Physics 598
Homework 5

Cunwei Fan

cfan11

Problem 1

For rarefaction, first do some definition. For any piston traveling with velocity, $-U_p$, we can define

$$U_p' = \begin{cases} U_p & U_p < U_{max} \\ U_{max} & U_p > U_{max} \end{cases}$$

Where $U_{max} = (2/(\gamma - 1))C_0$ for $C_0 = (\gamma P_0/\rho_0)^{1/2}$. Then the analytical solution is

	$x_i > C_0 t$ (region I)	$(C_0 - \frac{4}{3}U_p')t < x_i < C_0 t$ (region II)	$U_p' t < x_i < (C_0 - \frac{4}{3}U_p')t$ (region III)
u_i/C_0	0	$-3 + 3\left(r_i/(C_0 t)\right)^{1/4}$	$-U_p'$
C_i/C_0	1	$1 - \frac{1}{3}u_i/C_0$	$1 - \frac{1}{3}U_p'/C_0$
ρ_i/ρ_0	1	$[C/C_0(u_i)]^3$	$[C/C_0(U_p')]^3$
P_i/P_0	1	$[C/C_0(u_i)]^5$	$[C/C_0(U_p')]^3$

where, $\gamma = 5/3$ is used and the C/C_0 is a function of u. Also, the u_i, C_i, P_i and ρ_i are parameters of particle r_i, where r_i is the lagrangean index for each lagrangean particle. Besides, the equation for u_i/C_0 is by solving the differential equaiton

$$u = \dot{x} = -\frac{2}{\gamma + 1}(C_0 - x/t)$$

with initial condition $x_0 = r_i$ and $t = r_i/C_0$.

Equaitons

For the numerical simulation, the initial condition for the fluid is all particles r_i: $u_i = 0$, $P_i = 1$, $\rho_i = 1$ and $\epsilon = 1.5$. But to make the system to start evolving, the first particle or $r = 0$ particle has velocity as the piston's. For the followingsimulation, the r_0 particle is treated as the piston, i.e., with constant velocity U_p. Also, each particle has initial spacing 0.1. There are initially 100 particles.

The integration equations used are :

$$\begin{aligned}
u_j^{n+1} &= u_j^n - \Delta t A_j^n (P_j^n - P_{j-1}^n + q_j^n - q_{j-1}^n) \\
R_j^{n+1} &= R_j^n + \Delta t u_j^n \\
V_j^{n+1} &= V_j^0 \frac{(R_{j+1}^{n+1})^\alpha - (R_j^{n+1})^\alpha}{(r_{j+1})^\alpha - (r_j)^\alpha} \\
\epsilon^{n+1} &= [1 + \frac{1}{3}(V_j^{n+1} - V_j^n)/V_j^{n+1}]^{-1} [\epsilon_j^n - \frac{1}{2}(P_j^n + q_j^{n+1})(V_j^{n+1} - V_j^n)] \\
P_j^{n+1} &= \frac{2}{3}\epsilon_j^{n+1}/V_j^{n+1}
\end{aligned}$$

Where assumed each fluid element has mass $1 = \rho n_j V_j^n$ (that is why there is no dividing mass in the u equation). And A_j^n is the transverse area of particle r_j at iteration n which can be calculated as

$$A_j^n = \pi(2R_j^n)^\alpha = \frac{V_j^n + V_{j-1}^n}{2\Delta R}$$

Also,

$$\frac{V_j^0 + V_{j-1}^0}{2\Delta r} = \pi(2r_j)^{\alpha-1} \quad \Rightarrow \quad A_j^n = \frac{V^n j}{\Delta R} = \left(\frac{R_j^n}{r_j}\right)^{\alpha-1} \frac{V_j^0 + V_{j-1}^0}{2\Delta r}$$

where $\Delta r = 1/2(r_{j+1} - r_{j-1})$. The time step is chosen with some fraction of the critical time step:

$$\Delta t_{critical} = \min\{\frac{b(R_{j+1}^n - R_j^n)}{\gamma P_j^n V_j^n}\}$$

And the viscousity q is nonzero only when $u_{j+1}^n - u_j^n < 0$:

$$q_j^n = \frac{2a^2}{V_j^n + V_j^{n-1}}[u_{j+1}^n - u_j^n]^2$$

Where $a = 1.8$ in this simulation and $b = 0.2$.

