x=t^2-2$，$dx=2t\,dt$

原式 $=\int\frac{t}{1+t}\cdot 2t\,dt=\int\frac{2t^2+2t-2t}{1+t}dt=\int\left(2t-\frac{2t}{1+t}\right)dt$

$=\int\left(2t-\frac{2t+2-2}{1+t}\right)dt=\int\left(2t-2+\frac{2}{1+t}\right)dt=t^2-2t+2\ln|1+t|+C$

$=x+2-2\sqrt{x+2}+2\ln|1+\sqrt{x+2}|+C=x-2\sqrt{x+2}+2\ln|1+\sqrt{x+2}|+C$

(2) 令 $t=x^{\frac{1}{6}}$，$dt=\frac{1}{6t^5}dx$，$dx=6t^5dt$

原式 $=\int\frac{\sqrt{t^6}\cdot 6t^5}{1+(t^6)^{\frac{1}{3}}}dt=6\int\frac{t^8}{1+t^2}dt=6\int\frac{t^6(1+t^2)-t^4(1+t^2)+t^2(1+t^2)-(1+t^2)+1}{(1+t^2)}dt$

$=6\int\left(t^6-t^4+t^2-1+\frac{1}{1+t^2}\right)dt=6\left(\frac{t^7}{7}-\frac{t^5}{5}+\frac{t^3}{3}-t+\arctan t\right)+C$

$=\frac{6}{7}x^{\frac{7}{6}}-\frac{6}{5}x^{\frac{5}{6}}+2\sqrt{x}-6\sqrt[6]{x}+6\arctan\sqrt[6]{x}+C$

(3) 令 $t=\sqrt{x}$. $x=t^2$. $dx=2t\,dt$

原式 $=\int\frac{1}{1+t}2t\,dt=2\int\frac{t+1-1}{t+1}dt=2\int\left(1-\frac{1}{1+t}\right)dt=2t-2\ln|1+t|+C$

$=2\sqrt{x}-2\ln(1+\sqrt{x})+C$

(4). 令 $t=\sqrt[3]{3x+2}$，$x=\frac{t^3}{3}-\frac{2}{3}$，$dx=t^2dt$

原式 $=\int\frac{1}{\frac{t^3}{3}-t-\frac{2}{3}}t^2dt=\int\frac{3t^2}{t^3-3t-2}dt$

令 $\frac{3t^2}{t^3-3t-2}=\frac{3t^2}{(t+1)^2(t-2)}=\frac{A}{t+1}+\frac{B}{(t+1)^2}+\frac{C}{t-2}$.

$\Rightarrow A=\frac{5}{3}\qquad B=-1\qquad C=\frac{4}{3}$

原式 $=\int\frac{\frac{5}{3}}{t+1}d(t+1)+\int\frac{-1}{(t+1)^2}(t+1)+\frac{4}{3}\int\frac{1}{t-2}d(t-2)$

$=\frac{5}{3}\ln|1+t|+\frac{1}{t+1}+\frac{4}{3}\ln|t-2|+C$

$=\frac{5}{3}\ln|\sqrt[3]{3x+2}+1|+\frac{1}{\sqrt[3]{3x+2}+1}+\frac{4}{3}\ln|\sqrt[3]{3x+2}-2|+C$

(5) 令 $t=\sqrt{1-2x}$. 则 $x=\frac{1-t^2}{2}$ $(t\geq 0)$，$dx=-t\,dt$

原式 $=\int\frac{1-t^2}{2}\cdot t\,(-t\,dt)=\int\left(\frac{t^4}{2}-\frac{t^2}{2}\right)dt=\frac{t^5}{10}-\frac{t^3}{6}+C$

$=\frac{1}{10}(1-2x)^{\frac{5}{2}}-\frac{1}{6}(1-2x)^{\frac{3}{2}}+C$

(6). 令 $t=\sqrt{\frac{1+x}{x}}$，则 $x=\frac{1}{t^2-1}$，$dx=\frac{-2t}{(t^2-1)^2}dt$

原式 $=\int(t^2-1)\cdot t\cdot\frac{-2t}{(t^2-1)^2}dt=\int\frac{-2t^2}{t^2-1}dt=\int\frac{-2t^2+2-2}{t^2-1}dt=\int(-2-\frac{2}{t^2-1})dt$

