习题4-6.

1. (1) $\int_0^{+\infty} e^{-x}dx = -e^{-x}\Big|_0^{+\infty} = \lim\limits_{x\to+\infty}(-e^{-x})+1 = 1$

(2) $\int_1^{+\infty}\frac{dx}{x(x+1)} = \int_1^{+\infty}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = (\ln x - \ln(x+1))\Big|_1^{+\infty} = \ln\frac{x}{x+1}\Big|_1^{+\infty}$

$= \lim\limits_{x\to+\infty}\ln\frac{x}{x+1} - \ln\frac{1}{2} = \ln 2$

(3) $\int_{-\infty}^{-1}\frac{dx}{x^2(x^2+1)} = \int_{-\infty}^{-1}\left(\frac{1}{x^2}-\frac{1}{x^2+1}\right)dx = \left(-\frac{1}{x}-\arctan x\right)\Big|_{-\infty}^{-1}$

$= 1+\frac{\pi}{4} - \lim\limits_{x\to-\infty}\left(\frac{-1}{x}-\arctan x\right)$

$= 1+\frac{\pi}{4}-\frac{\pi}{2} = 1-\frac{\pi}{4}$

(4) $\int_0^{+\infty} xe^{-x^2}dx = \int_0^{+\infty}-\frac{1}{2}e^{-x^2}d(-x^2) = -\frac{1}{2}e^{-x^2}\Big|_0^{+\infty}$

$= 0-(-\frac{1}{2}) = \frac{1}{2}$

(5) $\int_1^{+\infty}\frac{\arctan x}{x^2}dx = -\frac{\arctan x}{x}\Big|_1^{+\infty} + \int_1^{+\infty}\frac{1}{x}\cdot\frac{1}{1+x^2}dx$

$= \frac{\pi}{4} + \int_1^{+\infty}\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx$

$= \frac{\pi}{4} + \ln|x|\Big|_1^{+\infty} - \frac{1}{2}\int\frac{1}{1+x^2}d(x^2)$

$= \frac{\pi}{4} + \left(\ln|x| - \frac{1}{2}\ln(1+x^2)\right)\Big|_1^{+\infty}$

$= \frac{\pi}{4} + \ln\frac{|x|}{\sqrt{1+x^2}}\Big|_1^{+\infty}$

$= \frac{\pi}{4} + \frac{\ln 2}{2}$

(6) $\int_0^{+\infty} e^{-ax}\cos bx\,dx = \frac{e^{-ax}}{a^2+b^2}(b\sin bx - a\cos bx)\Big|_0^{+\infty}$

$= 0 + \frac{a}{a^2+b^2}$

$= \frac{a}{a^2+b^2}$

(7) 令 $\sqrt{x}=u$，则 $x=u^2$，$dx=2udu$，$u:0\to+\infty$

$$原式=\int_0^{+\infty}2ue^{-u}du=-\int_0^{+\infty}2u\,de^{-u}$$

$$=-2ue^{-u}\Big|_0^{+\infty}+2\int_0^{+\infty}e^{-u}du$$

$$=(-2ue^{-u}-2e^{-u})\Big|_0^{+\infty}$$

$$=2$$

(8). 令 $x=\tan t$，$dx=\frac{1}{\cos^2 t}dt$，$t:(-\frac{\pi}{2},0)$

$$原式=\int_{-\frac{\pi}{2}}^{0}\frac{t\cos^3 t}{\cos^2 t}dt=\int_{-\frac{\pi}{2}}^{0}t\cos t\,dt=\int_{-\frac{\pi}{2}}^{0}t\,d\sin t$$

$$=t\sin t\Big|_{-\frac{\pi}{2}}^{0}-\int_{-\frac{\pi}{2}}^{0}\sin t\,dt=t\sin t\Big|_{-\frac{\pi}{2}}^{0}+\cos t\Big|_{-\frac{\pi}{2}}^{0}$$

