PHY407 Computational Physics Lab Assignment 6

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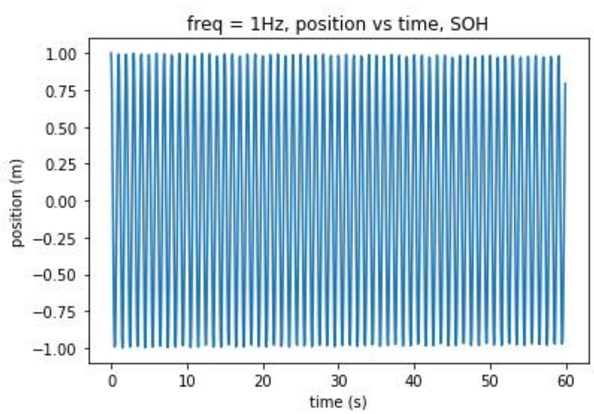
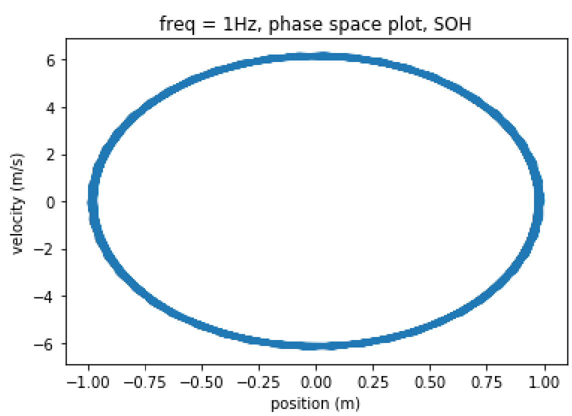
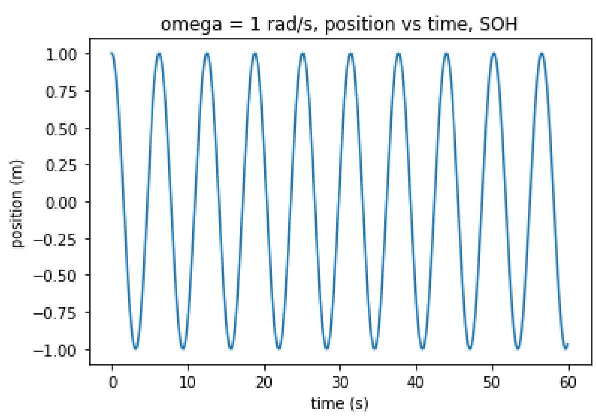
Question 1: The van der Pol oscillator

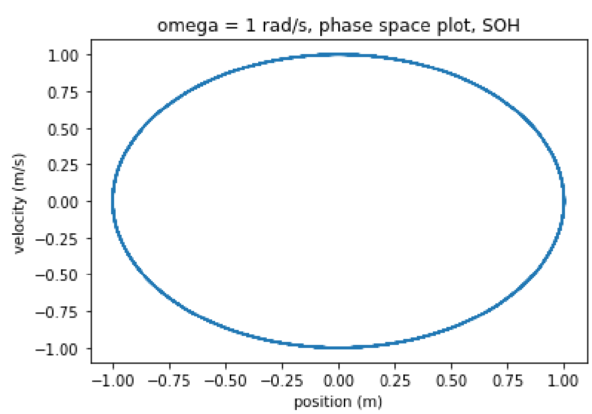
a) In this problem, we would like to convert a simple harmonic oscillator into 2 1st order ODEs in order to use the RK4 algorithm to integrate a 1 Hz pendulum for 60 s. To test our result, we double checked with a simpler known SHO.

