Algorithms for 2-player Nash Equilibrium

Zhengyang Liu, BIT

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Nash Equilibrium

Two-player version

- A pair of strategies $(\mathbf{x}, \mathbf{y})$ is NE iff neither can increase her payoff by deviating from her strategy unilaterally. That is

$$\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}'^T R\mathbf{y},\ \forall \mathbf{x}' \in \Delta_m;$$
$$\mathbf{x}^T C\mathbf{y} \geq \mathbf{x}^T C\mathbf{y}',\ \forall \mathbf{y}' \in \Delta_n\,.$$

- Or an equivalent definition
  - Support of $\mathbf{x}$: $\mathrm{supp}(\mathbf{x}) := \{i \in [n] \mid x_i \neq 0\}$.
  - Each action in the support of $\mathbf{x}$ (or $\mathbf{y}$) should be the best response to the other.

$$x_i > 0 \Rightarrow \mathbf{e}_i^T R\mathbf{y} \geq \mathbf{e}_k^T R\mathbf{y}, \forall k \in [m]$$
$$y_j > 0 \Rightarrow \mathbf{x}^T C\mathbf{e}_j \geq \mathbf{x}^T C\mathbf{e}_k, \forall k \in [n]$$

Normal Form Games

- NFG: $\langle n, (S_p)_{p \in [n]}, (u_p)_{p \in [n]} \rangle$
    - # of players in the game, $[n] = \{1, \ldots, n\}$
    - A set S_p of pure strategies of player $p \in [n]$
    - A utility function $u_p : \times_{p \in [n]} S_p \to \mathbb{R}$
- Recall RSP game…

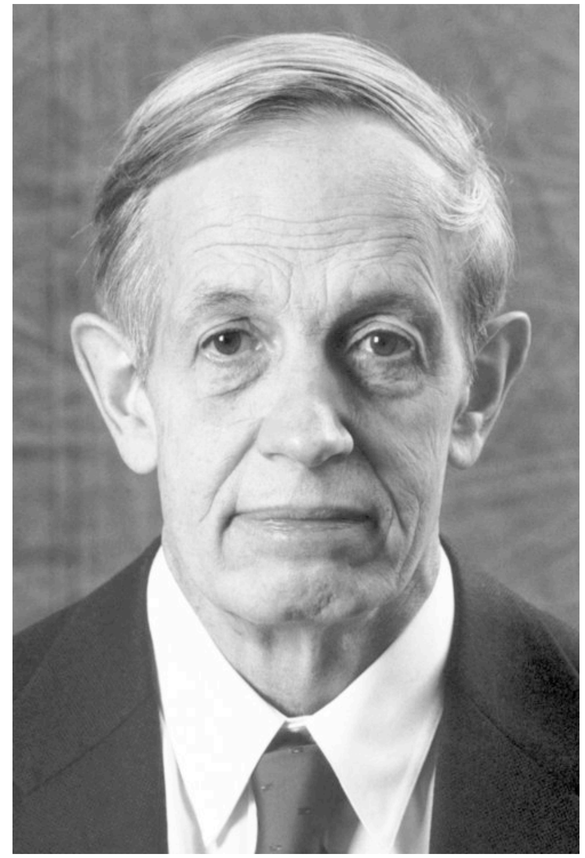
More math…

- The set Δ^{S_p} of mixed strategies to player p over S_p
- The set $S := \times_{p\in[n]} S_p$ of all the pure strategy profile.
  $\mathbf{s} = (s_1, \ldots, s_n) \sim S$
- The set $\Delta := \times_{p\in[n]} \Delta^{S_p}$ of all the mixed strategy profile.
  $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_n}) \sim \boldsymbol{\Delta}$
- Given $\mathbf{x} \in \Delta$, we define the expected payoff of player p is

$$u_p(\mathbf{x}) = \sum_{\mathbf{s}\in S} u_p(\mathbf{s}) \prod_{q\in[n]} \mathbf{x}_q(s_q) = \mathbb{E}_{\mathbf{s}\sim\mathbf{x}}\left[u_p(\mathbf{s})\right].$$

- NE $\mathbf{x} \in \Delta$ in multi-player games iff given any $\mathbf{x}'_p \in \Delta^{S_p}$

$$u_p(\mathbf{x}) \geq u_p(\mathbf{x}'_p; \mathbf{x}_{-p})$$



Nash's Theorem: "Every (finte) game has a Nash equilibrium."

