ECE470: Robot Modeling and Control
Study Guide

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1 Kinematics

1.1 Robotic Manipulators

- A robot manipulator with n joints has $n+1$ links.
- Joints are numbered 1 to n. Links are numbered 0 to n.
- Link 0, called the base, never moves.
- When joint i is actuated, link i moves.
- Joint $i+1$ is fixed with respect to link i.
- The joint variables are denoted q_i. If joint i is revolute (R), then $q_i = \theta_i$, an angle. If joint i is prismatic (P), then $q_i = d_i$, a length.

1.2 Rotation Matrices

- The *rotation matrix* of frame 1, $o_1x_1y_1z_1$, with respect to frame 0, $o_0x_0y_0z_0$, is given by

$$R_1^0 := \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 \end{bmatrix} = \begin{bmatrix} (x_1 \cdot x_0) & (y_1 \cdot x_0) & (z_1 \cdot x_0) \\ (x_1 \cdot y_0) & (y_1 \cdot y_0) & (z_1 \cdot y_0) \\ (x_1 \cdot z_0) & (y_1 \cdot z_0) & (z_1 \cdot z_0) \end{bmatrix}.$$

  Observe that, by definition, $R_1^0 = (R_0^1)^T$.

- If v is a vector in $\mathbb{R}^3$ and v^i denotes its coordinate vector w.r.t. frame i, then we have $v^0 = R_1^0 v^1$.
- Let v be a vector in $\mathbb{R}^3$ and let v^i denote its coordinate vector w.r.t. frame i. If we have three frames 0, 1 and 2, then $v^0 = R_1^0 v^1$ and $v^1 = R_2^1 v^2$. This implies $v^0 = R_1^0 v^1 = R_1^0 R_2^1 v^2$. But $v^0 = R_2^0 v^2$. Therefore, $R_2^0 = R_1^0 R_2^1$. This formula can be generalized to more coordinate frames.
- Let v be a vector in $\mathbb{R}^3$ and let v^i denote its coordinate vector w.r.t. frame i. If we have two frames 0 and 1, then $v^0 = R_1^0 v^1 = R_1^0 R_0^1 v^0$. Since v^0 is arbitrary, this implies $(R_1^0)(R_0^1) = I$, so $(R_1^0)^{-1} = (R_1^0)^T$.

- A matrix such that $R^T = R^{-1}$ is called *orthogonal*. If R is orthogonal, then its columns are mutually orthogonal unit vectors and its determinant is ± 1. If we choose $\det R = 1$, then its columns give rise to a right handed frame.

- We define the *special orthogonal group* to be

$$SO(3) := \left\{R \in \mathbb{R}^{3\times 3} \mid R^{-1} = R^T, \det(R) = 1\right\}.$$

1.3 Rotational Transformations

- Given $R \in SO(3)$ and a vector $v \in \mathbb{R}^3$, $w^0 = Rv^0$ is called a *rotational transformation* in frame 0.

- Suppose we are given two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$. Also we are given $R \in SO(3)$, a rotational transformation in frame 0. We want to find an expression for R in frame 1. To that end, let v, w be two vectors such that $w^0 = Rv^0$. Then

$$w^1 = R_0^1 w^0 = R_0^1 R v^0 = R_0^1 R R_1^0 v^1 = (R_1^0)^T R R_1^0 v^1 .$$

  We conclude the transformation in frame 1 is given by $\tilde{R} := (R_1^0)^T R R_1^0$.

- Suppose we are given a frame $o_0x_0y_0z_0$ and $R \in SO(3)$. Suppose we rotate each axis x_0, y_0, z_0 by R to generate a new frame 1. We want to find R_1^0. We have

$$R_1^0 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 \end{bmatrix} = \begin{bmatrix} Rx_0^0 & Ry_0^0 & Rz_0^0 \end{bmatrix} = RI = R ,$$

  where I is the 3×3 identity matrix. This calculation shows the relationship between rotation matrices and rotational transformations.

- Given frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$, and a rotational transformation $R \in SO(3)$ in frame 1. We generate a frame 2 by rotating frame 1 by R, and we want to find R_2^0. We know $v^0 = R_1^0 v^1$ and $v^1 = R_2^1 v^2$. Then we have $v^0 = R_1^0 R_2^1 v^2 = R_2^0 v^2$. Therefore, $R_2^0 = R_1^0 R_2^1 = R_1^0 R$.

