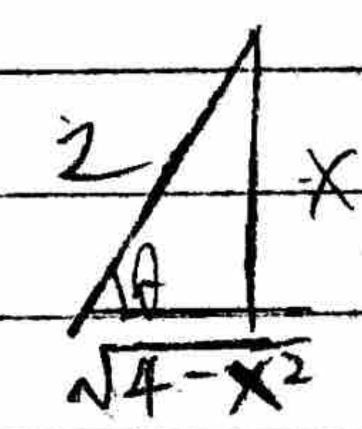
CSC 418 A1

① (a) $X(t) = 2\sin(t)$, $y(t) = 5\sin(t)\cos(t)$ $\quad (0 \le t \le 2\pi)$

$t = \sin^{-1}\left(\frac{x}{2}\right) \Rightarrow y = 5\sin\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)$

$\Rightarrow y = 5 \cdot \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} = \frac{5}{4} x \cdot \sqrt{4-x^2}$

$\Rightarrow$ Simplicit: $\boxed{0 = \frac{5}{4} x \sqrt{4-x^2} - y}$



(b) $\frac{dx}{dt} = 2\cos(t)$ $\quad \frac{dy}{dt} = 5\cos^2(t) - 5\sin^2(t) = 5\cos(2t)$

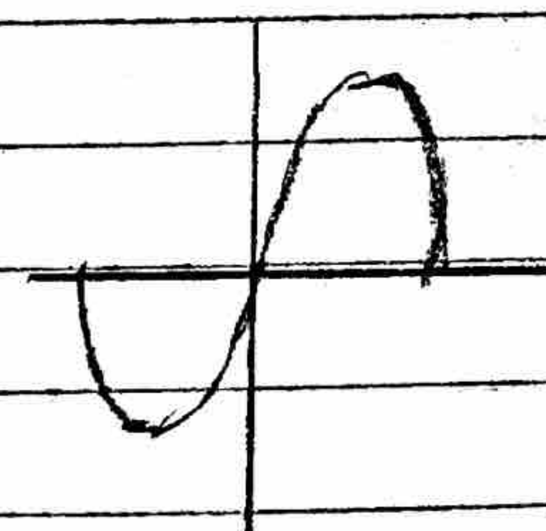
Tangent Vector $\langle x'(t), y'(t) \rangle = \boxed{\langle 2\cos(t), 5\cos(2t) \rangle}$

normal Vector $\langle y'(t), -x'(t) \rangle$ or $\langle -y'(t), x'(t) \rangle$

$= \boxed{\langle 5\cos(2t), -2\cos(t) \rangle}$ or $\boxed{\langle -5\cos(2t), 2\cos(t) \rangle}$

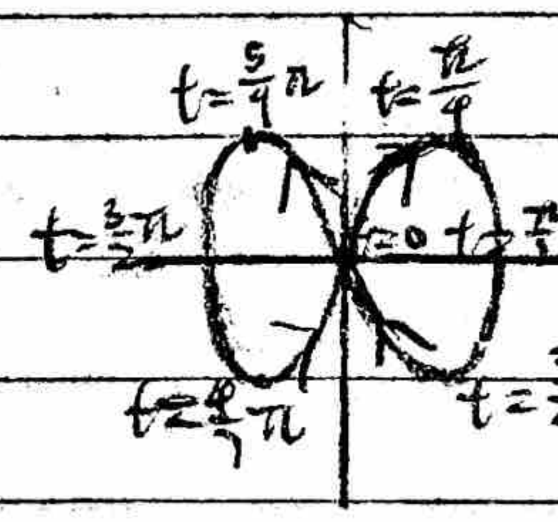
(c):

