Mathematical Physics

Chapter I

Sets and structures

1. Naive set theory.

1.1. Sets and subsets.

$S = \{x \mid P(x)\}$, where $P(x)$ is a proposition and x is a variable

- Although all mathematics can be reduced to set theory, set theory itself is not reducable to pure logic.
- Naive set theory leads to a collection of self-referential paradoxes, e.g. Russell's paradox:

$R = \{A \mid A \notin A\}$ $\quad R \in R \Rightarrow R \notin R$, $\; R \notin R \Rightarrow R \in R$ contradiction

These kinds of paradoxes are resolved in Zermelo-Fraenkel axiomatic system, e.g. some of the axioms:

$\forall P(x)\ \forall a\ (a \in \{x \mid P(x)\} \Leftrightarrow P(a))$ main axiom of set theory

$A = B \Leftrightarrow \forall a\ (a \in A \Leftrightarrow a \in B)$ axiom of extensionality

- We will be working with naive set theory, since its paradoxes supposedly are irrelevant to physics.

Examples:

1) $\{a\} = \{x \mid x = a\}$ singleton
2) $\{a_1, \dots, a_n\} = \{x \mid x = a_1 \vee \dots \vee x = a_n\}$ finite set
3) $\mathcal{A} = \{A_i \mid i \in I\}$, I is an indexing set family of indexed sets (collection)

$A \subseteq B \overset{def}{\Longleftrightarrow} (a \in A \Rightarrow a \in B)$ $\quad A$ is a subset of B (or B is a superset of A)

$A = B \Leftrightarrow (a \in A \Leftrightarrow b \in B) \Leftrightarrow A \subseteq B \wedge B \subseteq A$

Examples:

1) $\varnothing$: $\forall a\ (a \notin \varnothing)$ empty set

$\forall A\ (\varnothing \subseteq A)$

2) 2^A: the set of all subsets of A power set of A (alternative notation: $\mathbb{P}(A) = P(A) = 2^A$)

e.g. if $A = \{a_1, \dots, a_n\}$, then $|2^A| = \underset{\varnothing}{1} + \underset{\{a_1\},\dots,\{a_n\}}{\binom{n}{1}} + \underset{\{a_1,a_2\},\dots,\{a_{n-1},a_n\}}{\binom{n}{2}} + \dots + \underset{\{a_1,\dots,a_n\}}{\binom{n}{n}} = (1+1)^n = 2^n$

hence the symbol

1.2. Unions and intersections.

$A \cup B \overset{\text{def}}{=} \{x \mid x \in A \vee x \in B\}$ union

$A \cap B \overset{\text{def}}{=} \{x \mid x \in A \wedge x \in B\}$ intersection

$A \setminus B \overset{\text{def}}{=} \{x \mid x \in A \wedge x \notin B\}$ difference

Properties:

- A and B are disjoint: $A \cap B = \emptyset$
- $\bigcup \mathcal{A} \overset{\text{def}}{=} \{x \mid \exists A \in \mathcal{A} \, (x \in A)\}$ $\quad$ $\bigcap \mathcal{A} \overset{\text{def}}{=} \{x \mid \forall A \in \mathcal{A} \, (x \in A)\}$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\quad$ $A \cap \bigcup \mathcal{B} = \bigcup \{A \cap B_i \mid i \in I\}$

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad$ $A \cup \bigcap \mathcal{B} = \bigcap \{A \cup B_i \mid i \in I\}$
- $A \cap (B \cup C) = (A \cap B) \cup C \iff C \subseteq A$
- $A \cap B = (A^c \cup B^c)^c$

  $A \cup B = (A^c \cap B^c)^c$
- $A \setminus B = A \cap B^c$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ $\quad$ $A \setminus \bigcup \mathcal{B} = \bigcap \{A \setminus B_i \mid i \in I\}$

  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ $\quad$ $A \setminus \bigcap \mathcal{B} = \bigcup \{A \setminus B_i \mid i \in I\}$
- $2^A \cap 2^B = 2^{A \cap B}$ $\quad$ $\bigcap_{A \in \mathcal{A}} 2^A = 2^{\bigcap \mathcal{A}}$

  $2^A \cup 2^B \subseteq 2^{A \cup B}$ $\quad$ $\bigcup_{A \in \mathcal{A}} 2^A \subseteq 2^{\bigcup \mathcal{A}}$

② Relations.

2.1. Cartesian products.

Sets are not endowed with the notion of order, i.e. $\{a,b\} = \{b,a\}$, but we can artificially introduce this notion:

$(a,b) \overset{def}{=} \{a, \{a,b\}\}$ ordered pair

$(a_1, \dots, a_n) \overset{def}{=} (a_1, (a_2, \dots, a_n))$ ordered n-tuple

Now we can define the cartesian product of sets:

$A \times B \overset{def}{=} \{(a,b) \mid a \in A \wedge b \in B\}$

$A_1 \times \dots \times A_n \overset{def}{=} \{(a_1, \dots, a_n) \mid a_1 \in A_1 \wedge \dots \wedge a_n \in A_n\}$ (e.g. $\underbrace{A \times \dots \times A}_{n} \equiv A^n$)

Properties:

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$ $\bigcup \mathcal{A} \times \bigcup \mathcal{B} = \bigcup A_i \times B_j$
  $(A \cap B) \times C = (A \times C) \cap (B \times C)$ $\bigcap \mathcal{A} \times \bigcap \mathcal{B} = \bigcap A_i \times B_j$
- $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$

2.2. Relations.

$R \subset A^n$: n-ary relation on A

Consider a binary relation ($n=2$) on A. Common notation: aRb (instead of $(a,b) \in R$).

R is reflexive : $aRa \ \forall a \in A$
R is transitive : $(aRb \wedge bRc) \Rightarrow aRc \ \forall a,b,c \in A$
R is symmetric : $aRb \Rightarrow bRa \ \forall a,b \in A$
R is anti-symmetric: $(aRb \wedge bRa) \Rightarrow a = b \ \forall a,b \in A$

1. Equivalence.

R is an equivalence relation on A: R is reflexive, transitive, and symmetric.

Equivalence relation partitions A into disjoint equiv. classes $[a]_R \overset{def}{=} \{a' \in A \mid aRa'\}$, i.e. $\top(([a]=[b]) \vee ([a] \cap [b] = \emptyset))$. Now we can define the set of all equiv. classes of A w.r.t. R:

$A/R \overset{def}{=} \{[a]_R \mid a \in A\}$ factor space/set of A by R "A modulo R"

Examples:

1) $A = \mathbb{Z}$ $m = n \pmod p \Leftrightarrow m - n = kp,\ k,m,n \in \mathbb{Z}\ \ p \in \mathbb{N}$
   $\mathbb{Z}_p = \{[0]_p, [1]_p, \dots, [p-1]_p\}$ residue classes mod p
2) $A = \mathbb{R}^2$ $(x,y) \sim (x',y') \Leftrightarrow x' = x + m \ \ y' = y + n,\ n,m \in \mathbb{Z}$
   $T^2 = \mathbb{R}^2/\sim = \{[(x,y)] \mid 0 \leq x < 1 \ \ 0 \leq y < 1\}$

2. Order relations.

R is a partial order on A : R is reflexive, transitive, and antisymmetric.
R is a total order on A : R is a partial order and $\top(aRb \vee bRa)$

Then (A,R) is a poset (partially ordered set), e.g. $(\mathbb{R}, \leq)$, $(2^A, \subseteq)$.

③ Mappings.

3.1. Mappings.

$\varphi: X \to Y$ is a **mapping** from X to Y: $\varphi = \{(x,y) \mid \forall x \in X \ \exists! y \in Y\}$

i.e. $((x,y)\in\varphi \wedge (x,y')\in\varphi) \Rightarrow y = y'$

X: **domain** of φ
Y: **codomain** of φ (range)

$\varphi[X] \overset{\text{def}}{\equiv} \{y \in Y \mid y = \varphi(x),\ x \in X\}$ **image** of φ
$\varphi^{-1}[Y] \overset{\text{def}}{\equiv} \{x \in X \mid \varphi(x) \in Y\}$ **preimage** of φ (inverse image)

Examples:

1) $\sin: \mathbb{R} \to \mathbb{R}$ $\quad \sin^{-1}[\{0\}] = \{0, \pm\pi, \pm 2\pi, \dots\}$
$\quad \sin^{-1}[\{2\}] = \emptyset$
2) $\varphi: X_1 \times \dots \times X_n \to Y$ $\quad$ n-ary function
3) $pr_i: X_1 \times \dots \times X_n \to X_i$ $\quad$ projection on X_i
$\quad (x_1, \dots, x_n) \mapsto x_i$
4) $\psi \circ \varphi : X \to Z$ $\quad$ composition of $\varphi: X \to Y$ and $\psi: Y \to Z$
$\quad \psi\circ\varphi(x) = \psi(\varphi(x))$ $\quad$ satisfies $\alpha\circ(\beta\circ\gamma) = (\alpha\circ\beta)\circ\gamma$

$\varphi: X \to Y$ is **surjective** : $\forall y \in Y\ \exists x \in X\ (\varphi(x) = y)$ $\quad$ onto
$\varphi: X \to Y$ is **injective** : $\varphi(x) = \varphi(x') \Rightarrow x = x'$ $\quad$ one-to-one
$\varphi: X \to Y$ is **bijective** : $\forall y \in Y\ \exists! x \in X\ (\varphi(x) = y)$ $\quad$ one-to-one correspondence

- $\varphi: X \to X$ is a transformation of X: φ is bijective. e.g. $id_X: X \to X$, $x \mapsto x$
- If $\varphi: X \to Y$ is bijective, then $\exists \varphi^{-1}: Y \to X$ s.t. $\varphi^{-1}\circ\varphi = id_X$ and $\varphi\circ\varphi^{-1} = id_Y$.
- If $\varphi: X \to Y$ and $\psi: Y \to Z$ are bijective, then $(\psi\circ\varphi)^{-1} = \varphi^{-1}\circ\psi^{-1}$.
- $\varphi|_U$: restriction of $\varphi: X \to Y$ to $U \subset X$.
- $\forall \varphi: X \to Y$ defines equiv. relation on X: $xRx' \overset{\text{def}}{\Leftrightarrow} \varphi(x) = \varphi(x')$. Then $X/R = \{\varphi^{-1}(\{y\}) \mid y \in Y\}$.

Examples:

1) $i_U = id_X|_U$ $\quad$ inclusion map for $U \subset X$
Then $\forall \varphi: X \to Y$ $\ \varphi|_U = \varphi \circ i_U$
2) $\chi_U: X \to \{0,1\}$ $\quad$ characteristic function of $U \subset X$
$$\chi_U(x) = \begin{cases} 0, & x \notin U \\ 1, & x \in U \end{cases} \quad \Rightarrow \quad 2^X = \{\varphi \mid \varphi: X \to \{0,1\}\}$$
3) $\varphi: X \to X/\sim$ $\quad$ canonical map under $\sim$ (surjective)
$\quad \varphi(x) = [x]_\sim$

3.2. Characteristic function.

$$\chi_A(x) :\overset{\text{def}}{=} \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

e.g. Dirichlet function $\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

Properties:

1. $\chi_{A \cap B} = \min(\chi_A, \chi_B) = \chi_A \cdot \chi_B$
2. $\chi_{A \cup B} = \max(\chi_A, \chi_B) = \chi_A + \chi_B - \chi_A \cdot \chi_B$ (However, if $A \cap B = \emptyset$, then $\chi_{A \cup B} = \chi_A + \chi_B$)
3. $\chi_{A^c} = 1 - \chi_A$
4. Sometimes the term "indicator function" is used instead: $\mathbb{1}_A \equiv \chi_A$ (especially in probability theory)

3.3. Iverson bracket.

Equivalent notation for characteristic functions that also generalizes the Kronecker delta.

$$[P] :\overset{\text{def}}{=} \begin{cases} 1, & P \\ 0, & \neg P \end{cases}$$

e.g. $[i = j] = \delta_{ij}$

Properties:

1. $[P \wedge Q] = [P] \cdot [Q]$
2. $[P \vee Q] = [P] + [Q] - [P] \cdot [Q]$
3. $[\neg P] = 1 - [P]$

Usage:

1. $\sum_k f(k)[P(k)] = \sum_{P(k)} f(k)$ $\qquad \prod_k f(k)^{[P(k)]} = \prod_{P(k)} f(k)$

2. $\sum_{k \in A} f(k) + \sum_{k \in B} f(k) = \sum_k f(k)[k \in A] + \sum_k f(k)[k \in B] = \sum_k f(k)([k \in A] + [k \in B]) =$
$= \sum_k f(k)([k \in A \cup B] + [k \in A \cap B]) = \sum_{k \in A \cup B} f(k) + \sum_{k \in A \cap B} f(k)$ double-counting rule

3. $\sum_{j=1}^{n} \sum_{k=1}^{j} f(j,k) = \sum_{j,k} f(j,k)[1 \le j \le n][1 \le k \le j] = \sum_{j,k} f(j,k)[1 \le k \le j \le n] =$
$= \sum_{j,k} f(j,k)[1 \le k \le n][k \le j \le n] = \sum_{k=1}^{n} \sum_{j=k}^{n} f(j,k)$ summation interchange

Examples:

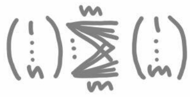
1. $\chi_A(x) = [x \in A]$
2. $\theta(x) = [x > 0]$ Heaviside step function
3. $R(x) = x[x \geq 0]$ Ramp function
4. $\operatorname{sgn}(x) = [x > 0] - [x < 0]$ signum
   $|x| = x \cdot \operatorname{sgn}(x)$
5. $\max(x, y) = x[x > y] + y[x \le y]$
   $\min(x, y) = x[x \le y] + y[x > y]$

④ Infinite sets.

A is finite : $\exists$ bijection $\varphi: A \to \{1,2,\dots,n\}$, where $|A| = n$: cardinality of A
A is infinite : A is not finite

Examples:

1) $B^A \overset{\text{def}}{=} \{\varphi \mid \varphi: A \to B\}$
Reasoning for notation : if A and B are finite, then $|B^A| = \underbrace{m \cdot \ldots \cdot m}_{n \text{ times}} = m^n = |B|^{|A|}$



2) $2^A = \{\varphi \mid \varphi: A \to \{0,1\}\} = \{0,1\}^A$
$|2^A| = 2^{|A|}$

A is countable : $\exists$ bijection $\varphi: A \to \mathbb{N}$
A is uncountable : A is not countable

- If A is countable, then $\forall\ A' \subset A$ is either countable or finite.
- If A and B are countable, then $A \times B$ is countable.
- If A is countable, then 2^A is uncountable.

Examples:

1) $\mathbb{Z}$ is countable , $\mathbb{Q} = \mathbb{Z} \times \mathbb{N}$ is countable
2) $\mathbb{R}$ is uncountable
$\forall\, x \in [0,1]$ $x = 0.\epsilon_1\epsilon_2\ldots$, where $\epsilon_i \in \{0,1\}$, then $[0,1] = \{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$, i.e. $|\mathbb{R}| = 2^{|\mathbb{N}|}$.
3) Cantor set is uncountable
$x = 0.\epsilon_1\epsilon_2\ldots$, where $\epsilon_i \in \{0,1,2\}$ but let $\epsilon_i \neq 1$ $\Rightarrow$ uncountable
This set is nowhere dense in $\mathbb{R}$ (i.e. $\forall\,(a,b) \subset \mathbb{R}$ it's not dense)

4) $S = \{f \mid f: \mathbb{R} \to \mathbb{R}\}$ is uncountable and $|S| > |\mathbb{R}|$

⑤ Physics.

Physical theories have two aspects : static and dynamic.

1. Static.
Background in which the theory is set, i.e. the mathematical structure.

Math. structures:
- algebraic (set + relations imposed on a set)
- geometric (set + relations imposed on the power set of a set)

2. Dynamic.
Laws of physics are usually written as differential equations.

⑥ Category theory.

C is a **category**: 1. $Ob(C)$ class of **objects**

2. $Hom(C)$ class of **morphisms** between the objects

3. $\circ : Hom(X,Y) \times Hom(Y,Z) \to Hom(X,Z)$ **composition** of morphisms

(assoc.) $h \circ (g \circ f) = (h \circ g) \circ f \quad \forall f,g,h$

(ident.) $\forall X \in Ob(C)\ \exists i_X \in Hom(X,X)$ s.t. $\forall f \in Hom(X,Y) \quad f \circ i_X = f$
$\forall g \in Hom(Y,X) \quad i_X \circ g = g$

Examples:

sets	mappings between sets
groups	homomorphisms
vector spaces	linear maps
algebras	algebra homomorphisms
topological spaces	homeomorphisms
smooth manifolds	diffeomorphisms

$f : X \to Y$ is a **monomorphism** : $\forall A \in Ob(C)\ \forall \alpha, \alpha' \in Hom(A,X)$
$f \circ \alpha = f \circ \alpha' \Rightarrow \alpha = \alpha'$

$f : X \to Y$ is an **epimorphism** : $\forall B \in Ob(C)\ \forall \beta, \beta' \in Hom(Y,B)$
$\beta \circ f = \beta' \circ f \Rightarrow \beta = \beta'$

$f : X \to Y$ is an **isomorphism** : $\exists f' \in Hom(Y,X)$ s.t. $f' \circ f = i_X$
$f \circ f' = i_Y$

$f : X \to X$ is an **endomorphism** : $f \in Hom(X,X)$

$f : X \to X$ is an **automorphism** : $f \in Hom(X,X)$ and isomorphic

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "f\circ\alpha"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

If f is monic, then α is uniquely defined

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\beta\circ f"'] & & Y \arrow[dl, "\beta"] \\ & B & \end{tikzcd}$$

If f is epic, then β is uniquely defined

- f is an isomorphism $\Rightarrow$ f is a monomorphism and an epimorphism
- f is a monomorphism and an epimorphism $\not\Rightarrow$ f is an isomorphism
- For the category of sets:

  f is a monomorphism $\iff$ f is injective

  f is an epimorphism $\iff$ f is surjective

  f is an isomorphism $\iff$ f is bijective

Chapter V
Measure theory and integration

1. Measurable spaces.

1.1. Overview.

- Topology ignores the notion of "size", measure theory is the area of mathematics concerned with these sorts of properties. A measure space is the structure that defines which sets are measurable (analogous to a topology, telling which sets are open).
- Firstly a measure space requires a σ-algebra imposed on the power set of the underlying space. Then a measure can be defined, it's a positive $\mathbb{R}$-valued function on the σ-algebra that is countably additive (i.e. the measure of a union of disjoint sets is the sum of their measures).
- There are other restrictions imposed on a measure. By general reckoning the broadest useful measure on $\mathbb{R}^n$ is the Lebesgue measure.

