4CM00: Control Engineering
Stability

Dr.ir. Gert Witvoet

Technische Universiteit
Eindhoven
University of Technology

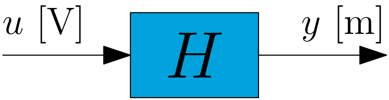
Where innovation starts

September 2020

Feedback and stability

Why feedback?

Consider a *linear* system $H(s)$:



Laplace transforms:
$\mathfrak{L}(u(t)) = U(s)$
$\mathfrak{L}(y(t)) = Y(s)$
$Y(s) = H(s)U(s)$

- $H(s)$ is a transfer function with polynomial numerator and denominator: $H(s) = \frac{p_1(s)}{q_1(s)}$

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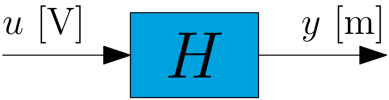

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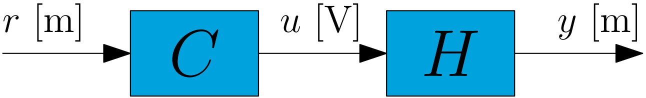
- $H(s)$ is a transfer function with polynomial numerator and denominator: $H(s) = \frac{p_1(s)}{q_1(s)}$
- Stability is governed by the *poles* λ_i of the system:
  - $H(s)$ is stable if and only if $\mathrm{Re}(\lambda_i) < 0$ for all λ_i
  - All poles must be in the LHP!
  - Poles are solutions $s = \lambda_i$ to $q_1(s) = 0$
- Note: discrepancy between units of input u and output y

Why feedback?

Add a controller $C(s)$ in series:



- Both input r and output y now in [m]
- Controller transfer function: $C(s) = \frac{p_2(s)}{q_2(s)}$

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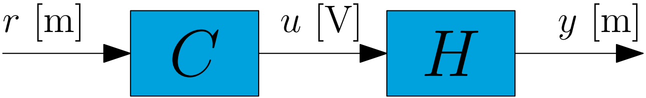
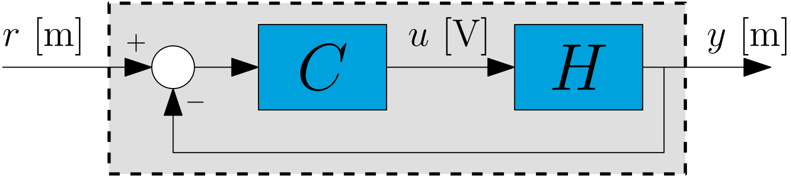


- Both input r and output y now in [m]
- Controller transfer function: $C(s) = \frac{p_2(s)}{q_2(s)}$
- Transfer from r to y:
  $\frac{y}{r} = L(s) = H(s)C(s) = \frac{p_1(s)}{q_1(s)} \cdot \frac{p_2(s)}{q_2(s)}$
- Poles are determined by $q_1(s)q_2(s) = 0$
- Hence, *poles remain unchanged!*

Furthermore, $C(s)$ does not 'see' the output y;
disturbances or errors are not detected nor compensated by $C(s)$!

Why feedback?

Solution: close the loop, feedback the error:



'Creates' a new system from r to y: $\frac{y}{r} = \frac{H(s)C(s)}{1+H(s)C(s)} = \frac{L(s)}{1+L(s)}$

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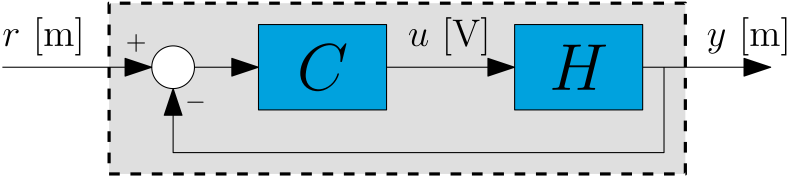

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- Poles are now determined by
  $1 + L(s) = 0 \quad \Rightarrow \quad \frac{q_1(s)q_2(s) + p_1(s)p_2(s)}{q_1(s)q_2(s)} = 0$
- Closed loop system has different poles than $H(s)$
- Controller $C(s)$ can be used to 'change' the poles

Stability tests

How to check closed loop stability?

1. (Re)compute the closed loop poles (Matlab?)

Drawbacks:

- Recompute after every controller adjustment
- Unclear how to change controller when poles are unsatisfactory
- Impossible to do when using an FRF measurement of the plant H (model of H is unknown)

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2. Graphically $\rightarrow$ Nyquist diagram
    - Checks stability as a function of frequency
    - No plant model required

Closed loop stability according to the Nyquist criterion

Open loop vs closed loop stability

Goal: checking **closed loop** stability.

