Computing Eigenvectors

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Computing Eigenvalues

Given a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

we can form the characteristic polynomial by computing the determinant of

$$\begin{bmatrix} \lambda - 1 & 2 & 3 \\ 0 & \lambda - 4 & 5 \\ 0 & 0 & \lambda - 6 \end{bmatrix}$$

In this case, since this matrix is in upper-triangular form, all other terms of the determinant vanish except for the first one, resulting in $(\lambda - 1)(\lambda - 4)(\lambda - 6) = 0$ which gives us the 3 eigenvalues.

Computing Eigenvectors

Once we have the eigenvalues, we plug them into the definition of eigenvalues and eigenvectors to compute the corresponding eigenvectors. We know from the definition of eigenvectors: $\mathbf{Ax} = \lambda\mathbf{x}$. For $\lambda = 1$, this implies:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For some vector whose components are (x_1, x_2, x_3). Solving this system of equations, we get $6x_3 = x_3$, $4x_2 + 5x_3 = x_2$, and $x_1 + 2x_2 + 3x_3 = x_1$. The first equation can be true only when $x_3 = 0$. This implies $x_2 = 0$ and x_1 can take any value. Customarily, we want the eigenvectors to be of unit length. So, this gives us an eigenvector of $(1, 0, 0)$ corresponding to $\lambda = 1$.

We can similarly solve for $\lambda = 6$. In that case, the equation becomes

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 \\ 6x_2 \\ 6x_3 \end{bmatrix}$$

Examining the 3rd equation, we note that x_3 can take any value and that will determine the rest of the values. From the 2nd equation, we get $x_2 = \frac{5}{2}x_3$. We can similarly solve $x_1 = \frac{8}{5}x_3$. Normalizing to unit length, we get the corresponding eigenvector as $(0.51, 0.80, 0.32)$.