$=-2t-2\int\frac{1}{t^2-1}dt=-2t-2\times\frac{1}{2}\int(\frac{1}{t-1}-\frac{1}{t+1})dt$

$=-2t-\ln|t-1|+\ln|t+1|+C.$

$=-2\sqrt{\frac{1+x}{x}}-\ln|\sqrt{\frac{1+x}{x}}-1|+\ln|\sqrt{\frac{1+x}{x}}+1|+C$

(7) 令 $\sqrt[4]{x}=t$，则 $x=t^4$，$\sqrt{x}=t^2$，$dx=4t^3dt$.

原式 $=\int\frac{1}{t^2+t}4t^3dt=\int\frac{4t^2}{1+t}dt=\int\frac{4(t+1)^2-8t-8+4}{1+t}dt.$

$=\int(4(t+1)-8+\frac{4}{1+t})dt=2(t+1)^2-8t+4\ln|1+t|+C$

$=2t^2+2-4t+4\ln|1+t|+C$

$=2\sqrt{x}-4\sqrt[4]{x}+4\ln|\sqrt[4]{x}+1|+C$

(8) 令 $t=\frac{x-1}{x+1}$，$x=\frac{1+t}{1-t}$，$dx=\frac{2dt}{(1-t)^2}$

原式 $=\int\frac{1}{\sqrt[3]{(\frac{1+t}{1-t}+1)^2(\frac{1+t}{1-t}-1)^4}}\cdot\frac{2}{(1-t)^2}dt=\int\frac{1}{\sqrt[3]{\frac{2^2\cdot 2^4\cdot t^4}{(1-t)^2(1-t)^4}}}\cdot\frac{2}{(1-t)^2}dt$

$=\frac{2}{4}\cdot\int\frac{1}{t^{\frac{4}{3}}}dt=\frac{1}{2}\cdot\frac{t^{1-\frac{4}{3}}}{1-\frac{4}{3}}+C=\frac{-3}{2t^{\frac{1}{3}}}+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C$

(9). 令 $t=\sqrt[3]{\frac{2-x}{2+x}}$，则 $x=\frac{2-2t^3}{1+t^3}$，$(2-x)^2=\frac{16t^6}{(1+t^3)^2}$. $dx=\frac{-12t^2}{(1+t^3)^2}dt$

原式 $=\int t\cdot\frac{(1+t^3)^2}{16t^6}\cdot\frac{1}{(1+t^3)^2}\cdot(-12)t^2dt$

$=\int\frac{-3}{4t^3}dt=\frac{3}{8}t^{-2}+C=\frac{3}{8}\sqrt[3]{(\frac{2+x}{2-x})^2}+C.$

(10) 令 $x=a\cos\theta$. $dx=-a\sin\theta d\theta$. $\sqrt{a^2-x^2}=a\sin\theta$, $\theta=\arccos\frac{x}{a}$. $\sin 2\theta=\frac{2x\sqrt{a^2-x^2}}{a^2}$.

原式 $=\int\frac{a^2\cos^2\theta}{a\sin\theta}(-a\sin\theta)d\theta=-\int a^2\cos^2\theta d\theta$

$=-\frac{a^2}{2}\int\frac{1+\cos 2\theta}{2}d2\theta=-\frac{a^2}{2}(\theta+\frac{\sin 2\theta}{2})+C$

$=-\frac{a^2\arccos\frac{x}{a}}{2}-\frac{x\sqrt{a^2-x^2}}{2}+C$.

(11) 原式 $=\int\frac{x\,dx}{x^2\sqrt{1-x^2}}=\frac{1}{2}\int\frac{1}{x^2\sqrt{1-x^2}}dx^2$; 令 $u=\sqrt{1-x^2}$, 则 $x^2=1-u^2$. $dx^2=-2u\,du$

原式 $=\frac{1}{2}\int\frac{1}{u(1-u^2)}(-2u)du=\int\frac{1}{u^2-1}du=\frac{1}{2}\ln\left|\frac{1-u}{1+u}\right|+C$

$=\frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right|+C=\frac{1}{2}\ln\left|\frac{(1-\sqrt{1-x^2})^2}{x^2}\right|+C$

$=\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|+C$.

