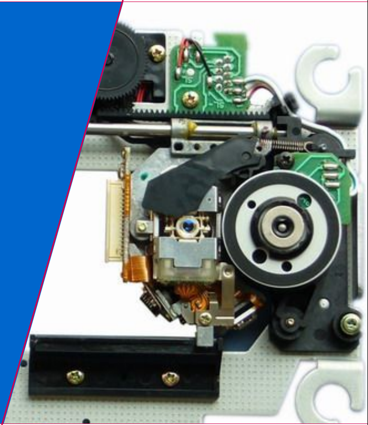
4CM00: Control Engineering

Background and prerequisites



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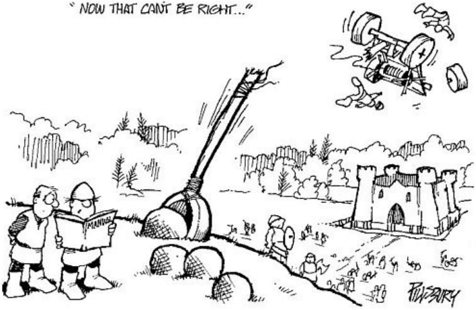
Where innovation starts

September 2020

What is this course all about?

- Control the output of a dynamic system by manipulating its input
  - via both feedback and feedforward techniques
  - requires knowledge and understanding of the dynamic system
- We consider only linear systems
- We consider only single-input-single-output (SISO) systems
- Because we're mechanical engineers, we apply our knowledge particularly to motion systems



Motion systems

What's so special about motion systems?

- they behave (fairly) linear
- they can be of very high order
- they're often hardly damped
  - they can have very sharp resonances
- they react fast, and can be sampled at a high rate
  - you can collect lot's of measurement data in a very short time

The theory of this course (4CM00) is

- in principle *generally* applicable to any linear system
- *particularly* useful for motion systems

Prerequisites; or things you should already know...

Linear dynamic systems

We consider dynamic systems with input u and output y

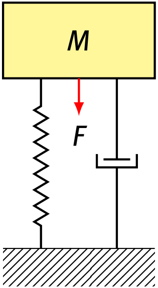
- The output of a dynamic system is *time dependent* and can be manipulated by its input
  - It can be interpreted as an operation on *signals*
  - It transforms signal $u(t)$ into another signal $y(t)$
- A dynamic system can be described by a *differential equation*

$$\begin{aligned} F\left(t, u, \frac{\mathrm{d}u}{\mathrm{d}t}, \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}, \ldots, \frac{\mathrm{d}^m u}{\mathrm{d}t^m}, y, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}t^n}\right) &= 0 \\ F\left(t, u, \dot{u}, \ddot{u}, \ldots, u^{(m)}, y, \dot{y}, \ddot{y}, \ldots, y^{(n)}\right) &= 0 \end{aligned}$$

- A dynamic system is *linear* if F is linear in its arguments, i.e. if the differential equation is *linear* in the (derivatives of) u and y

$$\begin{aligned} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = \\ b_m u^{(m)} + b_{m-1} u^{(m-1)} + \ldots + b_2 \ddot{u} + b_1 \dot{u} + b_0 u \end{aligned}$$

Example systems

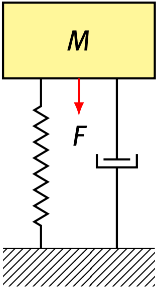


Mass supported by a spring and damper

$$M\ddot{x} + d\dot{x} + kx = F$$

- Input $u = F$, output $y = x$
- Linear system

Example systems



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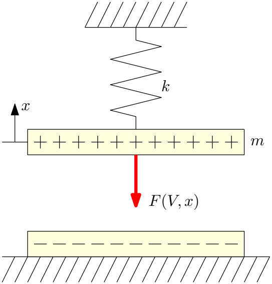


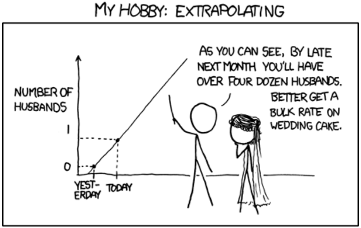
Charged plate suspended on a spring

$$m\ddot{x} + kx = -\alpha \frac{V^2}{(d_0 + x)^2}$$

- Input $u = V$, output $y = x$
- Non-linear system (both in u and in y)

Why linear systems?

All systems are non-linear...



...but in this course we focus on *linear systems* only. Why?

- Most systems behave linearly in their relevant operating range
- Most solutions are linear
  - linear controllers often do the job
- Linear systems are relatively easy to work with!
  - they have very useful properties

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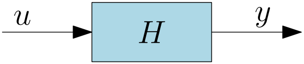
Linear systems: input–output behavior

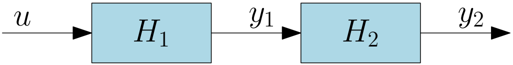
Linear systems have a unique *frequency response*



- when $u = \cos(\omega t)$, then $y = A\cos(\omega t + \phi)$
  - amplitude A and phase ϕ only depend on frequency ω
  - no other frequencies are induced

Linear systems have a superposition property:

1. When $u_1 \to y_1$ and $u_2 \to y_2$,
   then $\quad \alpha u_1 + \beta u_2 \to \alpha y_1 + \beta y_2 \quad$ for any (real) α and β
2. Consider the series interconnection



When $u = \cos(\omega t)$, then $y_1 = A_1 \cos(\omega t + \phi_1)$ and
$y_2 = A_2 A_1 \cos(\omega t + \phi_1 + \phi_2)$

Note that a *system* is actually an operation on one or more *signals*

- A time-domain signal $x(t)$ can be transformed to s-domain via the Laplace transform:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty e^{-st} x(t)\, dt \tag{1}$$

- Consequently, time derivatives of signals can be expressed as:

$$\begin{aligned}
\mathcal{L}\{\dot{x}(t)\} &= \int_0^\infty e^{-st} \dot{x}(t)\, dt = \left[e^{-st} x(t)\right]_0^\infty + s \int_0^\infty e^{-st} x(t)\, dt \\
&= sX(s) - x(0) \qquad (2) \\
\mathcal{L}\{\ddot{x}(t)\} &= s^2 X(s) - sx(0) - \dot{x}(0) \qquad (3)
\end{aligned}$$

- Since we're often only interested in input–output behavior, the transients $x(0)$ and $\dot{x}(0)$ are often omitted.

