

Basic maths

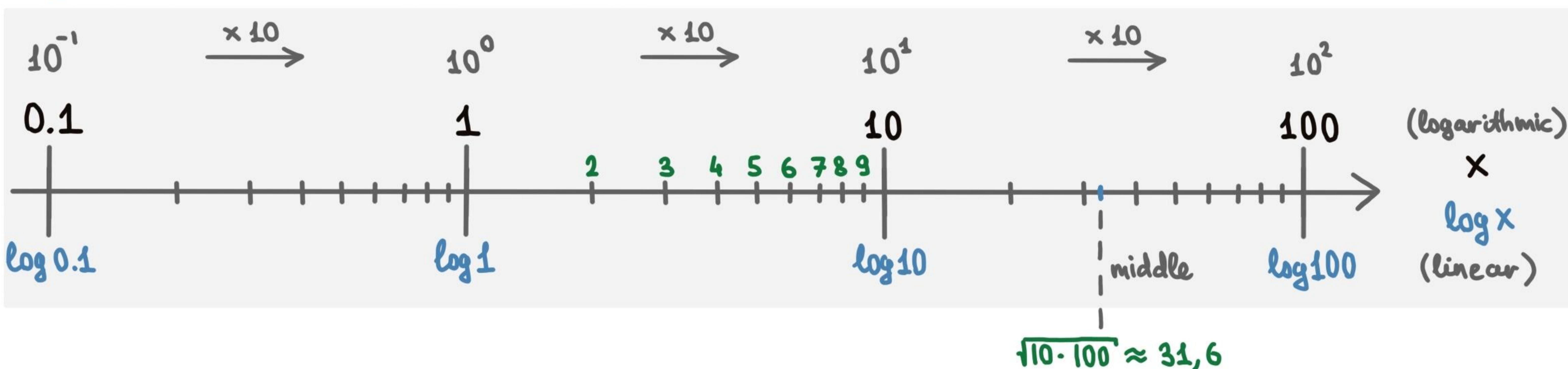
1. Logic symbols.

\Rightarrow	material implication	T	tautology (truth)
\Leftrightarrow	material equivalence	\perp	contradiction (falsity)
\neg	negation (not)	\forall	universal quantification
D	domain of discourse (universal set)	\exists	existential quantification
\wedge	conjunction (and)	$\exists!$	uniqueness quantification
\vee	disjunction (or)	\vdash	syntactically entails (proves)
\oplus	exclusive disjunction (xor)	\models	semantically entails (models)

2. Logarithms.

$$\begin{array}{lll} a^{\log_a x} = x & \log_a(x \cdot y) = \log_a x + \log_a y & \log_x y = (\log_y x)^{-1} \\ \log_a a^x = x & \log_a(x/y) = \log_a x - \log_a y & \log_{x^n}(y^m) = \frac{m}{n} \log_x y \\ & & \log_a b = \frac{\log_c b}{\log_c a} \end{array}$$

Log scale:



$$\begin{aligned} \log 2 &\approx 0.301 \\ \log 3 &\approx 0.477 \\ \log 4 &= 2 \log 2 \approx 0.602 \\ \log 5 &= \log 10 - \log 2 \approx 0.699 \end{aligned}$$

$$\begin{aligned} \log 6 &= \log 2 + \log 3 \approx 0.778 \\ \log 7 &\approx 0.845 \\ \log 8 &= 3 \cdot \log 2 \approx 0.903 \\ \log 9 &= 2 \cdot \log 3 \approx 0.954 \end{aligned}$$

3. Trigonometry.

$$\begin{aligned} \sin x &= \frac{e^{ix} - e^{-ix}}{2i} & \cos x &= \frac{e^{ix} + e^{-ix}}{2} & e^{ix} &= \cos x + i \sin x \\ \sinh x &= \frac{e^x - e^{-x}}{2} & \cosh x &= \frac{e^x + e^{-x}}{2} & e^x &= \sinh x + \cosh x \end{aligned}$$

$$\begin{aligned} \sin x &= -i \sinh(ix) \\ \sinh x &= -i \sin(ix) \\ \cos x &= \cosh(ix) \\ \cosh x &= \cos(ix) \end{aligned}$$

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 & \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\ \cosh^2 x - \sinh^2 x &= 1 & \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\ 1 + \tan^2 x &= \frac{1}{\cos^2 x} & \sin^2 x &= \frac{1 - \cos 2x}{2} & \sin 2x &= 2 \sin x \cos x \\ 1 + \cot^2 x &= \frac{1}{\sin^2 x} & \cos^2 x &= \frac{1 + \cos 2x}{2} & \cos 2x &= \cos^2 x - \sin^2 x \end{aligned}$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

4. Sequences.

4.1. Arithmetic progression.

$$a_n = a + (n-1)d \quad , \text{ where } d \text{ is a common difference}$$

- $a_n = a_m + (n-m)d$

- sum of a progression:

$$S_n = \sum_{k=1}^n (a + (k-1)d) = a + (a+d) + \dots + (a + (n-1)d)$$

$$S_n = \sum_{k=1}^n (a_n - (n-k)d) = (a_n - (n-1)d) + \dots + (a_n - d) + a_n$$

$$\Rightarrow 2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

- $S_{mn} = \sum_{k=m}^n (a + (k-1)d) = \frac{n-m}{2} (a_m + a_n)$

- $P_n = \prod_{k=0}^{n-1} (a + kd) = \prod_{k=1}^{n-1} d \left(\frac{a}{d} + k\right) = d^n \frac{\Gamma(\frac{a}{d} + n)}{\Gamma(\frac{a}{d})}$

- arithmetic mean: $A(a_1, \dots, a_n) = \frac{S_n}{n} = \frac{1}{2}(a_1 + a_n) \quad a_n = \frac{1}{2}(a_{n-1} + a_{n+1})$

4.2. Geometric progression.

$$b_n = b r^{n-1} \quad , \text{ where } r \text{ is a common ratio}$$

- $b_n = b_m r^{m-n}$

- sum of a progression:

$$S_n = \sum_{k=1}^n b r^{k-1} = b + br + br^2 + \dots + br^{n-1}$$

$$(1-r)S_n = (1-r)(1+r+r^2+\dots+r^{n-1})b$$

$$(1-r)S_n = (1+r+r^2+\dots+r^{n-1}-r-r^2-\dots-r^n)b \Rightarrow$$

$$S_n = \frac{b(1-r^n)}{1-r} = \frac{b_1 - b_{n+1}}{1-r}$$

- $S_{mn} = \sum_{k=m}^n b r^{k-1} = \frac{b(r^{m-1} - r^n)}{1-r}$

- $S_n^{(\text{even})} = \sum_{k=0}^n b r^{2k} = \sum_{k=0}^n b (r^2)^k = \frac{b(1-r^{2n+2})}{1-r^2}$

- $S_n^{(\text{odd})} = \sum_{k=0}^n b r^{2k+1} = \sum_{k=0}^n (br)(r^2)^k = \frac{br(1-r^{2n+2})}{1-r^2}$

- geometric series:

$$\sum_{k=0}^{\infty} br^k = \frac{b}{1-r} \quad , \quad |r| < 1$$

$$\sum_{k=0}^{\infty} br^{2k} = \frac{b}{1-r^2} \quad , \quad |r| < 1$$

$$\sum_{k=0}^{\infty} br^{2k+1} = \frac{br}{1-r^2} \quad , \quad |r| < 1$$

- $P_n = \prod_{k=0}^{n-1} br^k = b^n r^{1+2+\dots+(n-1)} = b^n r^{\frac{1}{2}(n-1)n} = (b \sqrt{r^{n-1}})^n = (\sqrt{b_1 \cdot b_n})^n$

- geometric mean: $G(b_1, \dots, b_n) = \sqrt[n]{P_n} = \sqrt{b_1 \cdot b_n} \quad b_n = \sqrt{b_{n-1} \cdot b_{n+1}}$

4.3. Arithmetico-geometric sequence.

$$t_n = a_n \cdot b_n, \text{ where } a_n = a + (n-1)d, b_n = b r^{n-1}$$

- sum of a progression:

$$S_n = \sum_{k=1}^n (a + (k-1)d) b r^{k-1} = ab + (a+d)br + (a+2d)b r^2 + \dots + (a+(n-1)d)b r^{n-1}$$

$$(1-r)S_n = ab + (a+d)br + (a+2d)b r^2 + \dots + (a+(n-1)d)b r^{n-1} -$$

$$- abr - (a+d)b r^2 - (a+2d)b r^3 - \dots - (a+(n-1)d)b r^n =$$

$$= ab + db(r + r^2 + \dots + r^{n-1}) - (a+(n-1)d)b r^n =$$

$$= ab + db(r + r^2 + \dots + r^{n-1} + r^n) - (db r^n + (a+(n-1)d)b r^n) =$$

$$= ab + db r(1 + r + \dots + r^{n-1}) - (a + nd)b r^n$$

$$S_n = \frac{ab - (a+nd)b r^n}{1-r} + \frac{db r(1-r^n)}{(1-r)^2} = \frac{a b_1 - a_{n+1} b_{n+1}}{1-r} + \frac{dr(b_1 - b_{n+1})}{(1-r)^2}$$

- $S = \sum_{k=1}^{\infty} t_n = \frac{ab}{1-r} + \frac{dr b}{(1-r)^2}, |r| < 1$

- examples:

- $t_n = (n-1) 2^{-(n-1)} \quad \left\{ \frac{0}{1}, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \dots \right\}$

