习题 2-4

1. 解：(1) $y''=[e^{-x^2}+xe^{-x^2}(-2x)]'=2e^{-x^2}(2x^3-3x)$

(2) $y''=\left(\dfrac{1+\frac{2x}{2\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}\right)'=-x(x^2-1)^{-\frac{3}{2}}$

(3) $y''=\left(e^{2x}\cdot 2\sin(2x+1)+e^{2x}\cos(2x+1)\cdot 2\right)'=8e^{2x}\cos(2x+1)$

(4) $y''=\left(\frac{1}{4}\cdot\frac{1-x}{1+x}\cdot\frac{1-x+(1+x)}{(1-x)^2}-\frac{1}{2}\frac{1}{1+x^2}\right)'=\dfrac{2x\cdot(1+x^4)}{(1-x^4)^2}$

(5) $y''=\left(\frac{1}{\ln x}\cdot\frac{1}{x}\right)'=\dfrac{-1-\ln x}{x^2\ln^2 x}$

(6). $y''=\left(4\sin^3 x\cos x-4\cos^3 x(-\sin x)\right)'=4\cos 2x$

(7) $y''=(e^{x\ln x})''=\left(e^{x\ln x}\cdot(\ln x+1)\right)'=x^x(\ln x+1)^2+x^{x-1}$

2. 解：(1). $y'=-f'(e^{-x})\cdot e^{-x}$

$y''=e^{-x}f'(e^{-x})+e^{-2x}f''(e^{-x})$

(2). $y'=\dfrac{1}{f(x)}f'(x)$

$y''=\dfrac{f''(x)f(x)-(f'(x))^2}{f^2(x)}$.

3. 解：(1) $f'(x)=e^x\cos x-e^x\sin x$，$f''(x)=e^x\cos x-e^x\sin x-e^x\sin x-e^x\cos x=-2e^x\sin x$

$f^{(3)}(x)=-2e^x\sin x-2e^x\cos x\qquad f^{(4)}(x)=-4e^x\cos x$

(2) $f^{(100)}(x)=x^{(0)}\cdot sh^{(100)}(x)+100(x^{(1)}sh^{(99)}(x))+\frac{99\times100}{2}(x^{(2)}sh^{(98)}x)+0$

$=x\,sh x+100\,ch x$

(3). $f^{(50)}(x)=x^2\sin^{(50)}2x+50\times2x\sin^{(49)}2x+\frac{50\times49}{2}\times2\times\sin^{(48)}2x$.

$=x^2(-\sin2x)\cdot2^{50}+100x\cos2x\cdot2^{49}+2450\sin2x\cdot2^{48}$

$=2^{50}\left(-x^2\sin2x+50x\cos2x+\frac{1225}{2}\sin2x\right)$

(4) $y^{(20)}=x^2(e^{2x})^{(20)}+20\times2x(e^{2x})^{(49)}+\frac{20\times19}{2}\times2\times(e^{2x})^{(48)}$

$=x^2e^{2x}\cdot2^{20}+40xe^{2x}2^{49}+380e^{2x}\cdot2^{48}$

$=2^{20}\cdot e^{2x}(x^2+20x+95)$.

4. 解：(1). $y = \ln(a+bx) - \ln(a-bx)$

则 $y' = \cancel{b} \frac{b}{a+bx} + \frac{b}{a-bx}$

则 $y^{(n)} = (y')^{(n-1)} = b\left(\frac{1}{a+bx}\right)^{(n-1)} + b\left(\frac{1}{a-bx}\right)^{(n-1)}$

$= b \cdot \frac{(-1)^{(n-1)} b^{n-1}(n-1)!}{(a+bx)^n} + b\,\frac{(-1)^{n-1}(-b)^{n-1}(n-1)!}{(a-bx)^n}$

$= b^n \cdot (n-1)! \cdot \left[\frac{(-1)^{n-1}}{(a+bx)^n} + \frac{1}{(a-bx)^n}\right]$

(2). $n=1$时. $y' = \ln x + 1$

$n \geq 2$时. $y^{(n)} = (y'')^{(n-2)} = \left(\frac{1}{x}\right)^{(n-2)} = \frac{(-1)^n (n-2)!}{x^{n-1}}$

(3). $y = \sin^2 x = \frac{1}{2}(1-\cos 2x)$

$y' = \left(\frac{1}{2}\right)^{\circ} x \cdot 2\sin 2x = \sin 2x$.

$y'' = 2\cos 2x = 2\sin(2x+\frac{\pi}{2})$

$y''' = -4\sin 2x = 4\sin(2x+\pi)$

$y^{(4)} = -8\cos 2x = 8\sin(2x+\frac{3}{2}\pi)$

由归纳法：

$y^{(n)} = 2^{n-1}\sin(2x+\frac{n-1}{2}\pi)$

(4). $y = \frac{x^3-1+1}{x-1} = x^2+x+1+\frac{1}{x-1}$

则 $\cancel{y^{(n)}}$ $y' = 2x+1-\frac{1}{x-1}$.

