Set Theory

Chapter 1

Axioms and Operations

1.1 Basic Axioms

For sets x and y, we write $x \in y$ to say that x is an element of y, and we write $x = y$ to say that x and y are equal. For simplicity, we write

$$\begin{aligned} x \notin y \quad &\Leftrightarrow \quad \neg(x \in y) \\ x \neq y \quad &\Leftrightarrow \quad \neg(x = y) \\ x \subseteq y \quad &\Leftrightarrow \quad \forall z(z \in x \to z \in y) \\ x \subsetneq y \quad &\Leftrightarrow \quad x \subseteq y \wedge x \neq y. \end{aligned}$$

Axiom I (Extensionality). Two sets are equal if they have exactly the same elements. That is,

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y).$$

Definition 1.1. The **empty set**, denoted by $\varnothing$, is the set that has no elements.

Axiom II (Pairing). For any two sets x and y, there is a set that consists of exactly x and y. That is,

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y)).$$

Definition 1.2. The **pair set** of two sets x and y, denoted by $\{x, y\}$, is the set that consists of exactly x and y.

Axiom III (Power Set). For any set x, there is a set whose members are exactly the subsets of x. That is,

$$\forall x \exists y \forall z (z \subseteq x \leftrightarrow z \in y).$$

Definition 1.3. The **power set** of a set x, denoted by $\mathcal{P}(x)$, is the set that consists of exactly the subsets of x.

Axiom IV (Separation Scheme). Let $\phi(z)$ be a formula. For any set x, there exists a set y such that for any set z, we have $z \in y$ if and only if both $z \in x$ and $\phi(z)$ hold. That is,

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \phi(z)).$$

Definition 1.4. Let x, y be sets and let $\phi(z)$ be a formula. If for any set z, we have $z \in y$ if and only if $z \in x$ and $\phi(z)$, then we write

$$y = \{z \in x : \phi(z)\}.$$

Definition 1.5. For sets x and y, we define

$$x \setminus y = \{z \in x : z \notin y\}.$$

Theorem 1.6. There is no set to which every set belongs. That is,

$$\forall x \exists y (y \notin x).$$

Proof. Let x be a set and let $y = \{z \in x : z \notin z\}$. Then

$$y \in y \quad \Leftrightarrow \quad y \in x \wedge y \notin y.$$

If $y \in x$, then

$$y \in y \quad \Leftrightarrow \quad y \notin y,$$

contradiction. Thus $y \notin x$, which completes the proof. $\square$

Axiom V (Union). For any set x, there exists a set whose elements are exactly the elements of the elements of x. That is,

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w)).$$

Definition 1.7. Let x be a set.

- We define the **union** of x, denoted by $\bigcup x$, to be the set that consists of the sets that belongs to at least one element of x. Formally, for any set z we have

$$z \in \bigcup x \quad \Leftrightarrow \quad \exists w (w \in x \wedge z \in w).$$

- If x is nonempty, we define the **intersection** of x, denoted by $\bigcap x$, to be the set that consists of the sets that belongs to all elements of x. Formally, for any set z we have

$$z \in \bigcap x \quad \Leftrightarrow \quad \forall w (w \in x \rightarrow z \in w).$$

For sets u and v, we define

$$u \cup v = \bigcup \{u, v\} \qquad \text{and} \qquad u \cap v = \bigcap \{u, v\}.$$

Chapter 2

Relations and Functions

2.1 Ordered Pairs

Definition 2.1. For sets x and y, we define

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

Lemma 2.2. Let x, y, y' be sets. If $\{x, y\} = \{x, y'\}$, then $y = y'$.

Proof. Suppose that $y \neq y'$. Since $y \in \{x, y\} = \{x, y'\}$ and $y \neq y'$, we have $y = x$. Then we have $y' \in \{x, y'\} = \{x, y\} = \{x\}$, implying $y' = x = y$, contradiction. Thus, $y = y'$. □

Theorem 2.3. For sets x, x', y, y', we have

$$\langle x, y\rangle = \langle x', y'\rangle$$

if and only if $x = x'$ and $y = y'$.

