4CM00: Control Engineering

Performance

Dr.ir. Gert Witvoet

TU/e Technische Universiteit
Eindhoven
University of Technology

Where innovation starts

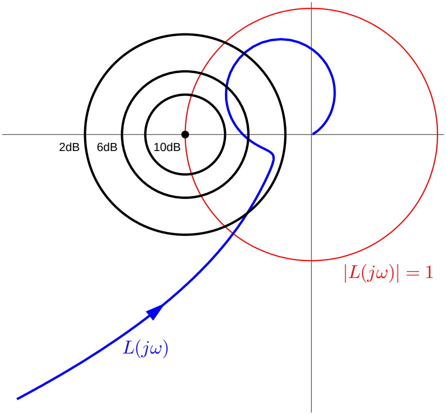
September 2020

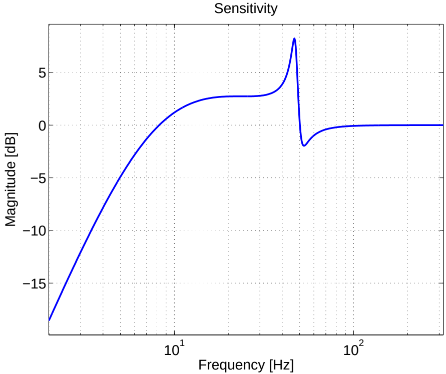
The importance of the Sensitivity function

The Sensitivity function

Remember from the previous lecture

- direct relation between Nyquist plot of $L(j\omega)$ and $|S(j\omega)|$
- modulus margin: $\max_{\omega} |S(j\omega)|$





The Sensitivity function

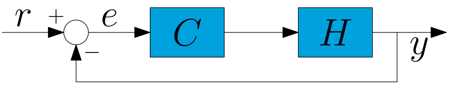
The use of the Sensitivity function:

- Robustness - ‘closeness to instability’
  - !! Small $|S|$ can still yield instability; the point (-1,0) can be passed on the *wrong* side with a large margin
  - !! Large $|S|$ can still come with stability
- Measure for performance
- Indicates the benefit of feedback

The Sensitivity function

The use of the Sensitivity function:

- Robustness - ‘closeness to instability’
  - !! Small $|S|$ can still yield instability; the point (-1,0) can be passed on the *wrong* side with a large margin
  - !! Large $|S|$ can still come with stability
- Measure for performance
- Indicates the benefit of feedback

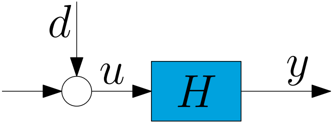


Error e resulting from an input r:

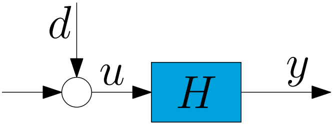
$$e = r - y = r - HCe \quad \Rightarrow \quad (1 + HC)\,e = r \quad \Rightarrow \quad \frac{e}{r} = \frac{1}{1 + HC} = S$$

The Sensitivity function determines the resulting *closed loop* error.

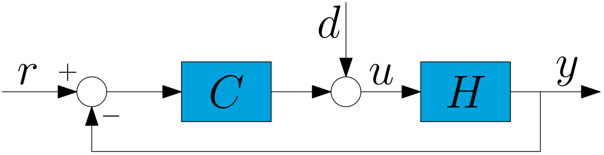
The benefit of feedback



Open loop: $y = Hu = Hd$



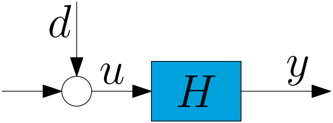
Open loop: $y = Hu = Hd$



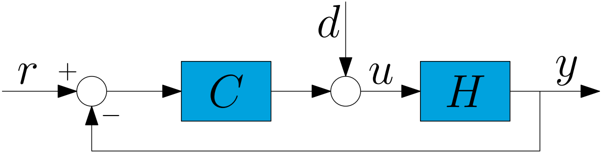
Closed loop:

$$
\begin{aligned}
y &= Hu = H(d + C(-y)) = Hd - HCy \\
(1 + HC)y &= Hd \\
y &= \frac{H}{1 + HC}\, d = S \cdot Hd
\end{aligned}
$$

The benefit of feedback



Open loop: $y = Hu = Hd$



Closed loop:

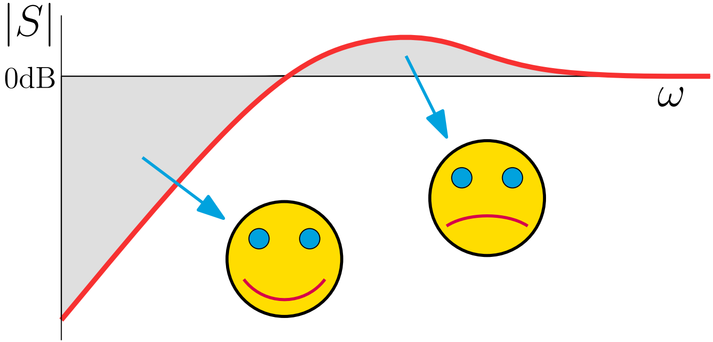
$$\begin{aligned} y &= Hu = H(d + C(-y)) = Hd - HCy \\ (1 + HC)y &= Hd \\ y &= \frac{H}{1 + HC}\, d = S \cdot Hd \end{aligned}$$

Improvement due to feedback: S

The benefit of feedback

Feedback is:

- advantageous when $|S| < 1$
- disadvantageous when $|S| > 1$



WaterBed Effect (1)

So, the goal of feedback is to get: $|S(j\omega)| < 1 \quad \forall\omega$, right?

...well, actually...

There's something known as the Bode Sensitivity Integral:

$$\int_0^\infty \ln|S(j\omega)| \, d\omega = \pi \sum_{i=1}^{N_p} \mathrm{Re}(p_i), \tag{1}$$

where $S(s) = \frac{1}{1+L(s)}$ and $L(s)$ has N_p RHP poles at locations p_i.

