PHY407 Computational Physics Lab Assignment 9
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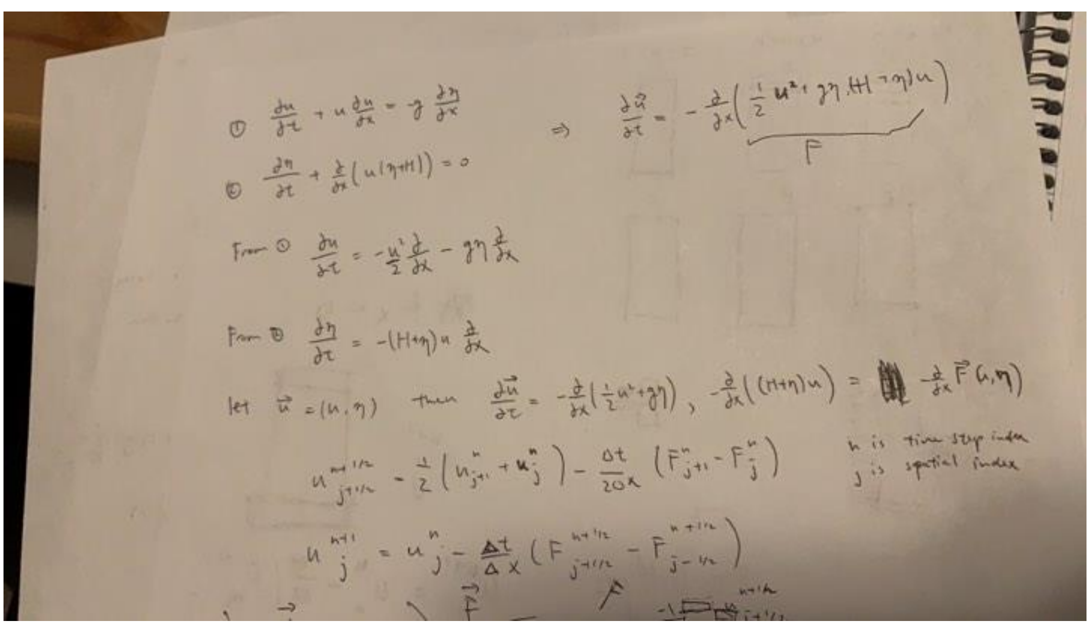
Question 1: Simulating the Shallow Water System

The Shallow Water Equations are used to describe the flow of fluid below a surface in situations where the depth of the fluid is much smaller in scale when compared to the horizontal length that the fluid covers. This has a range of applications in oceanic models because, as the lab manual illustrates, while the ocean is quite deep, it's also very expansive, therefore it's choosing a horizontal stretch that is sufficiently greater than the average depth in the stretch is very doable. For this particular question, we will be attempting to use the shallow water equations to model a 1-dimensional tsunami with relatively simplistic conditions. The equations we will be using will be only the 1-dimensional version, which is given in the lab manual as the following:

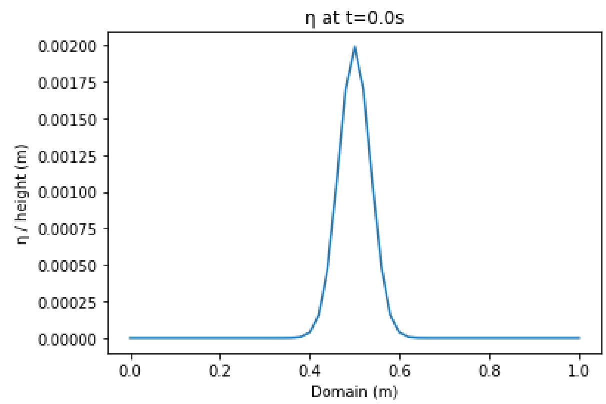
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(u(\eta + H)) = 0.$$