Results for $U_p = 0.1U_{max}$

Then the result for rarefaction with $U_p = 0.1U_{max}$ is shown in the figure below:

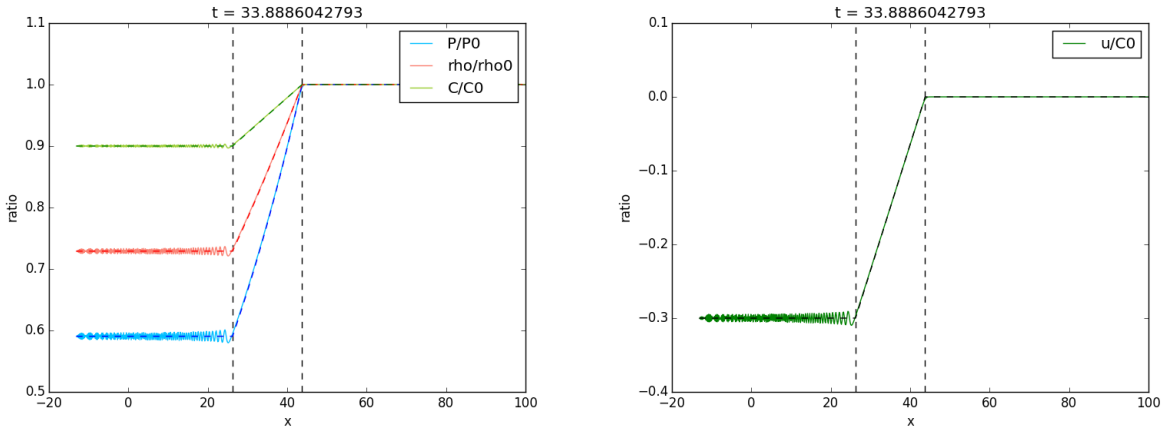

Figure 1: The **simulation** result is ploted wiht solid lines and the **analytical** ones are carepresented as dashed lines. The two vertical dashed lines are boundaries between region I and region II(right one) and region II and region III(left one). This plot is for all particles at time $t = 33.89$. The horizontal axis is space coordinates for each particle (Eulerian coordinates)

As can be seen, the numerical values is pretty accurate. As the the slid lines overlap with the dashed lines. And the C/C_0 converges to 0.9, ρ/ρ_0 converges to $0.9^3 = 0.729$, the P/P_0 converges to $0.9^5 = 0.591$ and u/C_0 converges to 0.3.

However, in this check, there is induced error as the horizonal x is used as the numerical results. Thus, to make a more convincing comparison, the parameters of one specific particle should be calculated.

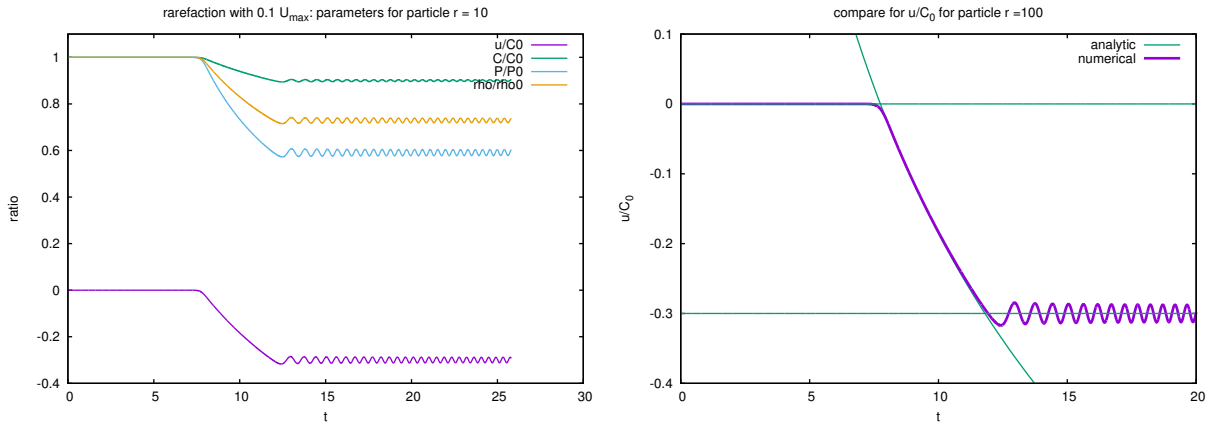

Figure 2: LEFT: all four parameters for particle 100 is plotted. RIGHT: Only the $u/C_0$is plotted the slimmer green line is the alanlytic solutions for three regions.