- Given frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$, and a rotational transformation $R \in SO(3)$ in frame 0. We generate a frame 2 by rotating frame 1 by R, and we want to find R_2^0. We know $v^0 = R_1^0 v^1$ and $v^1 = R_2^1 v^2$. Then we have $R_2^0 = R_1^0 R_2^1 = R_1^0 (R_1^0)^T R R_1^0 = R R_1^0$. Notice that the order reverses from the previous case.

- Let $c_\theta := \cos\theta$ and $s_\theta := \sin\theta$. The *elementary rotations* are rotations about the x-axis by angle θ, rotation about the y-axis by angle θ, and rotation about the z-axis by angle θ, given by:

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix} , \quad R_{y,\theta} = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} , \quad R_{z,\theta} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

- Suppose we rotate frame 0 by θ around x_0, then by ϕ around z_1. Find R_2^0. We have

$$R_2^0 = R_1^0 R_2^1 = R_{x,\theta} R_{z,\phi} .$$

- Suppose we rotate frame 0 by θ around x_0, then by ϕ around z_0, then by ψ around z_2. Find R_3^0. We have

$$R_3^0 = R_1^0 R_2^1 R_3^2 = R_{x,\theta}[R_{x,\theta}^T R_{z,\phi} R_{x,\theta}] R_{z,\psi} = R_{z,\phi} R_{x,\theta} R_{z,\psi} .$$

- Given $M_1, M_2, M_3 \in SO(3)$. Define frame 1 by rotating frame 0 by M_1. Define frame 2 by rotating frame 1 by M_2, represented as a rotation w.r.t. frame 0. Define frame 3 by rotating frame 2 by M_3, represented w.r.t. frame 1. Find R_3^0. We have that

$$\begin{aligned} R_1^0 &= M_1 \\ R_2^1 &= (R_1^0)^T M_2 R_1^0 = M_1^T M_2 M_1 \\ R_3^2 &= (R_2^1)^T M_3 R_2^1 = (M_1^T M_2 M_1)^T M_3 (M_1^T M_2 M_1) = M_1^T M_2^T M_1 M_3 M_1^T M_2 M_1 \,. \end{aligned}$$

  Therefore, $R_3^0 = R_1^0 R_2^1 R_3^2 = M_1 M_1^T M_2 M_1 M_1^T M_2^T M_1 M_3 M_1^T M_2 M_1 = M_1 M_3 M_1^T M_2 M_1$.

- Rotations can be parametrized using *Euler angles*. We use the ZYZ Euler angles (ϕ, θ, ψ) corresponding to

$$R_1^0 = R_{z,\phi} R_{y,\theta} R_{z,\psi} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & -s_\theta s_\psi & c_\theta \end{bmatrix} .$$

1.4 Rigid Motions

- Let $p \in \mathbb{R}^3$ be a point and consider a frame 0. A rigid motion is a function of the form $T(p^0) = Rp^0 + d^0$, where $R \in SO(3)$ and $d^0 \in \mathbb{R}^3$ is a constant vector. We can see that a rigid motion first rotates p^0 by R and then translates it by d^0.

- Rigid motions are used to change coordinate representations of points. Consider two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$. Notice o_0 and o_1 are not necessarily the same. Consider a point p, represented in frame i as p^i, where $i = 0, 1$. Then the relationship between p^0 and p^1 is:

$$p^0 = R_1^0 p^1 + o_1^0 \,.$$

- Suppose we have three frames $o_ix_iy_iz_i$, where $i = 0, 1, 2$. Then $p^0 = R_1^0 p^1 + o_1^0$ and $p^1 = R_2^1 p^2 + o_2^1$. Therefore, $p^0 = R_1^0 R_2^1 p^2 + R_1^0 o_2^1 + o_1^0$. We conclude

$$p^0 = R_2^0 p^2 + o_2^0 \,.$$

- If p^0 is a coordinate vector in frame 0, then $P^0 := (p^0, 1)$ is called the *homogeneous coordinates* of p^0.

