5．解：(1) $\lim f(x)=+\infty$，$\lim g(x)=+\infty$

则 $\lim[f(x)+g(x)]=\lim f(x)+\lim g(x)=+\infty$

$\lim(f(x)-g(x))$ 不确定.

(2) $\lim[f(x)+g(x)]$ 不确定.

$\lim[f(x)-g(x)]=+\infty$

(3) $\lim[f(x)+g(x)]$ 不确定

$\lim(f(x)-g(x))$ 不确定.

习题 1-6

1．(1) 解：$\lim\limits_{x\to0}\dfrac{\sin\alpha x}{\sin\beta x}=\lim\limits_{x\to0}\dfrac{\frac{\sin\alpha x}{\alpha x}}{\frac{\sin\beta x}{\beta x}}\cdot\dfrac{\alpha}{\beta}=\dfrac{1}{1}\times\dfrac{\alpha}{\beta}=\dfrac{\alpha}{\beta}$

(2) 解：$\lim\limits_{x\to0}\sqrt{x}\cot\sqrt{x}=\lim\limits_{x\to0}\dfrac{\sqrt{x}}{\tan\sqrt{x}}=1$

(3) 解：$\lim\limits_{x\to0}\dfrac{\arcsin5x}{\arctan3x}=\lim\limits_{x\to0}\dfrac{\frac{\arcsin5x}{5x}}{\frac{\arctan3x}{3x}}\cdot\dfrac{5}{3}=\dfrac{5}{3}$

(4) 解：$\lim\limits_{x\to0}\dfrac{1-\cos2x}{x\sin x}=\lim\limits_{x\to0}\dfrac{2x^2}{x\cdot x}=2$

(5) 解：$\lim\limits_{x\to0^+}\dfrac{x}{\sqrt{1-\cos x}}=\lim\limits_{x\to0^+}\dfrac{x}{\sqrt{2}\sin\frac{x}{2}}=\lim\limits_{x\to0^+}\dfrac{\frac{x}{2}}{\sqrt{2}\sin\frac{x}{2}}\times2=\dfrac{2}{\sqrt{2}}=2$

(6) 解：$\lim\limits_{x\to1}(1-x)\tan\dfrac{\pi x}{2}=\lim\limits_{x\to1}(1-x)\dfrac{\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}}=\lim\limits_{x\to1}\dfrac{-\sin\frac{\pi x}{2}+(1-x)\pi\cos\frac{\pi x}{2}}{\frac{-\pi\sin\frac{\pi x}{2}}{2}}=\dfrac{2}{\pi}$

(7) 解：$\lim\limits_{x\to0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2x}=\lim\limits_{x\to0}\dfrac{(\sqrt{2}-\sqrt{1+\cos x})(\sqrt{2}+\sqrt{1+\cos x})}{\sin^2x(\sqrt{2}+\sqrt{1+\cos x})}$

$=\lim\limits_{x\to0}\dfrac{1}{2\sqrt{2}}=\dfrac{\sqrt{2}}{8}$.

(8) 解：$\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}=\lim\limits_{x\to a}\dfrac{\cos x}{1}=\cos a$.

(9) 解：$\lim\limits_{x\to\pi}\dfrac{\sin(x-\pi)}{x^2-\pi^2}=\lim\limits_{x\to\pi}\dfrac{\sin(x-\pi)}{(x-\pi)(x+\pi)}=\lim\limits_{x\to\pi}\dfrac{1}{x+\pi}=\dfrac{1}{2\pi}$.

2. (1) 解：$\lim\limits_{x\to\infty}(1+\frac{3}{x})^x = \lim\limits_{x\to\infty}\left[(1+\frac{3}{x})^{\frac{x}{3}}\right]^3 = e^3$

(2) 解：$\lim\limits_{x\to0}(\frac{2+x}{2})^{\frac{2}{x}} = \lim\limits_{x\to0}(1+\frac{x}{2})^{\frac{2}{x}} = e$

(3) 解：$\lim\limits_{x\to\infty}(\frac{x}{1+x})^x = \lim\limits_{x\to\infty}\left[(1+\frac{1}{-1-x})^{-x-1}\cdot(1+\frac{1}{-1-x})\right]^{-1} = e^{-1}$

(4) 解：$\lim\limits_{x\to\infty}(\frac{x^2+2}{x^2+1})^{x^2+1} = \lim\limits_{x\to\infty}(1+\frac{1}{x^2+1})^{x^2+1} = e$

