Chapter 5: Lower bounds for Online Linear Optimization

5.1 Lower bounds for bounded OLO

Finding a lower bound accounts to find a strategy for the adversary that forces a certain regret onto the algorithm, no matter what the algorithm does.

The basic method relies on the fact that:

$$\sup_{x \in V} f(x) \geq E[f(z)]$$

Theorem 5.1. *Let $V \subset \mathbb{R}^d$ be any non-empty bounded closed convex subset. Let $D = \sup_{\boldsymbol{v},\boldsymbol{w} \in V} \|\boldsymbol{v} - \boldsymbol{w}\|_2$ be the diameter of V. Let $\mathcal{A}$ be any (possibly randomized) algorithm for OLO on V. Let T be any non-negative integer. There $\boxed{\text{exists}}$ a sequence of vectors $\boldsymbol{g}_1, \dots, \boldsymbol{g}_T$ with $\|\boldsymbol{g}_t\|_2 \leq L$ and $\boldsymbol{u} \in V$ such that the regret of algorithm $\mathcal{A}$ satisfies*

L-Lipschitz

D: diameter of V

$$\text{Regret}_T(\boldsymbol{u}) = \sum_{t=1}^T \langle \boldsymbol{g}_t, \boldsymbol{x}_t \rangle - \sum_{t=1}^T \langle \boldsymbol{g}_t, \boldsymbol{u} \rangle \geq \frac{\sqrt{2}LD\sqrt{T}}{4} .$$

x_t is generated by A

Proof:

Denote $\text{Regret}_T = \max_{u \in V} \text{Regret}_T(u)$ (the largest regret)

Let $v, w \in V$, such that $\|v - w\|_2 = D$

Let $z = \frac{v-w}{\|v-w\|_2}$, so that $\langle z, v-w \rangle = D$
(to let variable in the feasible set)

Let $\epsilon_1, \cdots, \epsilon_T$ be i.i.d. Rademacher random variables
i.e. $P[\epsilon_t = 1] = P[\epsilon_t = -1] = 0.5$

set $g_t = L \epsilon_t z$

$$\sup_{g_1,\dots,g_T} \text{Regret}_T$$

$$= \sup_{g_1,\dots,g_T} \max_{u \in V} \text{Regret}_T(u) \qquad (\text{definition})$$

$$\geq E\left[\sum_{t=1}^{T} L\epsilon_t \langle z, x_t \rangle - \min_{u \in V} \sum_{t=1}^{T} L\epsilon_t \langle z, u \rangle \right] \qquad \left(\sup_{x \in V} f(x) \geq E[f(z)] \right)$$

$$= E\left[- \min_{u \in V} \sum_{t=1}^{T} L\epsilon_t \langle z, u \rangle \right] \qquad (E[\epsilon_t] = 0)$$

$$= E\left[\max_{u \in V} \sum_{t=1}^{T} - L\epsilon_t \langle z, u \rangle \right]$$

$$= E\left[\max_{u \in V} \sum_{t=1}^{T} L\epsilon_t \langle z, u \rangle \right] \qquad (E[\epsilon_t] = E[-\epsilon_t])$$

$$= E\left[\max_{u \in \{v, w\}} \sum_{t=1}^{T} L\epsilon_t \langle z, u \rangle \right] \qquad (z\text{'s definition})$$

$$= E\left[\underbrace{\frac{1}{2} \sum_{t=1}^{T} L\epsilon_t \underbrace{\langle z, v+w \rangle}_{=\text{Constant}}}_{=0} + \frac{1}{2} \left| \sum_{t=1}^{T} L\epsilon_t \langle z, v-w \rangle \right| \right] \qquad \left(\max(a,b) = \frac{a+b}{2} + \frac{|a-b|}{2} \right)$$

$$= \frac{L}{2} E\left[\left| \sum_{t=1}^{T} \epsilon_t \underbrace{\langle z, v-w \rangle}_{D} \right| \right] = \frac{LD}{2} E\left[\left| \sum_{t=1}^{T} \epsilon_t \right| \right]$$

$$\geq \frac{\sqrt{2} L D \sqrt{T}}{4} \qquad (\text{Khintchine inequality})$$

Khintchine inequality

Let $\{\varepsilon_n\}_{n=1}^N$ be i.i.d. random variables with $P(\varepsilon_n = \pm 1) = \frac{1}{2}$ for $n = 1, \ldots, N$, i.e., a sequence with Rademacher distribution. Let $0 < p < \infty$ and let $x_1, \ldots, x_N \in \mathbb{C}$. Then