- Let p and q be two points in $\mathbb{R}^3$ and let $P^0 = (p^0, 1)$ and $Q^0 = (q^0, 1)$ be their homogeneous coordinates. Suppose that p and q are related by a rigid motion in frame 0: $q^0 = Rp^0 + d^0$. We define a matrix

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^4 ,$$

  where $R \in SO(3)$ and $d \in \mathbb{R}^3$. H is called a *homogeneous transformation*. Using this notation we have

$$Q^0 = HP^0 \,.$$

- We use homogeneous transformations to represent compositions of rigid motions as a sequence of matrix multiplications. Consider three frames 0, 1, and 2, and let p be a point in $\mathbb{R}^3$ with homogeneous coordinates $P^i = (p^i, 1)$, $i = 0, 1, 2$. Then $P^0 = H_1^0 P^1$ and $P^1 = H_2^1 P^2$, where

$$H_j^i := \begin{bmatrix} R_j^i & o_j^i \\ 0 & 1 \end{bmatrix} .$$

Therefore we have $P^0 = H_1^0 P^1 = H_1^0 H_2^1 P^2$. However, it is also true that $P^0 = H_2^0 P^2$ by using the formula $p^0 = R_2^0 p^2 + o_2^0$ derived above. We conclude that

$$H_2^0 = H_1^0 H_2^1 .$$

- We define the *special Euclidean group* to be the set of homogeneous transformations; that is

$$SE(3) := \left\{ \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \mid R \in SO(3), d \in \mathbb{R}^3 \right\}.$$

1.5 Denavit-Hartenberg Convention

The Denavit-Hartenberg (DH) Convention allows us to represent H_i^{i-1} using only four parameters. It relies on a careful assignment of coordinate frames $o_i x_i y_i z_i$ for $i = 0, \ldots, n$. These frames must adhere to two rules:

(DH1) Axis x_i is perpendicular to z_{i-1}.

(DH2) Axis x_i intersects z_{i-1}.

- The *DH frame assignment procedure* is:

  1. Assign $z_0, \ldots, z_{n-1}$ such that z_i is the axis of actuation of joint $i+1$. Specifically, if joint $i+1$ is revolute, then z_i is the axis of rotation; if joint $i+1$ is prismatic, then z_i is the axis of translation.
  2. Choose the base frame $o_0 x_0 y_0 z_0$ to form a right-handed orthogonal frame. Axes x_0 and y_0 can be chosen in any convenient manner.
  3. Frame i is chosen based on frame $i-1$. There are three cases:
     (a) z_{i-1} and z_i are not coplanar: there exists a unique shortest line segment from z_{i-1} to z_i perpendicular to both z_{i-1} and z_i. This defines x_i. The point where x_i intersects z_i is o_i.
     (b) z_{i-1} and z_i intersect: x_i is the normal vector to the plane formed by z_{i-1} and z_i. o_i is the point of intersection of z_{i-1} and z_i.
     (c) z_{i-1} and z_i are parallel: choose o_i anywhere along z_i. Then x_i is any vector normal to both z_{i-1} and z_i.
  4. Choose y_i, $i = 0, \ldots, n$ to form right-handed orthogonal frames.
  5. Choose the end effector frame with its base in a convenient location on the end effector. Choose x_n to satisfy rules (DH1)-(DH2). Typically z_{n-1} and z_n will coincide.

- The *DH parameters* are $(a_i, d_i, \alpha_i, \theta_i)$ where $a_i = |o_{i-1} o_i|$ along x_i, $d_i = |o_{i-1} o_i|$ along z_{i-1}, $\alpha_i = \angle z_{i-1} z_i$ about x_i, $\theta_i = \angle x_{i-1} x_i$ about z_{i-1}.

- The mapping from the DH parameters to the homogeneous transformations H_i^{i-1} is given by:

$$H_i^{i-1} = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1.6 Forward Kinematics Problem

- Recall the useful recursive formulas for $j > i$:

$$R_j^i = R_{i+1}^i \cdots R_j^{j-1}\,, \qquad\qquad o_j^i = o_{j-1}^i + R_{j-1}^i o_j^{j-1}\,.$$

- The *forward kinematics problem* is to find the position and orientation of the end effector as a function of the joint variables of the robot. The solution is:

$$H_n^0 = H_1^0 H_2^1 \cdots H_n^{n-1} = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix},$$

where the H_i^{i-1} are given above.