(5) 解：$\lim\limits_{x\to0}(1+x^2)^{\cot^2x} = \lim\limits_{x\to0}(1+x^2)^{\frac{1}{x^2}\cdot x^2\cancel{\frac{\sin^2x}{\cos^2x}}\frac{\cos^2x}{\sin^2x}}$

$= \lim\limits_{x\to0}(1+x^2)^{\frac{1}{x^2}\cdot\frac{x^2}{\sin^2x}\cos^2x} = e$

(6) 解：$\lim\limits_{x\to\infty}(\frac{x-1}{x})^{\frac{1}{\sin\frac{1}{x}}} = \lim\limits_{x\to\infty}(1-\frac{1}{x})^{(-x)\frac{1}{-x\sin\frac{1}{x}}} = \lim\limits_{x\to\infty}(1-\frac{1}{x})^{(-x)\frac{-\frac{1}{x}}{\sin\frac{1}{x}}} = e^{-1}$

(7) 解：$\lim\limits_{x\to\infty}(\frac{x^2}{x^2-1})^x = \lim\limits_{x\to\infty}(1+\frac{1}{x^2-1})^{(x^2-1)\frac{x}{x^2-1}} = \lim\limits_{x\to\infty}e^0 = 1.$

(8) 解：$\lim\limits_{n\to\infty}(1+\frac{5}{3^n})^{3^n} = \lim\limits_{n\to\infty}(1+\frac{5}{3^n})^{\frac{3^n}{5}\times5} = e^5$

3. (1) 解：$\lim\limits_{x\to0}\frac{\ln(1+x)}{\sin3x} = \lim\limits_{x\to0}\frac{x}{3x} = \frac{1}{3}$

(2) 解：$\lim\limits_{x\to0}\frac{e^{5x}-e^{2x}}{x} = \lim\limits_{x\to0}\frac{1+5x-(2x+1)}{x} = 3$

(3) 解：$\lim\limits_{x\to0}\frac{10^x-1}{2x} = \lim\limits_{x\to0}\frac{10^x\ln10}{2} = \frac{\ln10}{2}$

(4) 解：$\lim\limits_{x\to0}\frac{\ln(a+x)-\ln a}{x} = \lim\limits_{x\to0}\frac{\frac{1}{a+x}}{1} = \frac{1}{a}$

(5) 解：$\lim\limits_{x\to+\infty}x[\ln(1+x)-\ln x] = \lim\limits_{x\to\infty}x\ln\frac{1+x}{x} = \lim\limits_{x\to\infty}\ln(\frac{1+x}{x})^x = \ln e = 1$

4. 解：$A_n = n\cdot\frac{r}{2}\cdot r\sin\frac{2\pi}{n} = \frac{nr^2\sin\frac{2\pi}{n}}{2}$

$A = \lim\limits_{n\to\infty}A_n = \lim\limits_{n\to\infty}\frac{\pi r^2\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi r^2$

5. 解：$\lim\limits_{x\to0^+}f(x) = \lim\limits_{x\to0^+}(1+x^2)^{\frac{1}{x^2}\cdot3} = e^3$

$\lim\limits_{x\to0^-}f(x) = \lim\limits_{x\to0^-}\frac{e^{2x}-1}{x} = \lim\limits_{x\to0^-}\frac{e^{2x}-1}{2x}\times2 = 2$

所以$\lim\limits_{x\to0}f(x)$不存在.

6. 解：因为 $\lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^{kx}=\frac{1}{e}$ $\Leftrightarrow$ $\lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)^{kx}=\frac{1}{e}$

又因为 $\lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)^{\frac{x}{2}}=\frac{1}{e}$

$\Rightarrow k=\frac{1}{2}$

7. 解：(1) $\cancel{\lim\limits_{n\to\infty}} n\left(\frac{1}{n^2+\lambda}+\frac{1}{n^2+2\lambda}+\cdots+\frac{1}{n^2+n\lambda}\right)\le n\left(\frac{1}{n^2}+\frac{1}{n^2}+\cdots+\frac{1}{n^2}\right)=1$

$\cancel{\le \lim\limits_{n\to\infty} n\left(\frac{1}{n^2}+\frac{1}{n^2}+\cdots+\frac{1}{n^2}\right)=1}$

又 $n\left(\frac{1}{n^2+\lambda}+\frac{1}{n^2+2\lambda}+\cdots+\frac{1}{n^2+n\lambda}\right)\ge n\cdot\frac{n}{n^2+n\lambda}=\frac{n}{n+\lambda}$