①



$y = \frac{5}{4} x \sqrt{4-x^2}$

③

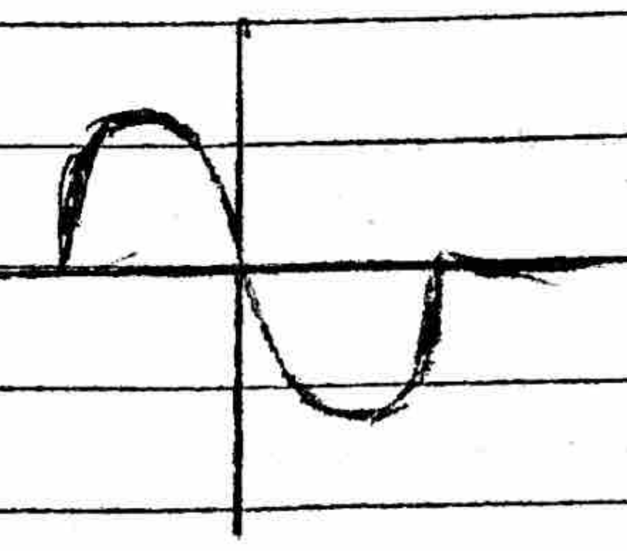


$X(t) = 2\sin(t)$

$y(t) = 5\sin(t)\cos(t)$

$0 \le t \le 2\pi$

②



$y = -\frac{5}{4} x \sqrt{4-x^2} \Rightarrow ① + ② = ③$

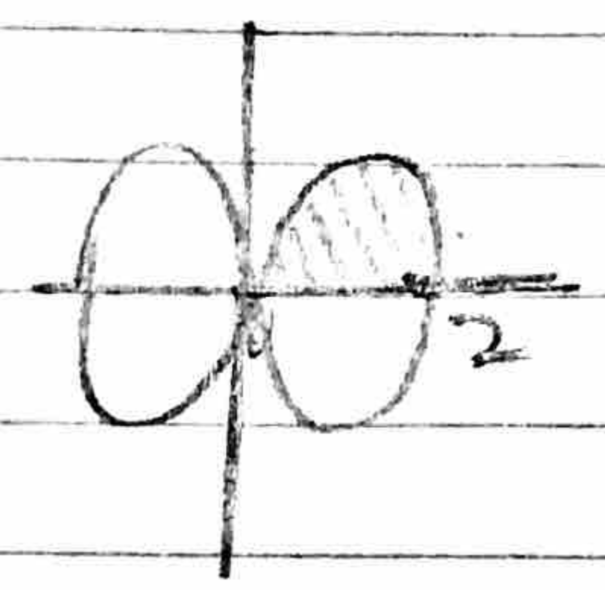
Therefore, for same x, we have opposite y (x, y) and $(x, -y)$

$\Rightarrow$ Symmetric around X-Axis

, for opposite x, we have same y (x, y) and $(-x, y)$

$\Rightarrow$ Symmetric around Y-Axis

$\Rightarrow$ The Curve is Symmetric around both x, y-Axis (d):



$$\text{Total Area} = 4 \cdot \int_{x=0}^{x=2} \frac{5}{4} x\sqrt{4-x^2}\,dx$$

$$= 5\int_0^2 x\sqrt{4-x^2}\,dx$$

$u = 4-x^2$
$du = -2x\,dx$

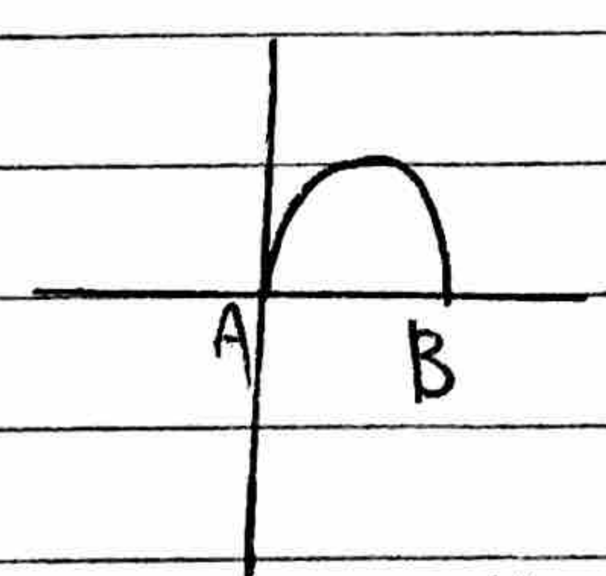
$$= 5\int_4^0 -\frac{1}{2}\cdot u^{\frac{1}{2}}\,du$$

$$= 5\int_4^0 -\frac{1}{2}\cdot\frac{2}{3}\, u^{\frac{3}{2}}\,du$$

$$= \frac{5}{3}\, u^{\frac{3}{2}}\Big|_0^4$$

$$= \frac{5}{3}\cdot 8 = \boxed{\frac{40}{3}}$$

(e): Total perimeter of the 'bow-tie' = 4 · length(curve AB)



$$\frac{dy}{dx} = \frac{5}{4}\sqrt{4-x^2} + \frac{5}{4}\cdot\frac{1}{2}\cdot x\cdot(4-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{5}{4}\sqrt{4-x^2} - \frac{5}{4}x^2(4-x^2)^{-\frac{1}{2}}$$

$$= 4\cdot\int_{x=0}^{x=2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\cdot\int_0^2\sqrt{1+\left(\frac{5}{4}\sqrt{4-x^2}-\frac{5}{4}x^2(4-x^2)^{-\frac{1}{2}}\right)^2}\,dx$$

We can use a program to approximate this length by increment x by 0.01 or 0.1 every step from 0 to 2 and then sum every output to get the final length

$\Rightarrow 4\sqrt{29}$ (by Wolfram Alpha)

② Transformation

(a)

$$T_1 = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} 1 & 0 & x_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_1 \cdot T_2 = \begin{bmatrix} 1 & 0 & x_2 + x_1 \\ 0 & 1 & y_2 + y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 \cdot T_1 = \begin{bmatrix} 1 & 0 & x_1 + x_2 \\ 0 & 1 & y_1 + y_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore T_1 \cdot T_2 = T_2 \cdot T_1 \Rightarrow$ Commute