1.2. Measurable spaces.

$\mathcal{M} \subseteq 2^X$ is a σ-algebra on X: (1) $\emptyset \in \mathcal{M}$ — $\emptyset$ is measurable

collection of measurable subsets of X

(2) $A \in \mathcal{M} \Rightarrow A^c = X \setminus A \in \mathcal{M}$ — $\mathcal{M}$ is closed under complementation in X

(3) $A_i \in \mathcal{M},\ i \in I \Rightarrow \bigcup_i A_i \in \mathcal{M}$ — $\mathcal{M}$ is closed under countable unions

$(X, \mathcal{M})$ is a measurable space : $\mathcal{M}$ is a σ-algebra on X.

Properties:

- (1) and (2) $\Rightarrow$ $X = \emptyset^c \in \mathcal{M}$
- $A \cap B = (A^c \cup B^c)^c \Rightarrow A \cap B \in \mathcal{M}$
- $A \setminus B = A \cap B^c = (A^c \cup B)^c \Rightarrow A \setminus B \in \mathcal{M}$
- difference between (meas. 2) and (top. 2): compl. of an open set is closed (most of the times)
- If $\mathcal{M}_i$, $i \in I$, are σ-algebras on X, then $\bigcap_{i \in I} \mathcal{M}_i$ is also a σ-algebra. Therefore, given $\mathcal{A} \subset 2^X$, there is a unique "smallest" σ-algebra $S \supseteq \mathcal{A}$, i.e. the intersection of all σ-algebras that contain $\mathcal{A}$. It's called the σ-algebra generated by $\mathcal{A}$.

  Borel sets on (X, τ) : σ-algebra on X generated by open sets. (includes open, closed, and clopen sets)

- If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable, then $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ is measurable, where $\mathcal{M} \otimes \mathcal{N}$ is a σ-algebra generated by all sets of the form $A \times B$, $A \in \mathcal{M}$ $B \in \mathcal{N}$.

Examples:

1) $\mathcal{M} = \{\emptyset, X\}$ $\mathcal{N} = 2^X$ are the smallest and the largest σ-algebras of X

2) $X = \mathbb{R}$ with the standard topology, i.e. open sets are countable unions of open intervals. Hence Borel sets are generated by $\{(a,b) \mid a < b\}$. Using the axioms (1) - (3) one can show that $(-\infty, a), (a, +\infty), (-\infty, a], [a, +\infty), [a, b], \{a\}$ are also Borel sets.

2 Measurable functions.

2.1. Measurable functions.

Consider measurable spaces $(X,\mathcal{M})$ and $(Y,\mathcal{N})$, then:

$f: X \to Y$ is **measurable**: $A \in \mathcal{N} \Rightarrow f^{-1}(A) \in \mathcal{M}$ i.e. preimage of every measurable set is measurable

- This definition mirrors the definition of a continuous function between topological spaces.
- If (X,τ_X) and (Y,τ_Y) are topological spaces and $\mathcal{M}$ and $\mathcal{N}$ are σ-algebras of Borel sets on them, then $\forall f \in C^0(X,Y)$ is measurable.
- Consider $f: X \to \mathbb{R}$. Since the family of Borel sets on $\mathbb{R}$ is generated by the intervals $(a,+\infty)$, we can formulate a criterion for measurability:

$f: X \to \mathbb{R}$ is measurable $\Leftrightarrow$ $\forall a \in \mathbb{R}$ $f^{-1}((a,+\infty)) = \{x \mid f(x) > a\}$ is measurable.

Example:

$\chi_A: X \to \mathbb{R}$, $A \subset X$, is measurable $\Leftrightarrow$ A is measurable since $\{x \mid \chi_A(x) > a\} = \begin{cases} X, & a<0 \\ A, & 0 \leq a < 1 \\ \emptyset, & a \geq 1 \end{cases}$

Properties:

- If $f: X \to Y$ and $g: Y \to Z$ are measurable, then $g \circ f: X \to Z$ is measurable.
- If $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are measurable, then $f+g$ and $f \cdot g$ are measurable.
- If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are measurable, then $h: \mathbb{R}^2 \to \mathbb{R}$ s.t. $(x,y) \mapsto f(x) \cdot g(y)$ is measurable.
- If $f: X \to \mathbb{R}$ is measurable, then $|f|$, f^a $(a>0)$, $\frac{1}{f}$ $(f(x) \neq 0)$ are measurable.
  (since each of these functions is a composition of a cont. function and a measurable one, e.g. $|f| = |\cdot| \circ f$)
- If X and Y are measurable spaces, then $pr_X: X \times Y \to X$ and $pr_Y: X \times Y \to Y$ are measurable.
  (since $pr_X^{-1}(A) = A \times Y$ $\forall A \subseteq X$ and $pr_Y^{-1}(B) = X \times B$ $\forall B \subseteq Y$)
- If $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are measurable and $E \subset X$ is measurable, then $h(x) = \begin{cases} f(x), & x \in E \\ g(x), & x \notin E \end{cases}$ is measurable. (since $\forall A \subset X$ $h^{-1}(A) = (f^{-1}(A) \cap E) \cup (g^{-1}(A) \cap E^c))$

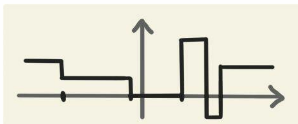
2.2. Simple functions.

$h: X \to \overline{\mathbb{R}}$ is a **simple function**: h is measurable and $h(X) = \{a_1, \dots, a_n\}$ $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ extended real line

$h = a_1 \chi_{A_1} + \dots + a_n \chi_{A_n}$, where $A_i = h^{-1}(\{a_i\})$ and $a_i \neq 0$ $\forall i \in I$

Example:

$h: \mathbb{R} \to \mathbb{R}$ step function ($\forall A_i$ is a union of disjoint intervals)



- If $f: X \to \overline{\mathbb{R}}$ and $g: X \to \overline{\mathbb{R}}$ are measurable, then $\overline{h} = \sup(f,g)$ and $\underline{h} = \inf(f,g)$ are measurable.
  $\overline{h}(x) = \begin{cases} f(x), & f(x) \geq g(x) \\ g(x), & g(x) > f(x) \end{cases}$ $\underline{h}(x) = \begin{cases} f(x), & f(x) < g(x) \\ g(x), & g(x) \leq f(x) \end{cases}$ $\left(\begin{aligned} \{x \mid h(x) > a\} &= \{x \mid f(x) > a \lor g(x) > a\} = \\ &= \{x \mid f(x) > a\} \cup \{x \mid g(x) > a\} \end{aligned} \right)$
- Consider $f: X \to \overline{\mathbb{R}}$. $f = f^+ - f^-$ and $|f| = f^+ + f^-$, where $f^+ = \sup(f,0)$ and $f^- = -\inf(f,0)$.
- If $\{f_n \mid f_n: X \to \overline{\mathbb{R}}, n \in \mathbb{N}\}$ are measurable, then $\sup f_n$ and $\inf f_n$ are measurable.
- If $\{f_n \mid f_n: X \to \overline{\mathbb{R}}, n \in \mathbb{N}\}$ are measurable and $f_n \to f$, then f is measurable.
  $f_n \to f \iff \lim\limits_{n\to\infty} f_n = \overline{\lim\limits_{n\to\infty}} f_n = \underline{\lim\limits_{n\to\infty}} f_n$, where $\begin{aligned} \overline{\lim_{n\to\infty}} f_n &\overset{def}{=} \lim_{n\to\infty} \sup f_n = \inf_{n \geq 1} \left(\sup_{k \geq n} f_k \right) \\ \underline{\lim_{n\to\infty}} f_n &\overset{def}{=} \lim_{n\to\infty} \inf f_n = \sup_{n \geq 1} \left(\inf_{k \geq n} f_k \right) \end{aligned}$
- If $f: X \to \mathbb{R}$ is measurable, then $\exists \{h_n\}$, h_n are simple, s.t. $h_n \to f$.

③ Measure spaces.

3.1. Measure.

$\mu : \mathcal{M} \to \overline{\mathbb{R}}$ is a **measure** on $(X, \mathcal{M})$: (4) $\mu(\emptyset) = 0$

(5) $\mu(A) \geq 0 \quad \forall A \subset \mathcal{M}$

(6) $\mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$, $A_i \cap A_j = \emptyset \quad \forall i \neq j$ — disjoint sets

i.e. μ is countably additive

$(X, \mathcal{M}, \mu)$ is a **measure space**: $\mathcal{M}$ is a σ-algebra on X and μ is a measure on $(X, \mathcal{M})$.

- If $B \subset A$, then $\mu(B) \leq \mu(A)$. (Since $\mu(A) = \mu(B) + \mu(A \setminus B) \geq \mu(B)$)
- If $\sum_{i \in I} \mu(A_i)$ diverges, then $\sum_{i \in I} \mu(A_i) = \infty$.
- Modularity: $\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B)$ (since $A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B)$)
- If $\{E_n \mid E_n \subset E_{n+1} \ \forall n \in \mathbb{N}\}$ are measurable, then $\mu(E) = \lim_{n \to \infty} \mu(E_n)$, where $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$.

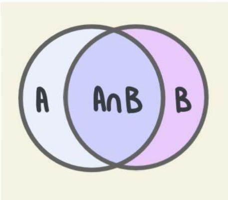
Example:

$\delta_a(A) = \begin{cases} 1, & a \in A \\ 0, & a \notin A \end{cases}$, $a \in X$ Dirac measure

3.2. Probability theory.

$(\Omega, \mathcal{M}, P)$ is a **probability space**: $P(\Omega) = 1$.

- Ω : a sample space (the set of all possible outcomes)
- $\mathcal{M}$: an event space (σ-algebra on Ω)
- P : a probability function (measure on $(\Omega, \mathcal{M})$)
- $P(A)$: the probability of an event A $(0 \leq P(A) \leq 1)$



$P(A) = 1$: certainty event

$P(A) = 0$: impossible event

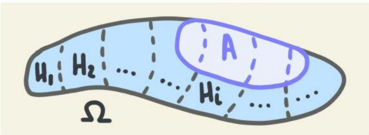
$P(A) + P(B) = P(A \cap B) + P(A \cup B)$

conditional probability of B given A: $P(B \mid A) \overset{\text{def}}{=} \dfrac{P(A \cap B)}{P(A)}$

events A and B are **independent**: $P(A \cap B) = P(A) \cdot P(B)$ — or $P(B \mid A) = P(B)$

$\{H_i \in \mathcal{M} \mid i \in I\}$ is a **hypothesis**: $H_i \cap H_j = \emptyset \ \forall i \neq j$ and $\Omega = \bigcup_{i \in I} H_i$ — i.e. partitioning of Ω

- $\forall A \in \mathcal{M}$ $\quad A = \bigcup_{i \in I} (H_i \cap A)$ $\quad P(A) = \sum_{i \in I} P(H_i \cap A) = \sum_{i \in I} P(A \mid H_i) P(H_i)$
- Bayes' formula: $P(H_i \mid A) = \dfrac{P(A \cap H_i)}{P(A)} = \dfrac{P(A \mid H_i) P(H_i)}{\sum_{k \in I} P(A \mid H_k) P(H_k)}$



④ Lebesgue measure.

4.1. Open intervals on $\mathbb{R}^n$.

First consider $\mathbb{R}$. Let's define a notion of lenght for open intervals:

$$\ell(I) \overset{\text{def}}{=} b - a \quad , \text{ where } I = (a,b)$$

- Notice that it works for closed and half-open intervals too:

  $(a-\varepsilon, b) = (a-\varepsilon, a) \cup [a,b) \Rightarrow b - a + \varepsilon = a - a + \varepsilon + \ell([a,b)) \Rightarrow \ell([a,b)) = b - a$

  $[a,b) = \{a\} \cup (a,b) \Rightarrow \ell(\{a\}) = 0$

  $(a,b] = (a,b) \cup \{b\} \Rightarrow \ell((a,b]) = b - a$

  $[a,b] = (a,b) \cup \{a\} \cup \{b\} \Rightarrow \ell([a,b]) = b - a$

- Any open set on $\mathbb{R}$ can be represented as a countable union of disjoint open intervals (since $\forall r \in \mathbb{Q}$ s.t. $r \in U$ lies in a maximal open interval $(a,b) \subseteq U$). And open sets generate Borel sets on $\mathbb{R}$.

- If $A \subset \mathbb{R}$ is s.t. $|A| = |\mathbb{N}|$, then $\ell(A) = 0$. (since $A = \bigcup_{k=1}^{\infty} \{a_k\}$)

In the case of $\mathbb{R}^n$ an open interval is a cartesian product of open intervals on $\mathbb{R}$. And we now define a notion of volume instead:

$$\text{vol}_n(I) \overset{\text{def}}{=} \prod_{k=1}^{n} \ell(I_k) = \prod_{k=1}^{n} (b_k - a_k) \quad , \text{ where } I = I_1 \times \ldots \times I_n \ , \ I_k = (a_k, b_k)$$

- Arguing as before one can show that the same definition works for closed and partly-open intervals too.

- Any open set on $\mathbb{R}^n$ can be also seen as a coutable union of disjoint open intervals, and open sets generate Borel sets on $\mathbb{R}^n$.

4.2. Outer measure.

To define a measure on $\mathbb{R}^n$ we will need an additional notion first:

$$\boxed{\mu^*(A) \overset{\text{def}}{=} \inf\{\text{vol}_n(U) \mid A \subseteq U,\ U \text{ is open}\}}$$

outer measure of $A \subset \mathbb{R}^n$

- μ^* is not a measure, it doesn't satisfy countable additivity (meas. 6), but it satisfies countable subadditivity:

  $\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu^*(A_k)$, where $A_i \cap A_j = \emptyset \ \forall i \neq j$ (disjoint sets)

- $\mu^*(\emptyset) = 0$

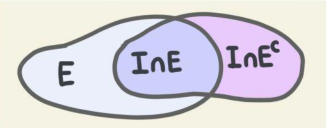
- $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

- $\mu^*(I) = \text{vol}_n(I)$, where I is an interval on $\mathbb{R}^n$

4.3. Lebesgue measure.

Now we can define a measure on $\mathbb{R}^n$:

E is **Lebesgue-measurable**: $\mu^*(I) = \mu^*(I \cap E) + \mu^*(I \cap E^c) \quad \forall$ open interval $I \subset \mathbb{R}^n$



i.e. when we cover mutually disjoint sets $I \cap E$ and $I \cap E^c$ with open intervals, the overlap between them can be made arbitrarily small

The set of all Lebesgue-measurable sets $\mathcal{L}$ form a σ-algebra on $\mathbb{R}^n$, and an outer measure μ^* defines a measure on it, i.e. $(\mathbb{R}^n, \mathcal{L}, \mu = \mu^*|_{\mathcal{L}})$ is a measure space:

1. $\emptyset \in \mathcal{L}$	4. $\mu(\emptyset) = 0$
2. $E \in \mathcal{L} \Rightarrow E^c \in \mathcal{L}$	5. $\mu(E) \geqslant 0$
3. $E_i \in \mathcal{L},\ i \in I \Rightarrow \bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{L}$	6. $\mu(\bigcup\limits_{i=1}^{\infty} E_i) = \sum\limits_{i=1}^{\infty} \mu(E_i),\ E_i \cap E_j = \emptyset\ \forall i \neq j$

- All open sets are Lebesgue-measurable (since any open set is a countable union of disjoint open intervals), therefore, Borel sets are included in $\mathcal{L}$ (but not vice versa!).
- $E \in \mathcal{L} \Leftrightarrow \forall \varepsilon > 0\ \exists$ open G and closed F s.t. $F \subset E \subset G$ and $\mu(G \setminus F) < \varepsilon$.
- If $\dim(E) < n$, then $\mu(E) = 0$ (e.g. any hyperplane in $\mathbb{R}^n$ has a measure zero).
- If $\mu(E_i) = 0 \quad \forall i \in I$, then $\mu(\bigcup\limits_{i=1}^{\infty} E_i) = 0$.
- If $\mu(N) = 0$, then $\forall E \quad \mu(E \cup N) = \mu(E \setminus N) = \mu(E)$
- Properties of the Lebesgue measure:
  - Lebesgue measure is complete (i.e. $\forall N' \subset N$, where $\mu(N) = 0$, is s.t. $\mu(N') = 0$).
  - If $E \in \mathcal{L}$, then $T_a(E) = \{x + a \mid x \in E,\ a \in \mathbb{R}^n\} \in \mathcal{L}$ and $\mu(T_a(E)) = \mu(E)$.
  - If $E \in \mathcal{L}$, then $\alpha E = \{\alpha x \mid x \in E,\ \alpha > 0\} \in \mathcal{L}$ and $\mu(\alpha E) = \alpha^n \mu(E)$.
  - If $E \in \mathcal{L}$ and $A \in GL_n(\mathbb{R})$, then $A(E) \in \mathcal{L}$ and $\mu(A(E)) = |\det A| \cdot \mu(E)$.
  - Lebesgue measure is a unique complete translation-invariant measure on $\mathbb{R}^n$ s.t. $\mu((0,1) \times \ldots \times (0,1)) = 1$.
- There are sets that are not Lebesgue-measurable.
- There is no infinite-dimensional analogue of Lebesgue measure.
- Inner measure:

$$\boxed{\mu_*(E) \overset{\text{def}}{=} \sup\{\operatorname{vol}_n(U) \mid U \subset E,\ U \text{ is open}\}}$$

inner measure of $E \subset \mathbb{R}^n$

- $\mu_*(E) \leqslant \mu^*(E)$
- $\mu(U) = \mu_*(U \cap E) + \mu^*(U \cap E^c) \quad \forall$ open $U \supset E$
- $\mu(E) < \infty \iff \mu_*(E) = \mu^*(E) < \infty$

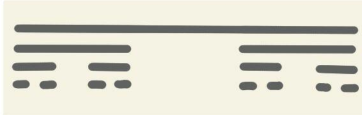
A property holds **almost everywhere**: holds everywhere except on N s.t. $\mu(N) = 0$.