Assume the open-loop transfer: $L(s) = H(s)C(s) = \frac{a(s)}{b(s)}$

- Closed loop transfers: $\frac{1}{1+L(s)}$, $\frac{L(s)}{1+L(s)}$, etc.
- Stability governed by denominator:
  $1 + L(s) = 1 + \frac{a(s)}{b(s)} = \frac{a(s)+b(s)}{b(s)} = 0$

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Hence:

- Poles of $1 + L(s) \Rightarrow b(s) = 0 \Rightarrow$ **open-loop** poles!
- Zeros of $1 + L(s) \Rightarrow a(s)+b(s) = 0 \Rightarrow$ **closed-loop** poles!

The transfer function $1 + L(s)$ is thus important as it contains both open loop and closed loop stability information!

Evaluating $1 + L(s)$: background

Let's analyze $1 + L(s)$ in the complex plane

Any linear system can be written as: $G(s) = \dfrac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$

- Zeros located at $-z_i$
- Poles located at $-p_i$
- z_i and p_i can be complex valued

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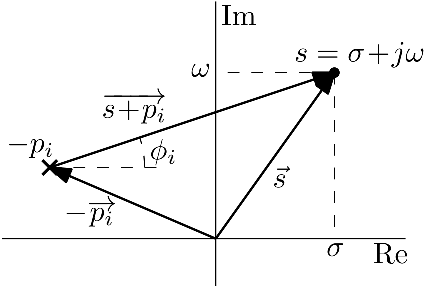
Magnitude and phase can be expressed as

- $|G(s)| = \dfrac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|}$
- $\angle G(s) = \displaystyle\sum_{i=1}^{m}\angle(s+z_i) - \sum_{i=1}^{n}\angle(s+p_i)$

Evaluating $1 + L(s)$: background

- The terms $s + z_i$ and $s + p_i$ are *complex valued*
- Thus they can be interpreted as *vectors* in the complex plane
- $s + z_i$: vector from the zero $-z_i$ to the value of $s = \sigma + j\omega$
  $s + p_i$: vector from the pole $-p_i$ to the value of $s = \sigma + j\omega$



Angle of $s + p_i$:

- Pole location: $-p_i$
- $s + p_i = \vec{s} - \overrightarrow{-p_i} = \overrightarrow{s + p_i}$
- $\angle \overrightarrow{s + p_i} = \phi_i$

Note: The 'image' $G(\sigma + j\omega)$ is also a complex number

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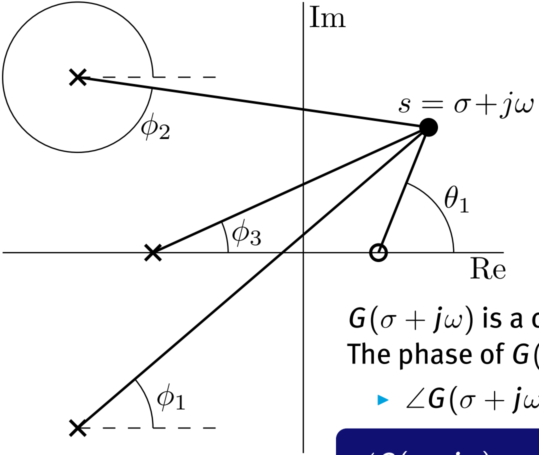
Evaluating $1 + L(s)$: background

For multiple poles and zeros:



$G(\sigma + j\omega)$ is a complex number.
The phase of $G(s)$ evaluated at $s = \sigma + j\omega$:

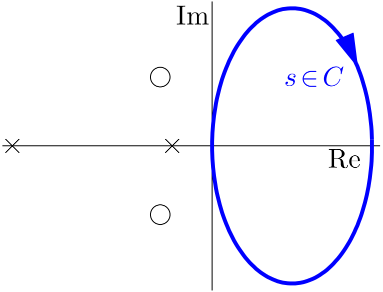
- $\angle G(\sigma + j\omega) = \theta_1 - \phi_1 - \phi_2 - \phi_3$

$$\angle G(\sigma + j\omega) = \angle \text{zeros} - \angle \text{poles}$$

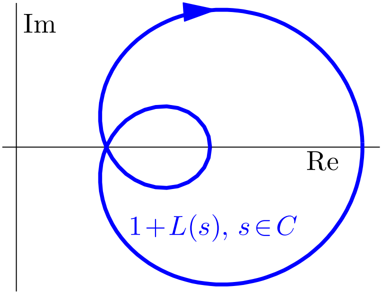
Evaluating $1 + L(s)$: along a closed contour

Now evaluate $1 + L(s)$ along a closed contour C in the complex plane.

