Graph Theory

Chapter 1

Graphs

1.1 Graphs

Definition 1.1. A **graph** is a pair

$$G = (V, E),$$

where V is a finite set of **vertices**, and E is a set of **edges**, which are 2-element subsets of V.

- The elements of an edge is called its **endpoints**, and we say that an edge **joins** its endpoints.
- We say that two vertices are **adjacent** if they are joined by an edge.

Remark. An edge $\{u, v\}$ is usually written as uv.

1.2 Paths

Definition 1.2. A **walk** of a graph $G = (V, E)$ is a sequence

$$(v_0, e_1, v_1, \ldots, e_k, v_k),$$

where $v_0, v_1, \ldots, v_k \in V$ and $e_1, \ldots, e_k \in E$ such that $e_i = v_{i-1}v_i$ for each $i \in \{1, \ldots, k\}$. The number of edges in a walk is called the **length** of the walk.

- A **trail** is a walk with no repeated edges.
- A **circuit** is a trail with starting vertex and ending vertex being the same.
- A **path** is a walk with no repeated vertices.
- A **cycle** is a circuit with no repeated internal vertices.

1.3 Connectivity

Definition 1.3. A graph is **connected** if any two vertices can be linked by a path. A graph is **disconnected** if it is not connected.

Definition 1.4. Let $G = (V, E)$ be a graph. We say that a set $S \subseteq V$ **separates** G if $G[V \setminus S]$ is disconnected.

Definition 1.5. Let $G = (V, E)$ be a graph and let k be a nonnegative integer. We say that G is ***k*-connected** if $|V| > k$ and every subset S of V with $|S| < k$ does not separate G.

Chapter 2

Planar Graphs

2.1 The Jordan Polygon Theorem

Definition 2.1. A **line segment** is a set of the form

$$\{\lambda x + (1 - \lambda y) : 0 \leq \lambda \leq 1\},$$

where x and y are distinct points in $\mathbb{R}^2$, called the **endpoints** of the line segment.

Definition 2.2. A **polygonal curve** C is a union of line segments such that there is a continuous bijection $\gamma : [0, 1] \to C$. The points $\gamma(0)$ and $\gamma(1)$ are called the **endpoints** of C.

Definition 2.3. A **polygon** P is a union of line segments such that there is a continuous bijection $\gamma : S^1 \to P$, where

$$S^1 = \{(x, y) : x^2 + y^2 = 1\}.$$

Definition 2.4. We say that $S \subseteq \mathbb{R}^2$ is **connected** if for any $x, y \in S$, there exists a polygonal curve $C \subseteq S$ whose endpoints are x and y. Furthermore, we say that S is a **region** of $Q \subseteq \mathbb{R}^2$ if S is a maximal connected subset of Q.

Theorem 2.5 (Jordan Polygon Theorem). Let P be a polygon. Then $\mathbb{R}^2 \setminus P$ has exactly two regions.

2.2 Plane Graphs

Definition 2.6. A **plane graph** is a pair $G = (V, E)$, where each components are as follows.

- V consists of a finite number of points (called **vertices**) in $\mathbb{R}^2$.
- E consists of polygonal curves (called **edges**) that connects vertices, such that no two edges have the same set of endpoints, and the interior of any edge does not contain any vertex or any point of other edges.

A region of $\mathbb{R}^2 \setminus \bigcup E$ is called a **face** of G. The set of faces of G is denoted by $F(G)$.

Remark. A plane graph defines a graph in a natural way. Thus, we usually use the same notation for both a plane graph and its corresponding graph.