$$A_p \left(\sum_{n=1}^N |x_n|^2 \right)^{1/2} \leq \left(\mathrm{E} \left| \sum_{n=1}^N \varepsilon_n x_n \right|^p \right)^{1/p} \leq B_p \left(\sum_{n=1}^N |x_n|^2 \right)^{1/2}$$

for some constants $A_p, B_p > 0$ depending only on p (see Expected value for notation). The sharp values of the constants A_p, B_p were found by Haagerup (Ref. 2; see Ref. 3 for a simpler proof). It is a simple matter to see that $A_p = 1$ when $p \geq 2$, and $B_p = 1$ when $0 < p \leq 2$.

Haagerup found that

$$A_p = \begin{cases} 2^{1/2-1/p} & 0 < p \leq p_0, \\ 2^{1/2}(\Gamma((p+1)/2)/\sqrt{\pi})^{1/p} & p_0 < p < 2 \\ 1 & 2 \leq p < \infty \end{cases}$$

and

$$B_p = \begin{cases} 1 & 0 < p \leq 2 \\ 2^{1/2}(\Gamma((p+1)/2)/\sqrt{\pi})^{1/p} & 2 < p < \infty \end{cases},$$

where $p_0 \approx 1.847$ and Γ is the Gamma function. One may note in particular that B_p matches exactly the moments of a normal distribution.

Thus, $\mathrm{E}\left|\sum_{t=1}^T \epsilon_t\right| \geq \sqrt{2}\left(\sum_{t=1}^T 1\right)^{\frac{1}{2}}$

Comment:

the lower bound $\frac{\sqrt{2} L D \sqrt{T}}{4}$ is also the upper bound for Online Subgradient Descent when $\eta_t = \frac{D}{L\sqrt{t}}$ or $\eta = \frac{D}{L\sqrt{T}}$

Theorem 2.13. *Let $V \subseteq \mathbb{R}^d$ a non-empty closed convex set with diameter D, i.e. $\max_{\boldsymbol{x},\boldsymbol{y} \in V} \|\boldsymbol{x} - \boldsymbol{y}\|_2 \leq D$. Let $\ell_1, \cdots, \ell_T$ an arbitrary sequence of convex functions $\ell_t : V \to \mathbb{R}$ differentiable in open sets containing V for $t = 1, \ldots, T$. Pick any $\boldsymbol{x}_1 \in V$ assume $\eta_{t+1} \leq \eta_t$, $t = 1, \ldots, T$. Then, $\forall \boldsymbol{u} \in V$, the following regret bound holds*

$$\sum_{t=1}^T (\ell_t(\boldsymbol{x}_t) - \ell_t(\boldsymbol{u})) \leq \frac{D^2}{2\eta_T} + \sum_{t=1}^T \frac{\eta_t}{2} \|\boldsymbol{g}_t\|_2^2 .$$

Moreover, if η_t is constant, i.e. $\eta_t = \eta$ $\forall t = 1, \cdots, T$, we have

$$\sum_{t=1}^T (\ell_t(\boldsymbol{x}_t) - \ell_t(\boldsymbol{u})) \leq \frac{\|\boldsymbol{u} - \boldsymbol{x}_1\|_2^2}{2\eta} + \frac{\eta}{2} \sum_{t=1}^T \|\boldsymbol{g}_t\|_2^2 .$$

5.2 Unconstrained Online Linear Optimization

5.2.1 Convex Analysis Bits : Fenchel conjugate

Before we move to the unconstrained case, we enrich our math toolbox.

