Revenue-Maximizing Auctions

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May 18, 2022

Recap

- Knapsack auctions, SW optimal in this auction is NP-hard
- State-of-the-art approximation algorithms for the welfare maximization problem may or may not induce monotone allocation rules.
- The revelation principle

Earn more revenue (in expectation).

Why Social Welfare Maximization?

- Relevant to many real-world scenarios, especially for the entire society.
- SW is special ... consider the single-parameter environments, it can generate a monotone allocation rule, hence DSIC.

One Bidder and One Item

- The private value is v.
- DSIC: posted pricing, take-it-or-leave-it price $r \geq 0$.
- How about SW? set $r = 0$ .. input-independent
- How about revenue? We need more information if we don't want to guess the value ..

Bayesian Analysis

- Single-parameter environment w/ players $[n]$ and feasible set $X \subseteq \mathbb{R}^n$.
- For each $i \in [n]$, there is a distribution F_i.
  - Assuming that F_i has support in $[0, v_{\max}]$ for some $v_{\max}$.
  - Let $F_i(z) = \Pr_{x \sim F_i}[x \leq z]$ denote the cumulative distribution function (CDF) of F_i.
  - Let f_i be the probability density function (PDF) of F_i, viz., $\int_0^z f_i(x)dx = F_i(z)$.
  - These distributions are public to the mechanism.
  - Bidder i has some private $v_i \sim F_i$.
- The "optimal" mechanism: max expected revenue over all DSIC mechanisms, where the expectation is taken over the distributions $F_1 \times \cdots \times F_n$.

Examples

Single-item Single-Bidder

- set price $r > 0$, the expected revenue is $r(1 - F(r))$.
- If $F \sim U([0,1])$, i.e., uniform over $[0,1]$, the max revenue is $r(1-r) \le 1/4$.

Single-item Two-Bidder

- $v_i \sim U([0,1])$, where $i = 1, 2$.
- Run a second-price auction: $1/3$
- What if a second-price auction with a reserve price r?
- bads: lose revenue when bids are less than r.
- goods: get more revenue if some bid is larger than r.
- say $r = 1/2$, revenue from $\frac{1}{3}$ to $\frac{5}{12}$!

Optimal DSIC Mechanisms

- Goal: Expected revenue-maximizing DSIC for every single-parameter environment and distributions $F_1, \ldots, F_n$.
- By the revelation principle, we restrict to direct-revelation mechanisms, and hence $\mathbf{b} = \mathbf{v}$.
- The expected revenue of a DSIC mechanism $(\mathbf{x}, \mathbf{p})$ is

$$\mathbb{E}_{\mathbf{v} \sim \mathbf{F}} \left[\sum_{i=1}^{n} p_i(\mathbf{v}) \right], \tag{1}$$

  where $\mathbf{F} = F_1 \times \cdots \times F_n$.
- Hard to solve? We use a *second* formula, which only depends on the allocation rule, not the payment rule.

Virtual Valuations

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- virtual value is

$$\varphi_i(v_i) := v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}. \tag{2}$$

- independent from others!
- e.g., if $F_i = U([0,1])$, then $\varphi_i(z) = 2z - 1 \in [-1, 1]$.

- the second term is known as the information rent
- $\varphi_i(v_i)$ is the slope of a "revenue curve" at v_i. (see Homework)

Exp. Revenue = Exp. Virtual Welfare

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The expected payment of an agent equals the expected virtual value earned by the agent.

分配 与他的payment

Lemma

For every single-parameter environment with valuation distributions $F_1, \ldots, F_n$, every DSIC mechanism $(\mathbf{x}, \mathbf{p})$, every agent i, and every value $\mathbf{v}_{-i}$, we have

sample from

$$\mathbb{E}_{v_i \sim F_i}[p_i(\mathbf{v})] = \mathbb{E}_{v_i \sim F_i}\left[\varphi_i(v_i) \cdot x_i(\mathbf{v})\right]. \tag{3}$$

Theorem

Foe every single-parameter environment with valuation distributions $F_1, \ldots, F_n$ and every DSIC mechanism $(\mathbf{x}, \mathbf{p})$,

$$\mathbb{E}_{\mathbf{v} \sim \mathbf{F}}\left[\sum_{i=1}^{n} p_i(\mathbf{v})\right] = \mathbb{E}_{\mathbf{v} \sim \mathbf{F}}\left[\sum_{i=1}^{n} \varphi_i(v_i) \cdot x_i(\mathbf{v})\right]. \tag{4}$$