(b) translation and rotation

$$T_1 = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

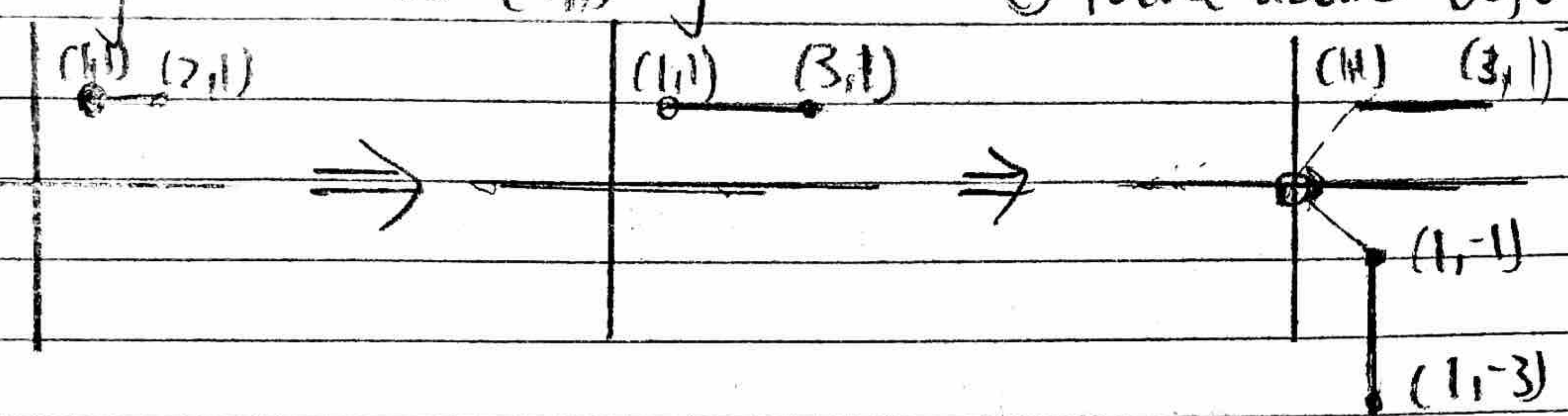
$$T_1 \cdot T_2 = \begin{bmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 \cdot T_1 = \begin{bmatrix} \cos\theta & -\sin\theta & x_1\cos\theta - y_1\sin\theta \\ \sin\theta & \cos\theta & x_1\sin\theta + y_1\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

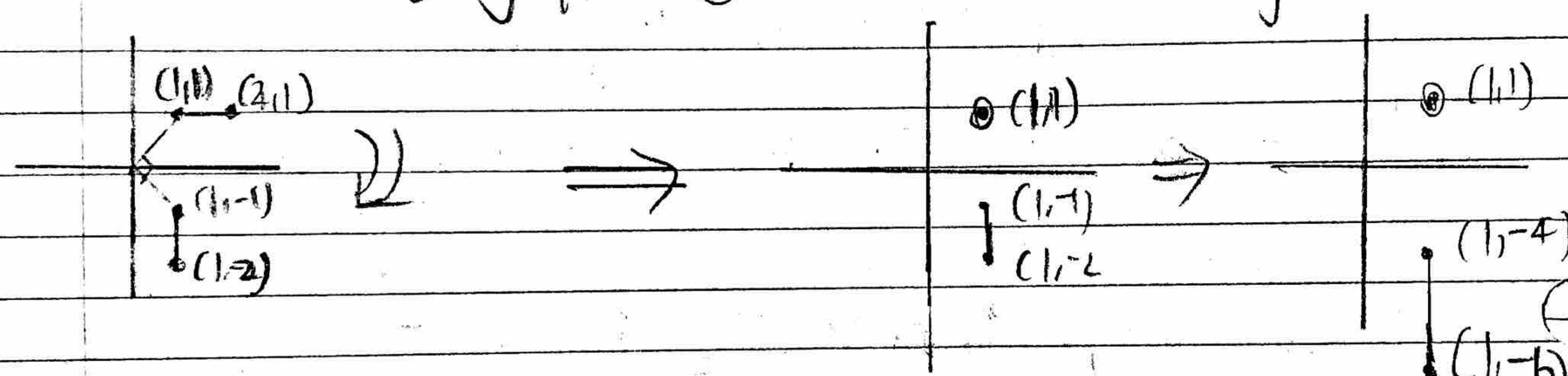
$T_1 \cdot T_2 \neq T_2 \cdot T_1 \Rightarrow$ Not Commute

(c) Scaling and rotation, having different fixed points.