Definition 5.2 (Closed Function). *A function $f : V \subseteq \mathbb{R}^d \to [-\infty, +\infty]$ is **closed** iff $\{\boldsymbol{x} : f(\boldsymbol{x}) \leq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.*

Example 5.3. *The indicator function of a set $V \subset \mathbb{R}^d$, is closed iff V is closed.*

$I : V \to \{0, 1\}$

Definition 5.4 (Fenchel Conjugate). *For a function $f : \mathbb{R}^d \to [-\infty, \infty]$, we define the **Fenchel conjugate** $f^\star : \mathbb{R}^d \to [-\infty, \infty]$ as*

$$f^\star(\boldsymbol{\theta}) = \sup_{\boldsymbol{x} \in \mathbb{R}^d} \langle \boldsymbol{\theta}, \boldsymbol{x} \rangle - f(\boldsymbol{x}) .$$

From the definition we immediately obtain the Fenchel-Young's inequality

$$\langle \boldsymbol{\theta}, \boldsymbol{x} \rangle \leq f(\boldsymbol{x}) + f^\star(\boldsymbol{\theta}), \ \forall \boldsymbol{x}, \boldsymbol{\theta} \in \mathbb{R}^d .$$

We have the following useful property for Fenchel conjugate

Theorem 5.5 ([Rockafellar, 1970, Corollary 23.5.1 and Theorem 23.5]). *Let $f : \mathbb{R}^d \to (-\infty, +\infty]$ be convex, proper, and closed. Then*

1. *$\boldsymbol{x} \in \partial f^\star(\boldsymbol{\theta})$ iff $\boldsymbol{\theta} \in \partial f(\boldsymbol{x})$.*
2. *$\langle \boldsymbol{\theta}, \boldsymbol{x} \rangle - f(\boldsymbol{x})$ achieves its supremum in $\boldsymbol{x}$ at $\boldsymbol{x} = \boldsymbol{x}^\star$ iff $\boldsymbol{x}^\star \in \partial f^\star(\boldsymbol{\theta})$.*

Example 5.6. *Let $f(x) = \exp(x)$, hence we have $f^\star(\theta) = \max_x \ x\theta - \exp(x)$. Solving the optimization, we have that $x^\star = \ln(\theta)$ if $\theta > 0$ and $f^\star(\theta) = \theta \ln \theta - \theta$, $f^\star(0) = 0$, and $f^\star(\theta) = +\infty$ for $\theta < 0$.*

$\nabla_x (x\theta - \exp(x)) = \theta - \exp(x)$

If $\theta > 0$, the supremum get when $x = \ln\theta$

Then $f^\star(\theta) = \theta \ln\theta - \theta$

If $\theta = 0$, $f^\star(\theta) = 0$ If $\theta < 0$, $f^\star(\theta) = +\infty$

Lemma 5.8. *Let f be a function and let $f^\star$ be its Fenchel conjugate. For $a > 0$ and $b \in \mathbb{R}$, the Fenchel conjugate of $g(\boldsymbol{x}) = af(\boldsymbol{x}) + b$ is $g^\star(\boldsymbol{\theta}) = af^\star(\boldsymbol{\theta}/a) - b$.*

Proof:

$$g^*(\theta) = \sup_{x \in \mathbb{R}^d} (\langle \theta, x \rangle - af(x) - b) \quad (\text{definition})$$

$$= -b + a \sup_{x \in \mathbb{R}^d} (\langle \tfrac{\theta}{a}, x \rangle - f(x))$$

$$= -b + af^*(\tfrac{\theta}{a})$$

Lemma 5.9 ([Bauschke and Combettes, 2011, Example 13.7]). *Let $f : \mathbb{R} \to (-\infty, +\infty]$ even. Then $(f \circ \|\cdot\|_2)^\star = f^\star \circ \|\cdot\|_2$.*

5.2.2 Lower Bound for the Unconstrained Case

The above lower bound applies only to the constrained setting. In the unconstrained setting, we proved that OSD with $\boldsymbol{x}_1 = \boldsymbol{0}$ and constant learning rate of $\eta = \frac{1}{L\sqrt{T}}$ gives a regret of $\frac{1}{2}L(\|\boldsymbol{u}\|_2^2 + 1)\sqrt{T}$ for any $\boldsymbol{u} \in \mathbb{R}^d$. Is this regret optimal? It is clear that the regret must be at least linear in $\|\boldsymbol{u}\|_2$, but is linear enough?

The approach I will follow is to *reduce the OLO game to the online game of betting on a coin*, where the lower bounds are known. So, let's introduce the coin-betting online game:

- Start with an initial amount of money $\epsilon > 0$.
- In each round, the algorithm bets a fraction of its current wealth on the outcome of a coin.
- The outcome of the coin is revealed and the algorithm wins or lose its bet, 1 to 1.

