Week 08: Connectivity

FanFly

April 26, 2020

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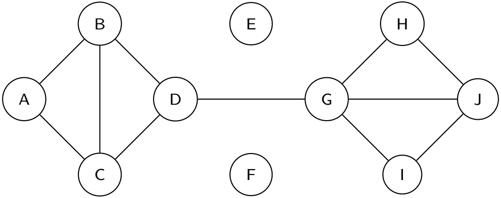
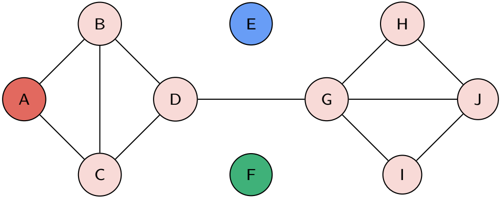


Connected Components (cont.)

For each connected components, we choose a representative to identify it.



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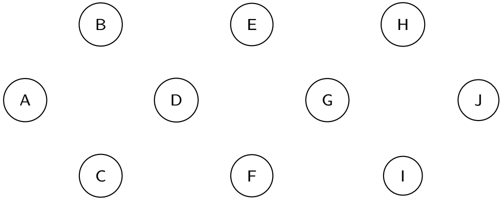


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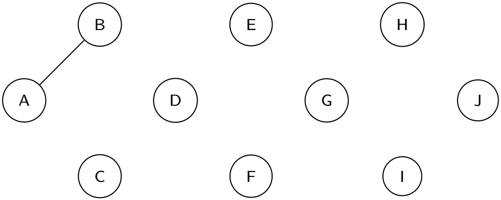


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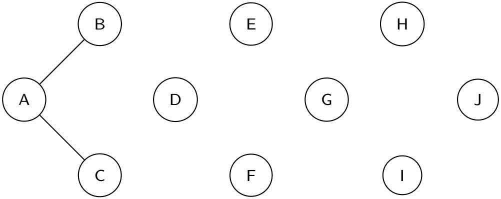


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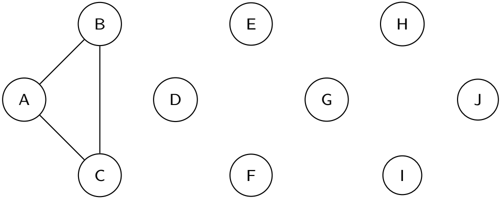


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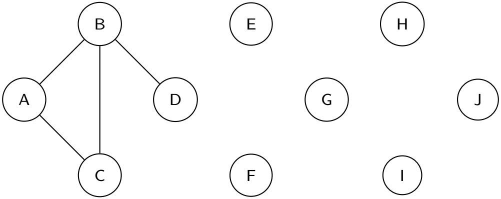


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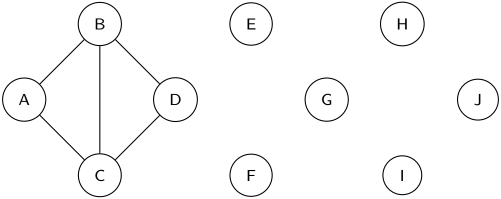


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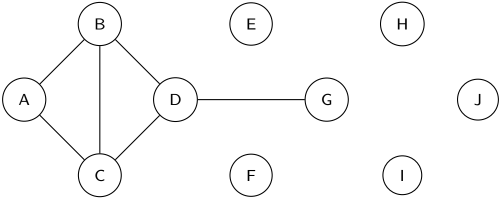


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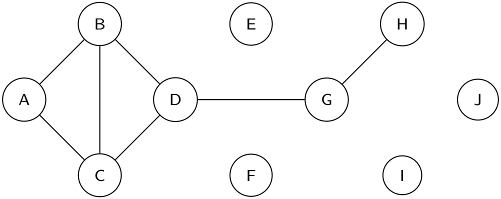


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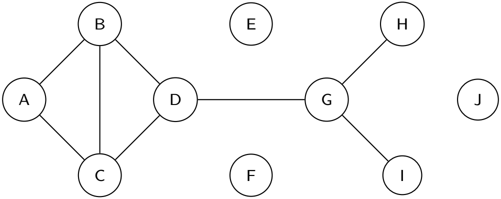