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Laplace transform table

Time domain $x(t)$	Laplace domain $X(s)$
$\delta(t)$	1
$1(t)$	$1/s$
t	$1/s^2$
t^2	$2!/s^3$
t^m	$m!/s^{m+1}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{(m-1)!}t^{m-1}e^{-at}$	$\frac{1}{(s+a)^m}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$

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Using Laplace transform on linear systems

- Consider the example

$$\ddot{y} + 2\dot{y} + 25y = u$$

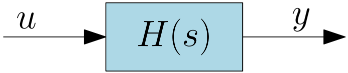
- Using the Laplace transform, this can be written as

$$s^2 Y(s) + 2sY(s) + 25Y(s) = U(s)$$

- The ratio between $U(s)$ and $Y(s)$ defines the transfer function $H(s)$

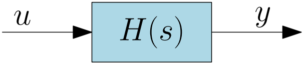
$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2s + 25}$$

- For linear systems the transfer function is always *rational*
  - i.e. both numerator and denominator are polynomials in s



The use of the Laplace transform (1)

How to determine the output $y(t)$ for a given $u(t)$?



- In time domain you'd need to solve the differential equation
- Or calculate the convolution

$$y(t) = h(t) \otimes u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\mathrm{d}\tau$$

where $h(\tau)$ is the *impulse response* of the system $H(s)$.

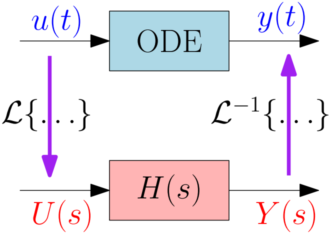
It's much easier via the Laplace domain (at least for linear systems)

- Convolution simplifies to

$$Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{h(t) \otimes u(t)\} = H(s) \cdot U(s)$$

The use of the Laplace transform (2)

Hence, $y(t) = \mathcal{L}^{-1}\left\{H(s) \cdot \mathcal{L}\{u(t)\}\right\}$, or schematically



Simple example: $H(s) = \frac{3}{s+3}$ with step input $u(t) = 1(t)$

- $U(s) = 1/s$
- $Y(s) = \frac{3}{s(s+3)} = \frac{1}{s} - \frac{1}{s+3}$
- $y(t) = 1 - e^{-3t}$

Note: Impulse response $h(t) = 3e^{-3t}$

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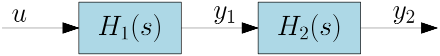
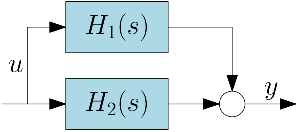
Using superposition

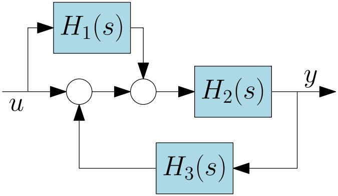
Linear systems + Laplace domain = easy combinations



$$y_2 = H_2(s)H_1(s)u$$



$$y = (H_1(s) + H_2(s))\, u$$



$$y = \frac{H_2(s)H_1(s) + H_2(s)}{1 - H_2(s)H_3(s)} u$$

Note: in this course we only consider SISO systems. For MIMO systems the order of multiplication matters, i.e. $H_2 H_1 \neq H_1 H_2$!

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Transfer functions: time domain

Poles and zeros: definition

What are poles and zeros?

- Poles: solutions of “denominator $= 0$”
- Zeros: solutions of “numerator $= 0$”

Example:

$$H(s) = \frac{s+16}{(s+2)(s+4)}$$

- Two poles: one at $s = -2$, one at $s = -4$
- One zero: at $s = -16$

But what do these poles and zeros mean?

Poles: interpretation

To understand, consider a step response of the same system.

- $U(s) = \mathcal{L}\{1(t)\} = \frac{1}{s}$
- $Y(s) = H(s)U(s) = \frac{s+16}{s(s+2)(s+4)}$

Using partial fraction expansion, we can write this output as

$$Y(s) = \frac{2}{s} - \frac{3.5}{s+2} + \frac{1.5}{s+4}$$

The step response in time domain thus becomes

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left[2 - 3.5e^{-2t} + 1.5e^{-4t}\right] \cdot 1(t)$$

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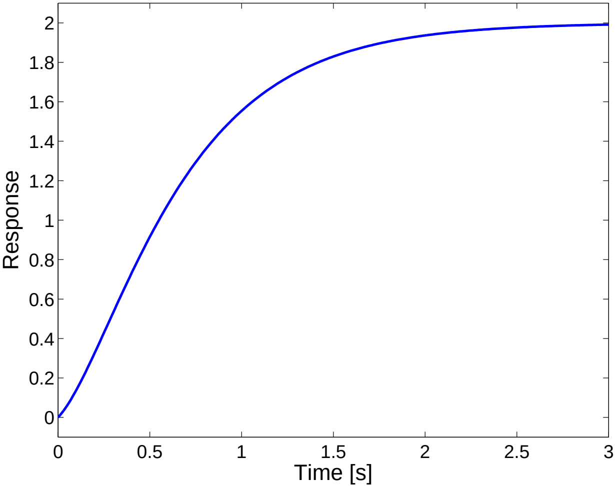
Note: the exponents are the same as the poles at $s = -2$ and $s = -4$.

Poles: interpretation



$$y(t) = \left[2 - 3.5e^{-2t} + 1.5e^{-4t}\right] \cdot 1(t)$$

Poles: interpretation

Another example:

$$H(s) = \frac{s+20}{s^2+4s+40}$$

- Poles: at $s = -2-6j$ and $s = -2+6j$.
- Zero: at $s = -20$

With a step input $U(s) = \frac{1}{s}$ the output becomes

$$\begin{aligned} Y(s) &= \tfrac{s+20}{s(s^2+4s+40)} \\ &= \tfrac{0.5}{s} - \tfrac{0.5s+1}{s^2+4s+40} = 0.5\tfrac{1}{s} - 0.5\tfrac{s+2}{(s+2)^2+6^2} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \left[0.5 - 0.5e^{-2t}\cos(6t)\right]\cdot 1(t) \end{aligned}$$

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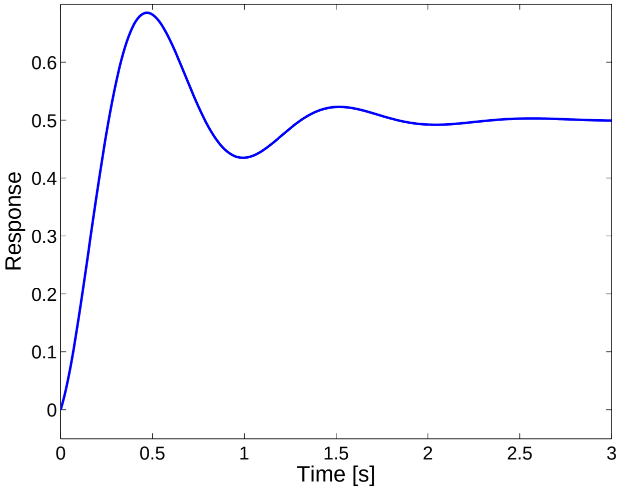
- exponent is real part of the pole, frequency is its imaginary part
- system is stable iff *all* exponents decay $\Leftrightarrow$ *all* poles are in LHP