Example:

$f = g$ a.e.: $N = \{x \mid f(x) \neq g(x)\}$ is s.t. $\mu(N) = 0$.

Examples:

1) The Cantor set (see Ch. I §4)



C is closed (since C^c is open) $\Rightarrow$ C is a Borel set $\Rightarrow$ $C \in \mathcal{L}$

$$\mu(C_n) = 1 - \frac{1}{3} - 2\left(\frac{1}{3}\right)^2 - 2^2\left(\frac{1}{3}\right)^3 - \dots - 2^{n-1}\left(\frac{1}{3}\right)^n$$

$$\mu(C) = \lim_{n\to\infty} \mu(C_n) = 1 - \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = 1 - \frac{1}{2}\left(\frac{1}{1-2/3} - 1\right) = 1 - \frac{1}{2}\cdot 2 = 0$$

i.e. $\mu(C) = 0$, but it's uncountable

2) The existence of sets that are not Lebesgue-measurable relies on the axiom of choice.

Let's define a relation on $I = (0,1)$:

$xQy \overset{\text{def}}{=} \{(x,y) \mid x-y \in \mathbb{Q},\ x,y \in I\}$ equiv. relation

$Q_x = \{y \mid y - x \in \mathbb{Q}\}$ equiv. classes

Axiom of choice $\Rightarrow$ $\exists T$ s.t. it has exactly one representative from each Q_x.

$T_r \overset{\text{def}}{=} \{x + r \mid x \in T,\ r \in \mathbb{Q}\ r \in (-1,1)\}$

$\forall y \in I\ \exists T_r$ s.t. $y \in T_r \overset{|r|<1}{\Longrightarrow} (0,1) \subset \bigcup_r T_r \subset (-1,2)$

$$1 \leq \mu\left(\bigcup_r T_r\right) \leq 3$$

$$1 \leq \sum_{k=1}^{\infty} \mu(T) \leq 3$$

(all the T_r are disjount and have the same measure)

If $\mu(T) = 0$, then $\sum_{k=1}^{\infty} \mu(T) = 0$.
If $\mu(T) > 0$, then $\sum_{k=1}^{\infty} \mu(T) = \infty$.
$\Rightarrow$ Contradiction $\Rightarrow$ T is non-measurable

(5.) Lebesgue integration.

5.1. Simple functions.

Consider a simple function $h: X \to \mathbb{R}$ on a measure space $(X, \mathcal{M}, \mu)$.

$\boxed{\int h \, d\mu \overset{\text{def}}{=} \sum_{i=1}^{n} a_i \mu(A_i)}$, where $h = \sum_{i=1}^{n} a_i \chi_{A_i}$, $a_i \in \mathbb{R}$

$A_i = h^{-1}(\{a_i\}) \subset X$ are measurable

- If $h_1: X \to \mathbb{R}$ and $h_2: X \to \mathbb{R}$ are simple, then $\alpha h_1 + \beta h_2$ $(\alpha, \beta \in \mathbb{R})$ is also simple and:
  1. $\alpha h_1 + \beta h_2 = \sum_{i,j} (\alpha a_i + \beta b_j) \chi_{A_i \cap B_j}$, $\alpha a_i + \beta b_j \neq 0$
  2. $\int (\alpha h_1 + \beta h_2) d\mu = \alpha \int h_1 d\mu + \beta \int h_2 d\mu$

  $X = \bigcup_i A_i = \bigcup_j B_j$

  i.e. partition of X by disjoint sets

- If $h_1: X \to \mathbb{R}$ and $h_2: X \to \mathbb{R}$ are simple and $h_1 \leq h_2$, then $\int h_1 d\mu \leq \int h_2 d\mu$.
  (since $h_2 - h_1 > 0$ implies that $\int (h_2 - h_1) d\mu > 0$)

Example:

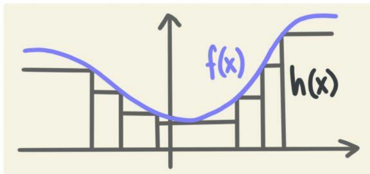
$h(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$ $\int h \, d\mu = \infty + (-\infty)$ is not well-defined

5.2. Lebesgue integral.

Consider $\mathbb{R}$ with Lebesgue measure on it. We define the integral of a non-negative measurable function $f^+: \mathbb{R} \to \mathbb{R}$ as:

$\boxed{\int f^+ d\mu \overset{\text{def}}{=} \sup \left\{ \int h \, d\mu \;\middle|\; h = \sum_{i=1}^{n} a_i \chi_{A_i} \text{ and } 0 \leq h \leq f^+ \right\}}$

Lebesgue integral of f^+



Th. 5.1. (Levi; monotone convergence th.):
If $\{f_n^+ \mid f_n^+ \geq 0 \; \forall n\}$ is an increasing sequence of measurable functions s.t. $f_n^+ \to f^+$, then $\lim_{n \to \infty} \int f_n^+ d\mu = \int f^+ d\mu$.

Th. 5.2.:
Suppose $f^+ \geq 0$ is measurable. Then $\int f^+ d\mu = 0 \iff f^+ = 0$ a.e.

Now we can extend integration to measurable functions that take negative values too:

$f: X \to \mathbb{R}$ is integrable w.r.t. μ: $f^+ = \sup(f, 0)$ and $f^- = -\inf(f, 0)$ are integrable w.r.t. μ.

$\boxed{\int f \, d\mu \overset{\text{def}}{=} \int f^+ d\mu - \int f^- d\mu}$

Lebesgue integral of f

$\boxed{\int_E f \, d\mu \overset{\text{def}}{=} \int f \chi_E \, d\mu}$, $f \chi_E = \begin{cases} f, & x \in E \\ 0, & x \notin E \end{cases}$, where $E \in \mathcal{M}$

integral of f over $E \subset X$

Properties:

- If $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are Lebesgue-integrable, then $\int(\alpha f+\beta g)d\mu = \alpha\int f d\mu + \beta\int g d\mu$.
- If $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are Lebesgue-integrable and $f\leq g$, then $\int f d\mu \leq \int g d\mu$.
- If $f:\mathbb{R}\to\mathbb{R}$ is Lebesgue-integrable, then $|f| = f^+ + f^-$ is Lebesgue-integrable and $|\int f d\mu| \leq \int |f| d\mu$. (since $|\int f d\mu| = |\int (f^+ - f^-) d\mu| \leq |\int (f^+ + f^-) d\mu| = \int |f| d\mu$)
- If $f:\mathbb{R}\to\mathbb{R}$ is Lebesgue-integrable, then $\psi(a) \overset{\text{def}}{=} \mu(\{x\in E \mid |f(x)| > a\}) \overset{a\to\infty}{=} O(a^{-1})$.
- Relation to Riemann integration:

  Lebesgue: the simple functions used to approximate f are arbitrary.
  Riemann : the simple functions used to approximate f are step functions.

  $(\forall \varepsilon > 0 \ \exists$ step functions $h_1 \leq f \leq h_2$ s.t. $\int_a^b (h_2(x) - h_1(x))dx < \varepsilon.)$

  - If f is Riemann-integrable, then f is Lebesgue-integrable and $\int_E f d\mu = \int_a^b f(x)dx$.
  - Lebesgue integration is defined for bounded and unbounded sets in the same way, and in the same way for bounded and unbounded integrands. This contrasts with Riemann integration.

Example:

$f:[0,1]\to\{0,1\}$ 1. $\int f d\mu = 1\cdot\mu([0,1]) = 1$ (since $f = 1$ a.e.)

$f(x) = \begin{cases} 1, & x\notin\mathbb{Q} \\ 0, & x\in\mathbb{Q}\end{cases}$ 2. $\int_0^1 f(x)dx$ is not defined (f cannot be approximated by step functions)

Lemma 5.3. (Fatou):
If $\{f_n \mid f_n \geq 0\}$ are measurable, then $\int \underline{\lim}_{n\to\infty} f_n \, d\mu \leq \int \overline{\lim}_{n\to\infty} f_n \, d\mu$.

Th. 5.4. (Lebesgue; dominated convergence th.):
Suppose $\{f_n\}$ are measurable and $f_n \to f$ a.e. If $\exists$ integrable $g: X\to\mathbb{R}$ s.t. $g > |f_n| \ \forall n$, then $\lim_{n\to\infty}\int f_n d\mu = \int f d\mu$.

Although the concept of an improper integral is not needed in Lebesgue theory, using Th.5.4. we now can define it.

Th. 5.5. (Fubini):
If $f:\mathbb{R}^2\to\mathbb{R}^2$ is Lebesgue-measurable, then $\forall x\in\mathbb{R}$ $f_x(y) = f(x,y)$ is Lebesgue-measurable, and $\forall y\in\mathbb{R}$ $f_y(x) = f(x,y)$ is Lebesgue-measurable, i.e.:
$\int f d\mu^2 = \int f(x,y)\,dxdy = \int\left(\int f_y(x)dx\right)dy = \int\left(\int f_x(y)dy\right)dx$

Example:

$\{\chi_{[n,n+1]}\}$ sequence of "unit humps" drifting to the right

$\lim_{n\to\infty}\int \chi_{[n,n+1]} d\mu = \lim_{n\to\infty}(1\cdot\mu([n,n+1])) = 1$

$\int \lim_{n\to\infty} \chi_{[n,n+1]} d\mu = \int 0 \, d\mu = 0 \neq 1$ (since $\{\chi_{[n,n+1]}\}$ has no dominating function)

Chapter VI
Distributions

1. Test functions.

1.1. Motivations.

In physics the Dirac δ-function is commonly defined as a function with the property
$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \quad \forall f \in C^0(\mathbb{R},\mathbb{R})$, i.e. $\delta(x) = \begin{cases} \infty, & x=0 \\ 0, & x \neq 0 \end{cases}$

However, this definition leads to contradictions: on the one hand, $\int_{-\infty}^{\infty}\delta(x)\,dx = 1$ (for $f(x)=1$), but on the other hand, $\int_{-\infty}^{\infty}\delta(x)\,dx = \delta(0)\cdot\mu(\{0\}) = 0$. To resolve this contradiction we should realize that $\delta(x)$ is not a function in a usual sense but a distribution (or a generalized function). So the definition above doesn't work.

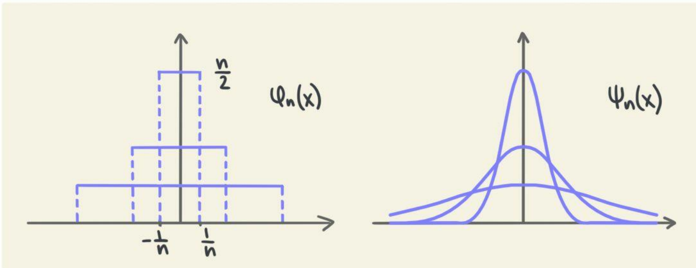
Two common ways of defining distributions:

1. As a limit of a sequence of functions.

<u>Example</u>:

$\varphi_n(x) = \frac{n}{2}\cdot\chi_{\{|x|\leq 1/n\}} \qquad \psi_n(x) = \frac{1}{\sqrt{\pi}\,n}\,e^{-x^2/n^2}$

$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\varphi_n(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\psi_n(x)\,dx = f(0)$



2. As a continuous linear functional on a space of regular test functions.

1.2. Spaces of test functions.

Consider $f: \mathbb{R}^n \to \mathbb{R}$, then:

$\mathrm{supp}(f) :\overset{\text{def}}{=} \mathrm{cl}\,\{x\in\mathbb{R}^n \mid f(x)\neq 0\}$ support of f, where cl is closure in $\mathbb{R}^n$

$f\in C^m(\mathbb{R}^n,\mathbb{R}) : \exists D_{\underline{m}}f \in C^0(\mathbb{R}^n,\mathbb{R}) \quad \forall \underline{m} = (m_1,\dots,m_n)$ s.t. $|\underline{m}| = m$ smooth function of order m

$D_{\underline{m}}f :\overset{\text{def}}{=} \frac{\partial^{|\underline{m}|}f}{\partial x_1^{m_1}\dots\partial x_n^{m_n}}$, where $\underline{m} = (m_1,\dots,m_n)$ is multi-index, $|\underline{m}| = \sum_{i=1}^{n} m_i$, $D_{(0,\dots,0)}f = f$ by convention

$C^0(\mathbb{R}^n,\mathbb{R})$: continuous
$C^1(\mathbb{R}^n,\mathbb{R})$: continuously differentable
...
$C^\infty(\mathbb{R}^n,\mathbb{R}) \equiv C(\mathbb{R}^n,\mathbb{R})$: infinitely differentable

We can construct a vector space from all smooth functions with compact support (on $\mathbb{R}^n$ that means that $\mathrm{supp}(f)$ is closed and bounded, i.e. $\exists R>0$ s.t. $f(x)=0 \; \forall x: |x|\geq R$).

$\mathcal{D}^m(\mathbb{R}^n) :\overset{\text{def}}{=} \{f\in C^m(\mathbb{R}^n,\mathbb{R}) \mid \mathrm{supp}(f) \text{ is compact}\}$ space of test functions of order m

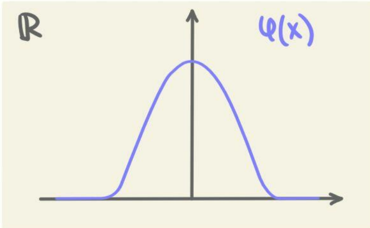
$\mathcal{D}(\mathbb{R}^n) :\overset{\text{def}}{=} \mathcal{D}^\infty(\mathbb{R}^n)$, i.e. $\mathcal{D}(\mathbb{R}^n) = \bigcap_{m=1}^{\infty}\mathcal{D}^m(\mathbb{R}^n)$ space of test functions

Examples:

1) Consider $f(x) = e^{-1/x} \cdot \chi_{\{x>0\}}$. $f \in C(\mathbb{R},\mathbb{R})$ and $f^{(n)}(x)\big|_{x=0} = 0 \ \forall n$ both from the left and from the right. Indeed, $\lim\limits_{x\to 0^+} f'(x) = \lim\limits_{x\to 0^+} \frac{e^{-1/x}}{x^2} = \lim\limits_{x\to 0^+} \frac{1}{x^2}(1 - \frac{1}{x} + \ldots) = 0 = \lim\limits_{x\to 0^-} f'(x)$.

$\varphi(x) :\overset{\text{def}}{=} f(x+a)\cdot f(-(x-a)) = e^{2a/x^2-a^2} \cdot \chi_{\{|x|<a\}}$

$\left.\begin{array}{l} \varphi \in C(\mathbb{R},\mathbb{R}) \\ \operatorname{supp}(\varphi) = [-a,a] \end{array}\right| \Rightarrow \varphi \in \mathcal{D}(\mathbb{R})$

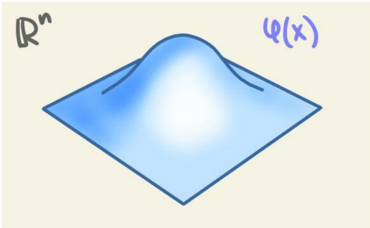


2) Counterpart in $\mathbb{R}^n$:

$\varphi(x) :\overset{\text{def}}{=} e^{2a/|x|^2-a^2} \cdot \chi_{\{|x|<a\}}$ $\quad x \in \mathbb{R}^n \quad |x| = \sqrt{x_1^2 + \ldots + x_n^2}$

$\quad a \in \mathbb{R} \quad a > 0$

$\left.\begin{array}{l} \varphi \in C(\mathbb{R}^n,\mathbb{R}) \\ \operatorname{supp}(\varphi) = B(0,a) \end{array}\right| \Rightarrow \varphi \in \mathcal{D}(\mathbb{R}^n)$



1.3. Convergence on $\mathcal{D}(\mathbb{R}^n)$.

Consider $\{\varphi_k \in \mathcal{D}(\mathbb{R}^n) \mid k \in \mathbb{N}\}$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We can define convergence on $\mathcal{D}(\mathbb{R}^n)$:

$\varphi_k \to \varphi$ to order m: 1. $\exists$ bounded $B \subset \mathbb{R}^n$ s.t. $\operatorname{supp}(\varphi) \subset B$ and $\operatorname{supp}(\varphi_k) \subset B \ \forall k$

2. $D_{\underline{m}}\varphi_k \underset{\text{uniformly}}{\rightrightarrows} D_{\underline{m}}\varphi \quad \forall \underline{m}$ s.t. $|\underline{m}| \le m$

- If $m = \infty$ we say that φ_k converges to φ (omitting "to the order of infinity").
- We can also define this convergence topologically. Consider the space of test functions on a compact set $K \subset \mathbb{R}^n$, i.e. $\mathcal{D}^m(K)$. We can induce topology by defining a norm:

$\|f\|_m :\overset{\text{def}}{=} \sup\limits_{x\in K} \sum\limits_{|\underline{m}|\le m} |D_{\underline{m}} f(x)|$

$U \subset \mathcal{D}^m(K)$ is open: $\forall f \in U \ \exists \varepsilon > 0 \ \exists$ compact K s.t. $f \in K$ and $\{g \in K \mid \|g-f\|_m < \varepsilon\} \subseteq U$

Then we say $\{\varphi_k \in \mathcal{D}^m(K) \mid k \in \mathbb{N}\}$ converges to $\varphi \in \mathcal{D}^m(K)$ to order m if $\varphi_k \to \varphi$ w.r.t. topology.
A similar treatment gives a topology on $\mathcal{D}(K)$ leading to convergence in all orders.