The plot of the simple harmonic oscillator with a frequency of 1 Hertz has a corresponding ω value of 6.28 rad/s. The 60 oscillations in 60 seconds is a bit cumbersome to look at, so we've also included a plot where ω = 1.0 rad/s for testing purposes, and will use this one for our further analysis. The following four plots (two position time and two phase space) were outputted:







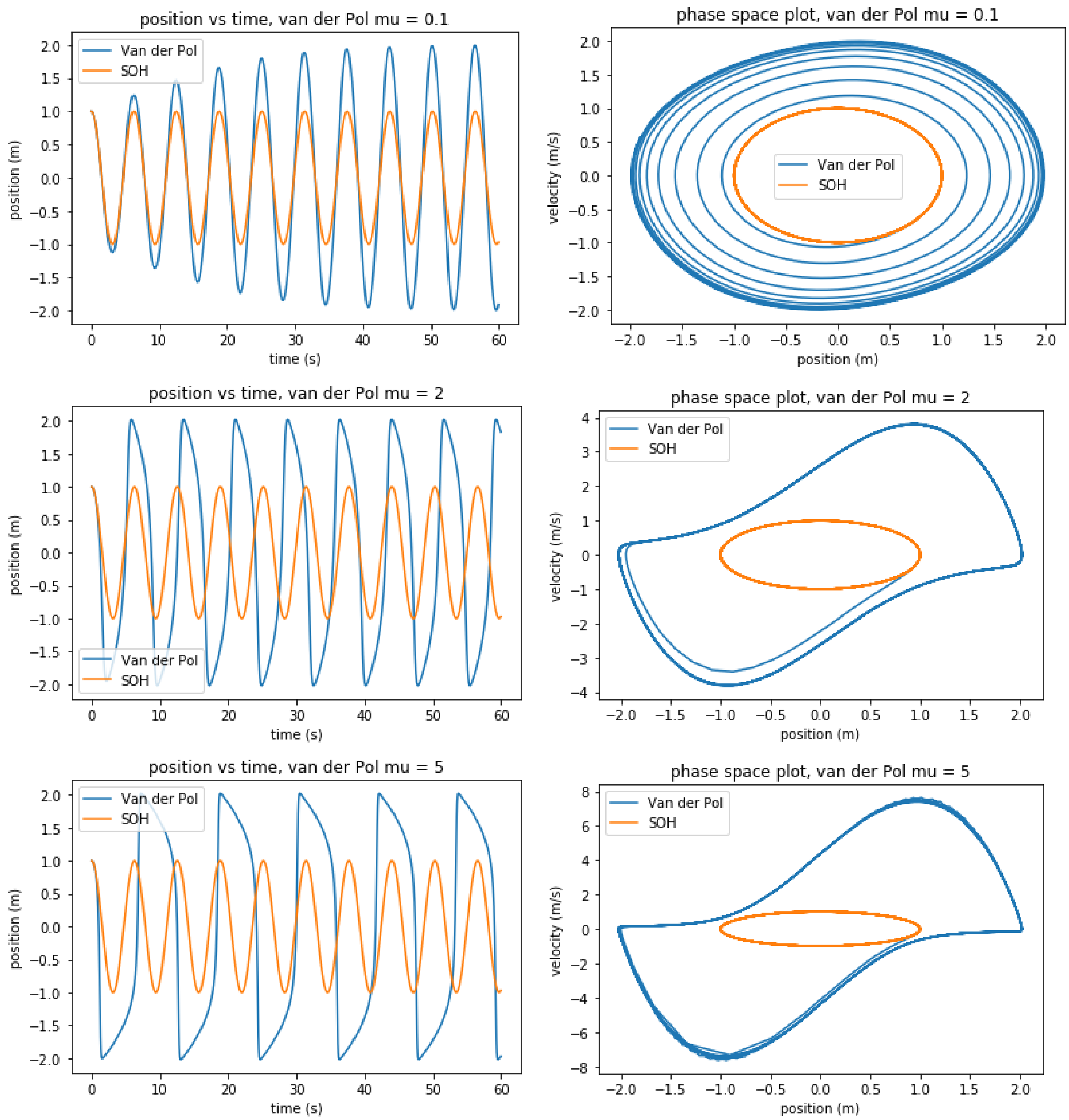


b) In this problem, we would like to modify our equations to reflect the van der Pol oscillator given in the equation:

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + \omega^2 x = 0$$

And explore how the results behave when we change μ to 0.1, 2 , and 5.