$$\begin{array}{ll} a = 0 & d = 1 \\ b = 1 & r = \frac{1}{2} \end{array}$$

- $t_n = (n-1) r^{n-1}, |r| < 1$ Gabriel's staircase

$$\begin{array}{ll} a = 0 & d = 1 \\ b = 1 & |r| < 1 \end{array}$$

$$\sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2}$$

4.4. Algebraic identities.

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n 2k = n(n+1)$$

$$\sum_{k=1}^n (2k-1) = n^2$$

$$(a+b)^n = \sum_{k=1}^n \binom{n}{k} a^{n-k} b^k$$

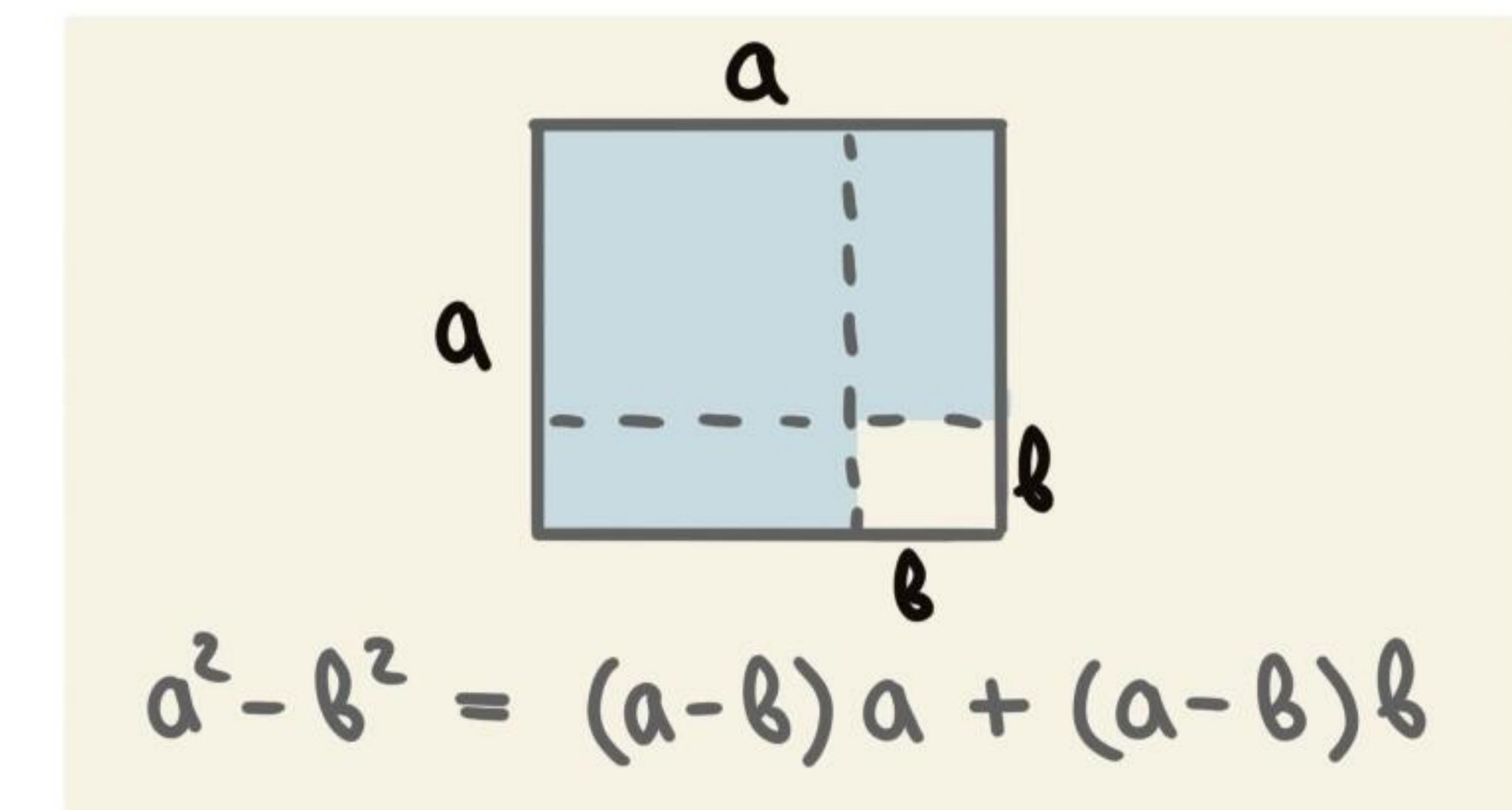
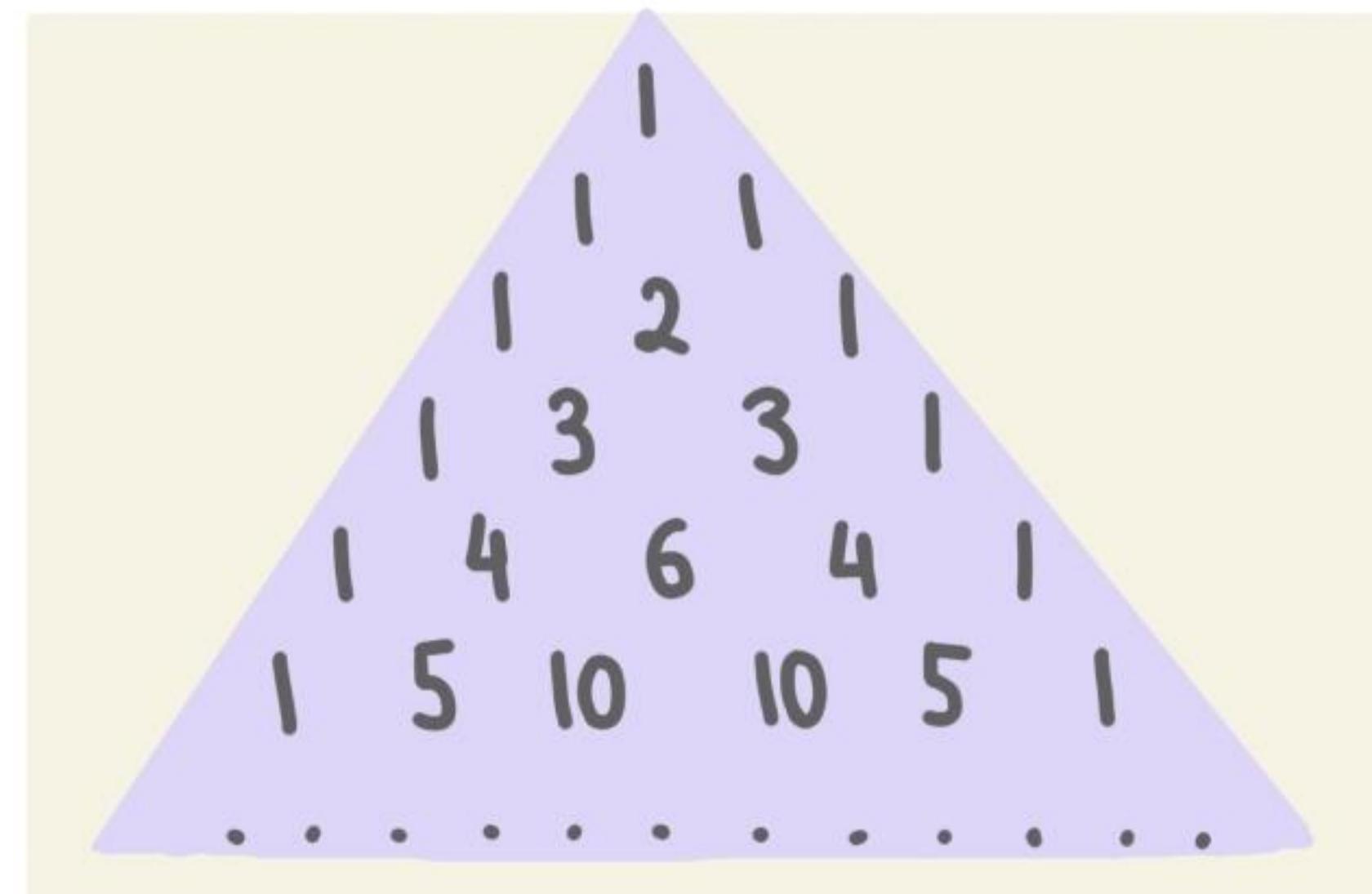
$$2^n = \sum_{k=1}^n \binom{n}{k}$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$$

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \dots - ab^{n-2} + b^{n-1}), n = 2k+1$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$



$$a^2 - b^2 = (a-b)a + (a-b)b$$

5. Differentiation.

$$(x^k)' = kx^{k-1}$$

$$(\sin x)' = \cos x \quad (\tan x)' = \frac{1}{\cos^2 x}$$

$$(\cos x)' = -\sin x \quad (\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\sinh x)' = \cosh x \quad (\tanh x)' = \frac{1}{\cosh^2 x}$$

$$(\cosh x)' = \sinh x \quad (\coth x)' = -\frac{1}{\sinh^2 x}$$

$$(a^x)' = a^x \ln a$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arcsin \frac{x}{a})' = \frac{1}{\sqrt{a^2 - x^2}}$$

$$(\arctan \frac{x}{a})' = \frac{a}{a^2 + x^2}$$

$$(\arccos \frac{x}{a})' = -\frac{1}{\sqrt{a^2 - x^2}}$$

$$(\operatorname{arccot} \frac{x}{a})' = -\frac{a}{a^2 + x^2}$$

$$(\operatorname{arcsinh} \frac{x}{a})' = \frac{1}{\sqrt{x^2 + a^2}}$$

$$(\operatorname{arctanh} \frac{x}{a})' = \frac{a}{a^2 - x^2}$$

$$(\operatorname{arccosh} \frac{x}{a})' = \frac{1}{\sqrt{x^2 - a^2}}$$

$$(\operatorname{arccoth} \frac{x}{a})' = \frac{a}{a^2 - x^2}$$

$$(u^v)' = (e^{v \ln u})'$$

6. Integration.

$$\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$\int \log_a x dx = \frac{x \ln x - x}{\ln a}$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int e^x dx = e^x$$

$$\int \ln x dx = x \ln x - x$$

$$\int \sin x dx = -\cos x$$

$$\int \tan x dx = -\ln|\cos x|$$

$$\int \sinh x dx = \cosh x$$

$$\int \tanh x dx = \ln|\cosh x|$$

$$\int \cos x dx = \sin x$$

$$\int \cot x dx = \ln|\sin x|$$

$$\int \cosh x dx = \sinh x$$

$$\int \coth x dx = \ln|\sinh x|$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}|$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|$$

7. Taylor series.

$$(1+x)^\alpha = \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-(k-1))}{k!} x^k + \dots, \quad \alpha \in \mathbb{R}, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$$

$$e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} \quad a^x = \sum_{k=1}^{\infty} \frac{(x \ln a)^k}{k!}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots$$

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \dots$$

Linear Algebra (from 3Blue1Brown and wiki)

1. Linear span of vectors.

$$\text{span}(X) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^k \lambda_i x_i \mid k \in \mathbb{N}, \lambda_i \in K, x_i \in X \right\}$$

span of vectors

, where $X = \{x_1, \dots, x_n\} \subset V(K)$
 (set of vectors) (vector space)

Properties:

1. $\text{span}(X)$ is the smallest linear subspace of V that contains X . (can take this as a definition of span)
2. Basis is the smallest subspace of V that spans V .

2. Linear transformations.

$$A(\vec{x}) = A\left(\sum_{i=1}^n x^i \vec{e}_i\right) = \sum_{i=1}^n x^i A(\vec{e}_i) = \begin{pmatrix} \sum_{i=1}^n x^i A^1(\vec{e}_i) \\ \dots \\ \sum_{i=1}^n x^i A^m(\vec{e}_i) \end{pmatrix} = \begin{pmatrix} \underbrace{A(\vec{e}_1)}_{A^1(\vec{e}_1)} & \dots & \underbrace{A(\vec{e}_n)}_{A^n(\vec{e}_n)} \\ \dots & \dots & \dots \\ A^m(\vec{e}_1) & \dots & A^m(\vec{e}_n) \end{pmatrix} \begin{pmatrix} x^1 \\ \dots \\ x^n \end{pmatrix}$$

Properties:

1. $A: \mathbb{R}^n \rightarrow \mathbb{R}^m$: 1.1. $A(\vec{0}) = \vec{0}$
 1.2. grid lines remain parallel and evenly spaced
2. Columns of A are a new basis $\{A(\vec{e}_i)\}$ in old coordinates.
3. If two columns or rows of A are linearly dependent (i.e. A is not full-ranked), then A squashes V to a space that is 1 dimension smaller.

Examples:

1) Rotations of $\mathbb{R}^2 (\pi/2)$	$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	
2) Shear of \mathbb{R}^2	$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
3) Squashing of \mathbb{R}^2	$A = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$	

3. Determinant

Consider an oriented volume in V and $A: V \rightarrow W$. Then $\det A$ represents a scale factor when transforming this volume under A .

Properties:

1. $\det A = 0$: A squashes space
2. $\det A \neq 0 \Leftrightarrow \exists A^{-1}$
3. $\text{rank } A \stackrel{\text{def}}{=} \dim(\text{im } A)$, where $\text{im } A$ is a span of column vectors of A (column space)
 (rank helps to tell when a solution to a system of linear eqns. exists)

Determinant formula for \mathbb{R}^2 :

$$1) S_{AB} = S_{A^\perp B}$$

$$\begin{aligned} S_{AB} &= |\vec{A}| \cdot |\vec{B}| \cdot \sin \varphi = |\vec{A}| |\vec{B}| \cos(\frac{\pi}{2} - \varphi) = \\ &= |\vec{A}^\perp| |\vec{B}| \cos \psi = \langle \vec{A}^\perp, \vec{B} \rangle = (-b \ a) \begin{pmatrix} c \\ d \end{pmatrix} = \\ &= ad - bc \end{aligned}$$

$$2) S_{AB} = (a+c)(b+d) - ab - dc - 2bc = ad - bc$$

4. 4 fundamental subspaces.

Consider $A: \mathbb{R}^n \rightarrow \mathbb{R}^m$, then:

$$\text{im } A : \dim(\text{im } A) = r \quad (\text{rank})$$

$$\text{coim } A \stackrel{\text{def}}{=} \text{im } A^T : \dim(\text{coim } A) = r \quad (\text{rank})$$

$$\ker A : \dim(\ker A) = n - r \quad (\text{nullity})$$

$$\text{coker } A \stackrel{\text{def}}{=} \ker A^T : \dim(\text{coker } A) = m - r \quad (\text{corank})$$

$$\dim(\text{im } A) + \dim(\ker A) = \dim V$$

rank-nullity theorem

$$\dim V \left\{ \begin{array}{c} \text{--- } V \\ \dim(\ker A) \{ \begin{array}{c} \xrightarrow{\text{ker } A} 0 \\ \xrightarrow{\text{im } A} \text{im } A \end{array} \} \end{array} \right\} \dim(\text{im } A)$$