Example:

Consider arbitrary $\varphi \in \mathcal{D}^1(\mathbb{R})$ and a sequence $\{\varphi_k \in \mathcal{D}^1(\mathbb{R}) \mid \varphi_k(x) = \frac{1}{k}\varphi(x)\sin(kx)\}$.

$\varphi_k \rightrightarrows 0$ but $\varphi_k' \not\rightrightarrows 0$ since $\varphi_k'(x) = \frac{1}{k}\varphi'(x)\sin(kx) + \varphi(x)\cos(kx) \not\rightrightarrows 0$

i.e. converges to order 0

② Distributions.

2.1. Spaces of distributions.

$T: \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$ is continuous to order m: $\varphi_k \to \varphi$ to order $m \Rightarrow T(\varphi_k) \to T(\varphi)$

$T: \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$ is linear: $T(a\varphi + b\psi) = aT(\varphi) + bT(\psi) \quad \forall \varphi, \psi \in \mathcal{D}(\mathbb{R}^n) \ \forall a, b \in \mathbb{R}$

$T: \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$ is a distribution of order m: 1. T is linear
2. T is continuous to order m

We can construct a vector space of distributions on $\mathbb{R}^n$. It is the dual space of $\mathcal{D}(\mathbb{R}^n)$.

$\mathcal{D}'^m(\mathbb{R}^n)$: space of distributions of order m
$\mathcal{D}'(\mathbb{R}^n)$: space of distributions $(m = \infty)$

Curious feature, characteristic of dual spaces:

$T \in \mathcal{D}'^{m_1}(\mathbb{R}^n) \Rightarrow T \in \mathcal{D}'^{m_2}(\mathbb{R}^n)$, $m_2 > m_1$
$\varphi \in \mathcal{D}^{m_2}(\mathbb{R}^n) \Rightarrow \varphi \in \mathcal{D}^{m_1}(\mathbb{R}^n)$, $m_2 > m_1$

2.2. Regular distributions.

$f: \mathbb{R}^n \to \mathbb{R}$ is locally integrable: f is integrable on $\forall$ compact $K \subset \mathbb{R}^n$

$T_f(\varphi) :\overset{\text{def}}{=} \int_{\mathbb{R}^n} \varphi f \, d\mu = \int \dots \int_{\mathbb{R}^n} \varphi(\bar{x}) f(\bar{x}) \, dx_1 \dots dx_n$, where f is density of T_f

- $T_f \in \mathcal{D}'(\mathbb{R}^n)$, indeed:
  - $T_f(\varphi)$ exists $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, since φ always vanishes outside some compact set
  - T_f is linear due to linearity of integration (Ch.V Th.5.4.)
  - T_f is continuous due to Lebesgue's dominated convergence th. and properties of the integral
- If $f = g$ a.e. and they are locally integrable, then $T_f = T_g$.
- If $T_f(\varphi) = T_g(\varphi) \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, then $f = g$ a.e.

By identifying f with T_f, locally integrable functions can be thought of as distributions. However, not all distributions arise in this way.

$T \in \mathcal{D}'(\mathbb{R}^n)$ is regular: $\exists$ locally integrable $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $T = T_f$

$T \in \mathcal{D}'(\mathbb{R}^n)$ is singular: $\nexists$ locally integrable $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $T = T_f$

We can also define weak convergence on $\mathcal{D}'(\mathbb{R}^n)$:

$T_k \to T$: $T_k(\varphi) \to T(\varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$

Th. 2.1:
If $f_k \rightrightarrows f$ on $\forall$ compact $K \subset \mathbb{R}^n$, then $T_{f_k} \to T_f$.

2.3. Dirac δ-function.

1. Case of $\mathbb{R}$.

$\delta_a(\varphi) :\overset{\text{def}}{=} \varphi(a)$, $a \in \mathbb{R}$ $\varphi \in \mathcal{D}(\mathbb{R})$

- $a = 0$: $\delta_0 \equiv \delta$, i.e. $\delta(\varphi) = \varphi(0)$
- δ_a is linear, since $\delta_a(A\varphi + B\psi) = A\varphi(a) + B\psi(a) = A\delta_a(\varphi) + B\delta_a(\psi)$
- δ_a is continuous, since $\varphi_k \to \varphi \Rightarrow \varphi_k(a) \to \varphi(a)$

So $\delta_a \in \mathcal{D}'(\mathbb{R}^n)$, but it's singular, i.e. it cannot correspond to any locally integrable function (see the reasoning in sec. 1.1.). Nevertheless, physicists and engineers often maintain the density notation, i.e.:

$$\delta(\varphi) = \int_{-\infty}^{\infty} \varphi(x)\,\delta(x)\,dx = \varphi(0)$$

Other common notation: $\delta_a(x) = \delta(x-a)$, so

$$\delta_a(\varphi) = \int_{-\infty}^{\infty} \varphi(x)\,\delta(x-a)\,dx = \int_{-\infty}^{\infty} \varphi(y+a)\,\delta(y)\,dy = \varphi(a)$$

2. Case of $\mathbb{R}^n$.

$\delta^n_a(\varphi) :\overset{\text{def}}{=} \varphi(a)$, $a \in \mathbb{R}^n$ $\varphi \in \mathcal{D}(\mathbb{R}^n)$

Same properties as for the 1-D case:

$\delta^n_a(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\,\delta^n(x-a)\,d^nx = \varphi(a)$, where $\delta^n(x-a) = \delta(x_1-a_1)\cdot \ldots \cdot \delta(x_n-a_n)$

__Examples:__

1) $T(\varphi) = \sum_n \lambda_n \varphi^{(n)}(0)$, $\lambda_n \in \mathbb{R}$ is a distribution (linear and cont.)

2) $T(\varphi) = \sum_n \lambda_n \varphi(x_n) = \sum_n \lambda_n \delta_{x_n}(\varphi)$, $\lambda_n \in \mathbb{R}$ is a distribution (linear and cont.)

3) $T(\varphi) = (\varphi(0))^2$ is not a distribution (nonlinear)

4) $T(\varphi) = \sup(\varphi)$ is not a distribution (nonlinear, since $\sup(\varphi+\psi) \neq \sup(\varphi) + \sup(\psi)$)

5) $T(\varphi) = \int_{-\infty}^{\infty} |\varphi(x)|\,dx$ is not a distribution (nonlinear, since $|\varphi(x)+\psi(x)| \neq |\varphi(x)| + |\psi(x)|$)

3. Operations on distributions.

3.1. Basic operations.

$\mathcal{D}'(\mathbb{R}^n)$ is a vector space, thus $(aT+bS)\in\mathcal{D}'(\mathbb{R}^n)$ $\forall T,S\in\mathcal{D}'(\mathbb{R}^n)$ $\forall a,b\in\mathbb{R}$. But there is no easy way of defining a product of two distributions (i.e. $\mathcal{D}'(\mathbb{R}^n)$ is not an algebra). For example, if we define the product as $ST(\varphi)\overset{def}{=}S(\varphi)\cdot T(\varphi)$, in general, it's nonlinear. However, we can define a product of a distribution with a smooth function:

$$\boxed{(\alpha T)(\varphi):\overset{def}{=}T(\alpha\varphi)}\quad,\ T\in\mathcal{D}'(\mathbb{R}^n)\quad \alpha\in C(\mathbb{R}^n,\mathbb{R})$$

If T is regular and α is locally integrable, then $\alpha T_f = T_{\alpha f}$.

__Examples:__

1) $\delta\in\mathcal{D}'^0(\mathbb{R})$ Consider $\alpha\in C^0(\mathbb{R},\mathbb{R})$, then:

$$\alpha\delta(\varphi)=\delta(\alpha\varphi)=\alpha(0)\cdot\varphi(0)\text{ , or }\int_{-\infty}^{\infty}\varphi(x)\alpha(x)\delta(x)\,dx=\alpha(0)\cdot\int_{-\infty}^{\infty}\varphi(x)\delta(x)\,dx$$

$$\alpha\delta=\alpha(0)\delta \qquad \alpha(x)\delta(x)=\alpha(0)\delta(x)$$

2) $\delta_a\in\mathcal{D}'^0(\mathbb{R})$ Consider $\alpha\in C^0(\mathbb{R},\mathbb{R})$, then:

$$\alpha\delta_a=\alpha(a)\delta_a \qquad \alpha(x)\delta(x-a)=\alpha(a)\delta(x-a) \qquad \text{e.g. } \alpha(x)=x:\ x\delta=0$$

3) $\delta_a^n\in\mathcal{D}'^0(\mathbb{R}^n)$ Consider $\alpha\in C(\mathbb{R}^n,\mathbb{R})$, then:

$$\alpha\delta_a^n=\alpha(a)\delta_a^n \qquad \alpha(x)\delta^n(x-a)=\alpha(a)\delta^n(x-a)$$

3.2. Differentiation of distributions.

Let $T_f\in\mathcal{D}'(\mathbb{R})$ be regular and f differentiable. Results from real analysis say that $f'=\frac{df}{dx}$ is locally integrable. Then $\forall\varphi\in\mathcal{D}^1(\mathbb{R})$ we have:

$$T_{f'}(\varphi)=\int_{-\infty}^{\infty}\varphi(x)f'(x)\,dx=\varphi(x)f(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\varphi'(x)f(x)\,dx=-T_f(\varphi')$$

We can extend this identity to singular distributions simply by demanding it as a definition:

$$\underbrace{\boxed{T'(\varphi):\overset{def}{=}-T(\varphi')}}_{\text{derivative of }T}\quad,\ \begin{matrix}T\in\mathcal{D}'^m(\mathbb{R}^n) & \varphi'\in\mathcal{D}^m(\mathbb{R}^n)\\ T'\in\mathcal{D}'^{m+1}(\mathbb{R}^n) & \varphi\in\mathcal{D}^{m+1}(\mathbb{R}^n)\end{matrix}$$

- $T'\in\mathcal{D}'^{m+1}(\mathbb{R}^n)$ (indeed, it's linear and continuous).
- For regular distributions T' corresponds to the derivative of its density, i.e. f'.
- $\forall T\in\mathcal{D}'^m(\mathbb{R}^n)$ is infinitely differentiable (see properties of distributions in sec. 2.1.).

Partial derivatives are defined in the same way:

$$\partial_k T(\varphi):\overset{def}{=}-T(\partial_k\varphi)\quad,\ T\in\mathcal{D}'^m(\mathbb{R}^n)\quad\varphi\in\mathcal{D}^{m+1}(\mathbb{R}^n)$$

$$D_{\underline{k}}T(\varphi):\overset{def}{=}(-1)^{|k|}T(D_{\underline{k}}\varphi)\quad,\ T\in\mathcal{D}'^m(\mathbb{R}^n)\quad\varphi\in\mathcal{D}^{m+|k|}(\mathbb{R}^n)$$

Usual properties of partial derivatives are true, e.g. $\partial_i\partial_j T=\partial_j\partial_i T$. Indeed:

$$\partial_i\partial_j T(\varphi)=T(\partial_i\partial_j\varphi)\underset{\varphi\text{ is cont.}}{=}T(\partial_j\partial_i\varphi)=\partial_j\partial_i T(\varphi)\quad\forall\varphi\in\mathcal{D}^{m+1}(\mathbb{R}^n)$$

Examples:

1) Heaviside step function $\theta(x) :\overset{def}{=} \chi_{\{x>0\}}$

It's locally integrable, thus $\theta(x)$ generates a regular distribution $T_\theta \in \mathcal{D}'^1(\mathbb{R})$. Then $\forall \varphi \in \mathcal{D}^1(\mathbb{R}^n)$:

$$T_\theta'(\varphi) = -T_\theta(\varphi') = -\int_{-\infty}^{\infty} \varphi'(x)\theta(x)dx = -\varphi(x)\Big|_0^\infty = \varphi(0) = \delta(\varphi), \text{ i.e. } \boxed{T_\theta' = \delta} \quad \theta'(x) = \delta(x)$$

$x=0$: inf. steep step

2) Derivatives of δ-function

Consider $\varphi \in \mathcal{D}^1(\mathbb{R})$, then:

$$\delta'(\varphi) = -\delta(\varphi') = -\varphi'(0)$$
$$\delta''(\varphi) = -\delta'(\varphi') = \delta(\varphi'') = \varphi''(0)$$
$$\dots$$
$$\boxed{\delta^{(k)}(\varphi) = (-1)^k \varphi^{(k)}(0)} \qquad \varphi(x)\,\delta^{(k)}(x) = (-1)^k \varphi^{(k)}(x)\,\delta(x)$$

3) Leibniz's rule

Consider $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\alpha \in C(\mathbb{R}^n, \mathbb{R})$.

$$(\alpha T)'(\varphi) = -(\alpha T)(\varphi') = T(-\alpha\varphi') = T((-\alpha\varphi)' + \alpha'\varphi) = T'(\alpha\varphi) + \alpha' T(\varphi) = \alpha T'(\varphi) + \alpha' T(\varphi)$$

$$\boxed{(\alpha T)' = \alpha' T + \alpha T'}$$

4) $\alpha(x) = x \quad T = \delta$

$$\left.\begin{aligned} (x\delta)' &= 0' = 0 \\ (x\delta)' &= \delta + x\delta' \end{aligned}\right| \Rightarrow \boxed{x\delta' = -\delta}$$

Indeed:

$$x\delta'(\varphi) = \delta'(x\varphi) = -\delta(\varphi + \varphi' x) = -\delta(\varphi) - \delta(\varphi' x) = -\delta(\varphi)$$

3.3. Surface δ-function.

The usual way of generalizing the Dirac δ-function to higher dimensions is to take the product of multiple δ-functions, i.e. $\delta(\vec{x}) = \delta(x_1) \cdot \ldots \cdot \delta(x_n)$. However, a different generalization is possible. We know that $\delta = \theta'$ and $\delta(x) \neq 0$ only on the boundary of the halfline $(0, +\infty)$. Thus we can define the Dirac δ for higher dimensions as a function that is non-vanishing only on the boundary of some domain $D \subset \mathbb{R}^n$. Or, more formally:

$\delta \rightarrow -\vec{n} \cdot \nabla \chi_{\{x \in D\}}$, $\vec{n}$ is outward normal vector to ∂D

$$\delta(\vec{x}) = \begin{cases} \infty, & x \in \partial D \\ 0, & x \notin \partial D \end{cases} \qquad \int_{\mathbb{R}^n} \delta(\vec{x})\, d^n x = S_{\partial D}$$

From here we can then define the derivative of δ-function:

$\delta' \rightarrow \nabla^2 \chi_{\{x \in D\}}$

$$\int_{-\infty}^{\infty} \frac{d^2}{dx^2} \chi_{\{a<x<b\}} \varphi(x)\,dx = \int_{-\infty}^{\infty} \chi_{\{a<x<b\}} \frac{d^2}{dx^2}\varphi(x)\,dx = \int_a^b \frac{d^2}{dx^2}\varphi(x)\,dx = \frac{d\varphi(x)}{dx}\Big|_a^b$$

$$\int_{\mathbb{R}^n} \nabla^2 \chi_{\{x \in D\}} \varphi(\vec{x})\, d^n x = \int_{\mathbb{R}^n} \chi_{\{x \in D\}} \nabla^2 \varphi(\vec{x})\, d^n x = \int_D \nabla^2 \varphi(\vec{x})\, d^n x = \int_{\partial D} \nabla\varphi(\vec{x}) \cdot d\vec{n}$$

4) Change of variable in δ-functions.

4.1. Change of variable in $\delta(f(x))$.

In applications it's common to consider functions of the form $\delta(f(x))$. This operation doesn't generalize to all distributions, but we can make sense of it for δ-function. Consider $f \in C^0(\mathbb{R},\mathbb{R})$ s.t. it's increasing monotonely and $f(\pm\infty) = \pm\infty$. Then $\forall \varphi \in \mathcal{D}(\mathbb{R})$ we have:

$$\int_{-\infty}^{\infty} \varphi(x)\,\delta(f(x))\,dx \underset{y=f(x)}{=} \int_{-\infty}^{\infty} \varphi(f^{-1}(y))\,\delta(y)\,\frac{dx}{dy}\,dy = \int_{-\infty}^{\infty} \frac{\varphi(f^{-1}(y))}{f'(f^{-1}(y))}\,\delta(y)\,dy = \frac{\varphi(a)}{f'(a)}, \text{ where } a = f^{-1}(0), \text{ i.e. } f(a) = 0$$

If f is decreasing monotonely, then the sign would change, so the formula is:

$$\boxed{\delta \circ f = \frac{\delta_a}{|f'(a)|}}, \; a = f^{-1}(0) \qquad \int_{-\infty}^{\infty} \varphi(x)\,\delta(f(x))\,dx = \frac{\varphi(a)}{|f'(a)|}$$

If $\varphi(x) = \psi(x)$ $\forall x \in [-\varepsilon,\varepsilon]$, then $\delta(\varphi) = \delta(\psi) = \varphi(0) = \psi(0)$. Hence δ can be regarded as a distribution on $\mathcal{D}([-\varepsilon,\varepsilon])$. So we can write the following: $\delta(\varphi) = \int_{-\varepsilon}^{\varepsilon} \varphi(x)\,\delta(x)\,dx$. Then we can generalize the results above even further. Consider $f \in C^0(\mathbb{R},\mathbb{R})$ s.t. it has zeros $\{a_1,\dots,a_n\}$ and it's monotone in the neighbourhoods $U(a_i)$ $\forall a_i$. Then, by restricting integration to a small neighbourhood of each zero, we obtain:

$$\boxed{\delta \circ f = \sum_k \frac{\delta_{a_k}}{|f'(a_k)|}}, \; a_k = f^{-1}(0) \; \forall k \qquad \int_{-\infty}^{\infty} \varphi(x)\,\delta(f(x))\,dx = \sum_k \frac{\varphi(a_k)}{|f'(a_k)|}$$

Examples:

1) $f(x) = -x$

Then $\forall \varphi \in \mathcal{D}(\mathbb{R})$ $\delta(-x) = \delta(x)$, i.e. δ is even (as expected).

