Graph Theory

Chapter 1

Graphs

1.1 Graphs

Definition 1.1. A **graph** is a pair

$$G = (V, E),$$

where V is a finite set of **vertices**, and E is a set of **edges**, which are 2-element subsets of V.

- The elements of an edge is called its **endpoints**, and we say that an edge **joins** its endpoints.
- We say that two vertices are **adjacent** if they are joined by an edge.

Remark. An edge $\{u, v\}$ is usually written as uv.

1.2 Paths

Definition 1.2. A **walk** of a graph $G = (V, E)$ is a sequence

$$(v_0, e_1, v_1, \ldots, e_k, v_k),$$

where $v_0, v_1, \ldots, v_k \in V$ and $e_1, \ldots, e_k \in E$ such that $e_i = v_{i-1}v_i$ for each $i \in \{1, \ldots, k\}$. The number of edges in a walk is called the **length** of the walk.

- A **trail** is a walk with no repeated edges.
- A **circuit** is a trail with starting vertex and ending vertex being the same.
- A **path** is a walk with no repeated vertices.
- A **cycle** is a circuit with no repeated internal vertices.

1.3 Connectivity

Definition 1.3. A graph is **connected** if any two vertices can be linked by a path. A graph is **disconnected** if it is not connected.

Definition 1.4. Let $G = (V, E)$ be a graph. We say that a set $S \subseteq V$ **separates** G if $G[V \setminus S]$ is disconnected.

Definition 1.5. Let $G = (V, E)$ be a graph and let k be a nonnegative integer. We say that G is **k-connected** if $|V| > k$ and every subset S of V with $|S| < k$ does not separate G.

Chapter 2

Planar Graphs

2.1 Topological Prerequisites

Definition 2.1. Let $x, y \in \mathbb{R}^2$ be different points.

- A **straight line segment** between x and y is a set $\ell \subseteq \mathbb{R}^2$ with

$$\ell = \{x + \lambda(y - x) : 0 \leq \lambda \leq 1\}.$$

- A **polygonal arc** between x and y is a set $\alpha \subseteq \mathbb{R}^2$ which is a union of finitely many straight line segments such that there is a homeomorphism $\varphi : [0, 1] \to \alpha$ with $\varphi(0) = x$ and $\varphi(1) = y$.

Definition 2.2. Let $S \subseteq \mathbb{R}^2$ beopen and let $\sim$ be the equivalence relation of being connected by a polygonal arc. The members of $S/{\sim}$ are called the **regions** of S.

Definition 2.3. Let $S \subseteq \mathbb{R}^2$. The **boundary** of S is the set of points whose every neighborhood consists of both a point in S and a point not in S.

2.2 Plane Graphs

Definition 2.4. A **plane graph** is a pair $G = (V, E)$ of finite sets such that the following properties hold, where the elements of V and those of E are called **vertices** and **edges**, respectively.

- V is a finite subset of $\mathbb{R}^2$.
- E is a finite set of simple curves between vertices.
- Different edges in E have different set of endpoints.
- The interior of an edge contains no vertex and no point of any other edge.

The **faces** of G are the regions of $\mathbb{R}^2 \setminus (V \cup \bigcup E)$, and we denote the set of faces of G by $F(G)$.

Remark. A plane graph defines a graph in a natural way. Thus, we usually use the same notation for both a plane graph and its corresponding graph.