(12) 原式 $=\int\frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}=\int\frac{1}{\sqrt{1+\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right)^2}}d\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right)=\arcsin\frac{2x+1}{\sqrt{5}}+C$.

(13) 原式 $=\int\frac{x\,dx}{\sqrt{(\sqrt{2}x-\sqrt{2})^2-2}}$. 令 $\sqrt{2}x-\sqrt{2}=\sqrt{2}\sec y$, $x-1=\sec y$. $dx=\sec y\cdot\tan y\,dy$

$\cos y=\frac{1}{x-1}$ $\sin y=\frac{\sqrt{x^2-2x}}{x-1}$

原式 $=\int\frac{(1+\sec y)}{\sqrt{2}\tan y}\cdot\sec y\cdot\tan y\,dy=\frac{1}{\sqrt{2}}\int(1+\sec y)\sec y\,dy$

$=\frac{1}{\sqrt{2}}\int(\sec^2 y+\sec y)dy=\frac{1}{\sqrt{2}}\tan y+\frac{1}{\sqrt{2}}\ln|\sec y+\tan y|+C$

$=\frac{1}{\sqrt{2}}\sqrt{x^2-2x}+\frac{1}{\sqrt{2}}\ln|x-1+\sqrt{x^2-2x}|+C$.

(14) 原式 $=\int\frac{x+1}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}dx$; 令 $x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan z$, $dx=\frac{\sqrt{3}}{2}\sec^2 z\,dz$, $\tan z=\frac{2x+1}{\sqrt{3}}$

$\sec z=\frac{2}{\sqrt{3}}(\sqrt{x^2+x+1})$

原式 $=\int\frac{\frac{\sqrt{3}}{2}\tan z-\frac{1}{2}+1}{\left|\frac{\sqrt{3}}{2}\sec z\right|}\frac{\sqrt{3}}{2}\sec^2 z\,dz=\int(\frac{\sqrt{3}}{2}\tan z+\frac{1}{2})\sec z\,dz=\frac{1}{2}\int(\sqrt{3}\sec z\tan z+\sec z)dz$

$=\frac{\sqrt{3}}{2}\sec z+\frac{1}{2}\ln|\sec z+\tan z|+C=\frac{\sqrt{3}}{2}\cdot\frac{2\sqrt{x^2+x+1}}{\sqrt{3}}+\frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\sqrt{x^2+x+1}+\frac{2x+1}{\sqrt{3}}\right|+C$

$=\sqrt{x^2+x+1}+\frac{1}{2}\ln|2x+1+2\sqrt{x^2+x+1}|+C$.

(15) 原式 $= \frac{1}{4}\int \frac{1}{\sqrt{1-x^8}}\,dx^4 = \frac{1}{4}\arcsin x^4 + C$

(16) 令 $t = x+2$；我们已知：$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln(x+\sqrt{x^2-a^2}) + C$

原式 $= \int (x-2)\sqrt{(x+2)^2-3}\,dx = \int (t-4)\sqrt{t^2-(\sqrt{3})^2}\,dt$

$= \int t\sqrt{t^2-(\sqrt{3})^2}\,dt - 4\int \sqrt{t^2-(\sqrt{3})^2}\,dt = \frac{1}{2}\int \sqrt{t^2-(\sqrt{3})^2}\,d(t^2-3) - 4\int \sqrt{t^2-(\sqrt{3})^2}\,dt$

$= \frac{1}{2}\times\frac{2}{3}(t^2-3)^{\frac{3}{2}} - \frac{4t}{2}\sqrt{t^2-3} + \frac{4\times 3}{2}\ln(t+\sqrt{t^2-3}) + C$

$= \frac{1}{3}(x^2+4x+1)^{\frac{3}{2}} - 2(x+2)\sqrt{x^2+4x+1} + 6\ln|x+2+\sqrt{x^2+4x+1}| + C$

(17) 令 $t = \sqrt{x^2-9}$，$t^2 = x^2-9$. $2t\,dt = x\,dx$.