$$=-\frac{\pi}{2}+1=1-\frac{\pi}{2}$$

(9). $$原式=-\frac{1}{x}(1-\ln x)\Big|_2^{+\infty}-\int_2^{+\infty}-\frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx$$

$$=\frac{\ln x-1}{x}\Big|_2^{+\infty}-\frac{1}{x}\Big|_2^{+\infty}$$

$$=\frac{\ln x}{x}\Big|_2^{+\infty}$$

$$=-\frac{1}{2}\ln 2$$

(10) $$原式=\int_2^{+\infty}\frac{1}{(\ln x)^k}d\ln x=\frac{1}{1-k}\int_2^{+\infty}d(\ln x)^{1-k}=\frac{1}{1-k}(\ln x)^{1-k}\Big|_2^{+\infty}$$

对于 $\lim\limits_{x\to+\infty}(\ln x)^{1-k}$，当 $1-k$ 小于0，即 $k>1$ 时，原积分收敛等于 $\frac{1}{k-1}(\ln 2)^{1-k}$

当 $1-k$ 大于等于0即 $k\le 0$ 时，原积分发散。

(11) $x=0$ 是瑕点，

$$\int_0^1\frac{dx}{\sqrt{x}}=2\sqrt{x}\Big|_0^1=2-\lim_{x\to0^+}2\sqrt{x}=2.$$

(12) $x=0$ 是瑕点，

$$\int_0^1\ln^2x\,dx=x(\ln x)^2\Big|_0^1-\int_0^1x\cdot2\ln x\cdot\frac{1}{x}dx=x(\ln x)^2\Big|_0^1-\int_0^1\ln x\,dx$$

$$=x(\ln x)^2\Big|_0^1-2x\ln x\Big|_0^1+2\int_0^1x\cdot\frac{1}{x}dx=x(\ln x)^2\Big|_0^1-2x\ln x\Big|_0^1+2x\Big|_0^1$$

$$=2$$

(13) $x=1$是瑕点

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}}d(1-x^2) = -\sqrt{1-x^2}\Big|_0^1$$

$$= \lim_{x\to 1^-}(-\sqrt{1-x^2})+1 = 1$$

(14) $x=a$是瑕点

$$\int_a^{2a}\frac{1}{(x-a)^{\frac{3}{2}}}dx = \int_a^{2a}(x-a)^{-\frac{3}{2}}d(x-a) = -2(x-a)^{-\frac{1}{2}}\Big|_a^{2a}$$

$$= -2a^{-\frac{1}{2}} - \lim_{x\to a^+}(-2(x-a)^{-\frac{1}{2}})$$

$$= +\infty$$

则原积分发散.

(15). $x=0$是瑕点

$$\int_0^1 \sin\ln x\,dx = x\sin\ln x\Big|_0^1 - \int_0^1 \cos\ln x\,dx = x\sin\ln x\Big|_0^1 - x\cos\ln x\Big|_0^1 - \int_0^1 \sin\ln x\,dx$$

$$\Rightarrow 2\int_0^1 \sin\ln x\,dx = x\sin\ln x\Big|_0^1 - x\cos\ln x\Big|_0^1$$

$$\Rightarrow \int_0^1 \sin\ln x\,dx = \frac{1}{2}\left(-\lim_{x\to 0^+}x\sin\ln x - 1 + \lim_{x\to 0^+}x\cos\ln x\right) \quad (\sin\ln x \text{ 有界}).$$

$$= -\frac{1}{2}.$$

(16) $x=1$是瑕点

$$\int_0^1 \frac{1}{(2-x)\sqrt{1-x}}dx \xrightarrow{t=1-x} \int_0^1 \frac{1}{(t+1)\sqrt{t}}dt \xrightarrow{u=\sqrt{t}} 2\int_0^1 \frac{1}{u^2+1}du$$

$$= 2\arctan u\Big|_0^1 \quad (\text{此时 } u=0 \text{ 是瑕点})$$

$$= \frac{\pi}{2}$$

(17) 令 $t=\sqrt{x+1}$, $dx=2tdt$. 则 $\int_1^{+\infty}\frac{dx}{x\sqrt{x+1}} = \int_{\sqrt{2}}^{+\infty}\frac{2tdt}{(t^2-1)t} = 2\int_{\sqrt{2}}^{+\infty}\frac{1}{t^2-1}dt$

