Advanced Motion Control

Part IV: Linear System Theory

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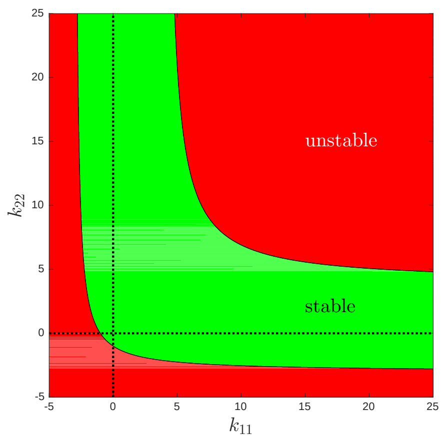
Introduction

Recall example

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \quad K(s) = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$$



What is going on?

- poles? stable?
- zeros? minimum phase?

Introduction

Recall example

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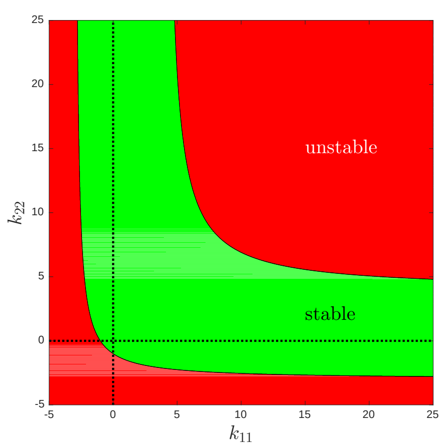


Further analysis

Assume $y_2 = 0$

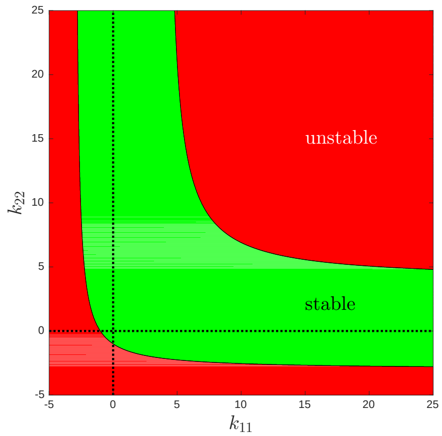
- perfect tracking in loop 2
- recall RGA

$$\begin{aligned} y_1 &= \underbrace{\left(g_{11} - \frac{g_{12}g_{21}}{g_{22}}\right)}_{=\hat{g}_{11}} u_1 \\ &= \frac{1}{s+1} - \frac{2(s+1)}{(s+3)(s+1)} \\ &= -\frac{s-1}{(s+1)(s+3)} \end{aligned}$$

- zero... RHP!
- where did it come from?

Recall example

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \quad K(s) = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$$



Further analysis

Other loop: $y_1 = 0$

- perfect tracking in loop 1

$$y_2 = -\frac{s-1}{(s+1)(s+3)}$$

- again RHP zero
- multivariable systems
  - need notion for zeros
  - stability theory

System descriptions

State space

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where

- $x(t) \in \mathbb{R}^n$: state vector
- $u(t) \in \mathbb{R}^m$: input vector
- $y(t) \in \mathbb{R}^l$: output vector

Equivalent

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

Shorthand notation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

System descriptions

State space

If $x(0) = 0$, then

$$sI_n x(s) = Ax(s) + Bu(s)$$
$$y(s) = Cx(s) + Du(s)$$

and thus

$$\underbrace{\begin{bmatrix} sI_n - A & B \\ -C & D \end{bmatrix}}_{\text{Rosenbrock (state-space) system matrix}} \begin{bmatrix} x(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -y(s) \end{bmatrix}$$

Elimination of $x(s)$

$$G(s) = C(sI_n - A)^{-1}B + D$$
$$= \frac{1}{\det(sI_n - A)}\left(C\operatorname{adj}(sI_n - A)B\right) + D \in \mathcal{R}^{l \times m}$$

key point: adj (adjugate matrix (Skogestad & Postlethwaite 2005,)) is polynomial $\Rightarrow$ poles result from $\det(sI_n - A)$

System descriptions

Domains

- Continuous time LTI
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
- Discrete time LTI
$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$
- Continuous time nonlinear time-invariant
$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$
- Continuous time nonlinear time-variant
$$\dot{x}(t) = f(t, x(t), u(t))$$
$$y(t) = g(t, x(t), u(t))$$

System descriptions

State space vs. transfer function matrices

- transfer function matrix contains ratio of polynomials
$$G(s) = C(sI_n - A)^{-1}B + D$$
- state space is ODE reformulated as 1^{st} order systems
$$G(s) \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$
- advantages state space
  - linear algebra vs. polynomial algebra
  - numerical reliability for large systems
  - easy to handle multivariable systems
- both representations are used throughout