$$\ker A = (\text{im } A^T)^\perp \quad \text{coker } A = (\text{im } A)^\perp$$

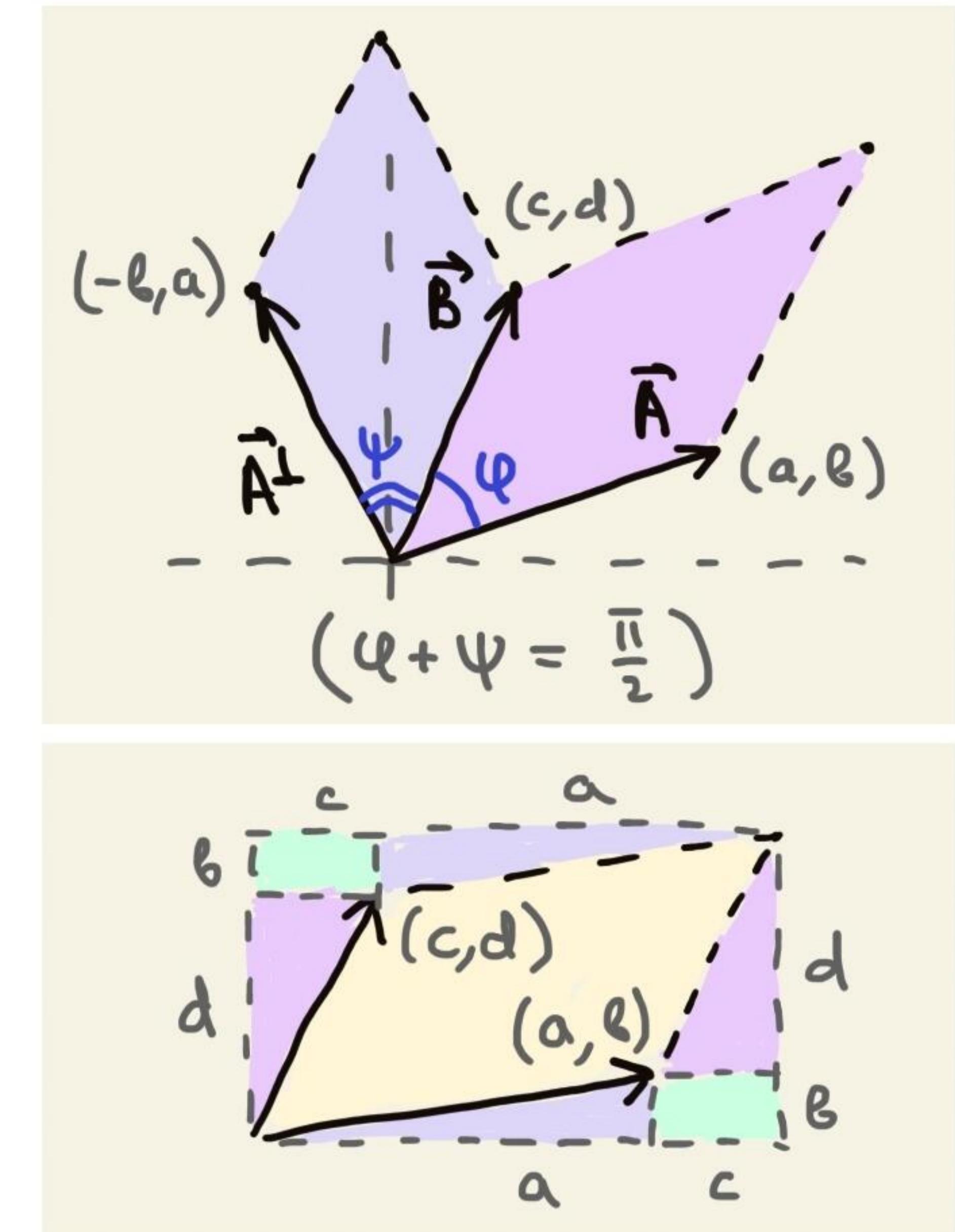
5. Bonus memes. (from 3B1B)

- cross product det formula derivation
- eigenvectors and eigenvalues (remain on a span after trans.)
- geometric interpretation of Kramer's rule
- derivative matrix for polynomials

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} = \frac{d}{dx} \quad \text{for } p_n(x) = \sum_{k=0}^n c_k x^k \quad (\{1, x, x^2, \dots\} \text{ is a basis})$$

$$(\vec{a} \cdot \vec{b})^2 + \|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \cdot \|\vec{b}\|^2$$

Lagrange's identity



Infinite series (from Zorich b.I §3.1.4)

① Numerical series.

$(\{a_n\}, \{S_n\})$ is a **series**: $S_n = \sum_{k=1}^n a_k \quad \forall n \in \mathbb{N}$, where $\{a_n\}$: terms of a series
 $\{S_n\}$: partial sums

$\sum_{n=1}^{\infty} a_n$ is **convergent**: $\{S_n\}$ converges

$S = \lim_{n \rightarrow \infty} S_n$: sum of a series

$\sum_{n=1}^{\infty} a_n$ is **divergent**: $\{S_n\}$ diverges (or $\exists \lim_{n \rightarrow \infty} S_n$)

Th.1.1 (Cauchy's conv. cr.):

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Leftrightarrow \forall \varepsilon > 0 \ \exists N \in \mathbb{N}: \forall k \geq l > N \quad \left| \sum_{m=l}^k a_m \right| < \varepsilon$$

Proof:

$$\{S_n\} \text{ conv.} \stackrel{\text{Cauchy}}{\Leftrightarrow} \forall \varepsilon > 0 \ \exists N \in \mathbb{N}: \forall k \geq l > N \quad |S_k - S_l| < \varepsilon$$

$$|S_k - S_l| = \left| \sum_{m=1}^k a_m - \sum_{m=1}^l a_m \right| = \left| \sum_{m=l+1}^k a_m \right|$$

Corollaries:

- if only a finite number of terms of a series are changed, the resulting new series will converge/diverge if the original series did.
- $\sum_{n=1}^{\infty} a_n$ conv. $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$ (necessary cond. for conv.)
- $\sum_{n=1}^{\infty} a_n$ conv. $\Rightarrow r_m = \sum_{n=m+1}^{\infty} a_n$ is s.t. $\lim_{m \rightarrow \infty} r_m = 0$ (m-th remainder of a series)
- $\sum_{n=1}^{\infty} C a_n$ conv. $\Rightarrow \sum_{n=1}^{\infty} a_n$ conv., where $C \neq 0$

Examples:

1) $\sum_{n=0}^{\infty} q^n$ geometric series

$$|q| \geq 1: |q^n| = |q|^n \geq 1 \Rightarrow \lim_{n \rightarrow \infty} q^n \neq 0 \Rightarrow \text{diverges}$$

$$|q| < 1: S_n = 1 + q + \dots + q^n = \frac{1 - q^n}{1 - q} \Rightarrow S = \lim_{n \rightarrow \infty} S_n = \frac{1}{1 - q} \text{ converges}$$

2) $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$S_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \quad |S_{2n} - S_n| = \frac{1}{n+1} + \dots + \frac{1}{n+n} > n \cdot \frac{1}{2n} = \frac{1}{2} \Rightarrow \text{diverges}$$

② Absolute and conditional convergence.

Consider $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - \dots$. It diverges since $\lim_{n \rightarrow \infty} a_n \neq 0$. But we can make it converge by inserting parentheses or moving the terms around:

$$\begin{aligned}(1 - 1) + (1 - 1) + \dots &\rightarrow 0 \\ 1 + (-1 + 1) + (-1 + 1) + \dots &\rightarrow 1 \\ 1 + 1 - 1 + 1 - \dots = 1 + (1 - 1) + (1 - 1) + \dots &\rightarrow 2\end{aligned}$$

So the usual laws for dealing with finite sums generally can't be extended to series. There is nevertheless a type of series that can be handled like finite sums, these are the so-called absolutely convergent series.

$\sum_{n=1}^{\infty} a_n$ is absolutely conv. : $\sum_{n=1}^{\infty} |a_n|$ converges

$\sum_{n=1}^{\infty} a_n$ is conditionally conv. : $\sum_{n=1}^{\infty} |a_n|$ diverges, but $\sum_{n=1}^{\infty} a_n$ conv.

Properties:

- $\sum_{n=1}^{\infty} a_n$ conv. abs. $\Rightarrow \sum_{n=1}^{\infty} a_n$ conv. (since $|a_1 + \dots + a_k| \leq |a_1| + \dots + |a_k| = ||a_1| + \dots + |a_k||$)
- $\sum_{n=1}^{\infty} a_n$ conv. abs. $\Rightarrow \# \text{ permutation } \delta : \sum_{n=1}^{\infty} a_n = S$ (Cauchy)
- $\sum_{n=1}^{\infty} a_n$ conv. cond. $\Rightarrow \# S \exists \text{ permutation } \delta : \sum_{n=1}^{\infty} a_n = S$ (Riemann)
- $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ conv. abs. $\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} (a_n + b_n)$ conv. abs. and $S^+ = A + B$
 $S^- = A \cdot B$

To better understand abs. convergence, let's investigate series $\sum_{n=1}^{\infty} p_n$ with only non-negative terms:

- $\{S_n\}$ for $\sum_{n=1}^{\infty} p_n$ is non-decreasing
- $\sum_{n=1}^{\infty} p_n$ conv. $\Leftrightarrow \{S_n\}$ is bounded above
- If $\exists N \in \mathbb{N} : p_n \leq \tilde{p}_n \ \forall n > N$, then
 1. $\sum_{n=1}^{\infty} \tilde{p}_n$ conv. $\Rightarrow \sum_{n=1}^{\infty} p_n$ conv.
 2. $\sum_{n=1}^{\infty} p_n$ diver. $\Rightarrow \sum_{n=1}^{\infty} \tilde{p}_n$ diver.direct comparison test
- If $\exists N \in \mathbb{N} : \frac{p_{n+1}}{p_n} \leq \frac{\tilde{p}_{n+1}}{\tilde{p}_n} \ \forall n > N$, then
 1. $\sum_{n=1}^{\infty} \tilde{p}_n$ conv. $\Rightarrow \sum_{n=1}^{\infty} p_n$ conv.
 2. $\sum_{n=1}^{\infty} p_n$ diver. $\Rightarrow \sum_{n=1}^{\infty} \tilde{p}_n$ diver.ratio comparison test
- If $\exists \lim_{n \rightarrow \infty} \frac{\tilde{p}_n}{p_n} = c$ s.t. $c < \infty$, then
 1. $\sum_{n=1}^{\infty} \tilde{p}_n$ conv. $\Leftrightarrow \sum_{n=1}^{\infty} p_n$ conv.
 2. $\sum_{n=1}^{\infty} p_n$ diver. $\Leftrightarrow \sum_{n=1}^{\infty} \tilde{p}_n$ diver.limit comparison test

Th. 2.1 (Weierstrass M-test):

Consider $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$. If $\exists N \in \mathbb{N} : |a_n| \leq b_n \ \forall n > N$, then $\sum_{n=1}^{\infty} b_n$ conv. $\Rightarrow \sum_{n=1}^{\infty} a_n$ abs. conv.
(i.e. $\sum a_n$ is majorized by $\sum b_n$)

Th.2.2 (Cauchy's root test):

Consider $\sum_{n=1}^{\infty} a_n$ and $\lambda = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ (if exists), then:

1. $\lambda < 1$: conv. abs.

2. $\lambda > 1$: diverges

3. $\lambda = 1$: ?

Th.2.3 (d'Alembert's ratio test):

Consider $\sum_{n=1}^{\infty} a_n$ and $\lambda = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ (if exists), then:

1. $\lambda < 1$: conv. abs.

2. $\lambda > 1$: diverges

3. $\lambda = 1$: ?

Th.2.4 (Maclaurin's integral test):

Consider non-increasing $f: [m, +\infty) \rightarrow \mathbb{R}$ s.t. $f(x) \geq 0$. Then $\sum_{n=m}^{\infty} f(n)$ conv. $\Leftrightarrow \exists \int_m^{\infty} f(x) dx$.

Th.2.5 (Leibniz's test):

Consider $\sum_{n=1}^{\infty} (-1)^{n+1} p_n$. Then if $\{p_n\}$ is non-increasing and $\lim_{n \rightarrow \infty} p_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} p_n$ conv.

Th.2.6 (Dirichlet-Abel's test):

Consider non-increasing $\{b_n\}$ s.t. $\lim_{n \rightarrow \infty} b_n = 0$ and $\{a_n\}$ that is bounded above, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Th.2.7 (Cauchy's condensation test):

Consider non-increasing $\{p_n\}$. Then $\sum_{n=1}^{\infty} p_n$ conv. $\Leftrightarrow \sum_{n=0}^{\infty} 2^n p_{2^n}$ conv.

Examples:

1) $\sum_{n=1}^{\infty} \frac{1}{n^{\lambda}}$ hyperharmonic series

• $f(x) = \frac{1}{x^{\lambda}}$ is non-increasing on $[1, +\infty)$ and $f(x) > 0$, then:

$$\int_1^{\infty} \frac{dx}{x^{\lambda}} = \begin{cases} \frac{x^{1-\lambda}}{1-\lambda} \Big|_1^{\infty}, & \lambda \neq 1 \\ \ln x \Big|_1^{\infty}, & \lambda = 1 \end{cases}, \text{then} \quad \begin{array}{ll} \lambda < 1: \text{diverges} \\ \lambda = 1: \text{diverges} \\ \lambda > 1: \text{conv. and } \sum_{n=1}^{\infty} \frac{1}{n^{\lambda}} = \frac{1}{\lambda-1} \end{array}$$

• $\{p_n\}$ is non-increasing for $\lambda \geq 0$ $\xrightarrow{\text{Cauchy}} \sum_{n=1}^{\infty} \frac{1}{n^{\lambda}}$ conv./diver. whenever $\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^{\lambda}}$ does

$$\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^{\lambda}} = \sum_{n=0}^{\infty} (2^{1-\lambda})^n, \text{then } d = \overline{\lim_{n \rightarrow \infty}} \sqrt[n]{(2^{1-\lambda})^n} = 2^{1-\lambda}$$

Series converges whenever $d < 1$ (Cauchy's test) $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{\lambda}}$ conv. if $\lambda > 1$

$$2) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\lambda}} = 1 - \frac{1}{2^{\lambda}} + \frac{1}{3^{\lambda}} - \dots, \lambda > 1$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^{\lambda}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\lambda}} \xrightarrow{\text{M-test}} \text{conv. abs.}$$

$$3) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$\{p_n\}$ is non-increasing and $\lim_{n \rightarrow \infty} p_n = 0 \xrightarrow{\text{Leibniz}}$ conv.

Fields on manifolds

(Zorich b.II § 15.3.6.)

