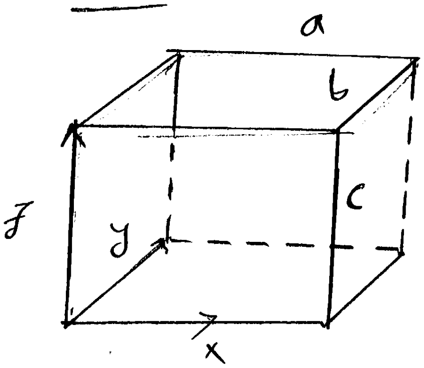
7_3 Assignment (5)



$$I_{xx} = \int_0^c \int_0^b \int_0^a (y^2 + z^2)\rho \, dx\, dy\, dz$$

$$= \tfrac{1}{3}\rho abc(b^2 + c^2)$$

since it is a uniform rectangular $\rho abc = m$

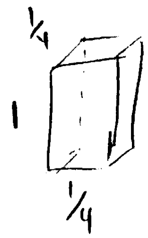
$$I_{xx} = \tfrac{m}{3}(b^2+c^2) \qquad I_{yy} = \tfrac{m}{3}(a^2+c^2) \qquad I_{zz} = \tfrac{m}{3}(a^2+b^2)$$

$$I_{xy} = I_{yx} = -\int_0^c \int_0^b \int_0^a xy\rho \, dx\, dy\, dz = -\tfrac{1}{4} a^2 b^2 c \rho = -\tfrac{1}{4} abc\rho\, ab = -\tfrac{m}{4}ab$$

& $I_{xz} = I_{zx} = -\frac{m}{4}ac \qquad I_{yz} = I_{zy} = -\frac{m}{4}bc$

$$\bar{I} = \begin{bmatrix} \frac{m}{3}(b^2+c^2) & -\frac{mab}{4} & -\frac{mac}{4} \\ -\frac{mab}{4} & \frac{m}{3}(a^2+c^2) & -\frac{mbc}{4} \\ -\frac{mac}{4} & -\frac{mbc}{4} & \frac{m}{3}(a^2+b^2) \end{bmatrix}$$

7_7



$$J_1 = \begin{bmatrix} \frac{m}{12}(b^2+c^2) & 0 & 0 \\ 0 & \frac{m}{12}(a^2+c^2) & 0 \\ 0 & 0 & \frac{m}{12}(a^2+b^2) \end{bmatrix}$$

$$J_1 = \begin{bmatrix} 1/96 & 0 & 0 \\ 0 & \frac{17}{192} & 0 \\ 0 & 0 & 17/192 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 17/192 & 0 & 0 \\ 0 & 1/96 & 0 \\ 0 & 0 & 17/192 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} 17/192 & 0 & 0 \\ 0 & 17/192 & 0 \\ 0 & 0 & 1/96 \end{bmatrix}$$

b) $V_{c_1} = J_{vc_1}\dot{q}$ $\quad J_{vc_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$

$V_{c_2} = J_{vc_2}\dot{q}$ $\quad J_{vc_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$V_{c_3} = J_{v_3}\dot{q}$ $\quad J_{vc_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$K = \frac{1}{2}\dot{q}^T\left\{ m_1 J_{vc_1}^T J_{vc_1} + m_2 J_{vc_2}^T J_{vc_2} + m_3 J_{vc_3}^T J_{vc_3} \right\}\dot{q}$$

$$\Rightarrow D = \begin{bmatrix} m_1+m_2+m_3 & 0 & 0 \\ 0 & m_2+m_3 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

c) Since D is constant, all christoffel symbols are zero

d) $P = g(m_1+m_2+m_3)q_1$ $\quad g_1 = \frac{\partial P}{\partial q_1} = g(m_1+m_2+m_3)$ $\quad g_2 = g_3 = 0$

$$\begin{bmatrix} m_1+m_2+m_3 & 0 & 0 \\ 0 & m_2+m_3 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} g(m_1+m_2+m_3) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

(a_1 is from DH Table)

7.8

$$J_{vc_1} = \begin{bmatrix} -a_1 s_1 & 0 \\ a_1 c_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad J_{vc_2} = \begin{bmatrix} -a_1 s_1 & s_1 \\ a_1 c_1 & c_2 \\ 0 & 0 \end{bmatrix}$$