1.7 Inverse Kinematics Problem

- The *inverse kinematics problem* is to find the joint variables such that the end effector achieves a desired position o_d^0 and orientation R_d. That is, given o_d^0 and R_d, we must solve the equation

$$H_n^0(q_1, \ldots, q_n) = \begin{bmatrix} R_d & o_d^0 \\ 0 & 1 \end{bmatrix}$$

for the unknowns $q_1, \ldots, q_n$.

- *Kinematic decoupling* is a method to solve the inverse kinematics problem in two steps. We assume: (i) the robot has $n = 6$ joints; and (ii) the last three joints form a spherical wrist.

  The *inverse position kinematics problem* is to solve for (q_1, q_2, q_3). Addition of angular velocities: given o_d^0 and R_d. Let o_c be the center of the wrist. From geometry we have

$$o_6^0 = o_c^0 + R_6^0 \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix}.$$

  We want $o_6^0 = o_d^0$ and $R_6^0 = R_d$. This gives

$$o_c^0(q_1, q_2, q_3) = o_d^0 - R_d \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix},$$

  and we solve this equation for (q_1, q_2, q_3).

  The *inverse orientation kinematics problem* is to solve for (q_4, q_5, q_6) given o_d^0, R_d, and (q_1, q_2, q_3). First, $R_d = R_6^0 = R_3^0(q_1, q_2, q_3) R_6^3(q_4, q_5, q_6)$. Solving for R_6^3, we get

$$R_6^3(q_4, q_5, q_6) = (R_3^0)^T R_d\,.$$

  Since R_d is given and (R_3^0) can be computed using (q_1, q_2, q_3) from the previous step, this equation can be solved for (q_4, q_5, q_6). If (q_4, q_5, q_6) correspond to the Euler angles (ϕ, θ, ψ), then we use the formulas to recover the Euler angles.

1.8 Velocity Kinematics

- Let $w = (w_1, w_2, w_3) \in \mathbb{R}^3$. We define the matrix

$$S(w) := \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} .$$

  Let $v \in \mathbb{R}^3$. Then

$$S(w)v = w \times v \,.$$

- The matrix $S(w)$ satisfies the following properties:

  (i) $S(w + \lambda v) = S(w) + \lambda S(v)$.

  (ii) $RS(w)R^T = S(Rw)$, where $R \in SO(3)$.

- A matrix M is *skew-symmetric* if $M = -M^T$. It is a fact that if M is skew-symmetric, then there exists a unique vector $w \in \mathbb{R}^3$ such that $M = S(w)$.

- Suppose we have two frames 0 and 1, and frame 1 rotates and translates w.r.t. frame 0. Then for a fixed point p in frame 1, we have

$$\begin{aligned} \dot{p}_1^0 &= \dot{o}_1^0 + w_1^0 \times (R_1^0 p^1) \\ \dot{R}_1^0 &= S(w_1^0) R_1^0 \,, \end{aligned}$$

  where $w_1^0 \in \mathbb{R}^3$ is the unique vector such that $S(w_1^0) = \dot{R}_1^0 (R_1^0)^T$. The vector w_1^0 is called the *angular velocity* of frame 1 w.r.t. frame 0.

- More generally, the angular velocity of frame j w.r.t. frame i is denoted w_j^i. A useful notation is:

$$w_{i,j}^k := R_i^k w_j^i \,.$$

  For n moving frames, we have

$$w_{0,n}^0 = \sum_{k=0}^{n-1} w_{k,k+1}^0 = \sum_{k=0}^{n-1} R_k^0 w_{k+1}^k \,.$$

  This formula shows how to sum angular velocity vectors.

- The *forward velocity kinematics problem* is to find the linear and angular velocities of the end effector as a function of the joint variable derivatives $\dot{q}_i$. The solution to the forward velocity kinematics problem is

$$\dot{o}_n^0 = J_v(q)\dot{q} \,, \qquad w_n^0 = J_w(q)\dot{q}$$

  where $J_v(q) \in \mathbb{R}^{3\times n}$ is the *linear velocity Jacobian* and $J_w(q) \in \mathbb{R}^{3\times n}$ is the *angular velocity Jacobian*.