则 $\frac{n}{n+\lambda}\le$ 原极限函数 ≤ 1

当 $n\to\infty$ 时，左边 $=\lim\limits_{n\to\infty}\frac{n}{n+\lambda}=1$

右边 $=1$

所以原极限 $=1$

(2) 当 $x\to0^+$ 时，$1-x\le x\left[\frac{1}{x}\right]\le 1$

$\lim\limits_{x\to0^+} 1-x=1$

所以 $\lim\limits_{x\to0^+} x\left[\frac{1}{x}\right]=1$

(3) $\lim\limits_{n\to\infty}\frac{5^n}{n!}=\lim\limits_{n\to\infty}\frac{5}{1}\times\frac{5}{2}\times\frac{5}{3}\times\frac{5}{4}\times\frac{5}{5}\times\frac{5}{6}\cdots\times\frac{5}{n}=0.$

(4) $\cancel{\lim\limits_{n\to\infty}}\ \frac{1}{n^3+1}+\frac{2^2}{n^3+4}+\cdots+\frac{n^2}{n^3+n^2}=\frac{1}{n^3+1}+\frac{2^2}{n^3+2^2}+\cdots+\frac{n^2}{n^3+n^2}\le\frac{1}{n^3+1}+\frac{2^2}{n^3+1}+\cdots+\frac{n^2}{n^3+1}$

则 $\frac{1+2^2+\cdots+n^2}{n^3+n^2}\le\frac{1}{n^3+1}+\frac{2^2}{n^3+2^2}+\cdots+\frac{n^2}{n^3+n^2}\le\frac{1+2^2+\cdots+n^2}{n^3+1}$

$\Leftrightarrow \frac{n(n+1)(2n+1)}{6(n^3+n^2)}\le\frac{1}{n^3+1}+\frac{2^2}{n^3+2^2}+\cdots+\frac{n^2}{n^3+n^2}\le\frac{n(n+1)(2n+1)}{6(n^3+1)}$

当 $n\to\infty$ 时，左边 $=\frac{1}{3}$ 右边 $=\frac{1}{3}$

则 $\lim\limits_{n\to\infty}\left(\frac{1}{n^3+1}+\frac{4}{n^3+4}+\cdots+\frac{n^2}{n^3+n^2}\right)=\frac{1}{3}$

8. (1) $y_1=10$, $y_2=4<y_1$.

设 $y_n<y_{n-1}$，则有 $y_{n+1}-y_n=\sqrt{6+y_n}-\sqrt{6+y_{n-1}}=\dfrac{y_n-y_{n-1}}{\sqrt{6+y_n}+\sqrt{6+y_{n-1}}}<0$

即 $y_{n+1}<y_n$，数列$\{y_n\}$单调减小。

~~$y_1=10$~~ 又 $y_1=10>0$. $y_{n+1}=\sqrt{6+y_n}>0$

则$\{y_n\}$有界．因此$\{y_n\}$有极限．

设 $\lim\limits_{n\to\infty}y_n=A\geq0$ 则 $\lim\limits_{n\to\infty}y_{n+1}=A\geq0$，对 $y_{n+1}=\sqrt{6+y_n}$ 两端取极限，得：

$A=\sqrt{6+A}$，解得 $A=3$.

故 $\lim\limits_{n\to\infty}y_n=3$

(2) $x_1=\frac{1}{2}$，$x_2=\frac{5}{8}>x_1$

~~设 $x_2>x_1$，则有~~

设 $x_n>x_{n-1}$，则有 $x_{n+1}-x_n=\dfrac{1+x_n^2}{2}-x_n=\dfrac{(x_n-1)^2}{2}\geq0$.

即 $x_{n+1}\geq x_n$，数列$\{x_n\}$单调递增.

又 $x_1=\frac{1}{2}<1$. 设 $x_n<1$，则有 $x_{n+1}=\dfrac{1+x_n^2}{2}<1$. 故$\{x_n\}$有界，因此$\{x_n\}$有极限.

设 $\lim\limits_{n\to\infty}x_n=A$ 则 $\lim\limits_{n\to\infty}x_{n+1}=A$. 对 $x_{n+1}=\dfrac{1+x_n^2}{2}$ 两端求极限，得：

$A=\dfrac{1+A^2}{2}\Rightarrow A=1$

故 $\lim\limits_{n\to\infty}x_n=3$

(3) $y_1=\sqrt{2}$，$y_2=\sqrt{2\sqrt{2}}<y_1$

设 $y_n<y_{n-1}$. 则有 $y_{n+1}-y_n=\sqrt{2y_n}-\sqrt{2y_{n-1}}=\dfrac{y_n-y_{n-1}}{\sqrt{2y_n}+\sqrt{2y_{n-1}}}<0$.