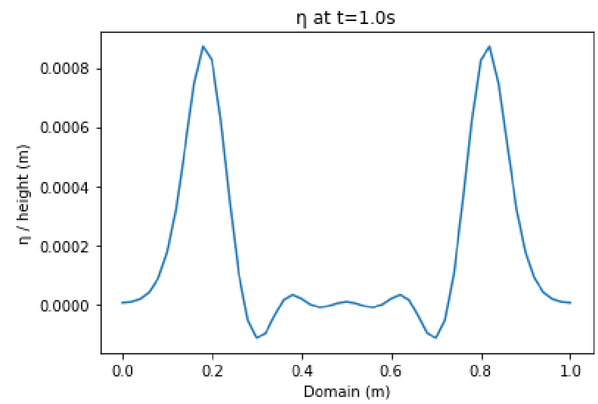
Where η is the surface displacement, g is the acceleration due to gravity, u is the velocity of the fluid in the x direction and H is a known function representing the varying depth from surface level to the ocean floor.

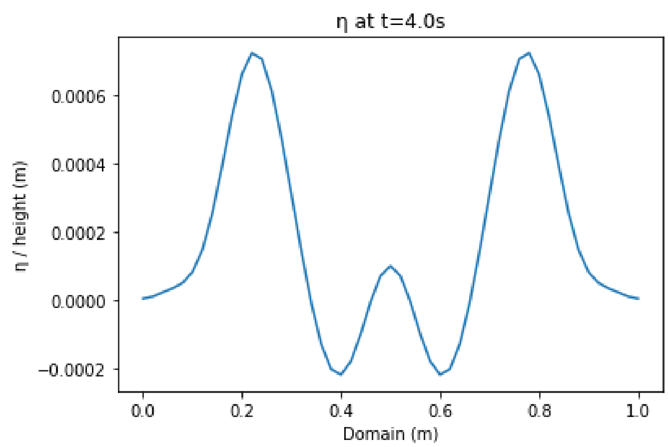
a) This first part asks us to rewrite these shallow water equations into flux-conservative form, and the use the Lax-Wendroff scheme to discretize the resulting equations. The derivations were done by hand so below is an image of the written calculations:



b) Now we want to implement the results in python. The goal here is to apply Lax-Wendroff scheme to produce a plot of our tsunami without any variation on the sea-floor (therefore a constant value of H). The conditions we were given are a horizontal range of 1 meter, as well as a constant depth of 0.01m. We were also given rigid boundary conditions and initial conditions for both the initial horizontal velocity as well as the initial surface displacement, which represent a Gaussian peak in the middle of our domain. Plotting for time values of 0, 1 and 4 seconds yield the following plots:

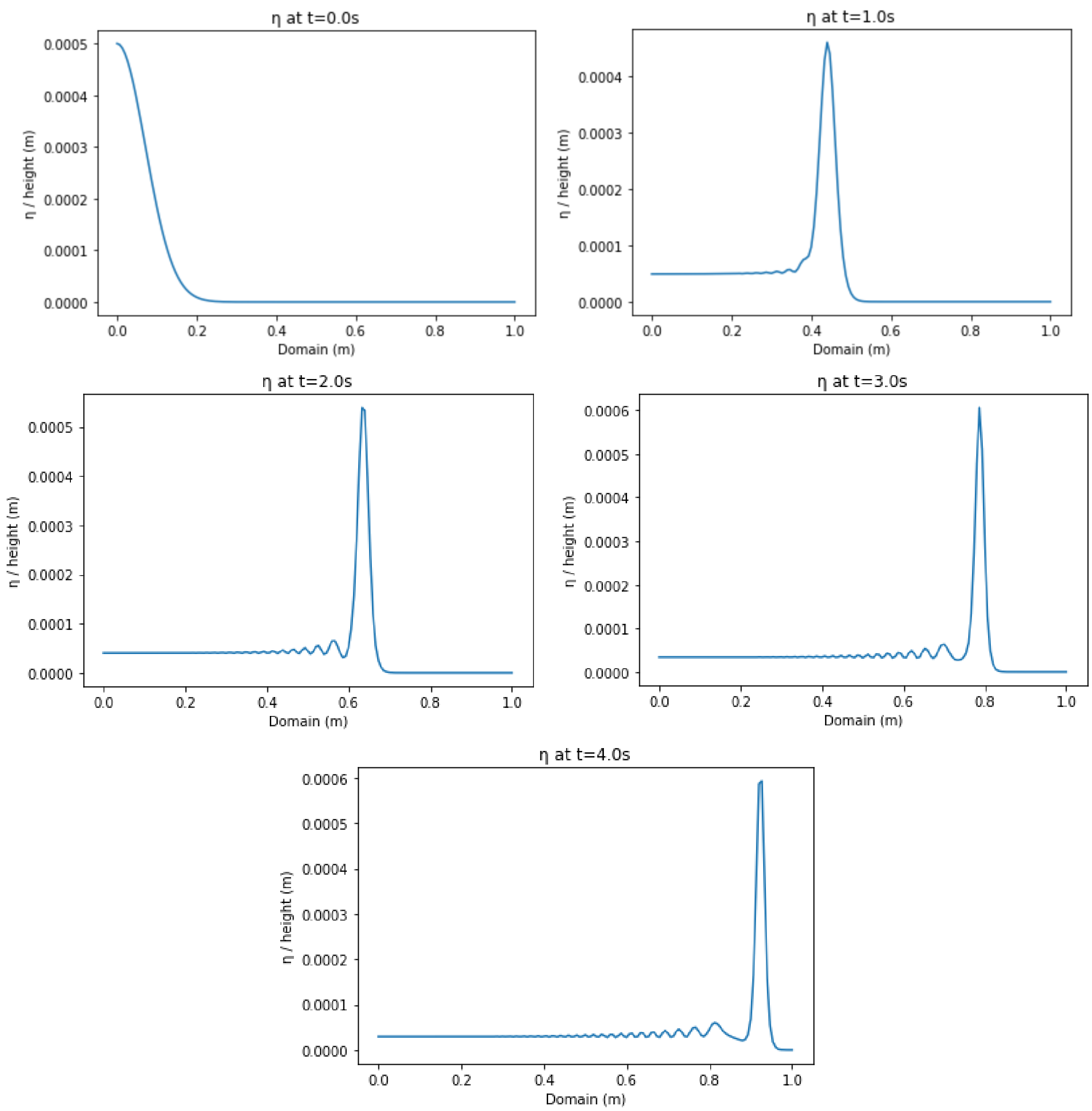






From these, we can easily observe the motion of the fluid. Starting with a peak in the center of our domain, the waves disperse towards the boundaries of the plot and reflect off, and continue oscillating in this way, shifting the peaks and interfering to create the ripples we see.

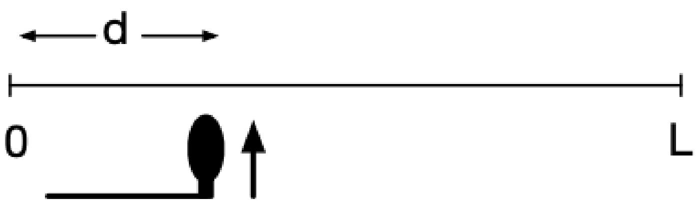
c) For this last part, we want to repeat most of what we did in the previous question only now using new initial conditions as well as a variable topography along the ocean floor. This means incorporating a function for H rather than leaving it as a constant value. We want to use 150 steps, and we want the topography at each point j to be used when calculating the associated flux. Plotting our new 1-dimentional tsunami for the times t = 0, 1, 2, 3, and 4 seconds, we get the following output:



These five figures represent the height of the wave as it moves to the right from deeper to shallower water. It's clear that the width of the wave decreases substantially as the water gets shallower, but the height above the surface line remains relatively constant. So, shape wise we see a thinner and thinner spike at the location of the wave. The velocity also appears to decrease with shallower water, as the spacing of the wave from the previous plot is shorter and shorter with every successive image while leaving the time steps constant at 1 second.