1. Commutator of vector fields

$$[X, Y] f \stackrel{\text{def}}{=} X(Yf) - Y(Xf), \quad X, Y : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$$

$$\begin{aligned} [X, Y] &= X^i \partial_i (Y^j \partial_j) - Y^i \partial_i (X^j \partial_j) = X^i \partial_i Y^j \partial_j + X^i Y^j \cancel{\partial_{ij}} - Y^i \partial_i X^j \partial_j - \\ &\quad - Y^i \cancel{X^j} \partial_{ij} = (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j \end{aligned}$$

2. Exterior derivative

$$d\omega(X, Y) = X(\omega Y) - Y(\omega X) - \omega[X, Y], \quad \omega : TM \rightarrow \mathbb{R}$$

Proof:

$$\begin{array}{lll} \omega = \omega_i dx^i & X = X^i \partial_i & \omega(X) = \omega_i X^i \\ d\omega = \partial_i \omega_j dx^i \wedge dx^j & Y = Y^i \partial_i & \omega(Y) = \omega_i Y^i \end{array}$$

$$\begin{aligned} 1) \quad X(\omega Y) - Y(\omega X) &= X^i \partial_i (\omega_j Y^j) - Y^i \partial_i (\omega_j X^j) = X^i (Y^j \partial_i \omega_j + \omega_j \partial_i Y^j) - \\ &\quad - Y^i (X^j \partial_i \omega_j + \omega_j \partial_i X^j) \end{aligned}$$

$$\omega[X, Y] = \omega_j (X^i \partial_i Y^j - Y^i \partial_i X^j)$$

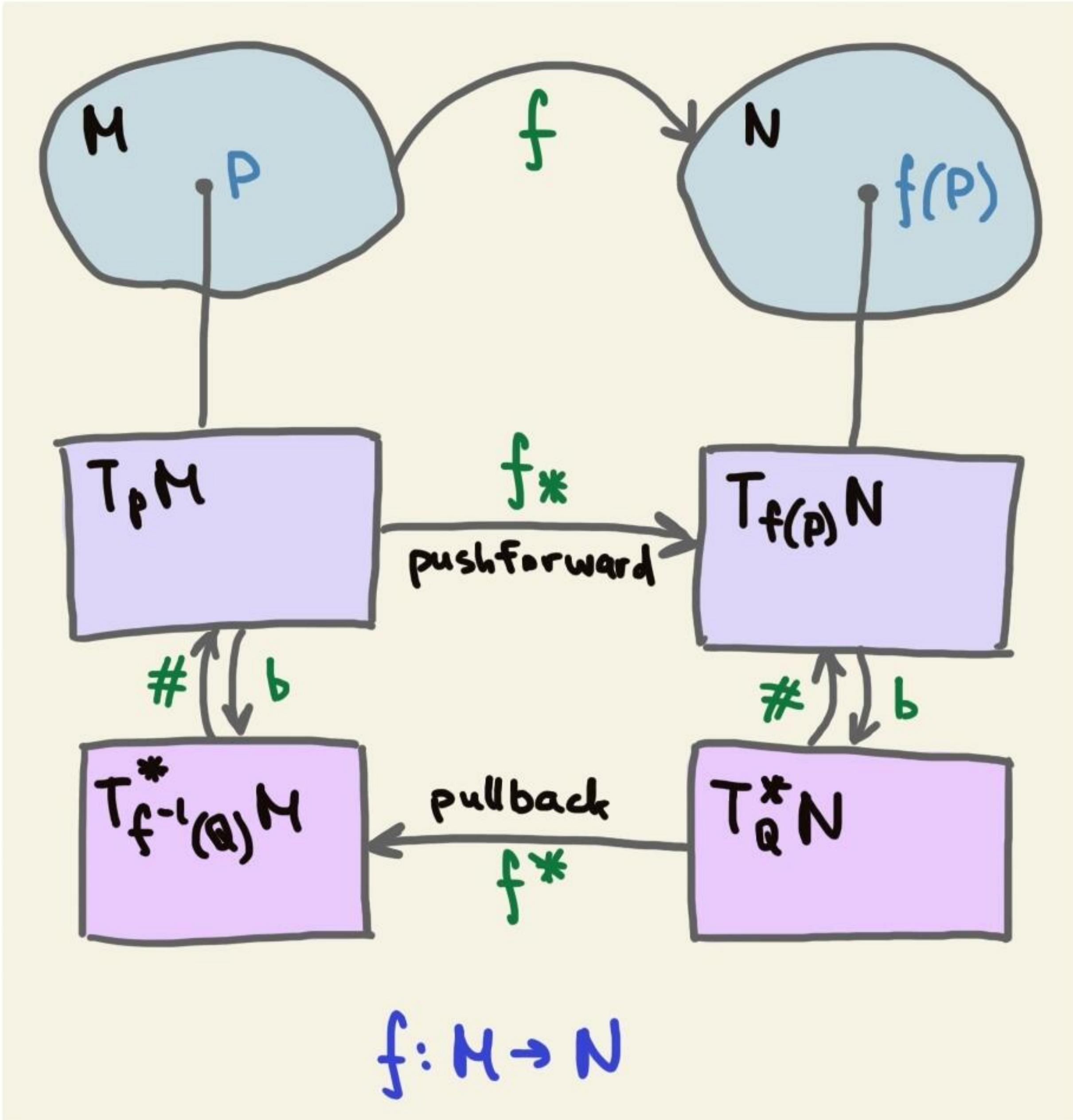
$$2) \quad X(\omega Y) - Y(\omega X) - \omega[X, Y] = \dots = \partial_i \omega_j (X^i Y^j - X^j Y^i) = \partial_i \omega_j \begin{vmatrix} X^i & X^i \\ Y^i & Y^i \end{vmatrix}$$

$$3) \quad d\omega(X, Y) = \partial_i \omega_j dx^i \wedge dx^j (X, Y) = \partial_i \omega_j \begin{vmatrix} X^i & X^i \\ Y^i & Y^i \end{vmatrix}$$

$$d\omega(X_1, \dots, X_{m+1}) = \sum_{i=1}^{m+1} (-1)^{i+1} X_i \omega(X_1, \dots, \hat{X}_i, X_{m+1}) + \sum_{1 \leq i < j \leq m+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{m+1})$$

Proof: (analogous)

3. Pullback and pushforward



$$f_* \bar{v}(\phi) \stackrel{\text{def}}{=} \bar{v}(\phi \circ f), \quad f_* \bar{v} \in T_{f(p)} N$$

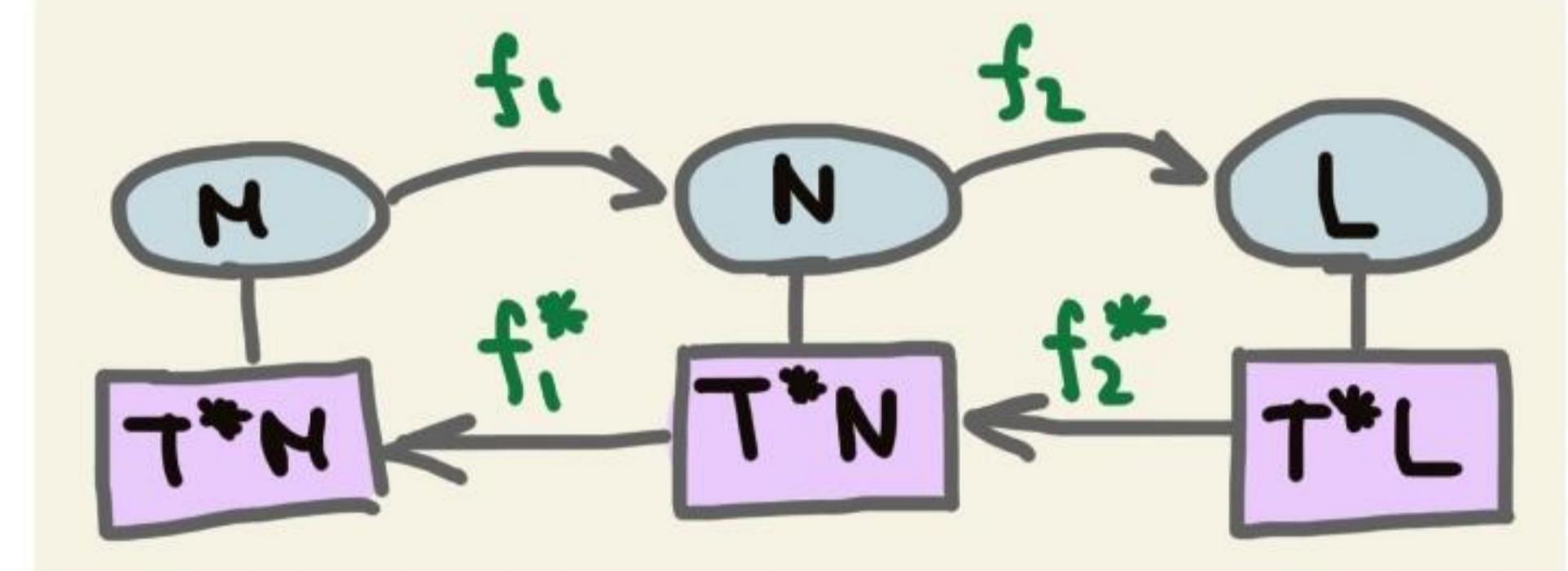
push forward

$$f^* \omega(\bar{v}) \stackrel{\text{def}}{=} \omega(f_* \bar{v}), \quad f^* \omega \in T_{f^{-1}(Q)}^* M$$

pullback

Properties:

1. $f_* \bar{v}(\phi) = \bar{v}(\phi \circ f) = v^i \partial_i (\phi \circ f) = v^i \frac{\partial \phi}{\partial f_j} \frac{\partial f_j}{\partial x^i} = \left(v^i \frac{\partial f_i}{\partial x^j} \frac{\partial \phi}{\partial f_j} \right) \phi$
2. $f_* [x, y] = [f_* x, f_* y]$
3. $f^*(\alpha_1 \omega_1 + \alpha_2 \omega_2) = \alpha_1 f^* \omega_1 + \alpha_2 f^* \omega_2$
4. $f^*(\omega_1 \wedge \omega_2) = f^* \omega_1 \wedge f^* \omega_2$
5. $f^* \circ d = d \circ f^*$
6. $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$



4. Lie derivative

$\vec{v} \rightarrow \{\varphi_t \mid t \in \Omega\}$ 1-parametric group of diffeomorphisms on M

$$D_{v(x)} f = v(f)(x) = \lim_{t \rightarrow 0} \frac{f(\varphi_t(x)) - f(x)}{t} = \lim_{t \rightarrow 0} \frac{\varphi_t^* f - f(x)}{t}$$

$$\mathcal{L}_X \omega \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} \frac{\varphi_t^* \omega - \omega}{t}(x), \quad \mathcal{L}_X : \Omega^k(M) \rightarrow \Omega^k(M) \text{ linear}$$

$$1. \mathcal{L}_X (\omega_1 \wedge \omega_2) = \mathcal{L}_X \omega_1 \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_X \omega_2$$

$$2. f \in \Omega^0(M) : \mathcal{L}_X f = Xf = df(X)$$

$$\mathcal{L}_X df = d(Xf) ?$$

5. Interior derivative (product)

$$i_X \omega (X_1, \dots, X_{k-1}) \stackrel{\text{def}}{=} \omega (X, X_1, \dots, X_{k-1}), \quad i_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M) \text{ linear}$$

$$\omega \in \Omega^k(M)$$

$$i_X \omega \in \Omega^{k-1}(M)$$

$$1. f \in \Omega^0(M) : i_X f = 0 \quad \forall f$$

$$i_X df = df(X) = X^i \partial_i f = Xf$$

$$\omega \in \Omega^1(M) : i_X \omega = \omega(X)$$

$$2. i_X \omega = \frac{1}{(k-1)!} X^i \omega_{i,i_2 \dots i_k} dx^{i_2} \wedge \dots \wedge dx^{i_k}$$

$$3. i_X (\omega_1 \wedge \omega_2) = i_X \omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge i_X \omega_2$$

$$\mathcal{L}_X = i_X d + d i_X$$

homotopy identity

$$4. \mathcal{L}_X \circ d = d \circ \mathcal{L}_X \quad 5. [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$$

$$\mathcal{L}_X \circ i_X = i_X \circ \mathcal{L}_X \quad [\mathcal{L}_X, i_Y] = i_{[X,Y]}$$

$$* \quad \nabla_X Y - \nabla_Y X = [X, Y], \text{ if } \Gamma_{jk}^i = \Gamma_{kj}^i$$

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= X^i \nabla_i (Y^j \partial_j) - Y^i \nabla_i (X^j \partial_j) = X^i (\nabla_i X^j \partial_j + Y^j \nabla_i \partial_j) - \\ &- Y^i (\nabla_i X^j \partial_j + X^j \nabla_i \partial_j) = X^i (\partial_i Y^j + \Gamma_{ik}^j Y^k) \partial_j + X^i Y^j \Gamma_{ji}^k \partial_k - \\ &- Y^i (\partial_i X^j + \Gamma_{ik}^j X^k) \partial_j - Y^i X^j \Gamma_{ji}^k \partial_k = (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j \end{aligned}$$

6. Geometry of k-vectors and k-forms

6.1. k-vectors (a.k.a. k-blades, multivectors, Clifford numbers)

$$u \in \Lambda^k V, \dim V = n$$

Components of $u = u_1 \wedge \dots \wedge u_k$ are oriented $(k-1)$ -volumes that are obtained from projecting a k -dimensional parallelepiped on the hyperplanes defined by the coordinate axes. And $\|u\|$ is the volume of this parallelepiped.