As can be seen, the acceleration for the analytical particles has infinite value at region boundaries but the simulation cannot do this. Besides, the difference is not big.

Energy check

For the energy check, the formula used is based on the equations of motion for the fluids:

$$\frac{1}{2}((u_j^{n+1})^2 - (u_j^n)^2) = \frac{1}{2}(u_j^{n+1} + u_j^n)(u_j^{n+1} - u_j^n) = \frac{1}{2}(u_j^{n+1} + u_j^n)(-\Delta t A_j^n (P_j^n - P_{j-1}^n + q_j^n - q_{j-1}^n))$$

Therefore:

$$\frac{1}{2}((u_j^{n+1})^2 - (u_j^n)^2) + \frac{1}{2}(u_j^{n+1} + u_j^n)(\Delta t A_j^n (P_j^n - P_{j-1}^n + q_j^n - q_{j-1}^n)) = 0$$

Thus, when this is summed up for all particles (sum over j's) and integrate over all time, the calculated quantity should not change with time, i.e:

$$\sum_j \frac{1}{2}(u_j^n)^2 + \sum_{j,n} \frac{1}{2}(u_j^{n+1} + u_j^n)\Delta t A_j^n (P_j^n - P_{j-1}^n + q_j^n - q_{j-1}^n) = \text{Constant}$$

Also, for the internal energy, we assumed adiabatic, therefore,

$$\epsilon_j^{n+1} - \epsilon_j^n = -(\frac{1}{2}(P_j^{n+1} + P_j n) + q_j^n)(V_j^{n+1} - V_j^n)$$

However, to check physically, the q terms should be eliminiated as it is introduced numerically so sum this equation for all j's and add the two equations together, we have :

$$\sum_j [\frac{1}{2}(u_j^n)^2 + \epsilon_j^n] + \sum_n \sum_j (\frac{1}{2}(P_j^{n+1} + P_j^n))(V_j^{n+1} - V_j^n) + \sum_n \sum_j \frac{1}{2}(u_j^{n+1} + u_j^n)\Delta t A_j^n (P_j^n - P_{j-1}^n) = \text{Constant}$$

If expressed in continues form, the equation is :

$$K(t) + T(t) + \int_0^\infty \int_{V_0}^{V(R,t)} P \, \mathrm{d}V \, \mathrm{d}R + \int_0^\infty \int_0^t uV \frac{\partial P}{\partial R} \, \mathrm{d}t \, \mathrm{d}R = \text{ Constant}$$

Then, the energy check for the rarefaction with $U_p = 0.1U_{max}$ for different time steps is

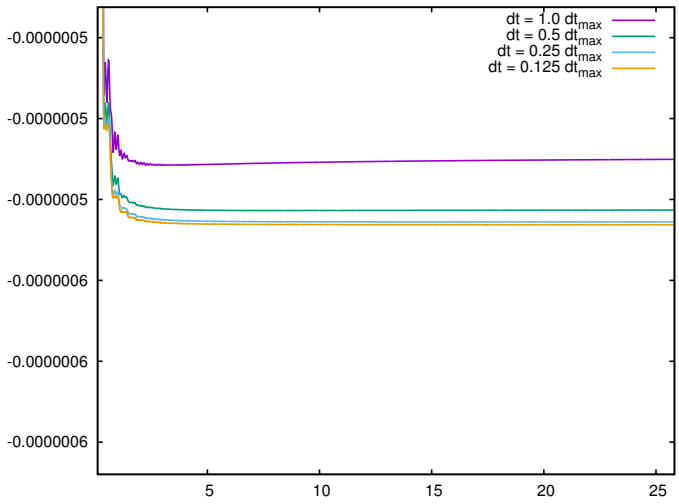

Figure 3: The energy check for different time steps, where dt_max represents the maximum stable time steps. The y-axis is as $E - E_0$ with $E_0 = 1498.5$ which is the energy at the first time