2) $f(x) = x^2 - a^2$, $a \neq 0$

1. Using general formula:

$$\delta(x^2-a^2) = \frac{\delta_{-a}}{|2x|}\Big|_{x=-a} + \frac{\delta_a}{|2x|}\Big|_{x=a} = \frac{1}{2a}(\delta_{-a} + \delta_a)$$

2. Using approximations:

$x = a$: $f(x) \approx 2a(x-a)$

$x = -a$: $f(x) \approx -2a(x+a)$

$$\delta(x^2-a^2) = \delta(2a(x-a)) + \delta(-2a(x+a)) = \frac{1}{2a}(\delta_a + \delta_{-a})$$

$$\delta(x^2-a^2) = \frac{1}{2a}(\delta(x-a) + \delta(x+a))$$

3) $f(x) = (x-a)(x-b)$, $a < b$

$$\delta((x-a)(x-b)) = \delta(x^2-(a+b)x+ab) = \frac{\delta_a}{|2x-(a+b)|}\Big|_{x=a} + \frac{\delta_b}{|2x-(a+b)|}\Big|_{x=b} = \frac{\delta_a}{|a-b|} + \frac{\delta_b}{|b-a|} \overset{a<b}{=}$$

$$= \frac{1}{b-a}(\delta_a + \delta_b)$$

$$\delta((x-a)(x-b)) = \frac{1}{b-a}(\delta(x-a) + \delta(x-b))$$

4.2. Change of variable in $\delta'(f(x))$.

First, consider a monotone function $f \in C^0(\mathbb{R},\mathbb{R})$ s.t. $f(\pm\infty) = \pm\infty$ and it has just one zero $x = a$.
Then $\forall \varphi \in \mathcal{D}(\mathbb{R})$ we have:

$$\int_{-\infty}^{\infty} \varphi(x)\, \delta'(f(x))\, dx \underset{y=f(x)}{=} \int_{-\infty}^{\infty} \varphi(x)\, \delta'(y) \frac{dy}{|f'(x)|} = -\int_{-\infty}^{\infty} \frac{d}{dy}\left(\frac{\varphi(x)}{|f'(x)|}\right) \delta(y)\, dy = -\int_{-\infty}^{\infty} \frac{1}{f'(x)} \frac{d}{dx}\left(\frac{\varphi(x)}{|f'(x)|}\right) \delta(y)\, dy = -\frac{1}{f'(x)} \frac{d}{dx}\left(\frac{\varphi(x)}{|f'(x)|}\right)\Bigg|_{x=a}$$

By the same reasoning we can derive a formula for the case of f with multiple zeros $\{a_1, \dots, a_N\}$, where f is monotone in the neighbourhoods of all zeros.

$$\boxed{\int_{-\infty}^{\infty} \varphi(x)\, \delta'(f(x))\, dx = -\sum_k \frac{1}{f'(x)} \frac{d}{dx}\left(\frac{\varphi(x)}{|f'(x)|}\right)\Bigg|_{x=a_k}} \quad , a_k = f^{-1}(0) \ \forall k$$

- We can expand this expression:

$$\frac{d}{dx}\left(\frac{\varphi(x)}{f'(x)}\right)\Bigg|_{x=a_k} = \frac{\varphi'(x) f'(x) - \varphi(x) f''(x)}{(f'(x))^2}\Bigg|_{x=a_k} = \frac{\varphi'(a_k)}{f'(a_k)} - \frac{\varphi(a_k) f''(a_k)}{(f'(a_k))^2}$$

  If we note that $|f'(x)| = f'(x)$ whenever f is increasing, and $|f'(x)| = -f'(x)$ whenever f is decreasing, then

$$\int_{-\infty}^{\infty} \varphi(x)\, \delta'(f(x))\, dx = -\sum_k \left(\frac{\varphi'(a_k)}{f'(a_k)|f'(a_k)|} - \frac{\varphi(a_k) f''(a_k)}{(f'(a_k))^3}\right) = \sum_k \left(\frac{\delta'_{a_k}(\varphi)}{f'(a_k)|f'(a_k)|} - \frac{\delta_{a_k}(\varphi) f''(a_k)}{(f'(a_k))^3}\right)$$

- Another useful in applications result:

  Consider $f \in C^0(\mathbb{R},\mathbb{R})$ s.t. it's increasing monotonely and $f(a) = 0$. Then $\forall \varphi \in \mathcal{D}(\mathbb{R})$ we have:

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \varphi(x) \frac{d}{dx} \delta(f(x))\, dx = -\int_{-\infty}^{\infty} \frac{d\varphi(x)}{dx} \delta(f(x))\, dx = -\frac{\varphi'(x)}{|f'(x)|}\Bigg|_{x=a} \\ &\int_{-\infty}^{\infty} \varphi(x) f'(x)\, \delta'(f(x))\, dx = -\left(\frac{1}{f'(x)} \frac{d}{dx}\left(\frac{\varphi(x) f'(x)}{|f'(x)|}\right)\Bigg|_{x=a}\right) = -\frac{\varphi'(x)}{|f'(x)|}\Bigg|_{x=a} \end{aligned}\right| \Rightarrow \boxed{\frac{d}{dx}\big(\delta(f(x))\big) = f'(x)\, \delta'(f(x))}$$

Examples:

1) Show that $\delta(f(x)) + f(x)\, \delta'(f(x)) = 0$

   Solution:

$$\left.\begin{aligned} \frac{d}{dx}\big(f(x)\, \delta(f(x))\big) = \frac{df(x)}{dx} \delta(f(x)) + f(x) \frac{d}{dx} \delta(f(x)) = f'(x)\, \delta(f(x)) + f(x) f'(x)\, \delta'(f(x)) \\ f(x)\, \delta(f(x)) = 0 \ \Rightarrow\ \frac{d}{dx}\big(f(x)\, \delta(f(x))\big) = 0 \end{aligned}\right| \Rightarrow \delta(f(x)) + f(x)\, \delta'(f(x)) = 0$$

2) Show that $\phi(x,y) = \delta(x^2 - y^2)$ is a solution of $x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + 2\phi = 0$.

   Solution:

$$\left.\begin{aligned} x\frac{\partial\phi}{\partial x} &= x\frac{\partial}{\partial x}\delta(x^2-y^2) = x \cdot 2x \cdot \delta'(x^2-y^2) \\ y\frac{\partial\phi}{\partial y} &= y\frac{\partial}{\partial y}\delta(x^2-y^2) = y \cdot (-2y) \cdot \delta'(x^2-y^2) \end{aligned}\right| \Rightarrow$$

$$x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + 2\phi = 2x^2\delta'(x^2-y^2) - 2y^2\delta'(x^2-y^2) + 2\delta(x^2-y^2) = 2(x^2-y^2)\,\delta'(x^2-y^2) + 2\delta(x^2-y^2) =$$
$$= -2\delta(x^2-y^2) + 2\delta(x^2-y^2) = 0$$

⑤ Fourier transform.

5.1. Basic properties.

Consider $\varphi : \mathbb{R} \to \mathbb{R}$. Then:

$$\mathcal{F}\varphi(y) \overset{\text{def}}{:=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \varphi(x)\, dx$$ Fourier transform of φ

$$\mathcal{F}^{-1}\varphi(x) \overset{\text{def}}{:=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \varphi(y)\, dy$$ inverse Fourier transform of φ

Th.3.1. (Fourier's integral th.):
If φ is s.t. $|\varphi|$ is locally integrable and φ is of bounded variation, then $\mathcal{F}^{-1}\mathcal{F}\varphi = \varphi$.

$$\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixy} \left(\int_{\mathbb{R}} e^{-ity} \varphi(t)\, dt \right) dy = \frac{1}{2\pi} \int_{\mathbb{R}} dt\, \varphi(t) \int_{\mathbb{R}} e^{iy(x-t)} dy$$

__Examples:__

1) $$\delta_a(x) = \frac{1}{2\pi} \int_{\mathbb{R}} dt\, \delta(t-a) \int_{\mathbb{R}} e^{iy(x-t)} dy = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iy(x-a)} dy$$

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixy} dy = \delta(-x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixy} dy$$

$$\delta_a(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x-a)y} dy$$

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixy} dy$$

2) $$\mathcal{F}\delta_a(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \delta(x-a)\, dx = \frac{1}{\sqrt{2\pi}} e^{-iay}$$

$$\mathcal{F}\delta(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \delta(x)\, dx = \frac{1}{\sqrt{2\pi}}$$

$$\mathcal{F}\delta_a(x) = \frac{1}{\sqrt{2\pi}} e^{-iax}$$

$$\mathcal{F}\delta(x) = \frac{1}{\sqrt{2\pi}}$$

3) $f(x) = \chi_{\{|x| \le a\}}$

$$\mathcal{F}f(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ixy} dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-ixy}}{(-iy)} \right|_{-a}^{a} = \frac{e^{iay} - e^{-iay}}{\sqrt{2\pi}\, iy} = \sqrt{\frac{2}{\pi}}\, \frac{\sin(ay)}{y}$$

4) $f(k) = e^{-a^2x^2/2}$

$$\mathcal{F}f(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} e^{-\frac{a^2x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx - \frac{a^2}{2}x^2} dx = \frac{1}{|a|} e^{-\frac{k^2}{2a^2}}$$

$$\left[\int_{\mathbb{R}} e^{iax - bx^2} dx = \sqrt{\frac{\pi}{b}}\, e^{-a^2/4b}, \quad a, b \in \mathbb{R} \quad b > 0 \right]$$ Gaussian integral (see Ex.11.3.8. (Has.))

5.2. Fourier transform of a distribution.

Consider a regular distribution $T_f \in \mathcal{D}'(\mathbb{R})$. We can define the Fourier transform of T_f as $\mathcal{F}T_f = T_{\mathcal{F}f}$. Then:

$$T_{\mathcal{F}f}(\varphi) = \int_{\mathbb{R}} \varphi(x)\,\mathcal{F}f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \varphi(x)\left(\int_{\mathbb{R}} e^{-ixy} f(y)\,dy\right)dx = \int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\varphi(x)\,dx\right)f(y)\,dy = T_f(\mathcal{F}\varphi)$$

We can take the last identity as a definction (so that it's true for singular distributions too).

$$\mathcal{F}T(\varphi) \overset{\text{def}}{:=} T(\mathcal{F}\varphi) \qquad \mathcal{F}^{-1}T(\varphi) \overset{\text{def}}{:=} T(\mathcal{F}^{-1}\varphi), \quad T \in \mathcal{D}'(\mathbb{R}) \quad \varphi \in \mathcal{D}(\mathbb{R})$$

There is, however, a problem with this definition: if $\text{supp}(\varphi)$ is bounded, then $\mathcal{F}\varphi$ is generally an entire function and $\text{supp}(\mathcal{F}\varphi)$ is unbounded (entire function that vanishes on any open set must vanish everywhere). To avoid these problems one should use the space of rapidly decreasing functions (or the Schwartz space) instead of the space of test functions. These are functions that approach zero as $|x| \to \infty$ faster than any inverse power $|x|^{-p}$.

$$S(\mathbb{R}) \overset{\text{def}}{:=} \{\varphi \in C(\mathbb{R},\mathbb{R}) \mid \sup_{x\in\mathbb{R}} |x^p \cdot \varphi^{(\ell)}(x)| < \infty \ \forall p,\ell\}$$ space of rapidly decreasing functions

We can also define convergence on $S(\mathbb{R})$:

$$\varphi_k \to \varphi : \ \lim_{k\to\infty} |x^p(\varphi_k^{(\ell)}(x) - \varphi^{(\ell)}(x))| = 0 \quad \forall p,\ell$$

Then we can introduce the space of continuous linear functionals on $S(\mathbb{R})$:

$S'(\mathbb{R})$: space of tempered distributions

- If $\varphi \in S(\mathbb{R})$, then $\mathcal{F}\varphi \in S(\mathbb{R})$, i.e. the Fourier transform is well defined for tempered distributions.
- Every test function is also rapidly decreasing, i.e. $\mathcal{D}(\mathbb{R}) \subset S(\mathbb{R})$.
- All the usual properties of Fourier transforms are true, e.g. $\mathcal{F}^{-1}\mathcal{F}T(\varphi) = T(\mathcal{F}\mathcal{F}^{-1}\varphi) = T(\varphi)$.

<u>Example:</u>

$$\mathcal{F}\delta_a(\varphi) = \delta_a(\mathcal{F}\varphi) = \mathcal{F}\varphi(a) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixa}\varphi(x)\,dx = T_{\frac{1}{\sqrt{2\pi}}e^{-ixa}}(\varphi), \quad \text{i.e. } f(x) = \frac{1}{\sqrt{2\pi}} e^{-ixa}$$

$$\mathcal{F}^{-1}\mathcal{F}\delta_a(\varphi) = \mathcal{F}\delta_a(\mathcal{F}^{-1}\varphi) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-ixa}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ixy}\varphi(y)\,dy\right)dx = \frac{1}{2\pi}\int_{\mathbb{R}} dy\,\varphi(y)\int_{\mathbb{R}} e^{ix(y-a)}\,dx =$$

$$= \int_{\mathbb{R}} \varphi(y)\,\delta(y-a)\,dx = \varphi(a) = \delta_a(\varphi)$$

6. Green's function.

Distribution theory may be used to find solutions of inhomogeneous linear PDEs (using the technique of Green's functions).

6.1. Poisson's equation.

$$\boxed{\Delta\varphi(\vec{x}) = -4\pi\rho(\vec{x})}\ , \quad \vec{x}\in\mathbb{R}^3$$

A solution is given by:

$$\varphi(\vec{x}) = -\int_{\mathbb{R}^3} 4\pi\rho(\vec{x}')\,G(\vec{x}-\vec{x}')\,d^3x' \ , \quad \text{where } \Delta G(\vec{x}-\vec{x}') = \delta(\vec{x}-\vec{x}')$$

Indeed:

$$\Delta\varphi(\vec{x}) = -\int_{\mathbb{R}^3} 4\pi\rho(\vec{x}')\,\Delta G(\vec{x}-\vec{x}')\,d^3x' = -\int_{\mathbb{R}^3} 4\pi\rho(\vec{x}')\,\delta(\vec{x}-\vec{x}')\,d^3x' = -4\pi\rho(\vec{x})$$

1. Finding $G(\vec{x}-\vec{x}')$.

$$G(\vec{x}-\vec{x}') = \mathcal{F}^{-1}(g(\vec{k})) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,g(\vec{k})\,d^3k$$

$$\Delta G(\vec{x}-\vec{x}') \overset{\circledast}{=} \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}(-k^2)\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,g(\vec{k})\,d^3k$$

$$\delta(\vec{x}-\vec{x}') = \mathcal{F}^{-1}\left(\frac{1}{(2\pi)^{3/2}}e^{-i\vec{k}\cdot\vec{x}'}\right) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,d^3k$$

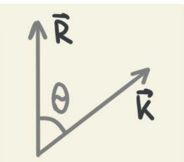
$$\Rightarrow \quad g(\vec{k}) = -\frac{1}{(2\pi)^{3/2}}\cdot\frac{1}{k^2}$$

$$G(\vec{x}-\vec{x}') = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{k^2}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,d^3k$$

$\circledast$ $$\Delta e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = e^{-i\vec{k}\cdot\vec{x}'}\cdot\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)e^{i(k_1x+k_2y+k_3z)} = e^{-i\vec{k}\cdot\vec{x}'}(-k_1^2-k_2^2-k_3^2)\,e^{i\vec{k}\cdot\vec{x}} = -k^2e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

2. Evaluating $G(\vec{x}-\vec{x}')$.

To evaluate $G(\vec{x}-\vec{x}')$ we will perform integration in polar coordinates (k,θ,φ) with the k_3-axis pointing along $\vec{R}=\vec{x}-\vec{x}'$.



$$\vec{k}\cdot(\vec{x}-\vec{x}') = \vec{k}\cdot\vec{R} = k\cdot R\cdot\cos\theta$$

$$d^3k = k^2\sin\theta\,dk\,d\theta\,d\varphi$$

$$G(\vec{R}) = -\frac{1}{(2\pi)^3}\int_0^\infty dk\int_0^\pi d\theta\int_0^{2\pi}\frac{1}{k^2}e^{ikR\cos\theta}\cdot k^2\sin\theta\,d\varphi = -\frac{1}{(2\pi)^2}\int_0^\infty dk\int_0^\pi e^{ikR\cos\theta}\cdot\sin\theta\,d\theta =$$

$$= +\frac{1}{(2\pi)^2}\int_0^\infty dk\int_1^{-1} e^{ikR\cos\theta}\,d(\cos\theta) = -\frac{1}{(2\pi)^2}\int_0^\infty\frac{1}{ikR}\,e^{ikRt}\Big|_{t=-1}^{t=1}dk = -\frac{1}{(2\pi)^2}\int_0^\infty\frac{e^{ikR}-e^{-ikR}}{ikR}\,dk =$$

$$= -\frac{1}{(2\pi)^2}\int_0^\infty\frac{2\sin(kR)}{kR}\,dk = \left[\int_0^\infty\frac{\sin x}{x}\,dx = \frac{\pi}{2}\right] = -\frac{1}{(2\pi)^2}\cdot\frac{\pi}{R} = -\frac{1}{4\pi R}$$

see CA Ex. 11.3.10. (Has.)

