Algorithmic Game Theory

Overview of Rank-1 Bimatrix Games: A Homeomorphism and a Polynomial Time Algorithm

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Abstract

As we all know, we can find a Nash equilibrium of zero-sum games in a polynomial time. The set of rank-1 bimatrix games (A+B), where rank (A+B) = 1, is the smallest extension of zero-sum games. This paper delves into rank-1 games, and it's[1] work can be divided into three parts.

Firstly, it constructs a suitable linear subspace of the rank-1 game space and show that this subspace is homeomorphic to its Nash equilibrium correspondence. Using this homeomorphism, authors give the first algorithm finds a Nash equilibrium of a rank-1 game in polynomial time.

Secondly, authors give another algorithm to enumerate all the Nash equilibria of a rank-1 game and finds at least one Nash equilibrium of a general bimatrix game.

Thirdly, authors extend the result to a fixed rank game. The homeomorphism and the fixed point formulation are piece-wise linear and considerably simpler than the classical constructions.

Keywords: Rank-1 games, Homeomorphism, Nash equilibrium

Contents

1. Introduction

1.1 Structure of the paper

In this section, I will introduce to you how the paper[1] organizes.

In Section 1 Introduction, authors introduce the previous achievements on finding Nash Equilibrium in non-cooperative games, and then describe their own contributions.

In Section 2 Games and Nash Equilibrium, authors introduce the concept of finite two-person game.

In Section 3 Rank-1 Space and Homeomorphism, authors prove that rank-1 game's NE correspondence is homeomorphic to the subspace.

In Section 4 Algorithm, authors present two algorithms which are finding a NE of a a non-degenerate rank-1 game and enumerating all NE of a rank-1 game.

In Section 5 Rank-k Space and Homeomorphism, authors extend the homeomorphism from rank-1 game to rank-k game, and come up with a piece-wise linear function where exists the fixed points in rank-k subspace.

Then came the conclusion section 6, where authors summarize their contribution.

1.2 My work

In my opinion, the core of authors' contribution is two polynomial time algorithms and the ideas to deduce that they are correct. So, in this overview you will see the two algorithms in Section 3 with how authors get to them in Section 2.

2. Problem Statement

2.1 Basic Settings and Definitions

In a general finite game with 2 players, their strategy sets be $S_1 = \{1, ..., m\}$ *and* $S_2 = \{1, ..., n\}$,$\Delta_1 = \{(x_1, ..., x_m)|x_i \geq 0, \forall i \in S_1, \sum_{i=1}^{m} x_i = 1\}$ *and* $\Delta_2 = \{(y_1, ..., y_n)|y_i \geq 0, \forall j \in S_2, \sum_{j=1}^{n} y_j = 1\}$, the two payoff matrices are $(A, B) \in \mathbb{R}^{mn} \times \mathbb{R}^{mn}$.

And the best response polytope of the row-player is P, column-player is Q. A_i means the i^{th} row and A^j means the j^{th} column.

$$P = \{(y, \pi_1) \in \mathbb{R}^{n+1} \mid A_i y - \pi_1 \leq 0, \forall i \in S_1; y_j \geq 0, \forall j \in S_2; \sum_{j=1}^{n} y_j = 1\}$$

$$Q = \{(x, \pi_2) \in \mathbb{R}^{m+1} \mid x_i \geq 0, \forall i \in S_1; x^T B^j - \pi_2 \leq 0, \forall j \in S_2; \sum_{i=1}^{m} x_i = 1\}$$

Since P and Q all contains $m+n$ inequalities, so we number them from 1 to $m+n$ and let function *label* $L(v)$ of a point v in the polytope be the set of indices of the tight inequalities at v. If a pair $(v, w) \in P \times Q$ satisfies $L(v) \cup L(w) = \{1, ..., m+n\}$, then it is called a $fully-labeled\ pair$. Then turns to the judging criteria of NE.

Lemma 1. *A strategy profile (x, y) is a NESP (Nash equilibrium strategy profile) of the game (A, B) iff $((y, \pi_1), (x, \pi_2)) \in P \times Q$ is a fully-labeled pair, for some π_1 and π_2.*

$E = \{(A, B, x, y) \in \mathbb{R}^{mn} \times \mathbb{R}^{mn} \times \Delta_1 \times \Delta_2| \ (x, y)\ is\ a\ NESP\ of\ the\ game (A, B)\}$.