$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{x^2-9}\,x\,dx}{x^2} = \int \frac{t^2\,dt}{t^2+9} = \int (1 - \frac{9}{t^2+9})\,dt$

$= t - 3\arctan\frac{t}{3} + C = \sqrt{x^2-9} - 3\arctan\frac{\sqrt{x^2-9}}{3} + C.$

(18) 令 $x = \sin u$，$\sqrt{1-x^2} = \cos u$，$dx = \cos u\,du$

原式 $= \int \frac{1}{\sin u+\cos u}\cdot\cos u\,du = \frac{1}{2}\int \frac{\cos u+\sin u+\cos u-\sin u}{\sin u+\cos u}\,du$

$= \frac{1}{2}\int (1 + \frac{\cos u-\sin u}{\sin u+\cos u})\,du = \frac{1}{2}\int 1\,du + \frac{1}{2}\int \frac{1}{\sin u+\cos u}\,d(\sin u+\cos u)$

$= \frac{1}{2}u + \frac{1}{2}\ln|\sin u+\cos u| + C$

$= \frac{1}{2}\arcsin x + \frac{1}{2}\ln|x+\sqrt{1-x^2}| + C$

(19) 原式 $= \int \frac{x\sqrt{1+\frac{2}{x}}}{x^2}\,dx = \int \frac{\sqrt{1+\frac{2}{x}}}{x}\,dx$，令 $t = \sqrt{1+\frac{2}{x}}$. $x = \frac{2}{t^2-1}$. $dx = \frac{-4t}{(t^2-1)^2}\,dt$

则原式 $= \int \frac{t}{\frac{2}{t^2-1}}\cdot\frac{-4t}{(t^2-1)^2}\,dt = \int \frac{-2t^2+2-2}{t^2-1}\,dt = \int (-2 - \frac{2}{t^2-1})\,dt$

$= -2t + \ln\left|\frac{t+1}{t-1}\right| + C$

$= -2\sqrt{1+\frac{2}{x}} + \ln\left|\frac{\sqrt{1+\frac{2}{x}}+1}{\sqrt{1+\frac{2}{x}}-1}\right| + C$

(20) 原式 $=\frac{1}{2}\int\frac{1+x^2-1}{(1+x^2)^{\frac{3}{2}}}d(x^2+1)$；令 $y=\sqrt{1+x^2}$. $d(x^2+1)=2ydy$

则原式 $=\frac{1}{2}\int\frac{y^2-1}{y^3}\cdot 2ydy=\int\frac{y^2-1}{y^2}dy=\int 1dy-\int\frac{1}{y^2}dy$

$=y+\frac{1}{y}+C=\sqrt{1+x^2}+\frac{1}{\sqrt{1+x^2}}+C$

(21) 令 $u=\sqrt{1+\ln x}$. $u^2=1+\ln x$. $2udu=\frac{1}{x}dx$.

原式 $=\int\frac{2u^2}{u^2-1}du=2\int(1+\frac{1}{u^2-1})du=2(u)+\ln\left|\frac{u-1}{u+1}\right|+C$

$=2\sqrt{1+\ln x}+\ln\left|\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right|+C$

(22). 原式 $=\int\frac{e^x}{\sqrt{3e^x-2}}de^x$，令 $\sqrt{3e^x-2}=t$，$e^x=\frac{t^2+2}{3}$，$de^x=\frac{2}{3}tdt$

原式 $=\int\frac{\frac{t^2+2}{2}}{t}\cdot\frac{2}{3}tdt=\int\frac{t^2+2}{3}dt=\frac{1}{9}t^3+\frac{2}{3}t+C$

$=\frac{1}{9}(3e^x-2)^{\frac{3}{2}}+\frac{2}{3}\sqrt{3e^x-2}+C$

(23). 考虑到 $\int\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C$

(这是前面的结论，可直接用).

对于本题：原式 $=\int\sqrt{(x+1)^2+2^2}\,d(x+1)$.

$=\frac{x+1}{2}\sqrt{x^2+2x+5}+2\ln\left|x+1+\sqrt{x^2+2x+5}\right|+C$

(24). 令 $u=1+x^3$. $du=3x^2dx$.

