# Set Theory

# Chapter 1

# Axioms and Operations

## 1.1 Basic Axioms

For sets $x$ and $y$, we write $x \in y$ to say that $x$ is an element of $y$, and we write $x = y$ to say that $x$ and $y$ are equal. Furthermore, we define

$$\begin{aligned} x \notin y \quad &\Leftrightarrow \quad \neg(x \in y) \\ x \neq y \quad &\Leftrightarrow \quad \neg(x = y). \end{aligned}$$

**Axiom I (Extensionality).** Two sets are equal if they have exactly the same elements. Formally,

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y).$$

**Definition 1.1.** Let $x$ and $y$ be sets. We say that $x$ is a **subset** of $y$, denoted by $x \subseteq y$, if every element of $x$ belongs to $y$. Formally,

$$x \subseteq y \quad \Leftrightarrow \quad \forall z (z \in x \to z \in y).$$

Furthermore, $x$ is a **proper subset** of $y$, denoted by $x \subsetneq y$, if both $x \subseteq y$ and $x \neq y$ hold.

**Definition 1.2.** The **empty set**, denoted by $\varnothing$, is the set that has no elements.

**Axiom II (Pairing).** For any two sets $x$ and $y$, there is a set that consists of exactly $x$ and $y$. Formally,

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \lor w = y)).$$

**Definition 1.3.** The **pair set** of two sets $x$ and $y$, denoted by $\{x, y\}$, is the set that consists of exactly $x$ and $y$.

**Axiom III (Power Set).** For any set $x$, there is a set whose memebers are exactly the subsets of $x$. Formally,

$$\forall x \exists y \forall z (z \subseteq x \leftrightarrow z \in y).$$

**Definition 1.4.** The **power set** of a set $x$, denoted by $\mathcal{P}(x)$, is the set that consists of exactly the subsets of $x$.

**Axiom IV (Separation Scheme).** Let $\phi(z)$ be a formula. For any set $x$, there exists a set $y$ such that for any set $z$, we have $z \in y$ if and only if both $z \in x$ and $\phi(z)$ hold. Formally,

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \land \phi(z)).$$

**Definition 1.5.** Let $x, y$ be sets and let $\phi(z)$ be a formula. If for any set $z$, we have $z \in y$ if and only if $z \in x$ and $\phi(z)$, then we write

$$y = \{z \in x : \phi(z)\}.$$

**Definition 1.6.** For sets $x$ and $y$, we define

$$x \setminus y = \{z \in x : z \notin y\}.$$

**Theorem 1.7.** There is no set to which every set belongs. Formally,

$$\forall x \exists y (y \notin x).$$

*Proof.* Let $x$ be a set and let $y = \{z \in x : z \notin z\}$. Then

$$y \in y \quad \Leftrightarrow \quad y \in x \land y \notin y.$$

If $y \in x$, then

$$y \in y \quad \Leftrightarrow \quad y \notin y,$$

contradiction. Thus $y \notin x$, which completes the proof. $\square$

**Axiom V (Union).** For any set $x$, there exists a set whose elements are exactly the elements of the elements of $x$. Formally,

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \land z \in w)).$$

**Definition 1.8.** Let $x$ be a set.

- We define the **union** of $x$, denoted by $\bigcup x$, to be the set that consists of the sets that belongs to at least one element of $x$. Formally, for any set $z$ we have

$$z \in \bigcup x \quad \Leftrightarrow \quad \exists w (w \in x \land z \in w).$$

- If $x$ is nonempty, we define the **intersection** of $x$, denoted by $\bigcap x$, to be the set that consists of the sets that belongs to all elements of $x$. Formally, for any set $z$ we have

$$z \in \bigcap x \quad \Leftrightarrow \quad \forall w (w \in x \to z \in w).$$

For sets $u$ and $v$, we define

$$u \cup v = \bigcup \{u, v\} \qquad \text{and} \qquad u \cap v = \bigcap \{u, v\}.$$

# Chapter 2

# Relations and Functions

## 2.1 Ordered Pairs

**Definition 2.1.** For sets $x$ and $y$, we define

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

**Lemma 2.2.** Let $x, y, y'$ be sets. If $\{x, y\} = \{x, y'\}$, then $y = y'$.

*Proof.* Suppose that $y \neq y'$. Since $y \in \{x, y\} = \{x, y'\}$ and $y \neq y'$, we have $y = x$. Then we have $y' \in \{x, y'\} = \{x, y\} = \{x\}$, implying $y' = x = y$, contradiction. Thus, $y = y'$. □

**Theorem 2.3.** For sets $x, x', y, y'$, we have

$$\langle x, y\rangle = \langle x', y'\rangle$$

if and only if $x = x'$ and $y = y'$.

