ECE 470

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1(a)

	a_i	α_i	d_i	θ_i
1	0	$-\frac{\pi}{2}$	x	0
2	a_2	0	0	Φ

*

$\begin{bmatrix} x \\ \Phi \end{bmatrix} = \begin{bmatrix} d_1 \\ \theta_2 \end{bmatrix}$

$l = l_2$

$$r_{c2} = \begin{bmatrix} l\cos\Phi \\ 0 \\ x - l\sin\Phi \end{bmatrix} \qquad \dot r_{c2} = \begin{bmatrix} -l\sin\Phi\,\dot\Phi \\ 0 \\ \dot x - l\cos\Phi\,\dot\Phi \end{bmatrix}$$

$$w_2^0 = \dot\Phi\, y_0 = \dot\Phi\, z_1^0 = \dot\Phi\, z_2^0 = \begin{bmatrix} 0 \\ \dot\Phi \\ 0 \end{bmatrix}$$

$$\frac{1}{2} w_{0,2}^{2} I_2\, w_{0,2}^{2} = \frac{1}{2} I_{zz}\dot\Phi^2$$

$$T = T_1 + T_2 = \frac{1}{2}m_1\dot x^2 + \frac{1}{2}m_2\left[(l\dot\Phi\sin\Phi)^2 + (\dot x - l\cos\Phi\,\dot\Phi)^2\right]$$

$$+\frac{1}{2}I_{zz}\dot\Phi^2 = \frac{1}{2}(m_1+m_2)\dot x^2 + \frac{1}{2}(m_2l^2 + I_2)\dot\Phi^2 - m_2\dot x\, l\cos\Phi\,\dot\Phi$$

$$U(q) = m_2 l g\cos\Phi$$

$$\mathcal{L}(q,\dot q) = T - U =$$

$$\frac{d}{dt}\nabla_{\dot q}\mathcal{L} - \nabla_q\mathcal{L} = \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\frac{\partial\mathcal{L}}{\partial\dot q_1} = \frac{\partial\mathcal{L}}{\partial\dot x} = m_1\dot x - m_2 l\cos\Phi\,\ddot\Phi + m_2 l\sin\Phi\,\dot\Phi^2$$

$$\frac{\partial\mathcal{L}}{\partial\dot q_2} = \frac{\partial\mathcal{L}}{\partial\dot\Phi} = (m_2l^2 + I_z)\dot\Phi - m_2 l\dot x\cos\Phi$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\Phi} = (m_2l^2 + I_2)\ddot\Phi - m_2 l\ddot x\cos\Phi + m_2 l\dot x\sin\Phi\,\dot\Phi$$

$$\frac{\partial\mathcal{L}}{\partial\Phi} = m_2 l\sin\Phi\,\dot\Phi\dot x + m_2 l g\sin\Phi$$

$$(m_2l^2 + I_z)\ddot\Phi - m_2 l\cos\Phi\,\ddot x + \cancel{m_2 l\dot x\sin\Phi\,\dot\Phi} - \cancel{m_2 l\sin\Phi\,\dot\Phi\dot x}$$

$$-m_2 l g\sin\Phi = 0$$

$$D(q)\ddot q + C(q,\dot q)\dot q + g(q) = \tau$$

$$D(q) = \begin{bmatrix} m_1 & -m_2 l_2\cos\theta_2 \\ -m_2 l_2\cos\theta_2 & I_2 + m_2 l_2^2 \end{bmatrix}$$

$$C(q,\dot q) = \begin{bmatrix} 0 & m_2 l_2 \sin\theta_2 \dot\theta_2 \\ 0 & 0 \end{bmatrix} \quad g(q) = \begin{bmatrix} 0 \\ -m_2 l_2 g \sin\theta_2 \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

iv, None of the above

(b) $\ddot{\tilde q} + k_d \dot{\tilde q} + k_p \tilde q = 0$

$$\tilde q = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$

$$\lambda_1 = -5 \quad \lambda_2 = -10$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = (\lambda_1 + 5)(\lambda + 10) = \lambda^2 + \underset{k_d}{15}\lambda + \underset{k_p}{50}$$