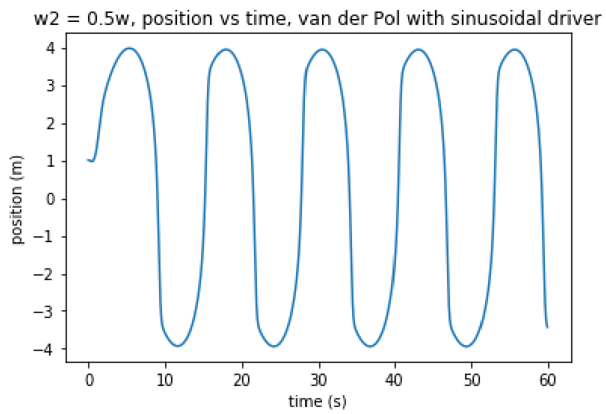
Modifying our original code to include the additional term altered both the phase space and position time graphs significantly. These changes are apparent in the outputs below:

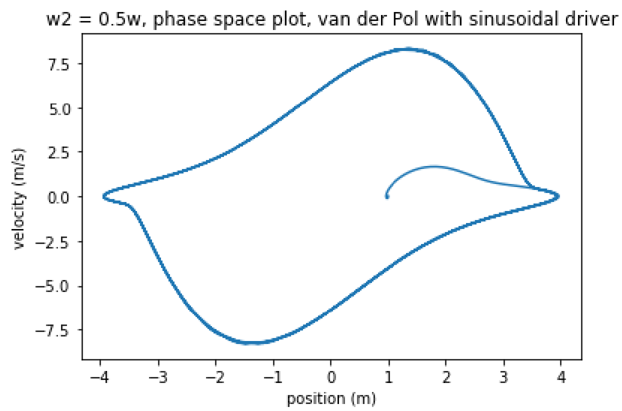


c) In this problem, we would like to investigate the effect of adding a driving term to the simulation. By adding a sinusoidal driving term to the van der Pol oscillator with the following conditions:

$$\omega_2 = 0.5\omega, \qquad \omega_2 = \omega, \qquad \omega_2 = 4\omega, \qquad \mu = 1, \qquad A = 8.53, \qquad \omega = 1$$

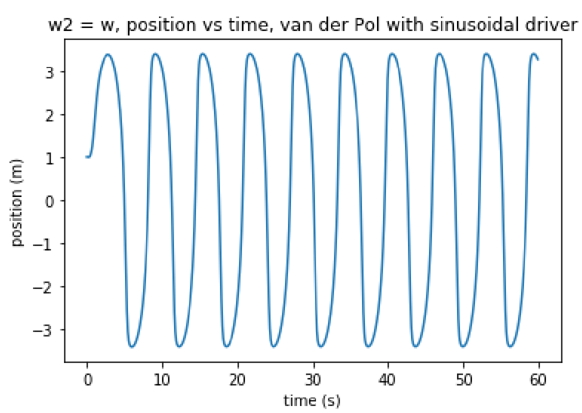
The first condition is adding a sinusoidal driver with angular frequency $\omega_2 = 0.5\omega$. This gives:

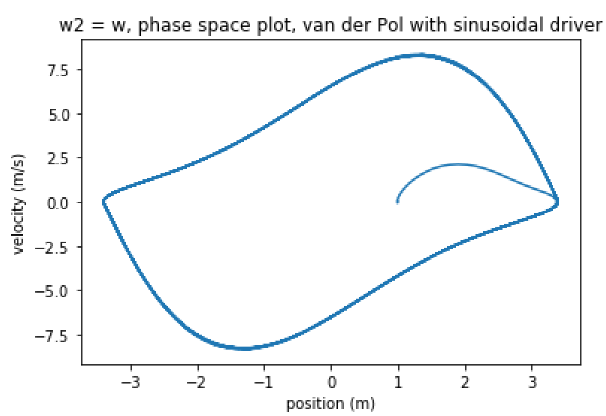




The addition of this driver has dramatically slowed the rate of oscillation, which is immediately apparent in the position time graph. We can see a much larger ampliude, or radius of oscillation (4m when compared to 2m without the driver). The oscillation also appears a bit awkward, spending more time at the extreme ends of the oscillation compared to the time spent in between transfering from one end to the other. The phase space plot is a bit more defined as well, including 'starting' point at (1,0) and sharper corners on the edges of the position axis.

