ASSIGNMENT 3

P4-10 In this problem, we investigate properties of e^S when $S \in \underbrace{so(3)}_{\text{the set of all } 3\times 3 \text{ skew-symmetric matrices}}$.

$$S \in so(3) \Longrightarrow S^T = -S. \quad (1)$$

$$(1) \Longrightarrow SS^T = S^TS = -S^2. \quad (2)$$

Note: $e^A e^B = e^B e^A = e^{A+B}$ iff $AB = BA$.

$$(\forall S \in so(3))\, e^S (e^S)^T = e^S e^{S^T} \underset{SS^T = S^TS}{=} e^{(S+S^T)} \underset{(1)}{=} e^0 = I. \quad (3)$$

Note: $e^S = \sum_{k=0}^{\infty} \frac{S^k}{k!} \Longrightarrow e^{-S} = \sum_{k=0}^{\infty} \frac{(-S)^k}{k!} \overset{(1)}{=} \sum_{k=0}^{\infty} \frac{(S^T)^k}{k!} = e^{S^T}. \quad (4)$

We can re-verify (3) using (4): $e^S (e^S)^T = e^S e^{S^T} \underset{(4)}{=} e^S e^{-S} = I.$

Finally we verify that $\det(e^S) = 1$.

Fact: $(\forall A \in \mathbb{R}^{n\times n})\ \det(e^A) = e^{tr(A)}$

Thus,

$$(\forall S \in so(3))\ \det(e^S) = e^{tr(S)} = e^0 = 1. \quad (4)$$

$$(3)\ \&\ (4) \Longrightarrow \boxed{(\forall S \in so(3))\ e^S \in SO(3)}$$

$\boxed{\text{P4-11}}$ Prove that $R_{k,\theta} = e^{S(k)\theta}$.

$\boxed{\text{Lemma:}}$ P4-25: $\boxed{R_{k,\theta} = I + S(k)\sin(\theta) + S^2(k)\underbrace{\text{vers}(\theta)}_{1-\cos\theta}}$

*** Proof of Lemma:**

$$I + S(k)\sin(\theta) + S^2(k)\,\text{vers}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -k_z s\theta & k_y s\theta \\ k_z s\theta & 0 & -k_x s\theta \\ -k_y s\theta & k_x s\theta & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} (-k_z^2 - k_y^2)v\theta & k_x k_y v_\theta & k_x k_z v\theta \\ k_x k_y v\theta & (-k_z^2 - k_x^2)v\theta & k_y k_z v\theta \\ k_x k_z v\theta & k_y k_z v\theta & (-k_y^2 - k_x^2)v\theta \end{bmatrix} \underset{k_x^2 + k_y^2 + k_z^2 = 1}{=} R_{k,\theta}.$$

$\boxed{\text{Note:}}$ $\underbrace{S^3(k) = -S(k)}_{\text{"verify by direct multiplication"}} \Longrightarrow \begin{cases} S^{2n+1}(k) = (-1)^n S(k), & n \geq 0 \\ S^{2n+2}(k) = (-1)^n S^2(k), & n \geq 0 \end{cases}$

* (Lemma) $\Longrightarrow R_{k,\theta} = I + S(k)\sin\theta + S^2(k)(1-\cos\theta) = I + S(k)\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$

$$+ S^2(k)\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+2}}{(2n+2)!} \overset{\text{Note}}{=} I + \sum_{n=0}^{\infty} \frac{(-1)^n S^{2n+1}(k)(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n S^{2n+2}(k)(-1)^n \theta^{2n+2}}{(2n+2)!} = I + \sum_{n=1}^{\infty} \frac{(S(k)\theta)^n}{n!} =$$

$$\sum_{n=0}^{\infty} \frac{(S(k)\theta)^n}{n!} = e^{S(k)\theta}. \Longrightarrow \boxed{R_{k,\theta} = e^{S(k)\theta}}$$