*Proof.* ($\Leftarrow$) Straightforward. ($\Rightarrow$) Suppose that $x \neq x'$. Since

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\},$$

either $\{x\} = \{x', y'\}$ or $\{x\} = \{x'\}$ holds. For both cases we all have $x' \in \{x\}$, implying $x' = x$, contradiction. Hence we have $x = x'$, and it follows that $\{x\} = \{x'\}$, implying $\{x, y\} = \{x', y'\}$, and thus $y = y'$. □

**Lemma 2.4.** If $x, y \in C$, then $\langle x, y\rangle \in \mathcal{P}(\mathcal{P}(C))$.

*Proof.* Since $\{x\}$ and $\{y\}$ are subsets of $C$, we have $\{x\}, \{x, y\} \in \mathcal{P}(C)$. It follows that $\{\{x\}, \{x, y\}\}$ is a subset of $\mathcal{P}(C)$, implying

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(C)).$$ □

**Theorem 2.5.** For any sets $A$ and $B$, there is a set whose members are exactly the pairs $(x, y)$ with $x \in A$ and $y \in B$.

*Proof.* Since $x, y \in A \cup B$, the set of pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$ can be given by

$$\{z \in \mathcal{P}(\mathcal{P}(A \cup B)) : z = \langle x, y\rangle \text{ for some } x \in A \text{ and } y \in B\}.$$ □

**Definition 2.6.** For any sets $A$ and $B$, the **Cartesian product** of $A$ and $B$, denoted by $A \times B$, is the set whose memebers are exactly the pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$.

## 2.2 Relations

**Definition 2.7.** A **relation** is a set of ordered pairs. For any relation $R$, the **domain** and the **range** of $R$, denoted by $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$, respectively, are defined as follows.

- $\mathrm{dom}(R)$ is the collection of sets $x$ with $\langle x, y \rangle \in R$ for some $y$.
- $\mathrm{ran}(R)$ is the collection of sets $y$ with $\langle x, y \rangle \in R$ for some $x$.

**Definition 2.8.** Let $R$ and $S$ be relations and let $X$ be a set.

- The **inverse** of $R$, denoted by $R^{-1}$, is the set of all pairs $\langle y, x \rangle$ with $\langle x, y \rangle \in R$.
- The **restriction** of $R$ to $X$, denoted by $R \upharpoonright X$, is the set of all pairs $\langle x, y \rangle \in R$ with $x \in X$.
- The **composition** of $R$ and $S$, denoted by $R \circ S$, is the set of all pairs $\langle x, z \rangle$ with $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in S$.

**Definition 2.9.** A **function** is a relation $f$ such that for any set $x \in \mathrm{dom}(f)$, there exists a unique set $y$ such that $\langle x, y \rangle \in f$. The unique set $y$ with respect to $x$ is called the **value** of $f$ at $x$ and is denoted $f(x)$.

- We say that $f$ is a function from $A$ to $B$, denoted by $f : A \to B$, if $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.
- $f$ is **one-to-one** if for any $y \in \mathrm{ran}(f)$, there exists a unique set $x \in \mathrm{dom}(f)$ with $f(x) = y$.

**Definition 2.10.** For any sets $A$ and $B$, the set of functions from $A$ to $B$ is denoted by $B^A$.

## 2.3 Equivalence Relations and Ordering Relations

**Definition 2.11.** Let $A$ be a set. An **equivalence relation** on $A$ is a relation $R \subseteq A \times A$ that satisfies the following three conditions.

- Reflexivity: $\langle x, x \rangle \in R$ for any $x \in A$.
- Symmetry: $\langle x, y \rangle \in R$ implies $\langle y, x \rangle \in R$ for any $x, y \in A$.
- Transitivity: $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ implies $\langle x, z \rangle \in R$ for any $x, y, z \in A$.

# Chapter 3

# Natural Numbers

## 3.1 Inductive Sets

**Definition 3.1.** The **successor** of a set $x$, denoted $x^+$, is defined by

$$x^+ = x \cup \{x\}.$$

**Definition 3.2.** A set $A$ is **inductive** if it has the empty set as member and is closed under successor. Formally,

$$A \text{ is inductive} \quad \Leftrightarrow \quad \varnothing \in A \land \forall x(x \in A \to x^+ \in A).$$

**Axiom VI (Infinity).** There exists an inductive set.

**Definition 3.3.** A **natural number** is a set $x$ that belongs to all inductive sets. The set of natural numbers is denoted by $\omega$. Formally,

$$x \in \omega \quad \Leftrightarrow \quad \forall A(A \text{ is inductive} \to x \in A)$$

**Theorem 3.4.** $\omega$ is inductive.

*Proof.* First, $\varnothing \in \omega$ since $\varnothing$ belongs to all inductive sets by definition. For any set $x \in \omega$, $x$ belongs to all inductive sets, implying that $x^+$ belongs to all inductive sets, and thus $x^+ \in \omega$. Thus, $\omega$ is inductive. $\square$

**Definition 3.5.** Let

$$0 = \varnothing, \quad 1 = \varnothing^+, \quad 2 = (\varnothing^+)^+, \quad 3 = ((\varnothing^+)^+)^+, \quad 4 = (((\varnothing^+)^+)^+)^+, \quad \ldots$$

denote the natural numbers.