iii $k_p > 0 \quad k_d > 10$

(c) $D(q)\ddot q + C\dot q + \nabla_q V = \begin{bmatrix} M\ddot d_1 - m_2 l \cos\theta_2 \ddot\theta_2 + m_2 l \sin\theta_2 \dot\theta_2^2 \\ -m_2 l \cos\theta_2 \ddot\theta_2 + (I + m_2 l^2)\ddot\theta_2 + m_2 l g \sin\theta_2 \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} \ddot q_1 & \cos q_2 \ddot q_2 + \sin q_2 \dot q_2^2 & 0 \\ 0 & -\cos q_2 \ddot q_1 - g \sin q_2 & \ddot q_2 \end{bmatrix}}_{Y(q,\dot q,\ddot q)} \underbrace{\begin{bmatrix} M \\ m_2 l \\ I_2 + m_2 l_2^2 \end{bmatrix}}_{P}$$

iv: None of the above

(d) $D(q) = \begin{bmatrix} \hat{M} & -\hat{m}_2 \hat{l} \cos q_2 \\ -\hat{m}_2 \hat{l} \cos q_2 & \hat{I} + \hat{m}_2 \hat{l}_2^2 \end{bmatrix}$; $C = \begin{bmatrix} 0 & \hat{m}_2 \hat{l}_2 \sin q_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix}$

$\nabla_q U = \begin{bmatrix} 0 \\ -\hat{m}_2 \hat{l} g \sin q_2 \end{bmatrix}$; $\hat{p} = \begin{bmatrix} \hat{M} \\ \hat{m}_2 l_2 \\ \hat{I} + \hat{m}_2 \hat{l}_2^2 \end{bmatrix}$

$Y(q, \dot{q}, a, v) = \begin{bmatrix} a_1 & -\cos q_2 a_2 + \sin q_2 \dot{q}_2 v_2 & 0 \\ 0 & -\cos q_2 a_1 - g \sin q_2 & a_2 \end{bmatrix}$

iv. None of the above

$$r = \begin{bmatrix} f(z) \\ h(z) \\ z \end{bmatrix} \Rightarrow \dot{r} = \begin{bmatrix} f'(z)\,\dot{z} \\ h'(z)\,\dot{z} \\ \dot{z} \end{bmatrix}$$

2. (a) $L(q,\dot{q}) = T(q,\dot{q}) - U(q)$ $\quad m = 1\text{kg}$

$T(q,\dot{q}) = \frac{1}{2} m \|\dot{r}\|^2 = \frac{1}{2}\left((f'(z)\dot{z})^2 + (h'(z)\dot{z})^2 + \dot{z}^2 \right)$

iv. None of the above

$U(q) = mgy = g z$

$L(q,\dot{q}) = T(q,\dot{q}) - U(q) = \frac{1}{2} m \left(\left(\frac{\partial f}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2 + 1 \right) \dot{z}^2 - mgz$

(b) EL: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau = 0$

*m = 1kg
∴ Omit m here

$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{z}} = f'(z)\dot{z} + h'(z)\dot{z} + \dot{z}$

$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial z} = f'(z) f''(z) \dot{z} + h'(z) h''(z) \dot{z} - g$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = f''(z)\dot{z}^2 + f'(z)\ddot{z} + h''(z)\dot{z}^2 + h'(z)\ddot{z} + \ddot{z}$

EoM:

$f''(z)\dot{z}^2 + f'(z)\ddot{z} + h''(z)\dot{z}^2 + h'(z)\ddot{z} + \ddot{z} - f'(z)f''(z)\dot{z}$
$- h'(z)h''(z)\dot{z} + g = 0$

$T(q,\dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q}$

$\Rightarrow D = m\left(\left(\frac{\partial f}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2 + 1 \right)$

$C(q,\dot{q}) = \left(-f'(z) f''(z)\dot{z} - h'(z) h''(z)\dot{z} \right) m$

$= -m\left(\frac{\partial f}{\partial z}(z) \frac{\partial^2 f}{\partial^2 z}(z) + \frac{\partial h}{\partial z}(z) \frac{\partial^2 h}{\partial^2 z}(z) \right)\dot{z}$