System descriptions

Impulse response matrix

$$g(t) = \begin{cases} 0 & t < 0 \\ Ce^{At}B + D\delta(t) & t \geq 0 \end{cases}$$

with $\delta(t)$ unit impulse delta function

With $x(0) = 0$, dynamic response to input $u(t)$

$$\begin{aligned} y(t) = g(t) * u(t) &= \int_0^t g(t-\tau)u(\tau)d\tau \\ &= C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}$$

Also,

$$G(s) = \int_0^\infty g(t)e^{-st}dt$$

Poles and stability

Definition (4.4)

A system is internally stable if there are no hidden unstable modes and bounded input signals result in bounded output signals

Remarks

- no hidden unstable modes: take

  $$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

  minimal, i.e., observable and controllable (see (Skogestad & Postlethwaite 2005, Definition 4.3))
- various notions of stability exist. For considered LTI case, no significant practical difference.

Poles and stability

Definition (4.6)

The poles p_i of a system with state space realization

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

are $\lambda_i(A)$, $i = 1, \ldots, n$. The characteristic polynomial (also: pole polynomial) is defined as

$$\phi(s) = \det(sI_n - A) = \prod_{i=1}^{n}(s - p_i)$$

Thus, the poles are the roots of the characteristic equation

$$\phi(s) = \det(sI_n - A) = 0$$

Q: what if A non-minimal?

Poles and stability

Stability is determined by poles.

Theorem (4.3)

The system $\dot{x} = Ax + Bu$ is stable iff all poles are in the open LHP, i.e., $\Re(\lambda_i(A)) < 0 \; \forall i$

If $G(s)$ is given instead of a state-space realization, then the following result applies.

Theorem (4.4)

The pole polynomial $\phi(s)$ corresponding to a minimal realization of $G(s)$ is the least common denominator of all non-identically zero minors of all orders of $G(s)$.

Definition

Minor of a matrix is the determinant of a sub-matrix by deleting certain rows and/or columns.

Poles and stability

Example

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

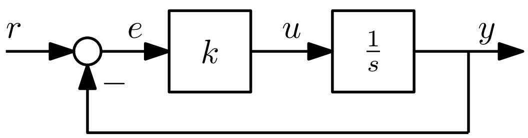
- Q: minor of order 2? A: $\det G(s) = \frac{2(s-1)^2-18}{(s+2)^2} = \frac{2(s-4)}{s+2}$
- Q: minors of order 1?
- Q: pole polynomial $\phi(s)$?

Poles and stability

Closed-loop stability follows similarly:

- all poles of the closed-loop system have to be in the open LHP

Example



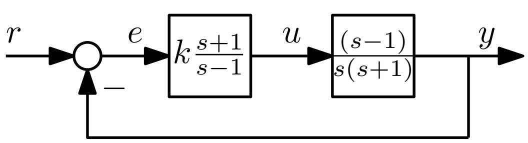
- Q: Stable?

Poles and stability

Closed-loop stability follows similarly:

- all poles of the closed-loop system have to be in the open LHP

Example



- Q: Sensitivity function?
- Q: Closed-loop stable?
- Q: Control sensitivity?

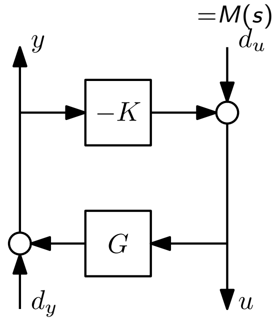
Poles and stability

Closed-loop stability follows similarly: all poles of the closed-loop system have to be in the open LHP.

Theorem (4.6)

Assume that G and K are minimal, i.e., no hidden unstable modes. Then, the feedback interconnection is internally stable iff $M(s)$ is stable, where

$$\begin{bmatrix} u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} I & K \\ -G & I \end{bmatrix}^{-1}}_{=M(s)} \begin{bmatrix} d_u \\ d_y \end{bmatrix} = \begin{bmatrix} S_I & -KS \\ GS_I & S \end{bmatrix} \begin{bmatrix} d_u \\ d_y \end{bmatrix}$$



Intuition: put an exogenous input and output between every component to avoid the creation of unstable hidden modes