$$= \ln\left|\frac{t-1}{t+1}\right|\Big|_{\sqrt{2}}^{+\infty}$$

$$= \lim_{t\to+\infty}\ln\left|\frac{1-\frac{1}{t}}{1+\frac{1}{t}}\right| - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|$$

$$= 2\ln(\sqrt{2}+1)$$

(18) $x=0$是瑕点

则原积分 $= \int_{-\frac{\pi}{4}}^{0^-} \frac{1}{x^2}\sin\frac{1}{x}\,dx + \int_{0^+}^{+\infty} \frac{1}{x^2}\sin\frac{1}{x}\,dx$

$= -\int_{-\frac{\pi}{4}}^{0^-} \sin\frac{1}{x}\,d(\frac{1}{x}) - \int_{0^+}^{+\infty} \sin\frac{1}{x}\,d(\frac{1}{x})$

$= \cos\frac{1}{x}\Big|_{-\frac{\pi}{4}}^{0^-} + \cos\frac{1}{x}\Big|_{0^+}^{+\infty}$

$= \lim\limits_{x\to 0^-}\cos\frac{1}{x} - \cos\frac{4}{\pi} + \lim\limits_{x\to+\infty}\cos\frac{1}{x} - \lim\limits_{x\to 0^+}\cos\frac{1}{x}$

$= \infty$

则原积分发散.

(19). $x=1$是瑕点.

则原积分 $= \int_0^{1^-} \frac{dx}{x^2-4x+3} + \int_{1^+}^{2} \frac{dx}{x^2-4x+3}$

$= \frac{1}{2}\left(\int_0^{1^-}(\frac{1}{x-3} - \frac{1}{x-1})dx + \int_{1^+}^{2}(\frac{1}{x-3} - \frac{1}{x-1})dx\right)$

$= \frac{1}{2}\left(\ln\left|\frac{x-3}{x-1}\right|\Big|_0^{1^-} + \ln\left|\frac{x-3}{x-1}\right|\Big|_{1^+}^{2}\right)$

$= \infty$

则原积分发散.

(20) $x=e$是瑕点

$\int_1^e \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int_1^e \frac{d\ln x}{\sqrt{1-(\ln x)^2}} = \arcsin\ln x\Big|_1^e = \lim\limits_{x\to e^-}\arcsin\ln x - 0$

$= \frac{\pi}{2}$

(21) $x=0$是瑕点,

$\int_0^{+\infty} \frac{1}{\sqrt{x}}e^{-\sqrt{x}}dx = -2\int_0^{+\infty} e^{-\sqrt{x}}d(-\sqrt{x}) = -2e^{-\sqrt{x}}\Big|_0^{+\infty}$

$= \lim\limits_{x\to+\infty}(-2e^{-\sqrt{x}}) - \lim\limits_{x\to 0^+}(-2e^{-\sqrt{x}})$

$= 0 - (-2)$

$= 2$

2. (1) 因为 $\frac{x^2}{x^4+\sqrt{x}+1} \leq \frac{x^2}{x^4} = \frac{1}{x^2}$ 在 $[1,+\infty)$ 成立

且 $\int_1^{+\infty}\frac{1}{x^2}dx$ 收敛.

则原积分 $=\int_0^1\frac{x^2}{x^4+\sqrt{x}+1}dx+\int_1^{+\infty}\frac{x^2}{x^4+\sqrt{x}+1}$，前者定积分，后者cgt. 则原积分cgt.

(2) 因为 $\frac{1}{\sqrt[3]{x^2+1}} > \frac{1}{(2x^2)^{\frac{1}{3}}}$ 在 $[1,+\infty)$ 成立.

且 $\int_1^{+\infty}\frac{1}{(2x^2)^{\frac{1}{3}}}dx$ dgt. (dgt代表发散).