2.2 Ideas

2.2.1 From General Game To m-dimensional Subspace

Kohlberg and Mertens[2] had proved that E is homeomorphic to the bimatrix game space $\mathbb{R}^{2mn}$. However, this general bimatrix game is not easy for us to extend the result to rank-1 game space. Therefore, authors define an $m-dimensional$ affine subspace Γ and its Nash equilibrium correspondence E_Γ.

$$\Gamma = \{(A, C + \alpha \cdot \beta^T) \mid \alpha \in \mathbb{R}^m\}$$

$$Q' = \{(x, \lambda, \pi_2) \in \mathbb{R}^{m+2} \mid x_i \geq 0, \forall i \in S_1; x^T C^j + \beta_j \lambda - \pi_2 \leq 0, \forall j \in S_2; \sum_{i=1}^{m} x_i = 1\}$$

$$E_\Gamma = \{(\alpha, x, y) \in \mathbb{R}^m \times \Delta_1 \times \Delta_2 \mid (x, y)\ is\ a\ NESP\ of\ the\ game\ G(\alpha) \in \Gamma\}$$
$$G(\alpha) = (A, C + \alpha \cdot \beta^T)$$

Let

$$\mathcal{N} = \{(v, w) \in P \times Q' \mid L(v) \cup L(w) = \{1, ..., m+n\}\}$$

$$H = \lambda - \sum_{i=1}^{m} \alpha_i x_i$$

$$H^+ : \lambda - \sum_{i=1}^{m} \alpha_i x_i \geq 0$$

$$H^- : \lambda - \sum_{i=1}^{m} \alpha_i x_i \leq 0$$

Then, we can know from Lemma 2 and 3 that E_Γ and $\mathcal{N}$ are closely related.

So, we can turn to study the structure of $\mathcal{N}$ which is showed by **Proposition 4**.

Proposition 4. *The set of fully-labeled points $\mathcal{N}$ admits the following decomposition into mutually disjoint connected components.* $\mathcal{N} = \mathcal{P} \ \cup \ \mathcal{C}_1 \cup ... \cup \mathcal{C}_k, k \geq 0$, *where $\mathcal{P}$ and $\mathcal{C}_i s$ respectively form a path and cycles on $1-skeleton\ of\ P \times Q'$.*

From **Proposition 4**, we realise that $\mathcal{N}$ at least contains the path $\mathcal{P}$, and we can also prove that the path $\mathcal{P}$ at least covers one NESP of the game $G(\alpha)$.

2.2.2 From m-dimensional Subspace to Rank-1 Space

However, in general bimatrix games, Γ and E_Γ are not homeomorphic while they turn out to be homeomorphic if Γ consists of only rank-1 games. That is when $C = -A$, then $rank(A + B) = rank(A + C + \alpha \cdot \beta^T) = rank(\alpha \cdot \beta^T) = 1$. So firstly, we define the subspace,

$$\Gamma = \{(A, -A + \alpha \cdot \beta^T) \mid \alpha \in \mathbb{R}^m, A\ and\ \beta\ are\ non-zero\}$$

To prove the homeomorphism, firstly we prove that $\mathcal{N}$ only contain one path $\mathcal{P}$. This conclusion tells us that $\mathcal{N}$ is a path, which is useful in the later demonstration.

Then, we can construct a function g to show the bijection between $\mathcal{N} \to \mathbb{R}$:

$$g((y, \pi_1), (x, \lambda, \pi_2)) = \beta^T \ \cdot \ y + \lambda$$

After definition, we can prove that function g is continuous and monotonically increases on the path $\mathcal{N}$ from $-\infty\ to\ +\infty$, which means a bijection. We know from **Lemma 2** that for every point in E_Γ, there is a unique point on the path $\mathcal{N}$, so finally we construct a function f which is continuous and bijective: $E_\Gamma \to \Gamma$:

$$f(\alpha, x, y) = (\beta^T \cdot y + \alpha^T \cdot x, \alpha_2 - \alpha_1, ..., \alpha_m - \alpha_1)^T$$

Then came to the result that Γ and E_Γ are homeomorphic in rank-1 game space.