Poles and stability

Corollary

- *if K stable, then feedback interconnection internally stable iff $P(I + KP)^{-1}$ stable*
- *if P stable, then feedback interconnection internally stable iff $K(I + PK)^{-1}$ stable*
- *if P, K stable, then feedback interconnection internally stable iff $(I + PK)^{-1}$ stable*

see (Zhou et al. 1996, Section 5.3) for details

Poles and stability

On all stabilizing controllers

- assume that $G(s)$ is stable and let $Q(s) = K(s)\,(I + G(s)K(s))^{-1}$
- then,

$$\begin{aligned} S_I &= I - QG \\ S &= I - GQ \\ KS &= Q \\ GS_I &= G(I - QG) \end{aligned}$$

Observations

- closed-loop stable iff Q stable, and becomes affine in Q
- $K = (I - QG)^{-1} Q$ parameterizes all stabilizing controllers
- known as Youla parameterization
  - has been a key result in automatic control, e.g., $\mathcal{H}_\infty$-optimization
  - will come back in a later lecture on robust control
  - if G unstable: use coprime factorizations
    - essentially: if you find a stabizing controller for G, you have found all!
    - if $G(s)$ is stable, then $K = 0$ is stabilizing

Zeros

Zeros are defined on the basis of transfer function matrices

Definition (4.7)

z_i is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$

Normal rank: rank of $G(s)$ at all values of s except at a finite number of singularities (the zeros)

To compute: write

- $G(s) = \frac{1}{d(s)} N(s)$
- $d(s)$ is the least common denominator of all elements of $G(s)$
- $N(s)$ is a polynomial matrix of rank $r = \operatorname{rank} G(s)$

Definition

The zero polynomial is the greatest common divisor of all numerators of all order r minors of $G(s)$, where these minors have been adjusted to have the pole polynomial $\phi(s)$ on Slide 12 as their denominator

Zeros

Zero directions

If $G(s)$ loses rank at $s = z$, then $\exists u_z, y_z \neq 0$, $u_z^H u_z = 1$, $y_z^H y_z = 1$ such that

$$G(z) u_z = 0 \cdot y_z$$

or equivalently

$$y_z^H G(z) = 0 \cdot u_z^H$$

- u_z is zero input direction
- y_z is zero output direction: gives indication which output (or linear combination of outputs) may be difficult to control
- note: directions are scaling-dependent: scale first

Example of slide 13 revisited

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

Normal rank = 2. We already computed

- minor of order 2 $= \det G(s) = \frac{2(s-1)^2-18}{(s+2)^2} = \frac{2(s-4)}{s+2}$
- pole polynomial $\phi(s) = s + 2$ (slide 12)

Q:

- zero polynomial? (slide 19)
- So $G(s)$ has a single RHP zero at $s = 4$

More examples

- $G(s) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \Rightarrow z(s) = s$
- $G(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^2 \end{bmatrix} \Rightarrow z(s) = s^2$
- $G(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \Rightarrow z(s) = s^2$
- $G(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \Rightarrow z(s) = 1$ (no zero)
- $G(s) = \begin{bmatrix} s(s+3) & 0 \\ 0 & s(s+5) \end{bmatrix} \Rightarrow z(s) = s^2(s+3)(s+5)$

Zeros

Diagonal form

- diagonal matrices have special properties
  $\Rightarrow$ poles and zeros are on the diagonal
- main idea: can we exploit this?
- answer: Smith-McMillan form
- required: technical result - unimodular matrices

Zeros

Unimodular matrices

Definition

A polynomial matrix $U(s)$ is called unimodular if $U^{-1}(s)$ also is polynomial.

Theorem

$U(s)$ is unimodular iff $\det U(s)$ is constant.

- are the following matrices unimodular?
  - $\begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}$
  - $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$

Zeros

The key trick: elementary operations on polynomial matrices

The following operations

- interchange two rows or columns
- multiplication of one row or column by a constant
- addition of a polynomial multiple of one row to another

can be represented by unimodular matrices.

Smith form

- any polynomial matrix $P(s)$ can be written as
$$\begin{aligned} S(s) &= U_1(s)P(s)U_2(s) \\ &= \operatorname{diag}\{\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_r(s), 0, 0, \ldots, 0\} \end{aligned}$$
$U_1(s)$, $U_2(s)$ unimodular
- rank r
- zero polynomial $z(s) = \prod_i \varepsilon_i(s)$

Zeros

Smith form example

- any polynomial matrix $P(s)$ can be written as
$$S(s) = U_1(s)P(s)U_2(s)$$
- let
$$P(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$
- then
$$U_1(s) = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$$
- leads to
$$S(s) = U_1(s)P(s) = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
- $z(s) = 1 \Rightarrow$ no zeros