① Scaling relative to (1,1) by 2 ② Rotate about (0,0) by 90°



Reverse: ① Rotate about (0,0) by 90° ② Scale about (1,1) by 2



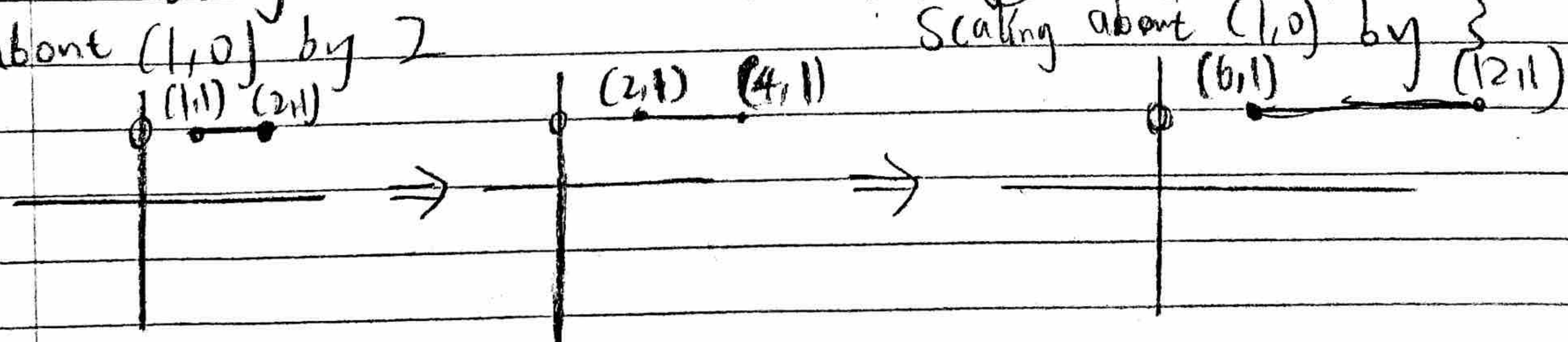
⇒ Not commute because at different positions of points

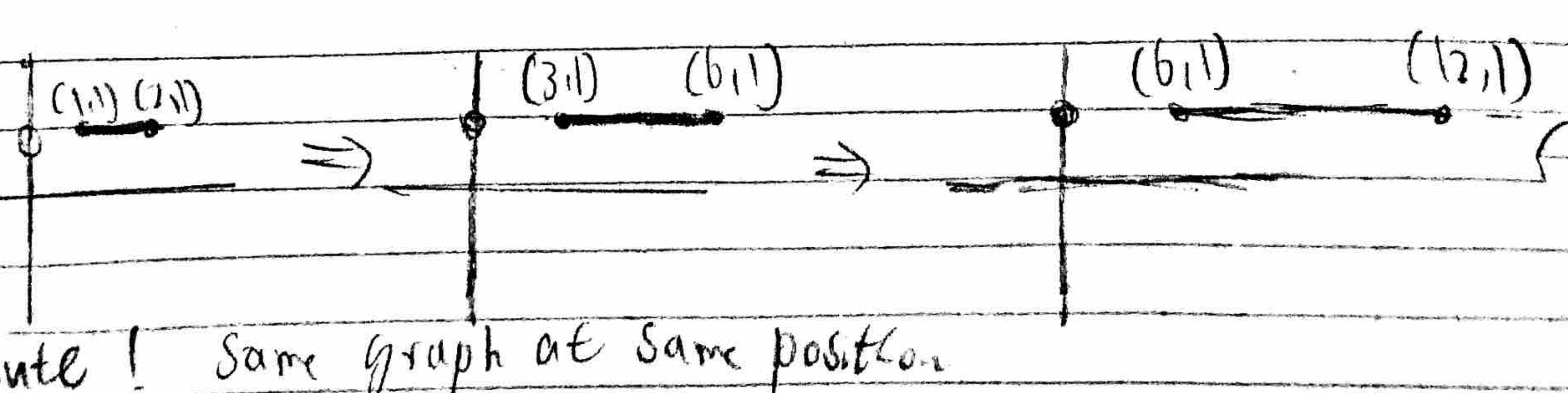
(d) Scaling and Scaling, having the same fixed point

Method ①

Scaling about (x_1,y_1) by a,b = translation (x_1,y_1) · Scaling about (x_1,y_1) by a,b · translation $(-x_1,-y_1)$

Scale about (1,0) by 2 Scaling about (1,0) by 3





⇒ Commute! Same graph at same position

Method ② (general mathematic way to prove)

Scale about any fixed point = $T(x_1, y_1)\, \text{scale}(a,b)\, T(-x_1,-y_1)$

$$T_1 = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & x_1 - ax_1 \\ 0 & b & y_1 - by_1 \\ 0 & 0 & 1 \end{bmatrix}$$

