Week 02: Sorting

 ${\sf FanFly}$

March 15, 2020

Time Complexity of a Problem

The time complexity T(n) of a problem P is the time complexity of the "best" algorithm that solves P.

- We have T(n) = O(f(n)) if there is an algorithm that solves P in O(f(n)) time.
- We have $T(n) = \Omega(f(n))$ if any algorithm that solves P needs $\Omega(f(n))$ time.

Week 02: Sorting March 15, 2020 2/16

The Primality Test Problem

The Primality Test Problem

- Input: An *n*-bit positive integer $m \ge 2$.
- Output: True if *m* is prime, false if *m* is composite.
- Last week, we found an $O(2^{n/2})$ -time algorithm that solves the primality test problem.
- An $O(n^{12}(\log_2 n)^{\epsilon})$ -time algorithm, called the AKS primality test, was proposed in 2002. (ϵ is a positive number.)
- In 2005, it is improved to run in $O(n^6(\log_2 n)^{\epsilon})$ time.
- Thus, now we know that the primality test problem can be solved in $O(n^6(\log_2 n)^{\epsilon})$ time, but no one knows if there is a faster algorithm than the ones above.

FanFly Week 02: Sorting March 15, 2020 3 / 16

The Champion Problem

The Champion Problem

- Input: An array A of n numbers (indexed from 0 to n-1).
- Output: An index i such that $A[i] \ge A[j]$ for any index j.

There is a O(n)-time algorithm that solves the problem as follows.

Also, note that we need at least n-1 comparisons to solve the problem, implying that the time complexity of the problem is $\Omega(n)$.

Thus, we have found an **optimal** algorithm for the champion problem.

The Sorting Problem

The Sorting Problem

- Input: An array A of n numbers (indexed from 0 to n-1).
- Output: A non-decreasing array that is reordered from A.

We have known that there is an algorithm, called merge sort, that can solve the sorting problem.

FanFly Week 02: Sorting March 15, 2020 5 / 16

Merging Sorted Arrays

First, we propose an algorithm merging two sorted arrays P and Q into a sorted array A.

```
def merge(P, Q, i, j):
    if j == len(Q):
        return P
    elif i == len(P):
        return Q
    else:
        if P[i] <= Q[j]:
            return [P[i]] + merge(P, Q, i + 1, j)
        else:
            return [Q[j]] + merge(P, Q, i, j + 1)</pre>
```



It can be shown that the algorithm runs in $\Theta(n)$ time.

Merge Sort

Then we can sort the array by repeating merging its subarrays.

```
def merge_sort(A, 1, r):
    if r - 1 <= 1:
        return a[1:r]
    else:
        m = (1 + r) // 2
        P = merge_sort(A, 1, m)
        Q = merge_sort(A, m, r)
        return merge(P, Q, 0, 0)</pre>
```

The time complexity of merge sort is

$$T(n) = egin{cases} O(1), & ext{if } n \leq 1 \ 2T(n/2) + \Theta(n), & ext{otherwise}. \end{cases}$$

What is the exact time complexity of merge sort?

FanFly Week 02: Sorting March 15, 2020 7 / 16

Master Theorem

Theorem (Master Theorem)

Let T(n) be a positive function satisfying the following recurrence relation.

$$T(n) = egin{cases} O(1), & \text{if } n \leq 1 \ aT(n/b) + M(n), & \text{otherwise}. \end{cases}$$

Let $c = \log_b a$.

- If $M(n) = O(n^k)$ with k < c, then $T(n) = \Theta(n^c)$.
- If $M(n) = \Theta(n^k)$ with k = c, then $T(n) = \Theta(n^c \log_2 n)$.
- If $M(n) = \Omega(n^k)$ with k > c, then $T(n) = \Theta(M(n))$.

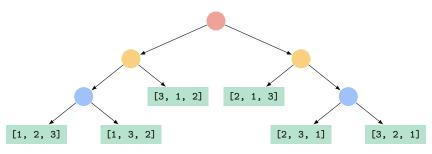
Thus, the time complexity of merge sort is $T(n) = \Theta(n \log_2 n)$ since

$$T(n) = egin{cases} O(1), & ext{if } n \leq 1 \ 2T(n/2) + \Theta(n), & ext{otherwise}. \end{cases}$$

FanFly Week 02: Sorting March 15, 2020 8 / 16

Lower Bound of Comparison-Based Sorting

Any comparison-based algorithm can be seen as a binary tree.



Since there are n! leaves, we know that the height of the binary tree is at least

$$h = \log_2(n!),$$

implying that any comparison-based algorithm runs in $\Omega(\log_2(n!))$ time.

