3. 证明：因为 $|\sin x| \le 1$，即 $\sin x$ 为有界函数.

又 $\lim\limits_{x\to 0} x = 0$

则 $x\to 0$ 时，$x\sin x$ 为无穷小.

4. 证明：因为 $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = A$，则 $\frac{f(x)}{g(x)} = A + \alpha(x)$，其中 $\alpha(x)$ 是无穷小.

则 $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0}\left(Ag(x) + \alpha(x)g(x)\right) = 0$.

证毕.

习题1-5：

(1) 解：$\lim\limits_{x\to -1}\frac{x^2+2x+4}{x^2+1} = \frac{(-1)^2+2(-1)+4}{(-1)^2+1} = \frac{3}{2}$.

(2) 解：$\lim\limits_{x\to 2}\frac{x^2-2}{\sqrt{x+7}} = \frac{2^2-2}{\sqrt{2+7}} = \frac{2}{3}$

(3) 解：$\lim\limits_{x\to 0}\frac{4x^3-2x^2+x}{3x^2+2x} = \lim\limits_{x\to 0}\frac{4x^2-2x+1}{3x+2} = \frac{1}{2}$

(4) 解：$\lim\limits_{x\to 1}\frac{x^2-2x+1}{x^2-1} = \lim\limits_{x\to 1}\frac{x-1}{x+1} = 0$

(5) 解：$\lim\limits_{x\to 3}\frac{2x^2-7x+3}{x^2+4x-21} = \lim\limits_{x\to 3}\frac{2x-1}{x+7} = \frac{1}{2}$.

(6) 解：$\lim\limits_{x\to\infty}\frac{3x^3-1}{4x^3+2x^2-5} = \lim\limits_{x\to\infty}\frac{3-\frac{1}{x^3}}{4+\frac{2}{x}-\frac{5}{x^3}} = \frac{3}{4}$

(7) 解：$\lim\limits_{x\to\infty}\frac{x^2+x}{x^4-x+2} = 0$

(8) 解：$\lim\limits_{x\to+\infty}\frac{\sqrt{x^2+1}}{5x+1} = \lim\limits_{x\to+\infty}\frac{\sqrt{1+\frac{1}{x^2}}}{5+\frac{1}{x}} = \frac{1}{5}$

(9) 解：$\lim\limits_{x\to a}\frac{\sin x-\sin a}{\sin\frac{x-a}{2}} = \lim\limits_{x\to a}\frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{\sin\frac{(x-a)}{2}} = \lim\limits_{x\to a}2\cos\left(\frac{x+a}{2}\right) = 2\cos a$

(10) 解：$\lim\limits_{x\to a^+}\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}} = \lim\limits_{x\to a^+}\frac{x-a}{\sqrt{x-a}(\sqrt{x}+\sqrt{a})} = \lim\limits_{x\to a^+}\frac{\sqrt{x-a}}{\sqrt{x}+\sqrt{a}} = 0$

(11) 解：$\lim\limits_{n\to\infty}\frac{1+2+3+\cdots+n}{n^2} = \lim\limits_{n\to\infty}\frac{n(n+1)}{2n^2} = \lim\limits_{n\to\infty}\frac{n+1}{2n} = \lim\limits_{n\to\infty}\frac{1+\frac{1}{n}}{2} = \frac{1}{2}$

(12) 解：$\lim\limits_{n\to\infty}\frac{1+2^2+3^2+\cdots+n^2}{n^3} = \lim\limits_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \lim\limits_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}$

$= \lim\limits_{n\to\infty}\frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} = \frac{2}{6} = \frac{1}{3}$.

(13) 解: $\lim\limits_{n\to\infty}\left(1+\frac{1}{3}+\cdots+\frac{1}{3^n}\right)=\lim\limits_{n\to\infty}\frac{\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}=\lim\limits_{n\to\infty}\frac{3}{2}\left(1-\frac{1}{3^n}\right)=\frac{3}{2}$

(14) 解: $\lim\limits_{x\to 2}\frac{\sqrt{x+2}-2}{\sqrt{x+7}-3}=\lim\limits_{x\to 2}\frac{\frac{1}{2\sqrt{x+2}}}{\frac{1}{2\sqrt{x+7}}}=\frac{3}{2}$.

(15) 解: $\lim\limits_{x\to 1}\frac{x^m-1}{x^n-1}=\lim\limits_{x\to 1}\frac{mx^{m-1}}{nx^{n-1}}=\frac{m}{n}$.