$g(q) = -mg$

iv. None of the above

(c) $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = 1$

(d) $C(z) = (\cos(z), \sin(z), z)$

$D(q) = m\left((-\sin z)^2 + (\cos(z))^2 + 1\right)$

$= 2m$

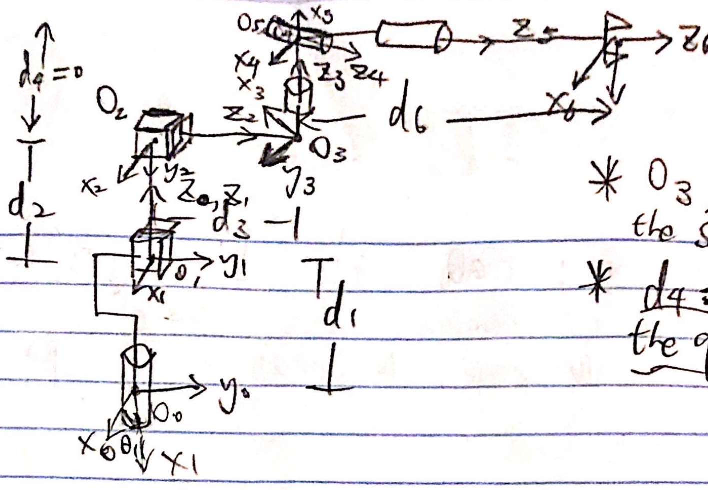
$C(q,\dot{q}) = -m\left((-\sin z)(-\cos z)^2 + (\cos z)(-\sin z)^2\right)\dot{z}$

$g(q) = -mg$

Eom $m\ddot{z} - m(-\sin z\cos^2 z + \cos z \sin^2 z)\dot{z}^2 - mgz = 1$

(d) None of the above

3. (a)



* O_3, O_4, O_5 are at the same position

* $d_4 = 0$ as the question given

LINK	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1
2	0	$-\frac{\pi}{2}$	d_2	0
3	0	$\frac{\pi}{2}$	d_3	0
4	0	$-\frac{\pi}{2}$	0	θ_4
5	0	$\frac{\pi}{2}$	0	θ_5
6	0	0	d_6	θ_6

IV, None of the above

(b) $$H_1^0 = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & 0 \\ s_{\theta_1} & c_{\theta_1} & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = \begin{bmatrix} c_{\theta_1} & 0 & -s_{\theta_1} & 0 \\ s_{\theta_1} & 0 & c_{\theta_1} & 0 \\ 0 & -1 & 0 & d_2+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & -s_{\theta_1} d_3 \\ s_{\theta_1} & c_{\theta_1} & 0 & d_3 c_{\theta_1} \\ 0 & 0 & 1 & d_2+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H_4^3 = \begin{bmatrix} c_{\theta_4} & 0 & -s_{\theta_4} & 0 \\ s_{\theta_4} & 0 & c_{\theta_4} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_5^4 = \begin{bmatrix} C_{\theta_5} & 0 & S_{\theta_5} & 0 \\ S_{\theta_5} & 0 & -C_{\theta_5} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H_6^5 = \begin{bmatrix} C_{\theta_6} & -S_{\theta_6} & 0 & 0 \\ S_{\theta_6} & C_{\theta_6} & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_6^3 = H_4^3 H_5^4 H_6^5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} C_{\theta_4} C_{\theta_5} C_{\theta_6} - S_{\theta_4} S_{\theta_6} & -C_{\theta_4} C_{\theta_5} S_{\theta_6} - S_{\theta_4} C_{\theta_6} & C_{\theta_4} S_{\theta_5} & C_{\theta_4} S_{\theta_5} d_6 \\ S_{\theta_4} C_{\theta_5} C_{\theta_6} + C_{\theta_4} S_{\theta_6} & -S_{\theta_4} C_{\theta_5} S_{\theta_6} + C_{\theta_4} C_{\theta_6} & S_{\theta_4} S_{\theta_5} & S_{\theta_4} S_{\theta_5} d_6 \\ -S_{\theta_5} C_{\theta_6} & S_{\theta_5} S_{\theta_6} & C_{\theta_5} & C_{\theta_5} d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_6^0 = H_3^0 H_6^3 =$$

