Myerson's Lemma

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Recap

- Single-item auctions
- "Ideal": DSIC, social welfare and computationally efficiency
- A general two-step approach to designing ideal auctions is to first assume truthful bids and understand how to allocate items to maximize the social welfare, and second to design selling prices that turn truthful bidding into a dominant strategy.

Two Important Definitions

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One can come up with a payment rule, such that $(\mathbf{x}, \mathbf{p})$ is DSIC.

Definition (Implementable Allocation Rule (IA))

An allocation rule $\mathbf{x}$ for a single-parameter environment is implementable if there is a payment rule $\mathbf{p}$ such that the direct-revelation mechanism $(\mathbf{x}, \mathbf{p})$ is DSIC.

Bidding higher can never get less stuff.

Definition (Monotone Allocation Rule (MA)) 单调

An allocation rule $\mathbf{x}$ for a single-parameter environment is monotone if for every agent i and bids $\mathbf{b}_{-i}$ by the other agents, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is non-decreasing in her bid z.

	highest	second	sponsored search
IA?	Yes	N/A	soon
MA?	Yes	No	Yes

可实现了
存在p使(x,p)DSIC

The Statement

Theorem (Myerson's Lemma)

For a single-parameter environment.

当且仅当 可求解

- *An allocation rule* $\mathbf{x}$ *is implementable iff it is monotone.*
- *If* $\mathbf{x}$ *is monotone, then there is a unique payment rule for which the direct-revelation mechanism* $(\mathbf{x}, \mathbf{p})$ *is DSIC and* $p_i(\mathbf{b}) = 0$ *whenever* $b_i = 0$.

前述提到的唯一的

- *The payment rule above is given by an explicit formula.*

显式结果

- IA is hard to solve and verify, while MA is more "operational"..
- Uniqueness for the payment rule. Furthermore, we have the formula ..

What we have now? $\mathbf{x}$ may or may not be monotone.. What should $\mathbf{p}$ like when $(\mathbf{x}, \mathbf{p})$ is DSIC?
The plan of the proof is to use the DSIC constraint to whittle the possibilities for $\mathbf{p}$ down to a single candidate. Given $(\mathbf{x}, \mathbf{p})$ is DSIC, and consider any $0 \le y < z$. For any i and $\mathbf{b}_{-i}$, we have

第i个人的utility　成本　=收益

估值

$$z \cdot x(z) - p(z) \ge z \cdot x(y) - p(y),$$
$$y \cdot x(y) - p(y) \ge y \cdot x(z) - p(z),$$

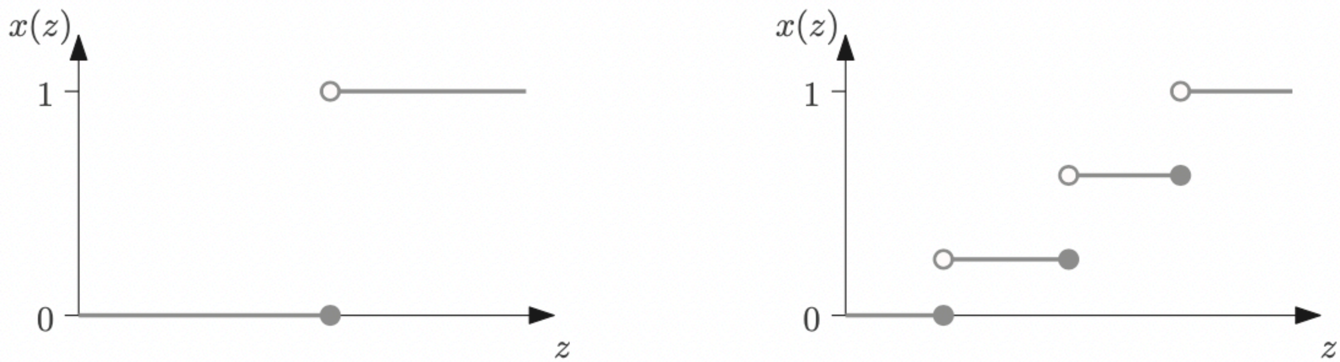
where $x(z) := x_i(z, \mathbf{b}_{-i})$ and $p(z) := p_i(z, \mathbf{b}_{-i})$. The above yields

$$z \cdot [x(y) - x(z)] \le p(y) - p(z) \le y \cdot [x(y) - x(z)]. \tag{1}$$

So every implementable allocation rule is monotone. (Why?)

