Advanced Motion Control

Part V: Nyquist

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2022-2023

Introduction

Motivation

- we now have notions for multivariable
  - poles
  - zeros
  - stability: open-loop & closed-loop
- in principle we can compute closed-loop stability for given/designed $K(s)$ using $G(s)$
- what if we do not have $G(s)$ but instead $G(j\omega)$?
  - e.g., from frequency response function identification
- aim:
  - develop frequency-response-based tests
  - generalized Nyquist stability theorem
  - requires
    - $G(j\omega)$, $K(j\omega)$
    - number of open-loop RHP poles of G and K
  - designs based on frequency response functions

Analysis: Nyquist stability theory

Summary and reading

Analysis: Nyquist stability theory

Intermezzo: basic results from complex analysis

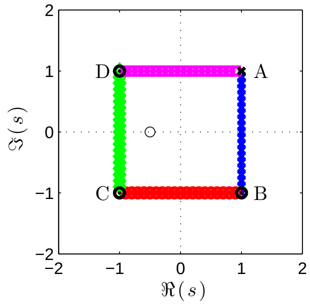
- Cauchy, principle of the argument, and conformal maps
- let us look at some examples
- let $g(s)$ be a function. So, $g(s)$ for

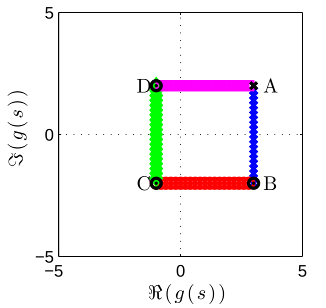
$$s = \sigma + j\omega$$

gives

$$g(s) = u + jv$$

Analysis: Nyquist stability theory





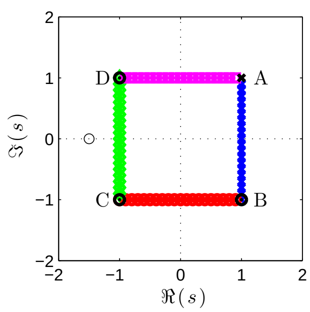
Intermezzo: example 1

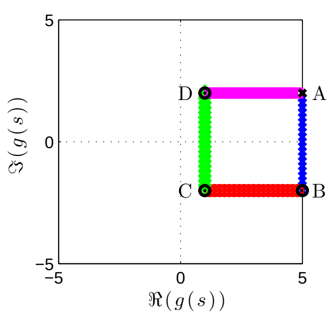
- let $g(s) = 2s + 1$
- $s = \sigma + j\omega$ gives $g(s) = \underbrace{2\sigma + 1}_{=u} + j\underbrace{2\omega}_{=v}$

Observations

- conformal mapping:
  - angles retained on infinitesimal scale
  - closed contour remains closed
- contour in s encircles zero $s = -\frac{1}{2}$ clockwise
- contour in $g(s)$ encircles origin clockwise

Analysis: Nyquist stability theory



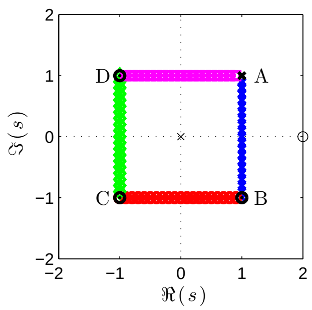


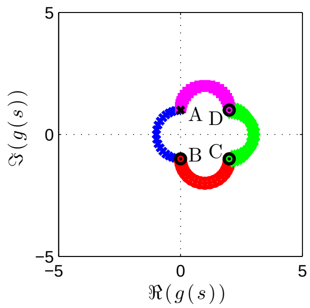
Intermezzo: example 2

- let $g(s) = 2(s + \frac{3}{2})$

Observations

- contour in s does not encircle zero $s = -\frac{3}{2}$
- contour in $g(s)$ does not encircle origin