The second condition is adding a sinusoidal driver with angular frequency $\omega_2 = \omega$. This gives:

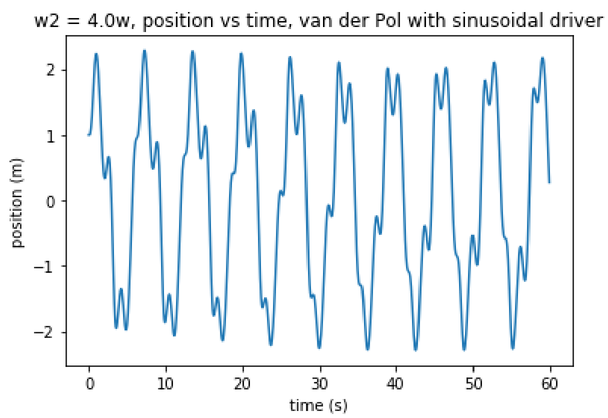


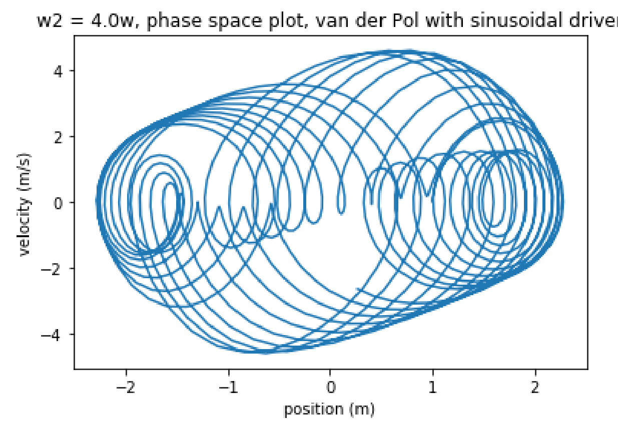


The plots shown here are very similar to those found in part b. Because the driving term has the same angular frequency as the van der Pol it's being applied to, we see a plot with similar frequencies to the van der Pol in part b (slightly different because here $\mu = 1$ while in b it was

0.1, 2 and 5) but with a much smoother pattern of oscillation than those shown in b. This looks more like the SOH than the van der Pol plots we saw before. The amplitue has been increased thanks to the addition of the sinusoidal driver, to 3 when compared to its original 2, and the phase plot exhibits a very similar shape to those found in part b, only sharper in its definition (thinner lines).

The third condition is adding a sinusoidal driver with angular frequency $\omega_2 = 4\omega$. This gives:





The position time plot shows a pretty messy oscillation, as the driver destructively interferes with the van der Pol oscillator throughout the time span. This results in a really interesting phase space plot, keeping a similar general shape (if you were to trace out the edges) to the previous two, but with a huge amount of fluctuation throughout as the sine driver and van der Pol compete through the oscillations. This causes the phase plot to take on the spiral form, because the actual size of the fluctuation varies with the interference of the two oscillators.

Question 2: The SEIR model:

a) In this problem, we would like to model system of ODEs in the SEIR model. The SEIR model simulates the spread of infectious disease through a population. This model divides the population into the categories: Susceptible(S), Exposed(E), Infectious(I), and Recovered(R). The rate at which the categories change is shown schematically below:

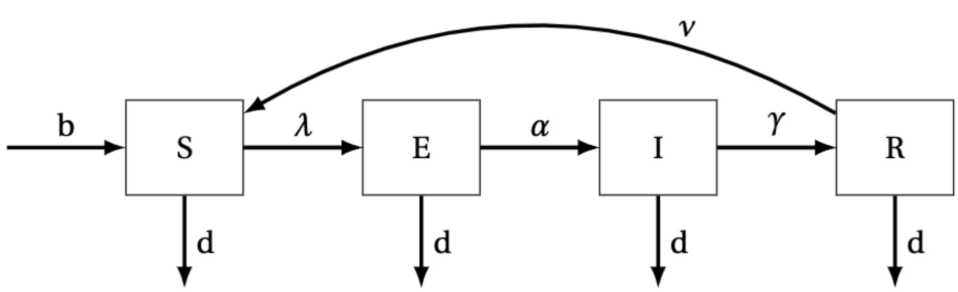


Figure 1: Population dynamics according to the SEIR model.