Results for $U_p = 10U_{max}$

For the comparison of the whole system :

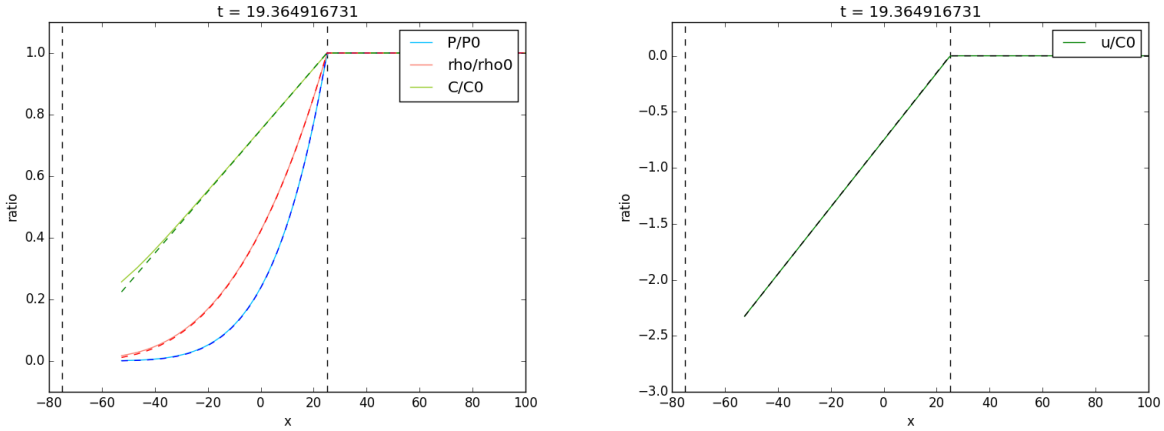

Figure 4: The **simulation** result is ploted wiht solid lines and the **analytical** ones are carepresented as dashed lines. The two vertical dashed lines are boundaries between region I and region II(right one) and region II and region III(left one). This plot is for all particles at time $t = 19.36$. The horizontal axis is space coordinates for each particle (Eulerian coordinates)

As can be seen in the plot, all particles cannor reach the piston which is the left dashed line (in this limmiting case, the $U_p' = U_{max}$, the boundary for III and II ovlerpas with the piston). Therefore, there is some vaccume between the piston and the fluid.

As before, check for one particle, but this time, to save the simulation power time, particle with $r = 0.1$ or the first good partical is checked.

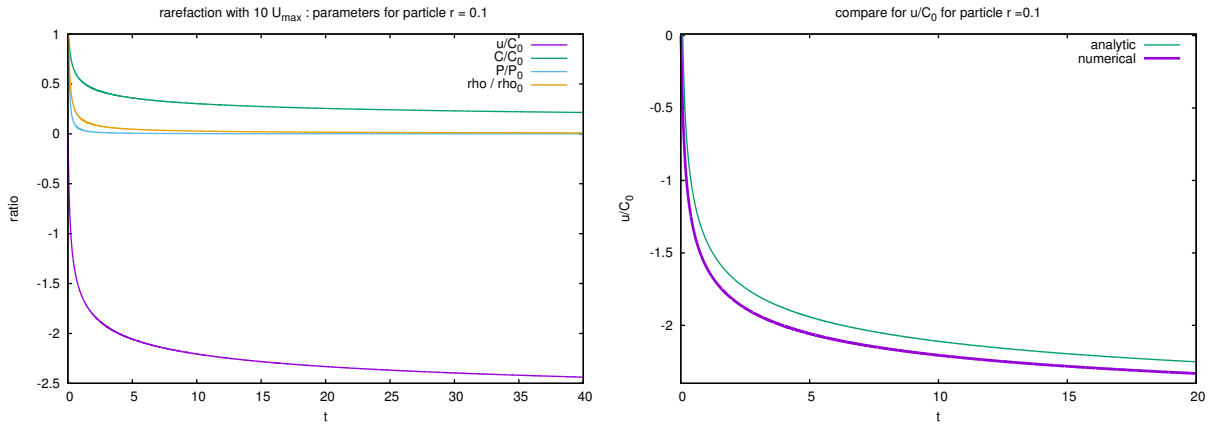

Figure 5: LEFT: all four parameters for particle $r = 0.1$ is plotted. RIGHT: Only the $u/C_0$is plotted the slimmer green line is the alanlytic solutions for three regions.