$\boxed{P4\text{-}13}$ Given $R = R_{z,\phi} R_{y,\theta} R_{z,\psi}$, show that $\frac{d}{dt}R = S(\omega)R$ with

$$\omega = \{c\phi s_\theta \dot\phi - s\phi\dot\theta\}i + \{s\phi s\theta\dot\theta + c\phi\dot\theta\}j + \{c\theta\dot\psi + \dot\phi\}k.$$

$$R = R_{z,\phi} R_{y,\theta} R_{z,\psi} \quad (*)$$

$\boxed{\text{Note:}}$ $\frac{d}{d\theta} R_{k,\theta} = S(k) R_{k,\theta}$, $\dot R = \frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = S(k)R_{k,\theta}\dot\theta = S(\dot\theta k)R_{k,\theta}$

$$\frac{d}{dt}(*) \Rightarrow \dot R = \dot R_{z,\phi} R_{y,\theta} R_{z,\psi} + R_{z,\phi}\dot R_{y,\theta} R_{z,\psi} + R_{z,\phi} R_{y,\theta}\dot R_{z,\psi} =$$

$$[S(\dot\phi k)R_{z,\phi}] R_{y,\theta} R_{z,\psi} + R_{z,\phi}[S(\dot\theta j)R_{y,\theta}]R_{z,\psi} + R_{z,\phi}R_{y,\theta}[S(\dot\psi k)R_{z,\psi}]. \quad (1)$$

Now, we use $\begin{cases} R^T_{z,\phi}R_{z,\phi} = I \\ (R_{z,\phi}R_{y,\theta})^T R_{z,\phi}R_{y,\theta} = I \end{cases}$ in (1) to get:

$$\dot R = S(\dot\phi k)R + R_{z,\phi}\, S(\dot\theta j) R^T_{z,\phi}\, R_{z,\phi}R_{y,\theta}R_{z,\psi} +$$

$$R_{z,\phi}R_{y,\theta}\, S(\dot\psi k)\,(R_{z,\phi}R_{y,\theta})^T R_{z,\phi}R_{y,\theta}R_{z,\phi} \overset{S(Ra) = RS(a)R^T}{=\!=}$$

$$S(\dot\phi k)R + S(R_{z,\phi}\dot\theta j)R + S(R_{z,\phi}R_{y,\theta}\dot\psi k)R =$$

$$S(\dot\phi k + R_{z,\phi}\dot\theta j + R_{z,\phi}R_{y,\theta}\dot\psi k)R = S(\omega)R. \quad (2)$$

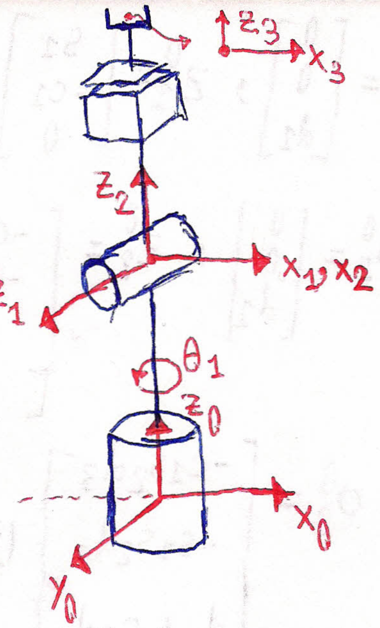
Thus,

$$\omega = \dot\phi k + R_{z,\phi}\dot\theta j + R_{z,\phi}R_{y,\theta}\dot\psi k = \begin{bmatrix}0\\0\\\dot\phi\end{bmatrix} + \begin{bmatrix}c\phi & -s\phi & 0\\ s\phi & c\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\\dot\theta\\0\end{bmatrix}$$

$$+ \begin{bmatrix}c\phi c\theta & -s\phi & c\phi s\theta\\ c\theta s\phi & c\phi & s\phi s\theta\\ -s\theta & 0 & c\theta\end{bmatrix}\begin{bmatrix}0\\0\\\dot\psi\end{bmatrix}$$

$$\Rightarrow \boxed{\omega = (c\phi s\theta\dot\psi - s\phi\dot\theta)i + (s\phi s\theta\dot\psi + c\phi\dot\theta)j + (c\theta\dot\psi + \dot\phi)k}$$

P4-18



DH Table

link	a_i	α_i	d_i	θ_i
1	0	$\frac{\pi}{2}$	d_1	θ_1
2	0	$\frac{\pi}{2}$	0	θ_2
3	0	0	d_3	0

revolute joints   Prismatic joint

$$J_{11} = \begin{bmatrix} z_0^0 \times (O_3^0 - O_0^0) & z_1^0 \times (O_3^0 - O_1^0) & z_2^0 \\ z_0^0 & z_1^0 & 0 \end{bmatrix} \quad (*)$$

