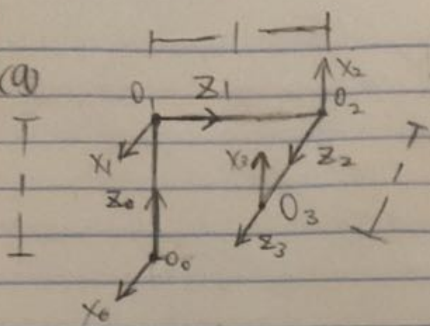
ECE 470 HW4

7-7 (a)



$$O_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad O_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad O_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

(b) $D(q) = \sum_{i=1}^{n} m_i J_{v_i}^T J_{v_i} + J_{w_i}^T I J_{w_i}$

$J_{v_1} = [\, z_0^0 \quad 0 \quad 0 \,] \qquad z_0^0 = [0\,0\,1]^T$

$J_{v_2} = [\, z_0^0 \quad z_1^0 \quad 0 \,] \qquad z_1^0 = [0\,1\,0]^T$

$J_{v_3} = [\, z_0^0 \quad z_1^0 \quad z_2^0 \,] \qquad z_2^0 = [1\,0\,0]^T$

$J_{w_1} = J_{w_2} = J_{w_3} = 0$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) The inertia matrix (D) is constant; therefore all christoffel symbols are zero. There is no centrifugal or coriolis effect for this dynamic system.

(d) $g(q) = g(m_1 + m_2 + m_3)\, q_1$

$$\frac{\partial g(q)}{q_1} = 3g \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} 3g \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\frac{\partial g(q)}{q_2} = 0$$

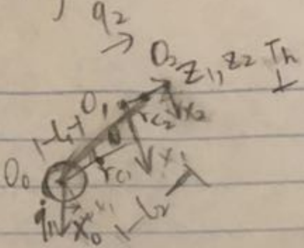
$$\frac{\partial g(q)}{q_3} = 0$$

$\theta = \tan^{-1}\left(\frac{h}{l_1+q_2}\right)$

$l_2 = \left((l_1+q_2)^2 + h^2\right)^{\frac{1}{2}}$

$r_{c2} = \begin{bmatrix} -l_2 \sin(q_1+\theta) \\ l_2 \cos(q_1+\theta) \\ 0 \end{bmatrix}$

7-8



$r_{c1} = \begin{bmatrix} -l_1 \sin q_1 \\ l_1 \cos q_1 \\ 0 \end{bmatrix}$ $\quad \dot{r}_{c1} = \begin{bmatrix} -l_1 \cos q_1 \dot{q}_1 \\ -l_1 \sin q_1 \dot{q}_1 \\ 0 \end{bmatrix}$

$T = T_1 + T_2 = \frac{1}{2} m_1 \dot{r}_{c1}^T \dot{r}_{c1} + \frac{1}{2} I_{zz_1} \dot{q}_1^2 + \frac{1}{2} m_2 \dot{r}_{c2}^T \dot{r}_{c2} + \frac{1}{2} I_{zz_2} (\dot{q}_1 + \dot{\theta})^2$

$U = -m_1 l_1 g \sin q_1 - m_2 l_2 g \sin(q_1+\theta)$

$\mathcal{L} = T - U$

$\dot{r}_{c2} = \begin{bmatrix} -\frac{1}{2}\left((l_1+q_2)^2+h^2\right)^{-\frac{1}{2}} 2 q_2 \dot{q}_2 \sin(q_1+\theta) - \left((l_1+q_2)^2+h^2\right)^{\frac{1}{2}} \cos(q_1+\theta) \dot{q}_1 \\ \frac{1}{2}\left((l_1+q_2)^2+h^2\right)^{-\frac{1}{2}} 2 q_2 \dot{q}_2 \cos(q_1+\theta) - \left((l_1+q_2)^2+h^2\right)^{\frac{1}{2}} \sin(q_1+\theta) \dot{q}_1 \\ 0 \end{bmatrix}$

$T = \frac{1}{2} m_1 l_1^2 + \frac{1}{2} \frac{m_1 l_1^2}{2} \dot{q}_1^2 + \frac{1}{2} m_2 \left(\left(\frac{q_2 \dot{q}_2}{l_2}\right)^2 + l_2 \dot{q}_1^2 \right) + \frac{1}{2} \frac{m_2 l_2^2}{2} \dot{q}_1^2$