As can be seen in the figure, the rho and P goes to zero which is predicted by the analytical solution. But the the error for the u is obvious so a detialed investagation is needed.

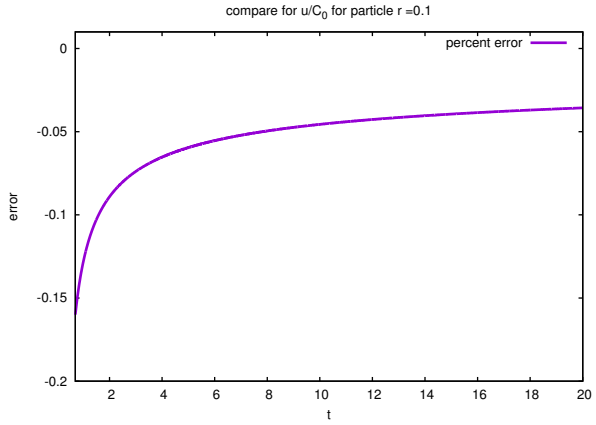

Figure 6: The percent error between the numerical and the analytical solution is calculated.

As can be seen from the figure, the error at starting time is prettey big since the particel's acceleration cannot make an abrupt change but as time goes on, the error stables at an acceptable value.

For the energy check:

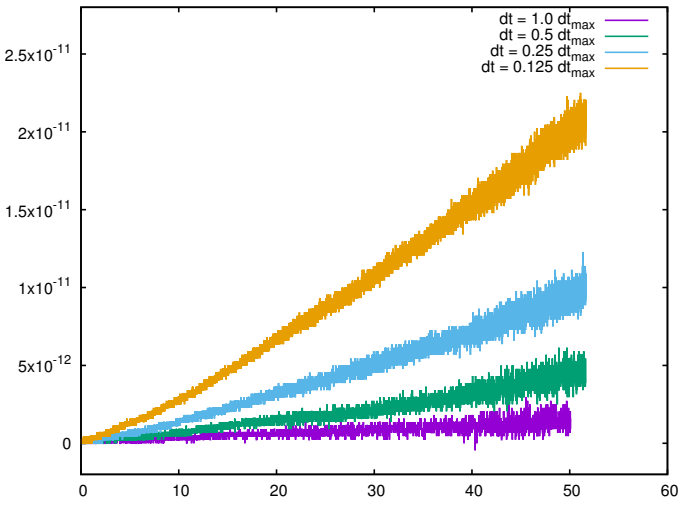

Figure 7: The energy check for different time steps, where dt_{max} represents the maximum stable time steps. The y-axis is as $E - E_0$ with $E_0 = 1498.5$ which is the energy at the first time

As seen, the error is much smaller. This is becuase, the particles never enter region III so that there is no visosity term which is discarded in our energy check. But the phenomeon that the finer the step is, the larger the error, makes the problem hard. For now, still there is no satisfactory answer.

Shocks

analytical solution

If the piston has velocity U_p, then to have conservation of mass it is needed that

$$U_{shock}\rho_0 = (U_{shock} - U_p)\rho_1 \text{ with } \frac{\rho_0}{\rho_1} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1}\frac{C_0^2}{u_0^2}$$

where $u_0 = u_{shock}$. This equation is solved as

$$u_0 = \frac{1}{4}(U_p(\gamma - 1) + \sqrt{16C_0^2 + U_p^2(1 + \gamma)^2})$$

Also with:

$$\frac{P_1}{P_0} = \frac{2\gamma}{\gamma + 1}\frac{u_0^2}{C_0^2} - \frac{\gamma - 1}{\gamma + 1}$$

Shock with $U_p = 0.1C_0$

For the simulation, to make the system stable, the first 30% of the simulation is done with very small time step 1.0×10^{-7}. The rest portion of the simulation is done with the previous time conditions.