So a solution of Poisson's equation:

$$\varphi(\vec{x}) = \int_{\mathbb{R}^3}\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' \quad , \text{ i.e. } \quad G(\vec{x}-\vec{x}') = -\frac{1}{4\pi|\vec{x}-\vec{x}'|}$$

Example:

$\rho(\vec{x}) = q\,\delta(\vec{x}-\vec{a})$ (point charge) $\qquad \varphi(\vec{x}) = \dfrac{q}{|\vec{x}-\vec{a}|}$ Coulomb potential

6.2. The wave equation. (inhomogeneous)

It's convinient to adopt relativistic notation:

$$\boxed{\Box\psi(x) = f(x)} \quad , \quad x = \begin{pmatrix}\vec{x}\\ ct\end{pmatrix} \quad \Box := \partial_\mu \partial^\mu = \Delta - \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}$$

Every Green's function generates a solution:

$$\psi_G(x) = \int_{\mathbb{R}^4} f(x')\, G(x-x')\, d^4x' \quad , \quad \text{where} \quad \Box G(x-x') = \delta(x-x')$$

Indeed:

$$\Box\psi_G(x) = \int_{\mathbb{R}^4} f(x')\,\Box G(x-x')\, d^4x' = \int_{\mathbb{R}^4} f(x')\,\delta(x-x')\, d^4x' = f(x)$$

1. Finding $G(x-x')$.

$$G(x-x') = \mathcal{F}^{-1}(g(k)) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i(\vec{k}\cdot(\vec{x}-\vec{x}') + \omega(t-t'))} g(k)\, d^4k \quad , \quad \text{where} \quad k = \begin{pmatrix}\vec{k}\\ \omega/c\end{pmatrix} \quad k^2 = k_\mu k^\mu$$

$$\Box G(x-x') \overset{\circledast}{=} -\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} k^2 e^{i(\vec{k}\cdot(\vec{x}-\vec{x}') + \omega(t-t'))} g(k)\, d^4k$$

$$\delta(x-x') = \mathcal{F}^{-1}\left(\frac{1}{(2\pi)^2} e^{-i(\vec{k}\cdot\vec{x}' + \omega t')}\right) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} e^{i(\vec{k}\cdot(\vec{x}-\vec{x}') + \omega(t-t'))} d^4k \qquad \Rightarrow$$

$$\circledast \quad \Box e^{i(\vec{k}\cdot(\vec{x}-\vec{x}') + \omega(t-t'))} = e^{-i(\vec{k}\cdot\vec{x}' + \omega t')}\cdot\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\cdot e^{i(\vec{k}\cdot\vec{x} + \omega t)} = -k^2\cdot e^{i(\vec{k}\cdot(\vec{x}-\vec{x}') + \omega(t-t'))}$$

$$\Rightarrow \quad g(k) = -\frac{1}{4\pi^2 k^2} \quad \Rightarrow \quad G(x-x') = -\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} \frac{e^{i(\vec{k}\cdot(\vec{x}-\vec{x}') + \omega(t-t'))}}{k^2} d^4k$$

2. Evaluating $G(x-x')$.

$$\vec{R} = \vec{x} - \vec{x}' \qquad k^2 = k_\mu k^\mu = K^2 - \left(\frac{\omega}{c}\right)^2$$
$$\tau = t - t' \qquad K^2 = |\vec{k}|^2$$

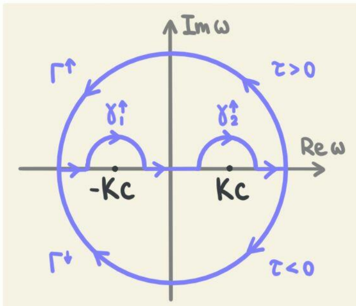
$$G(x-x') = \frac{c}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\tau}}{\omega^2 - (Kc)^2}\int_{\mathbb{R}^3} e^{i\vec{k}\cdot\vec{R}}\, d^3k$$

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}d\omega}{\omega^2-(Kc)^2} = -\frac{\pi}{Kc}\sin(Kc|\tau|)$$ (from CA, see Ex.11.3.13.(Has.))

However, it turns out that the physically interesting Green's functions are obtained not from the principle value, but from giving small imaginary parts to the poles, i.e. the poles are pushed in the LHP/UHP.

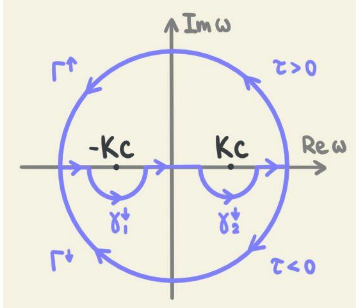
Advanced Green's function (ingoing wave condition):

- the contours $\gamma_1^\uparrow$ and $\gamma_2^\uparrow$ are used to go around the poles
- the integral vanishes for $\tau > 0$ (no poles inside the upper semicircle)
- solution depends only on the future sources, acausal (propagates signals only in the negative time direction)



Retarded Green's function (outgoing wave condition):

- the contours $\gamma_1^\downarrow$ and $\gamma_2^\downarrow$ are used to go around the poles
- the integral vanishes for $\tau < 0$ (no poles inside the lower semicircle)
- solution depends only on the past sources, causal (propagates signals only in the positive time direction)



Let's consider the retarded case:

$\tau = t - t' > 0$: the contour is completed with $\Gamma^{\uparrow}$, i.e. in the UHP

$$\int_{-\infty}^{\infty} \frac{e^{i\omega\tau} d\omega}{\omega^2 - (Kc)^2} = 2\pi i \left(\frac{e^{iKc\tau}}{2Kc} - \frac{e^{-iKc\tau}}{2Kc} \right) = -\frac{2\pi}{Kc} \sin(Kc\tau)$$

$\tau = t - t' < 0$: the contour is completed with $\Gamma^{\downarrow}$, i.e. in the LHP

$$\int_{-\infty}^{\infty} \frac{e^{i\omega\tau} d\omega}{\omega^2 - (Kc)^2} = 0$$

So we have:

$$\int_{-\infty}^{\infty} \frac{e^{i\omega\tau} d\omega}{\omega^2 - (Kc)^2} = -\frac{2\pi}{Kc} \theta(\tau) \sin(Kc\tau)$$, i.e. $G(x-x')$ vanishes for $\tau < 0$

To complete the computations we use polar coordinates (K, θ, φ) with k_3-axis parallel to $\vec{R}$:

$$G(\vec{R}, \tau) = -\frac{\theta(\tau)}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^{\infty} dK \int_0^{\pi} e^{iKR\cos\theta} \frac{\sin(Kc\tau)}{K} K^2 \sin\theta \, d\theta =$$

$$= -\frac{\theta(\tau)}{(2\pi)^3} \cdot 2\pi \cdot \int_0^{\infty} dK \int_0^{\pi} e^{iKR\cos\theta} \cdot (-K) \sin(Kc\tau) \, d(\cos\theta) =$$

$$= +\frac{\theta(\tau)}{(2\pi)^2} \int_0^{\infty} K \sin(Kc\tau) \left. \frac{e^{iKR\alpha}}{iKR} \right|_{\alpha=1}^{\alpha=-1} dK = -\frac{\theta(\tau)}{(2\pi)^2 R} \int_0^{\infty} 2 \sin(Kc\tau) \frac{e^{iKR} - e^{-iKR}}{2i} dK =$$

$$= -\frac{2\,\theta(\tau)}{(2\pi)^2 R} \int_0^{\infty} \frac{e^{iKc\tau} - e^{-iKc\tau}}{2i} \cdot \frac{e^{iKR} - e^{-iKR}}{2i} dK =$$

$$= \frac{\theta(\tau)}{(2\pi)^2 2R} \int_0^{\infty} \left(e^{iK(c\tau+R)} - e^{iK(c\tau-R)} - e^{-iK(c\tau-R)} + e^{-iK(c\tau+R)} \right) dK =$$

$$= \frac{\theta(\tau)}{(2\pi)^2 2R} \int_{-\infty}^{\infty} \left(e^{iK(c\tau+R)} - e^{iK(c\tau-R)} \right) dK = \frac{\theta(\tau)}{2\pi \cdot 2R} \left(\delta(c\tau+R) - \delta(c\tau-R) \right) =$$

$$= -\frac{1}{4\pi R} \delta(c\tau - R)$$ (since for $\tau > 0$ $\delta(c\tau+R) = 0$)

Then the solution generated by this Green's function:

$$\psi_G(x) = -\int_{\mathbb{R}^4} \frac{f(\vec{x}', ct')}{4\pi |\vec{x} - \vec{x}'|} \delta(c(t-t') - |\vec{x} - \vec{x}'|) \, d^4x' = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f^{(ret)}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} d^3x'$$, where $f^{(ret)}$ evaluated at the retarded time $t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$

$$\boxed{\psi_G(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f^{(ret)}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} d^3x'}$$, i.e. $$\boxed{G(x-x') = -\frac{1}{4\pi |\vec{x} - \vec{x}'|} \delta(c(t-t') - |\vec{x} - \vec{x}'|)}$$ (non-vanishing only on the future light cone $x = x'$)

Mathematical Physics (Exercises)

[Szek.]: P. Szekeres, "A Course in Modern Mathematical Physics"

1. Sets and mappings.

1.1. Sets.

Ex. 1.1.

1) $\{a\} = \{x \mid x = a\}$ singleton
2) $\{a_1, \dots, a_n\} = \{x \mid x = a_1 \vee \dots \vee x = a_n\}$ finite set
3) $\mathcal{A} = \{A_i \mid i \in I\}$, I is an indexing set family/collection of indexed sets

Ex. 1.2.

2^A: the set of all subsets of A power set of A (alternative notation: $\mathbb{P}(A) = P(A) = 2^A$)

e.g. if $A = \{a_1, \dots, a_n\}$, then $|2^A| = \underset{\emptyset}{1} + \underset{\{a_1\}, \dots, \{a_n\}}{\binom{n}{1}} + \underset{\{a_1, a_2\}, \dots, \{a_{n-1}, a_n\}}{\binom{n}{2}} + \dots + \underset{\{a_1, \dots, a_n\}}{\binom{n}{n}} = (1+1)^n = 2^n$ hence the symbol

1.2. Mappings.

Ex. 1.3.

1) $i_U = id_X|_U$ inclusion map for $U \subset X$

Then $\forall \varphi: X \to Y$ $\varphi|_U = \varphi \circ i_U$

2) $\chi_U: X \to \{0,1\}$ characteristic function of $U \subset X$

$\chi_U(x) = \begin{cases} 0, & x \notin U \\ 1, & x \in U \end{cases} \Rightarrow 2^X = \{\varphi \mid \varphi: X \to \{0,1\}\}$

3) $\varphi: X \to X/\sim$ canonical map under $\sim$ (surjective)

$\varphi(x) = [x]_\sim$

4) $B^A \overset{def}{=} \{\varphi \mid \varphi: A \to B\}$

Reasoning for notation: if A and B are finite, then $|B^A| = \underbrace{m \cdot \ldots \cdot m}_{n \text{ times}} = m^n = |B|^{|A|}$

5) $2^A = \{\varphi \mid \varphi: A \to \{0,1\}\} = \{0,1\}^A$

$|2^A| = 2^{|A|}$

Ex. 1.4.

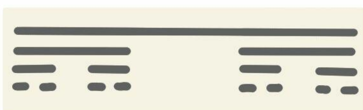
1) $\mathbb{Z}$ is countable, $\mathbb{Q} = \mathbb{Z} \times \mathbb{N}$ is countable
2) $\mathbb{R}$ is uncountable

$\forall x \in [0,1]$ $x = 0.\epsilon_1 \epsilon_2 \dots$, where $\epsilon_i \in \{0,1\}$, then $[0,1] = \{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$, i.e. $|\mathbb{R}| = 2^{|\mathbb{N}|}$.

3) Cantor set is uncountable

$x = 0.\epsilon_1 \epsilon_2 \dots$, where $\epsilon_i \in \{0,1,2\}$ but let $\epsilon_i \neq 1$ $\Rightarrow$ uncountable

This set is nowhere dense in $\mathbb{R}$ (i.e. $\forall (a,b) \subset \mathbb{R}$ it's not dense)



4) $S = \{f \mid f: \mathbb{R} \to \mathbb{R}\}$ is uncountable and $|S| > |\mathbb{R}|$

② Distributions.

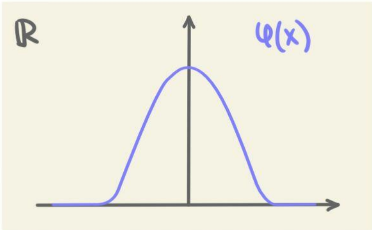
2.1. Test functions and distribution.

Ex. 2.1.

1) Consider $f(x) = e^{-1/x} \cdot \chi_{\{x>0\}}$. $f \in C(\mathbb{R}, \mathbb{R})$ and $f^{(n)}(x)\big|_{x=0} = 0$ $\forall n$ both from the left and from the right. Indeed, $\lim\limits_{x\to 0^+} f'(x) = \lim\limits_{x\to 0^+} \frac{e^{-1/x}}{x^2} = \lim\limits_{x\to 0^+} \frac{1}{x^2}\left(1 - \frac{1}{x} + \ldots\right) = 0 = \lim\limits_{x\to 0^-} f'(x)$.

$\varphi(x) \overset{\text{def}}{:=} f(x+a)\cdot f(-(x-a)) = e^{2a/x^2-a^2} \cdot \chi_{\{|x|<a\}}$

$$\left.\begin{array}{l} \varphi \in C(\mathbb{R}, \mathbb{R}) \\ \operatorname{supp}(\varphi) = [-a, a] \end{array}\right| \Rightarrow \varphi \in \mathcal{D}(\mathbb{R})$$



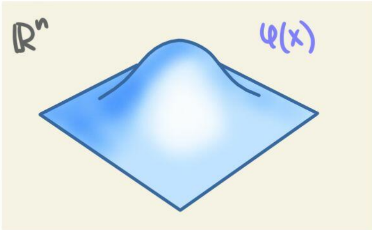
2) Counterpart in $\mathbb{R}^n$:

$\varphi(x) \overset{\text{def}}{:=} e^{2a/|x|^2-a^2} \cdot \chi_{\{|x|<a\}}$

$x \in \mathbb{R}^n$ $\quad |x| = \sqrt{x_1^2 + \ldots + x_n^2}$

$a \in \mathbb{R}$ $\quad a > 0$

$$\left.\begin{array}{l} \varphi \in C(\mathbb{R}^n, \mathbb{R}) \\ \operatorname{supp}(\varphi) = B(0, a) \end{array}\right| \Rightarrow \varphi \in \mathcal{D}(\mathbb{R}^n)$$



Ex. 12.13. (Szek.)

a) $T(\varphi) = \sum_n \lambda_n \varphi^{(n)}(0)$, $\lambda_n \in \mathbb{R}$ is a distribution (linear and cont.)

b) $T(\varphi) = \sum_n \lambda_n \varphi(x_n) = \sum_n \lambda_n \delta_{x_n}(\varphi)$, $\lambda_n \in \mathbb{R}$ is a distribution (linear and cont.)

c) $T(\varphi) = (\varphi(0))^2$ is not a distribution (nonlinear)

d) $T(\varphi) = \sup(\varphi)$ is not a distribution (nonlinear, since $\sup(\varphi+\psi) \neq \sup(\varphi) + \sup(\psi)$)

e) $T(\varphi) = \int_{-\infty}^{\infty} |\varphi(x)|\,dx$ is not a distribution (nonlinear, since $|\varphi(x)+\psi(x)| \neq |\varphi(x)| + |\psi(x)|$)

2.2. Dirac δ-function.

Ex. 12.4. (Szek.)

Heaviside step function $\theta(x) :\overset{\text{def}}{=} \chi_{\{x\geq 0\}}$

It's locally integrable, thus $\theta(x)$ generates a regular distribution $T_\theta \in \mathcal{D}'^{1}(\mathbb{R})$. Then $\forall \varphi \in \mathcal{D}^1(\mathbb{R}^n)$:

$$T_\theta'(\varphi) = -T_\theta(\varphi') = -\int_{-\infty}^{\infty} \varphi'(x)\theta(x)\,dx = -\varphi(x)\Big|_0^\infty = \varphi(0) = \delta(\varphi), \text{ i.e. } \boxed{T_\theta' = \delta} \quad \theta'(x) = \delta(x)$$

$x=0$: inf. steep step

Ex. 12.5. (Szek.)

Derivatives of δ-function

Consider $\varphi \in \mathcal{D}^1(\mathbb{R})$, then:

$$\delta'(\varphi) = -\delta(\varphi') = -\varphi'(0)$$

$$\delta''(\varphi) = -\delta'(\varphi') = \delta(\varphi'') = \varphi''(0)$$

...

$$\boxed{\delta^{(k)}(\varphi) = (-1)^k \varphi^{(k)}(0)} \qquad \boxed{\varphi(x)\,\delta^{(k)}(x) = (-1)^k \varphi^{(k)}(x)\,\delta(x)}$$

Ex. 12.5. (Szek.)

Leibniz's rule

Consider $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\alpha \in C(\mathbb{R}^n, \mathbb{R})$.

$$(\alpha T)'(\varphi) = -(\alpha T)(\varphi') = T(-\alpha\varphi') = T\left((-\alpha\varphi)' + \alpha'\varphi\right) = T'(\alpha\varphi) + \alpha' T(\varphi) = \alpha T'(\varphi) + \alpha' T(\varphi)$$

$$\boxed{(\alpha T)' = \alpha' T + \alpha T'}$$

Ex. 12.6. (Szek.)