则原积分 dgt.

(3). 因为 $\lim\limits_{x\to+\infty}\frac{\tan\frac{1}{x^2}}{\frac{1}{x^2}}=1<+\infty$，且 $\int_1^{+\infty}\frac{1}{x^2}dx$ 收敛

则由比较判别法极限形式.

原积分cgt.

(4). 因为 $\frac{x\arctan x}{1+x^3} \leq \frac{\frac{\pi}{2}x}{1+x^3} < \frac{\frac{\pi}{2}x}{x^3} = \frac{\pi}{2}\frac{1}{x^2}$，

且 $\int_1^{+\infty}\frac{1}{x^2}dx$ cgt

则原积分 cgt.

(5) 考虑 $x=1$ 是瑕点

由于 $\lim\limits_{x\to1^+}\frac{\frac{1}{(\ln x)^3}}{\frac{1}{(x-1)^3}} = \lim\limits_{x\to1^+}\left(\frac{x-1}{\ln x}\right)^3 = \lim\limits_{x\to1^+}\frac{3(x-1)^2x}{3(\ln x)^2\cdot\frac{1}{x}} = \lim\limits_{x\to1^+}\frac{2(x-1)x^2+(x-1)^2x}{2\ln x}$

$$=\lim_{x\to1^+}\frac{2x^2+4(x-1)x+2(x-1)x+(x-1)^2}{\frac{1}{x}}=2$$

又 $\int_1^2\frac{1}{(x-1)^3}dx$ dgt

则原积分dgt.

(6). $x=1$ 是瑕点，且 $\frac{x^4}{\sqrt{1-x^4}} \leq \frac{1}{\sqrt{1-x^4}} \leq \frac{1}{\sqrt{1-x}}$，

又因为 $\int_0^1\frac{dx}{(1-x)^{\frac{1}{2}}}$ cgt.

则原积分cgt

(7) $x=1$ 是瑕点. 因为 $1\le x\le 2$，则 $1-x<0$. 则 $\left|\frac{1}{\sqrt[3]{(x-1)(x-2)}}\right| = \frac{1}{\sqrt[3]{(x-1)(2-x)}} \le \frac{1}{(x-1)^{\frac{2}{3}}}$

又 $\int_1^2 \frac{1}{(x-1)^{\frac{2}{3}}}dx$ cgt，则 $\int_1^2 \left|\frac{1}{\sqrt[3]{x^2-3x+2}}\right|dx$ cgt

则原积分 cgt.

(8) 因 $x=0$，$x=\pi$ 都是瑕点

又 $\lim\limits_{x\to 0^+}\frac{\frac{1}{\sqrt{\sin x}}}{\frac{1}{\sqrt{x}}} = \lim\limits_{x\to 0}\frac{\sqrt{x}}{\sqrt{\sin x}} = \lim\limits_{x\to 0}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{\sin x}}\cos x} = \lim\limits_{x\to 0}\frac{\sqrt{\sin x}}{\sqrt{x}} = 1$.

$\lim\limits_{x\to \pi^-}\frac{\frac{1}{\sqrt{\sin x}}}{\frac{1}{\sqrt{x-\pi}}} = \lim\limits_{x\to \pi^-}\frac{\sqrt{x-\pi}}{\sqrt{\sin x}} = \lim\limits_{x\to \pi^-}\frac{\frac{1}{2\sqrt{x-\pi}}}{\frac{1}{2\sqrt{\sin x}}\cos x} = \lim\limits_{x\to \pi^-}\frac{\sqrt{\sin x}}{\sqrt{x-\pi}} = 1$

又由 $\int_0^\pi \frac{1}{\sqrt{x}}dx$ 与 $\int_0^\pi \frac{1}{\sqrt{x-\pi}}dx$ cgt.

则原积分 cgt.

(9) 因为 $\lim\limits_{x\to 0^+}\frac{\ln x}{1-x} = -\infty$. $\lim\limits_{x\to 1^-}\frac{\ln x}{1-x} = \lim\limits_{x\to 1^-}\frac{\frac{1}{x}}{-1} = -1$.