Poles ($\times$) and zeros ($\bigcirc$) of $1+L(s)$:



Resulting image:



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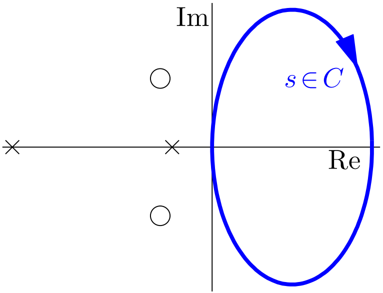

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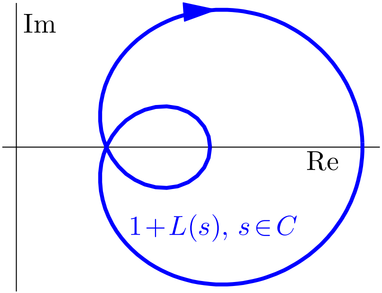

Total phase change after complete revolution: 0°

- No pole or zero encircled by C (all poles and zeros in LHP)

$\Rightarrow$ No encirclement of origin by $1 + L(s)$

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Evaluating $1 + L(s)$: along a closed contour

Now assume $1 + L(s)$ has one pole within the contour C.

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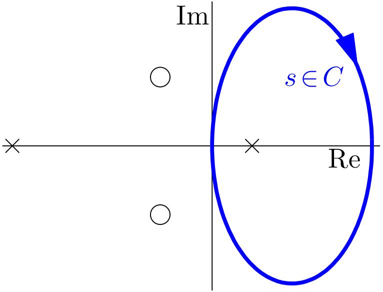

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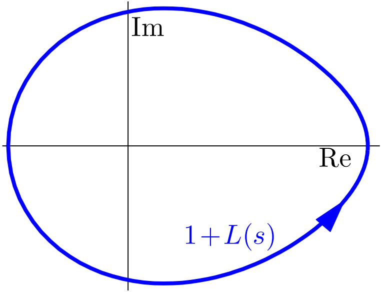

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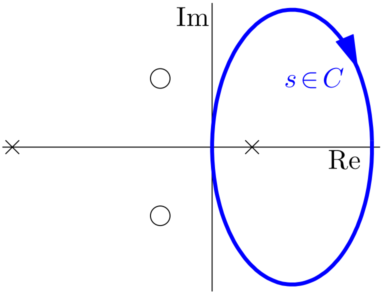

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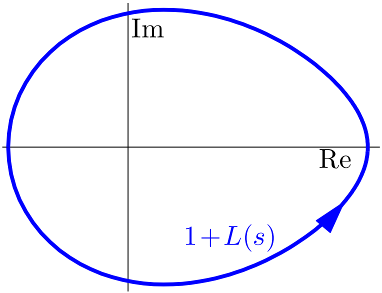


Total phase change after complete revolution: $+360°$

- Orientation w.r.t. enclosed RHP pole changes $360°$

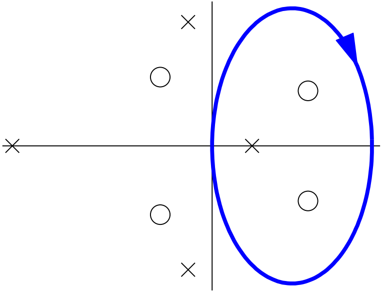
$\Rightarrow$ One *counterclockwise* encirclement of origin by $1 + L(s)$

Enclosed *zero* yields *clockwise* encirclement ($-360°$)

Evaluating $1 + L(s)$: along a closed contour

What if multiple poles and zeros are enclosed?