$y'' = 2+\frac{2}{(x-1)^3}$

$\cancel{y^{(n)}=}$ $y''' = \frac{-6}{(x-1)^4}$

则 $y^{(n)} = (y''')^{(n-3)} = \frac{\cancel{-6}(-1)^n n!}{(x-1)^{4+n-3}} = \frac{(-1)^n n!}{(x-1)^{n+1}}$ $(n \geq 3)$

(5). $y = \frac{x-1+1}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}$

$y^{(n)} = \frac{(-1)^n n!}{(x-1)^{n+1}} + \left(\frac{-1}{x-1}\right)^{(n+1)} = \frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^{n+1}(n+1)!(-1)^{n+1}}{(x-1)^{n+2}}$

$= \frac{n!(n+x)}{(1-x)^{n+2}}$

(6) $y=\frac{2x+2}{x^2+2x-3}=\frac{x-1+x+3}{(x-1)(x+3)}=\frac{1}{x-1}+\frac{1}{x+3}=(x-1)^{-1}+(x+3)^{-1}$

$$y^{(n)}=(-1)(-2)(-3)\cdots(-n)(x-1)^{-1-n}+(-1)(-2)(-3)\cdots(-n)(x+3)^{-1-n}$$
$$=(-1)^n n!(x-1)^{-n-1}+(-1)^n\cdot n!(x+3)^{-n-1}$$
$$=(-1)^n n!\left[(x-1)^{-n-1}+(x+3)^{-n-1}\right]$$

5. 解：$f'(x)=f^2(x)$，$f''(x)=2f(x)f'(x)=2f^3(x)$，

$f'''(x)=6f^2(x)f'(x)=6f^4(x)$.

由归纳法：

$f^{(n)}(x)=n![f(x)]^{n+1}$

6. 解：(1) 式子两边求导：

$2x-2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{x}{y}$，左式两边求导得：

$\frac{d^2y}{dx^2}=\frac{y-\frac{dy}{dx}x}{y^2}$，将$\frac{dy}{dx}=\frac{x}{y}$代入得：

$$\frac{d^2y}{dx^2}=\frac{y^2-x^2}{y^2}=\frac{-1}{y^2}$$

(2) 式子两边求导：

$e^{x+y}(1+\frac{dy}{dx})=y+x\cdot\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{y-e^{x+y}}{e^{x+y}-x}=\frac{y-xy}{xy-x}$

$$\frac{d^2y}{dx^2}=\frac{(\frac{dy}{dx}-y-x\frac{dy}{dx})(xy-x)-(y+x\frac{dy}{dx}-1)(y-xy)}{(xy-x)^2}$$

将$\frac{dy}{dx}=\frac{y-xy}{xy-x}$代入得：

$$\frac{d^2y}{dx^2}=\frac{-y[(x-1)^2+(y-1)^2]}{x^2(y-1)^3}$$

(3) 式子两求导：

$3x^2+3y^2y'-3ay-3axy'=0 \Rightarrow y'=\frac{3ay-3x^2}{3y^2-3ax}$，左式两边求导：

$\frac{d^2y}{dx^2}=\frac{(3ay'-6x)(3y^2-3ax)-(6yy'-3a)(3ay-3x^2)}{(3y^2-3ax)^2}$，将$y'=\frac{3ay-3x^2}{3y^2-3ax}$代入：

$$\frac{d^2y}{dx^2}=\frac{-2a^3xy}{(y^2-ax)^3}$$

(4). 两边求导得：$y' = \dfrac{1+y'}{\cos^2(x+y)}$

$\Rightarrow y' = \dfrac{1}{\cos^2(x+y)-1}$

则 $\dfrac{d^2y}{dx^2} = \dfrac{-2\cos(x+y)\sin(x+y)(1+y')}{[\cos^2(x+y)-1]^2}$，将 $y' = \dfrac{1}{\cos^2(x+y)-1}$ 代入

得：$\dfrac{d^2y}{dx^2} = -2\csc^2(x+y)\cot^3(x+y)$

(5). 两边求导得：$\dfrac{dy}{dx} = e^y + xe^y\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^y}{1-xe^y}$，再次求导：

$\dfrac{d^2y}{dx^2} = \dfrac{e^y\frac{dy}{dx}(1-xe^y) - e^y(-e^y - xe^y\frac{dy}{dx})}{(1-xe^y)^2}$，将 $\dfrac{dy}{dx}$ ~~代入~~ 和 $y=1+xe^y$ 代入：

$\dfrac{d^2y}{dx^2}\Big|_{x=0} = 2e^2$

(6). 两边求导得：$\cos y\cdot\dfrac{dy}{dx} + e^y + xe^y\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-e^y}{\cos y + xe^y}$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-e^y\frac{dy}{dx}(\cos y + xe^y) - (-e^y)(-\sin y\frac{dy}{dx} + e^y + xe^y\frac{dy}{dx})}{(\cos y + xe^y)^2}$

将 $\dfrac{dy}{dx} = \dfrac{-e^y}{\cos y + xe^y}$ 和 $\sin y = -xe^y$ 代~~入得~~，$x=0$ 代入.