Denote

the outcomes of the coin : $c_t \in \{-1, 1\}$, $t=1,\cdots,T$

fraction of money to bet : $\beta_t \in [-1, 1]$, $t=1,\cdots,T$

the initial money : ϵ

money won till the end of round t : r_t

total money :

$$\overbrace{r_t+\epsilon}^{\text{Money at the end of round } t} = \overbrace{r_{t-1}+\epsilon}^{\text{Money at the beginning of round } t} + \overbrace{c_t\beta_t(r_{t-1}+\epsilon)}^{\text{Money won or lost}} = \epsilon\prod_{i=1}^{t}(1+\beta_i c_i),$$

(use the fact $r_0=0$)

bet of the algorithm on round t : $x_t = \beta_t(\epsilon + r_{t-1})$

The best case : double money in each round : $\epsilon \cdot 2^T$

However, we can get a lower bound for max wealth

Theorem 5.10 ([Cesa-Bianchi and Lugosi, 2006, a simplified statement of Theorem 9.2]). *Let $T \geq 21$. Then, for any betting strategy with initial money $\epsilon > 0$ that bets fractions of its current wealth, there exists a sequence of coin outcomes c_t, such that*

$$\epsilon + \sum_{t=1}^{T} c_t x_t \leq \frac{1}{\sqrt{T}} \max_{-1\leq\beta\leq1} \epsilon\prod_{t=1}^{T}(1+\beta c_t) \leq \frac{\epsilon}{\sqrt{T}}\exp\left(\frac{\ln 2(\sum_{t=1}^{T} c_t)^2}{T}\right).$$

Now, connect the coin-betting game with OLO.

In OLO,

$$\sum_{t=1}^{T}\langle \boldsymbol{g}_t, \boldsymbol{x}_t\rangle - \sum_{t=1}^{T}\langle \boldsymbol{g}_t, \boldsymbol{u}\rangle \le \psi_T(\boldsymbol{u}),\ \forall \boldsymbol{u}\in\mathbb{R}^d,$$

Use Fenchel conjugate:

$$\sum_{t=1}^{T}\langle \boldsymbol{g}_t, \boldsymbol{x}_t\rangle \le \inf_{\boldsymbol{u}} \sum_{t=1}^{T}\langle \boldsymbol{g}_t, \boldsymbol{u}\rangle + \psi_T(\boldsymbol{u}) = -\sup_{\boldsymbol{u}} -\left\langle \sum_{t=1}^{T}\boldsymbol{g}_t, \boldsymbol{u}\right\rangle - \psi_T(\boldsymbol{u}) \overset{\text{definition}}{=} -\psi_T^\star\left(-\sum_{t=1}^{T}\boldsymbol{g}_t\right).$$

Without any other information, it can be challenging to guess what is the slowest increasing function ψ_T. So, we restrict our attention to online algorithms that guarantee a constant regret against the zero vector. This immediately imply the following important consequence.

Theorem 5.11. *Let $\epsilon(t)$ a non-decreasing function of the index of the rounds and $\mathcal{A}$ an OLO algorithm that guarantees $\text{Regret}_t(\mathbf{0}) \le \epsilon(t)$ for any sequence of $\boldsymbol{g}_1,\dots,\boldsymbol{g}_t \in \mathbb{R}^d$ with $\|\boldsymbol{g}_i\|_2 \le 1$. Then, there exists β_t such that $\boldsymbol{x}_t = \boldsymbol{\beta}_t(\epsilon(T) - \sum_{i=1}^{t-1}\langle \boldsymbol{g}_i, \boldsymbol{x}_i\rangle)$ and $\|\beta_t\|_2 \le 1$ for $t = 1,\dots,T$.*

Proof:

Define $r_t = -\sum_{i=1}^{t}\langle g_t, x_t\rangle$, the reward of the algorithm

(smaller, better)

So, we have

$$\text{Regret}_T(u) = \sum_{t=1}^{T}\langle g_t, x_t\rangle - \sum_{t=1}^{T}\langle g_t, u\rangle$$

$$= -r_T + \left\langle \sum_{t=1}^{T} g_t, u\right\rangle$$

Since we assume: $\text{Regret}_t(0) \le \epsilon(t)$

we always have: $\text{Regret}_t(0) = -r_t \le \epsilon(t)$

i.e. $\boxed{r_t \geq -\epsilon(t)}$

Then we prove : $\|x_t\|_2 \le r_{t-1} + \epsilon(t)$, for $t = 1, \cdots, T$

Proof:

Assume there's a sequence $g_1, \cdots g_{t-1}$, that $\|x_t\|_2 > r_{t-1} + \epsilon(t)$

We then set $g_t = \frac{x_t}{\|x_t\|_2}$

Thus we have
$$
\begin{aligned}
r_t &= r_{t-1} - g_t \cdot x_t \\
&= r_{t-1} - \|x_t\|_2 \\
&< -\epsilon(t)
\end{aligned}
$$

However, $r_t \ge -\epsilon(t)$ (get above)

Thus, contradict the assumption.

So, from the fact that

$$
\begin{aligned}
\|x_t\|_2 &\le r_{t-1} + \epsilon(t) \\
&\le r_{t-1} + \epsilon(T)
\end{aligned}
$$
(non-decreasing $\epsilon(t)$),

There exists β_t, $\|\beta_t\|_2 \le 1$, that

$$x_t = \beta_t (\epsilon(T) + r_{t-1})$$

This theorem informs us of something important: *any OLO algorithm that suffer a non-decreasing regret against the null competitor must predict in the form of a "vectorial" coin-betting algorithm.* This immediately implies the following.

Theorem 5.12. *Let $T \geq 21$. For any OLO algorithm, under the assumptions of Theorem 5.11, there exist a sequence of $\boldsymbol{g}_1, \dots, \boldsymbol{g}_T \in \mathbb{R}^d$ with $\|\boldsymbol{g}_t\|_2 \leq 1$ and $\boldsymbol{u}^\star \in \mathbb{R}^d$, such that*

$$\sum_{i=1}^{t} \langle \boldsymbol{g}_t, \boldsymbol{x}_t \rangle - \sum_{t=1}^{T} \langle \boldsymbol{g}_t, \boldsymbol{u}^\star \rangle \geq 0.8\|\boldsymbol{u}^\star\|_2 \sqrt{T} \left(\sqrt{0.3 \ln \frac{0.8\|\boldsymbol{u}^\star\|_2 T}{\epsilon(T)}} - 1 \right) + \epsilon(T) \left(1 - \frac{1}{\sqrt{T}} \right) .$$

Proof:

This proof works by reducing the OLO game to a coin-betting game and then use the upper bound to the reward for coin-betting games.

First, set $g_t = [-c_t, 0, 0, \cdots 0]$, where $c_t \in \{-1, 1\}$

c_t will be defined in the following, so that $\boxed{\langle g_t, x \rangle = -c_t x_1}$, for any $x \in \mathbb{R}^d$

Given Theorem 5.11,

Theorem 5.11. *Let $\epsilon(t)$ a non-decreasing function of the index of the rounds and $\mathcal{A}$ an OLO algorithm that guarantees $\text{Regret}_t(\mathbf{0}) \leq \epsilon(t)$ for any sequence of $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t \in \mathbb{R}^d$ with $\|\boldsymbol{g}_i\|_2 \leq 1$. Then, there exists β_t such that $\boldsymbol{x}_t = \boldsymbol{\beta}_t(\epsilon(T) - \sum_{i=1}^{t-1} \langle \boldsymbol{g}_i, \boldsymbol{x}_i \rangle)$ and $\|\beta_t\|_2 \leq 1$ for $t = 1, \dots, T$.*

we have the first coordinate of x_t must satisfy:

$$x_{t,1} = \beta_{t,1}\left(\epsilon(T) - \sum_{i=1}^{t-1} \langle g_i, x_i \rangle\right)$$

$$= \beta_{t,1}\left(\epsilon(T) + \sum_{i=1}^{t-1} c_i x_{i,1}\right)$$

for some $\beta_{t,1}$ that $|\beta_{t,1}| \leq 1$

Hence, that's a coin-betting algorithm that bets $x_{t,1}$ money on the outcome of a coin c_t, with initial money $\epsilon(T)$

