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1. i To get tangent vector, we take derivative of $(x(t), y(t))$ respect to t.
   $x'(t) = (4\cos(2\pi t) + \frac{\cos(32\pi t)}{16})' = -2\pi(4\sin(2\pi t) + \sin(32\pi t))$
   $y'(t) = (2\sin(2\pi t) + \frac{\sin(32\pi t)}{16})' = 2\pi(2\cos(2\pi t) + \cos(32\pi t))$
   Then the tangent vector is $(-(4\sin(2\pi t) + \sin(32\pi t)), (2\cos(2\pi t) + \cos(32\pi t)))$

   ii To get normal vector, we know that at any point normal vector is perpendicular to tangent vector; hence, normal vector is $((2\cos(2\pi t) + \cos(32\pi t)), (4\sin(2\pi t) + \sin(32\pi t)))$ or $(-(2\cos(2\pi t) + \cos(32\pi t)), -(4\sin(2\pi t) + \sin(32\pi t)))$.

   iii The curve is symmetric around X-axis and symmetric around Y-axis.
   We could notice that both $x(t)$ and $y(t)$ are periodic functions which repeat over intervals of 1 radians.
   Also if a function $f(x)$ is symmetric around X-axis means $f(x) = -f(x)$ and is symmetric around Y-axis means $f(x) = f(-x)$.
   Symmetric around X-axis:
   First, we know that $\cos(t) = \cos(-t)$ which means $x(t) = x(-t)$. Also, $\sin(t) = -\sin(-t)$ which means $y(t) = -y(-t)$. Then at t and $-t$ time, the curve has same x coordinates but opposite y coordinates; hence, the curve is symmetric around X-axis.
   Symmetric around Y-axis:
   We know that $\cos(t) = -\cos(\pi - t)$ which means $x(t) = -x(0.5 - t)$. Also, $\sin(t) = \sin(\pi - t)$ which means $y(t) = y(0.5 - t)$. Then at t and $0.5 - t$ time, the curve has opposite x coordinates but same y coordinates; hence, the curve is symmetric around Y-axis.

   iv Since both $x(t)$ and $y(t)$ are periodic functions which repeat over intervals of 1radians, and symmetric around both X-axis and Y-axis.
   Then perimeter:
$$p = 4\int_0^{0.25} \sqrt{(x'(t))^2 + (y'(t))^2}dt$$
$$= 4\int_0^{0.25} \sqrt{(2\pi(4\sin(2\pi t) + \sin(32\pi t)))^2 + (2\pi(2\cos(2\pi t) + \cos(32\pi t)))^2}dt$$

   v Define $f(t)$ as follow:
$$f(t) = \sqrt{(2\pi(4\sin(2\pi t) + \sin(32\pi t)))^2 + (2\pi(2\cos(2\pi t) + \cos(32\pi t)))^2}$$
   The way we approximate the perimeter is approximate the integral above. So we divide the interval $[0, 0.25]$ to n sub-interval $[x_0, x_1], [x_1, x_2]...[x_{n-1}, x_n]$ *where* $x_0 = 0, x_n = 0.25$. Then calculate sum, $perimeter = 4\sum_{i=0}^{i=n-1}[(f(x_{i+1}) - f(x_i))(x_{i+1} - x_i)]$.

2. Denote C_1 to be the circle with radius r_1 and C_2 to be the circle with radius r_2 ($r_2 > r_1$).

   i The area of donut is $A = \pi(r_2)^2 - \pi(r_1)^2 = \pi((r_2)^2 - (r_1)^2)$.

   ii There can be 0,1, 2 ,3 and 4 intersections.

   iii The distance from the centre (p_1) of the circles to line $p(\lambda) = p_0 + \lambda\overrightarrow{d}$ is
$$distance: \; d = \|(p_0 - p_1) - ((p_0 - p_1)\cdot\overrightarrow{n})\overrightarrow{n}\| \quad where \; \overrightarrow{n} = \frac{\overrightarrow{d}}{|\overrightarrow{d}|}$$