原式 $=\int\frac{1}{3}\cdot(u^{\frac{3}{2}}\cdot(u-1))du=\frac{1}{3}\int(u^{\frac{5}{2}}-u^{\frac{3}{2}})du$

$=\frac{2}{21}u^{\frac{7}{2}}-\frac{2}{15}u^{\frac{5}{2}}+C$

$=\frac{2}{21}(1+x^3)^{\frac{7}{2}}-\frac{2}{15}(1+x^3)^{\frac{5}{2}}+C$

(25) 令 $t=\sqrt{1+\sin^2 x}$. 则 $t^2-1=\sin^2 x$, $\sin x=\sqrt{t^2-1}$, $d\sin x=\frac{2t\,dt}{2\sqrt{t^2-1}}=\frac{t}{\sqrt{t^2-1}}dt$

$$原式=\int \frac{\sqrt{t^2-1}\cdot t}{t^2+1}\cdot\frac{t}{\sqrt{t^2-1}}dt=\int\frac{t^2}{t^2+1}dt=\int(1-\frac{1}{t^2+1})dt$$

$$=t-\arctan t+C$$

$$=\sqrt{1+\sin^2 x}-\arctan\sqrt{1+\sin^2 x}+C.$$

(26) 一般地: $\int\frac{dx}{x\sqrt{ax^2+bx+c}}=-\frac{1}{\sqrt{c}}\cdot\ln\frac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}+C$

对本题: $\int\frac{dx}{x\sqrt{3x^2+4x+1}}=-\ln\left|\frac{1+2x+\sqrt{3x^2+4x+1}}{x}\right|+C.$

5. (1) $原式=\frac{1}{3}\int x^2 de^{3x}=\frac{1}{3}x^2e^{3x}-\frac{1}{3}\int 2x\cdot e^{3x}dx$

$$=\frac{1}{3}x^2e^{3x}-\frac{2}{9}\int x\,de^{3x}=\frac{1}{3}x^2e^{3x}-\frac{2}{9}e^{3x}x+\frac{2}{9}\int e^{3x}dx$$

$$=\frac{1}{3}\cdot x^2e^{3x}-\frac{2}{9}xe^{3x}+\frac{2}{27}e^{3x}+C.$$

(2) $原式=\frac{1}{2}\int x(\cos 2x+1)dx+C=\frac{1}{2}(\int x\cos 2x\,dx+\int x\,dx)+C$

$$=\frac{1}{4}(\int x\,d\sin 2x+x^2)+C=\frac{1}{4}(x\sin 2x-\int\sin 2x\,dx+x^2)+C$$

$$=\frac{1}{8}(2x\sin 2x+\cos 2x+2x^2)+C.$$

(3) 令 $t=\arctan x$

$$原式=\int t\,d\tan t=t\tan t-\int\tan t\,dt=t\tan t-\int\frac{\sin t}{\cos t}dt$$

$$=t\tan t+\int\frac{1}{\cos t}d\cos t=t\tan t+\ln|\cos t|+C.$$

$$=x\arctan x+\ln|\cos\arctan x|+C$$

$$=x\arctan x+(-\frac{1}{2})\ln|1+x^2|+C.$$

(4) 原式 $= x(\ln x)^2 - \int x \cdot d(\ln x)^2 = x(\ln x)^2 - \int x^2 \cdot \ln x \cdot \frac{1}{x} dx$

$= x(\ln x)^2 - 2\int \ln x\, dx = x(\ln x)^2 - 2[x\ln x - \int x\, d\ln x]$

$= x(\ln x)^2 - 2x\ln x + 2\int x \cdot \frac{1}{x} dx = x(\ln x)^2 - 2x\ln x + 2x + C.$

(5). 原式 $= 2\int \ln x\, d\sqrt{1+x} = 2\ln x\sqrt{1+x} - 2\int \frac{\sqrt{1+x}}{x} dx.$

令 $u = \sqrt{1+x}$. $x = u^2-1$, $dx = 2u\,du$.