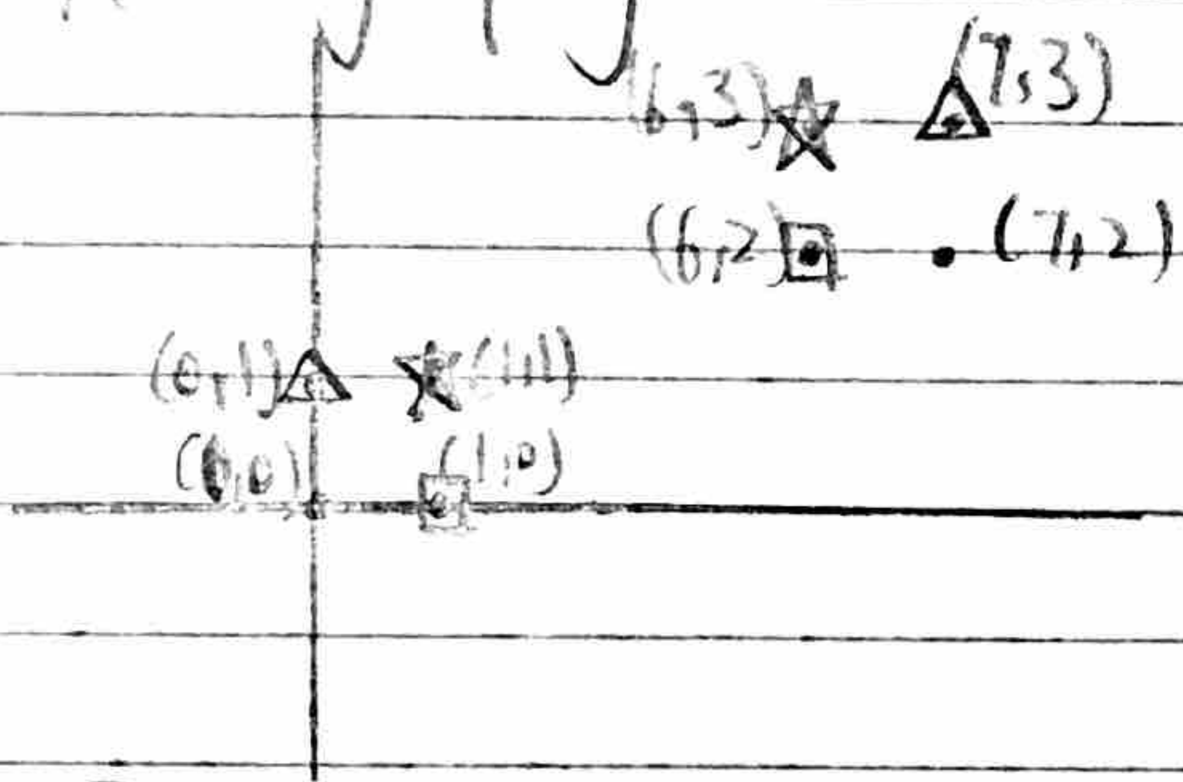
$$T_2 = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c & 0 & x_1 - cx_1 \\ 0 & d & y_1 - dy_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_1 \cdot T_2 = \begin{bmatrix} ac & 0 & a(x_1 - cx_1) + (x_1 - ax_1) \\ 0 & bd & b(y_1 - dy_1) + (y_1 - by_1) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ac & 0 & -cax_1 + x_1 \\ 0 & bd & -bdy_1 + y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

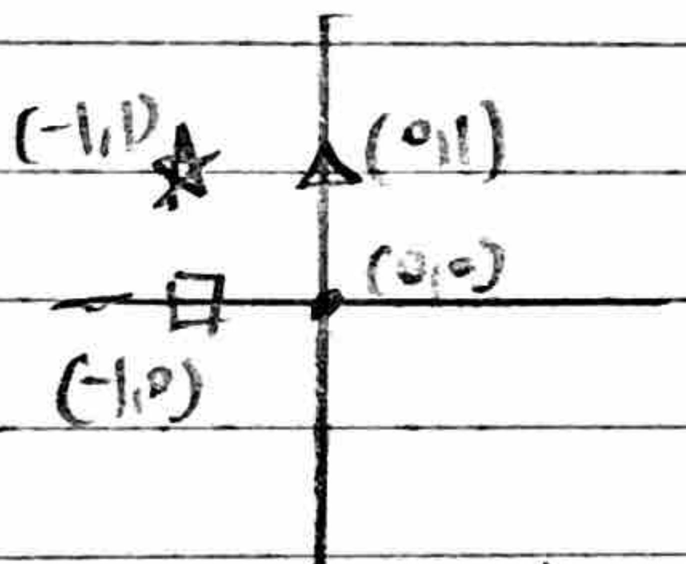
$$T_2 \cdot T_1 = \begin{bmatrix} ca & 0 & c(x_1 - ax_1) + (x_1 - cx_1) \\ 0 & db & d(y_1 - by_1) + (y_1 - dy_1) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ca & 0 & -cax_1 + x_1 \\ 0 & db & -bdy_1 + y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