Poles ($\times$) and zeros ($\bigcirc$) of $1+L(s)$:



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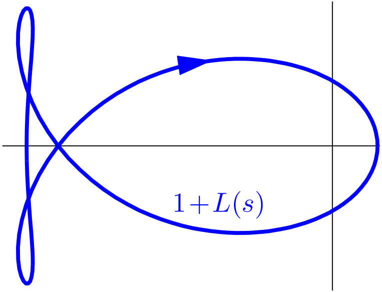

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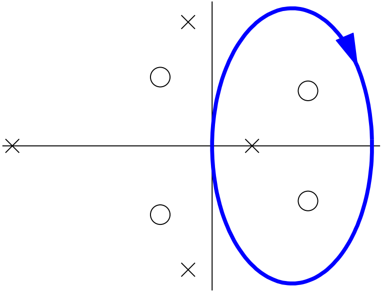

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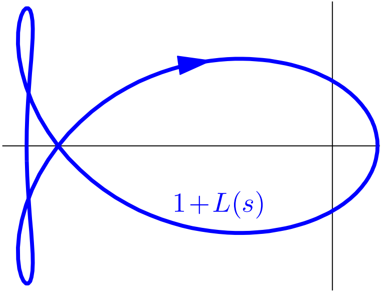


Net phase change after complete revolution: -360°

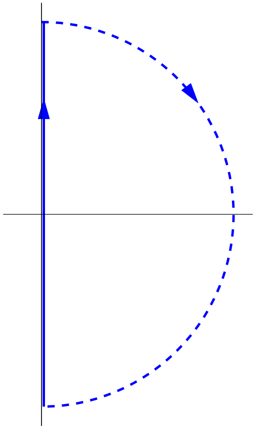
Hence, number of *clockwise* encirclements of the origin: $N = Z - P$

- Z = number of enclosed zeros
- P = number of enclosed poles

Evaluating $1 + L(s)$: contour around RHP

To check if there are poles and zeros in RHP:

- Closed contour must encircle whole RHP
- Called the *D-contour*



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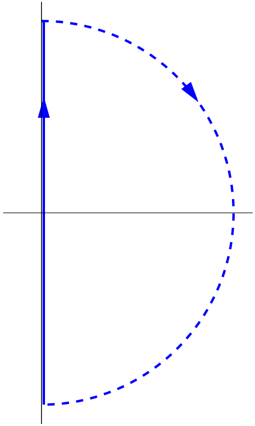

Assume **strictly proper** open loops $L(s)$:

- #poles $>$ #zeros
- $L(s) \to 0$ for $s \to \infty$
- $1 + L(s) = 1$ for all s on dashed line

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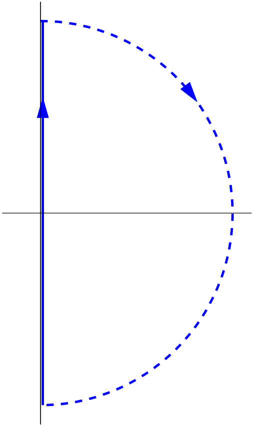

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Evaluation of $1+L(s)$ along D-contour **reduces to** evaluation of $1+L(s)$ along imaginary axis:

- $1 + L(j\omega)$ for $-\infty < \omega < \infty$

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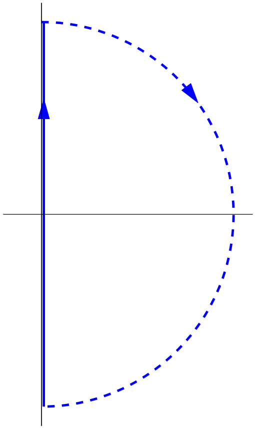

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clockwise encirclements of origin: $N = Z - P$

- $Z =$ # RHP zeros in $1+L(s)$
- $P =$ # RHP poles in $1+L(s)$

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Evaluating $1 + L(s)$: closed loop stability

- Goal: checking stability of **closed loop** $\frac{L(s)}{1+L(s)}$, $\frac{1}{1+L(s)}$, etc.
- Given: $L(s) = \frac{a(s)}{b(s)}$

Then:

- Poles of $1 + L(s) \quad \Rightarrow \quad b(s) = 0 \quad \Rightarrow$ **open-loop** poles!
- Zeros of $1 + L(s) \Rightarrow a(s)+b(s) = 0 \Rightarrow$ **closed-loop** poles!

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To test closed loop stability:

- Evaluate $1 + L(j\omega)$ for all $-\infty < \omega < \infty$
- Plot the the resulting image in the complex plane
- Count # clockwise encirclements of origin: $N = Z - P$
  - Z = # RHP *closed loop* poles
  - P = # RHP *open loop* poles (known!)