(16) 解: $\lim\limits_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)=\lim\limits_{x\to 1}\frac{1+x+x^2-3}{1-x^3}=\lim\limits_{x\to 1}\frac{(x-1)(x+2)}{-(x-1)(x^2+x+1)}=\cancel{x+2}\lim\limits_{x\to 1}\frac{x+2}{-x^2-x-1}=-1$

(17) 解: $\lim\limits_{x\to+\infty}\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)=\lim\limits_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}=\lim\limits_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{2}$

(18) 解: $\lim\limits_{x\to\infty}\frac{x+\cos x}{x-\arctan x}=\lim\limits_{x\to\infty}\frac{1+\frac{\cos x}{x}}{1-\frac{\arctan x}{x}}=1$

(19) 解: $\lim\limits_{n\to\infty}\frac{4^n-3^n}{4^n+2^n}=\lim\limits_{n\to\infty}\frac{1-\left(\frac{3}{4}\right)^n}{1+\left(\frac{1}{2}\right)^n}=1$

(20) 解: $\lim\limits_{n\to\infty}\frac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+n}}=\lim\limits_{n\to\infty}\frac{2n^2+1+2\sqrt{n^4+n^2}}{\sqrt[3]{n^6+n}}=\lim\limits_{n\to\infty}\frac{2+2\sqrt{1+\frac{1}{n^2}}+\frac{1}{n^2}}{\sqrt{1+\frac{1}{n}}}=\frac{2+2+0}{1}=4$

(21) 解: $\lim\limits_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2}=\lim\limits_{x\to 0}\frac{mn\left[(1+mx)^{n-1}-(1+nx)^{m-1}\right]}{2x}=\frac{mn(n-m)}{2}$.

(22) 解: $\lim\limits_{x\to 1}\frac{x+x^2+\cdots+x^n-n}{x-1}=\lim\limits_{x\to 1}\frac{n(1-x)-x(1-x^n)}{(1-x)^2}=\lim\limits_{x\to 1}\frac{-n-1+(n+1)x^n}{-2(1-x)}$

$=\lim\limits_{x\to 1}\frac{(n+1)nx^{n-1}}{2}=\frac{(n+1)n}{2}$.

(23) 解: $\lim\limits_{x\to 1^-}\frac{x^2-|x-1|-1}{|x-1|}=\lim\limits_{x\to 1^-}\frac{x^2-(1-x)-1}{1-x}=\lim\limits_{x\to 1^-}\frac{2x+1}{-1}=3$

$\lim\limits_{x\to 1^+}\frac{x^2-|x-1|-1}{|x-1|}=\lim\limits_{x\to 1^+}\frac{x^2-(x-1)-1}{x-1}=1$

所以 $\lim\limits_{x\to 1}\frac{x^2-|x-1|-1}{|x-1|}$ 不存在.

2. (1). 解: 因为 $0\le\left|x^3\left(\sin\frac{1}{x}+2\right)\right|\le\left|2x^3\right|$.

则$x\to 0$时, $0\le\lim\limits_{x\to 0}\left|x^3\left(\sin\frac{1}{x}+2\right)\right|\le\lim\limits_{x\to 0}\left|2x^3\right|$

则 $\lim\limits_{x\to 0}\left|x^3\left(\sin\frac{1}{x}+2\right)\right|=0$

则 $\lim\limits_{x\to 0}x^3\left(\sin\frac{1}{x}+2\right)=0$

(2) 解：因为 $\arctan x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$，即有界.

又 $\lim\limits_{x\to\infty} \frac{1}{x} = 0$

则 $\lim\limits_{x\to\infty} \frac{1}{x} \cdot \arctan x = 0$

(3) 解：因为 $\cos n \in [-1, 1]$，即有界.

又 $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$

则 $\lim\limits_{n\to\infty} \frac{1}{n^2} \cos n = 0$.

3. (1) 不存在. 取 $x_n = n\pi$，$n \in Z^+$. 则 $\lim\limits_{x_n\to\infty} (\frac{1}{x_n} + \sin x_n) = 0$

取 $x_n = 2n\pi + \frac{\pi}{2}$，$n \in Z^+$ 则 $\lim\limits_{n\to\infty} (\frac{1}{x_n} + \sin x_n) = 1$

则 $\lim\limits_{x\to\infty} (\frac{1}{x} + \sin x)$ 不存在.

(2) 不存在. $\lim\limits_{x\to+\infty} (\sqrt{x^2+1} - x) = \lim\limits_{x\to+\infty} \frac{1}{\sqrt{x^2+1}+x} = 0$

$\lim\limits_{x\to-\infty} (\sqrt{x^2+1} - x)$ ~~不存在~~ $= +\infty$.

则 $\lim\limits_{x\to\infty} (\sqrt{x^2+1} - x)$ 不存在

(3) 不存在. 因 $\lim\limits_{x\to+\infty} \frac{10^x+1}{10^x-1} = \lim\limits_{x\to+\infty} \frac{1+\frac{1}{10^x}}{1-\frac{1}{10^x}} = 1$

$\lim\limits_{x\to-\infty} \frac{10^x+1}{10^x-1} = -1$

则 $\lim\limits_{x\to\infty} \frac{10^x+1}{10^x-1}$ 不存在.

(4) 不存在. 因 $\lim\limits_{x\to1^+} (\frac{1}{x-1} - \frac{1}{|x-1|}) = \lim\limits_{x\to1^+} (\frac{1}{x-1} - \frac{1}{x-1}) = 0$.