Poles: interpretation



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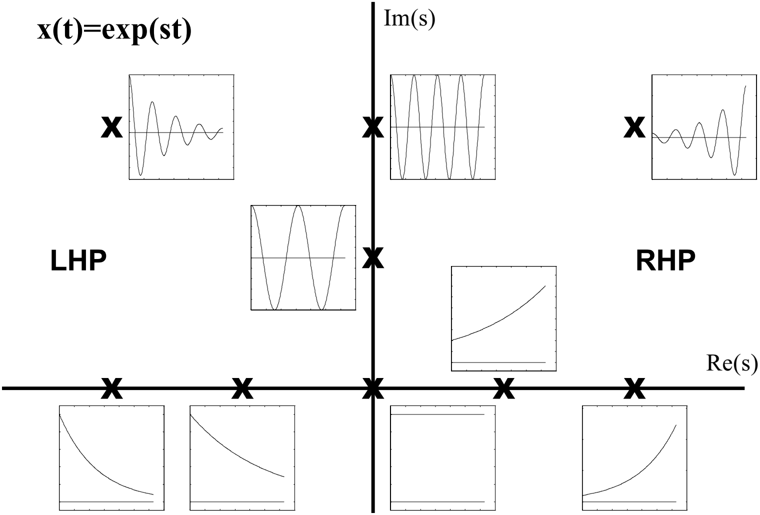
Pole locations: typical responses



Zeros: interpretation

The effect of the zero is much more subtle

- determines details of the time response
  - i.e. the gain of each exponent
  - i.e. how much each exponent contributes to the total response
- most evident when located in the right-half-plane (RHP)

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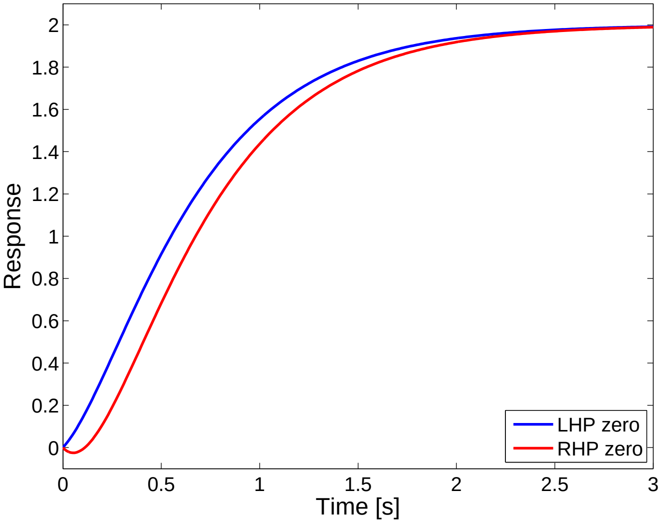
Example:

$$H(s) = \frac{16-s}{(s+2)(s+4)}$$

With a step input $U(s) = \frac{1}{s}$ the output becomes

$$\begin{aligned} Y(s) &= \tfrac{16-s}{s(s+2)(s+4)} \\ &= \tfrac{2}{s} - \tfrac{4.5}{s+2} + \tfrac{2.5}{s+4} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \left[2 - 4.5e^{-2t} + 2.5e^{-4t}\right] \cdot 1(t) \end{aligned}$$

Effect of RHP zeros



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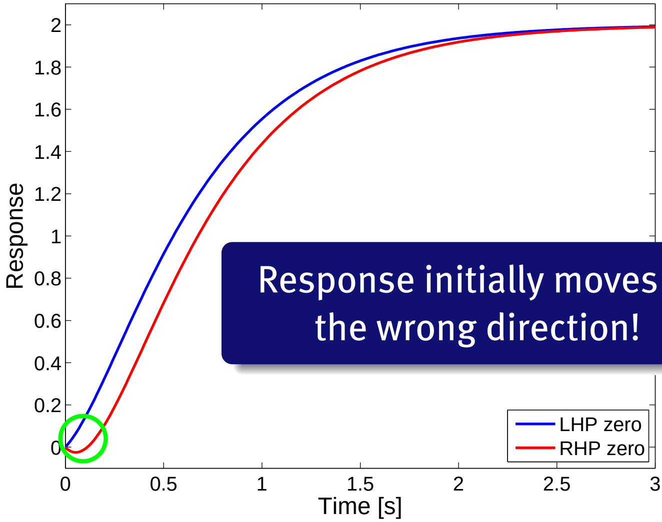

Response initially moves in the wrong direction!

Effect of RHP zeros

Another example:

$$H(s) = \frac{20 - s}{s^2 + 4s + 40}$$

- Poles: at $s = -2 - 6j$ and $s = -2 + 6j$.
- Zero: at $s = +20$

With a step input $U(s) = \frac{1}{s}$ the output becomes

$$\begin{aligned}
Y(s) &= \tfrac{s+20}{s(s^2+4s+40)} = \tfrac{0.5}{s} - \tfrac{0.5s+3}{s^2+4s+40} \\
&= \tfrac{1}{2} \cdot \frac{1}{s} - \tfrac{1}{2} \cdot \frac{s+2}{(s+2)^2 + 6^2} - \tfrac{1}{3} \cdot \frac{6}{(s+2)^2 + 6^2} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \left[\tfrac{1}{2} - \tfrac{1}{2} e^{-2t} \cos(6t) - \tfrac{1}{3} e^{-2t} \sin(6t)\right] \cdot 1(t)
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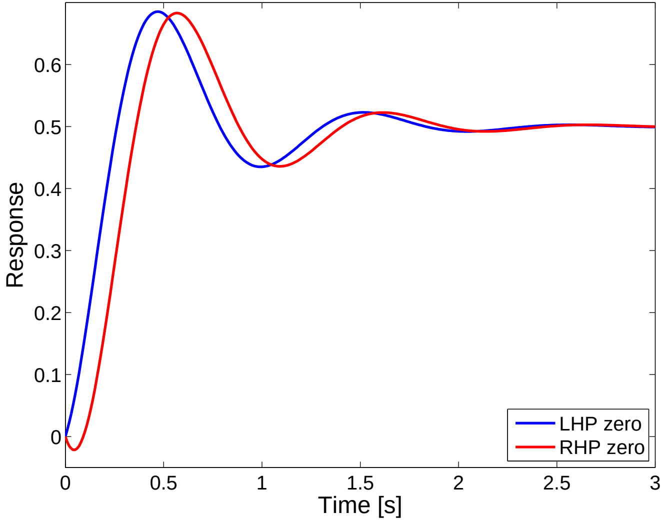

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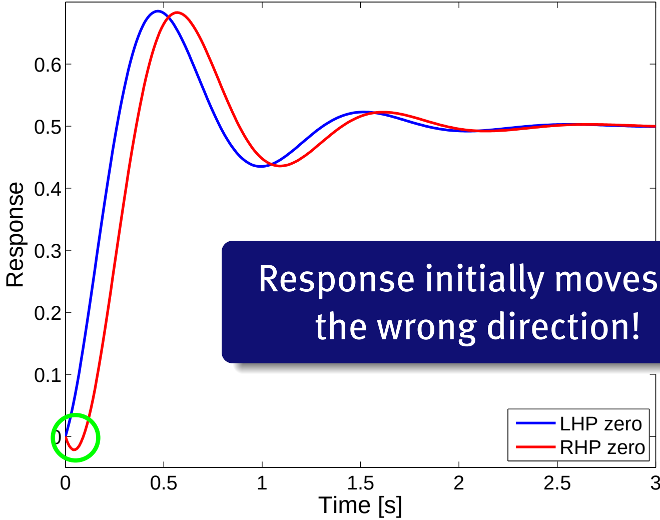

Response initially moves in the wrong direction!