(1) if $d > r_2$, then there is no intersection;
(2) if $d = r_2$, then there is 1 intersection, then solve the equation

$$\|(p_0 - p_1) - \lambda \overrightarrow{d}\| = r_2$$

get one solution λ_0 and $p(\lambda_0)$ is the location of the intersection.
(3) if $r_2 > d > r_1$, then there 2 intersections, then solve the equation

$$\|(p_0 - p_1) - \lambda \overrightarrow{d}\| = r_2$$

get two solutions λ_0, λ_1 and $p(\lambda_0), p(\lambda_1)$ are the location of the intersections.
(4) if $d = r_1$, then there are 3 intersections, then solve then solve the equation

$$\|(p_0 - p_1) - \lambda \overrightarrow{d}\| = r_2$$

$$\|(p_0 - p_1) - \lambda \overrightarrow{d}\| = r_1$$

get three solutions $\lambda_0, \lambda_1 \lambda_2$ and $p(\lambda_0), p(\lambda_1), p(\lambda_2)$ are the location of the intersections.
(5) if $d < r_1$, then there are 4 intersections, then solve then solve the equation

$$\|(p_0 - p_1) - \lambda \overrightarrow{d}\| = r_2$$

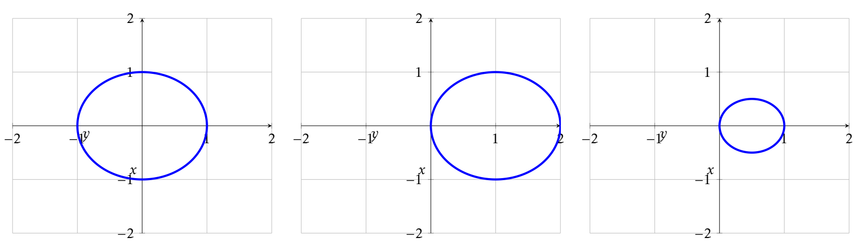
$$\|(p_0 - p_1) - \lambda \overrightarrow{d}\| = r_1$$

get three solutions $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ and $p(\lambda_0), p(\lambda_1), p(\lambda_2), p(\lambda_3)$ are the location of the intersections.

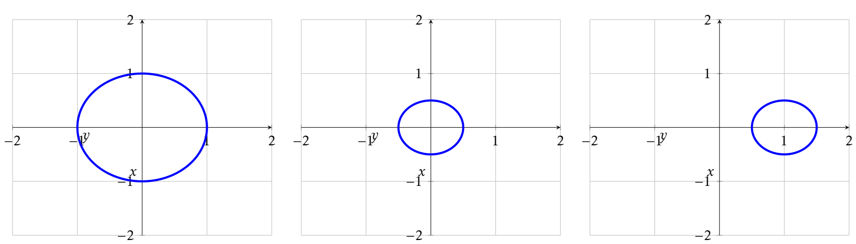
iv If both line and donut transformed by a non-uniform scale (s_x, s_y), then intersection won't change. Suppose the old location is (x, y), then location after scale it will be $(s_x x, s_y y)$.

v Suppose A(x, y) is a point on the donut. After scaling, it becomes A' $(s_x x, s_y y)$. Then $\Delta(A'A) = (A' - A)$, and $\theta = \arccos(\frac{\overrightarrow{A'-A} \cdot \overrightarrow{d}}{|\overrightarrow{A'-A}||\overrightarrow{d}|})$. Then if $\pi < \theta < 2\pi$ means after scaling A is closer to the line, if $0 < \theta < \pi$ means A is away from the line after scaling and $\theta = 0 \vee \theta = \pi$ means the distance from A to the line remains the same.

3. (a) Translation and uniform scaling do not commute First, translate by $(1, 0)$, then scaled by $(\frac{1}{2}, \frac{1}{2})$:

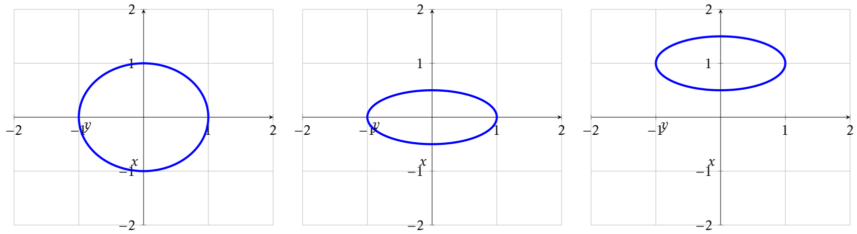


Now, apply it reversely, scaled by $(\frac{1}{2}, \frac{1}{2})$, then translate by $(1, 0)$:

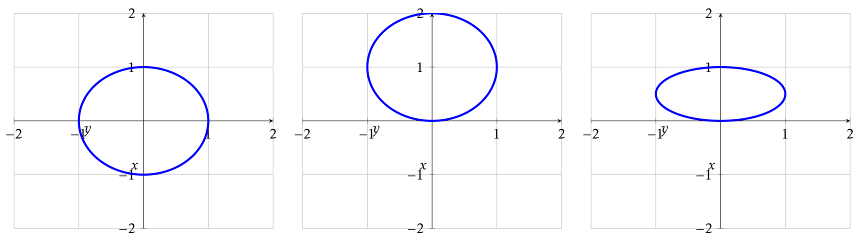


Hence, clearly translation and uniform scaling do not commute.

(b) Translation and non-uniform scaling do not commute. First we scaled by $(1, \frac{1}{2})$, then translate by $(0, 1)$:

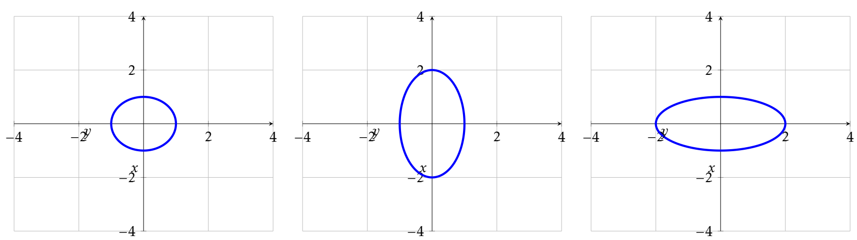


Now, we apply translation first, then scaling:

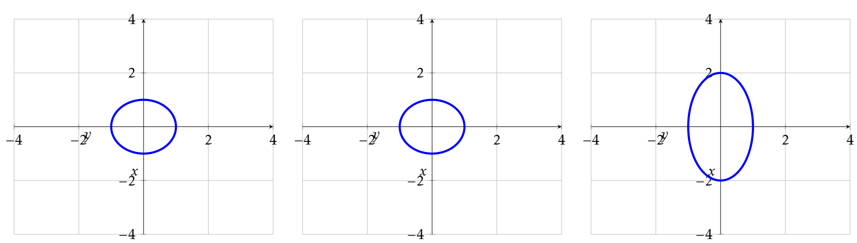


Hence, translation and non-uniform scaling do not commute.

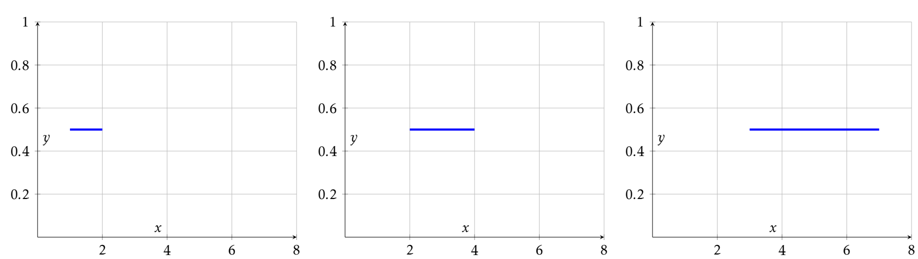
(c) Scaling and rotation, both having the same fixed points commute.
Scaling by $(1, 2)$, then rotating $\frac{\pi}{2}$:



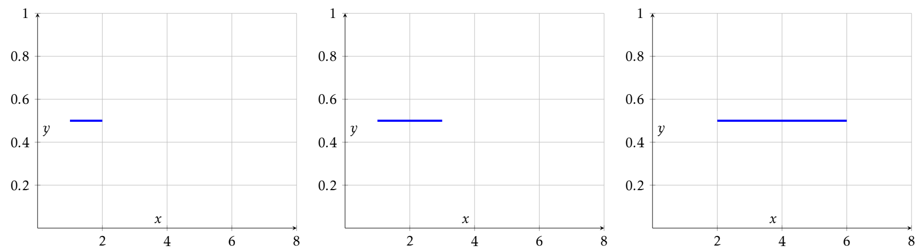
Now, rotating first then scaling



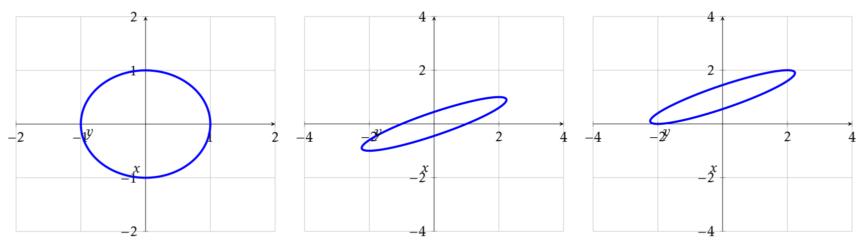
(d) Scaling and scaling, having different fixed points do not commute. First, Scaling around $(0, 0)$ by (2,0), then around $(1, 0)$ by (2, 0):