- Implies that area beneath $|S|$ is always non-negative
- Hence, there are always regions where $|S| > 1$
- The 'faster' the RHP poles are, the larger the area above $|S| = 1$
- That's the price of having to stabilize an unstable system

Note: holds for open loops $L(s)$ with relative degree 2 or higher

Technische Universiteit Eindhoven University of Technology

WaterBed Effect (2)

For stable open loops the Bode Sensitivity Integral simplifies to

$$\int_0^\infty \ln |S(j\omega)| \; d\omega = 0$$

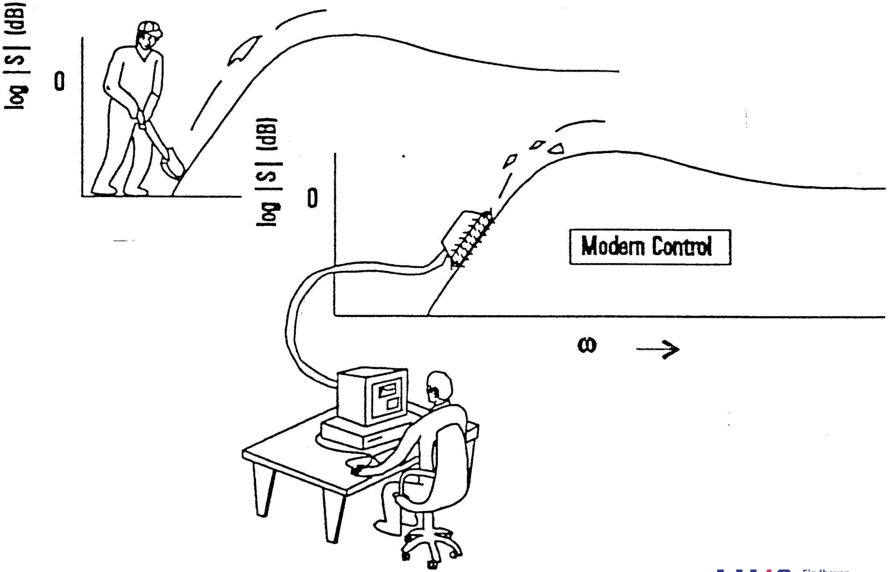
- Surface beneath $|S|$ is always zero
- Suppression of $|S|$ at certain frequency ω always results in an amplification elsewhere
- (Again only for open loops $L(s)$ with relative degree 2 or higher)

Also known as the *WaterBed Effect*

Fundamental limitation of linear control!

Control design: making S small only for those frequencies where it matters, and allowing large S where it doesn't matter.

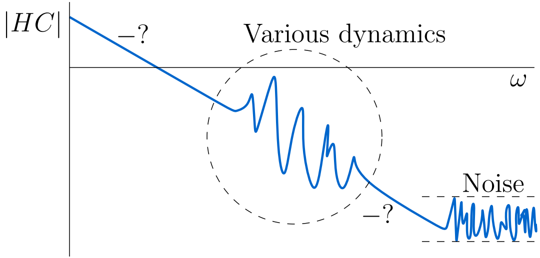
WaterBed Effect



Closed loop transfer functions

Assumption

We assume that the open loop is of the following form:

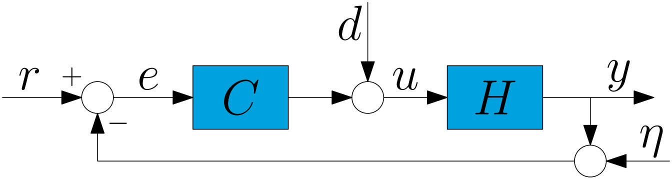


- Negative slope for low frequencies
- Various (anti-)resonances in the mid- to high-frequency range
- Flat (measurement) noise level for high frequencies

Note: Large H for $\omega \to 0$ and small H for $\omega \to \infty$

Note: This typically holds for **motion systems** and their open loop

Closed loop transfers



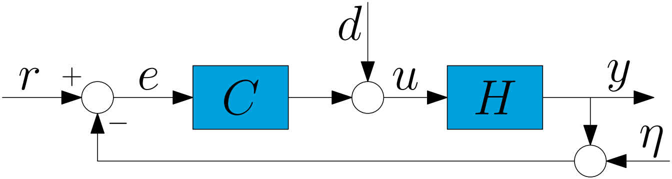
$$\frac{e}{r} = \frac{u}{d} =$$

$$\frac{y}{r} = \frac{y}{\eta} =$$

$$\frac{y}{d} =$$

$$\frac{u}{r} =$$

Closed loop transfers



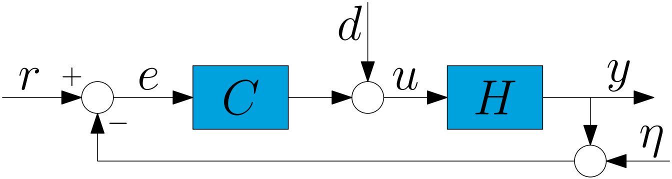
$$\frac{e}{r} = \frac{u}{d} = \quad \frac{1}{1+HC} = S : \quad \text{Sensitivity}$$

$$\frac{y}{r} = \frac{y}{\eta} =$$

$$\frac{y}{d} =$$

$$\frac{u}{r} =$$

Closed loop transfers



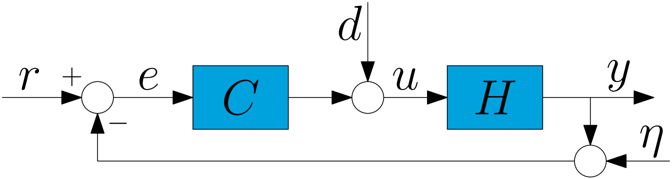
$$\frac{e}{r} = \frac{u}{d} = \quad \frac{1}{1+HC} = S: \quad \text{Sensitivity}$$

$$\frac{y}{r} = \frac{y}{\eta} = \quad \frac{HC}{1+HC} = T: \quad \text{Complementary sensitivity}$$

$$\frac{y}{d} =$$

$$\frac{u}{r} =$$

Closed loop transfers



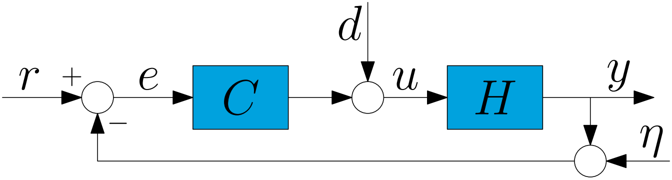
$$\frac{e}{r} = \frac{u}{d} = \frac{1}{1+HC} = S: \quad \text{Sensitivity}$$

$$\frac{y}{r} = \frac{y}{\eta} = \frac{HC}{1+HC} = T: \quad \text{Complementary sensitivity}$$

$$\frac{y}{d} = \frac{H}{1+HC} = HS: \quad \text{Process sensitivity}$$

$$\frac{u}{r} =$$

Closed loop transfers



$$\frac{e}{r} = \frac{u}{d} = \frac{1}{1+HC} = S : \quad \text{Sensitivity}$$

$$\frac{y}{r} = \frac{y}{\eta} = \frac{HC}{1+HC} = T : \quad \text{Complementary sensitivity}$$

$$\frac{y}{d} = \frac{H}{1+HC} = HS : \quad \text{Process sensitivity}$$

$$\frac{u}{r} = \frac{C}{1+HC} = CS : \quad \text{Control sensitivity}$$

Closed loop transfers

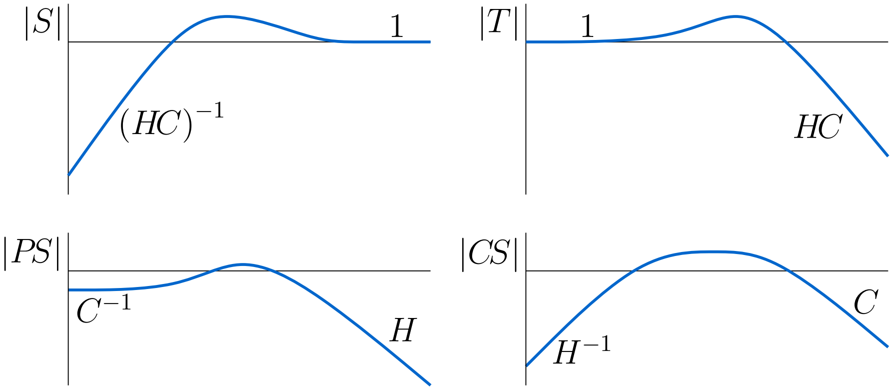
For simple controllers this yields the following closed loop transfers