## 3.2 Recursion

**Theorem 3.6 (Recursion Theorem).** For any sets $A$ and $e$ with $e \in A$ and any function $\Phi : A \to A$, there is a unique function $f : \omega \to A$ such that

$$f(\varnothing) = e \qquad \text{and} \qquad f(n^+) = \Phi(f(n))$$

for all $n \in \omega$.

*Proof.* We call a function $h \in \mathcal{P}(\omega \times A)$ a candidate function if it satisfies the following properties.

1. If $\varnothing \in \operatorname{dom}(h)$, then $h(\varnothing) = e$.
2. For every $n \in \omega$, if $n^+ \in \operatorname{dom}(h)$, then $n \in \operatorname{dom}(h)$ and $f(n^+) = \Phi(f(n))$.

Let $H$ denote the set of all candidate functions and let $f = \bigcup H$. First we show that $f \in \mathcal{P}(\omega \times A)$ is a function, i.e., the set

$$I = \{n \in \omega : \langle n, a\rangle, \langle n, a'\rangle \text{ implies } a = a' \text{ for all } a, a' \in A\}$$

is inductive. We have $\varnothing \in I$ by definition of candidate function. Now suppose that $n \in I$ and we prove that $n^+ \in I$. For any $h, h' \in H$ with $n^+ \in \operatorname{dom}(h)$ and $n^+ \in \operatorname{dom}(h')$, we have $h(n) = h'(n)$ by $n \in I$, implying

$$h(n^+) = \Phi(h(n)) = \Phi(h'(n)) = h'(n^+).$$

Thus $n^+ \in I$, and we conclude that $f$ is a function. Now we show that $\operatorname{dom}(f) = \omega$. We have $\varnothing \in \operatorname{dom}(f)$ since $\{\langle \varnothing, e\rangle\} \in H$. For any $n \in \operatorname{dom}(f)$, let $h \in H$ with $n \in \operatorname{dom}(h)$. If $n^+ \in \operatorname{dom}(h)$, then $n^+ \in \operatorname{dom}(f)$. Otherwise, since $h' = h \cup \{\langle n^+, \Phi(h(n))\rangle\} \in H$, we also have $n^+ \in \operatorname{dom}(f)$.

Now we show that $f \in H$. We have $f(\varnothing) = e$ since $\{\langle \varnothing, e\rangle\} \in H$. For any $n \in \operatorname{dom}(f)$, let $h \in H$ with $n \in \operatorname{dom}(h)$. If $n^+ \in \operatorname{dom}(h)$, then we have

$$f(n^+) = h(n^+) = \Phi(h(n)) = \Phi(f(n)).$$

Otherwise, let $h' = h \cup \{\langle n^+, \Phi(h(n))\rangle\} \in H$, and then we have

$$f(n^+) = h'(n^+) = \Phi(h(n)) = \Phi(f(n)).$$

Thus $f \in H$. For the uniqueness of $f$, let $g \in H$ with $\operatorname{dom}(g) = \omega$, and then since $g \subseteq f$, we have $g(n) = f(n)$ for all $n \in \omega$, implying $g = f$. This completes the proof. $\square$

## 3.3 Arithmetic

**Definition 3.7.** For $n, m \in \omega$, we define

$$n + 0 = n \qquad \text{and} \qquad n + m^+ = (n + m)^+$$

for all $n, m \in \omega$.

**Lemma 3.8.** For $n, m \in \omega$, we have the following properties.

(a) $0 + n = n$.

(b) $n^+ + m = (n + m)^+$.

*Proof.*

(a) The proof is by induction on $n$. We have $0 + 0 = 0$. Now let $n \in \omega$. If $0 + n = n$, then
$$0 + n^+ = (0 + n)^+ = n^+.$$

(b) The proof is by induction on $m$. We have $n^+ + 0 = n^+ = (n + 0)^+$ for all $n \in \omega$. Now let $m \in \omega$. If $n^+ + m = (n + m)^+$ for all $n \in \omega$, then
$$n^+ + m^+ = (n^+ + m)^+ = ((n + m)^+)^+ = (n + m^+)^+.$$
for all $n \in \omega$. □

**Theorem 3.9.** For $n, m, \ell \in \omega$, we have the following properties.

(a) $n + m = m + n$.