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\omega_1 = \dot{q}_1 K \qquad \omega_2 = q_1 K$$

The Kinetic Energy becomes:

$$K = \frac{1}{2} q^T \left\{ m_1 J_{vc_1}^T J_{vc_1} + m_2 J_{vc_2}^T J_{vc_2} \right\} q + I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} \phi & 0 \\ 0 & 0 \end{bmatrix}$$

$$= m_1 \begin{bmatrix} a_1^2 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} a_1^2 & -a_1 \\ -a_1 & 1 \end{bmatrix} + \begin{bmatrix} I_1 + I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} (m_1 + m_2) a_1^2 + I_1 + I_2 & -m_2 a_1 \\ -m_2 a_1 & m_2 \end{bmatrix}$$

$$P = P_1 + P_2 = \frac{1}{2} a_1 m_1 g \sin q_1 + m_2 g (q_2 + a_1) \sin q_1$$

$$g_1 = \frac{\partial P}{\partial q_1} = \frac{1}{2} a_1 m g \cos q_1 + m_2 g (q_2 + a_1) \cos q_1$$

$$g \quad \frac{\partial P}{} - m_2 g \sin q_1$$

$$\left((m_1+m_2)a_1^2 + I_1 + I_2\right)\ddot{q}_1 - m_2 a_1 \ddot{q}_2 + \frac{1}{2} m_1 g_1 a_1 \cos q_1 + m_2 g (q_2 + a_1)\cos q_1 = \tau_1$$

$$-m_2 a_1 \ddot{q}_1 + m_2 \ddot{q}_2 + m_2 g \sin q_1 = f_2$$

7-12

$$K = \frac{1}{2} \sum_{i,j}^{n} d_{ij}(q)\, \dot{q}_i\, \dot{q}_j$$

$$\Rightarrow P_K = \frac{\partial K}{\partial \dot{q}_K} = \sum_{j=1}^{n} d_{Kj} \dot{q}_j$$

Now,

$$\sum_{K=1}^{n} \dot{q}_K P_K = \sum_{K=1}^{n} \dot{q}_K \sum_{j=1}^{n} d_{Kj} \dot{q}_j = d_{Kj} \dot{q}_j \dot{q}_K = 2K$$

7-13

$$H = \sum_{k=1}^{n} \dot{q}_k P_k - L = \underset{\uparrow \text{ from (7-12)}}{2K} - (K - V) = K + V$$

b)

$$H = \sum_{k=1}^{n} \dot{q}_k P_k - L$$

$$\& \quad \dot{q}_K = \frac{\partial H}{\partial p_K} \qquad \&$$

$$\frac{\partial H}{\partial q_K} = -\frac{\partial L}{\partial q_K} = \tau_K - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_K} = \tau_K - \dot{P}_K$$

$$\Rightarrow \boxed{\dot{P}_k = \tau_k - \frac{\partial H}{\partial q_k}}$$

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$$H = K + V = \frac{1}{2}\dot{q}\,D(q)\,\dot{q} + P$$

$$\mathcal{L} = K - V = \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22}\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix} - P(q)$$

$$= \frac{1}{2}\left(d_{11}\dot{q}_1^2 + d_{12}\dot{q}_2\dot{q}_1 + d_{12}\dot{q}_1\dot{q}_2 + \dot{q}_2^2 d_{22}\right) - P(q)$$

$$= \frac{1}{2}\left(d_{11}\dot{q}_1^2 + 2d_{12}\dot{q}_1\dot{q}_2 + \dot{q}_2^2 d_{22}\right) - P(q)$$

$$\frac{\partial L}{\partial \dot{q}_1} = d_{11}\dot{q}_1 + d_{12}\dot{q}_2 \qquad \frac{\partial L}{\partial \dot{q}_2} = d_{22}\dot{q}_2 + d_{12}\dot{q}_1$$

$$\Rightarrow P = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22}\end{bmatrix}\dot{q} \Rightarrow \text{Solving for } \dot{q}$$

$$\dot{q} = \left(\begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22}\end{bmatrix}\right)^{-1} P = \frac{1}{d_{11}d_{22} - d_{12}^2}\begin{bmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11}\end{bmatrix} P$$

$$H = \sum \dot{q}_i P_i - L$$