Closed loop is stable when $Z = N + P = 0$

The Nyquist criterion

$$
\begin{array}{rcl}
1 + L(j\omega) \ \text{in } s\text{-plane} & \leftrightarrow & \text{encirclements of origin} \\
& \Updownarrow & \textit{is equivalent to} \\
L(j\omega) \ \text{in } s\text{-plane} & \leftrightarrow & \text{encirclements of point (-1,0)}
\end{array}
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Hence, for controller design: $N = Z - P$

- Controller $C(s)$ is designed *stable*
- Plant $H(s)$ has P unstable poles (known a priori)
- Requirement: stable closed loop, so $Z = 0$

Nyquist plot of $L(j\omega)$ should have P *counterclockwise* encirclements of (-1,0)

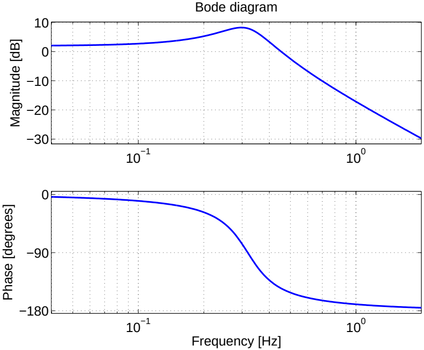
Note: $L(j\omega)$ for negative ω is mirror image of $L(j\omega)$ for positive ω w.r.t. real axis: $L(j\omega) = L^*(-j\omega)$

The Nyquist criterion: relationship with Bode

Closed loop stability determined by open loop $L(s) = H(s)C(s)$

- Evaluate $1+L(j\omega)$ for $-\infty<\omega<\infty$? $\Rightarrow$ Frequency response!
- Bode($L(j\omega)$) and Nyquist($L(j\omega)$) contain the same information!

Example: $L(s) = \frac{5}{s^2+s+4}$



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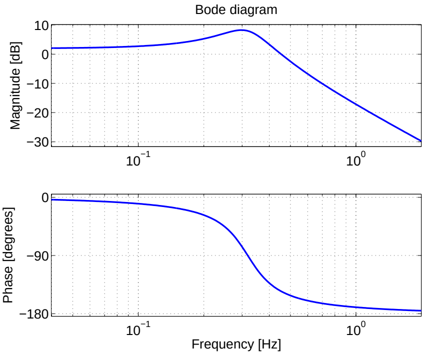
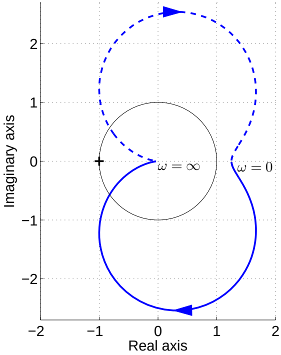


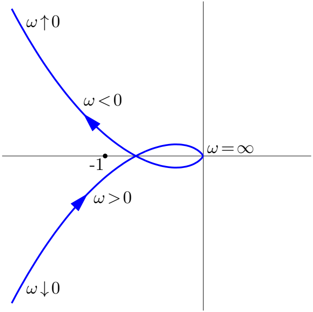


Note: Unit disc in Nyquist = 0dB line in Bode

The Nyquist criterion: pole at origin

What happens if $L(s)$ has an integrator / pole at the origin: $\frac{1}{s}$

- How do $\omega\uparrow 0$ and $\omega\downarrow 0$ connect?



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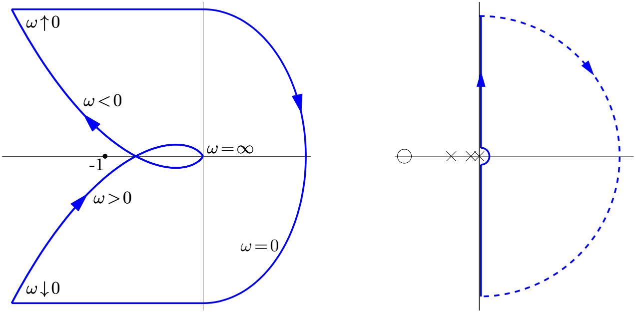

- Infinite gain at $\omega = 0$, i.e. $L(j\omega) \to \infty$ for $\omega \to 0$
- Rapid phase change of -180° at $\omega = 0$

$\Rightarrow$ $L(j\omega)$ travels half a circle *clockwise* at infinity

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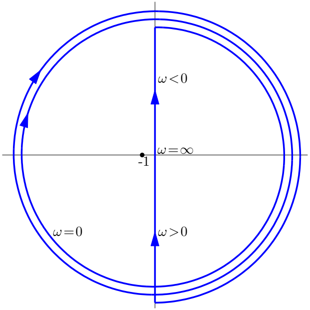
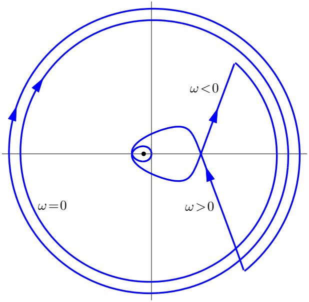
The Nyquist criterion: pole at origin