2.2.3 From Rank-1 Space To Rank-k Space

In this section, firstly we construct these basic settings:

$$\Gamma^k = \{(A, -A + \sum_{l=1}^{k} \alpha^l \cdot \beta^{l^T}) \in \mathbb{R}^{2mn} \mid \forall l \leq k,\ \alpha^l \in \mathbb{R}^m\}$$

$$E_{\Gamma^k} = \{(\alpha, x, y) \in \mathbb{R}^{km} \times \Delta_1 \times \Delta_2 \mid (x, y)\ is\ a\ NESP\ of\ the\ game\ G(\alpha) \in \Gamma^k\}$$

$$Q'^k = \{(x, \lambda, \pi_2) \in \mathbb{R}^{m+k+1} \mid x_i \geq 0, \forall i \in S_1; x^T(-A^j) + \sum_{l=1}^{k} \beta_j^l \lambda_l - \pi_2 \leq 0, \forall j \in S_2; \sum_{i=1}^{m} x_i = 1\}$$

$$\mathcal{N}^k = \{(v, w) \in P \times Q'^k \mid L(v) \cup L(w) = \{1, ..., m+n\}\}$$

And for any $\delta \in \mathbb{R}^k$, the parameterized linear program is $LP^k(\delta)$, and $OPT^k(\delta)$ is the set of optimal points of $LP^k(\delta)$.

$$\begin{aligned} LP^k(\delta) : max\ & \sum_{l=1}^{k} \delta_l(\beta^{l^T} \cdot y) - \pi_1 - \pi_2 \\ & (y, \pi_1) \in P \\ & (x, \lambda, \pi_2) \in Q'^k \\ & \lambda_l = \delta_l, \forall l \leq k \end{aligned}$$

As the result in Rank-1 name, we can prove that:
Theorem 20. *The Nash equilibrium correspondence E_{Γ^k} is homeomorphic to the game space Γ^k.*

Using the theorem above, we can construct a fixed point formulation, which appears in Nash's Theorem in the class.
Theorem 21. *Finding a Nash equilibrium of a game $G(\gamma) \in \Gamma^k$ reduces to finding a fixed point of a polynomially computable piece-wise linear function f:$[0,1]^k \to [0,1]^k$.*

Then we construct the piece-wise linear function f, let $\gamma_{min} = (\gamma^1_{min}, ..., \gamma^k_{min})$, where $\gamma^l_{min} = min_{i \in S_1} \gamma^l_i$ and $\gamma_{max} = (\gamma^1_{max}, ..., \gamma^k_{max})$, where $\gamma^l_{max} = max_{i \in S_1} \gamma^l_i$

$$\mathcal{B} = \{a \in \mathbb{R}^k \mid \gamma_{min} \leq a \leq \gamma_{max}\}$$

$$f(a) = (\sum_{i=1}^{m} \gamma^1_i x_i, ..., \sum_{i=1}^{m} \gamma^k_i x_i),\ (x, \lambda, \pi_2) = \{w \in Q'^k \mid (v, w) \in OPT^k(a),\ v \in P\}$$

Therefore, $f : \mathcal{B} \to \mathcal{B}$'s fixed points correspond to the Nash equilibria of the game $G(\gamma)$, and we also know the structure $\mathcal{N}^k$ in the previous proposition, which may be helpful in locate a fixed point of f in a polynomial time.

3. Algorithms

3.1 Rank-1 NE

Through **Lemma 13**, we can define a linear program **LP** and **OPT** which can switch the problem of finding NE into a linear program problem.
Lemma 13. *For all (v,w) =((y,π_1),(x,λ, π_2)) $\in P \times Q'$, we have* $\lambda(\beta^T \cdot y) - \pi_1 - \pi_2 \leq 0$*, and the equality holds iff (v,w)* $\in \mathcal{N}$.

So,

$$\begin{aligned} LP(\delta) : max\ \delta(\beta^T \cdot y) - \pi_1 - \pi_2 \\ (y, \pi_1) \in P \\ (x, \lambda, \pi_2) \in Q' \\ \lambda = \delta \end{aligned}$$

And $OPT(\delta)$ is the set of optimal points of $LP(\lambda)$, then we can prove that for any $a \in \mathbb{R}$, the set $OPT(a) = \{((y, \pi_1), (x, \lambda, \pi_2)) \in \mathcal{N} \mid \lambda = a\}$ is contained in $\mathcal{N}$. Firstly, we turn the problem of finding NE into finding a point in the intersection of $\mathcal{N}$ and the hyper-lane H. Then, the part of judging NE in this algorithm is:

IsNE(δ)
Find OPT(δ) by solving LP(δ)
$\overline{\mathrm{u,v}} \leftarrow$ The edge containing OPT(δ); $\mathcal{H} \leftarrow \{\mathrm{w} \in \overline{\mathrm{u,v}} \mid \mathrm{w} \in \mathrm{H}\}$
if $\mathcal{H} \neq \emptyset$ **then** Output $\mathcal{H}$; **return** 0;
else if $\overline{\mathrm{u,v}} \in \mathrm{H}^+$ **then return** 1;
else return -1;

Using the monotony, we can do a binary search to find a NE in a game $G(\gamma)$ which takes polynomial time:

BinSearch(*void*)
$a_1 \leftarrow min_{i \in S_1}\gamma_i$; $a_2 \leftarrow max_{i \in S_1}\gamma_i$;
if IsNE($\mathrm{a_1}$) = 0 **or** IsNE($\mathrm{a_2}$) = 0 **then return**;
while ture
  a $\leftarrow \dfrac{\mathrm{a_1} + \mathrm{a_2}}{2}$; flag $\leftarrow$ IsNE(a);
  if flag = 0 **then break**;
  else if flag < 0 **then** $\mathrm{a_1} \leftarrow$ a;
  else $\mathrm{a_2} \leftarrow$ a;
endwhile
return;

3.2 Rank-1 NE Enumeration

This algorithm is based on the continuity of the path $\mathcal{N}$ in the game $G(\gamma)$, and it is based on these result:
O_2. If $(v, w) \in \mathcal{N}$ and both v and w are vertices, then $|L(v) \cap L(w)| = 1$, and the element in the intersection is called the *duplicate label* of the pair (v, w).
O_3.If $v \in P$ is not a vertex, then $\varepsilon_v = \{w' \in Q' \mid (v, w') \in \mathcal{N}\}$ is either empty or it equals exactly one vertex of Q'.
O_4. If $v \in P$ is a vertex, then $\varepsilon_v = \{w' \in Q' \mid (v, w') \in \mathcal{N}\}$ is either empty or an edge of Q'.
O_5. Let $v \in P$ be a vertex and ε_v be an edge of Q'. If $w \in \varepsilon_v$ is a vertex, then (v, w) has a duplicate label. Let the duplicate label be i, then there exists a unique vertex $v' \in P$ adjacent to v such that $\overline{v, v'} \in \mathcal{N}^P = \{v \in P \mid \varepsilon_v \neq \emptyset\}$, where v' is obtained by relaxing the inequality i at v. This also implies that $\varepsilon_w = \overline{v, v'}$ and $\varepsilon_v \cap \varepsilon_{v'} = w$.

Then came the algorithm, which time depends on the numbers of edges on the path:

Enumeration(*void*)
 $\overline{u_1, v_1} \leftarrow$ *edge contain* $OPT(\gamma_{min})$;
 $\overline{u_2, v_2} \leftarrow$ *edge contain* $OPT(\gamma_{max})$;
 $\overline{u, u'} \leftarrow \overline{u_1, v_1}$;
 if $\overline{u, u'}$ *of type* (v, ε_v) **then** flag $\leftarrow 1$;
 else $flag \leftarrow 0$;
 while true
  $\mathcal{H} = \{\mathrm{w} \in \overline{\mathrm{u}, \mathrm{u}'} \mid \mathrm{w} \in \mathrm{H}\}$; Output $\mathcal{H}$;
  if $\overline{\mathrm{u}, \mathrm{u}'} = \overline{\mathrm{u}_2, \mathrm{v}_2}$ **then** break;
  if flag $= 1$ **then** $\overline{\mathrm{u}, \mathrm{u}'} \leftarrow (\varepsilon_{\mathrm{u}'}, \mathrm{u}')$; flag $\leftarrow 0$;
  else $\overline{\mathrm{u}, \mathrm{u}'} \leftarrow (\mathrm{u}', \varepsilon_{\mathrm{u}'})$; flag $\leftarrow 1$;
 endwhile
 return;

4. Conclusions

On account of my limited ability, if you want to see more proof of these lemmas in this overview, I recommend you read this paper where proofs are carefully presented. I have tried my best to exhibit the ideas and algorithms in a clear way.

For the first sight of this paper, it seems impossible for me to understand. However, after days and nights, I finally got the ideas of authors and could present it in a rather clear way. In the process of doing this assignment, I also felt the fun of thinking, which may be the significance of my signing up for this course.

References

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