即 $y_{n+1}<y_n$. 数列$\{y_n\}$单调递减.

又 $y_1=\sqrt{2}>0$. $y_{n+1}=\sqrt{2y_n}\geq0$，则$\{y_n\}$有界.

因此$\{y_n\}$有极限.

设 $\lim\limits_{n\to\infty}y_n=A\geq0$. 则 $\lim\limits_{n\to\infty}y_{n+1}=A>0$. 对 $y_{n+1}=\sqrt{2y_n}$ 两边取极限，得：

$A=\sqrt{2A}$. $\Rightarrow A=2$.

故 $\lim\limits_{n\to\infty}y_n=2$.

9. 证明：$x_n=\frac{1}{5+10}+\frac{1}{5^2+10}+\cdots+\frac{1}{5^n+10}=\sum_{i=1}^{n}\frac{1}{5^i+10}<\sum_{i=1}^{n}\frac{1}{5^i}=\frac{1-\frac{1}{5^n}}{1-\frac{1}{5}}=\frac{5}{4}(1-\frac{1}{5^n})<\frac{4}{5}$

又 $x_n-x_{n-1}=\frac{1}{5^n+10}>0$

故$\{x_n\}$单调递增且有界

由单调有界定理知$\{x_n\}$极限存在.

习题1-7.

1. (1) 解：$\lim_{x\to1}\frac{1-x}{1-x^3}=\frac{1}{3}$，同阶不等价

(2) 解：$\lim_{x\to1}\frac{1-x}{\frac{1}{2}(1-x^2)}=1$，等价

2. (1) 解：$\lim_{x\to0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$，同阶

(2) 解：$\lim_{x\to\infty}\frac{\sqrt{x^2+2}-\sqrt{x^2+1}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{x^2}{\sqrt{x^2+2}+\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{x}{\sqrt{1+\frac{2}{x^2}}+\sqrt{1+\frac{1}{x^2}}}=+\infty$

则 $\frac{1}{x^2}=o(\sqrt{x^2+2}-\sqrt{x^2+1})$

(3) 解：$\lim_{x\to1}\frac{\frac{1-x}{1+x}}{1-\sqrt{x}}=\lim_{x\to1}\frac{1+\sqrt{x}}{1+x}=1$. 等价

(4) 解：$\lim_{x\to0}\frac{x^2+x^3\sin\frac{1}{x}}{x^2}=\lim_{x\to0}(1+x\sin\frac{1}{x})=1$ 等价

(5) 解：$\lim_{x\to0}\frac{\sqrt{x+\sqrt{x}}}{\sqrt[8]{x}}=0$. 则 $\sqrt{x+\sqrt{x}}=o(\sqrt[8]{x})$

3. (1) $\lim_{x\to0}\frac{\sqrt{x}+\sin x}{x^{\frac{1}{2}}}=\lim_{x\to0}(1+\sqrt{x})=1$

则 $\sqrt{x}+\sin x$ 为 x 的 $\frac{1}{2}$ 阶无穷小，即阶为 $\frac{1}{2}$

(2) $\lim_{x\to0}\frac{x^{\frac{3}{4}}-x^{\frac{1}{3}}}{x^{\frac{1}{3}}}=\lim_{x\to0}(x^{\frac{5}{12}}-1)=-1$.

则 $x^{\frac{3}{4}}-x^{\frac{1}{3}}$ 为 x 的 $\frac{1}{3}$ 阶无穷小.

(3) $\lim_{x\to0}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^{\frac{1}{8}}}=\lim_{x\to0}\sqrt{\frac{x}{\sqrt{x}}+\sqrt{\sqrt{x}+1}}=1$

则原式为 x 的 $\frac{1}{8}$ 阶无穷小.

(4) $\lim_{x\to0}\frac{\sqrt[3]{\cos x}-1}{x^2}\cancel{=\lim_{x\to0}\sqrt[3]{1-\frac{1}{2}x^2}}=\lim_{x\to0}\frac{-\frac{1}{3}(\cos x)^{-\frac{2}{3}}\sin x}{2x}=\lim_{x\to0}\frac{(\cos x)^{-\frac{2}{3}}}{-6}=-\frac{1}{6}$.

则原式为 x 的 2 阶无穷小.