Week 07: Graphs

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The Seven Bridges of Königsberg

The **Seven Bridges of Königsberg** is considered to be the first problem of graph theory.

In this problem, people would like to know whether a citizen can take a walk through the town in such a way that each bridge would be crossed exactly once.

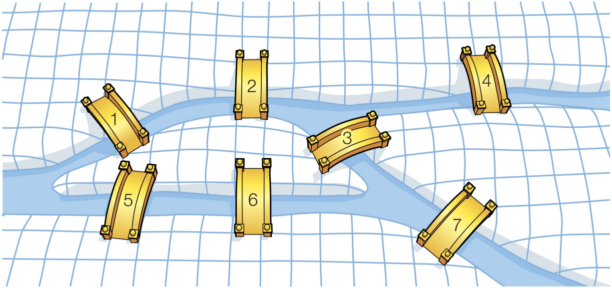
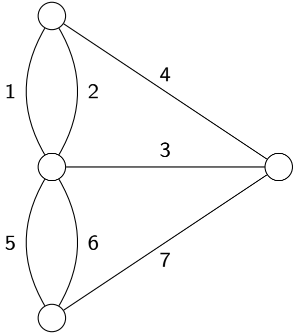


Figure: The Seven Bridges of Köingsberg

Abstraction

The towns and bridges can be seen as vertices and edges in a graph.



Graphs

In the following weeks, we are going to introduce some problems related to graphs.

The definition of a graph is as follows.

Definition

A **simple graph** is a pair $G = (V, E)$, where each component is as follows.

- V is a finite collection of **vertices**.
- E is a collection of **edges**, where each edge is of the form $e = \{u, v\}$ for some $u, v \in V$.

If we allow a graph to have more than one edges that have the same endpoints, then the graph is called a **multigraph**.

Eulerian Path Problem

Now we can formalize the Seven Bridges of Königsberg with new terminology.

- A **path** is a sequence of edges that joins a sequence of vertices.
- An **Eulerian path** is a path visiting each edge exactly once.

Then a walk through the town in such a way that each bridge would be crossed exactly once is exactly an Eulerian path.

Thus, we have the following problem.

Eulerian Path Problem

- Input: A multigraph $G = (V, E)$.
- Output: Whether G has an Eulerian path or not.

The Solution to the Eulerian Path Problem

In fact, we have the following theorem, which simply solves the Eulerian path problem.

Theorem

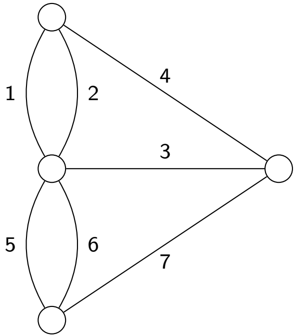
Let $G = (V, E)$ be a multigraph.

- If each vertex in G has an even degree, then G has an Eulerian path that starts and ends on the same vertex.
- If there are exactly two vertices of odd degree, then G has an Eulerian path that starts on one of them and ends at the other.
- If there are more than two vertices of odd degree, then G has no Eulerian path.

The **degree** of a vertex is the number of its incident edges.

The Solution to the Eulerian Path Problem (cont.)

Thus, no one can take a walk through the town in such a way that each bridge would be crossed exactly once.



Representation of Graphs

Now we can solve the Eulerian path problem.

But how can we “store” a graph in an computer? We need a **data structure**!

We will focus on the efficiency of the following operations.

- Initialize.
- Check if an edge exists.
- List all neighbors of a vertex.
- List all edges.

Attempt #1: Edge List

Edge List

```
n = 6
edges = [[0, 1], [0, 2], [1, 3], [2, 3], [4, 5]]
```

Attempt #2: Adjacency List

Adjacency List

```
n = 6
neighbor = [
    [1, 2],     # neighbor of vertex 0
    [0, 3],     # neighbor of vertex 1
    [0, 3],     # neighbor of vertex 2
    [1, 2],     # neighbor of vertex 3
    [5],        # neighbor of vertex 4
    [4]         # neighbor of vertex 5
]
```

Attempt #3: Adjacency Matrix

Adjacency Matrix

```
n = 6
matrix = [
    [False, True, True, False, False, False],
    [True, False, False, True, False, False],
    [True, False, False, True, False, False],
    [False, True, True, False, False, False],
    [False, False, False, False, False, True],
    [False, False, False, False, True, False]
]
```

Exercise #1

A **complete graph** is a graph in which each pair of vertices is connected by an edge.

What is the number of edges of an n-vertex complete graph?
(By the way, the n-vertex complete graph is denoted by K_n).

Solution

There are $n(n-1)/2$ edges in K_n.

Exercise #2

Let $G = (V, E)$ be a graph with $|V| = n$.

How much time does it take to find the degree of a vertex in V if G is stored as an adjacency matrix?

Solution

It takes $\Theta(n)$ time.

Exercise #3

A **coloring** of a graph is a labeling of vertices with colors such that no two adjacent vertices have the same color.

What is the minimum number of colors needed to color K_5?

Solution

We need 5 colors to color K_5.