Proof. ($\Leftarrow$) Straightforward. ($\Rightarrow$) Suppose that $x \neq x'$. Since

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\},$$

either $\{x\} = \{x', y'\}$ or $\{x\} = \{x'\}$ holds. For both cases we all have $x' \in \{x\}$, implying $x' = x$, contradiction. Hence we have $x = x'$, and it follows that $\{x\} = \{x'\}$, implying $\{x, y\} = \{x', y'\}$, and thus $y = y'$. □

Lemma 2.4. If $x, y \in C$, then $\langle x, y\rangle \in \mathcal{P}(\mathcal{P}(C))$.

Proof. Since $\{x\}$ and $\{y\}$ are subsets of C, we have $\{x\}, \{x, y\} \in \mathcal{P}(C)$. It follows that $\{\{x\}, \{x, y\}\}$ is a subset of $\mathcal{P}(C)$, implying

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(C)).$$ □

Theorem 2.5. For any sets A and B, there is a set whose members are exactly the pairs (x, y) with $x \in A$ and $y \in B$.

Proof. Since $x, y \in A \cup B$, the set of pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$ can be given by

$$\{z \in \mathcal{P}(\mathcal{P}(A \cup B)) : z = \langle x, y\rangle \text{ for some } x \in A \text{ and } y \in B\}.$$ □

Definition 2.6. For any sets A and B, the **Cartesian product** of A and B, denoted by $A \times B$, is the set whose memebers are exactly the pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$.

2.2 Relations

Definition 2.7. A **relation** is a set of ordered pairs. For any relation R, the **domain** and the **range** of R, denoted by $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$, respectively, are defined as follows.

- $\mathrm{dom}(R)$ is the collection of sets x with $\langle x, y\rangle \in R$ for some y.
- $\mathrm{ran}(R)$ is the collection of sets y with $\langle x, y\rangle \in R$ for some x.

Definition 2.8. Let R and S be relations and let X be a set.

- The **inverse** of R, denoted by R^{-1}, is the set of all pairs $\langle y, x\rangle$ with $\langle x, y\rangle \in R$.
- The **restriction** of R to X, denoted by $R \restriction X$, is the set of all pairs $\langle x, y\rangle \in R$ with $x \in X$.
- The **composition** of R and S, denoted by $R \circ S$, is the set of all pairs $\langle x, z\rangle$ with $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S$.

Definition 2.9. A **function** is a relation f such that for any set $x \in \mathrm{dom}(f)$, there exists a unique set y such that $\langle x, y\rangle \in f$. The unique set y with respect to x is called the **value** of f at x and is denoted $f(x)$.

- We say that f is a function from A to B, denoted by $f : A \to B$, if $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.
- f is **one-to-one** if for any $y \in \mathrm{ran}(f)$, there exists a unique set $x \in \mathrm{dom}(f)$ with $f(x) = y$.

Definition 2.10. For any sets A and B, the set of functions from A to B is denoted by B^A.

2.3 Equivalence Relations and Ordering Relations

Definition 2.11. Let A be a set. An **equivalence relation** on A is a relation $R \subseteq A \times A$ that satisfies the following three conditions.

- Reflexivity: $\langle x, x \rangle \in R$ for any $x \in A$.
- Symmetry: $\langle x, y \rangle \in R$ implies $\langle y, x \rangle \in R$ for any $x, y \in A$.
- Transitivity: $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ implies $\langle x, z \rangle \in R$ for any $x, y, z \in A$.