Near-pole-zero cancellation

What happens if poles and zeros are close to each other?

- Example:

$$H(s) = \frac{s + 1.92}{(s+2)(s+4)}$$

- Zero at $s = -1.92$ is very close to pole at $s = -2$

Near-pole-zero cancellation

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- Example:

$$H(s) = \frac{s + 1.92}{(s+2)(s+4)}$$

- Zero at $s = -1.92$ is very close to pole at $s = -2$

With a step input $U(s) = \frac{1}{s}$ the output becomes

$$\begin{aligned} Y(s) &= \frac{s + 1.92}{s(s+2)(s+4)} \\ &= \frac{0.24}{s} + \frac{0.02}{s+2} - \frac{0.26}{s+4} \\ y(t) &= \mathcal{L}\{Y(s)\} = \left[0.24 + 0.02e^{-2t} - 0.26e^{-4t}\right] \cdot 1(s) \end{aligned}$$

Hence, the pole at $s = -2$ hardly contributes to the time response.

Transfer functions: frequency domain

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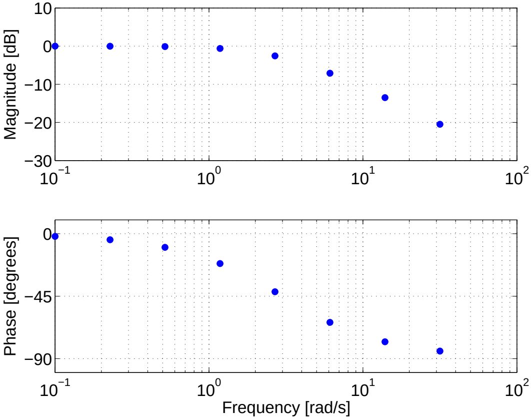
Linear systems: frequency response

Remember: linear systems have a unique *frequency response*

- when $u = \cos(\omega t)$, then $y = A\cos(\omega t + \phi)$

Hence, there is a unique *frequency response function* (FRF) $H(j\omega)$



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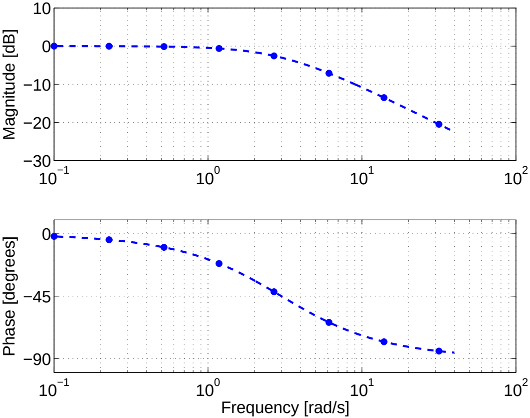

The Bode diagram

Frequency response $H(j\omega)$ is often represented in a *Bode diagram*

- considers *steady state* at each frequency
- gives amplitude amplification and phase shift at ω_1 in steady state
- superposition: also valid for a combination of frequencies ω_i

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FRF follows directly from the transfer function $H(s)$

- by substitution of $s = j\omega$ into $H(s)$
- yields a complex valued function $H(j\omega)$
- then $|H(j\omega)| = A(\omega)$ and $\angle H(j\omega) = \phi(\omega)$

Example: $H(s) = \frac{3}{s+3}$

- substituting $s = j\omega$ yields

$$H(j\omega) = \frac{3}{j\omega + 3} = \underbrace{\frac{9}{\omega^2 + 9}}_{\text{real part}} - \underbrace{\frac{3j\omega}{\omega^2 + 9}}_{\text{imaginary part}}$$

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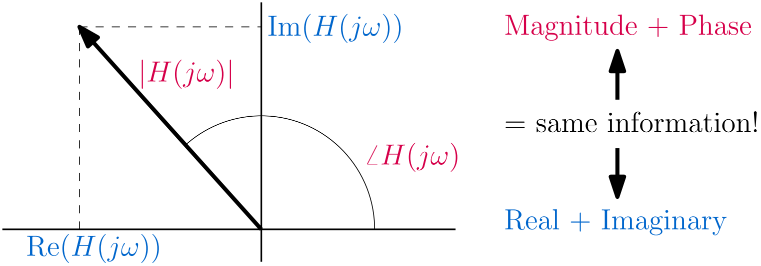
Frequency response and complex numbers

Hence, a frequency response function (FRF) is a *complex function*!

- Frequency response can also be drawn in the complex plane!



However, note that

- all input/output signals are of course still real valued
- the complex number represents the *operation* on these signals

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Bode basics

- frequency axis is logarithmic
- magnitude plot (vertical axis) is also logarithmic
  - amplitude A is expressed in decibel (dB):

$$\mathrm{dB} = 20 \cdot \log_{10} A$$

  - amplitude axis is thus linear on dB-scale
- consequence: simple powers ω^x are *straight lines*
  - E.g. mass line:

$$\begin{aligned} \log|H| &= \log\left|-\frac{1}{m\omega^2}\right| \\ &= \log(1/m) - 2 \cdot \log \omega \end{aligned}$$

- these asymptotes are therefore characterized by their *slopes*

Frequency response asymptotes

Another example: $H(s) = \frac{1}{s^2+2s+25}$

- substituting $s = j\omega$ yields

$$H(j\omega) = \frac{1}{25-\omega^2+2j\omega} = \underbrace{\frac{25-\omega^2}{(25-\omega^2)^2+4\omega^2}}_{\text{real part}} - \underbrace{\frac{2j\omega}{(25-\omega^2)^2+4\omega^2}}_{\text{imaginary part}}$$

But how to easily construct the Bode diagram from this?

The FRF approximates certain asymptotes, e.g.

- $\lim_{s\to 0} H(s) = \frac{1}{25}$ hence $\lim_{\omega\to 0} H(j\omega) = \frac{1}{25}$ $\Rightarrow$ 0-slope
- $\lim_{s\to \infty} H(s) = \frac{1}{s^2}$ hence $\lim_{\omega\to \infty} H(j\omega) = -\frac{1}{\omega^2}$ $\Rightarrow$ -2-slope

And what happens in between?