Note: $HC \to \infty$ for $\omega \to 0$ and $HC \to 0$ for $\omega \to \infty$

Closed loop transfers

For simple controllers this yields the following closed loop transfers

Note: $HC \to \infty$ for $\omega \to 0$ and $HC \to 0$ for $\omega \to \infty$



Technische Universiteit Eindhoven University of Technology

Relation S and T

Note that

$$S + T = \frac{1}{1 + HC} + \frac{HC}{1 + HC} = 1 \tag{2}$$

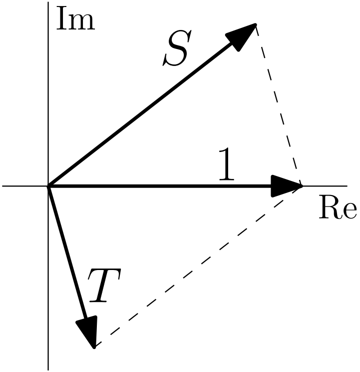
Relation S and T

Note that

$$S + T = \frac{1}{1+HC} + \frac{HC}{1+HC} = 1 \qquad (2)$$

Both S and T are complex valued

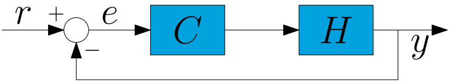
- S and T have different angles,
- so $|S|$ and $|T|$ can be larger than 1 at the same time!



Technische Universiteit Eindhoven University of Technology

How to achieve high performance?

High-gain feedback



Transfer from r to e:

$$\frac{e}{r} = \frac{1}{1+HC} = S$$

Transfer from r to y:

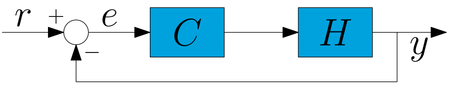
$$\frac{y}{r} = \frac{HC}{1+HC} = T$$

General goals:

- minimizing the error: $e = 0$,
- tracking the input: $y = r$.

TU/e Technische Universiteit Eindhoven University of Technology

High-gain feedback



Transfer from r to e:

$$\frac{e}{r} = \frac{1}{1+HC} = S$$

Transfer from r to y:

$$\frac{y}{r} = \frac{HC}{1+HC} = T$$

General goals:

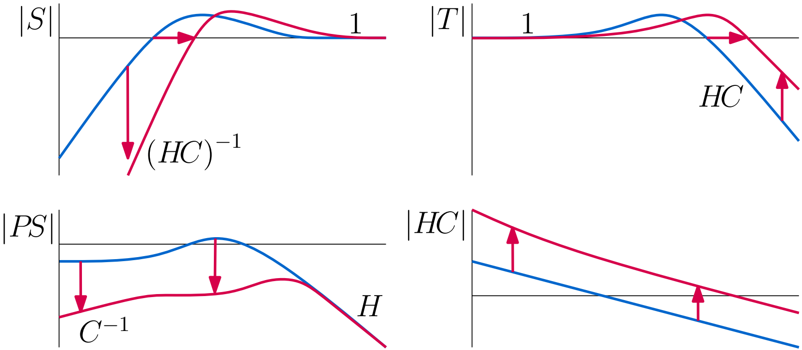
- minimizing the error: $e = 0$,
- tracking the input: $y = r$.

Can be achieved by *high-gain feedback*:

$$\text{maximize } HC \begin{cases} \dfrac{1}{1+HC} \to 0 \\ \dfrac{HC}{1+HC} \to \dfrac{HC}{HC} = 1 \end{cases}$$

High-gain feedback

Consequence of high-gain feedback:

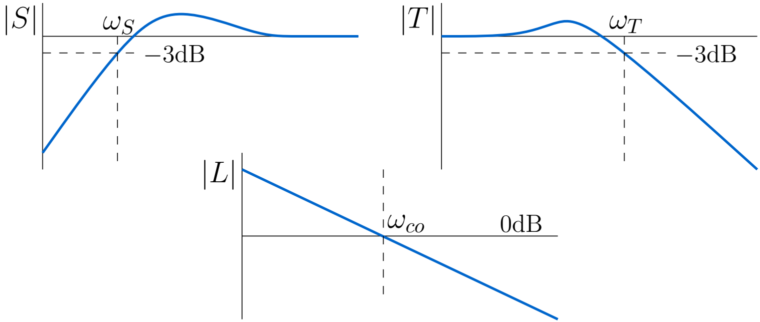


However, in reality the high-gain in *HC* is limited:

- High frequent measurement noise in *y* will be amplified
- High frequent gain in *T* should be small

Bandwidth

Around the world different definitions for bandwidth are used:



- Frequency up to where disturbances are suppressed: ω_S
- Frequency up to where references are tracked: ω_T
- Cross-over frequency of the open-loop L: ω_{co}

In practice: $\omega_S \leq \omega_{co} \leq \omega_T$ We take: bandwidth $\omega_b = \omega_{co}$

Bandwidth

Advantages of high bandwidth:

- $|S| < 1$ over large frequency band
  - $\Rightarrow$ small error e
  - $\Rightarrow$ more disturbance rejection
- $|T| = 1$ over large frequency band
  - $\Rightarrow$ good tracking performance

Disadvantages of high bandwidth

- Amplification of measurement noise
- Possibly large control signals (saturation?)