$\dfrac{d^2y}{dx^2}\Big|_{x=0} = 2$

7. 解：(1) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$.

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}(-\tan t)}{\frac{dx}{dt}} = \dfrac{\frac{-1}{\cos^2 x}}{-3a\cos^2 t\sin t} = \dfrac{1}{3a\cos^4 t\sin t}$.

(2) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^{2t}}{1-t}$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\frac{e^{2t}}{1-t}}{\frac{1-t}{e^t}} = \dfrac{\frac{2e^{2t}(1-t)+e^{2t}}{(1-t)^2}}{\frac{1-t}{e^t}} = \dfrac{e^{3t}(3-2t)}{(1-t)^3}$

(3) $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{\frac{1}{1+t^2}}{\frac{1}{1+t^2}\cdot 2t}=\dfrac{1}{2t}$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\frac{1}{2t}}{dx/dt}=\dfrac{\frac{-1}{2t^2}}{\frac{1}{1+t^2}\cdot 2t}=\dfrac{-1-t^2}{4t^3}$

(4) $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{\frac{6at(1+t^2)-3at^2\cdot 2t}{(1+t^2)^2}}{\frac{3a(1+t^2)-3a\cdot 2t}{(1+t^2)^2}}=\dfrac{2t(1+t^2)-2t^3}{1+t^2-2t}=\dfrac{2t}{(t-1)^2}$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\frac{2t}{(t-1)^2}}{dx/dt}=\dfrac{\frac{2(t-1)^2-(2t-2)2t}{(t-1)^4}}{\frac{3a+3at^2-6at}{(1+t^2)^2}}=\dfrac{2(1+t^2)^3}{3a(1-t^2)^3}$

(5) $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{f'(t)+tf''(t)-f'(t)}{f''(t)}=t.$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}t}{f''(t)}=\dfrac{1}{f''(t)}$

8、解：$\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{\cos t-\cos t+t\sin t}{\frac{1}{\cos t}\cdot(-\sin t)}=-t\cos t$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}(-t\cos t)}{\frac{1}{\cos t}(-\sin t)}=\dfrac{-\cos t+t\sin t}{-\tan t}$

则 $\dfrac{d^2y}{dx^2}\Big|_{t=\frac{\pi}{3}}=\dfrac{1}{6}(\sqrt{3}-\pi)$

9、解：$\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{x'e^{x}\sin t-\cos t\,e^{x}}{6t+2}$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}=\dfrac{\frac{(6t+2)(x'\cdot e^{x}\sin t-\cos t\,e^{x})'-6(x'e^{x}\sin t-\cos t\,e^{x})}{(6t+2)^2}}{6t+2}$

则 $\dfrac{d^2y}{dx^2}\Big|_{t=0}=\dfrac{e^3}{4}$

10、解：$f'(x)=2(x-a)\cdot g(x)+(x-a)^2g'(x)$

$f''(x)=2g(x)+2(x-a)g'(x)+2(x-a)g'(x)+g'(x)\,x(x-a)^2$

则 $f''(a)=2g(a)+0+0+0=2g(a)$

11. 解：因为$f''(0)$存在，则$f(x)$在$x=0$处连续，$f'(x)$在$x=0$处连续.

则 $\lim_{x\to0^-} f(x)=\lim_{x\to0^-} e^x=1$

$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}(ax^2+bx+c)=c$

则 $c=1$.

$f'_-(0)=\lim_{x\to0^-}\frac{e^x-1}{x-0}=1$.

$f'_+(0)=\lim_{x\to0^+}\frac{ax^2+bx+1-1}{x-0}=b$

则 $b=1$.

$f''_-(0)=\lim_{x\to0^-}\frac{f'(x)-f'(0)}{x-0}=\lim_{x\to0^-}\frac{e^x-1}{x}=1$

$f''_+(0)=\lim_{x\to0^+}\frac{f'(x)-f'(0)}{x-0}=\lim_{x\to0^+}\frac{2ax+1-1}{x}=2a$

则 $2a=1$

$\Rightarrow a=\frac{1}{2}$

综上：$a=\frac{1}{2}$，$b=1$，$c=1$

12. 解：

显然 $f(x)$是连续的. 则

$-\frac{\pi}{2}<x<0$时，$f'(x)=(-x\sin x)'=-\sin x-x\cos x$

$x=0$时. $\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=0=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}$

则 $f'(0)=0$.

$0<x<\frac{\pi}{2}$时. $f'(x)=(x\sin x)'=\sin x+x\cos x$.

则$-\frac{\pi}{2}<x<0$时，$f'(x)$是连续的. $f''(x)=-\cos x-\cos x+x\sin x=-2\cos x+x\sin x$

$x=0$时，由于 $\lim_{x\to0^-}\frac{f'(x)-f'(0)}{x-0}=-\infty\neq\lim_{x\to0^+}\frac{f'(x)-f'(0)}{x-0}=+\infty$.

则 $f''(0)$不存在.

$0<x<\frac{\pi}{2}$时. $f'(x)$是连续的. 则$f''(x)=\cos x+\cos x-x\sin x=2\cos x-x\sin x$.