Complex-valued poles (or zeros)

Complex-valued poles (or zeros) have an (anti-)resonance with *damping*

Normalized second-order denominator:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- ω_n: undamped natural frequency
- ξ: normalized damping

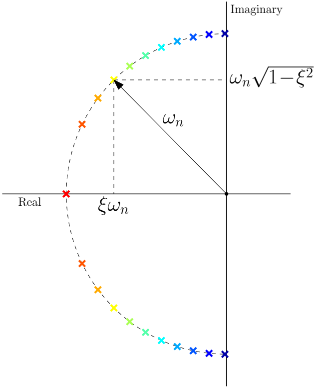
Poles are located at (using ABC-formula):

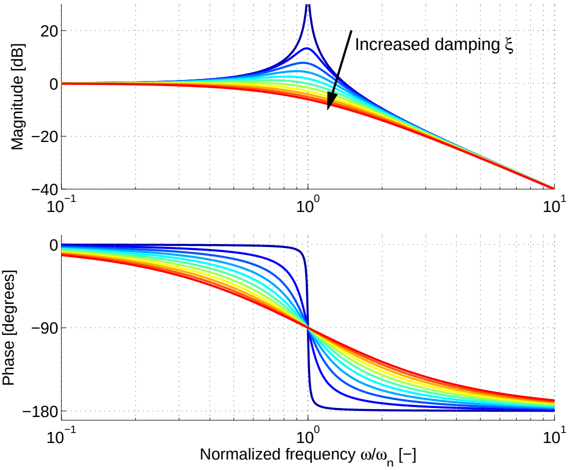
$$s = \underbrace{-\xi\omega_n}_{\text{exponential decay}} \pm j \underbrace{\omega_n\sqrt{1-\xi^2}}_{\text{oscillation frequency } \omega_d}$$

- for small damping $\omega_d \approx \omega_n$
- for $\xi \geq 1$ both poles are real

Complex-valued poles (or zeros)

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$





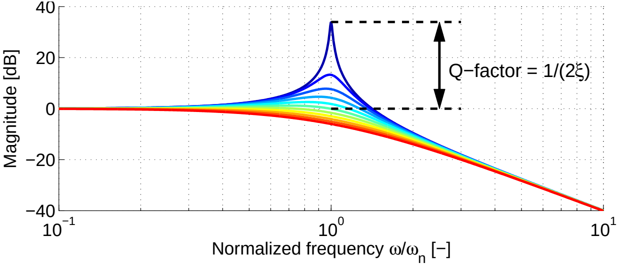
Complex-valued poles: Q-factor

How to estimate the damping from a Bode diagram?

- By evaluating the magnitude at the eigenfrequency ω_n, since

$$|H(j\omega_n)| = \left| \frac{\omega_n^2}{-\omega_n^2 + 2\xi\omega_n j\omega_n + \omega_n^2} \right| = \left| \frac{1}{j \cdot 2\xi} \right| = \frac{1}{2\xi}$$

- This is known as the *Q-factor* of a resonance.
- The damping is then $\xi = \frac{1}{2Q}$.



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Bode diagram example

Draw the frequency response function $H(j\omega)$

- for $H(s) = \frac{1}{s^2+2s+25}$

Bode diagram example

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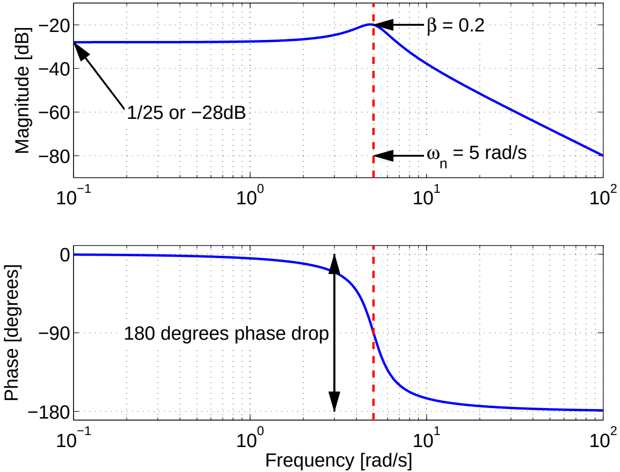

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From transfer function to Bode diagram

A transfer function

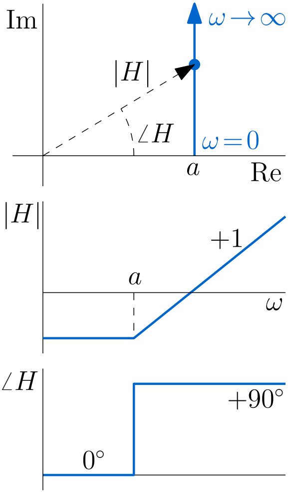
- consists of only first and second order terms in both numerator and denominator
- thus has real and complex valued zeros and poles

The shape of the Bode diagram is determined by the location of the poles and zeros of the transfer function

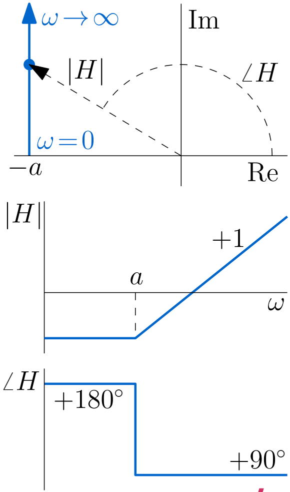
- a pole/zero at $s = \lambda$ manifests itself at frequency $f = |\lambda|$ rad/s
- at every zero the slope increases +1
- at every pole the slope decreases -1
- the phase depends on LHP or RHP

Effect of real LHP and RHP zeros

$$H(s) = s + a$$
$$H(j\omega) = j\omega + a$$



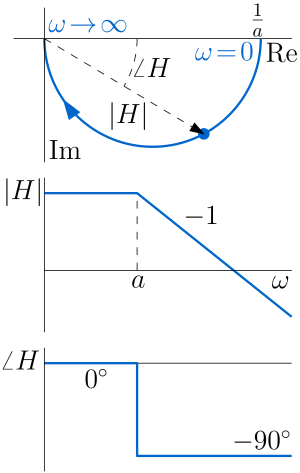
$$H(s) = s - a$$
$$H(j\omega) = j\omega - a$$



Effect of real LHP and RHP poles

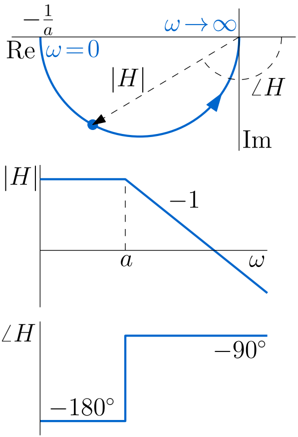
$$H(s) = \frac{1}{s+a}$$

$$H(j\omega) = \frac{1}{j\omega+a} = -\frac{j\omega-a}{\omega^2+a^2}$$



$$H(s) = \frac{1}{s-a}$$

$$H(j\omega) = \frac{1}{j\omega-a} = -\frac{j\omega+a}{\omega^2+a^2}$$



Effects of poles and zeros on Bode

Summarizing:

	Magnitude	Phase
LHP pole	slope decrease: -1	*phase lag*: -90°
RHP pole	slope decrease: -1	*phase lead*: $+90^\circ$
LHP zero	slope increase: $+1$	*phase lead*: $+90^\circ$
RHP zero	slope increase: $+1$	*phase lag*: -90°

Note: For complex conjugated poles or zeros (i.e. *two* poles or *two* zeros) the effect is doubled.