For performance reasons, bandwidth is normally chosen as high as possible. However, it is better to say:

"Choose the bandwidth as low as possible,
but such that the closed loop still meets its requirements"

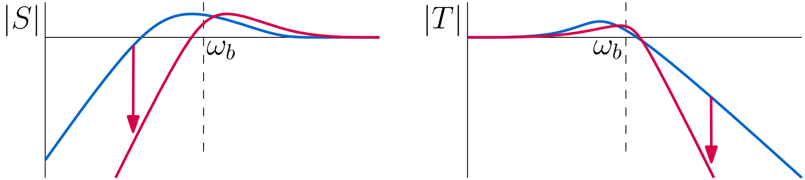
(I.M. Horowitz)

Control requirements

Control targets:

- Low frequencies (LF): track reference *r*, suppress disturbances *d*
  - $\Rightarrow$ small *S* for $\omega < \omega_b$
- High frequencies (HF): suppress measurement noise in *y*
  - $\Rightarrow$ small *T* for $\omega > \omega_b$

Desired closed loop transfers:



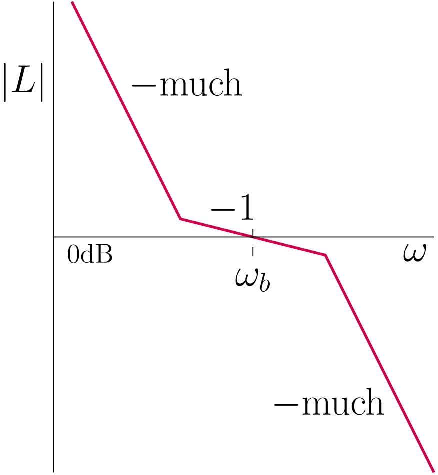
Control targets

Hence, the ideal open loop has the following shape:

- Large slope for $\omega \ll \omega_b$ and $\omega \gg \omega_b$
- Slope ≈ -1 at ω_b
  - $\Rightarrow$ Necessary for stability

However, keep the controller order limited to prevent

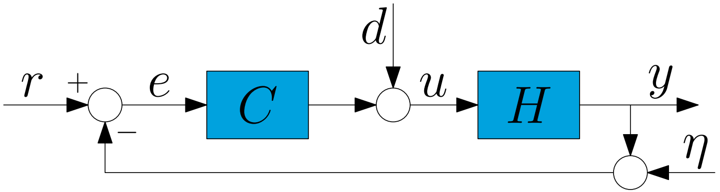
- large complexity
- large computation times
- risk of implementation errors



Feedback performance using the internal model principle

Tracking / disturbance attenuation

External signals enter at various points in the loop:



Closed loop performance (e.g. error e) is determined by

- Relevant closed loop transfer

  e.g. $e = -\frac{H}{1+HC}\, d$ (disturbance attenuation)

  e.g. $e = \frac{1}{1+HC}\, r$ (tracking)

- Specific frequency content of the input signals r, d and/or η

Tracking / disturbance attenuation

Perfect tracking / disturbance attenuation can be achieved when

- the input signal is exactly known;
- then the relevant transfer function should counteract the specific input signal.

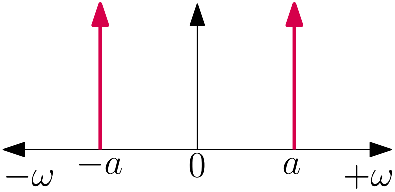
Tracking / disturbance attenuation

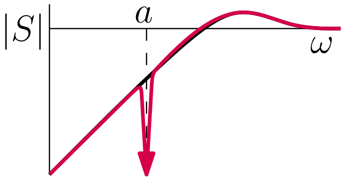
Perfect tracking / disturbance attenuation can be achieved when

- the input signal is exactly known;
- then the relevant transfer function should counteract the specific input signal.

Simple example:

- Suppose $r(t) = \sin(at)$
- In frequency domain: $R(s) = \mathfrak{L}(r(t))$ has infinite peak at $\omega = a$
- To get $e \to 0$ for $t \to \infty$ we need $\frac{e}{r} = S$ to be 0 at $\omega = a$





Technische Universiteit Eindhoven University of Technology

Note that:

Laplace transform of $r(t)$:
$R(s) = \mathfrak{L}(r(t)) = \frac{a}{s^2+a^2}$

$\Leftrightarrow$

Infinite dip in S means it should contain an **undamped anti-resonance** at a:
$S(s) \sim s^2 + a^2$

As a result, since $S = \frac{1}{1+L}$, we know that $e \to 0$ if

- L contains the term $\frac{1}{s^2+a^2}$,
- so if L contains the Laplace transform of the input r!

Tracking / disturbance attenuation

Note that:

Laplace transform of $r(t)$:
$R(s) = \mathfrak{L}(r(t)) = \frac{a}{s^2+a^2}$

$\Leftrightarrow$

Infinite dip in S means it should contain an **undamped anti-resonance** at a:
$S(s) \sim s^2 + a^2$

As a result, since $S = \frac{1}{1+L}$, we know that $e \to 0$ if

- L contains the term $\frac{1}{s^2+a^2}$,
- so if L contains the Laplace transform of the input r!

Similarly, if $r(t)$ is a step reference:

- Laplace transform: $R(s) = \frac{1}{s}$

$\Rightarrow$ For perfect tracking: $S(s) \sim s$ (+1 slope for $\omega \to 0$)

$\Rightarrow$ So perfect tracking if $L(s) \sim \frac{1}{s}$ (-1 slope for $\omega \to 0$)

Internal model principle

Summarizing:

Signal $r(t)$	Laplace $R(s)$	L should contain
$1(t)$ - step	$\frac{1}{s}$	$\frac{1}{s}$
$t \cdot 1(t)$ - ramp	$\frac{1}{s^2}$	$\frac{1}{s^2}$
$t^2 \cdot 1(t)$	$\frac{1}{s^3}$	$\frac{1}{s^3}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\frac{a}{s^2+a^2}$

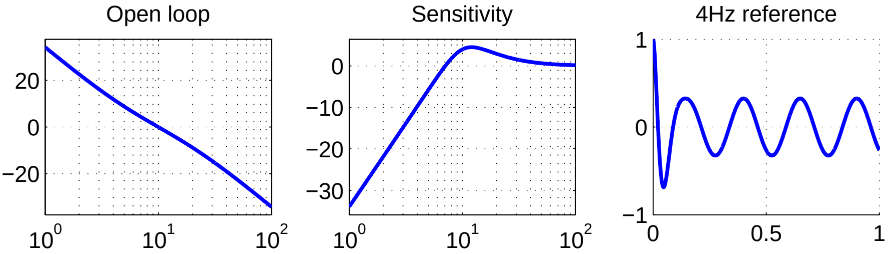
If $L(s)$ contains a model of the reference, perfect tracking can be guaranteed, so:

$$e \to 0 \quad \text{for} \quad t \to \infty$$

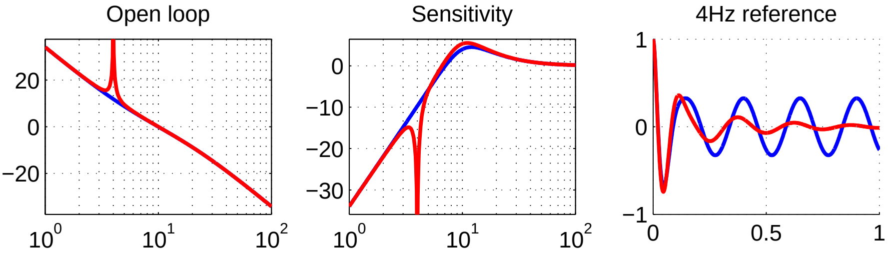
This is known as the Internal Model Principle.