- The angular velocity Jacobian is given by

$$J_w(q) = \begin{bmatrix} \rho_1 z_0^0 & \cdots & \rho_n z_{n-1}^0 \end{bmatrix}$$

  where

$$\rho_i := \left\{ \begin{array}{ll} 1 & \text{if joint } i \text{ is R} \\ 0 & \text{if joint } i \text{ is P} \end{array} \right. .$$

- The linear velocity Jacobian is given by

$$J_v(q) = \begin{bmatrix} J_{v1} & \cdots & J_{vn} \end{bmatrix}$$

where

$$J_{vj} = \begin{cases} z_{j-1}^0 \times (o_n^0 - o_{j-1}^0) & \text{if joint } j \text{ is R} \\ z_{j-1}^0 & \text{if joint } j \text{ is P} \end{cases} .$$

- Define $\xi = (\dot{o}_n^0, \omega_n^0)$ and $J(q) = \begin{bmatrix} J_v(q) \\ J_w(q) \end{bmatrix}$. Then we can write

$$\xi = J(q)\dot{q} \,.$$

2 Dynamics

2.1 Euler-Lagrange Equations

The Euler-Langrange equations provide an alternative method to Newton's Laws to obtain the equations of motion of a collection of masses. They are particularly useful for dealing with constraints. Key ideas in the derivation are: *holonomic constraints*, *virtual displacements*, *generalized coordinates*, and *constraint forces*.

- We consider N point masses in $\mathbb{R}^3$. Let r_i be the position of mass i. From Newton's 2nd Law each mass has dynamics

$$m_i \ddot{r}_i - f_i^c - f_i^e = 0 \,, \qquad i = 1, \dots N \,, \tag{1}$$

where f_i^e and f_i^c are the external and constraint forces on mass i. The masses are subject to l *holonomic constraints*

$$g(r_1, \dots, r_N) = 0 \,,$$

where $g : \mathbb{R}^{3N} \to \mathbb{R}^l$ such that rank $(\frac{\partial g}{\partial r}) = l$ (implying there are l independent constraints).

- Define $\delta r = (\delta r_1, \dots, \delta r_N)$ to be a *virtual displacement* if $\frac{\partial g}{\partial r} \delta r = 0$. Geometrically, this means a virtual displacement δr is tangent to the constraint set $\mathcal{C} := \{ r \in \mathbb{R}^{3N} \mid g(r) = 0 \}$ at a point r.

- We assume that the constraint forces are orthogonal the constraint set $\mathcal{C}$; that is $(f_i^c)^{\mathrm{T}} \delta r_i = 0$. Now take the dot product of each equation in (1) with δr_i and sum over i. We obtain the *d'Alembert Principle*:

$$\sum_{i=1}^{N} (m_i \ddot{r}_i - f_i^e)^{\mathrm{T}} \delta r_i = 0 \,.$$

- Let $dq = (dq_1, \dots, dq_N)$ be an arbitrary infinitesimal displacement. We need several facts from calculus (mostly by application of the chain rule):

$$\begin{aligned}
\delta r_i &= \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} dq_j \\
\dot{r}_i &= \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j \\
\frac{\partial \dot{r}_i}{\partial \dot{q}_j} &= \frac{\partial r_i}{\partial q_j} \\
\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) &= \sum_{k=1}^{n} \frac{\partial}{\partial q_k}\frac{\partial r_i}{\partial q_j}\dot{q}_k = \frac{\partial}{\partial q_j}\sum_{k=1}^{n}\frac{\partial r_i}{\partial q_k}\dot{q}_k = \frac{\partial \dot{r}_i}{\partial q_j} \\
\frac{d}{dt}\left(m_i \dot{r}_i^{\mathrm{T}} \frac{\partial r_i}{\partial q_j}\right) &= m_i \ddot{r}_i^{\mathrm{T}} \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i^{\mathrm{T}} \frac{\partial \dot{r}_i}{\partial q_j}
\end{aligned}$$

- Define the *kinetic energy*

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \|\dot{r}_i\|^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i^{\mathrm{T}} \dot{r}_i \,.$$

  Observe that

$$\begin{aligned}
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} &= \sum_{i=1}^{N} \frac{d}{dt}\big[m_i \dot{r}_i^{\mathrm{T}} \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\big] = \sum_{i=1}^{N} \frac{d}{dt}\big[m_i \dot{r}_i^{\mathrm{T}} \frac{\partial r_i}{\partial q_j}\big] \\
\frac{\partial T}{\partial q_j} &= \sum_{i=1}^{N} m_i \dot{r}_i^{\mathrm{T}} \frac{\partial \dot{r}_i}{\partial q_j} \,.
\end{aligned}$$

- Define the potential energy U such that the jth *generalized force* is given by

$$\psi_j = \sum_{i=1}^{N} (f_i^e)^{\mathrm{T}} \frac{\partial r_i}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \tau_j \,,$$

  where τ_j corresponds to the applied forces on mass j.