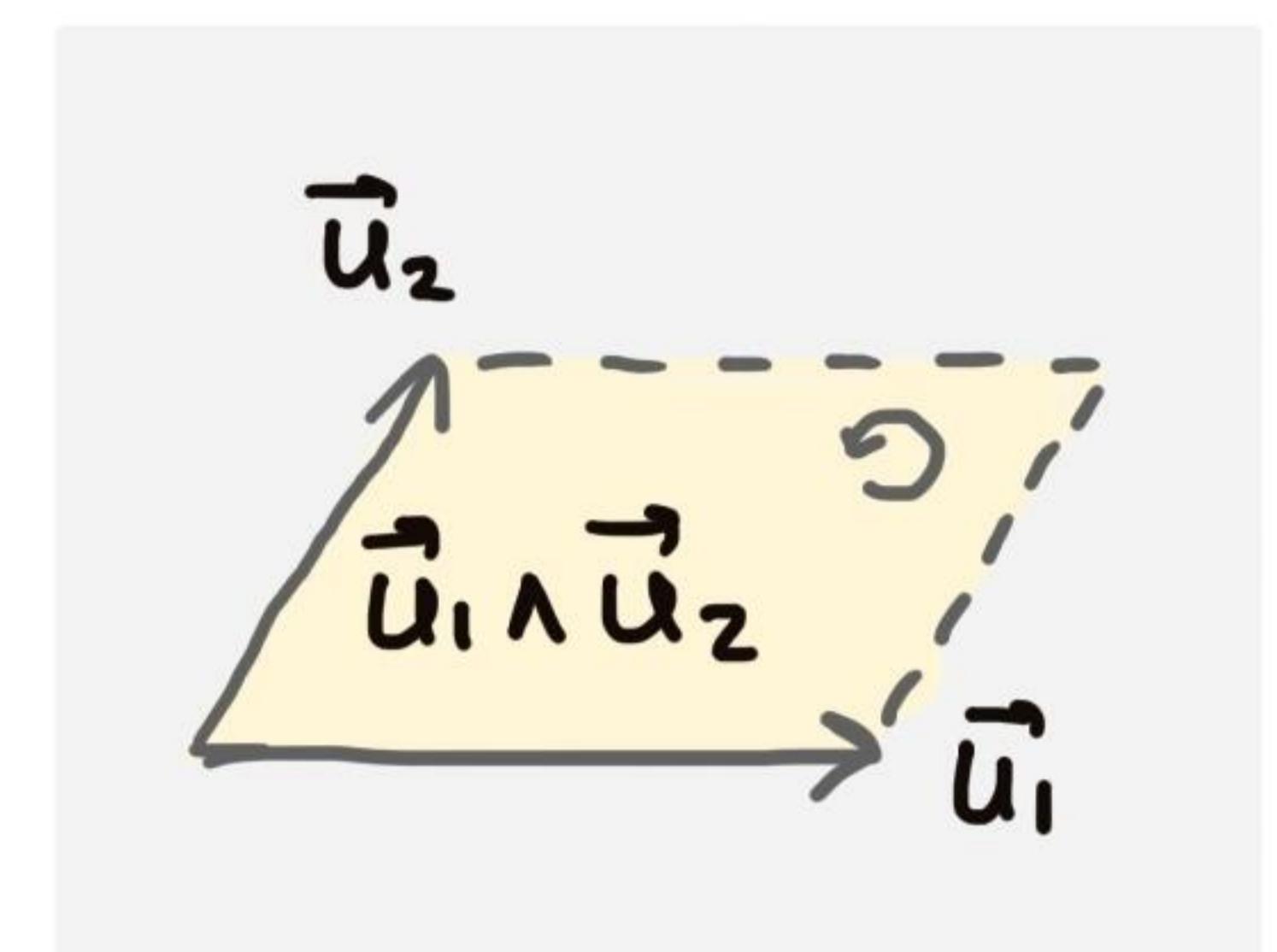
Examples:

1) $M = \mathbb{R}^2$

$$\begin{aligned} \vec{u}_1 &= u_1^1 \vec{e}_1 + u_1^2 \vec{e}_2 \\ \vec{u}_2 &= u_2^1 \vec{e}_1 + u_2^2 \vec{e}_2 \end{aligned} \quad \Rightarrow \quad \vec{u}_1 \wedge \vec{u}_2 = \begin{vmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{vmatrix} \vec{e}_1 \wedge \vec{e}_2$$

$\pm S$

$$\|\vec{u}_1 \wedge \vec{u}_2\| = S_{(\vec{u}_1, \vec{u}_2)}$$

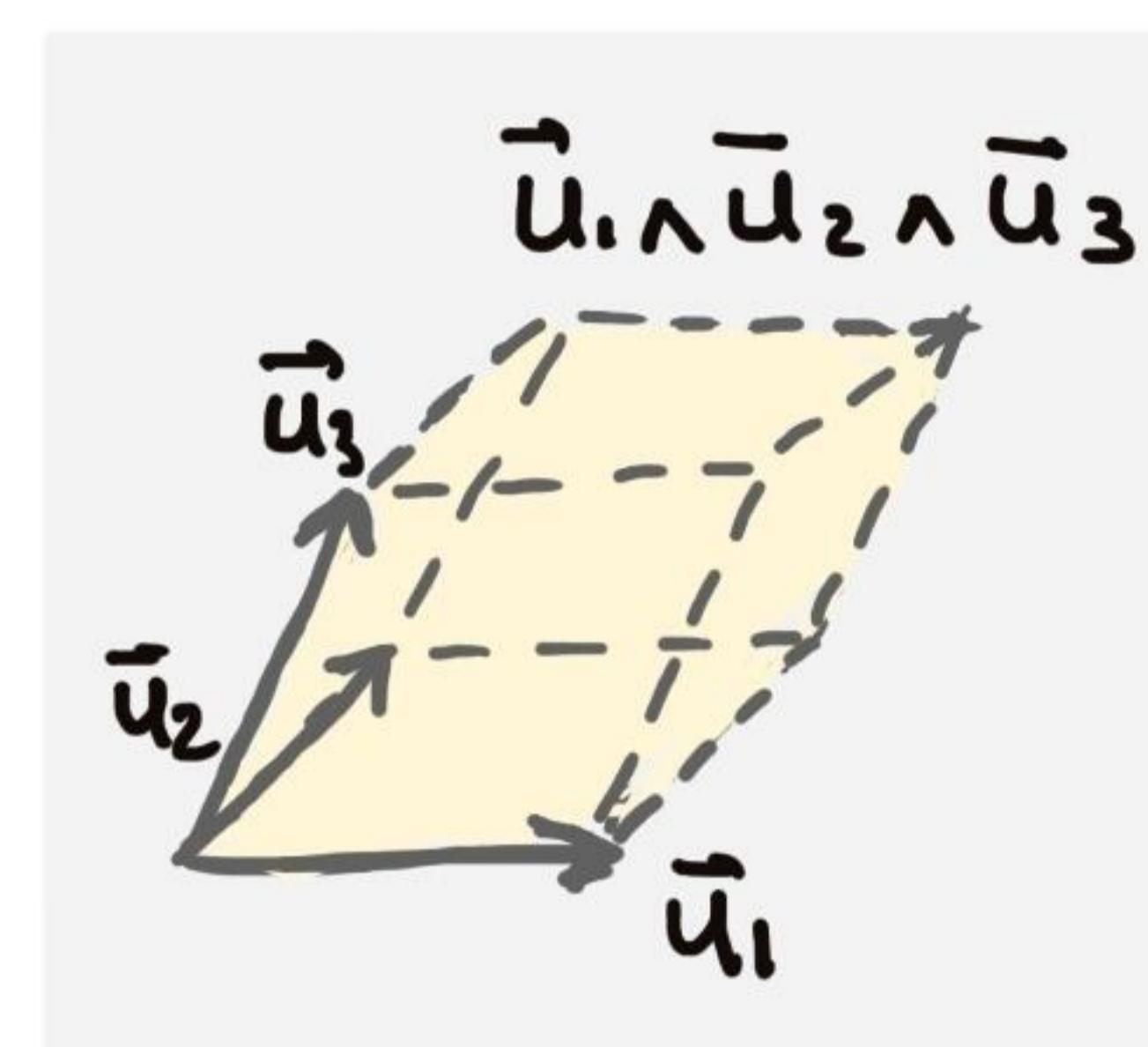


2) $M = \mathbb{R}^3 \quad \vec{u}_j = u_j^i \vec{e}_i$

$$\vec{u}_1 \wedge \vec{u}_2 = \underbrace{\begin{vmatrix} u_1^2 & u_2^2 \\ u_1^3 & u_2^3 \end{vmatrix}}_{\pm S_{yz}} \vec{e}_2 \wedge \vec{e}_3 + \underbrace{\begin{vmatrix} u_1^1 & u_2^1 \\ u_1^3 & u_2^3 \end{vmatrix}}_{\pm S_{xz}} \vec{e}_1 \wedge \vec{e}_3 + \underbrace{\begin{vmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{vmatrix}}_{\pm S_{xy}} \vec{e}_1 \wedge \vec{e}_2 \quad (\vec{u}_1 \wedge \vec{u}_2 \nleftrightarrow \vec{u}_1 \times \vec{u}_2)$$

$$\|\vec{u}_1 \wedge \vec{u}_2\| = S_{(\vec{u}_1, \vec{u}_2)} \quad (= \sqrt{S_{yz}^2 + S_{xz}^2 + S_{xy}^2})$$

$$\vec{u}_1 \wedge \vec{u}_2 \wedge \vec{u}_3 = \underbrace{\begin{vmatrix} u_1^1 & u_2^1 & u_3^1 \\ u_1^2 & u_2^2 & u_3^2 \\ u_1^3 & u_2^3 & u_3^3 \end{vmatrix}}_{\pm V} \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3$$

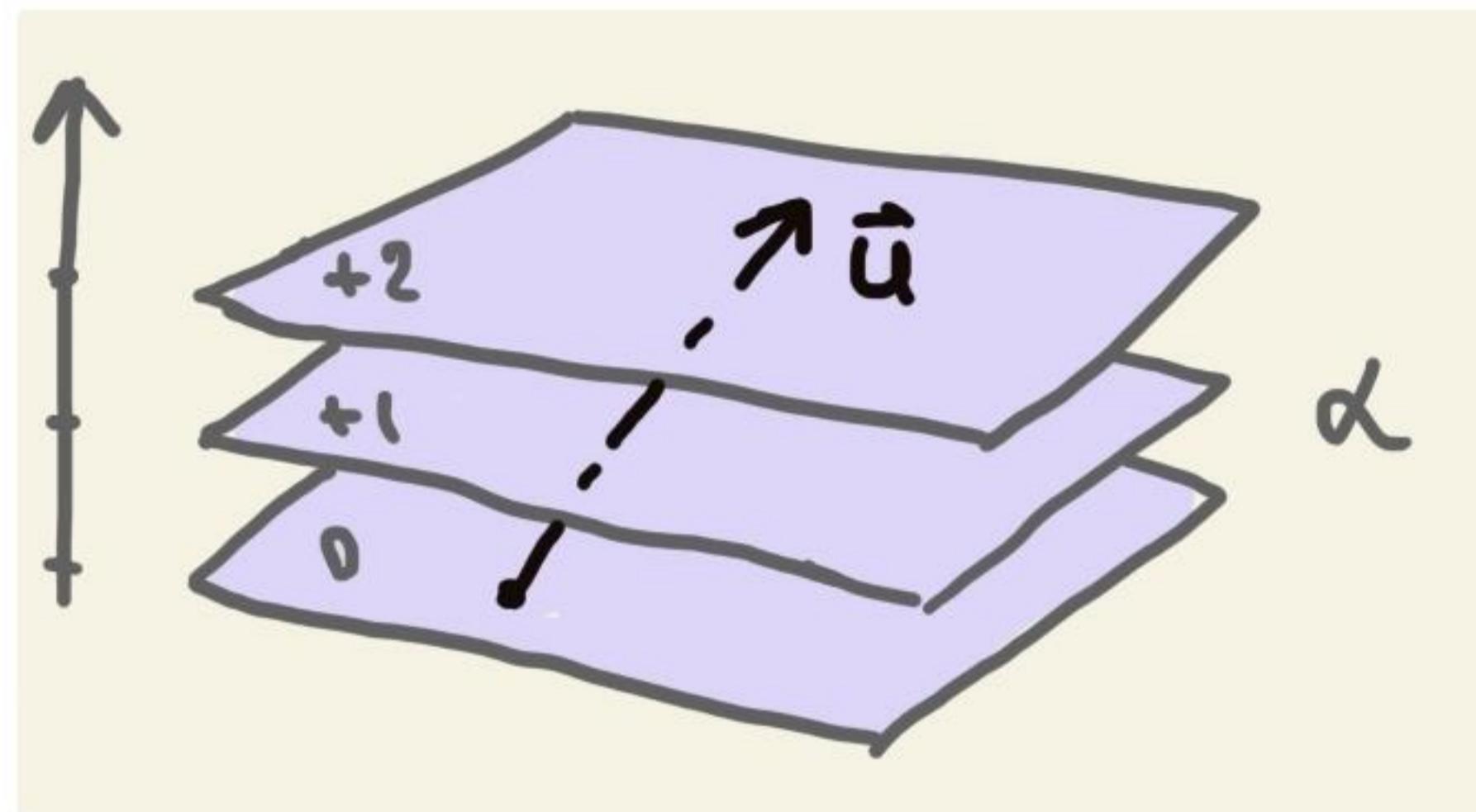


$$\|\vec{u}_1 \wedge \vec{u}_2 \wedge \vec{u}_3\| = V_{(\vec{u}_1, \vec{u}_2, \vec{u}_3)}$$