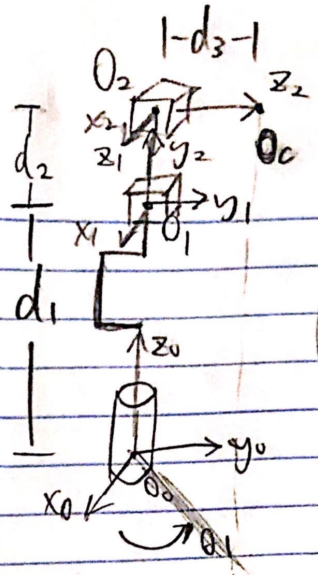
$$\begin{bmatrix} a_{11} C_{\theta_1} - S_{\theta_1} a_{21} - a_{41} S_{\theta_1} d_3 & a_{12} C_{\theta_1} - S_{\theta_1} a_{22} - a_{42} S_{\theta_1} d_3 & a_{13} C_{\theta_1} - S_{\theta_1} a_{23} - a_{43} S_{\theta_1} d_3 & a_{14} C_{\theta_1} - S_{\theta_1} a_{24} - a_{44} S_{\theta_1} \\ a_{11} S_{\theta_1} + a_{21} C_{\theta_1} + a_{41} d_3 C_{\theta_1} & a_{12} S_{\theta_1} + a_{22} C_{\theta_1} + a_{42} d_3 C_{\theta_1} & a_{13} S_{\theta_1} + a_{23} C_{\theta_1} + a_{43} d_3 \theta_1 & a_{14} S_{\theta_1} + a_{24} C_{\theta_1} + a_{44} d_3 \theta_1 \\ a_{31} + d_{41}(d_1 + d_2) & a_{32} + d_{42}(d_1 + d_2) & a_{33} + d_{43}(d_1 + d_2) & a_{34} + d_{44}(d_1 + d_2) \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$H_{11} = (C_4 C_5 C_6 - S_4 S_6) C_1 - S_1 (-C_4 C_5 S_6 - S_4 C_6)$$

$$= C_1 (C_4 C_5 C_6 - S_4 S_6) + S_1 C_4 C_5 S_6 + S_1 S_4 C_6$$

iv : None of the above

(c)



$$\theta_1 = aTan2(y_c, x_c) = aTan2(1,2)$$
$$d_2 = z_c - d_1 = 10 - d_6 - d_1$$
$$d_3 = \sqrt{x_c^2 + y_c^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$O_c^0(\theta_1, d_2, d_3) = O_d - R_d \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix}$$
$$= \begin{bmatrix} aTan2(1,2) \\ 1 \\ 10 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix} = \begin{bmatrix} aTan2(1,2) \\ 1 \\ 10 - d_6 \end{bmatrix}$$

$$R_d = R_6^0 = R_3^0 R_6^3$$
$$R_6^3(\theta_4, \theta_5, \theta_6) = R_3^{0T}(\theta_1, d_2, d_3) R_d \in SO(3)$$

H_6^3 From (a) $\rightarrow$ extract R_6^3

$\theta_4 = \Phi$ In the ZYZ Representation

$\theta_5 = \theta$

$\theta_6 = \psi$

If $(\theta_{13}^2 + \theta_{23}^2)^{\frac{1}{2}} \neq 0$

$$\theta_4 = aTan2(a_{23}, a_{13}) = aTan2(0,0)$$
$$\theta_5 = aTan2(\sqrt{1 - a_{33}^2}, a_{33}) = aTan2(0,0)$$
$$\theta_6 = aTan2(a_{32}, -a_{31}) = aTan2(0,0)$$

$$\therefore R_3^{0T} = \begin{bmatrix} C\theta_1 & S\theta_1 & 0 \\ -S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore R_6^3 = R_3^{0T} = \begin{bmatrix} C_2 & S_2 & 0 \\ -S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$* C_2 = \cos(aTan2(1,2))$

$* S_2 = \sin(aTan2(1,2))$

$$\boxed{\therefore d_3 = \sqrt{5} ;\ \theta_5 = 0}$$