- Putting it all together, we have

$$\sum_{i=1}^{N} (m_i \ddot{r}_i - f_i^e)^{\mathrm{T}} \delta r_i = 0 \tag{2}$$

$$\iff \sum_{i=1}^{N} m_i \ddot{r}_i \delta r_i = \sum_{j=1}^{n} \psi_j dq_j \tag{3}$$

$$\iff \sum_{i=1}^{N} \sum_{j=1}^{n} \left[\frac{d}{dt}\left[m_i \dot{r}_i^{\mathrm{T}} \frac{\partial r_i}{\partial q_j}\right] - m_i \dot{r}_i^{\mathrm{T}} \frac{\partial \dot{r}_i}{\partial q_j}\right] dq_j = \sum_{j=1}^{n} \psi_j dq_j \tag{4}$$

$$\iff \sum_{j=1}^{n} \left[\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j}\right] dq_j = \sum_{j=1}^{n} \psi_j dq_j \,. \tag{5}$$

- Since the dq_j's are arbitrary, we obtain

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j} = \psi_j\,, \qquad j = 1, \ldots, n\,.$$

- Define the *Lagrangian* $L = T - U$. Since $\frac{\partial U}{\partial \dot{q}_j} = 0$, the previous equation becomes

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} = \tau_j\,, \qquad j = 1, \ldots, n\,.$$

  These are the *Euler-Lagrange* equations.

2.2 Robot Dynamic Model

- The generalized coordinates for an n link robot are precisely the joint variables $(q_1, \ldots, q_n)$.
- The robot kinetic energy is

$$\begin{aligned} T &= \sum_{i=1}^{n} \frac{1}{2} m_i (\dot{r}_{c_i})^{\mathrm{T}} (\dot{r}_{c_i}) + \frac{1}{2} (w_i^0)^{\mathrm{T}} I_i (w_i^0) && (6) \\ &= \sum_{i=1}^{n} \frac{1}{2} \dot{q}^{\mathrm{T}} \left[\sum_{i=1}^{n} m_i \overline{J}_{v_i}^{\mathrm{T}} \overline{J}_{v_i} + \overline{J}_{w_i}^{\mathrm{T}} R_i^0 \overline{I}_i (R_i^0)^{\mathrm{T}} \overline{J}_{w_i} \right] \dot{q} && (7) \\ &= \frac{1}{2} \dot{q}^{\mathrm{T}} D(q) \dot{q}\,. && (8) \end{aligned}$$

  In this formula, $\overline{J}_{v_i}$ is the linear velocity Jacobian regarding the position r_{c_i} as a virtual end effector and is given by

$$\overline{J}_{v_i}(q) = \begin{bmatrix} \overline{J}_{v_i,1} & \cdots & \overline{J}_{v_i,i} & 0 & \cdots & 0 \end{bmatrix}$$

  where $\overline{J}_{v_i,j} = J_{vj}$, which was given in Section 1.8. Similarly, $\overline{J}_{w_i}$ is the angular velocity Jacobian regarding the position r_{c_i} as a virtual end effector and is given by

$$\overline{J}_{w_i}(q) = \begin{bmatrix} \rho_1 z_0^0 & \cdots & \rho_i z_{i-1}^0 & 0 & \cdots & 0 \end{bmatrix}\,.$$

- The robot potential energy

$$U(q) = -\sum_{i=1}^{n} m_i \overline{g}^{\mathrm{T}} r_{c_i}(q)\,,$$

  where $\overline{g}$ has the length $9.81 \frac{m}{s^2}$ and the direction of gravity in frame 0.
- The Euler-Langrange equations for the robot become

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla_q U(q) = \tau\,.$$

- The augmented model (including the motor dynamics) is:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B(q)\dot{q} + \nabla_q U(q) = u\,.$$

Assumption 2.1 *The augmented model satisfies the following assumptions.*

(A1) The matrix $M(q)$ is symmetric and positive definite.

