Set Theory

Chapter 1

Axioms and Operations

1.1 Basic Axioms

Axiom 1.1 (Extensionality). For any sets x and y, if for any set z, we have $z \in x$ if and only if $z \in y$, then we say that x and y are **equal**, denoted $x = y$.

Axiom 1.2 (Empty Set). There is a set x such that $y \notin x$ for each set y. The set x is called the **empty set** and is denoted by $\varnothing$.

Axiom 1.3 (Pairing). For any sets x and y, there is a set w such that for each set $z \in w$, either $z = x$ or $z = y$ holds. The set w is called the **pair set** of x and y and is denoted by $\{x, y\}$. If $x = y$, then we write $\{x\}$ for short.

Axiom 1.4 (Power Set). For any set x, there exists a set y such that for any set z, $z \in y$ if and only if $z \subseteq x$. The set y is called the **power set** of x and is denoted by $\mathcal{P}(x)$.

Axiom 1.5 (Subset). Let $\phi(z)$ be a first-order formula such that z is the only free variable in ϕ. For any set x, there exists a set y such that for any set z, $z \in y$ if and only if both $z \in x$ and $\phi(z)$ holds. The set y will be denoted by

$$y = \{z \in x : \phi(z)\}.$$

1.2 Arbitrary Unions and Intersections

Chapter 2

Relations and Functions

2.1 Ordered Pairs

Definition 2.1. For sets x and y, we define

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Lemma 2.2. Let x, y, y' be sets. If $\{x, y\} = \{x, y'\}$, then $y = y'$.

Proof. Suppose that $y \neq y'$. Since $y \in \{x, y\} = \{x, y'\}$ and $y \neq y'$, we have $y = x$. Then we have $y' \in \{x, y'\} = \{x, y\} = \{x\}$, implying $y' = x = y$, contradiction. Thus, $y = y'$. $\square$

Theorem 2.3. For sets x, x', y, y', we have

$$(x, y) = (x', y')$$

if and only if $x = x'$ and $y = y'$.

Proof. ($\Leftarrow$) Straightforward. ($\Rightarrow$) Suppose that $x \neq x'$. Since

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\},$$

either $\{x\} = \{x', y'\}$ or $\{x\} = \{x'\}$ holds. For both cases we all have $x' \in \{x\}$, implying $x' = x$, contradiction. Hence we have $x = x'$, and it follows that $\{x\} = \{x'\}$, implying $\{x, y\} = \{x', y'\}$, and thus $y = y'$. $\square$

Lemma 2.4. If $x, y \in C$, then $(x, y) \in \mathcal{P}(\mathcal{P}(C))$.

Proof. Since $\{x\}$ and $\{y\}$ are subsets of C, we have $\{x\}, \{x, y\} \in \mathcal{P}(C)$. It follows that $\{\{x\}, \{x, y\}\}$ is a subset of $\mathcal{P}(C)$, implying

$$(x, y) = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(C)).$$

$\square$

Theorem 2.5. For any sets A and B, there is a set whose members are exactly the pairs (x, y) with $x \in A$ and $y \in B$.

Proof. Since $x, y \in A \cup B$, the set of pairs (x, y) with $x \in A$ and $y \in B$ can be given by

$$\{z \in \mathcal{P}(\mathcal{P}(A \cup B)) : z = (x, y) \text{ for some } x \in A \text{ and } y \in B\}.$$

$\square$

Definition 2.6. For any sets A and B, the **Cartesian product** of A and B, denoted by $A \times B$, is the set whose memebers are exactly the pairs (x, y) with $x \in A$ and $y \in B$.

2.2 Relations

2.3 Functions

2.4 Equivalence Relations and Ordering Relations

Chapter 3

Natural Numbers

3.1 Inductive Sets

Definition 3.1. The **successor** of a set x, denoted x^+, is defined by

$$x^+ = x \cup \{x\}.$$

We say that a set A is **inductive** if it satisfies the following conditions.

- $\varnothing \in A$.
- For any $x \in A$, $x^+ \in A$.

Axiom 3.2 (Infinity). There exists an inductive set.

Definition 3.3. A **natural number** is a set belonging to all inductive sets. The set of natural numbers is denoted by ω.

3.2 Recursion

3.3 Arithmetic

3.4 Ordering