Poles of $L(s)$ in origin always give *clockwise* encirclements at infinity.
Examples:



$$L(s) = \frac{1}{s^5}$$

Two unstable closed loop poles



$$L(s) = \frac{4(s^2+0.8s+1)^2}{s^5}$$

No net encirclements of (-1,0)

$\Downarrow$

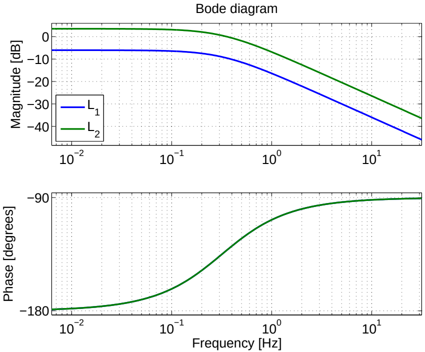
Stable closed loop

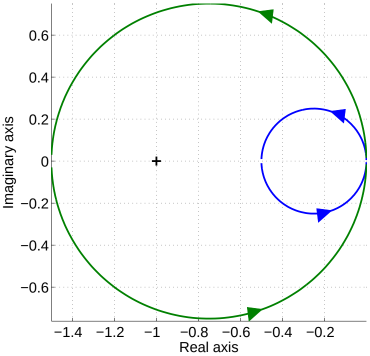
Some other examples...

Nyquist: examples

Example: unstable open loop ($P = 1$)

$$L_1(s) = \frac{1}{s-2} \qquad \text{and} \qquad L_2(s) = \frac{3}{s-2}$$



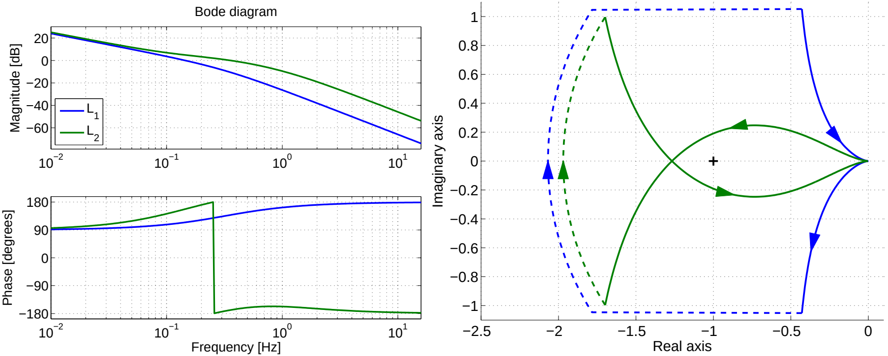


- L_1 has no encirclement $\rightarrow$ closed loop has 1 RHP pole
- L_2 has 1 counterclockwise encirclement $\rightarrow$ closed loop stable

Nyquist: examples

Example: unstable open loop ($P = 1$)

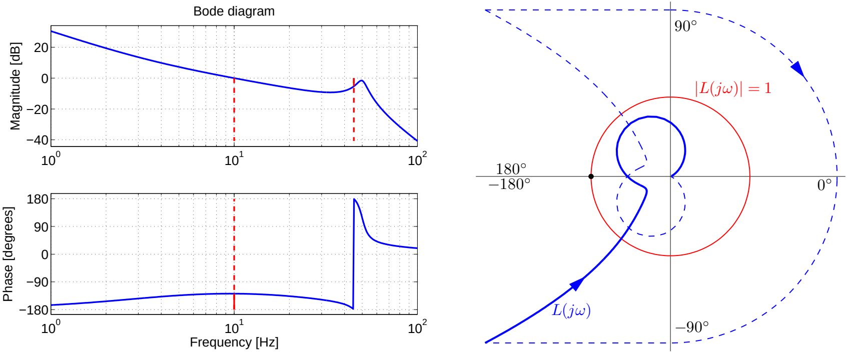
$$L_1(s) = \frac{2}{s(s-2)} \quad \text{and} \quad L_2(s) = \frac{10 \cdot 2}{s(s-2)} \cdot \frac{s+0.75}{s+6.75}$$



- L_1 has 1 clockwise encirclement $\rightarrow$ closed loop has 2 RHP poles
- L_2 has 1 counterclockwise encirclement $\rightarrow$ closed loop stable

Nyquist: examples

Specific motion system (stable, minimum phase):



Bode: $|L|$ and $\angle L$

Nyquist: Re(L) and Im(L)

2 integrators: 2 poles in the origin

- $L(j\omega)$ makes 1 clockwise (infinite length) encirclement at $\omega = 0$

Making Nyquist a little simpler...