Note: A similar reasoning can be applied for disturbances. Perfect disturbance attenuation can be achieved when PS, CS or T contains the inverse model of the disturbance d or η.

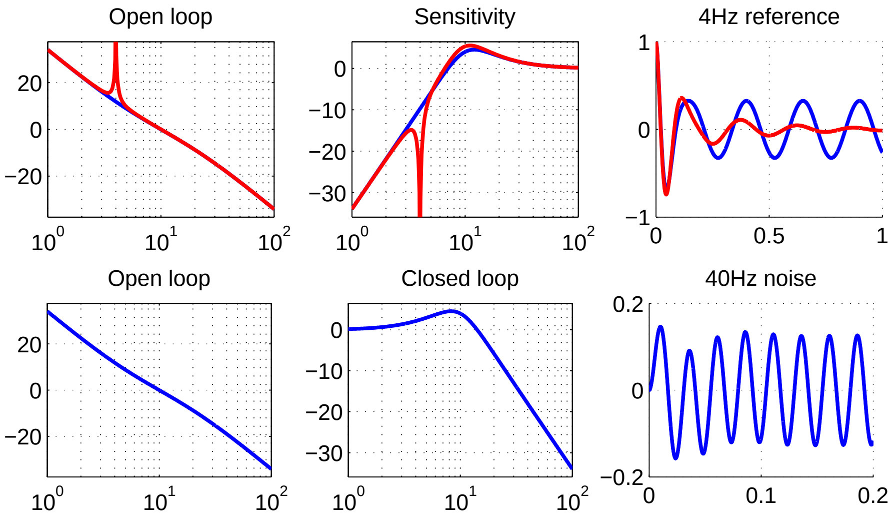
Internal model principe: examples



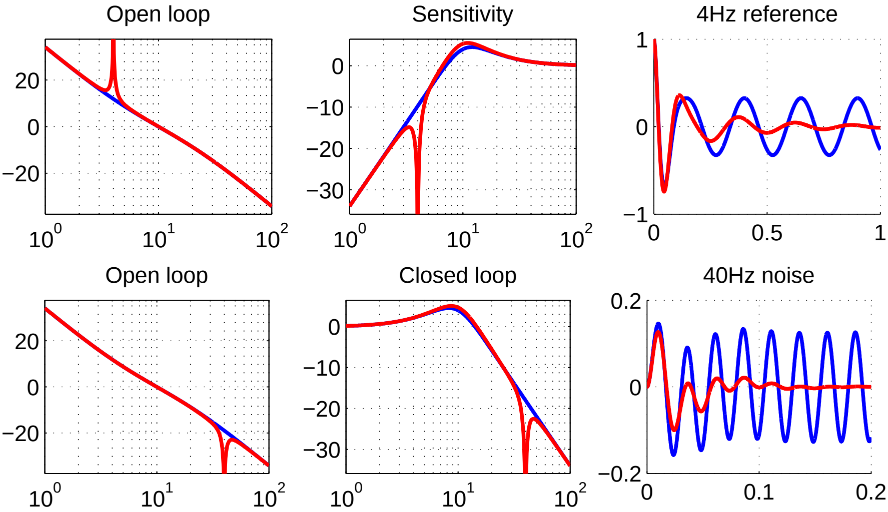
Internal model principe: examples



Internal model principe: examples



Internal model principe: examples



Basic building blocks of a linear controller

Controller blocks

Controller design = adding poles and zeros to $L(s)$.
Each controller element is either

- a first order term, or
- a second order term,

in the numerator and/or denominator.

Possible controller blocks:

- Integrator / PI-controller
- PD controller
- lead/lag filter
- second order filter (notch)
- low-pass filter (first or second order)

You should know the formulas of these blocks by heart!

Integrator / PI-controller

Pure integrator:

$$C = k \cdot \frac{1}{s}$$

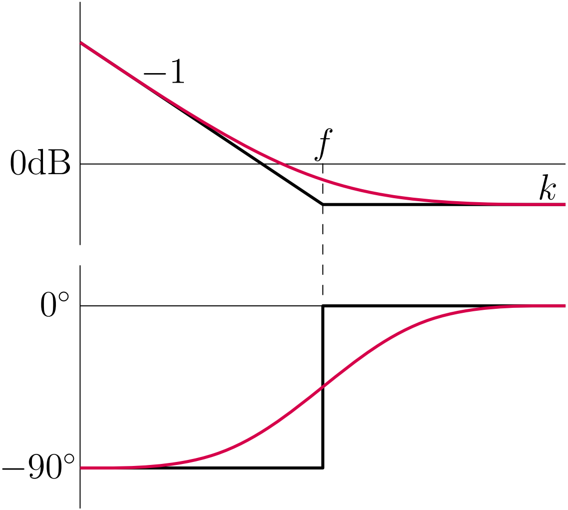
Cut-off integral action due to phase lag:

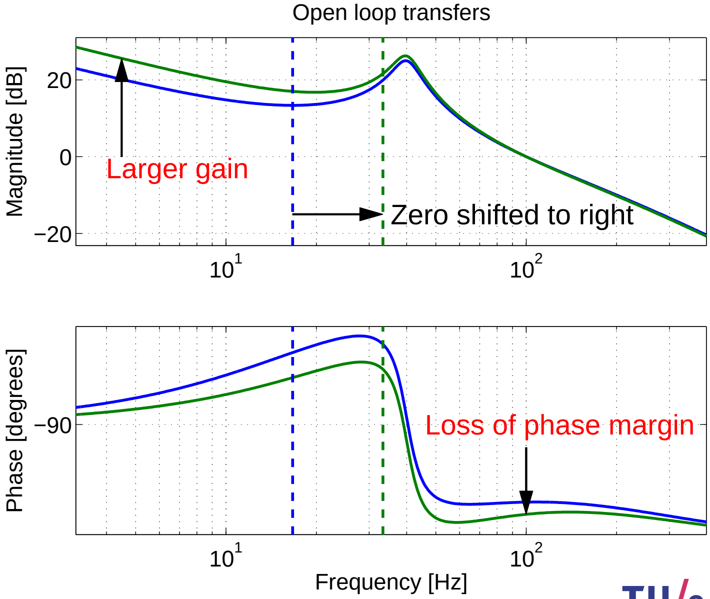
$$C = k \cdot \frac{s + 2\pi f}{s}$$

Same as a PI controller:

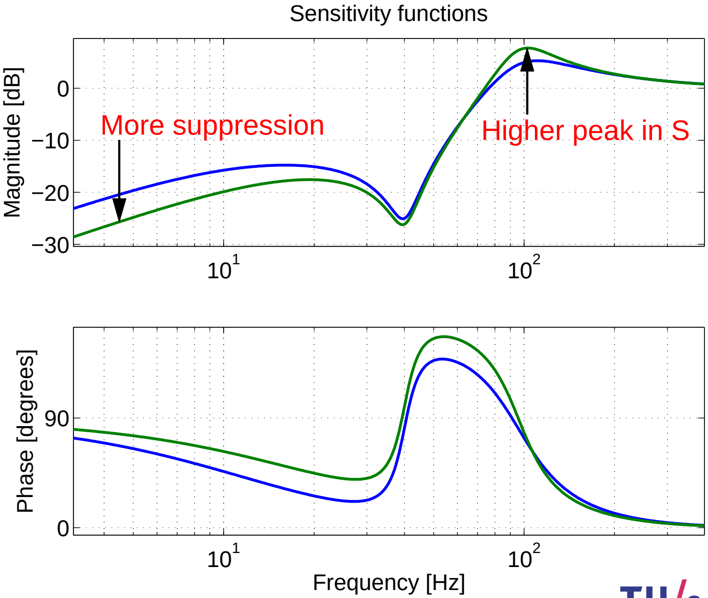
$$C = P + \frac{I}{s}$$

with $P = k$ and $I = 2\pi fk$.




Open loop transfers
Magnitude [dB]
20
0
−20
Larger gain
Zero shifted to right
10^1
10^2
Phase [degrees]
−90
Loss of phase margin
10^1
10^2
Frequency [Hz]

Integrator / PI-controller (cont'd)



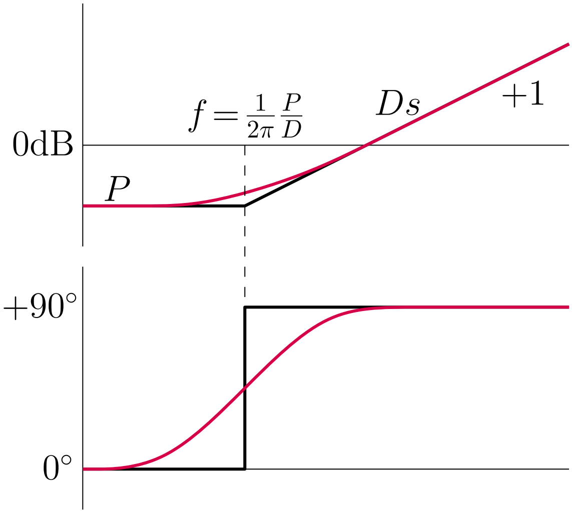
PD controller

Create phase lead:

$$\begin{aligned} C &= P + Ds \\ &= k\left(1 + \frac{1}{2\pi f}s\right) \end{aligned}$$

Note that:

$$P = k \quad \text{and} \quad D = \frac{k}{2\pi f}$$



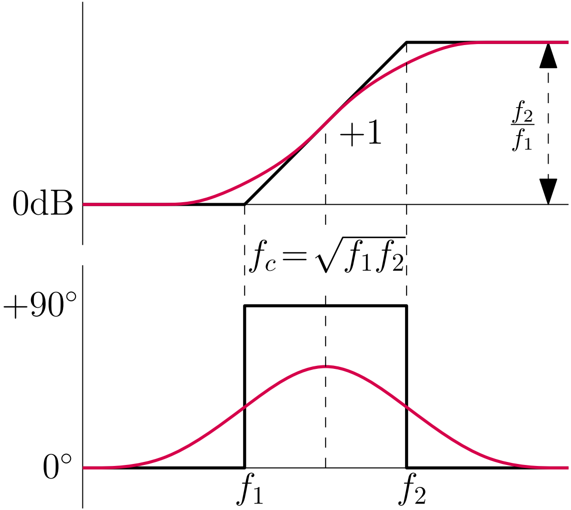
Lead/lag filter

$$C = \frac{\frac{1}{2\pi f_1}s + 1}{\frac{1}{2\pi f_2}s + 1}$$

If $f_2 > f_1$: phase *lead*
If $f_2 < f_1$: phase *lag*

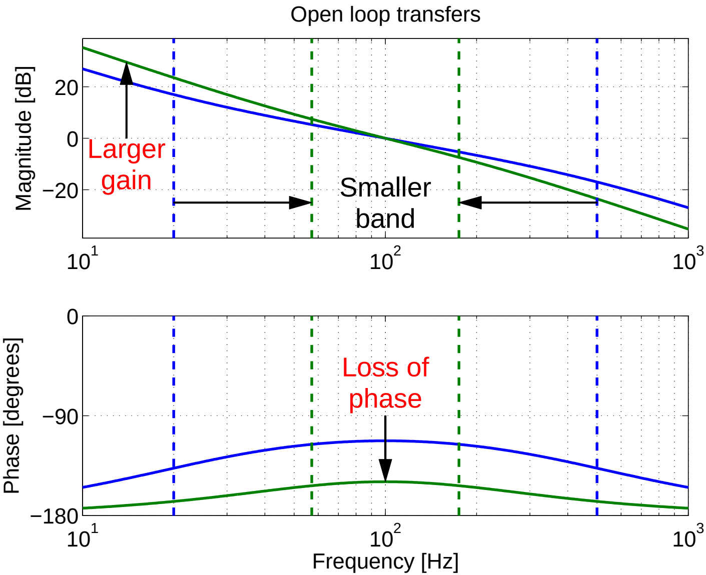
Maximum phase at:

$$f_c = \sqrt{f_1 f_2}$$

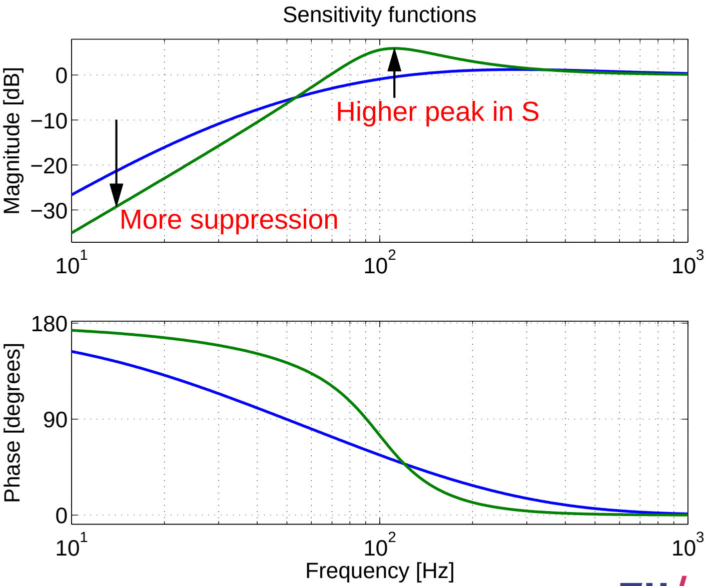


Technische Universiteit Eindhoven University of Technology

Lead/lag filter (cont'd)



Lead/lag filter (cont'd)