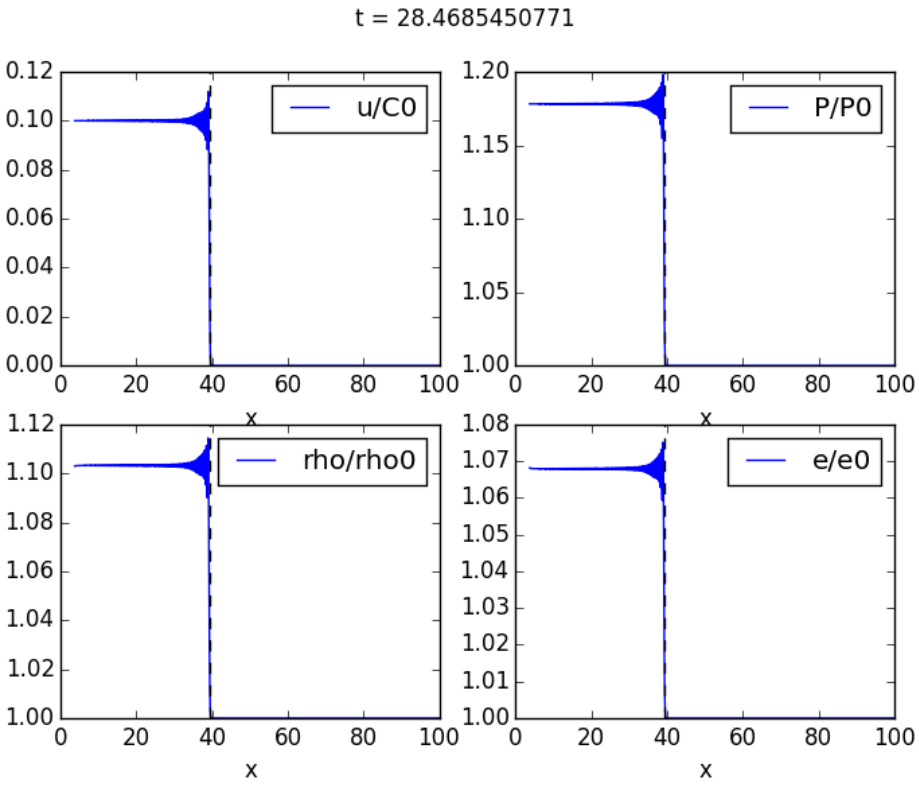

Figure 8: The dashed lines represent the analytical shock fronts and four important parameters are plotted

The analytical solution for u_1 is just U_p as the particles must move with the same speed as the piston's speed. The calculated $P/P_0 = 1.17814$ and $\rho/\rho_0 = 1.10321$, which comes from the analytical soluton for ρ/ρ_0 and the jump condition for P/P_0. As can be seen, the values are pretty close. Also the anlaytical shock fronts just overlap with the numerical one.

For the energy check, as can be preticeded, the discarded visusity term may introduce a great error.

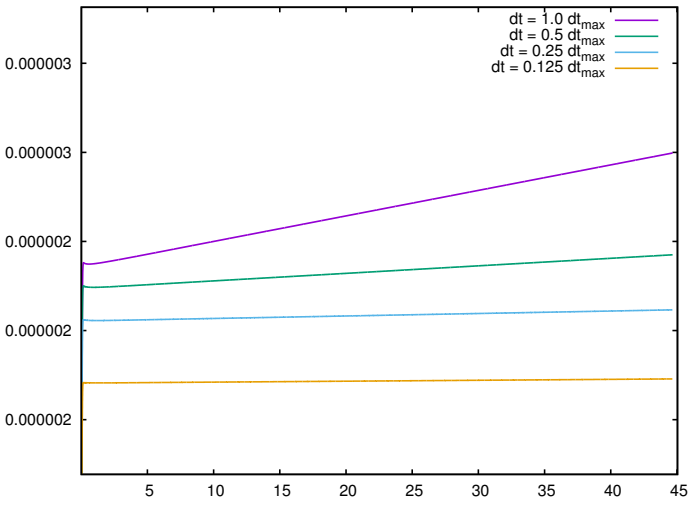

Figure 9: The energy check for different time steps, where dt_{max} represents the maximum stable time steps. The y-axis is as $E - E_0$ with $E_0 = 1498.5$ which is the energy at the first time

Shock with $U_p = 10C_0$

Fo the strong shock limit, the analytical values are: $P/P_0 = 224.4653$ and $\rho/\rho_0 = 3.9343$.

