Week 08: Connectivity

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Data Structures

Let us review some data structures we have learned.

List

- Create: $O(1)$ time (amortized).
- Search: $O(n)$ time.
- Update: $O(1)$ time.
- Delete: $O(n)$ time.

Dictionary / Set

- Create: $O(1)$ time (average-case).
- Search: $O(1)$ time (average-case).
- Update: $O(1)$ time.
- Delete: $O(1)$ time.

Data Structures for Graphs

We learned adjacency lists and adjacency matrix last week.

Adjacency List

- Preprocess: $O(n + m)$ time.
- Check if vertices u and v are adjacent: $O(d(u))$ time.

Adjacency Matrix

- Preprocess: $O(n^2)$ time.
- Check if vertices u and v are adjacent: $O(1)$ time.

Connectivity

In a graph $G = (V, E)$, vertices u and v are said to be **connected** if there is a path from u to v.

- Also, we say that a graph is **connected** if any two vertices in the graph are connected.

Today, we want to build a data structure satisfying the following properties.

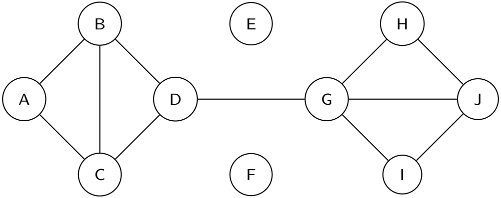
A Mysterious Data Structure

- Preprocess: $O(n + m)$ time.
- Check if vertices u and v are connected: $O(1)$ time.

Connected Components

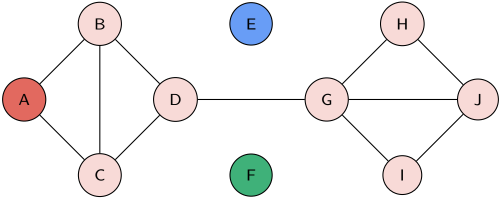
We want to label the vertices with the **connected components** they belong to.

- A connected component of a graph is a maximal set of vertices such that each pair of vertices are connected.



Connected Components (cont.)

For each connected components, we choose a representative to identify it.



The Graph Class

We are going to introduce a very useful "algorithm", which is called **depth-first search**.

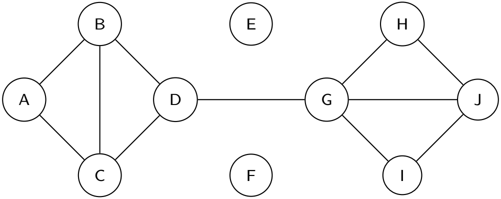
First we construct a class in Python called `Graph`.

```
class Graph:
    def __init__(self, vertices):
        self.vertices = list(vertices)
        self.adj = {u: [] for u in vertices}

    def add_edge(self, u, v):
        self.adj[u].append(v)
        self.adj[v].append(u)
```

The Graph Class (cont.)

For example, we can represent the graph as follows.




```
>>> G = Graph(["A", "B", "C", "D", "E", "F", "G", "H", "I", "J"])
>>> G.add_edge("A", "B")
>>> G.add_edge("A", "C")
>>> G.add_edge("B", "C")
>>> G.add_edge("B", "D")
>>> G.add_edge("C", "D")
>>> G.add_edge("D", "G")
>>> G.add_edge("G", "H")
>>> G.add_edge("G", "I")
>>> G.add_edge("G", "J")
>>> G.add_edge("H", "J")
>>> G.add_edge("I", "J")
```

Depth-First Search

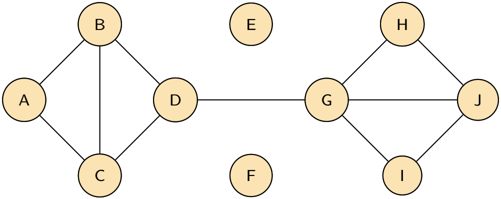
Now we introduce the depth-first search algorithm.

```
class Graph:
    ...
    def dfs(self):
        visited = set()
        # A recursive function used by depth first search
        def dfs_visit(u):
            visited.add(u)
            for v in self.adj[u]:
                if v not in visited:
                    dfs_visit(v)
        # Start depth first search
        for u in self.vertices:
            if u not in visited:
                dfs_visit(u)
```

In fact, the method `dfs()` does not do anything, but it traverses all the vertices exactly once in a specific order.