$x(y) < x(z)$

Piecewise Constant Function



(a) 0-1 monotone curve

(b) Piecewise constant monotone curve

分段的 常数 非减

WLOG, consider the case where x is a piecewise constant function (with finite “jumps”).

$$\text{jump in } p \text{ at } z = z \cdot [\text{jump in } x \text{ at } z].$$

Recall that $p(0) = 0$, the following payment formula is

$$p_i(b_i, \mathbf{b}_{-i}) = \sum_{j=1}^{l} z_j \cdot [\text{jump in } x_i(\cdot, \mathbf{b}_{-i}) \text{ at } z_j],$$

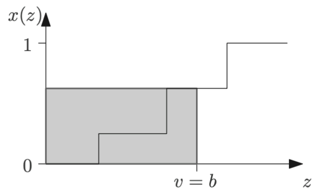
where $z_1, \ldots, z_l$ are the breakpoints of the allocation function $x_i(\cdot, \mathbf{b}_{-i})$ in range $[0, b_i]$. What is the payment like in the plot? Suppose that x is differentiable, we have

$$p_i(b_i, \mathbf{b}_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z, \mathbf{b}_{-i}) dz$$

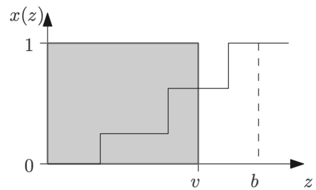
Have we finished the proof?

Proof III

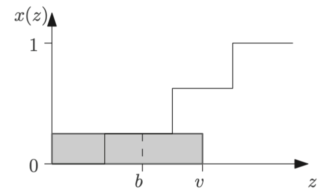
We have to check that $(\mathbf{x}, \mathbf{p})$ is indeed DSIC.



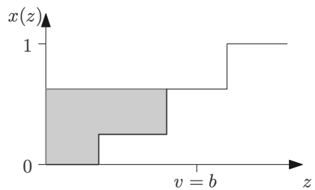
(a) $v \cdot x(v)$



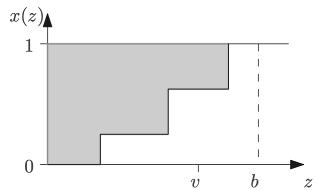
(b) $v \cdot x(b)$ with $b > v$



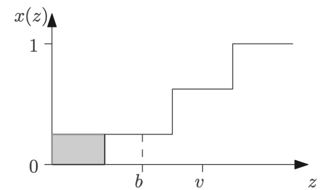
(c) $v \cdot x(b)$ with $b < v$



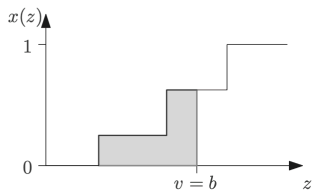
(d) $p(v)$



(e) $p(b)$ with $b > v$

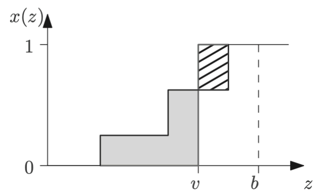


(f) $p(b)$ with $b < v$

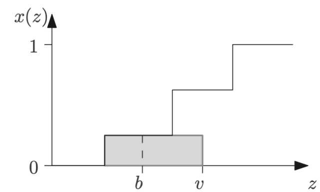


(g) utility with $b = v$

利润



(h) utility with $b > v$



(i) utility with $b < v$

Single-item auctions, revisit

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- The allocation rule: allocate the item to the highest bidder.
- Fixing a bidder i and bids $\mathbf{b}_{-i}$ by others, $x_i(z, \mathbf{b}_{-i})$ is 0 up to $B = \max_{j \neq i} b_j$, and 1 otherwise.
- A single breakpoint (a jump of 1 at B), recall $B = \max_{j \neq i} b_j$.
- So we have

$$p_i(b_i, \mathbf{b}_{-i}) = B.$$

It is the second-price auction!

- The allocation rule: assign the i-th highest bidder to the i-th best slot.

ctr↑越好

ctr越来越小

- k slots with $\alpha_1 \geq \cdots \geq \alpha_k$ and bidders with $b_1 \geq \cdots \geq b_n$.
- Consider the highest bidder, bidding from 0 to b_1, while others fixed.
- With jumps of $\alpha_j - \alpha_{j+1}$ at the point where z becomes the j-th highest bid in $(z, \mathbf{b}_{-i})$, that is b_{j+1}.
- We have

$$p_i(\mathbf{b}) = \sum_{j=i}^{k} b_{j+1} \cdot (\alpha_j - \alpha_{j+1})$$

达到就算钱

for the i-th highest bidder.

Sponsored search auctions, revisit

- The allocation rule: assign the i-th highest bidder to the i-th best slot.
- k slots with $\alpha_1 \geq \cdots \geq \alpha_k$ and bidders with $b_1 \geq \cdots \geq b_n$.
- Consider the highest bidder, bidding from 0 to b_1, while others fixed.
- With jumps of $\alpha_j - \alpha_{j+1}$ at the point where z becomes the j-th highest bid in $(z, \mathbf{b}_{-i})$, that is b_{j+1}.
- Generally, we have

点击时才算钱

$$p_i(\mathbf{b}) = \sum_{j=i}^{k} b_{j+1} \cdot \frac{\alpha_j - \alpha_{j+1}}{\alpha_i}$$

for the i-th highest bidder, since only interested in clicks.

Summary

- single-parameter environment 环境下
- Allocation and Payment
- implementable and monotone 等价
- Myerson's Lemma
- Applications: second-price auctions and sponsored search auctions

Q&A?

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