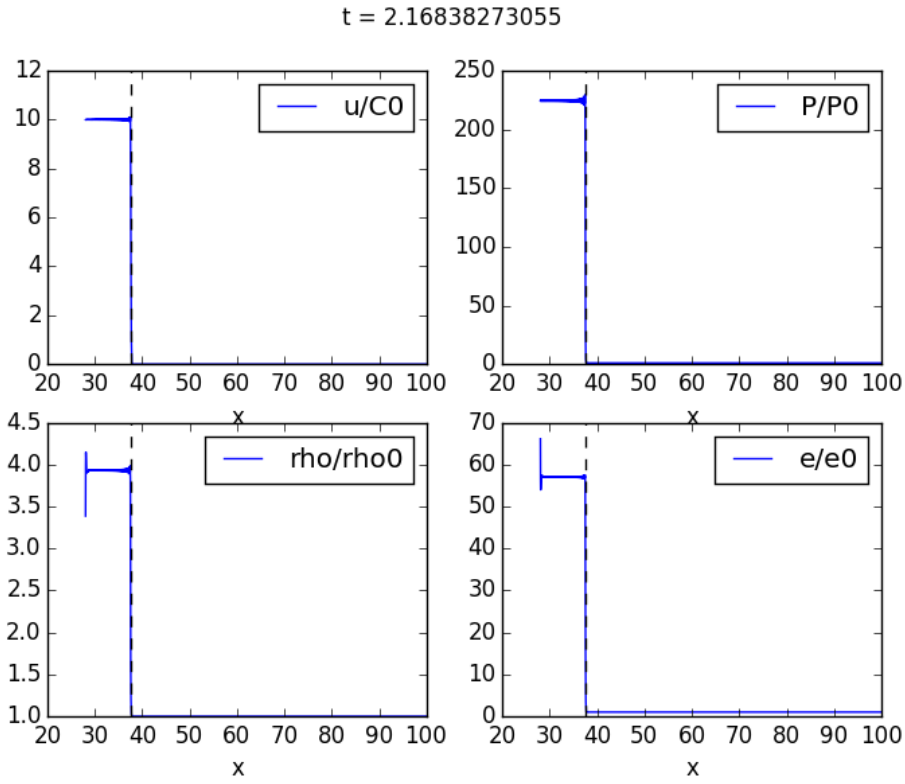

Figure 10: The dashed lines represent the analytical shock fronts and four important parameters are plotted

As can be seen from the figures, the values are pretty accurate.
For energy:

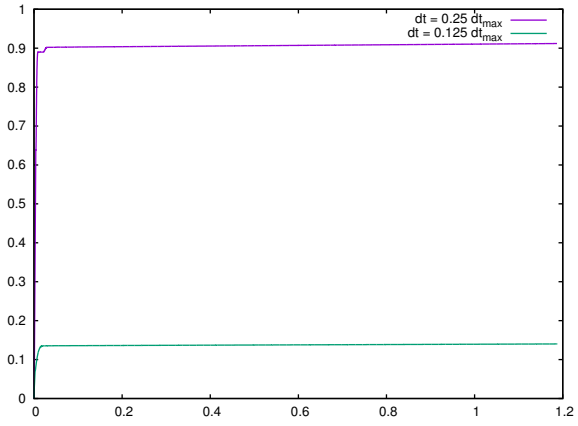

Figure 11: The energy check for different time steps, where dt_{max} represents the maximum stable time steps. The y-axis is as $E - E_0$ with $E_0 = 1498.5$ which is the energy at the first time

As seen, the error is big for larger time step, but if choose smaller time steps, the error is till in range of 0.15 which is good since the visusity will contribute some in the energy.