Set Theory

Chapter 1

Axioms and Operations

1.1 Basic Axioms

Axiom I (Extensionality). For any sets x and y, if for any set z, we have $z \in x$ if and only if $z \in y$, then we say that x and y are **equal**, denoted $x = y$.

Axiom II (Empty Set). There is a set x such that $y \notin x$ for each set y. The set x is called the **empty set** and is denoted by $\varnothing$.

Axiom III (Pairing). For any sets x and y, there is a set w such that for each set $z \in w$, either $z = x$ or $z = y$ holds. The set w is called the **pair set** of x and y and is denoted by $\{x, y\}$. If $x = y$, then we write $\{x\}$ for short.

Example. By axiom of pairing, $\{\varnothing\}$ and $\{\varnothing, \{\varnothing\}\}$ are sets.

Axiom IV (Power Set). For any set x, there exists a set y such that for any set z, $z \in y$ if and only if $z \subseteq x$. The set y is called the **power set** of x and is denoted by $\mathcal{P}(x)$.

Axiom V (Subset). Let $\phi(z)$ be a first-order formula such that z is the only free variable in ϕ. For any set x, there exists a set y such that for any set z, $z \in y$ if and only if both $z \in x$ and $\phi(z)$ holds. The set y will be denoted by

$$y = \{z \in x : \phi(z)\}.$$

Theorem 1.1. There is no set to which every set belongs. That is, for any set x, there exists a set y such that $y \notin x$.

Proof. Let $y = \{z \in x : z \notin z\}$. Then we have $y \in y$ if and only if $y \in x$ and $y \notin y$. If $y \in x$, then $y \in y$ if and only if $y \notin y$, contradiction. Thus, $y \notin x$. $\square$

1.2 Arbitrary Unions and Intersections

Axiom VI (Union). For any set x, there exists a set y whose elements are exactly the members of the members of x. That is,

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \land z \in w)).$$

The set y is called the **union** of x, denoted by $\bigcup x$.

Theorem 1.2. For any nonempty set x, there exists a unique set y such that for any set z, $z \in y$ if and only if z belongs to every member of x.

Proof. Since x is nonempty, there is a member w_0 of x. Then by a subset axiom there exists a set y such that

$$y = \{z \in w_0 : \forall w (w \in x \to z \in w)\},$$

and uniqueness of y follows from extensionality. □

Definition 1.3. For any nonempty set x, we define the **intersection** of x as the set y such that for any set z, $z \in y$ if and only if z belongs to every member of x. Let $\bigcap x$ denote the intersection of x.

Definition 1.4. For any sets x and y, we define

$$x \cup y = \bigcup \{x, y\}$$
$$x \cap y = \bigcap \{x, y\}.$$

Chapter 2

Relations and Functions

2.1 Ordered Pairs

Definition 2.1. For sets x and y, we define

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

Lemma 2.2. Let x, y, y' be sets. If $\{x, y\} = \{x, y'\}$, then $y = y'$.

Proof. Suppose that $y \neq y'$. Since $y \in \{x, y\} = \{x, y'\}$ and $y \neq y'$, we have $y = x$. Then we have $y' \in \{x, y'\} = \{x, y\} = \{x\}$, implying $y' = x = y$, contradiction. Thus, $y = y'$. $\square$

Theorem 2.3. For sets x, x', y, y', we have

$$\langle x, y\rangle = \langle x', y'\rangle$$

if and only if $x = x'$ and $y = y'$.

Proof. ($\Leftarrow$) Straightforward. ($\Rightarrow$) Suppose that $x \neq x'$. Since

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\},$$

either $\{x\} = \{x', y'\}$ or $\{x\} = \{x'\}$ holds. For both cases we all have $x' \in \{x\}$, implying $x' = x$, contradiction. Hence we have $x = x'$, and it follows that $\{x\} = \{x'\}$, implying $\{x, y\} = \{x', y'\}$, and thus $y = y'$. $\square$

Lemma 2.4. If $x, y \in C$, then $\langle x, y\rangle \in \mathcal{P}(\mathcal{P}(C))$.