Chapter 3

Natural Numbers

3.1 Inductive Sets

Definition 3.1. The **successor** of a set x, denoted x^+, is defined by

$$x^+ = x \cup \{x\}.$$

Definition 3.2. A set x is **inductive** if it has the empty set as member and is closed under successor. That is,

$$x \text{ is inductive} \quad \Leftrightarrow \quad \varnothing \in x \land \forall y(y \in x \to y^+ \in x).$$

Axiom VI (Infinity). There exists an inductive set. That is,

$$\exists x(\varnothing \in x \land \forall y(y \in x \to y^+ \in x)).$$

Definition 3.3. A **natural number** is a set x that belongs to all inductive sets. The set of natural numbers is denoted by ω. That is,

$$x \in \omega \quad \Leftrightarrow \quad \forall A(A \text{ is inductive} \to x \in A)$$

Theorem 3.4. ω is inductive.

Proof. First, $\varnothing \in \omega$ since $\varnothing$ belongs to all inductive sets by definition. For any set $x \in \omega$, x belongs to all inductive sets, implying that x^+ belongs to all inductive sets, and thus $x^+ \in \omega$. Thus, ω is inductive. $\square$

Definition 3.5. Let

$$0 = \varnothing, \quad 1 = \varnothing^+, \quad 2 = (\varnothing^+)^+, \quad 3 = ((\varnothing^+)^+)^+, \quad 4 = (((\varnothing^+)^+)^+)^+, \quad \ldots$$

denote the natural numbers.

3.2 Recursion

Theorem 3.6 (Recursion Theorem). For any sets A and e with $e \in A$ and any function $\Phi : A \to A$, there is a unique function $f : \omega \to A$ such that

$$f(\varnothing) = e \qquad \text{and} \qquad f(n^+) = \Phi(f(n))$$

for all $n \in \omega$.

Proof. We call a function $h \in \mathcal{P}(\omega \times A)$ a candidate function if it satisfies the following properties.

1. If $\varnothing \in \mathrm{dom}(h)$, then $h(\varnothing) = e$.

2. For every $n \in \omega$, if $n^+ \in \mathrm{dom}(h)$, then $n \in \mathrm{dom}(h)$ and $f(n^+) = \Phi(f(n))$.

Let H denote the set of all candidate functions and let $f = \bigcup H$. First we show that $f \in \mathcal{P}(\omega \times A)$ is a function, i.e., the set

$$I = \{n \in \omega : \langle n, a \rangle, \langle n, a' \rangle \text{ implies } a = a' \text{ for all } a, a' \in A\}$$

is inductive. We have $\varnothing \in I$ by definition of candidate function. Now suppose that $n \in I$ and we prove that $n^+ \in I$. For any $h, h' \in H$ with $n^+ \in \mathrm{dom}(h)$ and $n^+ \in \mathrm{dom}(h')$, we have $h(n) = h'(n)$ by $n \in I$, implying

$$h(n^+) = \Phi(h(n)) = \Phi(h'(n)) = h'(n^+).$$

Thus $n^+ \in I$, and we conclude that f is a function. Now we show that $\mathrm{dom}(f) = \omega$. We have $\varnothing \in \mathrm{dom}(f)$ since $\{\langle \varnothing, e \rangle\} \in H$. For any $n \in \mathrm{dom}(f)$, let $h \in H$ with $n \in \mathrm{dom}(h)$. If $n^+ \in \mathrm{dom}(h)$, then $n^+ \in \mathrm{dom}(f)$. Otherwise, since $h' = h \cup \{\langle n^+, \Phi(h(n)) \rangle\} \in H$, we also have $n^+ \in \mathrm{dom}(f)$.

Now we show that $f \in H$. We have $f(\varnothing) = e$ since $\{\langle \varnothing, e \rangle\} \in H$. For any $n \in \mathrm{dom}(f)$, let $h \in H$ with $n \in \mathrm{dom}(h)$. If $n^+ \in \mathrm{dom}(h)$, then we have

$$f(n^+) = h(n^+) = \Phi(h(n)) = \Phi(f(n)).$$

Otherwise, let $h' = h \cup \{\langle n^+, \Phi(h(n)) \rangle\} \in H$, and then we have

$$f(n^+) = h'(n^+) = \Phi(h(n)) = \Phi(f(n)).$$

Thus $f \in H$. For the uniqueness of f, let $g \in H$ with $\mathrm{dom}(g) = \omega$, and then since $g \subseteq f$, we have $g(n) = f(n)$ for all $n \in \omega$, implying $g = f$. This completes the proof. $\square$

3.3 Arithmetic

Definition 3.7. For $n, m \in \omega$, we define

$$n + 0 = n \qquad \text{and} \qquad n + m^+ = (n+m)^+$$

for all $n, m \in \omega$.

