Week 02: Sorting

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- We have $T(n) = O(f(n))$ if there is an algorithm that solves P in $O(f(n))$ time.
- We have $T(n) = \Omega(f(n))$ if any algorithm that solves P needs $\Omega(f(n))$ time.

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- In 2005, it is improved to run in $O(n^{6}(\log_2 n)^{\epsilon})$ time.
- Thus, now we know that the primality test problem can be solved in $O(n^{6}(\log_2 n)^{\epsilon})$ time, but no one knows if there is a faster algorithm than the ones above.

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Thus, we have found an **optimal** algorithm for the champion problem.

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    if j == len(Q):
        return P
    elif i == len(P):
        return Q
    else:
        if P[i] <= Q[j]:
            return [P[i]] + merge(P, Q, i + 1, j)
        else:
            return [Q[j]] + merge(P, Q, i, j + 1)
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P	2	3	5	7	11	13

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It can be shown that the algorithm runs in $\Theta(n)$ time.

Merge Sort

Then we can sort the array by repeating merging its subarrays.

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def merge_sort(A, l, r):
    if r - l <= 1:
        return a[l:r]
    else:
        m = (l + r) // 2
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The time complexity of merge sort is

$$T(n) = \begin{cases} O(1), & \text{if } n \leq 1 \\ 2T(n/2) + \Theta(n), & \text{otherwise.} \end{cases}$$

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What is the exact time complexity of merge sort?

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Let $T(n)$ be a positive function satisfying the following recurrence relation.

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Let $c = \log_b a$.

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Thus, the time complexity of merge sort is $T(n) = \Theta(n \log_2 n)$ since

$$T(n) = \begin{cases} O(1), & \text{if } n \le 1 \\ 2T(n/2) + \Theta(n), & \text{otherwise.} \end{cases}$$

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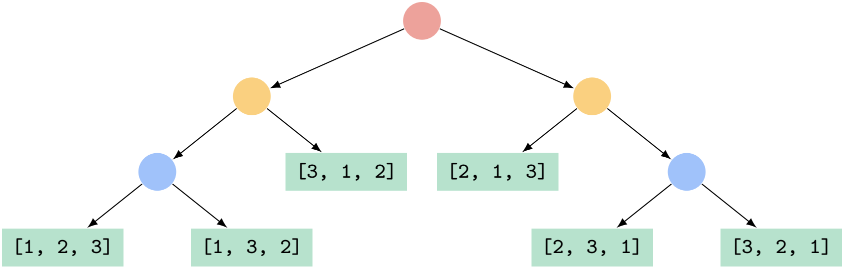

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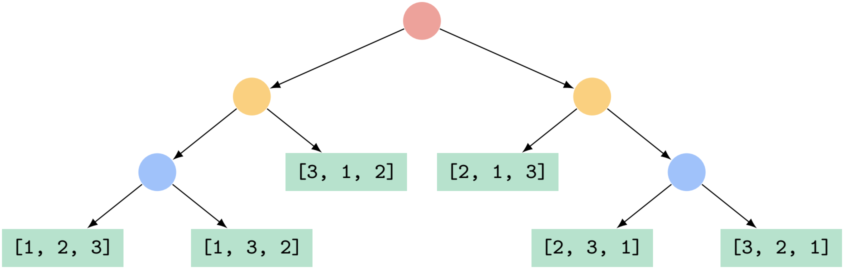

Since there are $n!$ leaves, we know that the height of the binary tree is at least

$$h = \log_2(n!),$$

implying that any comparison-based algorithm runs in $\Omega(\log_2(n!))$ time.

Lower Bound of Comparison-Based Sorting (cont.)

Note that

$$
\begin{aligned}
\log_2(n!) &= \log_2(n \times (n-1) \times \cdots \times 2 \times 1) \\
&\geq \log_2\left(n \times (n-1) \times \cdots \times \left\lceil \frac{n+1}{2} \right\rceil\right) \\
&\geq \frac{n}{2} \log_2\left(\frac{n}{2}\right) \\
&= \frac{n}{2}(\log_2 n - 1) \\
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Thus, any comparison-based algorithm runs in $\Omega(n \log_2 n)$ time.

Conclusion

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- Input: An array A of n numbers (indexed from 0 to $n-1$).
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- Any comparison-based algorithm that solves the sorting problem must run in $\Omega(n \log_2 n)$ time.

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- The merge sort algorithm runs in $O(n \log_2 n)$ time.
- Any comparison-based algorithm that solves the sorting problem must run in $\Omega(n \log_2 n)$ time.
- Thus, the merge sort algorithm is an optimal comparison-based algorithm that solves the sorting problem.

Exercise #1

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```python
def binary_search(A, val, l, r):
    if val <= A[l]:
        return l
    elif val > A[r - 1]:
        return r
    else:
        m = (l + r) // 2
        if val <= A[m]:
            return binary_search(A, val, l, m)
        else:
            return binary_search(A, val, m + 1, r)
```

Exercise #1 (cont.)

It can be shown that the time complexity of binary search is

$$T(n) = \begin{cases} O(1), & \text{if } n \leq 1 \\ T(n/2) + O(1), & \text{otherwise.} \end{cases}$$

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Please find the exact time complexity of binary search using the master theorem.

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Let $c = \log_b a$.

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Exercise #1 (cont.)

Solution

Let $c = \log_2 1 = 0$ and $M(n) = O(1)$. Since $M(n) = \Theta(n^c)$, we have

$$T(n) = \Theta(n^c \log_2 n) = \Theta(\log_2 n),$$

which means that binary search runs in logarithmic time.

Exercise #2

Suppose that f and g are functions such that

$$f(n) = \Theta(\log_2 n) \quad \text{and} \quad g(n) = \Theta(\log_e n),$$

and we know the fact that $2.718 < e < 2.719$.

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Solution

We have $f(n) = \Theta(g(n))$ since

$$\frac{\log_e n}{\log_2 n} = \log_e 2.$$

(From now on we'll use $O(\log n)$ instead of $O(\log_k n)$ to represent logarithm.)

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Solution

The magic sort runs in $O(n^2)$ time.