$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{m_1 l_1^2}{2} \dot{q}_1 + m_2 l_2 \dot{q}_1 + \frac{m_2 l_2^2}{2} \dot{q}_1 = \frac{1}{2} m_1 l_1^2 \dot{q}_1 + \frac{3 m_2 l_2}{2} \dot{q}_1$

$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{1}{2} m_1 l_1^2 \ddot{q}_1 + \frac{3}{2} m_2 l_2 \ddot{q}_1$

$\frac{\partial \mathcal{L}}{q_1} = -m_1 l_1 g \cos q_1 - m_2 l_2 g \cos(q_1+\theta)$

$\Rightarrow \frac{1}{2} m_1 h^2 \ddot{q}_1 + \frac{3}{2} m_2 l_2 \ddot{q}_1 + m_1 l_1 g \cos q_1 + m_2 l_2 g \cos(q_1+\theta) = u_1$

$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = m_2 \frac{q_2 \dot{q}_2}{l_2}$

$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \frac{m_2 \dot{q}_2^2}{l_2} + m_2 \frac{q_2 \ddot{q}_2}{l_2}$

$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{m_2}{l_2} \dot{q}_2$

$\Rightarrow \frac{m_2 \dot{q}_2^2}{l_2} + \frac{m_2 q_2 \ddot{q}_2}{l_2} - \frac{m_2}{l_2} \dot{q}_2 = u_2$

7-12

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{ij}^{n} d_{ij}\dot{q}_i\dot{q}_j$$

$$p_k = \frac{\partial K}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}\dot{q}_j$$

$$\sum_{k=1}^{n}\dot{q}_k p_k = \sum_{k=1}^{n}\dot{q}_k\sum_{j=1}^{n} d_{kj}\dot{q}_j = \sum_{k=1}^{n}\sum_{j=1}^{n} d_{kj}\dot{q}_k\dot{q}_j$$

$$= \sum_{kj}^{n} d_{kj}\dot{q}_k\dot{q}_j$$

$$= 2K$$

7-13 (a) from 7-12, $\underbrace{\sum_{k=1}^{n} \dot{q}_k p_k = 2K}$ & $L = K - V$

$$H = \sum_{k=1}^{n} \dot{q}_k p_k - L = 2K - (K - V) = \underbrace{K + V}$$

(b) $H = \sum_{k=1}^{n} \dot{q}_k p_k - L$

$$\Rightarrow \frac{\partial H}{\partial p_k} = \dot{q}_k - \frac{\partial L}{\partial p_k} \qquad \frac{\partial L(q,\dot{q})}{\partial p_k} = 0$$

$$= \underbrace{\dot{q}_k}$$

$$\Rightarrow \frac{\partial H}{\partial q_k} = \frac{\partial}{\partial q_k} \sum_{k=1}^{n} \dot{q}_k p_k - \frac{\partial L}{\partial q_k}$$

Recall Lagrange: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k \Rightarrow \frac{\partial L}{\partial q_k} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \tau_k$

$$0 = -\frac{\partial H}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \tau_k$$

$\Rightarrow$ prove $\underbrace{\dot{p}_k = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}}$

from 7-12 $p_k = \frac{\partial L}{\partial \dot{q}_k}$

$$\therefore \dot{p}_k = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$$

$$\therefore \begin{cases} \dot{q}_k = \frac{\partial H}{\partial p_k} \\ \dot{p}_k = -\frac{\partial H}{\partial q_k} + \tau_k \end{cases}$$

(c)

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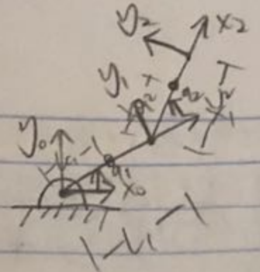

$$T = \frac{1}{2} m_1 \|\dot{r}_{c1}\|^2 + \frac{1}{2} m_2 \|\dot{r}_{c2}\|^2 + \frac{1}{2} w_1^{0\,T} I_1 w_1^0 + \frac{1}{2} w_2^{0\,T} I_2 w_2^0$$