Question 2: Solving the wave equation

For this question, we want to think about a piano string of length L, initially at rest, being struck by a piano hammer at a particular distance d from the end of the string, creating a vibration and therefore an associated wavefunction. The diagram of the situation is given in the lab manual as the following:



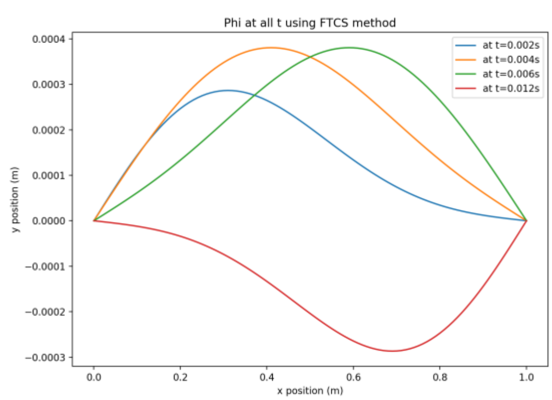
The goal for this question is to investigate the resulting wavefunction using 3 different methods for solving sets of first order differential equations. The FTCS method, the Crank-Nicolson method and the spectral method, and to analyze the differences in the corresponding results.

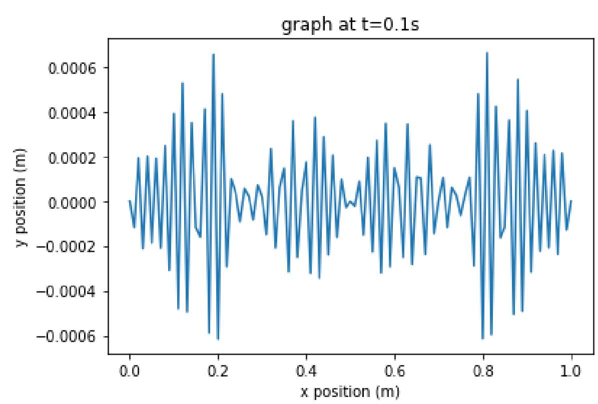
a) For this first part, we are asked to write a program which uses the FTCS method to solve equations 9.28 in the computational physics Mark Newman text using the initial conditions that phi(x) is zero everywhere, but psi(x) (velocity) is nonzero and behaves like:

$$\psi(x) = C\frac{x(L-x)}{L^2}\exp\left[-\frac{(x-d)^2}{2\sigma^2}\right]$$

The purpose of this is purely for comparison for future questions, as we expect this method to break eventually as time gets close to the end value.

b) Now we want to apply the FTCS method to our piano string. We want to demonstrate the wave motion of the string but without animating it, so to do so we plot position graphs for various times to get an idea how it's evolving. The times required are t = 2, 4, 6, 12 and 100ms. This last value should blow up using the FTCS method, and so it was plotted seperately.





The left plot shows the FTCS method plotting accurate results up to 12 ms. The wave begins at rest, is hit on the left end causing the blue peak at 2ms, after which the wave travels down the string and is reflected invertedly at 12 ms. Precisely the behaviour we would classically expect.