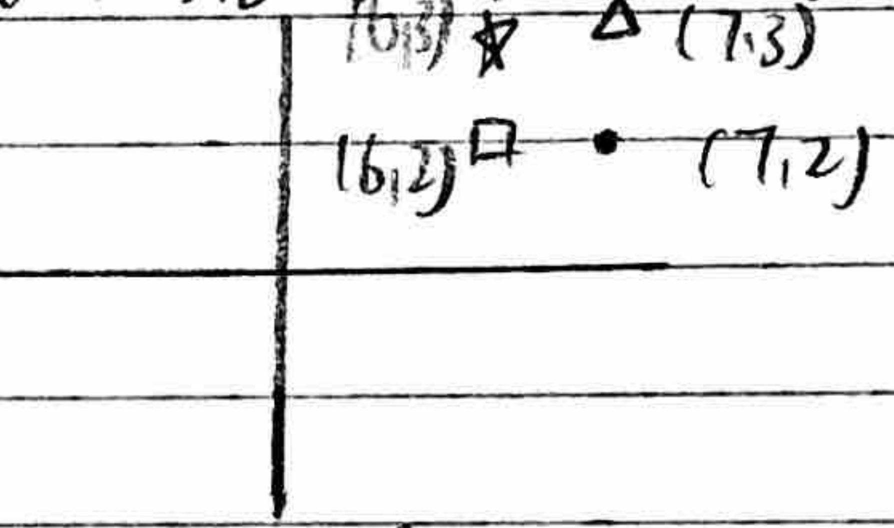
$\Rightarrow T_1 \cdot T_2 = T_2 \cdot T_1 \Rightarrow$ Commute!

3. Homography



① reflection about Y-Axis ② translation in X-Axis 7 units and in Y-Axis 2 units





$$T_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Affine Trans: $M = T_2 \cdot T_1 = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$(2,5) \rightarrow ?$

$$\begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -x+7 \\ y+2 \\ 1 \end{bmatrix}$$

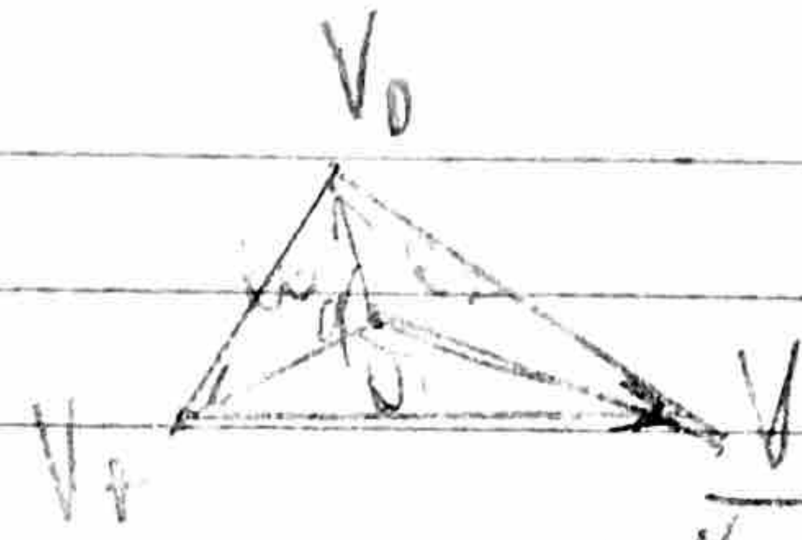
$x=2, y=5$

$$= \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$$

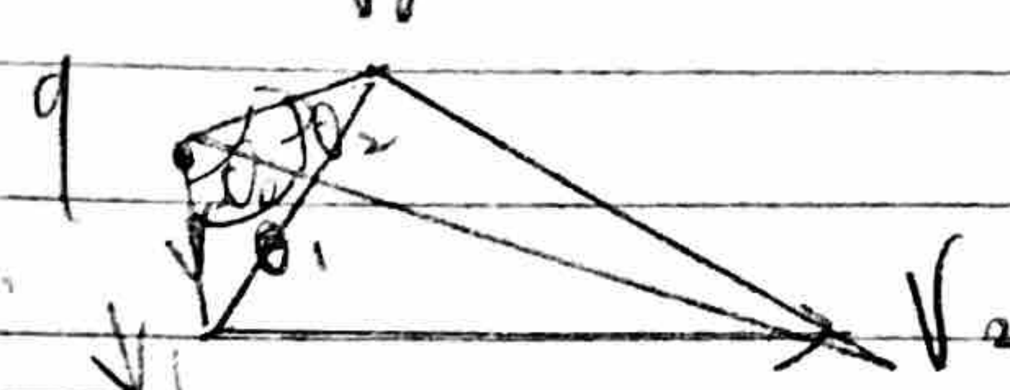
$$= (5,7)$$

(4)