6.2. k-forms

1-forms:

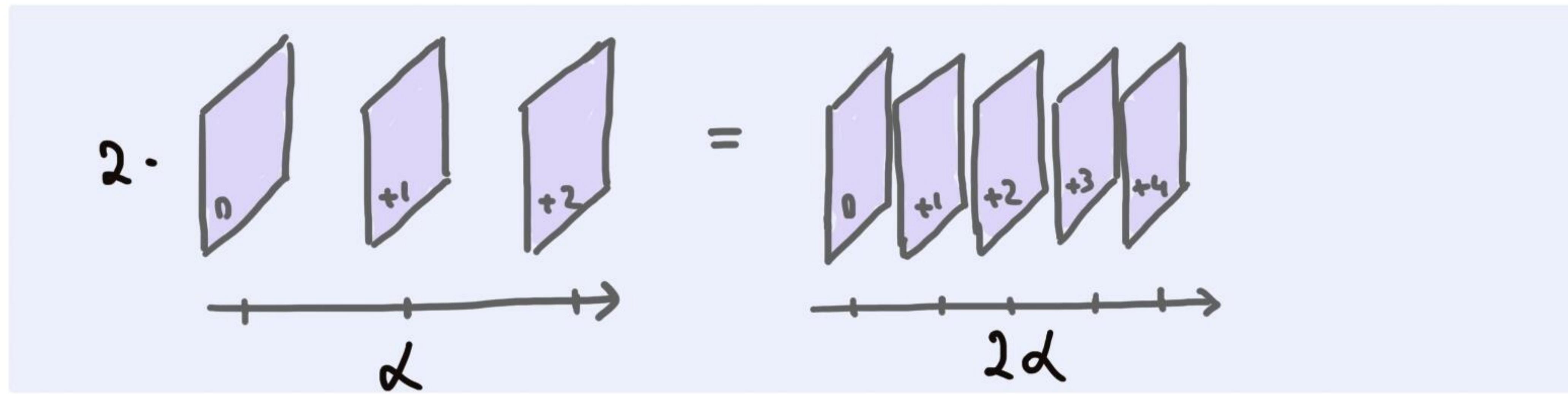
Geometrically 1-form can be viewed as a family of parallel hyperplanes with a chosen orientation. Then the result of a 1-form acting on a vector is a number of hyperplanes pierced by this vector (taking orientation into account).



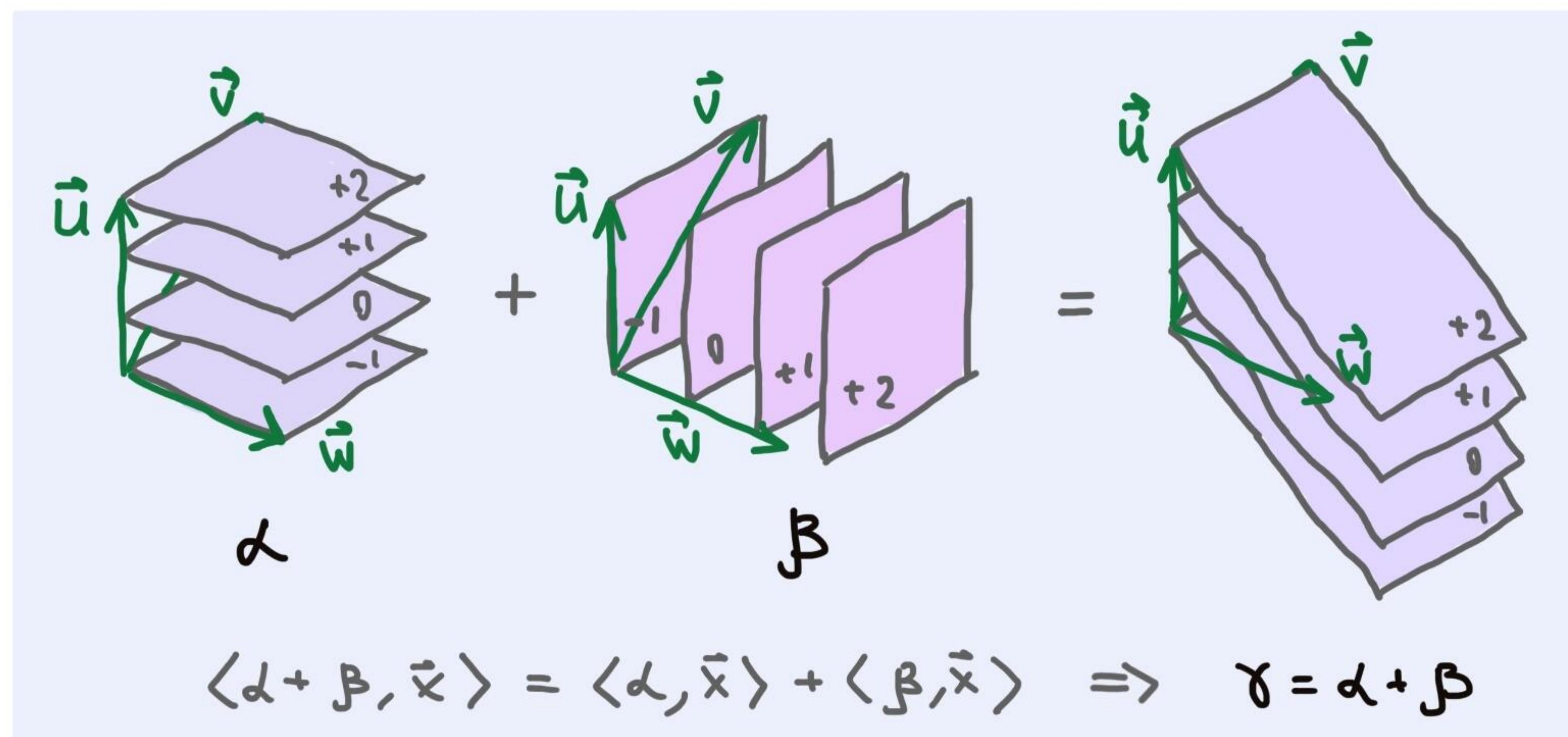
$\langle \alpha, \vec{u} \rangle$ number of intersections between \vec{u} and α

Geometrical interpretation of linear operations:

1. Scalar multiplication.



2. Addition.

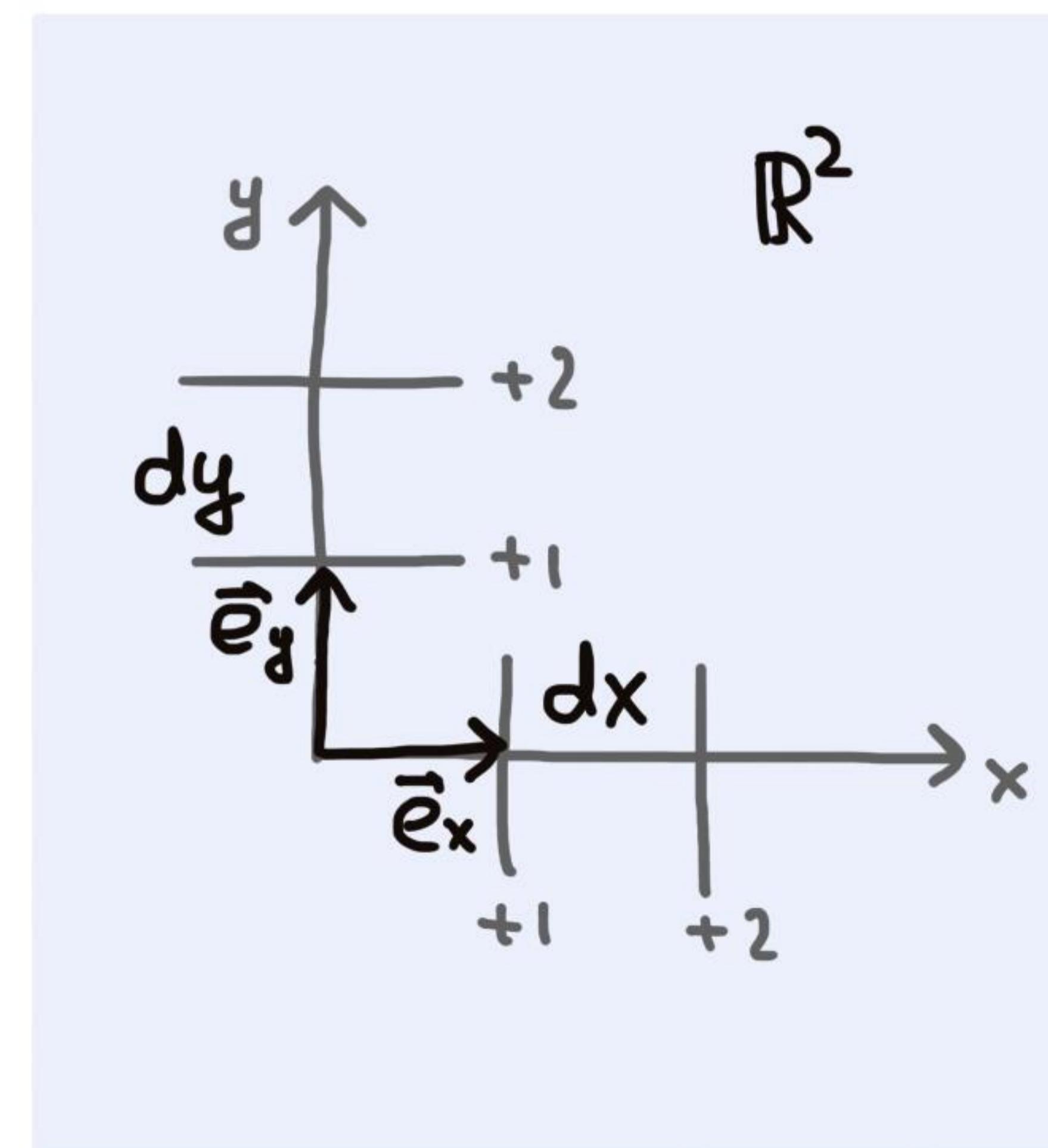
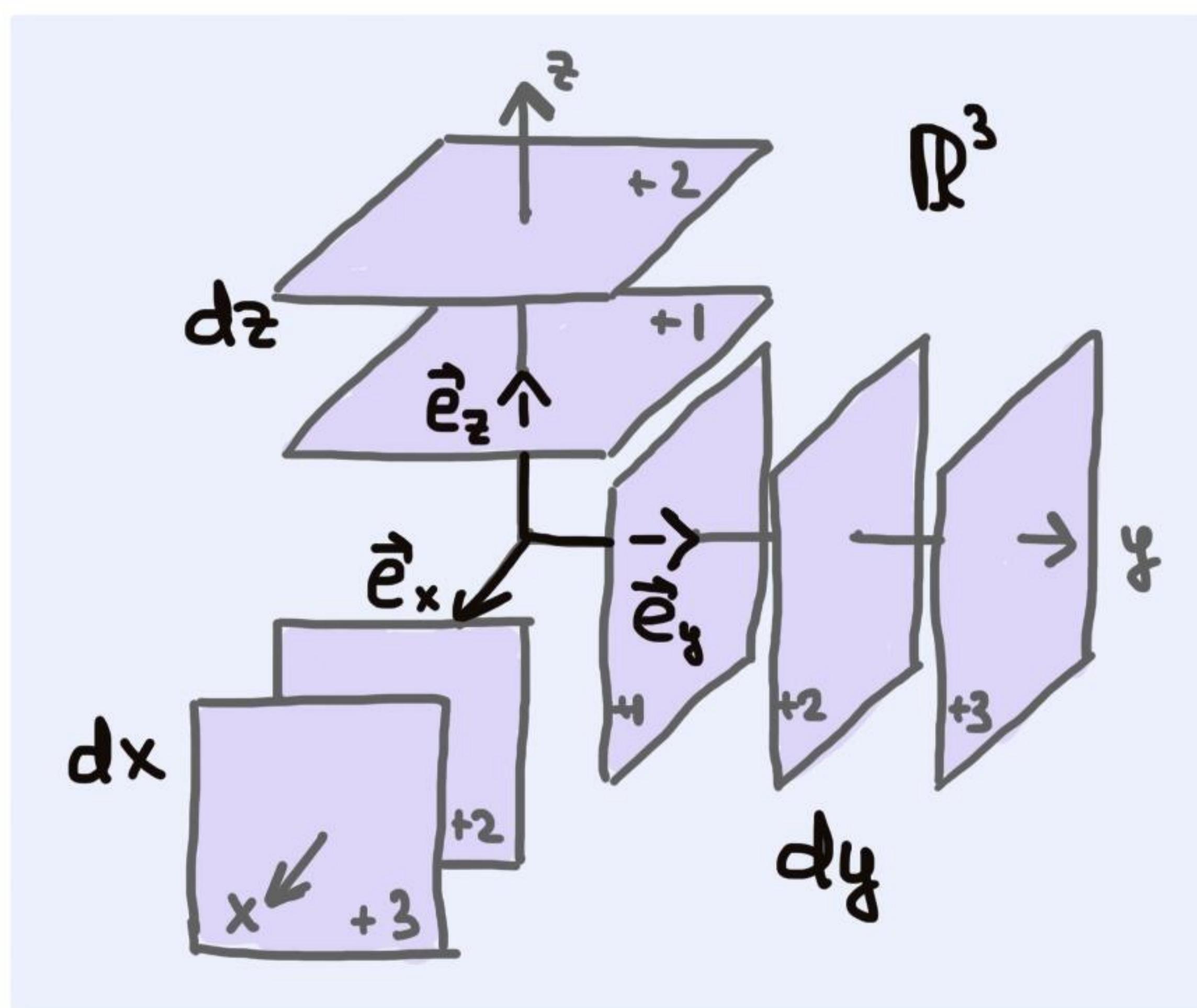