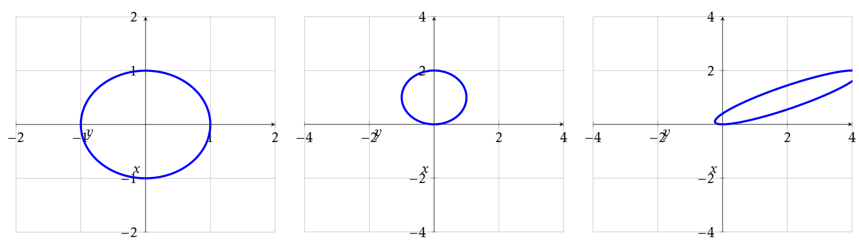
First, Scaling around $(1,0)$ by $(2, 0)$, then around $(0,0)$ by $(2,0)$:



(e) Translation and shear do not commute. First, shearing by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then translate by $(0,1)$:



Now, we first translate by $(0,1)$ then shearing by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$:



Hence, translation and shear do not commute.

4. i First we create 3 vectors $\overrightarrow{v_0 - q}, \overrightarrow{v_1 - q}, \overrightarrow{v_2 - q}$ using $\bar{v_0}, \bar{v_1}, \bar{v_2}$ and $\bar{q}$.

$$\theta_0 = \arccos\left(\frac{\overrightarrow{v_0 - q} \cdot \overrightarrow{v_1 - q}}{|\overrightarrow{v_0 - q}||\overrightarrow{v_1 - q}|}\right)$$

$$\theta_1 = \arccos\left(\frac{\overrightarrow{v_0 - q} \cdot \overrightarrow{v_2 - q}}{|\overrightarrow{v_0 - q}||\overrightarrow{v_2 - q}|}\right)$$

$$\theta_2 = \arccos\left(\frac{\overrightarrow{v_1 - q} \cdot \overrightarrow{v_2 - q}}{|\overrightarrow{v_1 - q}||\overrightarrow{v_2 - q}|}\right)$$

Then q is inside the triangle iff $(\theta_0 + \theta_1 + \theta_2 = 2\pi) \wedge (\theta_0 \neq \pi) \wedge (\theta_1 \neq \pi) \wedge (\theta_2 \neq \pi)$;
q is on the triangle iff $(\ (v_0 = q) \vee (v_1 = q) \vee (v_2 = q)\ )\ \vee\ (\ (\theta_0 + \theta_1 + \theta_2 = 2\pi) \wedge ((\theta_0 = \pi) \vee (\theta_1 = \pi) \vee (\theta_2 = \pi))\ )$;
Otherwise, q is outside the triangle.

ii Suppose we are given with 4 vertices $V = \{v_0, v_1, v_2, v_3\}$ and edge set $E = \{(v_0, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_0)\}$

Define:

$$\theta_0 = \arccos(\frac{\overrightarrow{v_1 - v_0} \cdot \overrightarrow{v_2 - v_0}}{|\overrightarrow{v_1 - v_0}||\overrightarrow{v_2 - v_0}|}) + \arccos(\frac{\overrightarrow{v_1 - v_0} \cdot \overrightarrow{v_3 - v_0}}{|\overrightarrow{v_1 - v_0}||\overrightarrow{v_3 - v_0}|}) + \arccos(\frac{\overrightarrow{v_2 - v_0} \cdot \overrightarrow{v_3 - v_0}}{|\overrightarrow{v_2 - v_0}||\overrightarrow{v_3 - v_0}|})$$