(b) $(n + m) + \ell = n + (m + \ell)$.

*Proof.*

(a) The proof is by induction on $m$. We have $n + 0 = n = 0 + n$ for all $n \in \omega$. Now let $m \in \omega$. If $n + m = m + n$ for all $n \in \omega$, then
$$n + m^+ = (n + m)^+ = (m + n)^+ = m^+ + n.$$
for all $n \in \omega$.

(b) The proof is by induction on $\ell$. We have $(n + m) + 0 = n + m = n + (m + 0)$. Now let $\ell \in \omega$. If $(n + m) + \ell = n + (m + \ell)$ for all $n, m \in \omega$, then
$$\begin{aligned}(n + m) + \ell^+ &= ((n + m) + \ell)^+ \\ &= (n + (m + \ell))^+ \\ &= n + (m + \ell)^+ \\ &= n + (m + \ell^+)\end{aligned}$$
for all $n, m \in \omega$.

□

**Definition 3.10.** For $n, m \in \omega$, we define

$$n \cdot 0 = 0 \qquad \text{and} \qquad n \cdot m^+ = n \cdot m + n$$

for all $n, m \in \omega$.

**Lemma 3.11.** For $n, m \in \omega$, we have the following properties.

(a) $0 \cdot n = 0$.

(b) $n^+ \cdot m = n \cdot m + m$.

*Proof.*

(a) The proof is by induction on $n$. We have $0 \cdot 0 = 0$. Now let $n \in \omega$. If $0 \cdot n = 0$, then
$$0 + n^+ = 0 \cdot n^+ + 0 = 0.$$

(b) The proof is by induction on $m$. We have $n^+ \cdot 0 = 0 = n \cdot 0 + 0$ for all $n \in \omega$. Now let $m \in \omega$. If $n^+ \cdot m = n \cdot m + m$ for all $n \in \omega$, then

$$\begin{aligned} n^+ \cdot m^+ &= n^+ \cdot m + n^+ \\ &= n \cdot m + m + n^+ \\ &= n \cdot m + n + m^+ \\ &= n \cdot m^+ + m^+ \end{aligned}$$

for all $n \in \omega$. $\square$

**Theorem 3.12.** For $n, m, \ell \in \omega$, we have the following properties.

(a) $n \cdot (m + \ell) = n \cdot m + n \cdot \ell$.

(b) $n \cdot m = m \cdot n$.

(c) $(n \cdot m) \cdot \ell = n \cdot (m \cdot \ell)$.

*Proof.*

(a) The proof is by induction on $\ell$. We have

$$n \cdot (m + 0) = n \cdot m = n \cdot m + 0 = n \cdot m + n \cdot 0$$

for all $n, m \in \omega$. Now let $\ell \in \omega$. If $n \cdot (m + \ell) = n \cdot m + n \cdot \ell$ for all $n, m \in \omega$, then

$$\begin{aligned} n \cdot (m + \ell^+) &= n \cdot (m + \ell)^+ \\ &= n \cdot (m + \ell) + n \\ &= (n \cdot m + n \cdot \ell) + n \\ &= n \cdot m + (n \cdot \ell + n) \\ &= n \cdot m + n \cdot \ell^+ \end{aligned}$$

for all $n, m \in \omega$.

(b) The proof is by induction on $m$. We have $n \cdot 0 = 0 = 0 \cdot n$ for all $n \in \omega$. Now let $m \in \omega$. If $n \cdot m = m \cdot n$ for all $n \in \omega$, then

$$n \cdot m^+ = n \cdot m + n = m \cdot n + n = m^+ \cdot n$$

for all $n \in \omega$.

(c) The proof is by induction on $\ell$. We have $(n \cdot m) \cdot 0 = 0 = n \cdot (m + 0)$. Now let $\ell \in \omega$. If $(n \cdot m) \cdot \ell = n \cdot (m \cdot \ell)$ for all $n, m \in \omega$, then

$$\begin{aligned}
(n \cdot m) \cdot \ell^+ &= (n \cdot m) \cdot \ell + n \cdot m \\
&= n \cdot (m \cdot \ell) + n \cdot m \\
&= n \cdot (m \cdot \ell + m) \\
&= n \cdot (m \cdot \ell^+)
\end{aligned}$$

for all $n, m \in \omega$. $\square$

## 3.4 Ordering

**Definition 3.13.** We define binary relations $<$ and $\leq$ over the set $\omega$ of natural numbers such that

$$n < m \quad \Leftrightarrow \quad n \in m$$

and

$$n \leq m \quad \Leftrightarrow \quad n \in m \text{ or } n = m$$

for $n, m \in \omega$.