$$\alpha \left\{ \begin{array}{l} \langle \alpha, \vec{u} \rangle = \langle \alpha, \vec{v} \rangle = 3 \\ \langle \alpha, \vec{w} \rangle = 0 \end{array} \right.$$

$$\beta \left\{ \begin{array}{l} \langle \beta, \vec{u} \rangle = \langle \beta, \vec{v} \rangle = 0 \\ \langle \beta, \vec{w} \rangle = 2,5 \end{array} \right.$$

$$\gamma \left\{ \begin{array}{l} \langle \gamma, \vec{u} \rangle = \langle \gamma, \vec{v} \rangle = 3 \\ \langle \gamma, \vec{w} \rangle = 2,5 \end{array} \right.$$

Coordinate 1-forms: ($\langle dx^\alpha, \partial_\beta \rangle = \delta_\beta^\alpha$)



1-forms in $\mathbb{R}^{1,3}$:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

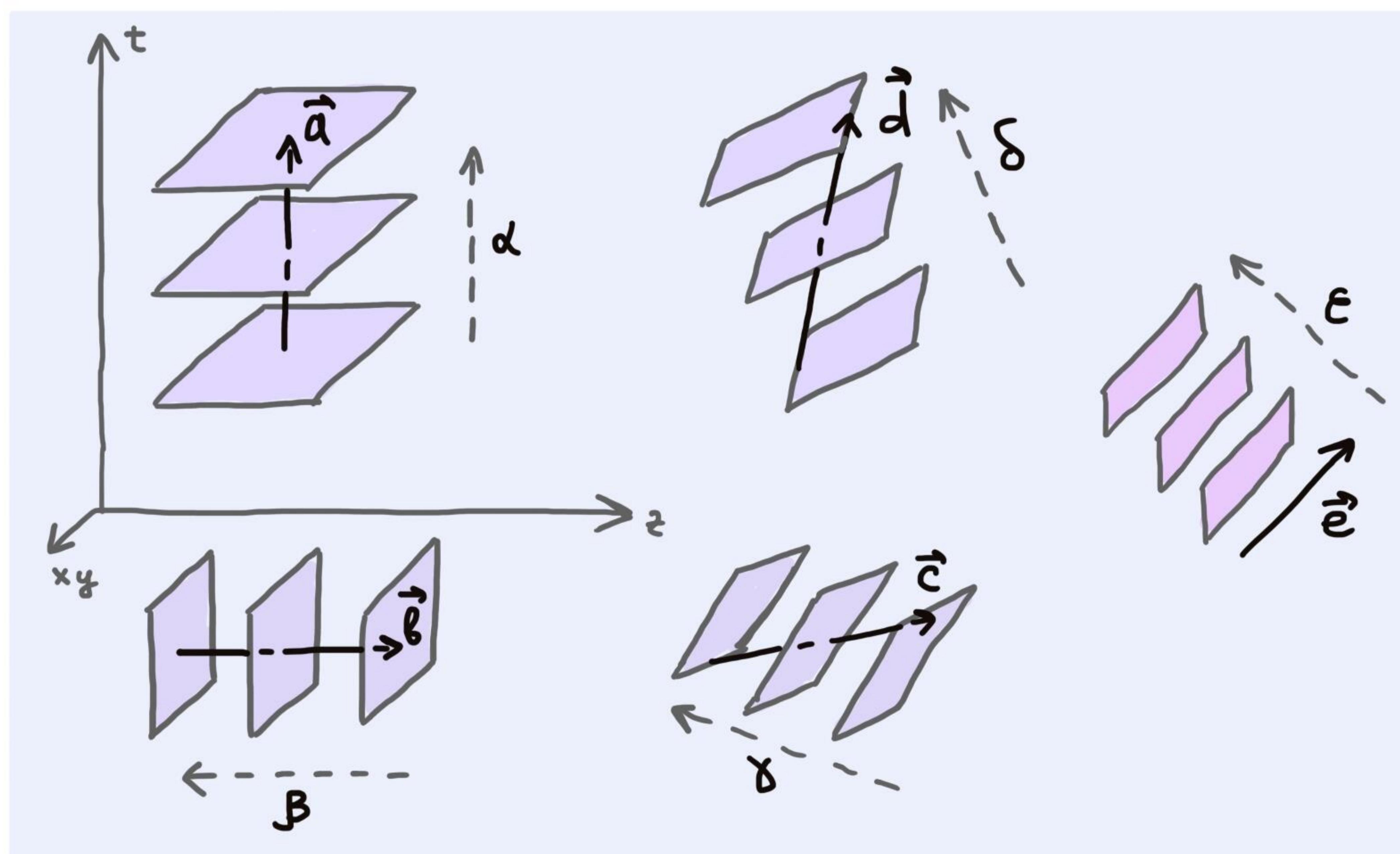
Let's find vectors corresponding to 1-forms:

$$\omega(\vec{u}) = \langle \vec{\omega}, \vec{u} \rangle \quad \forall \vec{u} \in TM$$

↑ covector ↑ vector

$\text{Pr}_{\vec{\omega}}(\vec{u}) = \omega(\vec{u})$. Then hypersurfaces of ω can be formed as spans of vectors perpendicular to $\vec{\omega}$. Consequently, $\omega(\vec{\omega}) = \langle \vec{\omega}, \vec{\omega} \rangle$ (number of intersections between ω and $\vec{\omega}$).

Examples:



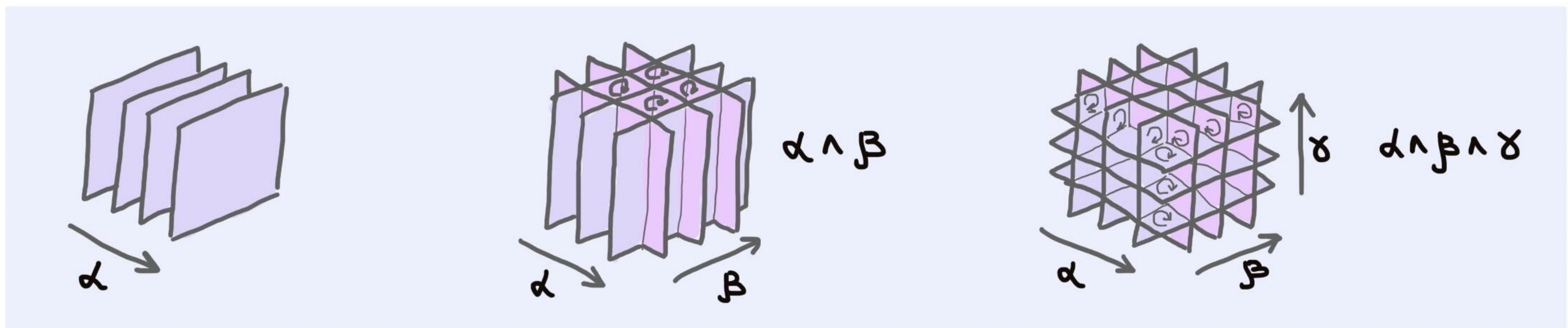
here hyperplanes ($k=3$) are depicted as planes ($k=2$)

$$(\langle \vec{e}, \vec{e} \rangle = 0)$$

k -forms ($k > 1$):

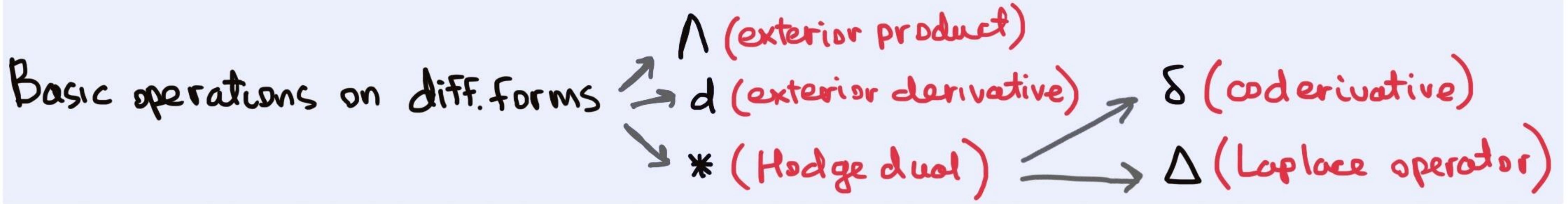
k -forms can be viewed as collections of intersecting 1-forms (taking orientation into account). Then the result of a k -form $\omega = \omega^1 \wedge \dots \wedge \omega^k$ acting on a k -vector $\vec{u}_1 \wedge \dots \wedge \vec{u}_k$ is a number of k -form structures intersected by a k -parallelipiped defined by the k -vector $\vec{u}_1 \wedge \dots \wedge \vec{u}_k$.

For example, in the case of $\omega \in \Omega^2$ in \mathbb{R}^3 2-form is a honeycomb-like structure and $\langle \omega, \vec{u}_1 \wedge \vec{u}_2 \rangle$ is a number of tubes cut by an oriented parallelogram $\vec{u}_1 \wedge \vec{u}_2$.



Hodge Duality

(Tevian Dray, 1999)
+ Flanders "Diff. forms"



V : vector space ($\dim V = n$)

g : metric ($g^{ij} = g(\delta^i, \delta^j) = \pm \delta^{ij}$, where $\{\delta^i \in V, i=1, \dots, n\}$ orthonormal basis)

s : signature of g (# of "-" signs in g , i.e. nonstandard definition)

$$\bigwedge^0 V = \mathbb{R} \quad \bigwedge^1 V = V \dots \bigwedge^p V = \emptyset, p > n ; \quad \dim \bigwedge^p V = \binom{n}{p}$$

$\{\delta^I \mid \delta^I = \delta^{i_1} \wedge \dots \wedge \delta^{i_p}; I = \{i_k \mid k=1, \dots, p; 1 \leq i_1 < \dots < i_p \leq n\}\}$ basis in $\bigwedge^p V$

$\alpha \in \bigwedge^p V$ decomposable : $\exists \alpha^i \in V : \alpha = \alpha^1 \wedge \dots \wedge \alpha^p$

$\langle \alpha, \beta \rangle \stackrel{\text{def}}{=} \det \langle \alpha^i, \beta^j \rangle$, where $\alpha, \beta \in \bigwedge^p V$
 inner product of p-vectors
 α, β - decomposable with $\{\alpha^i\}$ and $\{\beta^j\}$

1) $\langle \delta^I, \delta^J \rangle = \left(\prod_{k=1}^p \langle \delta^{i_k}, \delta^{j_k} \rangle \right) \delta^{IJ} = \pm \delta^{IJ}$, i.e. $\{\delta^I\}$ - orthonorm basis in $\bigwedge^p V$
 (Note: $\langle \cdot, \cdot \rangle$ same notation for inner product in different $\bigwedge^p V$)

2) $\langle \omega, \omega \rangle = \prod_{k=1}^n \langle \delta^{i_k}, \delta^{i_k} \rangle = (-1)^s$, where $\omega = \delta^1 \wedge \dots \wedge \delta^n \in \bigwedge^n V$
 (n -vector which defines orientation)