Proof of the lemma

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Recall Myerson's Lemma, fix i and $\mathbf{v}_{-i}$, denote $p(\cdot) := p_i(\cdot, \mathbf{v}_{-i})$, the similar with $x(\cdot)$.

$$p(v_i) = \int_0^{v_i} z \cdot x'(z) dz. \tag{5}$$

Step 1: rewriting the payment in terms of the allocation rule.

$$\begin{aligned}\mathbb{E}_{v_i \sim F_i}[p(v_i)] &= \int_0^{v_{\max}} p(v_i) f(v_i) dv_i \\ &= \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x'(z) dz \right] f(v_i) dv_i.\end{aligned}$$

Step 2: reversing the integration order.

$$\int_0^{v_{\max}} \left[\int_z^{v_{\max}} f(v_i) dv_i \right] z \cdot x'(z) dz = \int_0^{v_{\max}} (1 - F(z)) \cdot z \cdot x'(z) dz.$$

Step 3: Dealing with integration by parts.

$$\int_0^{v_{\max}} \underbrace{(1-F(z))\cdot z}_{g(z)} \cdot \underbrace{x'(z)}_{h'(z)} dz$$

$$= (1-F(z))\cdot z\cdot x(z)\big|_0^{v_{\max}} - \int_0^{v_{\max}} x(z)\cdot(1-F(z)-z\cdot f(z))dz$$

$$= \int_0^{v_{\max}} \underbrace{\left(z-\frac{1-F(z)}{f(z)}\right)}_{\varphi(z)} x(z)f(z)dz$$

Step 4: rewrite in terms of expectation.

$$\mathbb{E}_{v_i\sim F_i}[p_i(\mathbf{v})] = \mathbb{E}_{v_i\sim F_i}\left[\varphi_i(v_i)\cdot x_i(\mathbf{v})\right]. \tag{6}$$

Maximizing Expected Virtual Welfare

- We focus on an optimization problem with only the allocation rule instead of calculating the revenue directly.
- We can choose $\mathbf{x}$ for each valuation profile $\mathbf{v}$. So we can maximize it "pointwise", called virtual welfare-maximizing allocation rule.
- What if all the virtual valuations are negative? GIVE UP!
- The allocation rule maxs the expected virtual welfare over all allocation rules, monotone or not. If so, we can use Myerson's Lemma.

Regular Distributions

Monotonicity of the allocation depends on the valuation distributions. But we know the following.

Definition (Regular Distribution)

A distribution F is regular if the corresponding virtual valuation function $v - \frac{1-F(v)}{f(v)}$ is non-decreasing.

One can show that regular distributions bring us the monotonicity.

Virtual Welfare Maximizer (VWM)

1. Transform the (truthfully reported) valuation v_i of agent i into the corresponding virtual valuation $\varphi_i(v_i)$.
2. Choose the feasible allocation $(x_1, \ldots, x_n)$ that maximizes the virtual welfare $\sum_{i=1}^{n} \varphi_i(v_i) x_i$.
3. Charge payments according to Myerson's payment formula.

Theorem (VWM is optimal)

For every single-parameter environment and regular distributions $F_1, \ldots, F_n$, the corresponding virtual welfare maximizer is a DSIC mechanism with the maximum-possible expected revenue.

It gives a satisfying solution to the problem of expected revenue-maximizing mechanism design, in the form of a relatively explicit and easy-to-implement optimal mechanism. However, these mechanisms are not easy to interpret ..

Optimal Single-Item Auctions

- Consider single-item auctions
- bidders are i.i.d., meaning that they have a common valuation distribution F and hence φ.
- F is strictly regular, meaning that φ is a strictly increasing function.
- VWM allocates the item to the bidder with the highest virtual valuation, in our case, the highest valuation.
- The allocation rule is the same as that of a second-price auction with a reverse price of $\varphi^{-1}(0)$.
- e.g., $v_i \sim U([0,1])$, the second-price auction with reverse $\varphi^{-1}(0) = .5$ is optimal.

Summary

- Virtual Valuations
- Expected revenue equals expected virtual welfare
- Regular distributions
- revenue-maximization in single-item auction: second-price auction with a reserve price.

Q&A?