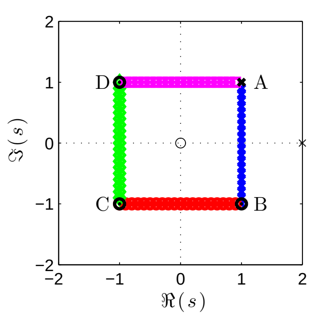
Intermezzo: example 3

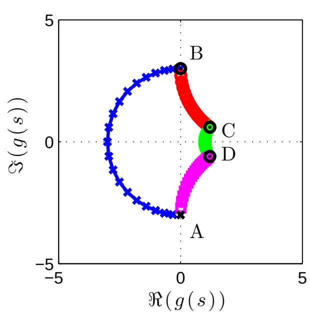
- let $g(s) = \frac{s-2}{s}$

Observations

- contour in s encircles pole $s = 0$ clockwise
- contour in $g(s)$ encircles origin counter-clockwise

Analysis: Nyquist stability theory





Intermezzo: example 4

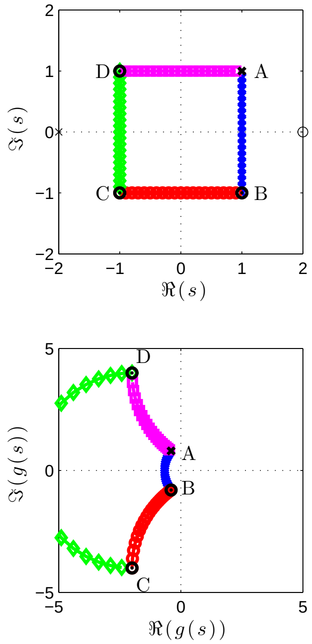
- let $g(s) = 5\frac{s}{s-2}$

Observations

- contour in s encircles zero $s = 0$ clockwise
- contour in $g(s)$ encircles origin clockwise

Tom Oomen - AMC handout

Analysis: Nyquist stability theory



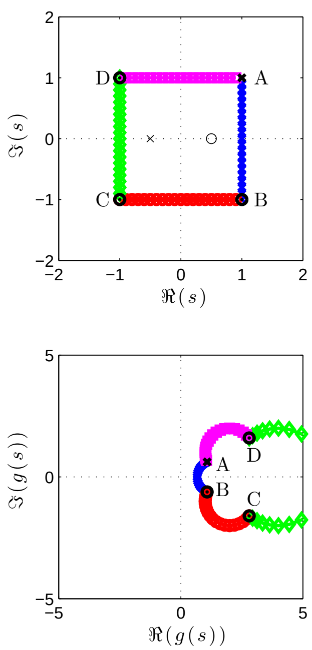
Intermezzo: example 5

- let $g(s) = 2\frac{s-2}{s+2}$

Observations

- contour in s does not encircle zero $s = 2$ nor pole at $s = -2$
- contour in $g(s)$ does not encircles origin

Analysis: Nyquist stability theory



Intermezzo: example 6

- let $g(s) = 2\frac{s-\frac{1}{2}}{s+\frac{1}{2}}$

Observations

- contour in s encircles both zero $s = \frac{1}{2}$ and pole at $s = -\frac{1}{2}$
- contour in $g(s)$ does not encircles origin

Intermezzo: Cauchy's theorem - principle of the argument

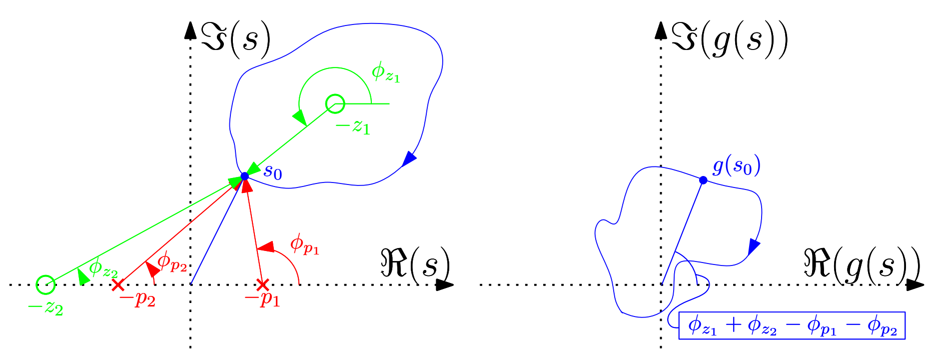
- let $g(s)$ satisfy

$$g(s) = k\frac{\prod_{i=1}^{n_z}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