Th.: $\forall f \in V^*$ $\exists \beta \in V : f(\alpha) = \langle \alpha, \beta \rangle \quad \forall \alpha \in V$ ($\langle \cdot, \cdot \rangle$ defined by metric in V)

Col.: $\forall \lambda \in \bigwedge^p V \quad \exists f_\lambda \in (\bigwedge^{n-p} V)^* : \lambda \wedge \theta = f_\lambda(\theta) \omega \quad \forall \theta \in \bigwedge^{n-p} V; \omega \in \bigwedge^n V$
 $\uparrow \qquad \qquad \qquad \uparrow$
 $*\lambda \in \bigwedge^{n-p} V \quad \lambda \wedge \theta = (-1)^s \langle \theta, *\lambda \rangle \omega$

Therefore:

$\lambda \wedge \theta = (-1)^s \langle \theta, *\lambda \rangle \omega$, $* : \bigwedge^p V \rightarrow \bigwedge^{n-p} V$ (or $\Omega^p \rightarrow \Omega^{n-p}$)
 $\omega = e_1 \wedge \dots \wedge e_n \in \bigwedge^n V$

$\bigwedge^p V \cong \bigwedge^{n-p} V$
 because
 $\binom{n}{p} = \binom{n}{n-p}$

e.g. $\bigwedge^2(\mathbb{R}^3) \xleftrightarrow{*} \bigwedge^1(\mathbb{R}^3) : u \wedge v \xleftrightarrow{*} u \times v$
 (\wedge -product \leftrightarrow \times -product) (bivector \leftrightarrow pseudovector)

Practical definition:

Consider $\lambda = \beta^1 \wedge \dots \wedge \beta^p$, then if basis vector $\beta^I \in \bigwedge^{n-p} V$ (orthonormal) $\lambda \wedge \beta^I = (-1)^s \langle \beta^I, * \lambda \rangle \omega$

$$\begin{aligned} \lambda \wedge \beta^I &= 0 \quad \text{if } I \neq \{p+1, \dots, n\} \\ \langle \beta^I, * \lambda \rangle &= (*\lambda)_I \quad (\text{I-component of } *\lambda) \end{aligned}$$

$\Rightarrow * \lambda = c \beta^{p+1} \wedge \dots \wedge \beta^n, c \in \mathbb{R}$

Therefore:

$$\omega = (-1)^s \langle \beta^I, c \beta^I \rangle \omega, I' = \{p+1, \dots, n\}$$

$$c = \frac{(-1)^s}{\langle \beta^{I'}, \beta^{I'} \rangle} = \frac{\langle \omega, \omega \rangle}{\langle \beta^{I'}, \beta^{I'} \rangle} = \langle \lambda, \lambda \rangle = \langle \beta^1, \beta^1 \rangle \cdot \dots \cdot \langle \beta^p, \beta^p \rangle, \text{ then:}$$

$$*(\beta^1 \wedge \dots \wedge \beta^p) = \langle \beta^1, \beta^1 \rangle \dots \langle \beta^p, \beta^p \rangle \beta^{p+1} \wedge \dots \wedge \beta^n$$

, or $\{i_1, \dots, i_n\}$ any permutation, but:
 LHS = RHS for even perm.
 LHS = - RHS for odd perm.

Examples:

$$1) V = \mathbb{R}^2 \quad ds^2 = dx^2 + dy^2 \quad \omega = dx \wedge dy \quad (s=0)$$

$$*dx = dy$$

$$*dy = -dx$$

$$*1 = dx \wedge dy$$

$$*(dx \wedge dy) = 1$$

$$2) V = \mathbb{R}^{1,2} \quad ds^2 = -dt^2 + dx^2 + dy^2 \quad \omega = dt \wedge dx \wedge dy \quad (s=1)$$

$$*1 = dt \wedge dx \wedge dy$$

$$*dt = -dx \wedge dy$$

$$*dx = dy \wedge dt$$

$$*dy = dt \wedge dx$$

$$*(dt \wedge dx) = -dy$$

$$*(dx \wedge dy) = dt$$

$$*(dy \wedge dt) = -dx$$

$$*(dt \wedge dx \wedge dy) = -1$$

$$3) V = \mathbb{R}^{1,3} \quad ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad \omega = dt \wedge dx \wedge dy \wedge dz \quad (s=3)$$

$$*dt = dx \wedge dy \wedge dz$$

$$*dx = dy \wedge dz \wedge dt$$

$$*dy = -dz \wedge dt \wedge dx$$

$$*dz = dt \wedge dx \wedge dy$$

$$*1 = dt \wedge dx \wedge dy \wedge dz$$

$$*(dt \wedge dx \wedge dy \wedge dz) = -1$$

$$*(dt \wedge dx) = -dy \wedge dz$$

$$*(dt \wedge dy) = -dz \wedge dx$$

$$*(dt \wedge dz) = -dx \wedge dy$$

$$*(dx \wedge dy) = -dz \wedge dt$$

$$*(dy \wedge dz) = -dx \wedge dt$$

$$*(dz \wedge dx) = -dy \wedge dt$$

$$\text{e.g. } *(dx \wedge dy) = \eta^1 \cdot \eta^2 \cdot (-1)^I dz \wedge dt, \text{ where } (-1)^I = \text{sign} \left\{ \begin{smallmatrix} dx & dy & dz & dt \\ 1 & 2 & 3 & 0 \end{smallmatrix} \right\} = -1$$

$$*(dx \wedge dy) = (-1)^3 dz \wedge dt$$

$$*(dx \wedge dy) = -dz \wedge dt$$

$\begin{matrix} dx & dy & dz & dt \\ 1 & 2 & 3 & 0 \end{matrix}$
 odd perm.

Properties of Hodge *-operator:

$$1) \quad *1 = \omega \quad *\omega = (-1)^s, \quad 1 \in \Lambda^0 V, \quad \omega \in \Lambda^n V$$

Proof:

$$\begin{aligned} \exists \sigma^I = \sigma^1 \wedge \dots \wedge \sigma^p & \Rightarrow * \sigma^I = \langle \sigma^I, \sigma^I \rangle \sigma^I \quad \text{and} \quad \langle 1, 1 \rangle = 1 \\ \exists \sigma^J = \sigma^{p+1} \wedge \dots \wedge \sigma^n & \Downarrow \\ * \alpha = \|\alpha\|^2 \beta, \quad \alpha \in \Lambda^p V, \quad \beta \in \Lambda^{n-p} & \end{aligned}$$

$$2) \quad **\alpha = (-1)^{p(n-p)+s} \alpha, \quad \text{formally should write } *^{-1} *$$

$$\begin{cases} n = 2k+1: p(n-p) \text{ even} \neq p \\ n = 2k: p(n-p) \text{ even for even } p \\ p(n-p) \text{ odd for odd } p \end{cases}$$

Proof:

$$\sigma^J \wedge \sigma^I = (-1)^{p(n-p)} \sigma^I \wedge \sigma^J = (-1)^{p(n-p)} \omega \Rightarrow \sigma^J \wedge ((-1)^{p(n-p)} \sigma^I) = \omega$$

Therefore:

$$*\sigma^J = \langle \sigma^J, \sigma^J \rangle (-1)^{p(n-p)} \sigma^I$$

$$**\sigma^I = *(\langle \sigma^I, \sigma^I \rangle \sigma^J) = \underbrace{\langle \sigma^I, \sigma^I \rangle}_{\langle \omega, \omega \rangle = (-1)^s} \langle \sigma^J, \sigma^J \rangle (-1)^{p(n-p)} \sigma^I = (-1)^{n(n-p)+s} \sigma^I$$

$$3) \quad \alpha \wedge * \beta = \langle \alpha, \beta \rangle \omega, \quad \alpha, \beta \in \Lambda^p V$$

, one can take this as a definition of *

(but be carefull with signs)

Proof:

$$\lambda \wedge \theta \stackrel{\text{def}}{=} (-1)^s \langle \theta, * \lambda \rangle \omega$$

$$\begin{aligned} \exists \lambda = * \beta \in \Lambda^{n-p} V & \Rightarrow * \beta \wedge \alpha = (-1)^s \langle \alpha, * \beta \rangle \omega \\ \theta = \alpha \in \Lambda^p V & \\ (-1)^{p(n-p)} \alpha \wedge * \beta &= (-1)^s (-1)^{p(n-p)+s} \langle \alpha, \beta \rangle \omega \\ \alpha \wedge * \beta &= \langle \alpha, \beta \rangle \omega \end{aligned}$$

$$4) \quad \langle \alpha, \beta \rangle = (-1)^s * (\alpha \wedge * \beta), \quad \alpha, \beta \in \Lambda^p V$$

, one can take this as a definition of $\langle \cdot, \cdot \rangle$

Proof:

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle \omega \Rightarrow \langle \alpha, \beta \rangle = (-1)^s * (\alpha \wedge * \beta)$$

Examples:

1) $M = \mathbb{R}^3$ $ds^2 = dx^2 + dy^2 + dz^2$ ($s=0$)

$$f \in \Omega^0 : **f = f \quad \beta \in \Omega^2 : **\beta = \beta$$

$$\alpha \in \Omega^1 : *\alpha = \alpha \quad \omega \in \Omega^3 : **\omega = \omega$$

$**\alpha = (-1)^{p(n-p)+s} \alpha, \alpha \in \Omega^p(M)$

2) $M = \mathbb{R}^{1,3}$ $ds^2 = dt^2 - d\vec{r}^2$ ($s=3$)

$$J \in \Omega^1 : **J = J \quad \omega \in \Omega^4 : **\omega = -\omega$$

$$F \in \Omega^2 : **F = -F \quad f \in \Omega^0 : **f = -f$$

$$B \in \Omega^3 : **B = B$$

3) $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = E_x dt \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz - B_x dy \wedge dz - B_y dz \wedge dx - B_z dx \wedge dy$ (Faraday 2-form)

$$*F = -B_x dt \wedge dx - B_y dt \wedge dy - B_z dt \wedge dz - E_x dy \wedge dz - E_y dz \wedge dx - E_z dx \wedge dy$$

$$j = \delta dt - j_x dx - j_y dy - j_z dz \quad (\text{current 3-form, or vice versa: current 1-form})$$

could use any definition; $*J = j$

$$J \stackrel{\text{def}}{=} *j = \delta dx \wedge dy \wedge dz - j_x dt \wedge dy \wedge dz - j_y dt \wedge dz \wedge dx - j_z dt \wedge dx \wedge dy$$

Then:

$dF = 0$
$d^*F = 4\pi J$

$$\Rightarrow dJ = 0 \quad (\text{charge conservation})$$

Maxwell eqn-ns

4) $M = \mathbb{R}^{1,3}$ (or $M = \mathbb{R}^4$)

$$\omega \in \Omega^0 \cup \Omega^1 \cup \Omega^2 \cup \Omega^3 \cup \Omega^4 : \# \omega \text{ decomposable}$$

$$\omega \in \Omega^2 : \omega \text{ decomposable} \Leftrightarrow \omega \wedge \omega = 0 \quad (\text{proof on stack exchange; just direct computation})$$

$$\delta \stackrel{\text{def}}{=} (-1)^{n(k-1)+s+1} *d*$$

codifferential

$$\begin{aligned}\delta: \Omega^{k+1} &\rightarrow \Omega^k \\ d: \Omega^k &\rightarrow \Omega^{k+1}\end{aligned}$$

(wikipedia)

Properties:

$$1. (\eta, \delta\xi) = (d\eta, \xi), \quad \begin{matrix} \eta \in \Omega^k(M) \\ \xi \in \Omega^{k+1}(M) \end{matrix} \quad \boxed{(\alpha, \beta) \stackrel{\text{def}}{=} \int_M \alpha \wedge * \beta, \quad \alpha, \beta \in \Omega^k(M)}$$

(i.e. δ and d are adjoint operators)

Proof:

$$\begin{aligned}\int_M d(\eta \wedge * \xi) &= \int_M (d\eta \wedge * \xi + (-1)^k \eta \wedge d* \xi) = \int_M (d\eta \wedge * \xi + (-1)^k \eta \wedge * d* \xi) = \\ &= \int_M (d\eta \wedge * \xi - (-1)^{k+1} \eta \wedge * \delta \xi) = (d\eta, \xi) - (\eta, \delta \xi) = 0\end{aligned}$$

↑
from Stokes' theorem

$$2. \delta^2 = 0 \quad (\text{from def. and } d^2 = 0)$$

$$\Delta \stackrel{\text{def}}{=} (\delta + d)^2 = \delta d + d\delta$$

Laplace-de Rham operator

$$1. (\Delta\eta, \xi) = (\eta, \Delta\xi)$$

$$2. (\Delta\eta, \eta) \geq 0$$

$$3. V = \mathbb{R}^3, \text{ then:}$$

$$\begin{array}{ll} df \leftrightarrow \nabla f & *d\vec{A} \leftrightarrow \nabla \times \vec{A} \\ \delta df \leftrightarrow \Delta f & \delta \vec{A} \leftrightarrow \nabla \cdot \vec{A} \end{array}$$