From the diagram, we extracted the rate of change for each category, where the arrows pointing into the box means it is adding more population to that category (+) while the arrows pointing away from the box means that the population are moving away from that category (-).

$$\frac{dS}{dt} = bN - \lambda S + vR - dS$$

$$\frac{dE}{dt} = \lambda S - \alpha E - dE$$

$$\frac{dI}{dt} = \alpha E - \gamma I - dI$$

$$\frac{dR}{dt} = \gamma I - vR - dR$$

To check our sanity, we need to make sure if the death rate is equal to birth rate, the total population should remain constant:

$$if\ b = d, show\ that\ \frac{dN}{dt} = 0$$

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

$$= bN - \lambda S + vR - dS + \lambda s - \alpha E - dE + \alpha E - \gamma I - dI + \gamma I - vR - dR$$

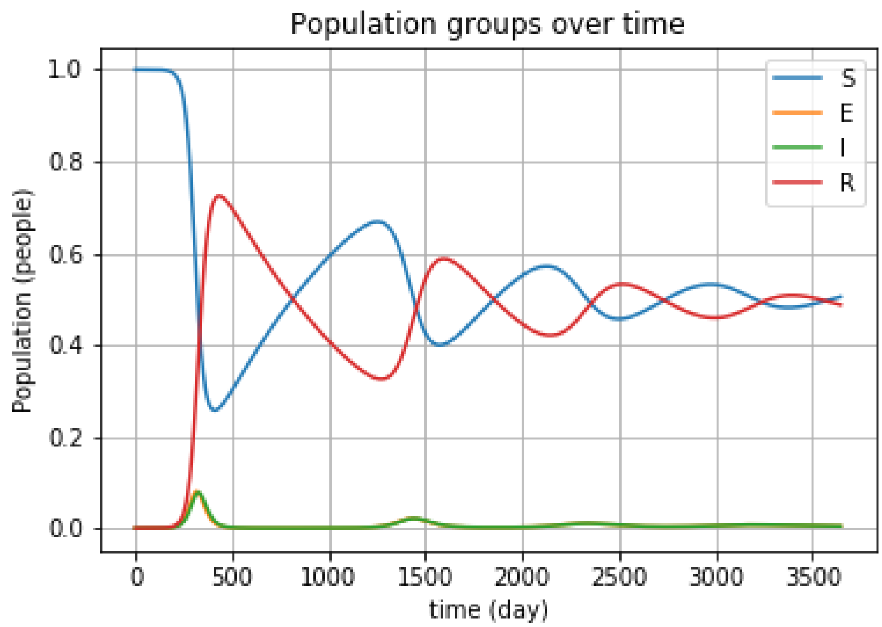
$$= bN - dS - dE - dI - dR$$

$$= bN - dN = 0\ when\ b = d$$

b) Nothing to submit

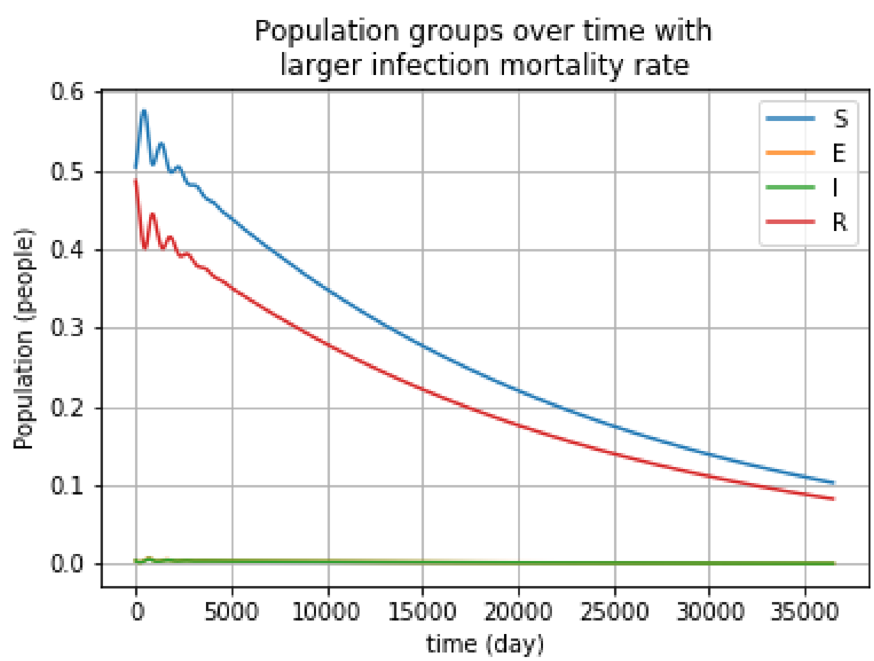
c) We can now simulate the SEIR model with the following parameters: β = 0.2, γ = 0.1, b = 10−4, d = 10−4, α = 0.1, ν = 10−3. Initial conditions are $I_o = 10^{-6}$, $N_o = 1$, and $S_o = N_o - I_o$. Then we simulated for 10 years (3650 days) and here are the results:

Using the required parameters, the following is a plot of the four functions of population percentage over time in days, associated with each of the population fractions in the SEIR model of disease.



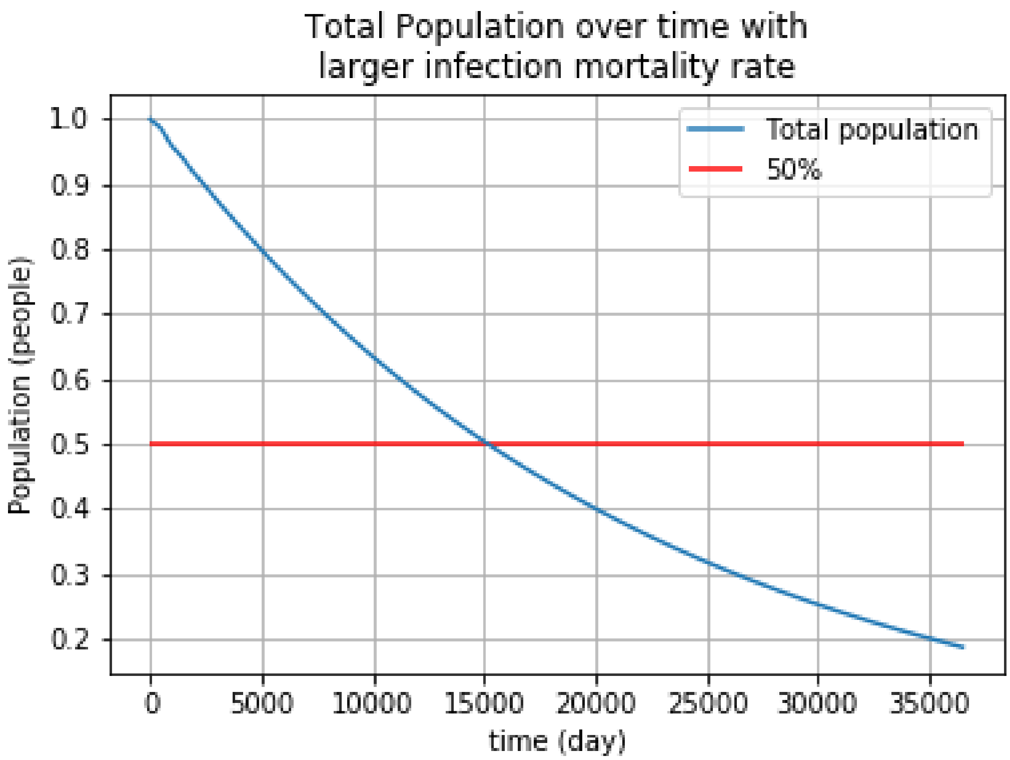
We can clearly see the count of susceptible individuals oscillating with the count of recovered individuals, eventually averaging out at around 50%, as well as small spikes in the infectious count, eventually flattening out as well.

d) We would like to investigate what happens when the mortality rate for the infected are 100 times higher, meaning $d = 10^{-2}$ just for the infected population when the other population having the same mortality rate as before. We would like to find when would the population to fall by half:



Here, we see the susceptible and recovered count fall over time rather than oscillate. This is because the new mortality rate causes the population to die out more frequently than they can recover, and thus less people can recover or become susceptible over time because they are just dying.

