Advanced Motion Control

Part II: Multivariable System Theory

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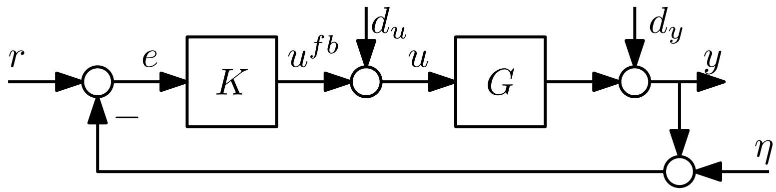
Transfer functions for MIMO systems

Frequency response for MIMO systems

Directions in MIMO systems

Summary and reading

Block diagrams again



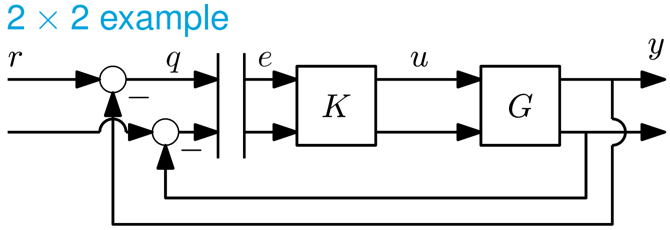
- Q: which input-output relation corresponds to the transfer function

$$G\,(I + GK)^{-1}$$

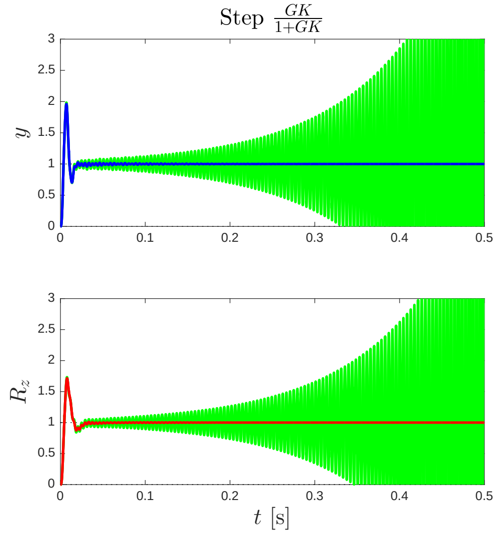
MIMO systems

- $G(s)$: $l \times m$ transfer function matrix
- u: $m \times 1$ vector
- y: $l \times 1$ vector
- $L(s) = G(s)K(s)$: open-loop
- $T(s) = L(s)(I_l + L(s))^{-1}$: closed-loop
- $F_o(s) = I_l + L(s) = I + G(s)K(s)$: return difference
  $=$ difference between original signal e and returned signal $q = -Le$

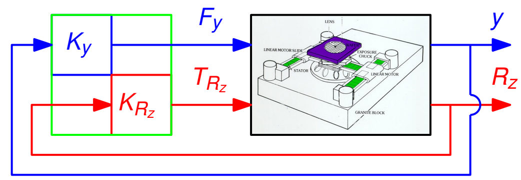
2×2 example



Controller for y and R_z-direction



Recap:



- only implement K_y
- only implement K_{R_z}
- implement K_y and K_{R_z}
- reason: interaction!
  - stability, but also ...
  - directionality
  - poles
  - zeros
  - frequency response
  - ...

Example

- directionality plays a key role in controller design for MIMO systems
- consider the system and controller

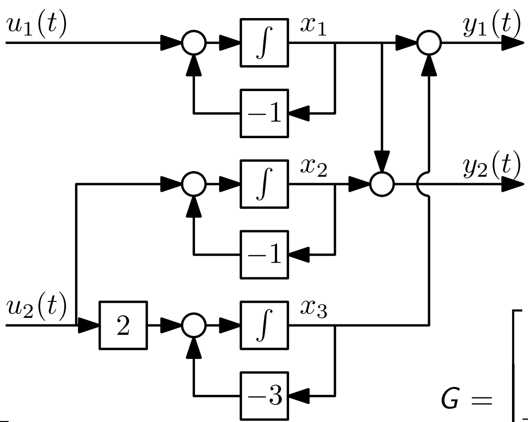
$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \quad K(s) = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}, \quad k_{11}, k_{22} \in \mathbb{R}$$

- Q: what is the 'order' of the system?
- the SISO loops first... Q: for which k_{11} is loop 1 stable?