(A2) The matrix $\dot{M} - 2C$ is skew symmetric, i.e. $(\dot{M} - 2C)^{\mathrm{T}} = (\dot{M} - 2C)$.

(A3) The matrix $B(q)$ is symmetric and positive semi-definite; i.e. $v^{\mathrm{T}} B v \geq 0$ for all v.

3 Control

3.1 Lyapunov Stability

Theorem 3.1 (Lyapunov Theorem) *Consider a system $\dot{x} = f(x)$ such that $f(\overline{x}) = 0$. Suppose there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that V is positive definite at $\overline{x}$. If $\dot{V} = \frac{\partial V}{\partial x} f(x) \leq 0$, then $\overline{x}$ is a stable equilibrium. If moreover $\dot{V}$ is negative definite at $\overline{x}$, then $\overline{x}$ is an asymptotically stable equilibrium.*

Theorem 3.2 (LaSalle Invariance Principle) *Consider a system $\dot{x} = f(x)$ such that $f(\overline{x}) = 0$. Suppose there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that V is positive definite at $\overline{x}$ and $\dot{V} = \frac{\partial V}{\partial x} f(x) \leq 0$. Then $\dot{V}(x(t)) \to 0$ as $t \to 0$. Moreover, if the statement $\dot{V}(x(t)) \equiv 0$ for all t implies $x(t) \equiv \overline{x}$, then $\overline{x}$ is an asymptotically stable equilibrium.*

3.2 Feedback Linearization

This method is also called the *computed torque method* in the robotics literature. Consider the augmented model

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B(q)\dot{q} + \nabla_q U(q) = u\,.$$

Assuming that the model is perfectly known, we define the feedback linearizing controller

$$u = M(q)w + C(q,\dot{q})\dot{q} + B(q)\dot{q} + \nabla_q U(q)\,,$$

where $w \in \mathbb{R}^n$ is a new control input. The closed-loop system is

$$M(q)\ddot{q} = M(q)w\,.$$

Because of (A1), M is invertible. Therefore, the closed-loop system is:

$$\ddot{q} = w\,.$$

Now we design a controller w to make q track a reference signal $q^r(t)$ according to the following steps:

- Define the *tracking error* for the ith joint:

  $$e_i := q_i^r - q_i\,.$$

- Take two derivatives to get the *error model*:

  $$\ddot{e}_i = \ddot{q}_i^r - w_i\,.$$

- Choose the PD controller

  $$w_i = \ddot{q}_i^r + K_d \dot{e}_i + K_p e_i$$

  such that $s^2 + K_d s + K_p$ is a Hurwitz polynomial; i.e. its roots are in the open left-half complex plane (OLHP). For example, we can simply take $K_d, K_p > 0$.

- The closed-loop dynamics for the ith error are:

  $$\ddot{e}_i + K_d \dot{e}_i + K_p e_i = 0\,.$$

  Therefore $e_i(t) \longrightarrow 0$, so $q_i(t) \longrightarrow q_i^r(t)$, as desired.

3.3 Adaptive Control with Gravity Compensation

Consider the augmented model

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B(q)\dot{q} + \nabla_q U(q) = u\,.$$

We can rewrite the model in the form

$$Y(q,\dot{q},\ddot{q})\Phi = u\,,$$

where Y is a matrix-valued function and Φ is a vector of unknown parameters.

Assumption 3.1 *The model and controller satisfy the following assumptions:*

(A1) The matrix $M(q)$ is symmetric and positive definite.

(A2) The matrix $\dot{M} - 2C$ is skew symmetric, i.e. $(\dot{M} - 2C)^{\mathrm{T}} = (\dot{M} - 2C)$.

(A3) The matrix $B(q)$ is symmetric and positive semi-definite; i.e. $v^{\mathrm{T}}Bv \geq 0$ for all v.

(A4) The matrices K and Γ are symmetric and positive definite.

(A5) The matrix Λ is diagonal and positive definite.