则 $x=0$ 是唯一瑕点.

对于 $\lim\limits_{x\to 0^+}\frac{\frac{\ln\frac{1}{x}}{(1-x)}}{\frac{1}{\sqrt{x}}} = \lim\limits_{x\to 0^+}\frac{x\cdot(-\frac{1}{x^2})}{-\frac{1}{x}\cdot\frac{1}{2\sqrt{x}}} = 0$

又 $\int_0^1 \frac{1}{\sqrt{x}}dx$ cgt. 则 $\int_0^1 \left|\frac{\ln x}{1-x}\right|dx = \int_0^1 \frac{\ln\frac{1}{x}}{1-x}dx$ cgt.

则 $\int_0^1 \frac{\ln x}{1-x}dx$ cgt

(10). $x=0$，$x=\frac{\pi}{2}$ 是瑕点

我们先计算 $\int\frac{dx}{\sin^2 x\cos^2 x} = 4\int\frac{1}{\sin^2 2x}dx = 2\int\csc^2 2x\,d2x = -2\cot 2x + C$

则 $\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^2 x\cos^2 x} = -2\cot 2x\Big|_0^{\frac{\pi}{2}} = \infty$

则原积分发散.

(11) $\lim\limits_{x\to 0^+} x^\alpha\ln\frac{1}{x} = \lim\limits_{x\to 0^+}\frac{\ln\frac{1}{x}}{\frac{1}{x^\alpha}} = \lim\limits_{x\to 0^+}\frac{-x\cdot\frac{1}{x^2}}{-\alpha x^{-\alpha-1}} = \lim\limits_{x\to 0^+}\frac{x^\alpha}{\alpha} = 0$

则 $\int_0^1 x^\alpha\ln\frac{1}{x}dx = \int_0^1 |x^\alpha\ln x|dx$ cgt

则原积分 cgt.

3. 解：$\frac{2x^2+bx+a}{x(2x+a)}-1=\frac{(b-a)x+a}{x(2x+a)}=\frac{1}{x}+\frac{b-a-2}{2x+a}$

则 $\int\left(\frac{2x^2+bx+a}{x(2x+a)}-1\right)dx=\ln|x|+\frac{b-a-2}{2}\ln|2x+a|$

则 $\int_1^{+\infty}\left(\frac{2x^2+bx+a}{x(2x+a)}-1\right)dx=\int_1^{(-\frac{a}{2})^-}\left(\frac{1}{x}+\frac{b-a-2}{2x+a}\right)dx+\int_{(-\frac{a}{2})^+}^{+\infty}\left(\frac{1}{x}+\frac{b-a-2}{2x+a}\right)dx$

$=\left(\ln|x|+\frac{b-a-2}{2}\ln|2x+a|\right)\Big|_1^{(-\frac{a}{2})^-}+\left(\ln|x|+\frac{b-a-2}{2}\ln|2x+a|\right)\Big|_{(-\frac{a}{2})^+}^{+\infty}$

要使原积分存在，则 $\lim\limits_{x\to+\infty}\left(\ln|x|+\frac{b-a-2}{2}\ln|2x+a|\right)$ 存在.

上式 $=\lim\limits_{x\to+\infty}\ln|2x+a|^{\frac{b-a-2}{2}}\cdot|x|$，要存在，则 $\frac{b-a-2}{2}=-1$.

则 $b=a$.

此时 $\lim\limits_{x\to+\infty}\ln\frac{|2x+a|}{|x|}=\ln\frac{1}{2}$

又 $\lim\limits_{x\to(-\frac{a}{2})^-}(\ln|x|-\ln|2x+a|)=\lim\limits_{x\to(-\frac{a}{2})^+}(\ln|x|-\ln(|2x+a|))$

则原积分 $=\ln\frac{1}{2}+\ln|2+a|=\ln\frac{2+a}{2}=1$

$\Rightarrow a=2e-2$

则 $b=2e-2$