- Q: idem for loop 2?
- Q: what if k_{11} and k_{22} are implemented simultaneously?

Example - continued

- we can analyse this through known state-space results . . .

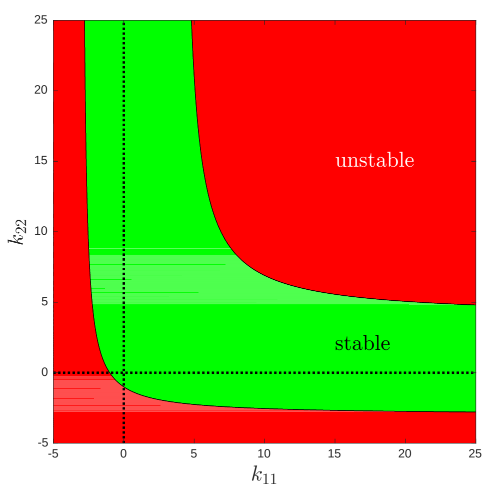


$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

$G =$

Example - continued

- given the state-space description, closed-loop stability requires $\lambda(A - BKC)$ in OLHP



Observation

- cannot have high gain k_{11}, k_{22} simultaneously
- directions play a key role

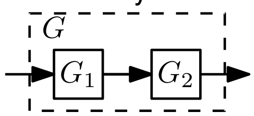
Aim

- develop a transfer function/frequency response function based framework
  - assess stability
    - SISO: roots $1 + g(s)k(s)$
    - MIMO: roots $\det F_o = \det(I + L)$

Transfer functions for MIMO systems

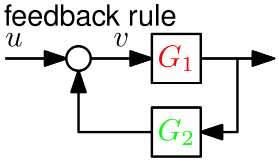
Basic rules transfer function matrices

- cascade system



$G = G_2 G_1 (\neq G_1 G_2$ in general)

- feedback rule



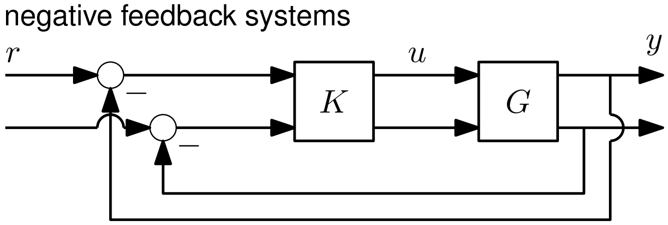
$v = (I - L)^{-1} u$, where $L = G_2 G_1$ is gain around the loop

- push-through rule
  $G_1 (I - G_2 G_1)^{-1} = (I - G_1 G_2)^{-1} G_1$ (green and red always alternate)
- Q: validity push-through for non-square G_1, G_2?
- Q: prove the push-through rule

Transfer functions for MIMO systems

Basic rules transfer function matrices

- negative feedback systems

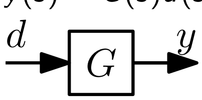


  - open loop at output: $L = GK$
  - output sensitivities: $S = (I + L)^{-1}, \quad T = L(I + L)^{-1}$
  - input sensitivities: $S_I = (I + L_I)^{-1}, \quad T_I = L_I(I + L_I)^{-1}, \quad L_I = KG$
  - in general: $S_I \neq S$ due to $AB \neq BA$ for matrices A and B
  - useful relationships using push-through rule:
    - $G(I + KG)^{-1} = (I + GK)^{-1} G$ (called process sensitivity)
    - $GK(I + GK)^{-1} = G(I + KG)^{-1} K = (I + GK)^{-1} GK$
  - Q: given a minimal state-space realization of L. How to compute a minimal state-space realization of T?