$\lim\limits_{x\to1^-} (\frac{1}{x-1} - \frac{1}{|x-1|}) = \lim\limits_{x\to1^-} \frac{2}{x-1} = -\infty$.

则 $\lim\limits_{x\to1} (\frac{1}{x-1} - \frac{1}{|x-1|})$ 不存在.

4. (1) $\lim\limits_{x\to\frac{\pi}{4}} (\sin x)^{\cos x} = \left(\lim\limits_{x\to\frac{\pi}{4}} \sin x\right)^{\lim\limits_{x\to\frac{\pi}{4}} \cos x} = \left(\frac{\sqrt{2}}{2}\right)^{\frac{\sqrt{2}}{2}}$

(2) $\lim\limits_{x\to\infty} \left(\frac{3x^2+x-1}{x^2+2}\right)^{\frac{4x^3+x}{x^3-3}} = \lim\limits_{x\to\infty} \left(\frac{3+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{2}{x^2}}\right)^{\lim\limits_{x\to\infty} \frac{4+\frac{1}{x^2}}{1-\frac{3}{x^3}}} = 3^4 = 81.$

(3) $\lim\limits_{x\to x_0} f(x)^{g(x)} = \lim\limits_{x\to x_0} e^{g(x)\ln f(x)} = e^{\lim\limits_{x\to x_0} g(x) \cdot \lim\limits_{x\to x_0} \ln f(x)} = e^{B\ln A} = A^B.$

5．解：(1) $\lim f(x)=+\infty$，$\lim g(x)=+\infty$

则 $\lim[f(x)+g(x)]=\lim f(x)+\lim g(x)=+\infty$

$\lim(f(x)-g(x))$ 不确定。

(2) $\lim[f(x)+g(x)]$ 不确定。

$\lim[f(x)-g(x)]=+\infty$

(3) $\lim[f(x)+g(x)]$ 不确定

$\lim(f(x)-g(x))$ 不确定。

习题 1-6

1．(1) 解：$\lim\limits_{x\to0}\dfrac{\sin\alpha x}{\sin\beta x}=\lim\limits_{x\to0}\dfrac{\frac{\sin\alpha x}{\alpha x}}{\frac{\sin\beta x}{\beta x}}\cdot\dfrac{\alpha}{\beta}=\dfrac{1}{1}\times\dfrac{\alpha}{\beta}=\dfrac{\alpha}{\beta}$

(2) 解：$\lim\limits_{x\to0}\sqrt{x}\cot\sqrt{x}=\lim\limits_{x\to0}\dfrac{\sqrt{x}}{\tan\sqrt{x}}=1$

(3) 解：$\lim\limits_{x\to0}\dfrac{\arcsin5x}{\arctan3x}=\lim\limits_{x\to0}\dfrac{\frac{\arcsin5x}{5x}}{\frac{\arctan3x}{3x}}\cdot\dfrac{5}{3}=\dfrac{5}{3}$

(4) 解：$\lim\limits_{x\to0}\dfrac{1-\cos2x}{x\sin x}=\lim\limits_{x\to0}\dfrac{2x^2}{x\cdot x}=2$

(5) 解：$\lim\limits_{x\to0^+}\dfrac{x}{\sqrt{1-\cos x}}=\lim\limits_{x\to0^+}\dfrac{x}{\sqrt{2}\sin\frac{x}{2}}=\lim\limits_{x\to0^+}\dfrac{\frac{x}{2}}{\sqrt{2}\sin\frac{x}{2}}\times2=\dfrac{2}{\sqrt{2}}=2$

(6) 解：$\lim\limits_{x\to1}(1-x)\tan\dfrac{\pi x}{2}=\lim\limits_{x\to1}(1-x)\dfrac{\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}}=\lim\limits_{x\to1}\dfrac{-\sin\frac{\pi x}{2}+(1-x)\pi\cos\frac{\pi x}{2}}{\frac{-\pi\sin\frac{\pi x}{2}}{2}}=\dfrac{2}{\pi}$

(7) 解：$\lim\limits_{x\to0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2x}=\lim\limits_{x\to0}\dfrac{(\sqrt{2}-\sqrt{1+\cos x})(\sqrt{2}+\sqrt{1+\cos x})}{\sin^2x(\sqrt{2}+\sqrt{1+\cos x})}$

$=\lim\limits_{x\to0}\dfrac{1}{2\sqrt{2}}=\dfrac{\sqrt{2}}{8}$.

(8) 解：$\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}=\lim\limits_{x\to a}\dfrac{\cos x}{1}=\cos a$.

(9) 解：$\lim\limits_{x\to\pi}\dfrac{\sin(x-\pi)}{x^2-\pi^2}=\lim\limits_{x\to\pi}\dfrac{\sin(x-\pi)}{(x-\pi)(x+\pi)}=\lim\limits_{x\to\pi}\dfrac{1}{x+\pi}=\dfrac{1}{2\pi}$.