Zeros

Smith-McMillan form

- another key idea: use this for rational systems
- let $G = \frac{1}{d}N$
- then,

$$M(s) = U_1(s)G(s)U_2(s)$$
$$= \mathrm{diag}\{\frac{\varepsilon_1(s)}{\psi_1(s)}, \frac{\varepsilon_2(s)}{\psi_2(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, 0, \ldots, 0\}$$

$U_1(s)$, $U_2(s)$ unimodular

and ε_i, ψ_i coprime (i.e., no common factors)

denotes the Smith-McMillan form

- zero polynomial $z(s) = \prod_i \varepsilon_i(s)$
- pole polynomial $\phi(s) = \prod_i \psi_i(s)$
- directly extendable to nonsquare systems

Zeros

Smith-McMillan example - slide 21 revisited

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix} \xrightarrow{\text{pre}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}} \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 9 & 4(s-1) \end{bmatrix}$$

$$\xrightarrow{\text{post}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \frac{1}{s+2}\begin{bmatrix} 4 & s-1 \\ 4(s-1) & 9 \end{bmatrix} \xrightarrow{\text{pre}\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}} \frac{1}{s+2}\begin{bmatrix} 4 & s-1 \\ -4 & -s^2+s+9 \end{bmatrix}$$

$$\xrightarrow{\text{pre}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \frac{1}{s+2}\begin{bmatrix} 4 & s-1 \\ 0 & -s^2+2s+8 \end{bmatrix} \xrightarrow{\text{post}\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}} \frac{1}{s+2}\begin{bmatrix} 1 & s-1 \\ 0 & -(s+2)(s-4) \end{bmatrix}$$

$$\xrightarrow{\text{post}\begin{bmatrix} 1 & s-1 \\ 0 & -1 \end{bmatrix}} \frac{1}{s+2}\begin{bmatrix} 1 & 0 \\ 0 & (s+2)(s-4) \end{bmatrix} \to \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & s-4 \end{bmatrix} \to \begin{matrix} z(s) = s-4 \\ \phi(s) = s+2 \end{matrix}$$

Zeros

Smith-McMillan summary

- systematic approach for computing the poles and zeros of transfer function matrices
- uses polynomial algebra
- poles are "easy": s_i is a pole iff it is a pole of G_{ij}
- zeros are tricky
  - example: $G(s) = \begin{bmatrix} 1 & \frac{1}{s} \\ 0 & 1 \end{bmatrix}$
  - if $G(s)$ square and invertible, then zeros of G equal poles of $G^{-1}(s)$
  - $G^{-1}(s) = \begin{bmatrix} 1 & -\frac{1}{s} \\ 0 & 1 \end{bmatrix} \Rightarrow s = 0$ is a zero of $G(s)$!
  - also: $s = 0$ is a pole and a zero at the same time
  - Q: can you use $\det G(s)$ to compute poles?
    A: no! fails in case of pole/zero cancellations!
  - theory is based on transfer function matrices & polynomial algebra
    - Matlab? $\Rightarrow$ consider state-space approach (next)

Zeros

State space

Theorem

Let

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be minimal. Then, the transmission zeros (see slide 19) are the values $s = z$ for which the Rosenbrock matrix (see slide 6)

$$\begin{bmatrix} sI_n - A & B \\ -C & D \end{bmatrix}$$

loses rank.

- See, e.g., (Bosgra et al. 2005, Lemma 3.2.8) for a proof
- numerically: generalized eigenvalue problem (`eig.m`, `tzero.m`):

$$\left(s \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right)$$

Zeros

State space - simplified result

- if D is invertible, then

$$\begin{aligned}
&\det\left(\begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix}\begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} sI - (A - BD^{-1}C) & -B \\ 0 & -D \end{bmatrix}\right)
\end{aligned}$$

- since D is invertible, zeros are eigenvalues of $A - BD^{-1}C$
- thus: zeros depend on all matrices (A, B, C, D), while poles only depend on A

Zeros

State space - summary on zeros

- using state-space techniques we can compute zeros using linear algebra techniques
  - generalized eigenvalue problem
- we tacitly assumed transmission zeros
- if state space is non-minimal, then invariant zeros will arise
  - use minimal realizations!

Summary and reading

Take-home messages

- notions for poles and zeros: Smith-McMillan form
- multivariable poles: easy to see, difficult to count
- multivariable zeros: difficult to see

Next

- stability analysis using frequency response functions

Summary and reading

Reading

- general material linear system theory: (Skogestad & Postlethwaite 2005, Chapter 4)

Additional reading material

- Smith-McMillan form: Maciejowski (1989, Section 2.2) for many details and Hespanha (2009, Lecture 18) for a condensed overview

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