Recall: $z_i^0 =$ third column of R_i^0, $H_i^0 = \begin{bmatrix} R_i^0 & O_i^0 \\ 0 & 1 \end{bmatrix}$, $A_i = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = A_1 \Rightarrow \quad o_1^0 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}, \quad z_1^0 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix} \qquad (1)$$

$$H_2^0 = A_1 A_2 \Rightarrow \quad o_2^0 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}, \quad z_2^0 = \begin{bmatrix} -c_1 s_2 \\ -s_1 s_2 \\ c_2 \end{bmatrix} \qquad (2)$$

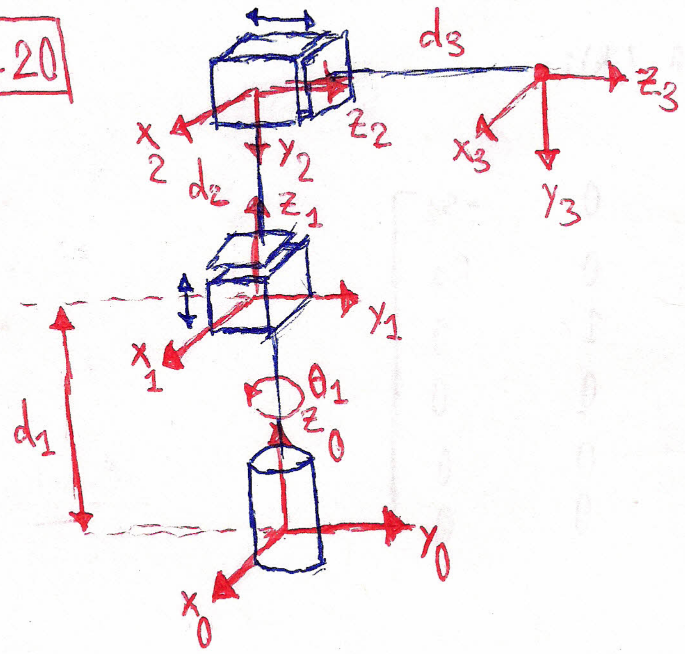
$$H_3^0 = A_1 A_2 A_3 \Rightarrow \quad o_3^0 = \begin{bmatrix} -c_1 s_2 d_3 \\ -s_1 s_2 d_3 \\ d_1 + c_2 d_3 \end{bmatrix} \qquad (3)$$

Also, $o_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad z_0^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (4)$

Replacing (1), (2), (3), (4) in (*)

$$J_{11} = \begin{bmatrix} s_1 s_2 d_2 & -c_1 c_2 d_2 & -s_2 c_1 \\ c_1 s_2 d_2 & s_1 c_2 d_2 & -s_2 s_1 \\ 0 & s_2 d_2 & c_2 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

P4-20



DH Table

link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1
2	0	$-\pi/2$	d_2	0
3	0	0	d_3	0

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

revolute joint

prismatic joints

$$J = \begin{bmatrix} z_0^0 \times (O_3^0 - O_0^0) & z_1^0 & z_2^0 \\ z_0^0 & 0 & 0 \end{bmatrix} \quad (*)$$

$$O_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad z_0^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad z_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (1)$$

$$H_2^0 = A_1 A_2 \Rightarrow z_2^0 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad (2)$$

$$H_3^0 = A_1 A_2 A_3 \Rightarrow z_3^0 = \begin{bmatrix} -s_1 d_3 \\ c_1 d_3 \\ d_1 + d_2 \end{bmatrix} \quad (3)$$