Lemma 3.8. For $n, m \in \omega$, we have the following properties.

(a) $0 + n = n$.

(b) $n^+ + m = (n+m)^+$.

Proof.

(a) The proof is by induction on n. We have $0 + 0 = 0$. Now let $n \in \omega$. If $0 + n = n$, then
$$0 + n^+ = (0+n)^+ = n^+.$$

(b) The proof is by induction on m. We have $n^+ + 0 = n^+ = (n+0)^+$ for all $n \in \omega$. Now let $m \in \omega$. If $n^+ + m = (n+m)^+$ for all $n \in \omega$, then

$$n^+ + m^+ = (n^+ + m)^+ = ((n+m)^+)^+ = (n+m^+)^+.$$

for all $n \in \omega$. □

Theorem 3.9. For $n, m, \ell \in \omega$, we have the following properties.

(a) $n + m = m + n$.

(b) $(n+m) + \ell = n + (m + \ell)$.

Proof.

(a) The proof is by induction on m. We have $n + 0 = n = 0 + n$ for all $n \in \omega$. Now let $m \in \omega$. If $n + m = m + n$ for all $n \in \omega$, then

$$n + m^+ = (n+m)^+ = (m+n)^+ = m^+ + n.$$

for all $n \in \omega$.

(b) The proof is by induction on ℓ. We have $(n+m) + 0 = n + m = n + (m+0)$. Now let $\ell \in \omega$. If $(n+m) + \ell = n + (m+\ell)$ for all $n, m \in \omega$, then

$$\begin{aligned}(n+m) + \ell^+ &= ((n+m)+\ell)^+ \\ &= (n + (m+\ell))^+ \\ &= n + (m+\ell)^+ \\ &= n + (m + \ell^+)\end{aligned}$$

for all $n, m \in \omega$.

□

Definition 3.10. For $n, m \in \omega$, we define

$$n \cdot 0 = 0 \qquad \text{and} \qquad n \cdot m^+ = n \cdot m + n$$

for all $n, m \in \omega$.

Lemma 3.11. For $n, m \in \omega$, we have the following properties.

(a) $0 \cdot n = 0$.

(b) $n^+ \cdot m = n \cdot m + m$.

Proof.

(a) The proof is by induction on n. We have $0 \cdot 0 = 0$. Now let $n \in \omega$. If $0 \cdot n = 0$, then

$$0 + n^+ = 0 \cdot n^+ + 0 = 0.$$

(b) The proof is by induction on m. We have $n^+ \cdot 0 = 0 = n \cdot 0 + 0$ for all $n \in \omega$. Now let $m \in \omega$. If $n^+ \cdot m = n \cdot m + m$ for all $n \in \omega$, then

$$\begin{aligned} n^+ \cdot m^+ &= n^+ \cdot m + n^+ \\ &= n \cdot m + m + n^+ \\ &= n \cdot m + n + m^+ \\ &= n \cdot m^+ + m^+ \end{aligned}$$

for all $n \in \omega$. $\square$

Theorem 3.12. For $n, m, \ell \in \omega$, we have the following properties.

(a) $n \cdot (m + \ell) = n \cdot m + n \cdot \ell$.

(b) $n \cdot m = m \cdot n$.

(c) $(n \cdot m) \cdot \ell = n \cdot (m \cdot \ell)$.

Proof.