Bode gain-phase relation

If all poles and zeros are in LHP we have:

$$\text{angle}(H) = \text{slope}(H) \times 90^\circ$$

- known as Bode gain-phase relation

Does not hold for:

- unstable systems (RHP poles)
- non-minimum phase systems (RHP zeros)

Fortunately, most *motion* systems are stable and minimum phase:

- simplifies stability analysis
- simplifies controller design

Note1: Poles or zeros in RHP can easily be identified from the Bode diagram.

Note2: Often, you can even determine stability by close examination of the Bode diagram.

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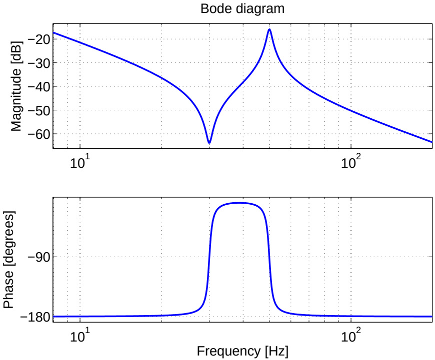
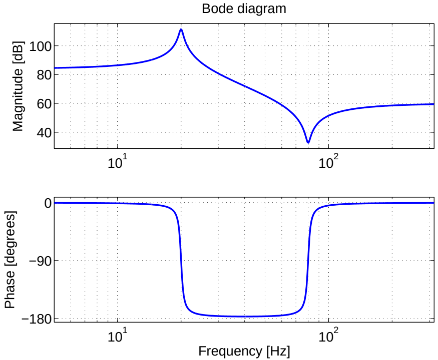
Bode diagram

Poles, zeros and resonances can be read from the Bode diagram.

Some examples:

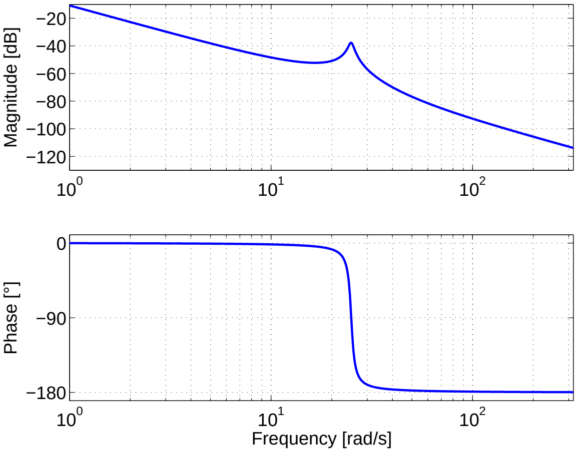




Note: $\text{angle}(H) = \text{slope}(H) \times 90^\circ$

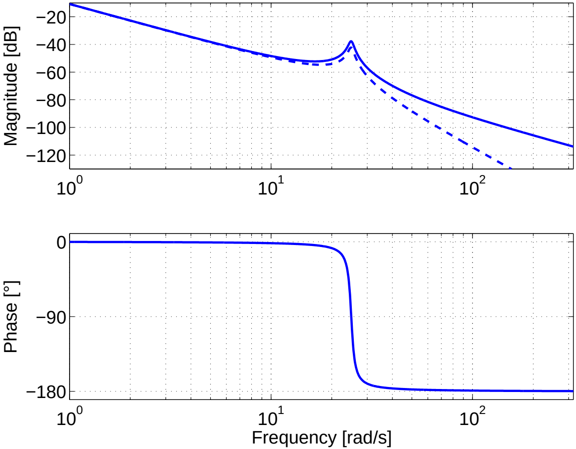
Bode diagram - interpretation

Another example:



How many poles and zeros are there, and are they LHP or RHP?

Another example:



How many poles and zeros are there, and are they LHP or RHP?

Bode contains more information:

- Amplitude for $\omega \to 0$ gives the steady state gain in the output for a *constant* input
- Amplitude for $\omega \to \infty$ denotes the *direct feedthrough*: it denotes the output gain *at* a step on the input

The first one obviously follows from the *Final Value Theorem*:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

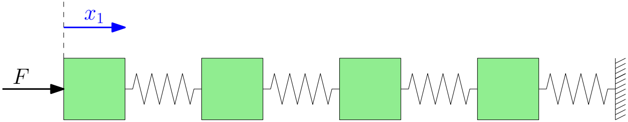
- for a step input $U(s) = \frac{1}{s}$ this yields

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{H(s)}{s} = H(0).$$

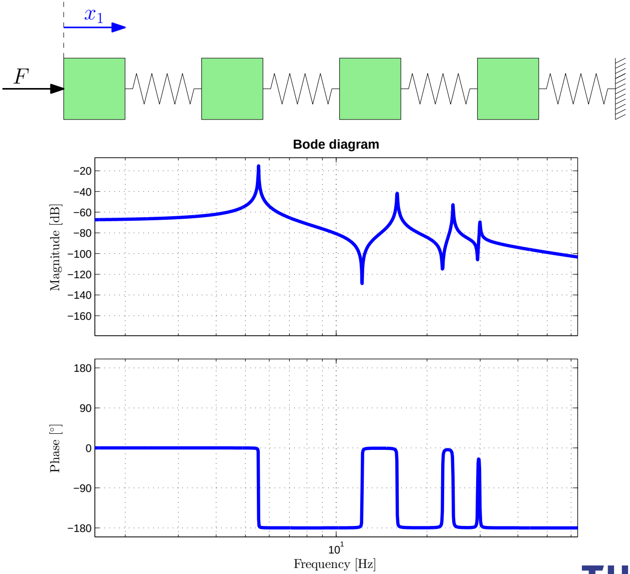
Note: This holds for any transfer function, including closed loops!