Frequency response for MIMO systems

System

- $y(s) = G(s)d(s)$



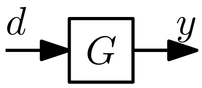
- given $G(s)$, the frequency response function matrix is given by $G(j\omega)$
- main point: for a given frequency ω, the response $G(j\omega)$ is matrix valued
- how to interpret this?

Tom Oomen - AMC handout

Frequency response for MIMO systems

System

- $y(s) = G(s)d(s)$



- in this section, d denotes input (U will be used in the SVD)
- $g_{ij}(j\omega)$: sinusoidal response from input j to output i
- input $d_j(t) = d_{jo}\sin(\omega t + \alpha_j)$
  leads to
  output $y_i(t) = y_{io}\sin(\omega t + \beta_i)$
- gain: $\frac{y_{io}}{d_{io}} = |g_{ij}(j\omega)|$
- phase shift: $\beta_i - \alpha_j = \arg(g_{ij}(j\omega))$
- phasor notation:
  - $d_j(\omega) = d_{jo}e^{j\alpha_j}$
  - $y_i(\omega) = y_{io}e^{j\beta_i}$

Frequency response for MIMO systems

MIMO: use superposition

- $y_1(\omega) = g_{11}(\omega)d_1(\omega) + g_{12}(\omega)d_2(\omega) + \ldots + g_{1m}(\omega)d_m(\omega)$
- in matrix notation

$$\underbrace{\begin{bmatrix} y_1(\omega) \\ y_2(\omega) \\ \vdots \\ y_l(\omega) \end{bmatrix}}_{y(\omega)} = \underbrace{\begin{bmatrix} g_{11}(\omega) & g_{12}(\omega) & \ldots & g_{1m}(\omega) \\ g_{21}(\omega) & g_{22}(\omega) & \ldots & g_{2m}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ g_{l1}(\omega) & g_{l2}(\omega) & \ldots & g_{lm}(\omega) \end{bmatrix}}_{G(j\omega)} \underbrace{\begin{bmatrix} d_1(\omega) \\ d_2(\omega) \\ \vdots \\ d_m(\omega) \end{bmatrix}}_{d(\omega)}$$

- here:

$$\begin{bmatrix} d_1(\omega) \\ d_2(\omega) \\ \vdots \\ d_m(\omega) \end{bmatrix} = \begin{bmatrix} d_{10}e^{j\alpha_1} \\ d_{20}e^{j\alpha_2} \\ \vdots \\ d_{m0}e^{j\alpha_m} \end{bmatrix}$$

Transfer functions for MIMO systems

Frequency response for MIMO systems

Directions in MIMO systems

Summary and reading

Gain of a system

- the gain at a given frequency for a SISO system is given by

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

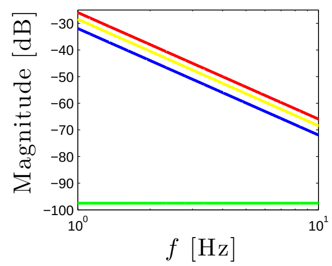
- for MIMO systems, what is the gain of

$$\begin{bmatrix} g_{11}(\omega) & g_{12}(\omega) & \dots & g_{1m}(\omega) \\ g_{21}(\omega) & g_{22}(\omega) & \dots & g_{2m}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ g_{l1}(\omega) & g_{l2}(\omega) & \dots & g_{lm}(\omega) \end{bmatrix}?$$

Directions in MIMO systems

Example

- dynamics $m\ddot{x} = F_1 + F_2$
- thus: $G(s) = \begin{bmatrix} \frac{1}{ms^2} & \frac{1}{ms^2} \end{bmatrix}$
- suppose only Fred is running (F_1)
- now suppose both Fred and Wilma run ($F_1 = F_2$)
- now Fred is runs opposite to Wilma ($F_1 = -F_2$)
- now Fred is tired and performs at 50% ($F_2 = 0.5F_1$)
- infinitely many Bode diagrams... how to define "gain"?