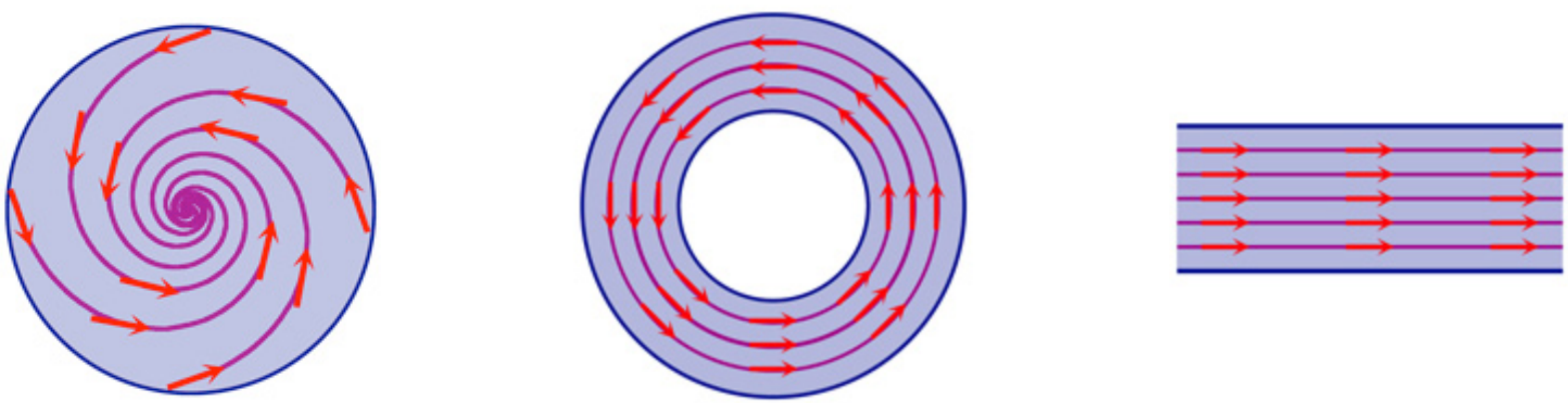
John Forbes Nash Jr.

Proof of Nash's Theorem

An reduction to fixed point.

- The idea is to construct a reduction from the problem of finding an NE in a NFG to the problem of finding a fixed point in a well-defined domain.

- [Brouwer's Fixed Point Thm] Let D be a good (convex, compact) subset of $\mathbf{R}^n$. If a function $f: D \to D$ is continuous, then there exists an $x \in D$ such that $f(x) = x$.



- Let's make a mapping between these two problems.
- So the question is how to construct the continuous function f.
    - It's a good choice to set $f : \Delta \to \Delta$.
- We define a gain function $G_{p,s_p}(\mathbf{x}) := \max\{u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}), 0\}$.
    - Can you increase your utility when only using s_p instead of $\mathbf{x}_p$?
- We define $\mathbf{y} = f(\mathbf{x})$, where $y_{p,s_p} := \dfrac{x_{p,s_p} + G_{p,s_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} G_{p,s'_p}(\mathbf{x})}$.
- f is well-defined, continuous and Δ is good enough $\Rightarrow$ Bingo!
- Next we will show that any fixed point of f is an NE of the game.

$$y_{p,s_p} := \frac{x_{p,s_p} + G_{p,s_p}(\mathbf{x})}{1 + \sum_{s_p' \in S_p} G_{p,s_p'}(\mathbf{x})}$$

- Given $\mathbf{x} = f(\mathbf{x})$, sufficient to show that $G_{p,s_p}(\mathbf{x}) = 0$, $\forall p, s_p$
- Proof by contradiction!
  - Assume that there exists p, s_p such that $G_{p,s_p}(\mathbf{x}) > 0$
    - $x_{p,s_p} > 0$, otherwise $x_{p,s_p} = 0$ but $y_{p,s_p} > 0$
    - There exists some other pure strategy s_p' such that $x_{p,s_p'} > 0$ and $u_p(s_p'; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0$
      - By $u_p(\mathbf{x}) = \sum_{s \in S_p} x_{p,s} \cdot u_p(s; \mathbf{x}_{-p})$
    - We have $y_{p,s_p'} < x_{p,s_p'}$, so $\mathbf{x}$ is not a fixed point!

Algorithms for 2-player NE

We will never discuss the multi-player case in the future…

Overview

- Support Enumeration Algorithm
- The Lipton-Markakis-Mehta (LMM) Approximation Algorithm
- The Lemke-Howson (LH) Algorithm (Next lecture)

Support Enumeration Algorithm

What if we know the supports of an NE?

- Let (R, C) be a two-player game, where $R, C \in \mathbb{R}^{m\times n}$.
- Someone tells us the supports S_R and S_C of their NE $(\mathbf{x}, \mathbf{y})$, that is $S_R = \mathrm{supp}(\mathbf{x})$ and $S_C = \mathrm{supp}(\mathbf{y})$.
- How many possible pairs of S_R and S_C? So the running time is not good…

$$
\begin{aligned}
& \max 0 \\
\text{s.t.} \quad & \mathbf{e}_i^T R\mathbf{y} \geq \mathbf{e}_j^T R\mathbf{y}, \forall i \in S_R, j \in [m] \\
& \mathbf{x}^T C\mathbf{e}_i \geq \mathbf{x}^T C\mathbf{e}_j, \forall i \in S_C, j \in [n] \\
& \mathbf{x}^T\mathbf{1} = 1, \mathbf{y}^T\mathbf{1} = 1 \\
& x_i = 0, \forall i \notin S_R, y_j = 0, \forall j \notin S_C
\end{aligned}
$$

Relax the goal!

