Homogeneous transformations combine rotation and translation. In the three-dimensional case, a homogeneous transformation has the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, R \in SO(3),\ d \in \mathbb{R}^3$$

The set of all such matrices comprises the set $SE(3)$, and these matrices can be used to perform coordinate transformations, analogous to rotational transformations using rotation matrices.

Homogeneous transformation matrices can be used to perform coordinate transformations between frames that differ in orientation and translation. We derived rules for the composition of rotational transformations as

$$H_2^0 = H_1^0 H$$

for the case where the second transformation, H, is performed relative to the current frame and

$$H_2^0 = H H_1^0$$

for the case where the second transformation, H, is performed relative to the fixed frame.

PROBLEMS

2-1 Using the fact that $v_1 \cdot v_2 = v_1^T v_2$, show that the dot product of two free vectors does not depend on the choice of frames in which their coordinates are defined.

2-2 Show that the length of a free vector is not changed by rotation, that is, that $\|v\| = \|Rv\|$.

2-3 Show that the distance between points is not changed by rotation, that is, $\|p_1 - p_2\| = \|Rp_1 - Rp_2\|$.

2-4 If a matrix R satisfies $R^T R = I$, show that the column vectors of R are of unit length and mutually perpendicular.

2-5 If a matrix R satisfies $R^T R = I$, then
a) Show that $\det R = \pm 1$
b) Show that $\det R = +1$ if we restrict ourselves to right-handed coordinate frames.

2-6 Verify Equations (2.3)–(2.5).

2-7 A **group** is a set X together with an operation $*$ defined on that set such that

- $x_1 * x_2 \in X$ for all $x_1, x_2 \in X$
- $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
- There exists an element $I \in X$ such that $I * x = x * I = x$ for all $x \in X$
- For every $x \in X$, there exists some element $y \in X$ such that $x * y = y * x = I$

Show that SO(n) with the operation of matrix multiplication is a group.

2-8 Derive Equations (2.6) and (2.7).

2-9 Suppose A is a 2×2 rotation matrix. In other words $A^T A = I$ and $\det A = 1$. Show that there exists a unique θ such that A is of the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2-10 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the current z-axis.
3. Rotate by ψ about the world y-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-11 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the world z-axis.
3. Rotate by ψ about the current x-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-12 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.

2. Rotate by θ about the current z-axis.
3. Rotate by ψ about the current x-axis.
4. Rotate by α about the world z-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-13 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the world z-axis.
3. Rotate by ψ about the current x-axis.
4. Rotate by α about the world z-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-14 If the coordinate frame $o_1x_1y_1z_1$ is obtained from the coordinate frame $o_0x_0y_0z_0$ by a rotation of $\frac{\pi}{2}$ about the x-axis followed by a rotation of $\frac{\pi}{2}$ about the fixed y-axis, find the rotation matrix R representing the composite transformation. Sketch the initial and final frames.

2-15 Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$, and $o_3x_3y_3z_3$ are given, and suppose

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad R_3^1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the matrix R_3^2.

2-16 Derive equations for the roll, pitch, and yaw angles corresponding to the rotation matrix $R = (r_{ij})$.

2-17 Verify Equation (2.43).

2-18 Verify Equation (2.45).

2-19 If R is a rotation matrix show that $+1$ is an eigenvalue of R. Let k be a unit eigenvector corresponding to the eigenvalue $+1$. Give a physical interpretation of k.

2-20 Let $k = \frac{1}{\sqrt{3}}[1, 1, 1]^T$, $\theta = 90°$. Find $R_{k,\theta}$.

2-21 Show by direct calculation that $R_{k,\theta}$ given by Equation (2.43) is equal to R given by Equation (2.47) if θ and k are given by Equations (2.48) and (2.49), respectively.

2-22 Compute the rotation matrix given by the product

$$R_{x,\theta}R_{y,\phi}R_{z,\pi}R_{y,-\phi}R_{x,-\theta}$$

2-23 Suppose R represents a rotation of 90° about y_0 followed by a rotation of 45° about z_1. Find the equivalent axis/angle to represent R. Sketch the initial and final frames and the equivalent axis vector k.

2-24 Find the rotation matrix corresponding to the Euler angles $\phi = \frac{\pi}{2}$, $\theta = 0$, and $\psi = \frac{\pi}{4}$. What is the direction of the x_1 axis relative to the base frame?

2-25 Section 2.5.1 described only the Z-Y-Z Euler angles. List all possible sets of Euler angles. Is it possible to have Z-Z-Y Euler angles? Why or why not?

2-26 Unit magnitude complex numbers $a + ib$ with $a^2 + b^2 = 1$ can be used to represent orientation in the plane. In particular, for the complex number $a + ib$, we can define the angle $\theta = \text{Atan2}(a, b)$. Show that multiplication of two complex numbers corresponds to addition of the corresponding angles.

2-27 Show that complex numbers together with the operation of complex multiplication define a group. What is the identity for the group? What is the inverse for $a + ib$?