Directions in MIMO systems

Directionality and gain

- MIMO systems: vector-valued inputs and outputs
- measure the size of a signal: vector 2-norm (frequency dependent)

$$\|d(\omega)\|_2 = \sqrt{\sum_j |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \dots}$$

$$\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \dots}$$

- for a given input the gain of G

$$\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\sqrt{y_{10}^2+y_{20}^2+\dots}}{\sqrt{d_{10}^2+d_{20}^2+\dots}}$$

- can we quantify minimum and maximum gain? Answer: SVD!

Directions in MIMO systems

Singular value decomposition (SVD)

- from transfer function matrix $G(s)$ to complex matrix: $G(j\omega)$
- for given matrix $G \in \mathbb{C}^{l \times m}$ with $\text{rank}(G) = k \leq \min(l, m)$,

$$G = U\Sigma V^*$$

denotes its SVD, with

  - $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$
  - $\Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k), \ \sigma_1 \geq \sigma_2 \geq \sigma_k > 0$
  - $U = \begin{bmatrix} u_1, \ldots, u_l \end{bmatrix}$, $U^* U = I_l$
  - $V = \begin{bmatrix} v_1, \ldots, v_m \end{bmatrix}$, $V^* V = I_m$

Directions in MIMO systems

Singular value decomposition (SVD)

- for given matrix $G \in \mathbb{C}^{l \times m}$ with $\text{rank}(G) = k \leq \min(l, m)$,

$$G = U\Sigma V^*$$

- $U = \begin{bmatrix} u_1, \ldots, u_l \end{bmatrix}$, $U^* U = I_l$
- $V = \begin{bmatrix} v_1, \ldots, v_m \end{bmatrix}$, $V^* V = I_m$
- maximum gain: $\max\limits_{d \neq 0} \dfrac{\|Gd\|_2}{\|d\|_2} = \max\limits_{\|d\|_2 = 1} \dfrac{\|Gd\|_2}{\|d\|_2} = \bar{\sigma}$

  select $d = v_1 \Rightarrow \ \|d\|_2 = 1$
  then $Gd = u_1 \sigma_1 \ \Rightarrow \ \|Gd\|_2 = \sigma_1 = \bar{\sigma}$
- minimum gain $\min\limits_{d \neq 0} \dfrac{\|Gd\|_2}{\|d\|_2} = \min\limits_{\|d\|_2 = 1} \dfrac{\|Gd\|_2}{\|d\|_2} = \underline{\sigma}$
  - $\underline{\sigma} = 0$ if G not of full column rank, e.g., if $l > m$
  - $\underline{\sigma} > 0$ if G square and invertible

Singular value decomposition (SVD) - example

- let $G \in \mathbb{R}^{2\times 2}$, then

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}}_{V^T}$$

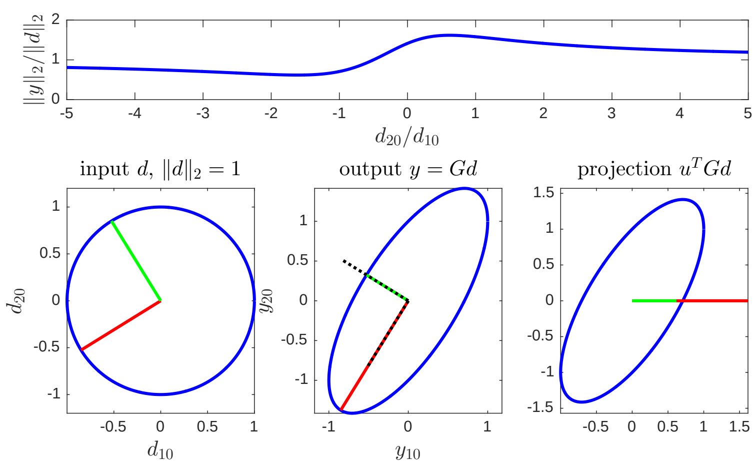
- interpretation
  - rotate
  - scale
  - rotate