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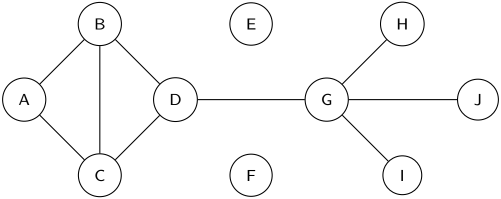


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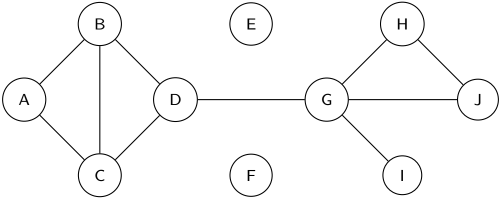


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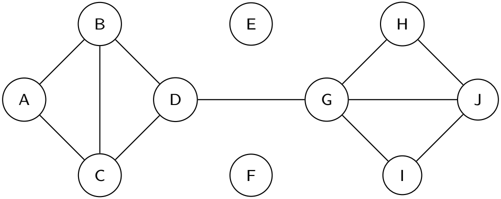


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In fact, the method `dfs()` does not do anything, but it traverses all the vertices exactly once in a specific order.

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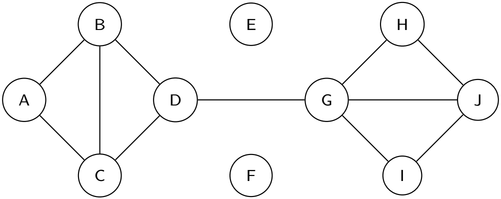

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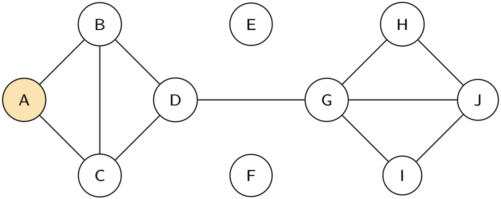

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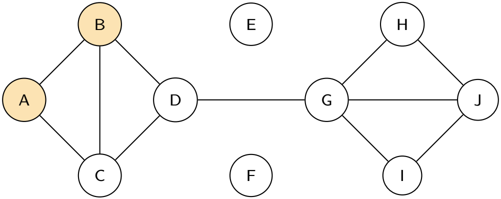

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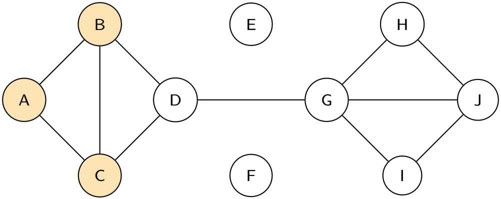

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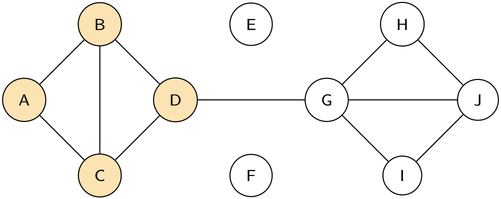

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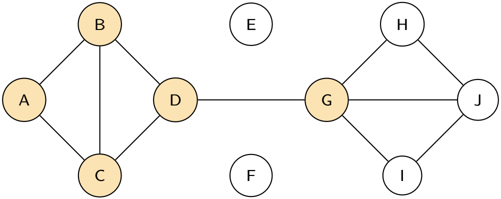

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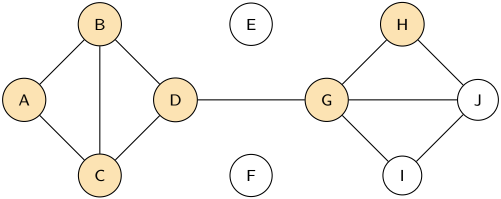

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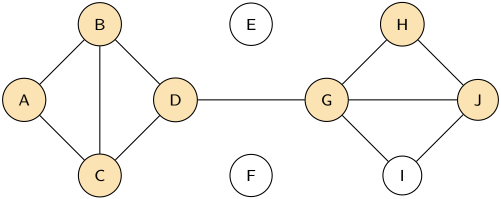

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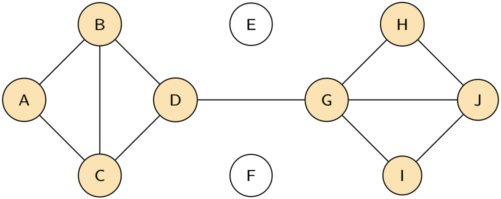