2-28 Complex numbers can be generalized by defining three independent square roots for -1 that obey the multiplication rules

$$\begin{aligned}
-1 &= i^2 = j^2 = k^2, \\
i &= jk = -kj, \\
j &= ki = -ik, \\
k &= ij = -ji
\end{aligned}$$

Using these, we define a **quaternion** by $Q = q_0 + iq_1 + jq_2 + kq_3$, which is typically represented by the 4-tuple (q_0, q_1, q_2, q_3). A rotation by θ about the unit vector $n = [n_x, n_y, n_z]^T$ can be represented by the unit quaternion $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$. Show that such a quaternion has unit norm, that is, $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

2-29 Using $Q = (\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2})$, and the results from Section 2.5.3, determine the rotation matrix R that corresponds to the rotation represented by the quaternion (q_0, q_1, q_2, q_3).

2-30 Determine the quaternion Q that represents the same rotation as given by the rotation matrix R.

2-31 The quaternion $Q = (q_0, q_1, q_2, q_3)$ can be thought of as having a scalar component q_0 and a vector component $q = [q_1, q_2, q_3]^T$. Show that the product of two quaternions, $Z = XY$ is given by

$$\begin{aligned} z_0 &= x_0 y_0 - x^T y \\ z &= x_0 y + y_0 x + x \times y, \end{aligned}$$

Hint: Perform the multiplication $(x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3)$ and simplify the result.

2-32 Show that $Q_I = (1, 0, 0, 0)$ is the identity element for unit quaternion multiplication, that is, $QQ_I = Q_I Q = Q$ for any unit quaternion Q.

2-33 The conjugate Q^* of the quaternion Q is defined as

$$Q^* = (q_0, -q_1, -q_2, -q_3)$$

Show that Q^* is the inverse of Q, that is, $Q^* Q = QQ^* = (1, 0, 0, 0)$.

2-34 Let v be a vector whose coordinates are given by $[v_x, v_y, v_z]^T$. If the quaternion Q represents a rotation, show that the new, rotated coordinates of v are given by $Q(0, v_x, v_y, v_z)Q^*$, in which $(0, v_x, v_y, v_z)$ is a quaternion with zero as its real component.

2-35 Let the point p be rigidly attached to the end effector coordinate frame with local coordinates (x, y, z). If Q specifies the orientation of the end effector frame with respect to the base frame, and T is the vector from the base frame to the origin of the end effector frame, show that the coordinates of p with respect to the base frame are given by

$$Q(0, x, y, z)Q^* + T \tag{2.68}$$

in which $(0, x, y, z)$ is a quaternion with zero as its real component.

2-36 Verify Equation (2.60).

2-37 Compute the homogeneous transformation representing a translation of 3 units along the x-axis followed by a rotation of $\frac{\pi}{2}$ about the current z-axis followed by a translation of 1 unit along the fixed y-axis. Sketch the frame. What are the coordinates of the origin o_1 with respect to the original frame in each case?

2-38 Consider the diagram of Figure 2.13. Find the homogeneous transformations H_1^0, H_2^0, H_2^1 representing the transformations among the three frames shown. Show that $H_2^0 = H_1^0, H_2^1$.

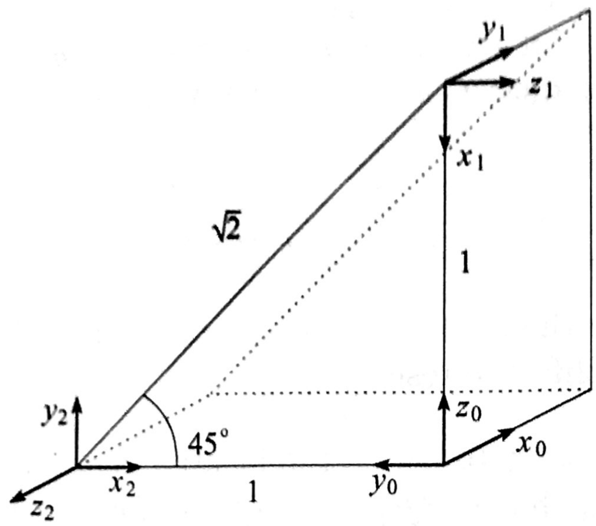

Figure 2.13: Diagram for Problem 2-38.

2-39 Consider the diagram of Figure 2.14. A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. A frame $o_1x_1y_1z_1$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2 meters above the table top with frame $o_3x_3y_3z_3$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $o_0x_0y_0z_0$. Find the homogeneous transformation relating the frame $o_2x_2y_2z_2$ to the camera frame $o_3x_3y_3z_3$.

2-40 In Problem 2-39, suppose that, after the camera is calibrated, it is rotated 90° about z_3. Recompute the above coordinate transformations.

2-41 If the block on the table is rotated 90° about z_2 and moved so that its center has coordinates $[0, .8, .1]^T$ relative to the frame $o_1x_1y_1z_1$, compute the homogeneous transformation relating the block frame to the camera frame; the block frame to the base frame.