Poles and zeros in physical systems

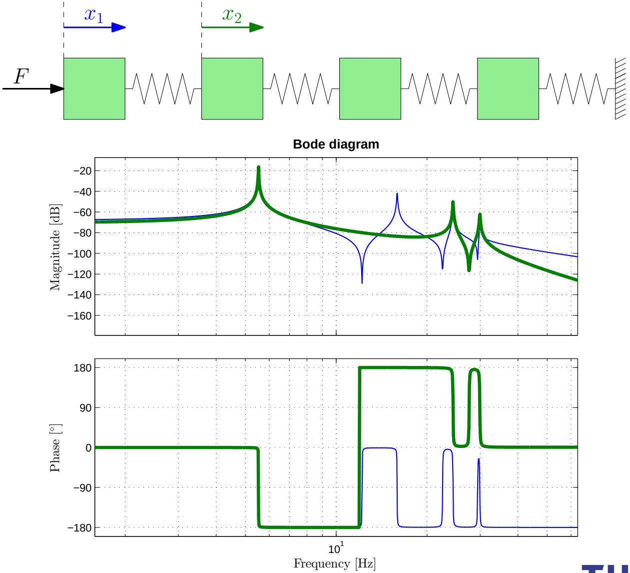
Poles and zeros: 1-D example



Poles and zeros: 1-D example



Poles and zeros: 1-D example



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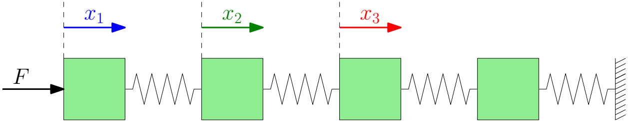
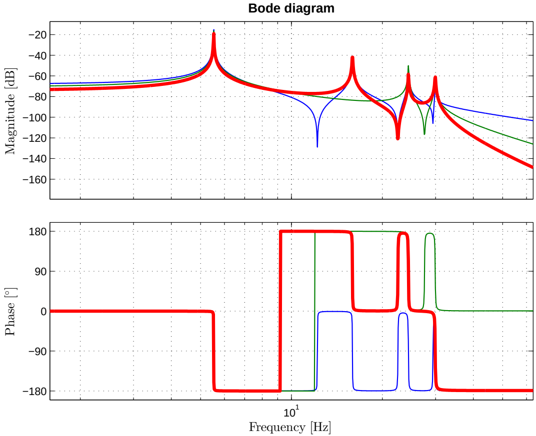
Poles and zeros: 1-D example





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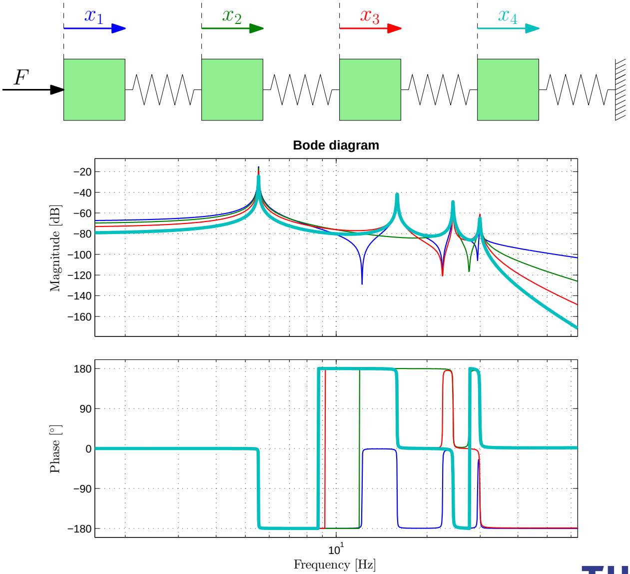


Poles and zeros in physical systems

Summarizing, the example shows that

- all plants had exactly the same resonances, regardless of sensor position;
- each plant had different anti-resonances; even the number of anti-resonances varies.

This is an important fundamental difference between poles and zeros.

- Poles are a *system property*
- Zeros are the result of specific actuator and/or sensor placement

Note that zeros can make poles *unobservable* or *uncontrollable*, but they cannot remove or shift poles.
This can be understood by looking at the State-Space (SS) notation of dynamical systems.

State-space system representations

Most general form of linear dynamical system is State Space (SS):

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- The transfer function is then given by

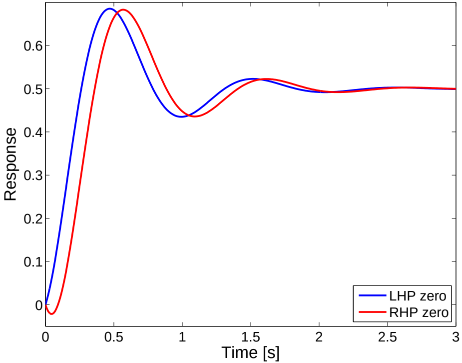
$$Y(s) = \underbrace{\left[C\left(sI - A\right)^{-1} B + D \right]}_{H(s)} \cdot U(s)$$

- The poles of the system are the eigenvalues of A
- The zeros follow from B, C and D, which are determined by the sensor and actuator locations
- System matrix A is unaffected by sensor/actuator locations!

Here we only use the Transfer Function (TF) representation $H(s)$.

When do we get RHP zeros in practice?

Remember how we recognized RHP zeros in the time domain?



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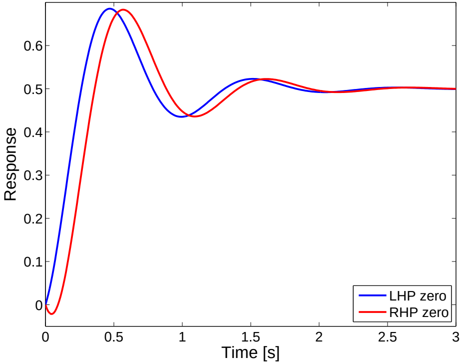

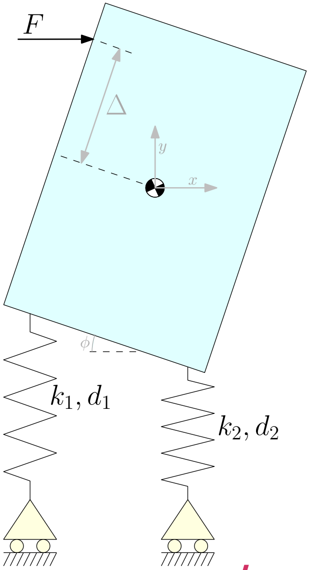

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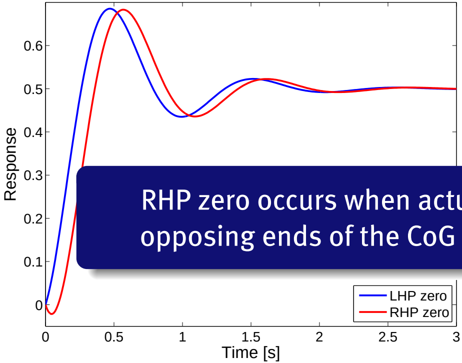
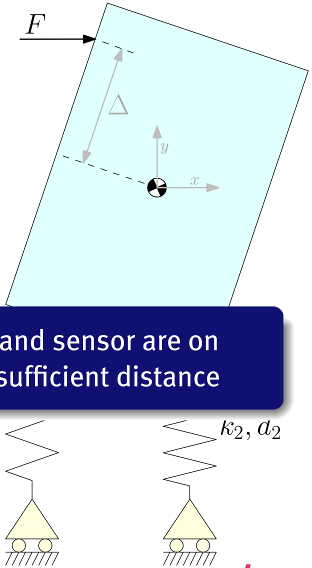