Depth-First Search (cont.)

```
>>> for u in G.vertices:
...     print(u, G.adj[u])
...
A ["B", "C"]
B ["A", "C", "D"]
C ["A", "B", "D"]
D ["B", "C", "G"]
E []
F []
G ["D", "H", "I", "J"]
H ["G", "J"]
I ["G", "J"]
J ["G", "H", "I"]
```



Time Complexity of Depth-First Search

```
class Graph:
    ...
    def dfs(self):
        visited = set()
        # A recursive function used by depth first search
        def dfs_visit(u):
            visited.add(u)
            for v in self.adj[u]:
                if v not in visited:
                    dfs_visit(v)
        # Start depth first search
        for u in self.vertices:
            if u not in visited:
                dfs_visit(u)
```

What is the time complexity of depth-first search?

- The function `dfs_visit()` is called exactly once for each vertex.
- During the execution of `dfs_visit()` for vertex u, the `for` loop runs in $\Theta(d(u))$ time.
- Thus, the overall running time is $\Theta(n + m)$.

Finding Connected Components

Depth-first search can be used to find connected components.

- When `dfs()` calls `dfs_visit()`, a connected component is found.
- Thus, with little revision we can use `dfs()` to label the vertices with the connected component they belong to.

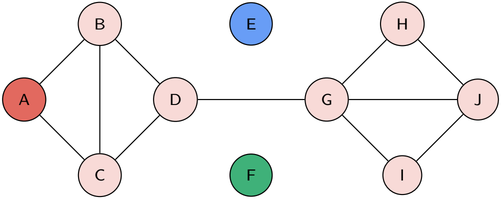
Finding Connected Components (cont.)

We use a dictionary `label` to store the representative of the connected component that a vertex belongs to.

```python
class Graph:
    ...
    def dfs(self):
      visited = set()
      label = {}
      # A recursive function used by depth first search
      def dfs_visit(u, r):
          label[u] = r
          visited.add(u)
          for v in self.adj[u]:
              if v not in visited:
                  dfs_visit(v, r)
      # Start depth first search
      for u in self.vertices:
          if u not in visited:
              dfs_visit(u, u)
      return label
```

Finding Connected Components (cont.)

```
>>> label = G.dfs()
>>> for u in G.vertices:
...     print(u, label[u])
...
A A
B A
C A
D A
E E
F F
G A
H A
I A
J A
```



Conclusion

With depth-first search we can know the connected component a vertex belongs to, and thus we have the following data structure.

Adjacency List with Label

- Preprocess: $O(n + m)$ time.
- Check if u and v are connected: $O(1)$ time.

Iterative Method

In fact, we can use a **stack** to eliminate recursion!

```python
class Graph:
    ...
    def dfs(self):
        visited = set()
        # A non-recursive function used by depth first search
        def dfs_visit(u):
            stack = [u]
            while stack:
                u = stack.pop()
                if u not in visited:
                    visited.add(u)
                    for v in self.adj[u]:
                        stack.append(v)
        # Start depth first search
        for u in self.vertices:
            if u not in visited:
                dfs_visit(u)
```

Exercise #1

Consider the following problem.

Connected Component Problem

- Input: A graph G with n vertices and m edges.
- Output: The number of connected components of G.

What is the time complexity of the connected component problem?

- $\Theta(n+m)$
- $\Theta(n^2)$
- $\Theta((n+m)\log n)$
- $\Theta(2^n)$

Solution

An $O(n+m)$-time depth-first search can solve the connected component problem. Thus, its time complexity is $\Theta(n+m)$ since the input size is $\Omega(n+m)$.

Exercise #2

Let $G = (V, E)$ with n vertices and m edges. Let u, v, w be vertices in G. Which of the following statements is true?

- A vertex can have at most n neighbors.
- m should not be less than n.
- If u and v are adjacent and v and w are adjacent, then u and w are adjacent.
- If u and v are connected and v and w are connected, then u and w are connected.

Solution

Only the last statement is correct.

Exercise #3

Let $G = (V, E)$ with $|V| = n$ and $|E| = m$. Furthermore, let A and C be $n \times n$ matrices such that

$$A_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

and

$$C_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are connected} \\ 0, & \text{otherwise.} \end{cases}$$

Let $B = I + A + A^2 + \cdots + A^{n-2} + A^{n-1}$.
What is the relation between B and C?

Solution

$C_{ij} = 1$ if and only if $B_{ij} \geq 1$.