Inside

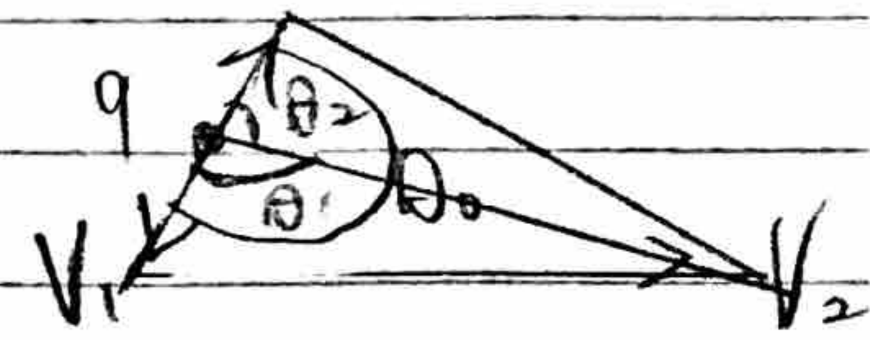


Outside

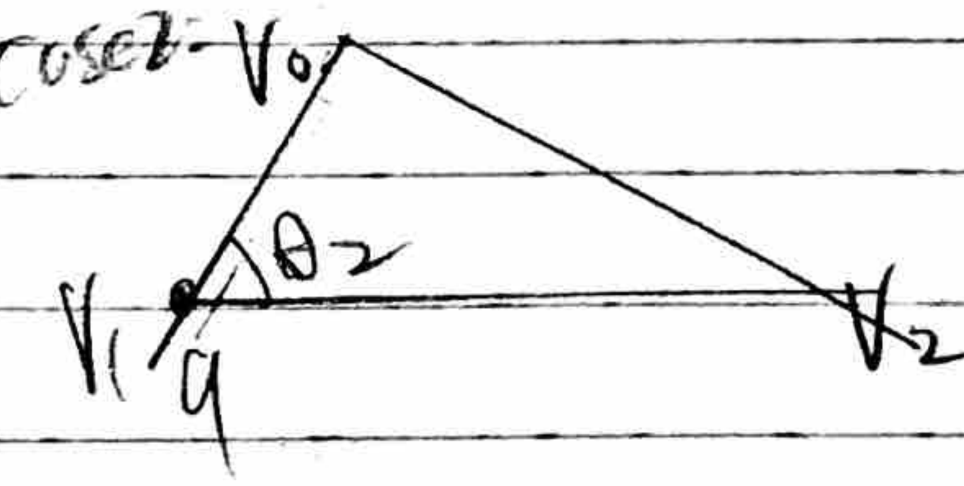


On edge

Case 1: V_0



Case 2: V_0



Def:

$$\theta_0 = \cos^{-1}\left(\frac{\overrightarrow{V_0-q}\cdot\overrightarrow{V_1-q}}{\|V_0-q\|\cdot\|V_1-q\|}\right)$$

$$\theta_1 = \cos^{-1}\left(\frac{\overrightarrow{V_1-q}\cdot\overrightarrow{V_2-q}}{\|V_1-q\|\cdot\|V_2-q\|}\right)$$

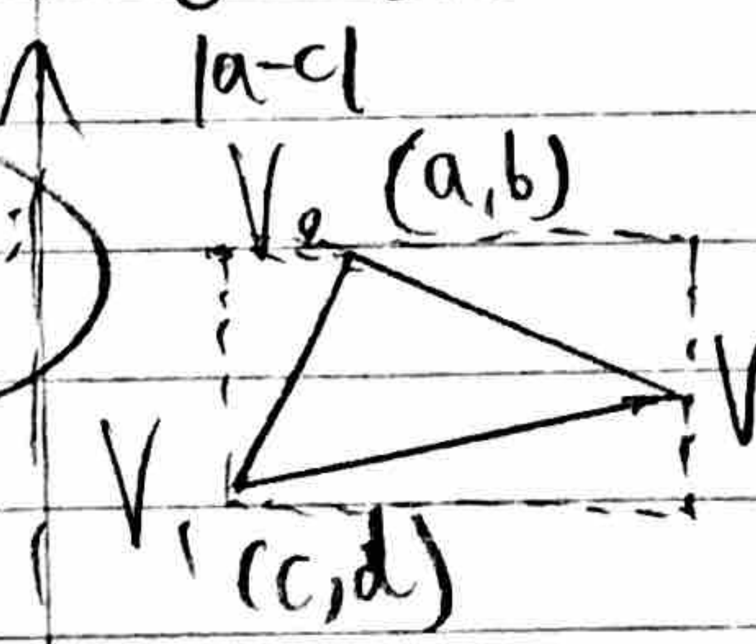
$$\theta_2 = \cos^{-1}\left(\frac{\overrightarrow{V_0-q}\cdot\overrightarrow{V_2-q}}{\|V_0-q\|\cdot\|V_2-q\|}\right)$$