Approximate NE

- In the literature, we have two different definition of approximate NE. (We assume that $R, C \in [0,1]^{n\times n}$)

- "ϵ-Approximate" NE: given any $\epsilon > 0$,

$$\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}'^T R\mathbf{y} - \epsilon, \ \forall \mathbf{x}' \in \Delta_n;$$
$$\mathbf{x}^T C\mathbf{y} \geq \mathbf{x}^T C\mathbf{y}' - \epsilon, \ \forall \mathbf{y}' \in \Delta_n.$$

- "ϵ-Well-Supported" NE: given any $\epsilon > 0$,

$$x_i > 0 \Rightarrow \mathbf{e}_i^T R\mathbf{y} \geq \mathbf{e}_k^T R\mathbf{y} - \epsilon, \forall k \in [n]$$
$$y_j > 0 \Rightarrow \mathbf{x}^T C\mathbf{e}_j \geq \mathbf{x}^T C\mathbf{e}_k - \epsilon, \forall k \in [n]$$

- ϵ-WSNE $\Rightarrow$ ϵ-ANE; one can prove that $\epsilon^2/8$-ANE $\Rightarrow$ ϵ-WSNE

The LMM Algorithm

Lipton, R. J., Markakis, E., and Mehta, A. (2003). Playing large games using simple strategies. (EC'03)

Theorem 1 (Liption et al.)
For any $\epsilon \in (0,1)$, there exists an ϵ-ANE where each player plays only $k = O(\log n/\epsilon^2)$ actions with positive probability.

- We use probabilistic method, that is, $\Pr[A] > 0 \Rightarrow A$ exists.
- Idea: approximating the original NE $(\mathbf{x}, \mathbf{y})$ with large enough samples from $\mathbf{x}, \mathbf{y}$. What is the sample complexity?

Theorem 2 (Chernoff Bound)
Let $X_1, \ldots, X_m$ be m random variables over $[0,1]$. For any $\epsilon > 0$ and X be the mean of $\{X_i\}_{i\in[m]}$, we have $\Pr\left[\,|X - \mathbb{E}[X]| \geq \epsilon\right] \leq 2\exp\left(-2m\epsilon^2\right)$.

Proof of Theorem 1

- Let $(\mathbf{x}, \mathbf{y})$ be any NE of our instance.
- Take k i.i.d. samples (actions) $(r_1, \ldots, r_k)$ from the distribution $\mathbf{x}$.
- Let $\tilde{\mathbf{x}}$ be the "empirical" strategy which plays r_i uniformly at random. Similarly with $\tilde{\mathbf{y}}$.
- We will show, when k is large enough, below could happen: $|\mathbf{e}_i^T R\mathbf{y} - \mathbf{e}_i^T R\tilde{\mathbf{y}}| \leq \epsilon/2$ and $|\mathbf{x}^T C\mathbf{e}_j - \tilde{\mathbf{x}}^T C\mathbf{e}_j| \leq \epsilon/2$ where $i, j \in [n]$.
- If so, we have

$$\mathbf{e}_i^T R\tilde{\mathbf{y}} \leq \mathbf{e}_i^T R\mathbf{y} + \epsilon/2 \leq \frac{1}{k}\sum_{j=1}^{k} \mathbf{e}_{r_j}^T R\mathbf{y} + \epsilon/2 \leq \frac{1}{k}\sum_{j} \mathbf{e}_{r_j}^T R\tilde{\mathbf{y}} + \epsilon = \tilde{\mathbf{x}}^T R\tilde{\mathbf{y}} + \epsilon$$

$\blacksquare$

Proving that $|\mathbf{e}_i^T R\mathbf{y} - \mathbf{e}_i^T R\tilde{\mathbf{y}}| \leq \epsilon/2$

Theorem 3 (The Union Bound)
$\Pr[A_1 \cup A_2] \leq \Pr[A_1] + \Pr[A_2]$

- We focus on a bad case that $|\mathbf{e}_i^T R\mathbf{y} - \mathbf{e}_i^T R\tilde{\mathbf{y}}| > \epsilon/2$ for fixed i
- By Chernoff bound, we have (by setting $X_j = \mathbf{e}_i^T R\mathbf{e}_{r_j}$)
  $\Pr[|\mathbf{e}_i^T R\mathbf{y} - \mathbf{e}_i^T R\tilde{\mathbf{y}}| > \epsilon/2] \leq 2\exp(-k\epsilon^2/2)$.
- By the union bound, we have $2n$ bad cases, so the probability that any of the bad cases happens is at most $4n\exp(-k\epsilon^2/2)$.
- For $k > 2\log(4n)/\epsilon^2$, the probability above is less than 1!

Remark for LMM algo

- One can approximate any NE w.r.t. ANE
- The running time is $\binom{n}{k}^2 = n^{O(\frac{\log n}{\epsilon^2})}$.
- With reasonable assumption (ETH for PPAD), Rubinstein (FOCS'16) proved that LMM is optimal, that is, finding an ϵ-ANE needs at least $n^{\log^{1-o(1)} n}$.

Comments

- 师者，所以传道受业解惑也
- 大学与中学的区别
- 《实践论》和《矛盾论》

Q&A?

zhengyang@bit.edu.cn