Then we can use Theorem 5.10

Theorem 5.10 ([Cesa-Bianchi and Lugosi, 2006, a simplified statement of Theorem 9.2]). *Let $T \geq 21$. Then, for any betting strategy with initial money $\epsilon > 0$ that bets fractions of its current wealth, there exists a sequence of coin outcomes c_t, such that*

$$\epsilon + \sum_{t=1}^{T} c_t x_t \leq \frac{1}{\sqrt{T}} \max_{-1\leq\beta\leq 1} \epsilon \prod_{t=1}^{T}(1+\beta c_t) \leq \frac{\epsilon}{\sqrt{T}} \exp\left(\frac{\ln 2(\sum_{t=1}^{T} c_t)^2}{T}\right) .$$

Then,

$$\sum_{t=1}^{T} \langle g_t, x_t \rangle - \epsilon(T) = -\sum_{t=1}^{T} c_t x_{t,1} - \epsilon(T)$$

$$\geq \frac{-\epsilon(T)}{\sqrt{T}} \exp\left(\frac{\ln 2 (\sum_{t=1}^{T} c_t)^2}{T} \right)$$

$$= -f\left(\sum_{t=1}^{T} c_t\right) \qquad \left(\text{let } f(x) = \frac{\epsilon(T)}{\sqrt{T}} \exp\left(\frac{x^2 \ln 2}{T}\right) \right)$$

$$= -\sum_{t=1}^{T} c_t u_1^* + f^*(\|u^*\|_2) \qquad (\text{Theorem 5.5 part 2})$$

$$\left(f^*(\theta) = \sup_x [\theta x - f(x)] \right)$$

$$\left(u^* = f'\left(\sum_{t=1}^{T} c_t\right) \right)$$

$$= \sum_{t=1}^{T} \langle g_t, u^* \rangle + f^*(\|u^*\|_2) \qquad (\langle g_t, x \rangle = -c_t x_1)$$

Theorem 5.5 ([Rockafellar, 1970, Corollary 23.5.1 and Theorem 23.5]). *Let $f : \mathbb{R}^d \to (-\infty, +\infty]$ be convex, proper, and closed. Then*

1. $\boldsymbol{x} \in \partial f^\star(\boldsymbol{\theta})$ *iff* $\boldsymbol{\theta} \in \partial f(\boldsymbol{x})$.

2. $\langle \boldsymbol{\theta}, \boldsymbol{x} \rangle - f(\boldsymbol{x})$ *achieves its supremum in* $\boldsymbol{x}$ *at* $\boldsymbol{x} = \boldsymbol{x}^\star$ *iff* $\boldsymbol{x}^\star \in \partial f^\star(\boldsymbol{\theta})$.

Then by Lemma A.3.

$$f(x) = \frac{\epsilon(T)}{\sqrt{T}} \exp\left(\frac{x^2 \ln 2}{T}\right)$$

Theorem A.3. *Let $a, b > 0$. Then, the Fenchel conjugate of $f(x) = b\exp(x^2/(2a))$ is*

$$f^\star(\theta) = \sqrt{a}|\theta|\sqrt{W(a\theta^2/b^2)} - b\exp\left(\frac{W(a\theta^2/b^2)}{2}\right) = \sqrt{a}|\theta|\left(\sqrt{W(a\theta^2/b^2)} - \frac{1}{\sqrt{W(a\theta^2/b^2)}}\right).$$

Moreover,

$$f^\star(\theta) \le \sqrt{a}|\theta|\sqrt{\ln(a\theta^2/b^2+1)} - b\,.$$

We already have:

$$\sum_{t=1}^{T} \langle g_t, x_t \rangle - \epsilon(T) \geq \sum_{t=1}^{T} \langle g_t, u^* \rangle + f^*(\|u^*\|_2)$$

we have

$$\sum_{t=1}^{T} \langle g_t, x_t \rangle - \sum_{t=1}^{T} \langle g_t, u^* \rangle$$

$$\geq \epsilon(T) + 0.8\|\boldsymbol{u}^\star\|_2\sqrt{T}\left(\sqrt{0.3\ln\frac{0.8\|\boldsymbol{u}^\star\|_2 T}{\epsilon(T)}} - 1\right)$$

$$\geq \epsilon(T)\left(1 - \frac{1}{\sqrt{T}}\right) + 0.8\|\boldsymbol{u}^\star\|_2\sqrt{T}\left(\sqrt{0.3\ln\frac{0.8\|\boldsymbol{u}^\star\|_2 T}{\epsilon(T)}} - 1\right)$$