Simplifying Nyquist

- Summarizing: a necessary requirement for stable closed loop:

Nyquist: (-1,0) is on the *left side* of $L(j\omega) = H(j\omega)C(j\omega)$, where $0 < \omega < \infty$ (only positive frequencies)

$\Updownarrow$

Bode: slope at first crossover frequency ω_{co} is approx -1, i.e. $\angle L(j\omega)$ at ω_{co} is between -180° and 0°

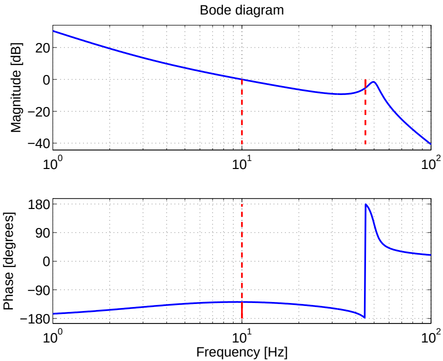
- When there are no more crossovers and Bode gain-phase holds, then this is also **sufficient**, i.e. when $L(s)$:
  - is stable and minimum phase (all poles and zeros in LHP),
  - and $|L(j\omega)| < 1, \quad \forall \omega > \omega_{co}$

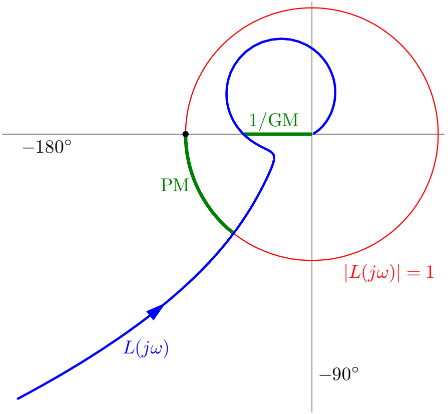
  then all 'rotations' have been compensated by the time $L(j\omega)$ hits the unit disc, and $L(j\omega)$ stays inside this disc for $\omega > \omega_{co}$.
- If $|L(j\omega)| = 1$ at multiple frequencies, then also count the possible encirclements between subsequent crossovers.

Crossover ω_{co}: Frequency where $|L(j\omega)| = 1$, i.e. where $L(j\omega)$ crosses 0dB line.

Nyquist and stability margins

Stability margins





- Infinite gain and phase$= -180^\circ$ for small ω
- Small phase lead PM at crossover ω_{co}
- Resonance loop (with phase shift) around 50Hz

Robustness

Robustness: the quality of being able to withstand changes or variations in circumstances or dynamics

- A robust closed loop stays stable after small variations in dynamics
- A Nyquist curve far from -1 is more robust than a curve close to -1

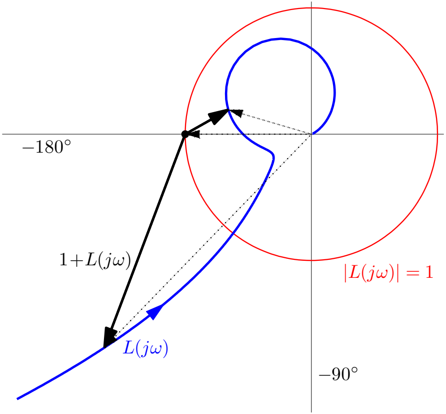
Robustness margins:

Phase margin (PM): Phase lead above -180° at the frequency where $|L(j\omega)| = 0\text{dB}$

Gain margin (GM): The inverse of GM is the gain at the frequency where $\angle L(j\omega) = -180^\circ$

Modulus margin (MM): The inverse of the smallest distance of $L(j\omega)$ over all ω to the point -1

Modulus margin



Distance between a point on $L(j\omega)$ and (-1,0) is a simple vector operation:

$$L(j\omega) - (-1) = 1 + L(j\omega)$$

- This is the inverse of Sensitivity $S(j\omega) = \frac{1}{1+L(j\omega)}$!
- Peak in S gives smallest distance to (-1,0)