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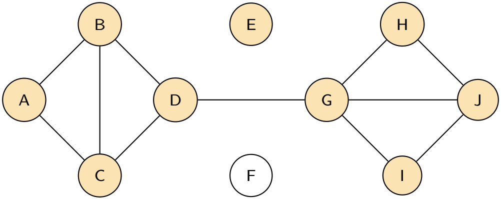
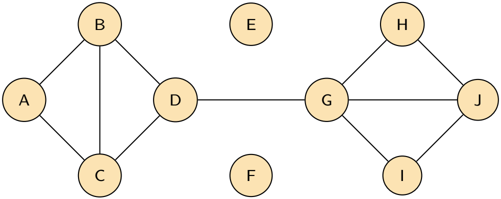
J ["G", "H", "I"]
```



Depth-First Search (cont.)

```
>>> for u in G.vertices:
...     print(u, G.adj[u])
...
A ["B", "C"]
B ["A", "C", "D"]
C ["A", "B", "D"]
D ["B", "C", "G"]
E []
F []
G ["D", "H", "I", "J"]
H ["G", "J"]
I ["G", "J"]
J ["G", "H", "I"]
```



Time Complexity of Depth-First Search

```python
class Graph:
    ...
    def dfs(self):
        visited = set()
        # A recursive function used by depth first search
        def dfs_visit(u):
            visited.add(u)
            for v in self.adj[u]:
                if v not in visited:
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        # Start depth first search
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- During the execution of `dfs_visit()` for vertex u, the `for` loop runs in $\Theta(d(u))$ time.
- Thus, the overall running time is $\Theta(n + m)$.

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- Thus, with little revision we can use `dfs()` to label the vertices with the connected component they belong to.

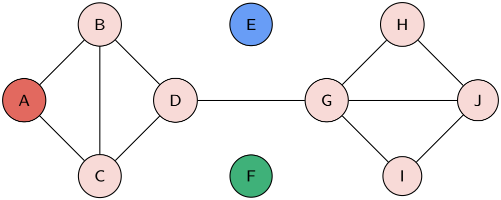
Finding Connected Components (cont.)

We use a dictionary `label` to store the representative of the connected component that a vertex belongs to.

```
class Graph:
    ...
    def dfs(self):
      visited = set()
      label = {}
      # A recursive function used by depth first search
      def dfs_visit(u, r):
          label[u] = r
          visited.add(u)
          for v in self.adj[u]:
              if v not in visited:
                  dfs_visit(v, r)
      # Start depth first search
      for u in self.vertices:
          if u not in visited:
              dfs_visit(u, u)
      return label
```

Finding Connected Components (cont.)

```
>>> label = G.dfs()
>>> for u in G.vertices:
...     print(u, label[u])
...
A A
B A
C A
D A
E E
F F
G A
H A
I A
J A
```



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Adjacency List with Label

- Preprocess: $O(n + m)$ time.
- Check if u and v are connected: $O(1)$ time.

Iterative Method

In fact, we can use a **stack** to eliminate recursion!

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```
class Graph:
    ...
    def dfs(self):
        visited = set()
        # A non-recursive function used by depth first search
        def dfs_visit(u):
            stack = [u]
            while stack:
                u = stack.pop()
                if u not in visited:
                    visited.add(u)
                    for v in self.adj[u]:
                        stack.append(v)
        # Start depth first search
        for u in self.vertices:
            if u not in visited:
                dfs_visit(u)
```

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- Output: The number of connected components of G.

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Solution

An $O(n+m)$-time depth-first search can solve the connected component problem. Thus, its time complexity is $\Theta(n+m)$ since the input size is $\Omega(n+m)$.

Exercise #2

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- m should not be less than n.
- If u and v are adjacent and v and w are adjacent, then u and w are adjacent.
- If u and v are connected and v and w are connected, then u and w are connected.

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- m should not be less than n.
- If u and v are adjacent and v and w are adjacent, then u and w are adjacent.
- If u and v are connected and v and w are connected, then u and w are connected.

Solution

Only the last statement is correct.

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Let $G = (V, E)$ with $|V| = n$ and $|E| = m$.

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$$A_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

and

$$C_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are connected} \\ 0, & \text{otherwise.} \end{cases}$$

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Solution

$C_{ij} = 1$ if and only if $B_{ij} \geq 1$.