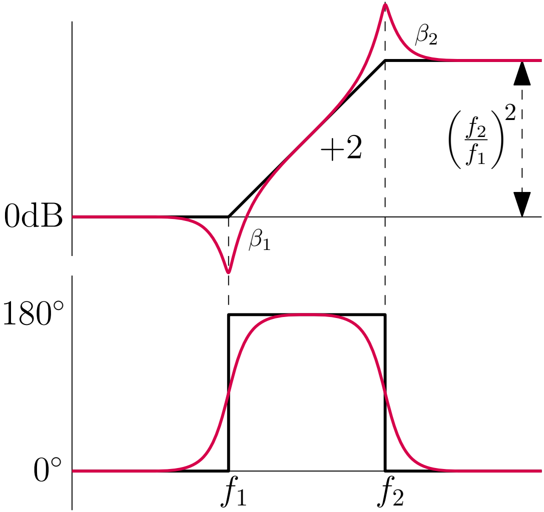


General second order filter (skewed notch)

$$C = \frac{\frac{1}{(2\pi f_1)^2}s^2 + \frac{2\beta_1}{2\pi f_1}s + 1}{\frac{1}{(2\pi f_2)^2}s^2 + \frac{2\beta_2}{2\pi f_2}s + 1}$$

gain at high frequencies:

$$\left(\frac{f_2}{f_1}\right)^2$$

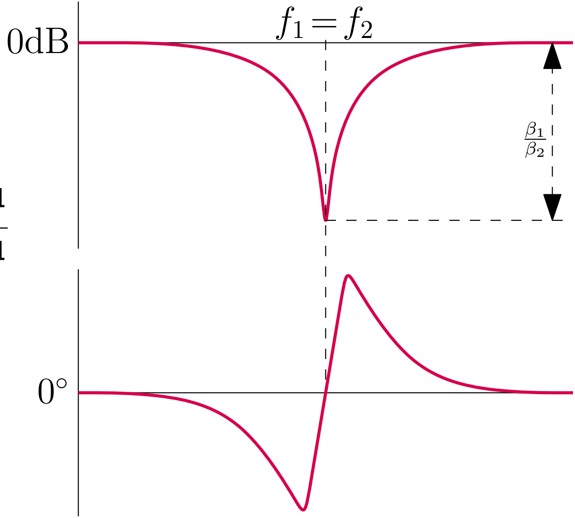


Second order filter (notch)

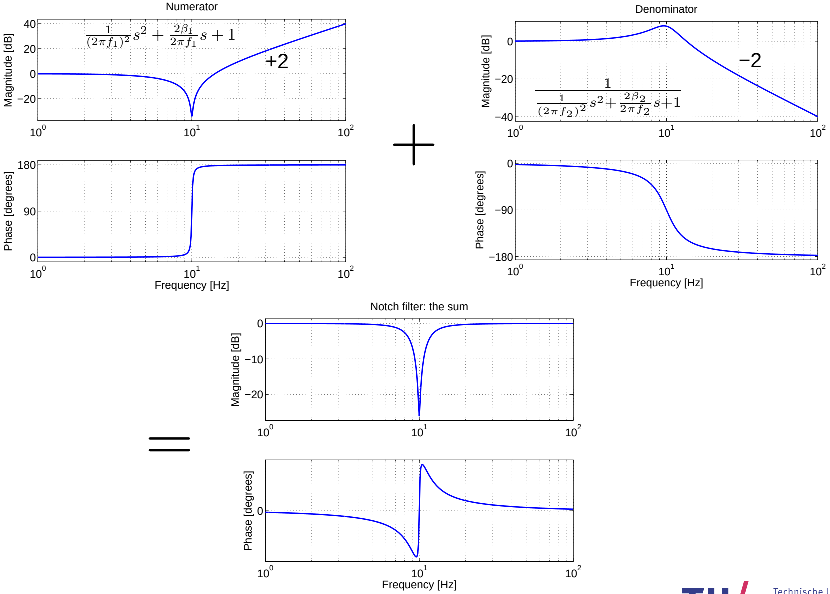
$$C = \frac{\frac{1}{(2\pi f_1)^2}s^2 + \frac{2\beta_1}{2\pi f_1}s + 1}{\frac{1}{(2\pi f_2)^2}s^2 + \frac{2\beta_2}{2\pi f_2}s + 1}$$

with

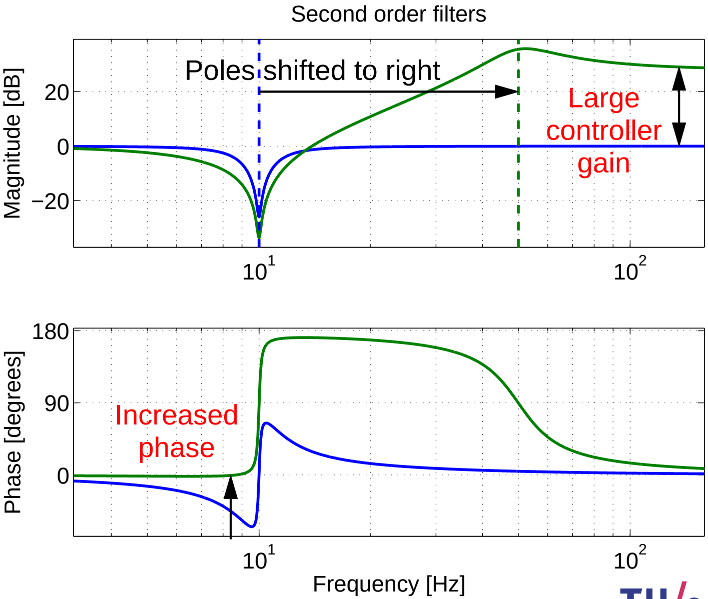
$$f_1 = f_2$$



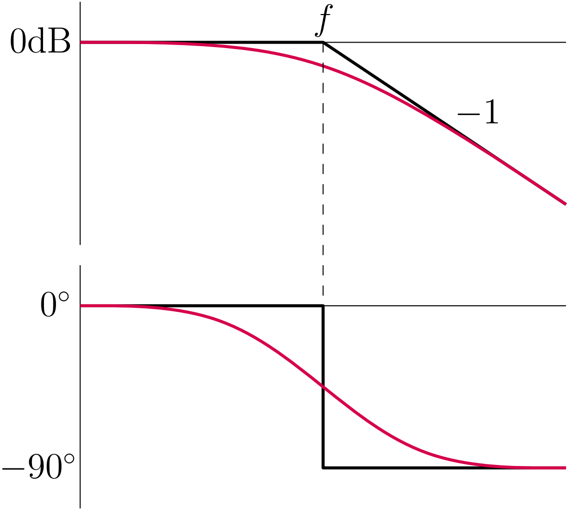
Second order filter (cont'd)



Second order filter (cont'd)