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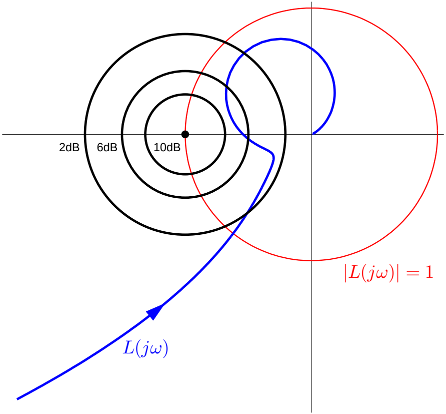
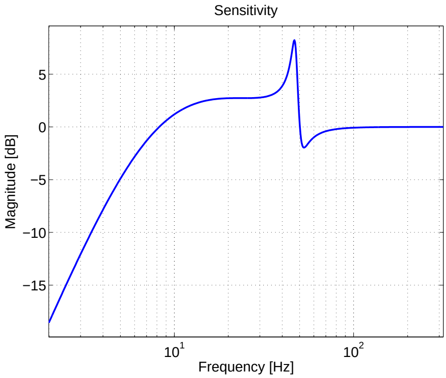
Modulus margin

Circles around (-1,0) represent bounds on

- ‘distance from instability’ $\Rightarrow$ robustness
- maximum value in S





What does this mean for controller design?

Stabilization

General procedure to design stabilizing controller:

- choose desired crossover frequency ω_{co}
- tune $L(j\omega)$ for low frequencies $\omega \leq \omega_{co}$
  - Bode: change slope of $L(j\omega)$ at ω_{co} to approx -1,
    i.e. create phase between -180° and 0°
    Nyquist: leave (-1,0) on the left of $L(j\omega)$
  - if RHP zeros or poles in open loop:
    check encirclements in Nyquist
    $\Downarrow$
    if unstable, try different ω_{co}
- tune $L(j\omega)$ for higher frequencies $\omega > \omega_{co}$
  - check phase at other 0dB crossings and correct if necessary
- make adjustments to meet desired margins

Stabilization tools

Basic control methods to stabilize closed loop:

- Open loop gain
  - increase or decrease controller gain globally

  $\Rightarrow$ Bode: move frequency response of $L(j\omega)$ up or down to change ω_{co} and the corresponding phase

  $\Rightarrow$ Nyquist: enlarge or reduce the Nyquist plot of $L(j\omega)$ w.r.t. (0,0) to position $L(j\omega)$ around (-1,0)
- Controller phase
  - add LHP zero(s) (or pole(s)) to the controller to increase (or decrease) the phase locally
  - normally using **lead filter** or **PD**

  $\Rightarrow$ Bode: place LHP zero(s) somewhat before ω_{co} to increase the phase around ω_{co}

  $\Rightarrow$ Nyquist: rotate a part of $L(j\omega)$ to pull it to right of (-1,0)
  - Note 1: this also changes the gain locally!
  - Note 2: negative gain changes phase with 180° globally

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Some final stabilization tips

For stable minimum-phase open loops:

- Use simplified Nyquist criterion

$\Rightarrow$ Leave -1 to the left (i.e. phase lead above -180° at cross-over)

For unstable open loops (P RHP poles):

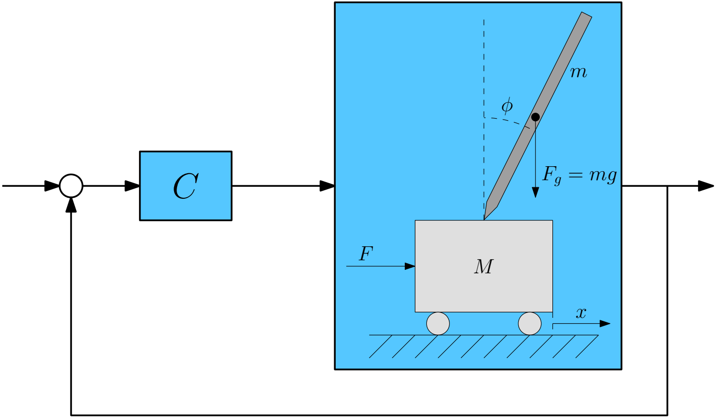
- Use full Nyquist criterion
- Create P counterclockwise encirclements (i.e. extra phase)
- Use phase lead of RHP pole to create extra phase *outside* unit disc

$\Rightarrow$ Choose crossover ω_{co} above unstable poles

For non-minimum phase open loops:

- Use full and/or simplified Nyquist criterion
- RHP zero creates undesired phase lag
- Allow this phase lag only *inside* the unit disc

$\Rightarrow$ Choose crossover ω_{co} below RHP zeros



Let's stabilize!