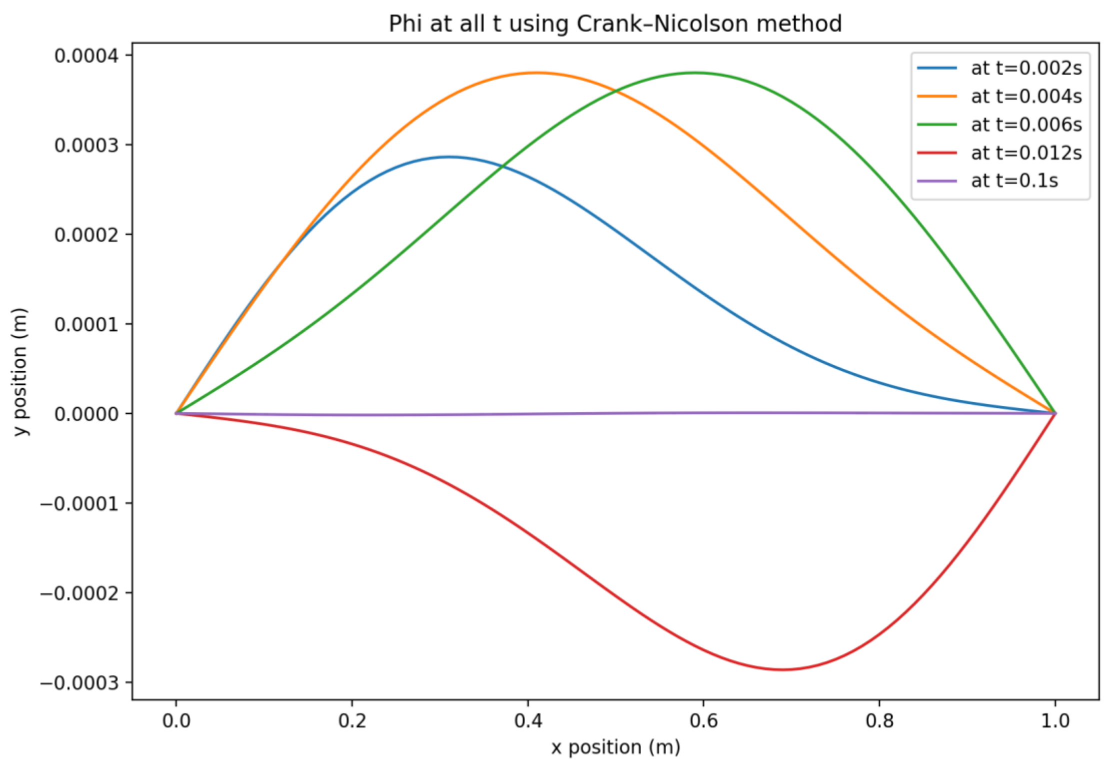
However, at t = 100ms, we see the structure break entirely leaving a completely redundant solution.

c) For this question, we first substitue (9.49b) into (9.49a) and rearrange so that the LHS has terms at t+h while the RHS has terms at t.
Then we observed that we can use a near tri-diagonal matrix with the diagonal being -2 and its neighbours as 1, we can represent the part of equations underlined in red into a matrix multiplication.

$$\phi(x,t+h) - \tfrac{1}{2}h\psi(x,t+h) = \phi(x,t) + \tfrac{1}{2}h\psi(x,t), \tag{9.49a}$$

$$\psi(x,t+h) - h\frac{v^2}{2a^2}\underline{\left[\phi(x+a,t+h)+\phi(x-a,t+h)-2\phi(x,t+h)\right]} = \psi(x,t) + h\frac{v^2}{2a^2}\underline{\left[\phi(x+a,t)+\phi(x-a,t)-2\phi(x,t)\right]}. \tag{9.49b}$$

Then, after implementing the Crank-Nicolson method. Here is our result:



d) For this question, we want to show that given the expansion of the initial condistions of our piano string into a fourier series, that the general solution for phi(x) is the following

$$\phi(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x/L)\left[\bar{\phi}_{0,k}\cos(\omega_k t) + \frac{\bar{\psi}_{0,k}}{\omega_k}\sin(\omega_k t)\right]$$

This was done by hand using the methods in the Newman text on pages 435-437.

Q2 d)

$$\varphi_0(x) = \sum_{k=1}^{\infty} \tilde{\varphi}_{0,k} \sin(k\pi x/L), \quad \psi_0(x) = \sum_{k=1}^{\infty} \tilde{\psi}_{0,k} \sin(k\pi x/L)$$

WTS $\varphi(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x/L)\left[\tilde{\varphi}_{0k}\cos(\omega_k t) + \frac{\tilde{\psi}_{0k}}{\omega_k}\sin(\omega_k t)\right]$

(3) $\frac{\partial^2\varphi}{\partial t^2} = v^2 \frac{\partial^2\varphi}{\partial x^2}$

(4) $\varphi(x=0,t) = \varphi(x=L,t) = 0$ (zero at bounds)