with poles $-p_i$ and zeros $-z_i$ and let C denote a closed contour.
- assume that
  - $g(s)$ is analytic along C, i.e., no poles on C
  - $g(s)$ has Z zeros inside C
  - $g(s)$ has P poles inside C
- then the image $g(s)$ as s traverses in clockwise direction along C encircles the origin $N = Z - P$ times in a clockwise direction.

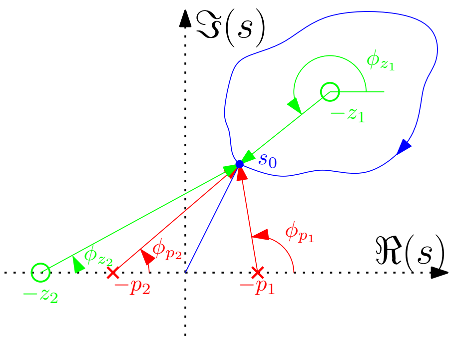
Intermezzo: principle of the argument - explanation

$$\begin{aligned}\text{Let } g(s) &= \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}\\ &= \frac{|s+z_1||s+z_2|}{|s+p_1||s+p_2|}\left(\angle(s+z_1)+\angle(s+z_2)-\angle(s+p_1)-\angle(s+p_2)\right)\\ &= |g(s)|\left(\phi_{z_1}+\phi_{z_2}-\phi_{p_1}-\phi_{p_2}\right)\end{aligned}$$



Intermezzo: principle of the argument - explanation

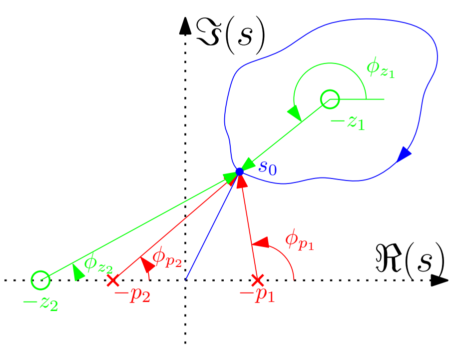
$$\text{Let } g(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$$
$$= \frac{|s+z_1||s+z_2|}{|s+p_1||s+p_2|}\left(\angle(s+z_1)+\angle(s+z_2)-\angle(s+p_1)-\angle(s+z_2)\right)$$
$$= |g(s)|\left(\phi_{z_1}+\phi_{z_2}-\phi_{p_1}-\phi_{p_2}\right)$$

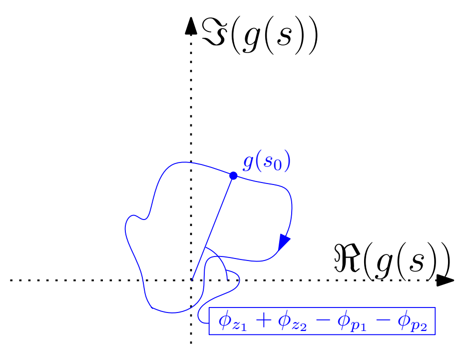


- s traverses along C
- net change of angle for
  - $\phi_{z_2} = \phi_{p_1} = \phi_{p_2} = 0$
  - $\phi_{z_1} = 360° = 2\pi$
  - $g(s) = 2\pi$

Intermezzo: principle of the argument - explanation

- general case $g(s) = k\frac{\prod_{i=1}^{n_z}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$
- if Z zeros and P poles of $g(s)$ enclosed by C, then net resultant angle $\phi_g = 2\pi(Z-P)$
- thus $\phi_g = \phi_Z - \phi_P \Rightarrow N = Z - P$
- and number of encirclements of origin by $g(s)$ equals N





Analysis: Nyquist stability theory

So we have insight in "principle of the argument", how can we exploit this for stability analysis?