To isolate when the population falls below 50%, we look at the plot of total population percentage over time for a period of about 100 years (36500 days):



From the intersection of the total population function and the linear function along the 50% line, we see the population dips below 50% after a little over 15000 days, or a little over 41 years.

e) Now we would like to investigate the role of vaccination by adding a vaccinated population after 15 years. Below is the new SEIR model with the vaccinated population (V) with a birth rate of P, which assume all newborn after 15 years are vaccinated:

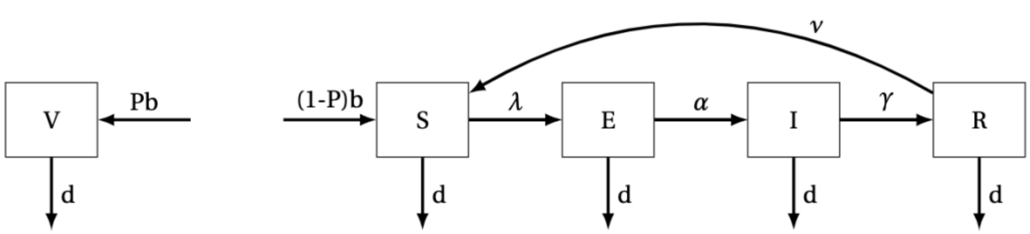


Figure 2: Population dynamics according to the SEIRV model with newborn vaccinations.

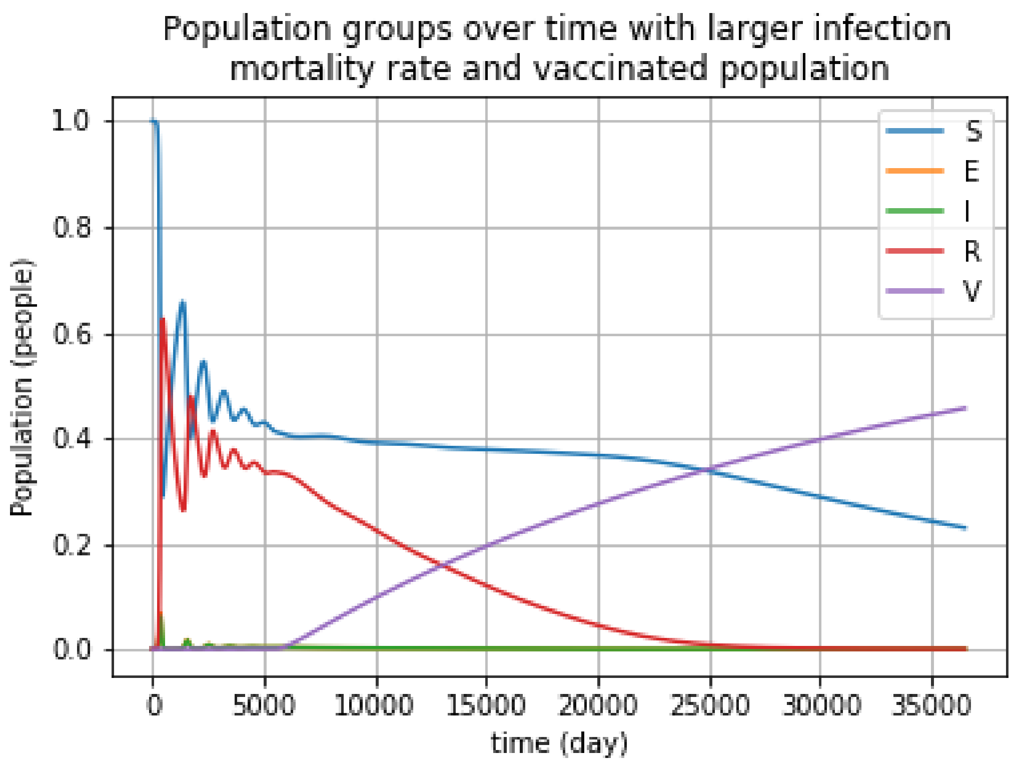
We added a new rate and modified the Susceptible population rate of change:

$$\frac{dV}{dt} = PbN - dV$$

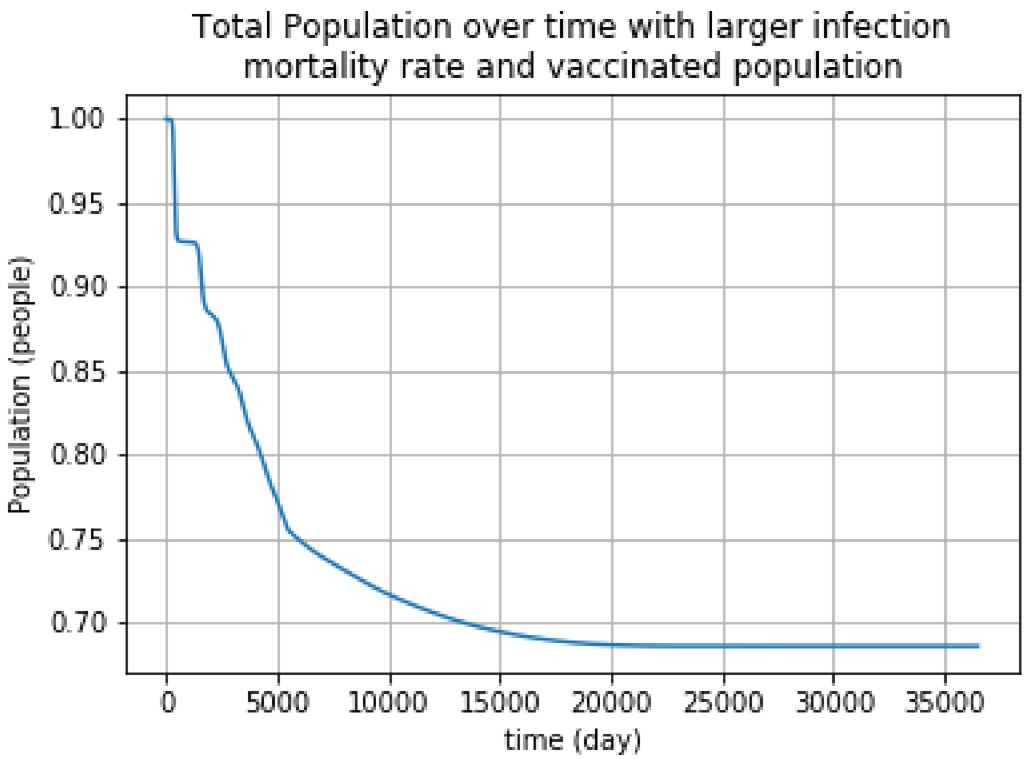
$$\frac{dS}{dt} = (1 - P)bN - \lambda S + vR - dS$$

While the rest of the system remains the same, including the increased mortality rate for the infected population.

First, the vaccination function was implemented into the model. Plotting the new model including the vaccination, with the same axis as the population percent functions in the previous two parts, we output the following graph:



The main thing to take away is the increasing vaccination function throughout the model. It begins increasing right when the susceptible count decreases, which makes sense as those individuals are likely now exposed or infected, and helps balance out the decrease. Taking a look at just how well it balances the decrease out, we plotted the total population percentage over time for the same period as in part d (100 years) to observe the behavior with the vaccination added in:



At around the same time as the function in part d reached 50% of the popuation remaining (15000 days), we observe the vaccinated population to steady out at just under 70% of the population. It seems to hit a horizontal asymptote at around 68% of the population remaining at the same time as the previous function hit 50% population remainaing, and sticks that mark with exceptional stability from the 45 year mark (16425 days) onwards. Therefore we can conclude that the new population has a very stable size thanks to vaccination.