$\alpha(x) = x \quad T = \delta$

$$\left.\begin{aligned} (x\delta)' &= 0' = 0 \\ (x\delta)' &= \delta + x\delta' \end{aligned}\right| \Rightarrow \boxed{x\delta' = -\delta}$$

Indeed:

$$x\delta'(\varphi) = \delta'(x\varphi) = -\delta(\varphi + \varphi' x) = -\delta(\varphi) - \delta(\varphi' x) = -\delta(\varphi)$$

2.3. Change of variable in $\delta(f(x))$.

Ex. 12.6. (Szek.)

Evaluate integrals.

a) $\int\limits_{-\infty}^{\infty} e^{at}\sin(bt)\,\delta^{(n)}(t)\,dt$ for $n=0,1,2$

$n=0$: $\int\limits_{-\infty}^{\infty} e^{at}\sin(bt)\,\delta(t)\,dt = e^{at}\sin(bt)\Big|_{t=0} = 0$

$n=1$: $\int\limits_{-\infty}^{\infty} e^{at}\sin(bt)\,\delta'(t)\,dt = -\left(e^{at}\sin(bt)\right)'\Big|_{t=0} = -\left(ae^{at}\sin(bt)+be^{at}\cos(bt)\right)\Big|_{t=0} = -b$

$n=2$: $\int\limits_{-\infty}^{\infty} e^{at}\sin(bt)\,\delta''(t)\,dt = \left(e^{at}\sin(bt)\right)''\Big|_{t=0} = \ldots = 2ab$

b) $\int\limits_{-\infty}^{\infty} (\cos t+\sin t)\,\delta^{(n)}(t^3+t^2+t)\,dt$ for $n=0,1$

$n=0$: $\int\limits_{-\infty}^{\infty} (\cos t+\sin t)\,\delta(t^3+t^2+t)\,dt = \dfrac{\cos t+\sin t}{|3t^2+2t+1|}\Big|_{t=0} = 1$

$n=1$: $\int\limits_{-\infty}^{\infty} (\cos t+\sin t)\,\delta'(t^3+t^2+t)\,dt = \int\limits_{-\varepsilon}^{\varepsilon} \dfrac{\cos t+\sin t}{|3t^2+2t+1|}\,\delta'(y)\,dy = -\int\limits_{-\varepsilon}^{\varepsilon} \dfrac{d}{dy}\left(\dfrac{\cos t+\sin t}{|3t^2+2t+1|}\right)\delta(y)\,dy =$

$$= -\int\limits_{-\varepsilon}^{\varepsilon} \frac{1}{dy/dt}\,\frac{d}{dt}\left(\frac{\cos t+\sin t}{|3t^2+2t+1|}\right)\delta(y)\,dy =$$

$$= -\int\limits_{-\varepsilon}^{\varepsilon} \frac{1}{3t^2+2t+1}\,\frac{d}{dt}\left(\frac{\cos t+\sin t}{|3t^2+2t+1|}\right)\delta(y)\,dy =$$

$$= -\frac{1}{3t^2+2t+1}\,\frac{d}{dt}\left(\frac{\cos t+\sin t}{|3t^2+2t+1|}\right)\Big|_{t=0} =$$

$$= -\frac{(\cos t-\sin t)(3t^2+2t+1)-(6t+2)(\cos t+\sin t)}{(3t^2+2t+1)^2}\Big|_{t=0} = 1$$

Ex. 12.7. (Szek.)

Proove identities:

0) $\delta(x^2-a^2) = \frac{1}{2a}\left(\delta(x-a)+\delta(x+a)\right)$, $a\neq 0$

Proof:

1. Using general formula:

$$\delta(x^2-a^2) = \frac{\delta_{-a}}{|2x|}\Big|_{x=-a} + \frac{\delta_a}{|2x|}\Big|_{x=a} = \frac{1}{2a}(\delta_{-a}+\delta_a)$$

2. Using approximations:

$x=a$: $f(x)\approx 2a(x-a)$

$x=-a$: $f(x)\approx -2a(x+a)$

$$\delta(x^2-a^2) = \delta(2a(x-a)) + \delta(-2a(x+a)) = \frac{1}{2a}(\delta_a+\delta_{-a})$$

a) $\delta((x-a)(x-b)) = \frac{1}{b-a}\left(\delta(x-a)+\delta(x-b)\right)$, $a<b$

Proof:

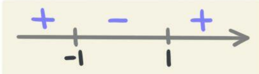
$$\delta((x-a)(x-b)) = \delta(x^2-(a+b)x+ab) = \frac{\delta_a}{|2x-(a+b)|}\Big|_{x=a} + \frac{\delta_b}{|2x-(a+b)|}\Big|_{x=b} = \frac{\delta_a}{|a-b|} + \frac{\delta_b}{|b-a|} \overset{a<b}{=}$$

$$= \frac{1}{b-a}(\delta_a+\delta_b)$$

b) $\frac{d}{dx}\theta(x^2-1) = 2x\,\delta(x^2-1) = \delta(x-1) - \delta(x+1)$

Proof:

$\theta(x^2-1) = 0$: $\quad x^2-1<0 \quad (x-1)(x+1)<0$



$\theta(x^2-1) = \chi_{\{x<-1\}} + \chi_{\{x>1\}}$

Consider $\varphi \in \mathcal{D}(\mathbb{R})$. Then:

$$\int_{-\infty}^{\infty}\varphi(x)\frac{d}{dx}\theta(x^2-1)\,dx = -\int_{-\infty}^{\infty}\varphi'(x)\theta(x^2-1)\,dx = -\left(\int_{-\infty}^{-1}\varphi'(x)\,dx + \int_{1}^{\infty}\varphi'(x)\,dx\right) = \varphi(1) - \varphi(-1)$$

$$\int_{-\infty}^{\infty}\varphi(x)\,2x\,\delta(x^2-1)\,dx = \int_{-\infty}^{\infty}\varphi(x)\,2x\cdot\frac{1}{2}(\delta(x-1)+\delta(x+1))\,dx = x\varphi(x)\Big|_{x=1} + x\varphi(x)\Big|_{x=-1} = \varphi(1) - \varphi(-1)$$

c) $\frac{d}{dx}\delta(x^2-1) = \frac{1}{2}(\delta'(x-1) + \delta'(x+1))$

Proof:

$$\int_{-\infty}^{\infty}\varphi(x)\frac{d}{dx}\delta'(x^2-1)\,dx = -\int_{-\infty}^{\infty}\varphi'(x)\,\delta(x^2-1)\,dx = -\frac{1}{2}\int_{-\infty}^{\infty}\varphi'(x)\left(\delta(x-1)+\delta(x+1)\right)dx = -\frac{1}{2}\left(\varphi'(1)+\varphi'(-1)\right)$$

d) $\delta'(x^2-1) = \frac{1}{4}\left(\delta'(x-1) - \delta'(x+1) + \delta(x-1) + \delta(x+1)\right)$

Proof:

$$\int_{-\infty}^{\infty}\varphi(x)\,\delta'(x^2-1)\,dx \overset{y=x^2-1}{=} \int_{-\infty}^{\infty}\varphi(x)\frac{d}{dy}\delta'(y)\frac{dy}{|2x|} = -\int_{-\infty}^{\infty}\frac{d}{dy}\left(\frac{\varphi(x)}{|2x|}\right)\delta(y)\,dy = -\left(\frac{d}{dy}\left(\frac{\varphi(x)}{2x}\right)\Big|_{x=-1} + \frac{d}{dy}\left(\frac{\varphi(x)}{2x}\right)\Big|_{x=1}\right) =$$

$$= -\frac{1}{2x}\left(\frac{\varphi'(x)\,|2x| - 2\varphi(x)\operatorname{sgn}(x)}{4x^2}\Big|_{x=-1} + \frac{\varphi'(x)\,|2x| - 2\varphi(x)\operatorname{sgn}(x)}{4x^2}\Big|_{x=1}\right) =$$

$$= -\frac{1}{4}\left(-(\varphi'(-1)+\varphi(-1)) + (\varphi'(1)-\varphi(1))\right) = \frac{1}{4}\left(\varphi'(-1) - \varphi'(1) + \varphi(-1) + \varphi(1)\right)$$

Ex. 12.9. (Szek.)

1) Show that $\frac{d}{dx}\left(\delta(f(x))\right) = f'(x)\,\delta'(f(x))$

Solution:

Consider $f \in C^{\circ}(\mathbb{R},\mathbb{R})$ s.t. it's increasing monotonely and $f(a)=0$. Then $\forall \varphi \in \mathcal{D}(\mathbb{R})$ we have:

$$\int_{-\infty}^{\infty}\varphi(x)\frac{d}{dx}\delta(f(x))\,dx = -\int_{-\infty}^{\infty}\frac{d\varphi(x)}{dx}\delta(f(x))\,dx = -\frac{\varphi'(x)}{|f'(x)|}\Big|_{x=a}$$

$$\int_{-\infty}^{\infty}\varphi(x)\,f'(x)\,\delta'(f(x))\,dx = -\left(\frac{1}{f'(x)}\frac{d}{dx}\left(\frac{\varphi(x)f'(x)}{|f'(x)|}\right)\Big|_{x=a}\right) = -\frac{\varphi'(x)}{|f'(x)|}\Big|_{x=a}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\delta(f(x))\right) = f'(x)\,\delta'(f(x))$$

2) Show that $\delta(f(x)) + f(x)\,\delta'(f(x)) = 0$

Solution:

$$\frac{d}{dx}\left(f(x)\,\delta(f(x))\right) = \frac{df(x)}{dx}\delta(f(x)) + f(x)\frac{d}{dx}\delta(f(x)) = f'(x)\,\delta(f(x)) + f(x)\,f'(x)\,\delta'(f(x))$$

$$f(x)\,\delta(f(x)) = 0 \;\Rightarrow\; \frac{d}{dx}\left(f(x)\,\delta(f(x))\right) = 0$$

$$\Rightarrow \quad \delta(f(x)) + f(x)\,\delta'(f(x)) = 0$$

3) Show that $\phi(x,y) = \delta(x^2-y^2)$ is a solution of $x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + 2\phi = 0$.

Solution:

$$x\frac{\partial\phi}{\partial x} = x\frac{\partial}{\partial x}\delta(x^2-y^2) = x\cdot 2x\cdot\delta'(x^2-y^2)$$

$$y\frac{\partial\phi}{\partial y} = y\frac{\partial}{\partial y}\delta(x^2-y^2) = y\cdot(-2y)\cdot\delta'(x^2-y^2)$$

$$\Rightarrow$$

$$x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + 2\phi = 2x^2\delta'(x^2-y^2) - 2y^2\delta'(x^2-y^2) + 2\delta(x^2-y^2) = 2(x^2-y^2)\,\delta'(x^2-y^2) + 2\delta(x^2-y^2) \overset{\text{from (2)}}{=}$$

$$= -2\delta(x^2-y^2) + 2\delta(x^2-y^2) = 0$$

③ Fourier transforms.

3.1. Fourier transforms of functions.

Ex. 12.10. (Szek.)

Find the Fourier transforms of the functions:

1) $f(x) = \chi_{\{|x|\le a\}}$

$$\mathcal{F}f(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ixy}\,dx = \frac{1}{\sqrt{2\pi}}\left.\frac{e^{-ixy}}{(-iy)}\right|_{-a}^{a} = \frac{e^{iay}-e^{-iay}}{\sqrt{2\pi}\,iy} = \sqrt{\frac{2}{\pi}}\,\frac{\sin(ay)}{y}$$

2) $g(x) = \left(1-\frac{|x|}{2}\right)\cdot\chi_{\{|x|\le a\}}$

$$\mathcal{F}g(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy} g(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ixy}\left(1-\frac{|x|}{2}\right)dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{0} e^{-ixy}\left(1+\frac{x}{2}\right)dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{a} e^{ixy}\left(1-\frac{x}{2}\right)dx =$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin(ay)}{y} + \frac{1}{\sqrt{2\pi}}\int_{-a}^{0} e^{-ixy}\frac{x}{2}\,dx - \frac{1}{\sqrt{2\pi}}\int_{0}^{a} e^{-ixy}\frac{x}{2}\,dx = \ldots = \sqrt{\frac{2}{\pi}}\,\frac{\sin(ay)}{y} + \frac{1}{2\sqrt{2\pi}}\left.\left(\frac{x\cdot e^{-ixy}}{(-iy)} + \frac{x\cdot e^{-ixy}}{y^2}\right)\right|_{-a}^{0} -$$

$$- \frac{1}{2\sqrt{2\pi}}\left.\left(\frac{x\cdot e^{-ixy}}{(-iy)} + \frac{x\cdot e^{-ixy}}{y^2}\right)\right|_{0}^{a} = \ldots = \sqrt{\frac{2}{\pi}}\,\frac{\sin(ay)}{y} + \frac{ia\cos(ay)}{\sqrt{2\pi}\,y} + \frac{1-\cos(ay)}{\sqrt{2\pi}\,y^2}$$

Ex. 12.11. (Szek.)

Show that $\mathcal{F}\left(e^{-a^2x^2/2}\right) = \frac{1}{|a|}e^{-k^2/2a^2}$.

Solution:

$$\mathcal{F}f(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx} e^{-\frac{a^2x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx-\frac{a^2}{2}x^2}\,dx = \frac{1}{|a|}e^{-\frac{k^2}{2a^2}}$$

$\left[\int_{\mathbb{R}} e^{iax-bx^2}\,dx = \sqrt{\frac{\pi}{b}}\,e^{-a^2/4b},\quad a,b\in\mathbb{R}\quad b>0\right.$ Gaussian integral (see Ex. 11.3.8. (Has.)) $\left.\right]$

3.2. Fourier transforms of distributions.

Ex. 3.1.

Using the identity $\mathcal{F}^{-1}\mathcal{F}\varphi = \varphi$ find the expression for δ-function. Also find $\mathcal{F}\delta$ and $\mathcal{F}\delta_a$.

Solution:

$$\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixy}\left(\int_{\mathbb{R}} e^{-ity}\varphi(t)\,dt\right)dy = \frac{1}{2\pi}\int_{\mathbb{R}} dt\,\varphi(t)\int_{\mathbb{R}} e^{iy(x-t)}\,dy$$

1) $\delta_a(x) = \frac{1}{2\pi}\int_{\mathbb{R}} dt\,\delta(t-a)\int_{\mathbb{R}} e^{iy(x-t)}\,dy = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iy(x-a)}\,dy$

$\delta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixy}\,dy = \delta(-x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixy}\,dy$

$$\delta_a(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-a)y}\,dy$$

$$\delta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixy}\,dy$$

2) $\mathcal{F}\delta_a(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\delta(x-a)\,dx = \frac{1}{\sqrt{2\pi}}e^{-iay}$

$\mathcal{F}\delta(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\delta(x)\,dx = \frac{1}{\sqrt{2\pi}}$

$$\mathcal{F}\delta_a(x) = \frac{1}{\sqrt{2\pi}}e^{-iax}$$

$$\mathcal{F}\delta(x) = \frac{1}{\sqrt{2\pi}}$$

Ex. 12.8. (Szek.)

Find the Fourier transform of δ_a using the definition $\mathcal{F}T(\varphi) = T(\mathcal{F}\varphi)$. Check that $\mathcal{F}^{-1}\mathcal{F}\delta_a = \delta_a$.

Solution:

$$\mathcal{F}\delta_a(\varphi) = \delta_a(\mathcal{F}\varphi) = \mathcal{F}\varphi(a) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixa}\varphi(x)\,dx = T_{\frac{1}{\sqrt{2\pi}}e^{-ixa}}(\varphi), \text{ i.e. } f(x) = \frac{1}{\sqrt{2\pi}}e^{-ixa}$$

$$\mathcal{F}^{-1}\mathcal{F}\delta_a(\varphi) = \mathcal{F}\delta_a(\mathcal{F}^{-1}\varphi) = \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}e^{-ixa}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ixy}\varphi(y)\,dy\right)dx = \frac{1}{2\pi}\int_{\mathbb{R}} dy\,\varphi(y)\int_{\mathbb{R}} e^{ix(y-a)}\,dx =$$

$$= \int_{\mathbb{R}}\varphi(y)\delta(y-a)\,dx = \varphi(a) = \delta_a(\varphi)$$

Ex. 12.12. (Szek.)

Evaluate Fourier transforms of the following distributions:

a) $f(x) = \delta(x-a)$

$$\mathcal{F}f(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\delta(x-a)\,dx = \boxed{\frac{1}{\sqrt{2\pi}}e^{-iay}}$$

b) $f(x) = \delta'(x-a)$

$$\mathcal{F}f(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\delta'(x-a)\,dx = -\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\left(e^{-ixy}\right)\Big|_{x=a} = \boxed{\frac{iy}{\sqrt{2\pi}}e^{-iay}}$$

c) $f(x) = \delta^{(n)}(x-a)$

$$\mathcal{F}f(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\delta^{(n)}(x-a)\,dx = (-1)^n\frac{1}{\sqrt{2\pi}}\frac{d^n}{dx^n}\left(e^{-ixy}\right)\Big|_{x=a} = \boxed{\frac{(iy)^n}{\sqrt{2\pi}}e^{-iay}}$$

d) $f(x) = \delta(x^2-a^2)$

$$\mathcal{F}f(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\delta(x^2-a^2)\,dx \overset{\text{Ex. 12.7 (a)}}{=} \frac{1}{\sqrt{2\pi}}\frac{1}{2a}\left(e^{iay} - e^{-iay}\right) = \boxed{\frac{1}{\sqrt{2\pi}\,a}\cos(ay)}$$

e) $f(x) = \delta'(x^2-a^2)$

$$\mathcal{F}f(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\delta'(x^2-a^2)\,dx = -\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2x}\frac{d}{dx}\left(\frac{e^{-ixy}}{|2x|}\right)\Big|_{x=-a} + \frac{1}{2x}\frac{d}{dx}\left(\frac{e^{-ixy}}{|2x|}\right)\Big|_{x=a}\right) =$$

$$= \frac{1}{2a\sqrt{2\pi}}\left(\frac{-iy\cdot e^{-ixy}\cdot(-2x) + 2e^{-ixy}}{4x^2}\Big|_{x=-a} - \frac{-iy\cdot e^{-ixy}\cdot 2x - 2e^{-ixy}}{4x^2}\Big|_{x=a}\right) =$$

$$= \ldots = \boxed{\frac{1}{2a^2\sqrt{2\pi}}\left(y\sin(ay) + \frac{1}{a}\cos(ay)\right)}$$

Ex. 12.13. (Szek.)