Recap SISO

- let $L(s) = k\frac{z(s)}{\phi_{ol}(s)}$
- $z(s)$, $\phi_{ol}(s)$ coprime
- $z(s)$ open-loop zero polynomial
- $\phi_{ol}(s)$ open-loop characteristic (or pole) polynomial
- then the closed-loop is given by

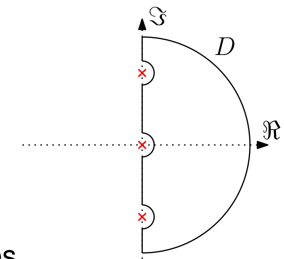
$$S(s) = \frac{1}{1+L(s)} = \frac{\phi_{ol}}{kz(s)+\phi_{ol}} = \frac{\phi_{ol}}{c\cdot\phi_{cl}} \Rightarrow F(s) = 1 + L(s) = \frac{c\cdot\phi_{cl}}{\phi_{ol}}$$

Key idea

- we know number of open-loop poles of $L(s)$ in RHP
- closed-loop is stable if there are no closed-loop poles in RHP
- apply Cauchy to $F(s)$ for a smart contour C!

Analysis: Nyquist stability theory

Nyquist theorem



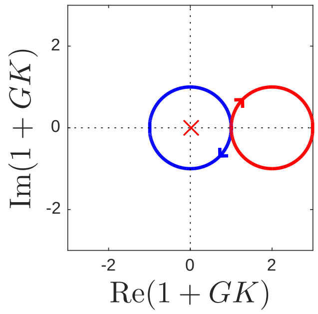
$$L(s) = k\frac{\prod_{i=1}^{n_z}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

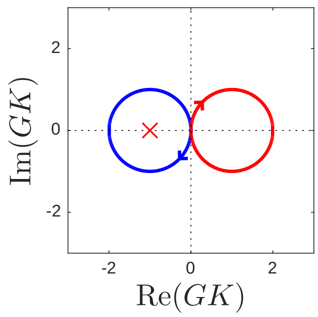
$$F(s) = 1 + L(s) = \frac{c\cdot\phi_{cl}}{\phi_{ol}}$$

- P = number of poles of F in D = number open-loop RHP poles
- Z = number of zeros of F in D = number of closed-loop RHP poles
- N = number of clockwise encirclements of F
- e.g., follow (Skogestad & Postlethwaite 2005, Section 4.9.2): take Nyquist D-contour
- closed-loop stable if $Z = 0$

Closed-loop stable if image of F makes $N = Z - P = -P$ clockwise encirclements (= P counter-clockwise encirclements) around the origin, and $F(s)$ does not pass through the origin.
note: equivalent to stating that $L(s)$ makes encirclements around point -1

Analysis: Nyquist stability theory





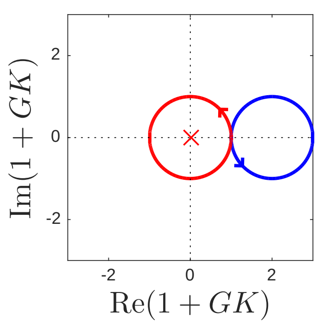
Reconsider earlier example

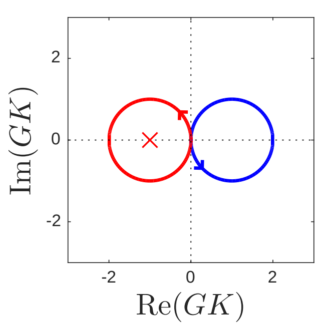
- consider first diagonal element $g(s) = \frac{1}{s+1}$

Nyquist analysis

- feedback $k \in \mathbb{R}$
- $L(s) = \frac{k}{s+1}$
- $S(s) = \frac{s+1}{s+1+k}$, stable if $k > -1$
- Q: P? A: 0
- Let $k = -2$
  - Q: N? A: 1
  - Q: stable? A: $1 = N \neq -P = 0 \Rightarrow$ no
- Let $k = 2$
  - Q: N? A: 0
  - Q: stable? A: $0 = N = -P = 0 \Rightarrow$ yes