原式 $= 2\ln x\sqrt{1+x} - 2\int \frac{2u^2}{u^2-1} du = 2\ln x\sqrt{1+x} - 2\int (2 + \frac{1}{u-1} - \frac{1}{u+1}) du$

$= 2\ln x\sqrt{1+x} - 2(2u + \ln\frac{u-1}{u+1} + C)$

$= 2\ln x\sqrt{1+x} - 2(2\sqrt{1+x} + \ln\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1} + C)$

$= 2\sqrt{1+x}(\ln x - 2) - 2\ln\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1} + C$

(6). 原式 $= 2\int \arcsin\sqrt{x}\, d\sqrt{x} = 2\sqrt{x}\arcsin\sqrt{x} - 2\int \sqrt{x}\, d\arcsin\sqrt{x}$

$= 2\sqrt{x}\arcsin\sqrt{x} - 2\int \frac{\sqrt{x}}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} dx$

$= 2\sqrt{x}\arcsin\sqrt{x} - \int \frac{1}{\sqrt{1-x}} dx$

$= 2\sqrt{x}\arcsin\sqrt{x} + 2\sqrt{1-x} + C$

(7) 利用 $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$，直接得出：

原式 $= -\frac{e^{-x}}{5}(\sin 2x + 2\cos 2x) + C.$

(8) 令 $\sqrt{x} = t$. 原式 $= 2\int t\sin t\, dt = -2\int t\, d\cos t = -2t\cos t + 2\int \cos t\, dt$

$= -2t\cos t + 2\sin t + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

(9) 令 $t = \arctan x$. $x = \tan t$. 则：

原式 $= \int \frac{t\tan t}{\sec t} d\tan t = \int t\sec t\tan t\, dt = \int t\, d\sec t = t\sec t - \int \sec t\, dt$

$= t\sec t - \int \frac{1}{\cos t} dt = t\sec t - \int \frac{1}{\cos^2 t} d\sin t = t\sec t - \int \frac{1}{1-\sin^2 t} d\sin t$

$= t\sec t - \frac{1}{2}\int \frac{1}{1+\sin t} + \frac{1}{1-\sin t} d\sin t = t\sec t - \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right| + C$

$= t\sqrt{1+\tan^2 t} - \ln(\sqrt{1+\tan^2 t} + \tan t) + C$

$= \sqrt{1+x^2}\cdot\arctan x - \ln(\sqrt{1+x^2}+x) + C$

(10) 原式 $= \frac{1}{3}\int \arctan x\, dx^3 = \frac{1}{3}x^3\arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx$

$= \frac{1}{3}x^3\arctan x - \frac{1}{6}\int \frac{x^2}{1+x^2}dx^2$

$= \frac{1}{3}x^3\arctan x - \frac{1}{6}\int [1-\frac{1}{1+x^2}]dx^2$

$= \frac{1}{3}x^3\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$

(11) 原式 $= \frac{1}{2}\int \ln(x^2+1)dx^2 = \frac{1}{2}\int \ln(1+x^2)d(1+x^2)$

$= \frac{1}{2}(1+x^2)\ln(1+x^2) - \frac{1}{2}\int(1+x^2)d\ln(1+x^2)$

$= \frac{1}{2}(1+x^2)\cdot\ln(1+x^2) - \frac{1}{2}\int(1+x^2)\frac{1}{1+x^2}d(1+x^2)$

$= \frac{1}{2}(1+x^2)\cdot\ln(1+x^2) - \frac{1}{2}\int dx^2$

$= \frac{1}{2}(1+x^2)\cdot\ln(1+x^2) - \frac{1}{2}x^2 + C$

(12) 原式 $= \int x(\sec x)^2 dx = \int x\, d\tan x$

$= x\tan x - \int \tan x\, dx = x\tan x + \ln|\cos x| + C$

(13) 原式 $= \int \ln^3 x\, d(-\frac{1}{x}) = -\frac{\ln^3 x}{x} + \int \frac{1}{x}\cdot 3(\ln x)^2\cdot\frac{1}{x}dx$

$= -\frac{\ln^3 x}{x} + 3\int\frac{(\ln x)^2}{x^2}dx$

$= -\frac{\ln^3 x}{x} + 3\int(\ln x)^2 d(-\frac{1}{x})$

$= \frac{-\ln^3 x}{x} - 3\frac{(\ln x)^2}{x} + 6\int(\ln x)/x^2 dx$

$= \frac{-\ln^3 x}{x} - \frac{3\ln^2 x}{x} + 6\int \ln x\, d\frac{1}{x}$

$= \frac{-\ln^3 x}{x} - \frac{3\ln^2 x}{x} - \frac{6\ln x}{x} + 6\int\frac{1}{x^2}dx$