1) Proove that $x^m\delta^{(n)}(x) = (-1)^m\frac{n!}{(n-m)!}\delta^{(n-m)}(x)$, $n \geqslant m$.

Solution:

$$\int_{\mathbb{R}}\varphi(x)\,x^m\delta^{(n)}(x)\,dx = (-1)^n\frac{d^n}{dx^n}\left(x^m\varphi(x)\right)\Big|_{x=0} = (-1)^n\binom{n}{n-m}m!\,\varphi^{(n-m)}(0) =$$

$$\frac{d^n}{dx^n}\left(x^m\varphi(x)\right) = \binom{n}{n-m}m!\,\varphi^{(n-m)}(x) + \binom{n}{n-m+1}m!\,x\,\varphi^{(n-m+1)}(x) + \ldots + x^m\varphi^{(n)}(x)$$

$$= (-1)^n\frac{n!}{(n-m)!\,m!}m!\,\varphi^{(n-m)}(0) = (-1)^n\frac{n!}{(n-m)!}(-1)^{n-m}\varphi^{(n-m)}(0)\,(-1)^{m-n} =$$

$$= (-1)^m\frac{n!}{(n-m)!}\delta^{(n-m)}(\varphi)$$

2) Show that $\mathcal{F}\left(\sqrt{2\pi}\frac{k!}{(m+k)!}x^m\delta^{(m+k)}(-x)\right) = (-iy)^k$, $m,k \geqslant 0$.

Solution:

$$\mathcal{F}f(y) = \int_{-\infty}^{\infty}e^{-ixy}\frac{k!}{(m+k)!}x^m\delta^{(m+k)}(-x)\,dx \overset{-x\to x}{=} \int_{\infty}^{-\infty}e^{ixy}\frac{k!}{(m+k)!}(-1)^m\cdot x^m\delta^{(m+k)}(x)\cdot(-1)\,dx =$$

$$= \int_{-\infty}^{\infty}e^{ixy}\frac{k!}{(m+k)!}(-1)^{2m}\frac{(m+k)!}{k!}\delta^{(k)}(x)\,dx = \int_{-\infty}^{\infty}e^{ixy}\delta^{(k)}(x)\,dx = (-1)^k\frac{d^k}{dx^k}\left(e^{ixy}\right)\Big|_{x=0} = (-iy)^k$$

Ex. 12.14. (Szek.)

a) Show that $\mathcal{F}(\delta + \delta_a + \delta_{2a} + \ldots + \delta_{(2n-1)a})$ is a distribution with density $f(y) = \frac{1}{\sqrt{2\pi}}\frac{\sin(nay)}{\cos(\frac{1}{2}ay)}e^{-(n-\frac{1}{2})iay}$.

Solution:

$$\mathcal{F}(\delta + \delta_a + \delta_{2a} + \ldots + \delta_{(2n-1)a}) \overset{\text{Ex. 12.12(a)}}{=} \frac{1}{\sqrt{2\pi}}\left(1 + e^{-iay} + e^{-2iay} + \ldots + e^{-(2n-1)iay}\right) =$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1-e^{-2niay}}{1-e^{-iay}} = \frac{1}{\sqrt{2\pi}}\frac{e^{\frac{1}{2}iay}}{e^{niay}}\cdot\frac{e^{niay}-e^{-niay}}{2i}\cdot\frac{2i}{e^{\frac{1}{2}iay}-e^{-\frac{1}{2}iay}} =$$

$$= \frac{1}{\sqrt{2\pi}}e^{(\frac{1}{2}-n)iay}\frac{\sin(nay)}{\sin(\frac{1}{2}ay)}$$

b) Show that $\mathcal{F}^{-1}(f(y)e^{iby}) = (\mathcal{F}^{-1}f)(x+b)$. Hence find $\mathcal{F}^{-1}g(x)$, where $g(y) = \frac{\sin(nay)}{\sin(\frac{1}{2}ay)}$.

Solution:

$$\mathcal{F}^{-1}(f(y)e^{iby}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixy}f(y)e^{iby}\,dy = (\mathcal{F}^{-1}f)(x+b)$$

$$\mathcal{F}^{-1}(g(y)) = \left[b = (n-\tfrac{1}{2})a\right] = \mathcal{F}^{-1}\left(\sqrt{2\pi}\,e^{iby}\frac{1}{\sqrt{2\pi}}e^{-iby}\frac{\sin(nay)}{\sin(\frac{1}{2}ay)}\right) =$$

$$= \sqrt{2\pi}\left(\delta_{-b} + \delta_{a-b} + \delta_{2a-b} + \ldots + \delta_{(2n-1)a-b}\right) =$$

$$= \sqrt{2\pi}\left(\delta(x+(n-\tfrac{1}{2})a) + \delta(x+(n-\tfrac{3}{2})a) + \ldots + \delta(x-(n-\tfrac{1}{2})a)\right)$$

4. Green's function.

Ex. 12.15. (Szek.)

Solve the time-independent Klein-Gordon equation $(\nabla^2 - m^2)\varphi(\vec{x}) = \rho(\vec{x})$. Find a solution corresponding to a point source $\rho(\vec{x}) = q\,\delta(\vec{x})$.

Solution:

$$\varphi(\vec{x}) = \int_{\mathbb{R}^3} \rho(\vec{x}')\,G(\vec{x}-\vec{x}')\,d^3x' \quad , \text{ where } \quad (\nabla^2 - m^2)\,G(\vec{x}-\vec{x}') = \delta(\vec{x}-\vec{x}')$$

1) Finding $G(\vec{x}-\vec{x}')$.

$$G(\vec{x}-\vec{x}') = \mathcal{F}^{-1}(g(\vec{k})) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,g(\vec{k})\,d^3k \qquad k^2 = |\vec{k}|^2$$

$$(\nabla^2 - m^2)\,G(\vec{x}-\vec{x}') = -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (k^2+m^2)\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,g(\vec{k})\,d^3k$$

$$\delta(\vec{x}-\vec{x}') = \mathcal{F}^{-1}\left(\frac{1}{(2\pi)^{3/2}}\,e^{-i\vec{k}\cdot\vec{x}'}\right) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,d^3k \quad \Rightarrow$$

$$\Rightarrow \quad g(\vec{k}) = -\frac{1}{(2\pi)^{3/2}}\,\frac{1}{k^2+m^2} \quad \Rightarrow \quad G(\vec{x}-\vec{x}') = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{k^2+m^2}\,d^3k$$

2) Evaluating $G(\vec{x}-\vec{x}')$.

To evaluate $G(\vec{x}-\vec{x}')$ we use polar coordinates (k,θ,φ) with k_3-axis pointing along $\vec{R} = \vec{x}-\vec{x}'$, so that $\vec{k}\cdot\vec{R} = kR\cos\theta$.

$$G(\vec{R}) = -\frac{1}{(2\pi)^3} \int_0^{\infty} dk \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{e^{ikR\cos\theta}}{k^2+m^2}\,k^2 \sin\theta\,d\theta = +\frac{1}{(2\pi)^2} \int_0^{\infty} dk \int_1^{-1} \frac{k^2}{k^2+m^2}\,e^{ikR\cos\theta}\,d(\cos\theta) =$$

$$= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{k^2}{k^2+m^2} \cdot \frac{e^{ikR\alpha}}{ikR}\bigg|_{\alpha=1}^{\alpha=-1} dk = \frac{1}{(2\pi)^2 Ri} \int_0^{\infty} \frac{k}{k^2+m^2}\left(e^{-ikR} - e^{ikR}\right) dk = G_1$$

On the other hand, if we make the change $k \to -k$:

$$G(\vec{R}) = \frac{1}{(2\pi)^2 Ri} \int_0^{-\infty} \frac{-k}{k^2+m^2}\left(e^{ikR} - e^{-ikR}\right) d(-k) = \frac{1}{(2\pi)^2 Ri} \int_{-\infty}^{0} \frac{k}{k^2+m^2}\left(e^{-ikR} - e^{ikR}\right) dk = G_2$$

Then, taking the average of G_1 and G_2:

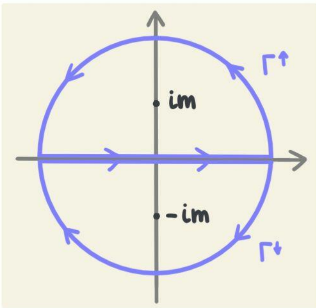
$$G(\vec{R}) = \frac{1}{(2\pi)^2\,2Ri} \int_{-\infty}^{\infty} \frac{k}{k^2+m^2}\left(e^{-ikR} - e^{ikR}\right) dk = I_1 + I_2$$

I_1: complete the contour using $\Gamma^{\downarrow}$ (LHP)
I_2: complete the contour using $\Gamma^{\uparrow}$ (UHP)

The residue theorem gives us:

$$G(\vec{R}) = \frac{1}{(2\pi)^2\,2Ri} \cdot 2\pi i \left(\frac{k\,e^{-ikR}}{k-im}\bigg|_{k=-im} - \frac{k\,e^{ikR}}{k+im}\bigg|_{k=im}\right) = \frac{1}{4\pi R}\left(-\frac{1}{2}e^{-mR} - \frac{1}{2}e^{-mR}\right) = -\frac{1}{4\pi R}\,e^{-mR}$$

$$\varphi(\vec{x}) = -\int_{\mathbb{R}^3} \frac{\rho(\vec{x}')\,e^{-m|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}\,d^3x' \quad , \text{ i.e. } \quad G(\vec{x}-\vec{x}') = -\frac{1}{4\pi|\vec{x}-\vec{x}'|}\,e^{-m|\vec{x}-\vec{x}'|}$$



Consider point source $\rho(\vec{x}) = q\,\delta(\vec{x})$:

$$\varphi(\vec{x}) = -\int_{\mathbb{R}^3} \frac{q\,\delta(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}\,e^{-m|\vec{x}-\vec{x}'|}\,d^3x' = -\frac{q}{4\pi|\vec{x}|}\,e^{-m|\vec{x}|}$$

Ex. 12.16. (Szek.)

Find the Green's function for the diffusion equation in 1, 2, and 3 dimensions. For the 1D case write out the corresponding solution of the inhomogeneous equation.

Solution:

1. 1D case.

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{æ}\frac{\partial}{\partial t}\right)\psi(x,t) = f(x,t)$$

$$G(x,t) = \mathcal{F}^{-1}\big(g(k,\omega)\big) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kx+\omega t)}\, g(k,\omega)\, dk\, d\omega$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{æ}\frac{\partial}{\partial t}\right)G(x,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(k^2 + i\frac{\omega}{æ}\right)e^{i(kx+\omega t)}\, g(k,\omega)\, dk\, d\omega$$

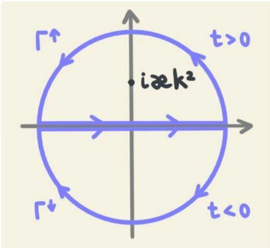
$$\delta(x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kx+\omega t)}\, dk\, d\omega \qquad \Rightarrow$$

$$\Rightarrow\ g(k,\omega) = -\frac{1}{2\pi}\frac{1}{k^2 + i\omega/æ} \quad \Rightarrow\ G(x,t) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk\, e^{ikx}\int_{-\infty}^{\infty}\frac{e^{i\omega t}\, d\omega}{k^2 + i\omega/æ}$$

First we evaluate the integral $\int_{-\infty}^{\infty}\frac{e^{i\omega t}\, d\omega}{k^2 + i\omega/æ}$ (using the residue th.).

$t>0$: complete the contour with $\Gamma^{\uparrow}$ (UHP)

$t<0$: complete the contour with $\Gamma^{\downarrow}$ (LHP)



$$\int_{-\infty}^{\infty}\frac{e^{i\omega t}\, d\omega}{k^2 + i\omega/æ} = \theta(t)\cdot 2\pi i\cdot(-iæ)\, e^{i\omega t}\Big|_{\omega = iæk^2} = \theta(t)\cdot 2\pi æ\, e^{-k^2 æ t}$$

Then we have:

Ex. 12.11. (Szek.)

$$G(x,t) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\theta(t)\cdot 2\pi æ\cdot e^{ikx - k^2 æ t}\, dk = -\frac{\theta(t)}{2\pi}\, æ\cdot\sqrt{\frac{\pi}{æ t}}\, e^{-\frac{x^2}{4æt}} = -\theta(t)\sqrt{\frac{æ}{4\pi t}}\, e^{-\frac{x^2}{4æt}}$$

And a solution is given by:

$$\psi(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',t')\, G(x-x', t-t')\, dx'dt' = -\sqrt{\frac{æ}{4\pi}}\int_{-\infty}^{\infty} dx'\int_{-\infty}^{t}\frac{f(x',t')}{\sqrt{t-t'}}\, e^{-\frac{(x-x')^2}{4æ(t-t')}}\, dt'$$

$$G(x-x', t-t') = -\theta(t-t')\sqrt{\frac{æ}{4\pi(t-t')}}\, e^{-\frac{(x-x')^2}{4æ(t-t')}}$$

2. 2D case.

$$\left(\nabla^2 - \frac{1}{\varkappa}\frac{\partial}{\partial t}\right)\psi(\vec{x},t) = f(\vec{x},t) \quad , \vec{x} \in \mathbb{R}^2$$

$$G(\vec{x},t) = \mathcal{F}^{-1}(g(\vec{k},\omega)) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(\vec{k}\cdot\vec{x}+\omega t)} g(\vec{k},\omega)\, d^2k\, d\omega \qquad k^2 = |\vec{k}|^2$$

$$\left(\nabla^2 - \frac{1}{\varkappa}\frac{\partial}{\partial t}\right)G(\vec{x},t) = -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(k^2 + i\frac{\omega}{\varkappa}\right) e^{i(\vec{k}\cdot\vec{x}+\omega t)}\, d^2k\, d\omega$$

$$\delta(\vec{x},t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i(\vec{k}\cdot\vec{x}+\omega t)}\, d^2k\, d\omega \qquad \Rightarrow$$

$$\Rightarrow \quad g(\vec{k},\omega) = -\frac{1}{(2\pi)^{3/2}} \frac{1}{k^2 + i\omega/\varkappa} \quad \Rightarrow \quad G(\vec{x},t) = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} d^2k\, e^{i\vec{k}\cdot\vec{x}} \int_{\mathbb{R}} \frac{e^{i\omega t}\, d\omega}{k^2 + i\omega/\varkappa}$$

Arguing as before:

$$G(\vec{x},t) = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} \theta(t)\cdot 2\pi\varkappa \cdot e^{i\vec{k}\cdot\vec{x} - k^2\varkappa t}\, d^2k = -\frac{\theta(t)}{(2\pi)^2}\varkappa \cdot \frac{\pi}{\varkappa t}\, e^{-\frac{|\vec{x}|^2}{4\varkappa t}} = -\frac{\theta(t)}{4\pi t}\, e^{-\frac{|\vec{x}|^2}{4\varkappa t}}$$

$$\boxed{G(\vec{x}-\vec{x}', t-t') = -\frac{\theta(t-t')}{4\pi(t-t')}\, e^{-\frac{|\vec{x}-\vec{x}'|^2}{4\varkappa(t-t')}}}$$

3. 3D case.

$$\left(\nabla^2 - \frac{1}{\varkappa}\frac{\partial}{\partial t}\right)\psi(\vec{x},t) = f(\vec{x},t) \quad , \vec{x} \in \mathbb{R}^3$$

$$G(\vec{x},t) = \mathcal{F}^{-1}(g(\vec{k},\omega)) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{i(\vec{k}\cdot\vec{x}+\omega t)} g(\vec{k},\omega)\, d^3k\, d\omega \qquad k^2 = |\vec{k}|^2$$

$$\left(\nabla^2 - \frac{1}{\varkappa}\frac{\partial}{\partial t}\right)G(\vec{x},t) = -\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left(k^2 + i\frac{\omega}{\varkappa}\right) e^{i(\vec{k}\cdot\vec{x}+\omega t)}\, d^3k\, d\omega$$

$$\delta(\vec{x},t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i(\vec{k}\cdot\vec{x}+\omega t)}\, d^3k\, d\omega \qquad \Rightarrow$$

$$\Rightarrow \quad g(\vec{k},\omega) = -\frac{1}{(2\pi)^2} \frac{1}{k^2 + i\omega/\varkappa} \quad \Rightarrow \quad G(\vec{x},t) = -\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} d^3k\, e^{i\vec{k}\cdot\vec{x}} \int_{\mathbb{R}} \frac{e^{i\omega t}\, d\omega}{k^2 + i\omega/\varkappa}$$

Arguing as before:

$$G(\vec{x},t) = -\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \theta(t)\cdot 2\pi\varkappa \cdot e^{i\vec{k}\cdot\vec{x} - k^2\varkappa t}\, d^3k = -\frac{\theta(t)}{(2\pi)^3}\varkappa \cdot \left(\frac{\pi}{\varkappa t}\right)^{3/2} e^{-\frac{|\vec{x}|^2}{4\varkappa t}} = -\frac{\theta(t)}{(4\pi)^{3/2}\sqrt{\varkappa}}\, t^{-3/2} e^{-\frac{|\vec{x}|^2}{4\varkappa t}}$$

$$\boxed{G(\vec{x}-\vec{x}', t-t') = -\frac{\theta(t-t')}{(4\pi)^{3/2}\sqrt{\varkappa}}\, (t-t')^{-3/2}\, e^{-\frac{|\vec{x}-\vec{x}'|^2}{4\varkappa(t-t')}}}$$