Analysis: Nyquist stability theory





Modified example

- consider modified first diagonal element $g(s) = \frac{1}{s-1}$

Nyquist analysis

- feedback $k \in \mathbb{R}$
- $L(s) = \frac{k}{s-1}$
- $S(s) = \frac{s-1}{s-1+k}$, stable if $k > 1$
- Q: P? A: 1
- Let $k = -2$
  - Q: N? A: 0
  - Q: stable? A: $0 = N \neq -P = -1 \Rightarrow$ no
- Let $k = 2$
  - Q: N? A: -1
  - Q: stable? A:$-1 = N = -P = -1 \Rightarrow$ yes
- key point: # RHP open-loop poles matter!

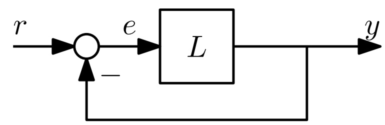
Analysis: Nyquist stability theory

Towards Nyquist tests for MIMO systems

- assume $L(s) = C_{ol}(sI - A_{ol})^{-1}B_{ol} + D_{ol}$ is minimal with

$$L(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_{ol} & B_{ol} \\ \hline C_{ol} & D_{ol} \end{array}\right]$$

- characteristic polynomial (see earlier slide set): $\phi_{ol} = \det(sI - A_{ol})$
- next, $y = C_{ol}x + D_{ol}e$



$$\begin{aligned} e &= r - y \\ &= r - C_{ol}x - D_{ol}e \\ &= (I + D_{ol})^{-1}(r - C_{ol}x) \\ \dot{x} &= A_{ol}x + B_{ol}(I + D_{ol})^{-1}(r - C_{ol}x) \\ &= \underbrace{(A_{ol} - B_{ol}(I + D_{ol})^{-1}C_{ol})}_{A_{cl}}x + B_{ol}(I + D_{ol})^{-1}r \end{aligned}$$

- hence $\phi_{cl} = \det(sI - A_{cl}) = \det(sI - A_{ol} + B_{ol}(I + D_{ol})^{-1}C_{ol})$

Analysis: Nyquist stability theory

Towards Nyquist tests for MIMO systems

- so both characteristic polynomials involve determinants
- key idea: try taking the determinant of the return difference

$$\det(I + L(s)) = \det(I + C_{ol}(sI - A_{ol})^{-1}B_{ol} + D_{ol})$$

Trick

$$\begin{aligned} &\begin{vmatrix} I + D_{ol} & -C_{ol} \\ B_{ol} & sI - A_{ol} \end{vmatrix} \\ &= \underbrace{|I + D_{ol}|}_{c}\underbrace{|sI - A_{ol} + B_{ol}(I + D_{ol})^{-1}C_{ol}|}_{\phi_{cl}} \\ &= \underbrace{|sI - A_{ol}|}_{\phi_{ol}}\underbrace{|I + D_{ol} + C_{ol}(sI - A_{ol})^{-1}B_{ol}|}_{\det(I + L(s))} \end{aligned}$$

Thus: $\det(I + L(s)) = c\dfrac{\phi_{cl}}{\phi_{ol}}$

Schur's formula:

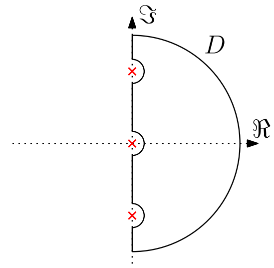
$$\begin{aligned} &\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \\ &= \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) \\ &= \det(A_{22})\det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \end{aligned}$$

(with A_{11}, A_{22} invertible)

Key idea

apply principle of the argument to identity

$$\det(I + L(s)) = c\frac{\phi_{cl}}{\phi_{ol}}$$



Theorem (4.9 Generalized (MIMO) Nyquist theorem)