Singular value decomposition (SVD) - example

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = U\Sigma V^T$$

$\sigma_1 = 1.6$
$\sigma_2 = 0.6$

condition number: $\frac{\bar{\sigma}}{\underline{\sigma}} = 2.6$



$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

Singular value decomposition (SVD) - properties

- gain bound: for any d,

$$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G)$$

- matrix norm
  - $\|G_1 + G_2\| \leq \|G_1\| + \|G_2\|$ (triangle inequality)
    $\Rightarrow \bar{\sigma}(G_1 + G_2) \leq \bar{\sigma}(G_1) + \bar{\sigma}(G_2)$
  - $\|G_1 G_2\| \leq \|G_1\|\|G_2\|$ (multiplicative)
    $\Rightarrow \bar{\sigma}(G_1 G_2) \leq \bar{\sigma}(G_1)\bar{\sigma}(G_2)$

Singular value decomposition (SVD) vs eigenvalues

- SVD is a measure of gain
- eigenvalues are a poor measure of gain: example
  - $G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}$
  - Q: eigenvalues?
  - apply input $d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Q: output?
- reason: $\rho(G)$ is not a norm
  - spectral radius $\rho(G) = \max_i |\lambda_i(G)|$, λ_i satisfies $\lambda_i x_i = G x_i$
  - Let $G = G_1 + G_2$, then $\rho(G) \not\leq \rho(G_1) + \rho(G_2)$
  - example: $G_1 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $G_2 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$
- lower bound for any norm: $\rho(G) \leq \|G\|$

Directions in MIMO systems

Singular value decomposition (SVD) vs. eigenvalues

- computing the SVD: a connection to eigenvalues
- let $G = U\Sigma V^*$
- note that

$$GG^* = U\Sigma \underbrace{V^* V}_{=I} \Sigma^* U^* \tag{1}$$

with

$$\Sigma\Sigma^* = \begin{bmatrix} \sigma_1^2 & 0 & \dots \\ 0 & \sigma_2^2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

- thus (1) is an eigenvalue decomposition of GG^* with eigenvalues $\sigma_1^2, \sigma_2^2, \dots$

Directions in MIMO systems

SVD for transfer function matrices $G(s)$

- maximum gain as function of frequency

$$y(\omega) = G(j\omega)d(\omega) = U(j\omega)\Sigma(j\omega)V^*(j\omega)d(\omega)$$

- very suitable for performance:

$$\underline{\sigma}(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}(S(j\omega))$$

- $\mathcal{H}_\infty$ norm: $\|G(s)\|_\infty = \sup_\omega \bar{\sigma}(G(j\omega))$
- suppose we want a small tracking error for any direction of r:

$$\bar{\sigma}(S(j\omega)) < \frac{1}{|w_P(j\omega)|} \quad \forall\omega \qquad \Leftrightarrow \bar{\sigma}(w_P(j\omega)S(j\omega)) < 1 \quad \forall\omega \Leftrightarrow \|w_P S\|_\infty < 1$$

will be used in a later lecture!

Transfer functions for MIMO systems

Frequency response for MIMO systems

Directions in MIMO systems

Summary and reading

Summary and reading

Take-home messages

- transfer functions for multivariable systems: lack of commutation makes life harder
- FRF is matrix valued for multivariable systems
- we have to rethink the notion of gain for multivariable systems: key role for SVD

Next

- how to use these tools to analyse multivariable systems?
- are there special cases when a multivariable system a collection of SISO systems that allow multi-loop SISO control?

Summary and reading

Reading

- general material: Skogestad & Postlethwaite (2005, Chapter 3.1-3.5)

Additional reading material

- SVD: Skogestad & Postlethwaite (2005, Appendix A.3), excellent overview of alternative definitions: Vinnicombe (2001, Appendix B)
- spectral radius: Skogestad & Postlethwaite (2005, Appendix A.5.3)

References I

Skogestad, S. & Postlethwaite, I. (2005), *Multivariable Feedback Control: Analysis and Design*, second edn, John Wiley & Sons, West Sussex, United Kingdom.

Vinnicombe, G. (2001), *Uncertainty and Feedback: $\mathcal{H}_\infty$ loop-shaping and the ν-gap metric*, Imperial College Press, London, United Kingdom.