The control design proceeds along the following steps:

- Define the intermediate variables

$$\tilde{q} = q^r - q\,, \quad \dot{\tilde{q}} = \dot{q}^r - \dot{q}\,, \quad r = \dot{\tilde{q}} + \Lambda\tilde{q}\,, \quad v = \dot{q}^r + \Lambda\tilde{q}\,.$$

- Define the controller

$$u \;=\; \hat{M}(q)\dot{v} + \hat{C}(q,\dot{q})v + \hat{B}(q)\dot{q} + \nabla_q \hat{U}(q) + Kr =: \overline{Y}(q,\dot{q},\dot{v},v)\hat{\Phi} + Kr\,.$$

- The closed loop system is

$$M(q)\dot{r} + C(q,\dot{q})r + Kr = \overline{Y}(q,\dot{q},\dot{v},v)\widetilde{\Phi}\,,$$

  where $\widetilde{\Phi} = \Phi - \hat{\Phi}$.

- Define the Lyapunov function

$$V = \frac{1}{2}r^{\mathrm{T}}M(q)r + \tilde{q}^{\mathrm{T}}\Lambda K\tilde{q} + \frac{1}{2}\widetilde{\Phi}^{\mathrm{T}}\Gamma\widetilde{\Phi}\,.$$

  If we substitute the expression for r, this function can be seen to be positive definite at $(\tilde{q}, \dot{\tilde{q}}, \widetilde{\Phi}) = (0,0,0)$.

- We compute the Lie derivative

$$\begin{aligned}
\dot{V} &= r^{\mathrm{T}}M(q)\dot{r} + \frac{1}{2}r^{\mathrm{T}}\dot{M}r + 2\tilde{q}^{\mathrm{T}}\Lambda K\dot{\tilde{q}} + \widetilde{\Phi}^{\mathrm{T}}\Gamma\dot{\widetilde{\Phi}} \\
&= r^{\mathrm{T}}\left[-C(q,\dot{q})r - Kr + \overline{Y}(q,\dot{q},\dot{v},v)\widetilde{\Phi}\right] + \frac{1}{2}r^{\mathrm{T}}\dot{M}r + 2\tilde{q}^{\mathrm{T}}\Lambda K\dot{\tilde{q}} + \widetilde{\Phi}^{\mathrm{T}}\Gamma\dot{\widetilde{\Phi}} \\
&= -r^{\mathrm{T}}Kr + r^{\mathrm{T}}\overline{Y}\widetilde{\Phi} + \frac{1}{2}r^{\mathrm{T}}\left[\dot{M} - 2C\right]r + 2\tilde{q}^{\mathrm{T}}\Lambda K\dot{\tilde{q}} + \widetilde{\Phi}^{\mathrm{T}}\Gamma\dot{\widetilde{\Phi}} \\
&= -r^{\mathrm{T}}Kr + 2\tilde{q}^{\mathrm{T}}\Lambda K\dot{\tilde{q}} + \widetilde{\Phi}^{\mathrm{T}}\overline{Y}^{\mathrm{T}}r + \widetilde{\Phi}^{\mathrm{T}}\Gamma\dot{\widetilde{\Phi}} \\
&= -\begin{bmatrix}\tilde{q}^{\mathrm{T}} & \dot{\tilde{q}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\Lambda K\Lambda & 0 \\ 0 & K\end{bmatrix}\begin{bmatrix}\tilde{q}^{\mathrm{T}} \\ \dot{\tilde{q}}^{\mathrm{T}}\end{bmatrix} + \widetilde{\Phi}^{\mathrm{T}}\left(\overline{Y}^{\mathrm{T}}r + \Gamma\dot{\widetilde{\Phi}}\right)\,.
\end{aligned}$$

- We choose the adaptation law $\dot{\hat{\Phi}} = \Gamma^{-1}\overline{Y}(q, \dot{q}, \dot{v}, v)^{\mathrm{T}} r$. Then

$$\dot{V} = -\begin{bmatrix} \tilde{q}^{\mathrm{T}} & \dot{\tilde{q}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \Lambda K \Lambda & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \tilde{q}^{\mathrm{T}} \\ \dot{\tilde{q}}^{\mathrm{T}} \end{bmatrix}.$$

- Because the matrix $\begin{bmatrix} \Lambda K \Lambda & 0 \\ 0 & K \end{bmatrix}$ is symmetric and positive definite, we get $\dot{V} \leq 0$. By the LaSalle Invariance Principle, $\dot{V} \longrightarrow 0$ along solutions of the closed-loop system. By the form of $\dot{V}$, this implies $(\tilde{q}(t), \dot{\tilde{q}}(t)) \longrightarrow 0$, as desired.

- Notice there is no statement that $\widetilde{\Phi} \longrightarrow 0$.