Lowpass filter: 1st order



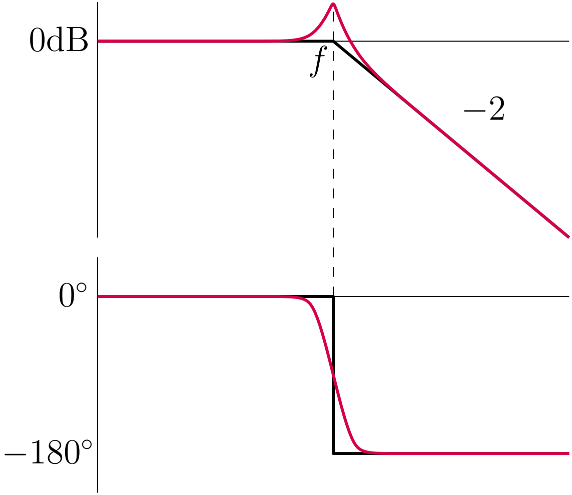
$$C = \frac{1}{\frac{1}{2\pi f}s + 1}$$

Low frequent gain: 0dB

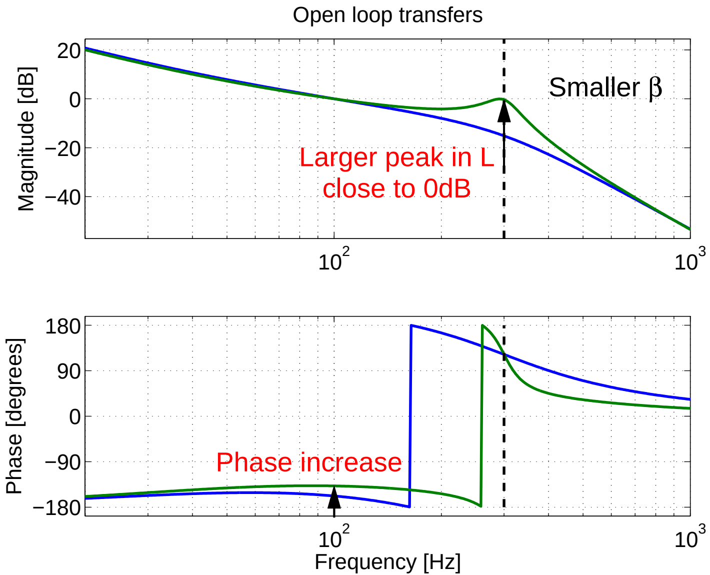
Lowpass filter: 2nd order

$$C = \frac{1}{\frac{1}{(2\pi f)^2}s^2 + \frac{2\beta}{2\pi f}s + 1}$$

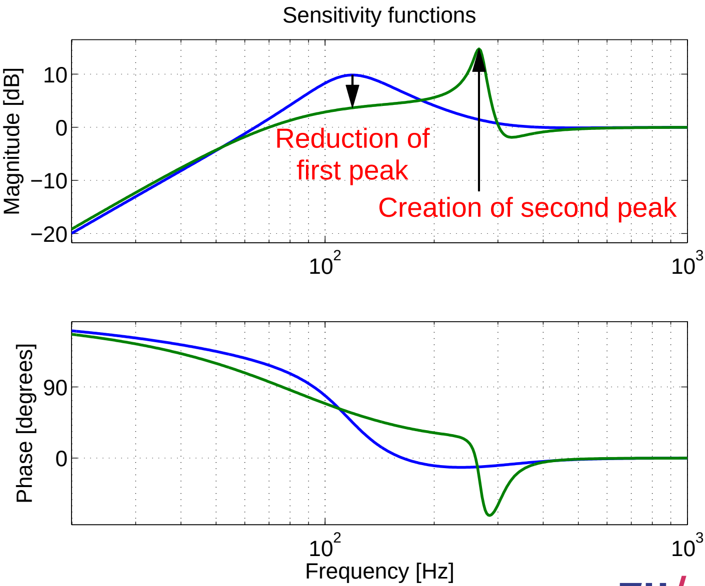
Low frequent gain: 0dB



Lowpass filter: 2nd order (cont'd)



Lowpass filter: 2nd order (cont'd)



Technische Universiteit
Eindhoven
University of Technology

Loop shaping motion systems

'Shaping' the open loop transfer, until stability and satisfactory performance is achieved.

1. Stabilize the plant
   - make the right number of encirclements of (-1,0)
   - create phase lead at bandwidth

   ROT: add lead filter with zero at bandwidth/3 and pole at bandwidth*3; adjust gain
2. Meeting the margins and/or shape the loop
   - remove resonances *if necessary* (e.g. for stability margins)
   - use notch or second order filters (skewed notch)
3. Increase performance
   - add integrator (if necessary or desired)

   ROT: choose zero at bandwidth/5
   - add other filter blocks (e.g. notches) to shape specific closed loop transfer functions in specific frequency regions

Loop shaping motion systems (cont'd)

4. Cut-off high frequent controller gain
   - add low-pass filter

   ROT: choose poles at bandwidth*6 (and beta of 0.5 in case of second order low-pass)

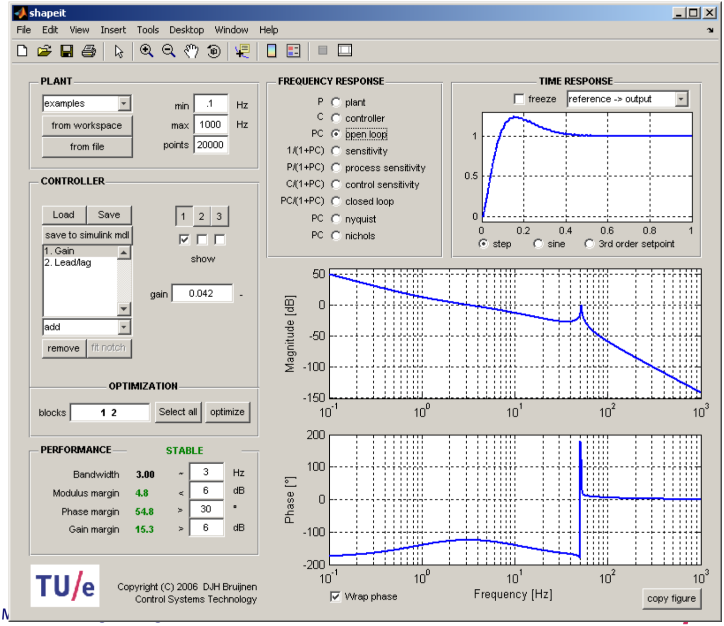
If desired, increase bandwidth.

The procedure is iterative; adjust the poles and zeros while shaping the loop / tuning the bandwidth. Make sure to check all relevant transfer functions!

ROT = Rule Of Thumb
All parameter choices are *indicative*, adjust them to your specific situation!

Technische Universiteit Eindhoven University of Technology

Controller design example



Technische Universiteit
Eindhoven
University of Technology