(a) The proof is by induction on ℓ. We have

$$n \cdot (m + 0) = n \cdot m = n \cdot m + 0 = n \cdot m + n \cdot 0$$

for all $n, m \in \omega$. Now let $\ell \in \omega$. If $n \cdot (m + \ell) = n \cdot m + n \cdot \ell$ for all $n, m \in \omega$, then

$$\begin{aligned} n \cdot (m + \ell^+) &= n \cdot (m + \ell)^+ \\ &= n \cdot (m + \ell) + n \\ &= (n \cdot m + n \cdot \ell) + n \\ &= n \cdot m + (n \cdot \ell + n) \\ &= n \cdot m + n \cdot \ell^+ \end{aligned}$$

for all $n, m \in \omega$.

(b) The proof is by induction on m. We have $n \cdot 0 = 0 = 0 \cdot n$ for all $n \in \omega$. Now let $m \in \omega$. If $n \cdot m = m \cdot n$ for all $n \in \omega$, then

$$n \cdot m^+ = n \cdot m + n = m \cdot n + n = m^+ \cdot n$$

for all $n \in \omega$.

(c) The proof is by induction on ℓ. We have $(n \cdot m) \cdot 0 = 0 = n \cdot (m + 0)$. Now let $\ell \in \omega$. If $(n \cdot m) \cdot \ell = n \cdot (m \cdot \ell)$ for all $n, m \in \omega$, then

$$\begin{aligned}(n \cdot m) \cdot \ell^+ &= (n \cdot m) \cdot \ell + n \cdot m \\ &= n \cdot (m \cdot \ell) + n \cdot m \\ &= n \cdot (m \cdot \ell + m) \\ &= n \cdot (m \cdot \ell^+)\end{aligned}$$

for all $n, m \in \omega$. □

3.4 Ordering

Definition 3.13. We define binary relations $<$ and $\leq$ over the set ω of natural numbers such that

$$n < m \quad \Leftrightarrow \quad n \in m$$

and

$$n \leq m \quad \Leftrightarrow \quad n \in m \text{ or } n = m$$

for $n, m \in \omega$.

Chapter 4

Real Numbers

4.1 Integers

Definition 4.1. Let

$$\sim_{\mathbb{Z}} = \{\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in (\omega \times \omega) \times (\omega \times \omega) : a + d = b + c\}.$$

Theorem 4.2. The relation $\sim_{\mathbb{Z}}$ is an equivalence relation.

Definition 4.3. Let $\mathbb{Z} = (\omega \times \omega)/\sim_{\mathbb{Z}}$.

4.2 Rational Numbers

Definition 4.4. Let $\mathbb{Z}' = \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$.

Definition 4.5. Let

$$\sim_{\mathbb{Q}} = \{\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in (\mathbb{Z} \times \mathbb{Z}') \times (\mathbb{Z} \times \mathbb{Z}') : a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c\}.$$

Theorem 4.6. The relation $\sim_{\mathbb{Q}}$ is an equivalence relation.

Definition 4.7. Let $\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}')/\sim_{\mathbb{Q}}$.

Chapter 5

Equinumerosity

5.1 Equinumerosity

Definition 5.1. We say that A is **equinumerous** to B, denoted by $A \approx B$, if there exists a one-to-one function from A onto B.

Theorem 5.2. The following statements hold for any sets A, B, C.

(a) $A \approx A$.

(b) $A \approx B$ implies $B \approx A$.

(c) $A \approx B$ and $B \approx C$ implies $A \approx C$.

Proof. To be completed. □

Theorem 5.3. For any set A, we have $A \not\approx \mathcal{P}(A)$.

5.2 Finite Sets

Definition 5.4. A set is **finite** if it is equinumerous to a natural number. A set is **infinite** if it is not finite.

Theorem 5.5 (Pigeonhole Principle). If A is finite and $B \subsetneq A$, then $A \prec B$.

Chapter 6

Ordinal Numbers

6.1 Transfinite Induction and Recursion