$$r_{c1} = \begin{bmatrix} l_{c1}\cos q_1 \\ l \sin q_1 \\ 0 \end{bmatrix} \qquad \dot{r}_{c1} = \begin{bmatrix} -l_{c1}\sin q_1 \dot{q}_1 \\ l_{c1}\cos q_1 \dot{q}_1 \\ 0 \end{bmatrix}$$

$$r_{c2} = \begin{bmatrix} l_1 \cos q_1 + l_{c2}\cos(q_1+q_2) \\ l_1 \sin q_1 + l_{c2}\sin(q_1+q_2) \\ 0 \end{bmatrix} \qquad \dot{r}_{c2} = \begin{bmatrix} -l_1 \sin q_1 \dot{q}_1 - l_{c2}\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ l_1\cos q_1 \dot{q}_1 + l_{c2}\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ 0 \end{bmatrix}$$

$$w_1^0 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \qquad w_2^0 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1+\dot{q}_2 \end{bmatrix}$$

$$T = \frac{1}{2} m_1 (l_{c1}^2 \dot{q}_1)^2 + \frac{1}{2} m_2 \Big(l_1^2 \dot{q}_1^2 + l_{c2}^2 (\dot{q}_1+\dot{q}_2)^2 + 2 l_1 \sin q_1^2 \dot{q}_1 l_{c2} \sin (q_1+q_2)(\dot{q}_1+\dot{q}_2) + 2 l_1 \cos q_1 \dot{q}_1 l_{c2}\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \Big) + \frac{1}{2} I_{z1} \dot{q}_1^2 + \frac{1}{2} I_{z2} (\dot{q}_1+\dot{q}_2)^2$$

$$U = m_1 g l_{c1} \sin(q_1) + m_2 g \big(l_1 \sin q_1 + l_{c2} \sin(q_1+q_2) \big)$$

$$\mathcal{L} = T - U$$

$$H = T + U \qquad p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial U}{\partial \dot{q}_k}$$

$$\frac{\partial H}{\partial p_k} = \frac{\partial T}{\partial p_k} + \frac{\partial U}{\partial p_k}$$

$$\frac{\partial T}{\partial \dot{q}_1} = m_1 l_{c1}^2 \dot{q}_1 + \frac{1}{2} m_2 \big(2 l_1^2 \dot{q}_1 + 2 l_{c2}^2 (\dot{q}_1+\dot{q}_2) + 4 \dot{q}_1 l_1 \sin q_1^2 l_{c2} \sin (q_1+q_2) + 4 l_1 \cos q_1 l_{c2} \cos(q_1+q_2) \dot{q}_1 + 2 l_1 \cos q_1$$
$$+ 2 l_1 \sin q_1^2 l_{c2} \sin (q_1+q_2) \dot{q}_2 + l_{c2} \cos (q_1+q_2) \dot{q}_2$$

$$\frac{\partial H}{\partial_1} =$$

$$\frac{\partial U}{\partial \dot{q}_1} = 0 \qquad \dot{q}_1 = \frac{\partial H}{\partial p_1}$$

Similarly $\dot{q}_2 = \dfrac{\partial H}{\partial p_2}$

$$\frac{\partial H}{\partial q_1} = \frac{1}{2} m_2 \Big(2 l_1 \cos q_1^2 \; 2 q_1 \dot{q}_1 l_{c2} \sin(q_1 + q_2)(\dot{q}_1 + q_2)$$
$$+ 2 l_1 \sin q_1^2 \dot{q}_1 l_{c2} \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) - 2 l_1 \sin q_1 \dot{q}_1 l_{c2}$$
$$\cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) - 2 l_1 \cos q_1 \dot{q}_1 l_{c2} \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) + m_1 g l_{c1} \cos(q_1)$$
$$+ m_2 g \left(l_1 \cos q_1 + l_{c2} \cos(q_1 + q_2) \right) \Big)$$

$$\frac{\partial H}{\partial q_2} = \frac{1}{2} m_2 \Big(2 l_1 \sin q_1^2 \dot{q}_1 l_{c2} \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) + 2 l_1 \cos q_1 \dot{q}_1$$
$$- \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \Big) + m_2 g \, l_{c2} \cos(q_1 + q_2)$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} + \tau_1 \qquad \dot{p}_2 = -\frac{\partial H}{\partial q_2} + \tau_2$$