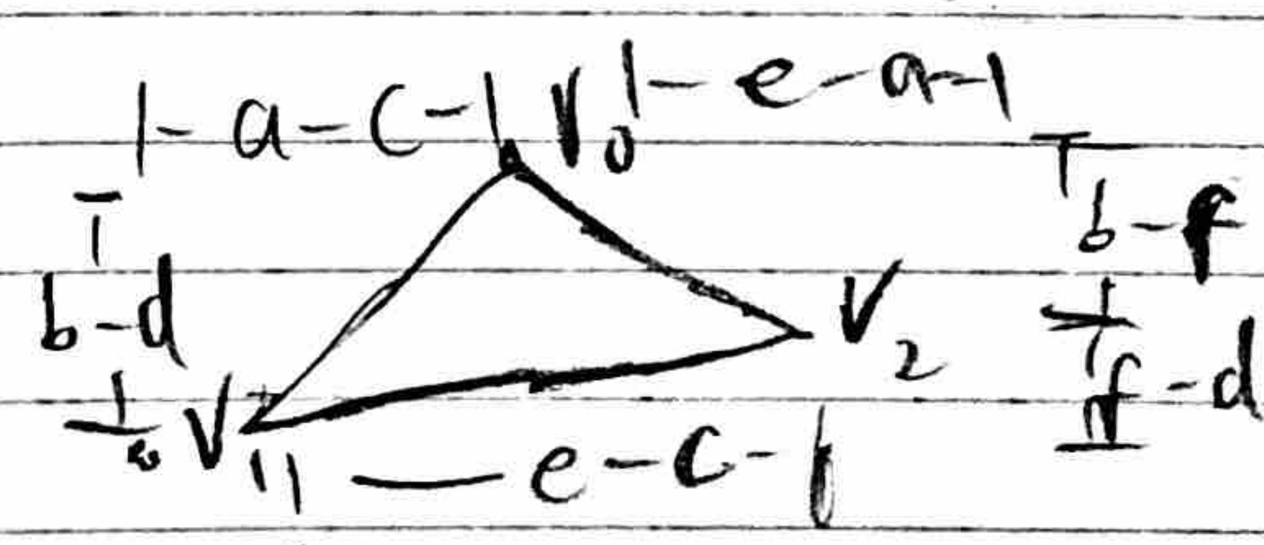
Inside: if $(\theta_0+\theta_2+\theta_1 = 360°)$ and $(\theta_0 \neq 180°)$ and $(\theta_1 \neq 180°)$ and $(\theta_2 \neq 180°)$

one edge: if $((\theta_0+\theta_2+\theta_1 = 360°)$ and $(\theta_0 = 180°$ or $\theta_1 = 180°$ or $\theta_2 = 180°))$ or $(q = V_0$ or $q = V_1$ or $q = V_2)$

Outside: other cases then inside and onedge

Area





By using the vertice coordinate of each vertices

We can compute the lengths of V_0V_1, V_0V_2, V_1V_2 by pythagorean theorem

$$V_0V_1 = \sqrt{(a-c)^2+(b-d)^2} \quad V_1V_2 = \sqrt{(f-d)^2+(e-c)^2}$$

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$$V_0V_2 = \sqrt{(e-a)^2+(b-f)^2}$$

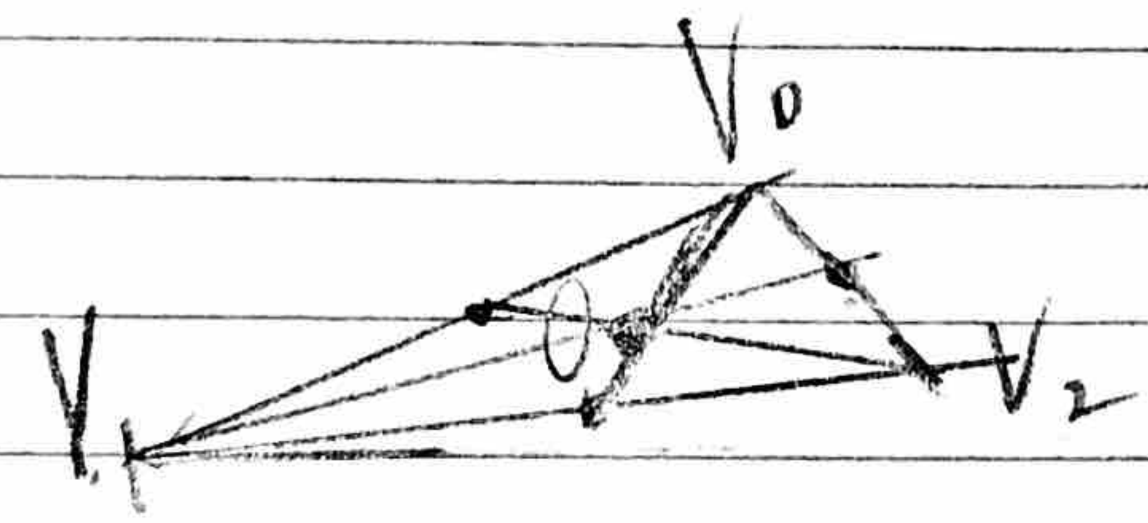
Then we can use Heron's Formula

$S = \frac{1}{2}(a+b+c)$; Area $= \sqrt{S(S-a)(S-b)(S-c)}$

$S = \frac{1}{2}(V_0V_1 + V_1V_2 + V_0V_2)$

Area $= \sqrt{S(S - V_0V_1)(S - V_1V_2)(S - V_0V_2)}$

Centroid:



Given the coordinate of the three vertices of a triangle $V_0V_1V_2$, the centroid O coordinates are given by

$$O_x = \frac{V_{0x} + V_{1x} + V_{2x}}{3} \qquad O_y = \frac{V_{0y} + V_{1y} + V_{2y}}{3}$$