FanFly Week 02: Sorting March 15, 2020 9 / 16

Lower Bound of Comparison-Based Sorting (cont.)

Note that

$$\log_{2}(n!) = \log_{2}(n \times (n-1) \times \cdots \times 2 \times 1)$$

$$\geq \log_{2}\left(n \times (n-1) \times \cdots \times \left\lceil \frac{n+1}{2} \right\rceil\right)$$

$$\geq \frac{n}{2}\log_{2}\left(\frac{n}{2}\right)$$

$$= \frac{n}{2}(\log_{2}n - 1)$$

$$= \Omega(n\log_{2}n).$$

Thus, any comparison-based algorithm runs in $\Omega(n \log_2 n)$ time.

FanFly Week 02: Sorting March 15, 2020 10 / 16

Conclusion

The Sorting Problem

- Input: An array A of n numbers (indexed from 0 to n-1).
- Output: A non-decreasing array that is reordered from A.

For any computational problem, the final goal is to find an optimal algorithm that solves the problem.

- The merge sort algorithm runs in $O(n \log_2 n)$ time.
- Any comparison-based algorithm that solves the sorting problem must run in $\Omega(n \log_2 n)$ time.
- Thus, the merge sort algorithm is an optimal comparison-based algorithm that solves the sorting problem.

FanFly Week 02: Sorting March 15, 2020 11 / 16

Exercise #1

We have learned the binary search algorithm, which can search a value in a sorted array.

```
def binary_search(A, val, 1, r):
    if val <= A[1]:
        return 1
    elif val > A[r - 1]:
        return r
    else:
        m = (1 + r) // 2
        if val <= A[m]:
            return binary_search(A, val, 1, m)
        else:
            return binary_search(A, val, m + 1, r)</pre>
```

FanFly Week 02: Sorting March 15, 2020 12 / 16

Exercise #1 (cont.)

It can be shown that the time complexity of binary search is

$$T(n) = \begin{cases} O(1), & \text{if } n \leq 1 \\ T(n/2) + O(1), & \text{otherwise.} \end{cases}$$

Please find the exact time complexity of binary search using the master theorem.

Theorem (Master Theorem)

Let T(n) be a positive function satisfying the following recurrence relation.

$$T(n) = \begin{cases} O(1), & \text{if } n \leq 1\\ aT(n/b) + M(n), & \text{otherwise.} \end{cases}$$

Let $c = \log_b a$.

- If $M(n) = O(n^k)$ with k < c, then $T(n) = \Theta(n^c)$.
- If $M(n) = \Theta(n^k)$ with k = c, then $T(n) = \Theta(n^c \log_2 n)$.
- If $M(n) = \Omega(n^k)$ with k > c, then $T(n) = \Theta(M(n))$.

FanFly Week 02: Sorting March 15, 2020 13 / 16

Exercise #1 (cont.)

Solution

Let $c = \log_2 1 = 0$ and M(n) = O(1). Since $M(n) = \Theta(n^c)$, we have

$$T(n) = \Theta(n^c \log_2 n) = \Theta(\log_2 n),$$

which means that binary search runs in logarithmic time.

FanFly Week 02: Sorting March 15, 2020 14 / 16

Exercise #2

Suppose that f and g are functions such that

$$f(n) = \Theta(\log_2 n)$$
 and $g(n) = \Theta(\log_e n)$,

and we know the fact that 2.718 < e < 2.719.

Which of the following is true?

- f(n) = O(g(n)) and $f(n) \neq \Omega(g(n))$.
- $f(n) = \Theta(g(n))$.
- $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$.

Solution

We have $f(n) = \Theta(g(n))$ since

$$\frac{\log_e n}{\log_2 n} = \log_e 2.$$

(From now on we'll use $O(\log n)$ instead of $O(\log_k n)$ to represent logarithm.)

FanFly Week 02: Sorting March 15, 2020 15 / 16

Exercise #3

Please find the time complexity of the following algorithm using big-O notation.

```
def magic_sort(A):
    P = list(A)
    Q = []
    while P:
        Q.append(P.pop(index_min(P)))
    return Q
```

We have the following assumption, where n is the length of 1s.

- index_min(ls) returns the index of the smallest item in ls in O(n) time.
- ls.append(val) adds val to the end of ls in O(1) time.
- ls.pop(i) removes the item with index i in ls in O(n) time.

Solution

The magic sort runs in $O(n^2)$ time.

FanFly Week 02: Sorting March 15, 2020 16 / 16