Let P_{ol} denote the number of unstable (Smith-McMillan) poles in $L(s)$. The closed-loop system is stable iff the Nyquist plot of $\det(I + L(s))$

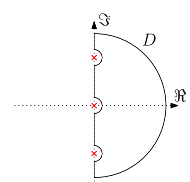
i) *makes P_{ol} anti-clockwise encirclements of the origin*

ii) *does not pass through the origin*

note: similar D-contour with indentations as in the SISO case - yet care has to be taken in the MIMO case, see exercises

Second version: note that

$$\det(I + L(s)) = \prod_i (1 + \lambda_i(L(s)))$$



Theorem (Generalized Nyquist theorem via characteristic loci)

Let $L(s)$ have P unstable (Smith-McMillan) poles. The closed-loop system is stable iff the characteristic loci $1 + \lambda_i(L(j\omega))$, taken together,

i) *make P anti-clockwise encirclements of the origin*

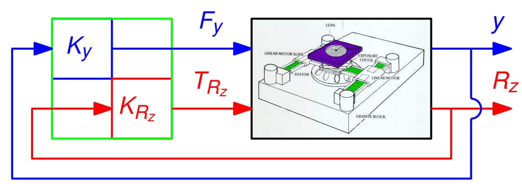
ii) *does not pass through the origin*

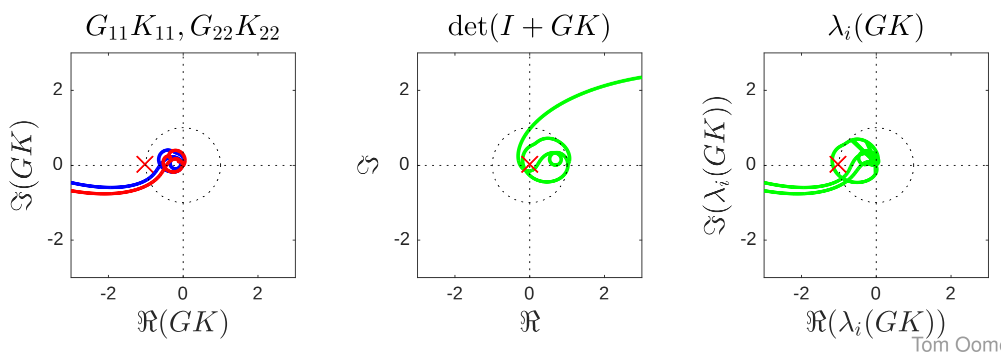
- characteristic loci $\lambda_i(L(j\omega))$ need not form a closed curve. Taken together, they form a closed curve.
- shifting: count encirclements of $\lambda_i(L(j\omega))$ around -1
- gain/phase margin: simultaneous parameter changes in all loops

Analysis: Nyquist stability theory

Example revisited

- independent SISO loops: stable
- $\det(I + GK)$: MIMO unstable
- $\lambda_i(GK)$: MIMO unstable





Analysis: Nyquist stability theory

Summary and reading

Summary and reading

Take-home messages

- Nyquist stability test for MIMO systems: uses frequency response function data
  - if you only have frequency response data: use (generalized) Nyquist
  - if you have a parametric model: compute closed-loop poles and avoid checking encirclements

Multivariable design procedure

1. interaction analysis
2. decouple
3. try independent SISO design + MIMO stability analysis

$\Rightarrow$ all can be based solely on frequency response functions

Next

- what if there is too much interaction to guarantee closed-loop stability?

Summary and reading I

Additional reading material

- Nyquist stability theory: Maciejowski (1989, Section 2.9)
- characteristic loci: Skogestad & Postlethwaite (2005, Section 4.9.3)

References I

Maciejowski, J. M. (1989), *Multivariable Feedback Design*, Addison-Wesley, Wokingham, United Kingdom.

Skogestad, S. & Postlethwaite, I. (2005), *Multivariable Feedback Control: Analysis and Design*, second edn, John Wiley & Sons, West Sussex, United Kingdom.