$= \frac{-\ln^3 x}{x} - \frac{3\ln^2 x}{x} - \frac{6\ln x}{x} - \frac{6}{x} + C.$

(14) $\int \cos\ln x\,dx = x\cos\ln x - \int x\cdot\frac{1}{x}\cdot(-\sin\ln x)\,dx$

$= x\cos\ln x + \int \sin\ln x\,dx$

$= x\cos\ln x + x\sin\ln x - \int x\cdot\frac{1}{x}\cdot(\cdot\cos\ln x)\,dx$

$= x\cos\ln x + x\sin\ln x - \int \cos\ln x\,dx$

则 $2\int \cos\ln x\,dx = x\cos\ln x + x\sin\ln x$

$\Rightarrow \int \cos\ln x\,dx = \frac{x}{2}\cdot(\cos\ln x + \sin\ln x)$

(15) 原式 $= x(\arcsin x)^2 - \int x\cdot 2\arcsin x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$

$= x(\arcsin x)^2 - \int \frac{2x}{\sqrt{1-x^2}}\arcsin x\,dx$

$= x(\arcsin x)^2 + \int \frac{2\arcsin x}{2\sqrt{1-x^2}}\,d(1-x^2)$

$= x(\arcsin x)^2 + 2\int \arcsin x\,d\sqrt{1-x^2}$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2\int \sqrt{1-x^2}\,d\arcsin x$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2\int \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$.

(16) $\int e^x\sin^2 x\,dx = \frac{1}{2}\cdot\int e^x(1-\cos 2x)\,dx$

$= \frac{1}{2}\cdot\int (e^x - e^x\cos 2x)\,dx$

$= \frac{1}{2}\int e^x\,dx - \frac{1}{2}\int e^x\cos 2x\,dx$

$= \frac{1}{2}\cdot e^x - \frac{1}{2}\cdot\int e^x\cos 2x\,dx$

由(27)结论: $\int e^x\cos 2x\,dx = \frac{1}{5}\cdot e^x(\cos 2x + 2\sin 2x) + C$

则原式 $= \frac{1}{2}e^x - \frac{1}{10}(e^x\cos 2x + 2e^x\sin 2x) + C$

$= \frac{e^x}{2} - \frac{e^x}{5}\sin 2x - \frac{1}{10}e^x\cos 2x + C$

(21) 原式 $=\int \frac{(x^2+1-1)\arctan x}{1+x^2}dx$

$=\int \arctan x\,dx-\int \frac{\arctan x}{1+x^2}dx$

$=x\arctan x-\int \frac{x}{1+x^2}dx-\int \arctan x\,d(\arctan x)$

$=x\arctan x-\frac{1}{2}\int \frac{1}{1+x^2}dx^2-\frac{1}{2}(\arctan x)^2$

$=x\arctan x-\frac{1}{2}\ln(1+x^2)-\frac{1}{2}(\arctan x)^2+C$

(22) 原式 $=\int \sec^2 x\ln\cos x\,dx$

$=\int \ln\cos x\,d\tan x$

$=\tan x\ln\cos x-\int \tan x\,d[\ln\cos x]$

$=\tan x\cdot\ln\cos x-\int \tan x\cdot\frac{1}{\cos x}\cdot(-\sin x)dx$

$=\tan x\ln(\cos x)+\int \tan^2 x\,dx$

$=\tan x\cdot\ln\cos x+\int(\sec^2 x-1)dx$

$=\tan x\cdot\ln\cos x+\tan x-x+C$

6. 证：用分部积分求 $\int f^{-1}(x)dx=xf^{-1}(x)-\int x\,d[f^{-1}(x)]$

由反函数的微分：$d[f^{-1}(x)]=\frac{1}{f'(x)}dx$

又 $f(x)$ 是单调函数，则 $f[f^{-1}(x)]=f^{-1}[f(x)]=x$

$\int f^{-1}(x)dx=xf^{-1}(x)-\int x\,d[f^{-1}(x)]$

$=xf^{-1}(x)-\int f[f^{-1}(x)]\,d[f^{-1}(x)]$

$=xf^{-1}(x)-F[f^{-1}(x)]+C$