$$\theta_1 = \arccos(\frac{\overrightarrow{v_0 - v_1} \cdot \overrightarrow{v_2 - v_1}}{|\overrightarrow{v_0 - v_1}||\overrightarrow{v_2 - v_1}|}) + \arccos(\frac{\overrightarrow{v_0 - v_1} \cdot \overrightarrow{v_3 - v_1}}{|\overrightarrow{v_0 - v_1}||\overrightarrow{v_3 - v_1}|}) + \arccos(\frac{\overrightarrow{v_2 - v_1} \cdot \overrightarrow{v_3 - v_1}}{|\overrightarrow{v_2 - v_1}||\overrightarrow{v_3 - v_1}|})$$

$$\theta_2 = \arccos(\frac{\overrightarrow{v_1 - v_2} \cdot \overrightarrow{v_3 - v_2}}{|\overrightarrow{v_1 - v_2}||\overrightarrow{v_3 - v_2}|}) + \arccos(\frac{\overrightarrow{v_1 - v_2} \cdot \overrightarrow{v_0 - v_2}}{|\overrightarrow{v_1 - v_2}||\overrightarrow{v_0 - v_2}|}) + \arccos(\frac{\overrightarrow{v_0 - v_2} \cdot \overrightarrow{v_3 - v_2}}{|\overrightarrow{v_0 - v_2}||\overrightarrow{v_3 - v_2}|})$$

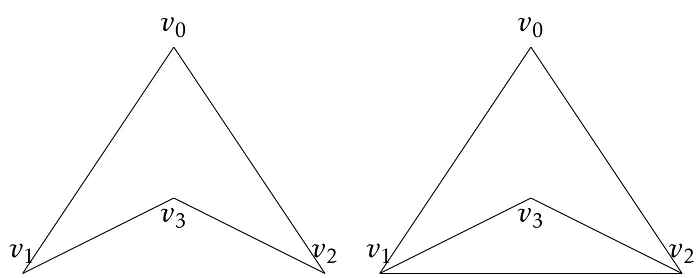
$$\theta_3 = \arccos(\frac{\overrightarrow{v_1 - v_3} \cdot \overrightarrow{v_2 - v_3}}{|\overrightarrow{v_1 - v_3}||\overrightarrow{v_2 - v_3}|}) + \arccos(\frac{\overrightarrow{v_1 - v_3} \cdot \overrightarrow{v_0 - v_3}}{|\overrightarrow{v_1 - v_3}||\overrightarrow{v_0 - v_3}|}) + \arccos(\frac{\overrightarrow{v_0 - v_3} \cdot \overrightarrow{v_2 - v_3}}{|\overrightarrow{v_0 - v_3}||\overrightarrow{v_2 - v_3}|})$$

Check if any $\theta_0, \theta_1, \theta_2, \theta_3$ equals 2π, if $\theta_x = 2\pi$ we know that quadrilateral is concave. And based on (i) we also know that v_x is inside the triangle with vertices $V \setminus v_x$. Then connect v_x, v_y where $(v_y \in V \setminus v_x) \wedge ((v_x, v_y) \notin E)$, and quadrilateral is now split into the union of two triangles.

If none of $\theta_0, \theta_1, \theta_2, \theta_3$ equals 2π which means the quadrilateral is convex. Then just connect one of two diagonals (i.e (v_0, v_2) or (v_1, v_3)), the quadrilateral is now split into the union of two triangles.

iii Suppose we are given with n vertices $V = \{v_0, v_1, v_2, v_3 ... v_(n-1)\}$ and edge set $E = \{(v_0, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_4) ..., (v_{n-2}, v_{n-1}), (v_{n-1}, v_0)\}$. pick any vertex v_i in the vertices set V (i.e v_0), then connect v_i to the rest of the vertices if there no edge between them.

iv Counterexample, this algorithm won't work for concave, for example if we choose $v_i = v_2$ in the following graph.



v For any point q, we connect q with every vertex in the vertices set V, and define

$$\theta_i = \arccos(\frac{\overrightarrow{v_i - q} \cdot \overrightarrow{v_{i+1\,(mod\ n)} - q}}{|\overrightarrow{v_i - q}||\overrightarrow{v_{i+1\,(mod\ n)} - q}|}) \quad 0 \le i \le n-1$$

Also, define the line function for each edge i is l_i. Then q is inside the polygon iff $\sum_{i=0}^{n-1} \theta_i = 2\pi$ and q doesn't satisfy any line function l_i. q is on the polygon iff $\sum_{i=0}^{n-1} \theta_i = 2\pi$ and satisfy some line function l_i. Otherwise, q is outside the polygon.