RHP zero occurs when actuator and sensor are on opposing ends of the CoG with sufficient distance

Drawback of RHP zeros

Other classic examples:

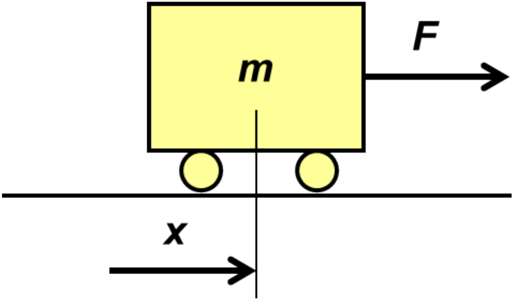
- Throwing coal on a fire: temperature first drops before it rises
- Parallel parking: car front initially goes to the left when you steer to the right

But what's wrong with RHP zeros?

- It takes a while before the system moves in the right direction
- Time domain: the response effectively becomes 'slower' or delayed
- Frequency domain: the system looses extra phase (non-minimum phase)
- A controller has to 'wait' for that delayed response
- Hence it limits the achievable bandwidth
  - will become clear when we look at it in frequency domain (Nyquist)

Basic controller design

General motion system



$$F = m \cdot a = m\ddot{x}$$

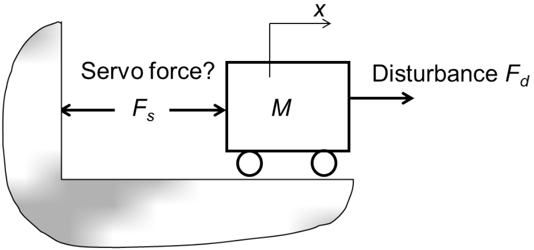
$$\Downarrow$$

$$X(s) = \frac{1}{ms^2} \cdot F(s)$$

Motion system example (1)

Problem: there is a disturbance force F_d on the mass

Goal: how to maintain a certain position x in the presence of this disturbance?

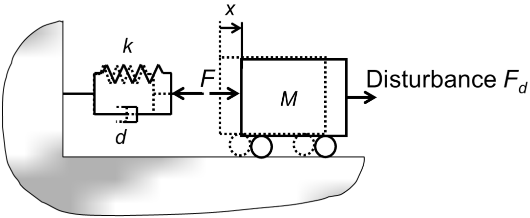


In other words, what must F_s look like?

Motion system example (2)

Mechanical solution:

- make a stiff connection between mass and base



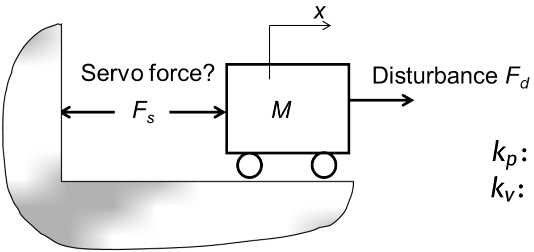
Resulting force on mass: $F = -k \cdot x - d \cdot \dot{x}$

Resulting eigenfrequency: $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{M}}$

Motion system example (2)

Electrical solution / servo analogy:

- measure distance x between mass and base
- compute servo force based on x and its velocity $\dot{x}$
  - $\Rightarrow$ PD-controller



k_p: servo stiffness
k_v: servo damping

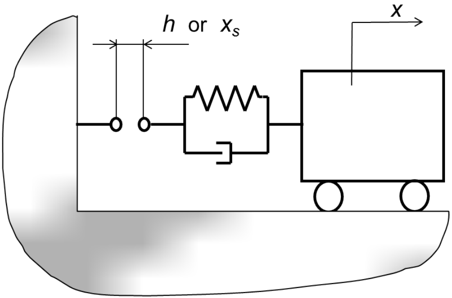
Resulting force on mass: $F_s = -k_p \cdot x - k_v \cdot \dot{x}$

Resulting eigenfrequency: $f = \frac{1}{2\pi}\sqrt{\frac{k_p}{M}}$

Motion control example (3)

To follow a non-zero setpoint:

- make a 'stiff connection' between x and the setpoint



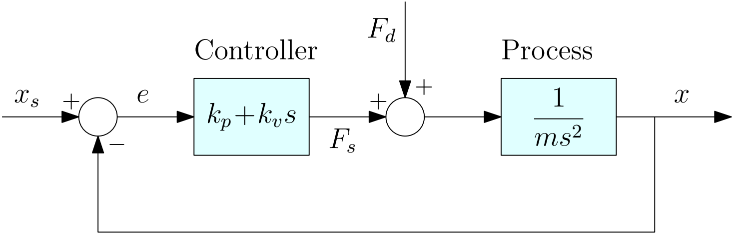
Mechanical: $F = k(h - x) + d(\dot{h} - \dot{x})$

Servo: $F_s = k_p(x_s - x) + k_v(\dot{x}_s - \dot{x}) = k_p e + k_v \dot{e}$

Motion system example (4)

What are we actually doing?

⇒ designing a PD-controller!



What is the transfer function from F_d to x?

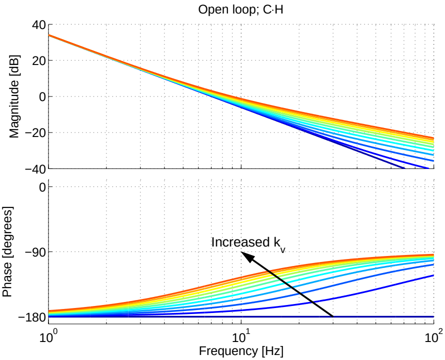
Open loop:

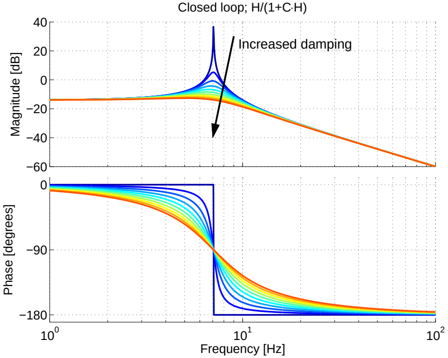
$$\frac{x}{F_d} = \frac{1}{ms^2}$$

Closed loop:

$$\frac{x}{F_d} = \frac{\frac{1}{ms^2}}{1 + \frac{k_p + k_v s}{ms^2}} = \frac{1}{ms^2 + k_v s + k_p}$$

Motion system example: frequency response





Suggested reading

- all lecture slides
- book: “Feedback control of dynamic systems” (Franklin, Powell, Emami-Naeini)
- document: “Summary_Control_Engineering”
- document: “Tips and tricks”