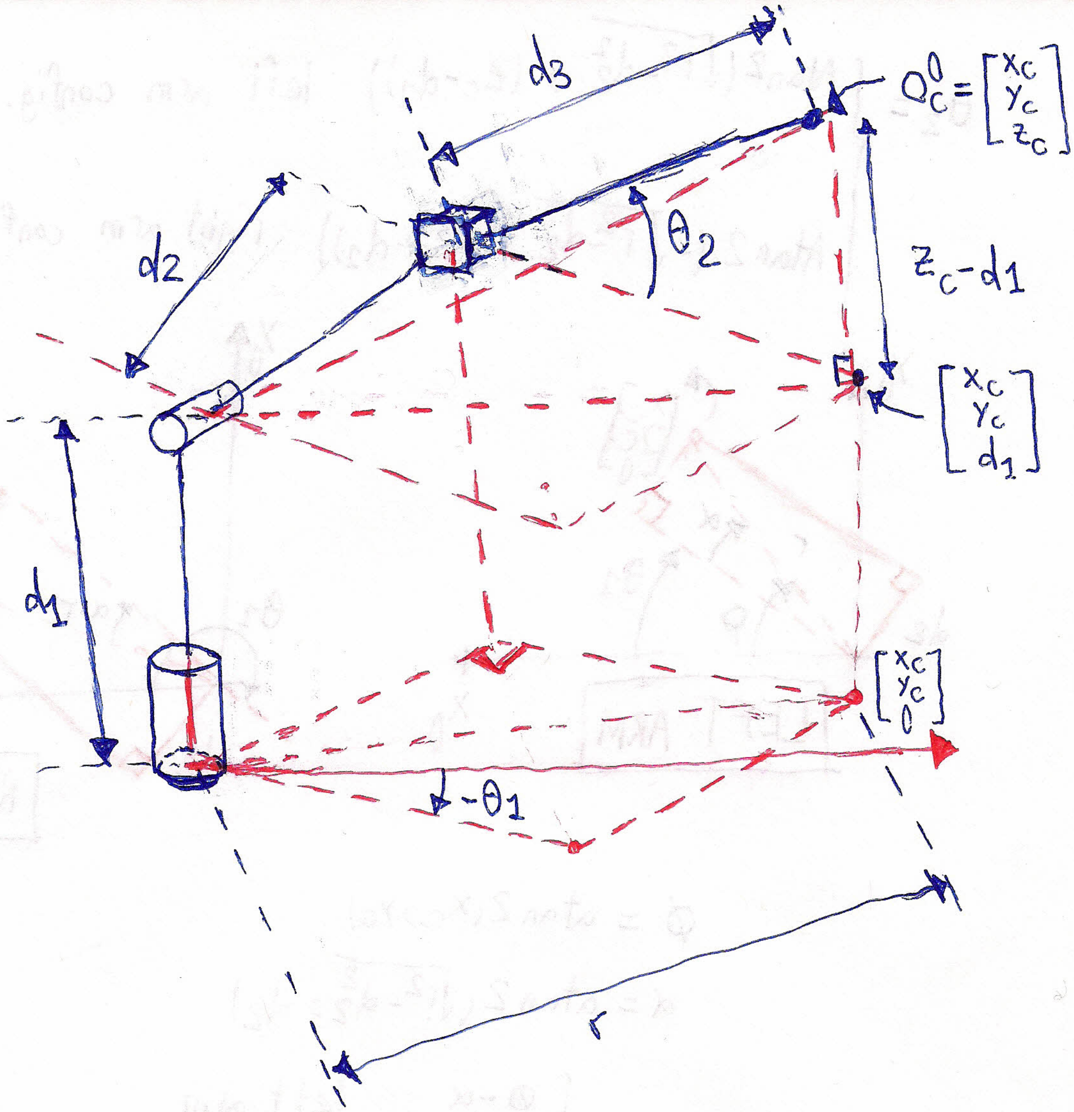
Replacing (1), (2) and (3) in (*):

$$J = \begin{bmatrix} -c_1 d_3 & 0 & -s_1 \\ -s_1 d_3 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Note that $\det(J(1{:}3, 1{:}3)) = c_1^2 d_3 + s_1^2 d_3 = d_3$

thus, we need $d_3 \neq 0$.

P3.18



□ Finding the desired coordinates of the wrist center:

$$O_c^0 = d - R\begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix} \quad (1)$$

□ inverse position kinematics:

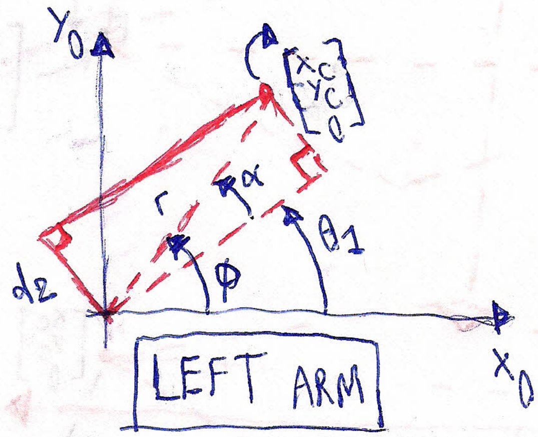
$$r^2 := x_c^2 + y_c^2 \quad (2)$$

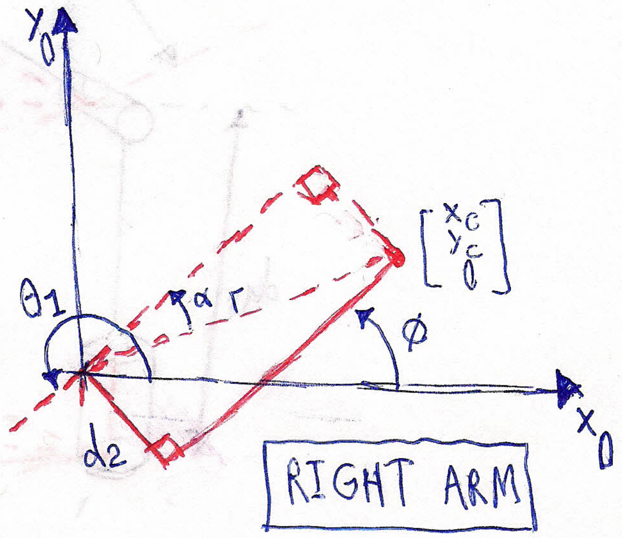
$$\ell^2 := d_2^2 + d_3^2 \quad (3)$$

$$\ell^2 = (z_c - d_1)^2 + r^2 \quad (4)$$

$$(2) \,\&\, (3) \,\&\, (4) \Rightarrow d_3 = \sqrt{(z_c - d_1)^2 + x_c^2 + y_c^2 - d_2^2} \quad (5)$$

$$\theta_2 = \begin{cases} \operatorname{Atan2}\left(\sqrt{r^2-d_2^2},\ (z_c-d_1)\right) & \text{left arm config.} \\ \operatorname{Atan2}\left(-\sqrt{r^2-d_2^2},\ (z_c-d_1)\right) & \text{right arm config.} \end{cases}$$





$$\phi = \operatorname{atan2}(x_c, y_c)$$

$$\alpha = \operatorname{atan2}\left(\sqrt{r^2-d_2^2},\ d_2\right)$$

$$\theta_1 = \begin{cases} \phi - \alpha & \text{left arm} \\ \phi + \alpha + \pi & \text{right arm} \end{cases}$$

☐ inverse orientation kinematics:

$$R_3^0 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 \\ s_1c_2 & c_1 & s_1s_2 \\ -s_2 & 0 & c_2 \end{bmatrix}$$

$$R_6^3 = (R_3^0)^T R = \begin{bmatrix} c_1c_2r_{11}+s_1c_2r_{21}-s_2r_{31} & c_1c_2r_{12}+s_1c_2r_{22}-s_2r_{32} & \mathrm{I} \\ -s_1r_{11}+c_1r_{21} & -s_1r_{13}+c_1r_{23} & \mathrm{II} \\ c_1s_2r_{11}+s_1s_2r_{21}+c_2r_{31} & c_1s_2r_{13}+s_1s_2r_{23}+c_2r_{23} & \mathrm{III} \end{bmatrix}$$

$\mathrm{I} = c_1c_2r_{13}+s_1s_2r_{32}-s_2r_{33}$

$\mathrm{III} = c_1s_2r_{13}+s_1s_2r_{23}+c_2r_{33}$

$\mathrm{II} = -s_1r_{13}+c_1r_{23}$

$$c_4 s_5 = s_1 c_1 r_{13} + s_1 c_2 r_{32} - s_2 r_{33} \quad (1)$$

$$s_4 s_5 = s_1 r_{13} + c_1 r_{23} \quad (2)$$

$$c_5 = c_1 s_2 r_{13} + s_1 s_2 r_{23} + c_2 r_{33} \quad (3)$$

If (1) & (2) are not both zero:

$$s_5 = \pm\sqrt{1 - (3)^2}$$

If $s_5 > 0$

$$\theta_5 = \text{Atan2}\left((3), \sqrt{1-(3)^2}\right)$$

$$\theta_4 = \text{Atan2}\left((1), (2)\right)$$

$$\theta_6 = \text{Atan2}\left(-(4), (5)\right)$$

where $(4) = c_1 s_2 r_{11} + s_1 s_2 r_{21} + c_2 r_{31}$

$(5) = c_1 s_2 r_{12} + s_1 s_2 r_{22} + c_2 r_{32}$

If $s_5 < 0$

$$\theta_5 = \text{Atan2}\left((3), -\sqrt{1-(3)^2}\right)$$

$$\theta_4 = \text{Atan2}\left(-(1), -(2)\right)$$

$$\theta_6 = \text{Atan2}\left((4), (5)\right)$$

If (1) & (2) are both zero $\rightarrow$ we have infinitely many solutions.