Proof. Since $\{x\}$ and $\{y\}$ are subsets of C, we have $\{x\}, \{x, y\} \in \mathcal{P}(C)$. It follows that $\{\{x\}, \{x, y\}\}$ is a subset of $\mathcal{P}(C)$, implying

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(C)). \qquad \square$$

Theorem 2.5. For any sets A and B, there is a set whose members are exactly the pairs (x, y) with $x \in A$ and $y \in B$.

Proof. Since $x, y \in A \cup B$, the set of pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$ can be given by

$$\{z \in \mathcal{P}(\mathcal{P}(A \cup B)) : z = \langle x, y\rangle \text{ for some } x \in A \text{ and } y \in B\}. \qquad \square$$

Definition 2.6. For any sets A and B, the **Cartesian product** of A and B, denoted by $A \times B$, is the set whose memebers are exactly the pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$.

2.2 Relations

Definition 2.7. A **relation** is a set of ordered pairs. For any relation R, the **domain** and the **range** of R, denoted by $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$, respectively, are defined as follows.

- $\mathrm{dom}(R)$ is the collection of sets x with $\langle x, y\rangle \in R$ for some y.
- $\mathrm{ran}(R)$ is the collection of sets y with $\langle x, y\rangle \in R$ for some x.

Definition 2.8. Let R and S be relations and let X be a set.

- The **inverse** of R, denoted by R^{-1}, is the set of all pairs $\langle y, x\rangle$ with $\langle x, y\rangle \in R$.
- The **restriction** of R to X, denoted by $R \restriction X$, is the set of all pairs $\langle x, y\rangle \in R$ with $x \in X$.
- The **composition** of R and S, denoted by $R \circ S$, is the set of all pairs $\langle x, z\rangle$ with $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S$.

Definition 2.9. A **function** is a relation f such that for any set $x \in \mathrm{dom}(f)$, there exists a unique set y such that $\langle x, y\rangle \in f$. The unique set y with respect to x is called the **value** of f at x and is denoted $f(x)$.

- We say that f is a function from A to B, denoted by $f : A \to B$, if $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.
- f is **one-to-one** if for any $y \in \mathrm{ran}(f)$, there exists a unique set $x \in \mathrm{dom}(f)$ with $f(x) = y$.

Definition 2.10. For any sets A and B, the set of functions from A to B is denoted by B^A.

2.3 Equivalence Relations and Ordering Relations

Definition 2.11. Let A be a set. An **equivalence relation** on A is a relation $R \subseteq A \times A$ that satisfies the following three conditions.

- Reflexivity: $\langle x, x \rangle \in R$ for any $x \in A$.
- Symmetry: $\langle x, y \rangle \in R$ implies $\langle y, x \rangle \in R$ for any $x, y \in A$.
- Transitivity: $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ implies $\langle x, z \rangle \in R$ for any $x, y, z \in A$.

Chapter 3

Natural Numbers

3.1 Inductive Sets

Definition 3.1. The **successor** of a set x, denoted x^+, is defined by

$$x^+ = x \cup \{x\}.$$

We say that a set A is **inductive** if $\varnothing \in A$ and for any $x \in A$, we have $x^+ \in A$.

Axiom VII (Infinity). There exists an inductive set.

Definition 3.2. A **natural number** is a set belonging to all inductive sets. The set of natural numbers is denoted by ω.

Theorem 3.3. ω is inductive.

Proof. First, $\varnothing \in \omega$ since $\varnothing$ belongs to all inductive sets by definition. For any set $x \in \omega$, x belongs to all inductive sets, implying that x^+ belongs to all inductive sets, and thus $x^+ \in \omega$. Thus, ω is inductive. $\square$

3.2 Recursion

3.3 Arithmetic

3.4 Ordering