(5) $\varphi_0(x) = \varphi(x,t=0)$, $\psi_0(x) = \psi(x,t=0) = \frac{\partial\varphi(x,t=0)}{\partial t}$

consider $\varphi_k(x,t) = \sin\left(\frac{\pi k x}{L}\right)e^{i\omega t}$

where $\omega = \frac{\pi v k}{L}$, for interger k

this satisfies $\frac{\partial^2\varphi}{\partial t^2} = v^2\frac{\partial^2\varphi}{\partial x^2}$

as $\frac{\partial^2\varphi}{\partial t^2} = -\omega^2\sin\left(\frac{\pi k x}{L}\right)e^{i\omega t}$

and $v^2\frac{\partial^2\varphi}{\partial x^2} = v^2\left(-\frac{\pi^2 k^2}{L^2}\left(\sin\left(\frac{\pi k x}{L}\right)e^{i\omega t}\right)\right)$

and $\frac{-v^2\pi^2 k^2}{L^2} = -\omega^2$

for a linear eqn, that means

$$\varphi(x_n,t) = \sum_{k=1}^{N} b_k \sin\left(\frac{\pi k n}{N}\right)e^{i\omega t}$$

for individual segments in N intervals

writing $b_k = \alpha_k - \eta_k$ as in the text we have

$$\varphi(x_n,t) = \sum_{k=1}^{N-1}\left[\alpha_k\cos\left(\frac{\pi v k t}{L}\right) - \eta_k\sin\left(\frac{\pi v k t}{L}\right)\right]\sin\left(\frac{\pi k n}{N}\right)$$

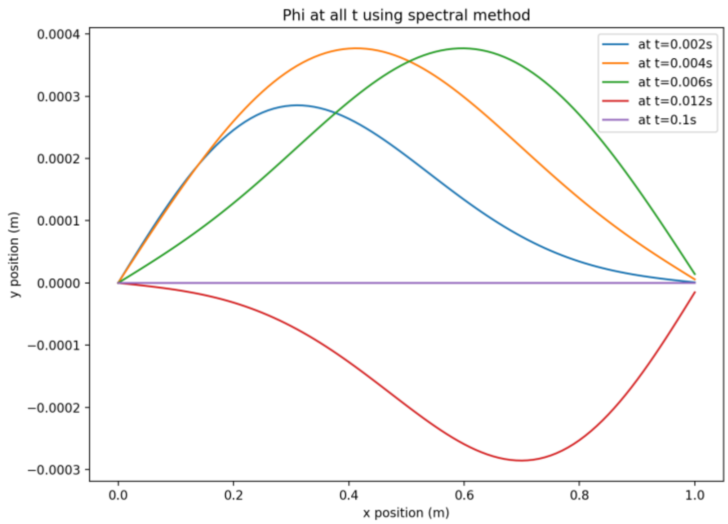
substituting $\alpha_k = \tilde{\varphi}_{0,k}$ and η_k as $\frac{-\tilde{\psi}_{0k}}{\omega_k}$

and $\omega = \frac{\pi v k}{L}$

this gives

$$\boxed{\varphi(x,t) = \sum_{k=1}^{\infty}\sin(k\pi x/L)\left[\tilde{\varphi}_{0k}\cos(\omega_k t) + \frac{\tilde{\psi}_{0,k}}{\omega_k}\sin(\omega_k t)\right]}$$

e) We now want to apply this method to our string once again to produce a plot of the wave function at the same times as in the previous questions. This method is exceptionally easy to code using the dst and idst functions provided by the dcst.py resource link. To do so, we applied the dst function to our psi wave to discretize it. Next, divided it by omega and multiplied it by sin(omega*t) to replicate the coefficient in the square bracket in the above solution. We can ignore the term with a phi on it because the initial position is 0. Finally, we get our solution by taking the inverse discrete sine transform of the found coefficient. The plot that resulted for the desired time values is the following:



f) The first four lines match up with our other results perfectly, and the final t=0.1s line didn't break, displaying the effectiveness of this method at higher values of t. The FTCS does hold the advantage of being quicker than the other two, but that's about as far as its advantages go. The spectral method was substantially easier to code than the FTCS and Crank-Nicolson, as well as far more accurate at higher time values than the FTCS. However, a key drawback is that this method is limited in its applications, we were fortunate in that this problem met the requirements of simple boundary conditions (rigid edges in our case). This method also only works for linear differential equations because we can't sum a set of solutions for a nonlinear system and expect a correct result as we did here. So overall, the spectral method seems ideal for this particular sort of problem, however it is more limited in usefulness when compared to the other two.