# Theory of Computation

# Chapter 1

# Regular Languages

## 1.1 Strings and Languages

**Definition 1.1.** An **alphabet** is a finite set of symbols. A **string** over alphabet $\Sigma$ is a finite sequence

$$w = a_1 a_2 \cdots a_n$$

with $a_1, \ldots, a_n \in \Sigma$, where $n$ is called the **length** of the string $w$. The **empty string** is the unique string of length zero, which is denoted by $\epsilon$.

**Definition 1.2.** The **concatenation** of strings

$$u = a_1 a_2 \cdots a_n \quad \text{and} \quad v = b_1 b_2 \cdots b_m$$

is defined as the string

$$uv = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m.$$

**Definition 1.3.** Let $\Sigma^*$ denote the set of all strings over $\Sigma$. A subset of $\Sigma^*$ is called a **language** over $\Sigma$.

**Definition 1.4.** The **concatenation** of languages $L_1$ and $L_2$ is

$$L_1 L_2 = \{w_1 w_2 : w_1 \in L_1 \text{ and } w_2 \in L_2\}.$$

For any language $L$, we define $L^0 = \{\epsilon\}$ and $L^{n+1} = L^n L$ for all integers $n \geq 0$. Also, we define $L^* = \bigcup_{n \geq 0} L^n$.

## 1.2 Deterministic Finite State Automata

**Definition 1.5.** A **deterministic finite state automaton (DFA)** is a 5-tuple

$$M = (Q, \Sigma, \delta, s, F),$$

where each component is as follows.

- $Q$ is a finite set of **states**.
- $\Sigma$ is an alphabet.
- $\delta : Q \times \Sigma \to Q$ is a **transition function**.
- $s \in Q$ is the **start state**.
- $F \subseteq Q$ is the set of **final states**.

**Definition 1.6.** Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

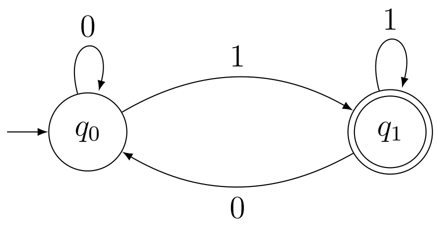
- We say that $A$ accepts a string $w \in \Sigma^*$ if $\delta_w(q_0) \in F$.
- The **language** of $A$, denoted $L(A)$, is defined as the set of strings that are accepted by $A$.

**Definition 1.7.** A language $L$ is **regular** if there exists a DFA $A$ such that $L(A) = L$.

**Example.** Let $A = (\{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{q_1\})$ be a DFA, where the transition function $\delta$ is as follows.

| | 0 | 1 |
|---|---|---|
| $q_0$ | $q_0$ | $q_1$ |
| $q_1$ | $q_0$ | $q_1$ |

Instead of using the formal definition, one can also draw a state diagram of $A$ as follows.




It can be shown that a string $w \in \{0, 1\}^*$ is accepted by $A$ if and only if $w$ ends with 1. Thus, the language $L = \{w \in \{0, 1\}^* : w \text{ ends with } 1\}$ is regular.

**Theorem 1.8.** If $L$ is a regular language over $\Sigma$, then $\Sigma^* \setminus L$ is also regular.

*Proof.* Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA with $L = L(A)$. Let $A' = (Q, \Sigma, \delta, q_0, Q \setminus F)$. Then for each $w \in \Sigma^*$, we have

$$w \in L(A') \quad \Leftrightarrow \quad \delta_w(q_0) \in Q \setminus F \quad \Leftrightarrow \quad w \notin L(A).$$

Thus, $L(A') = \Sigma^* \setminus L(A)$, implying that $\Sigma^* \setminus L$ is regular. $\square$

**Theorem 1.9.** If $L_1$ and $L_2$ are regular languages over $\Sigma$, then $L_1 \cup L_2$ is also regular.

*Proof.* Let

$$A_1 = (Q_1, \Sigma, \delta^{(1)}, q_1, F_1) \quad \text{and} \quad A_2 = (Q_2, \Sigma, \delta^{(2)}, q_2, F_2)$$

be DFAs with $L_1 = L(A_1)$ and $L_2 = L(A_2)$. We construct the DFA

$$A = (Q_1 \times Q_2, \Sigma, \delta, (q_1, q_2), F)$$

as follows.

- $\delta((p, q), a) = (\delta^{(1)}(p, a), \delta^{(2)}(q, a))$ for each $p \in Q_1$, $q \in Q_2$ and $a \in \Sigma$.
- $F = \{(p, q) : p \in F_1 \text{ or } q \in F_2\}$.

It can be shown that for each string $w \in \Sigma^*$, we have

$$\begin{aligned} w \in L(A) \quad &\Leftrightarrow \quad \delta_w((q_1, q_2)) \in F \\ &\Leftrightarrow \quad \delta_w^{(1)}(q_1) \in F_1 \text{ or } \delta_w^{(2)}(q_2) \in F_2 \\ &\Leftrightarrow \quad w \in L(A_1) \text{ or } w \in L(A_2). \end{aligned}$$

Thus, $L(A) = L(A_1) \cup L(A_2)$, implying that $L_1 \cup L_2$ is regular. $\square$

**Corollary 1.10.** If $L_1$ and $L_2$ are regular languages over $\Sigma$, then $L_1 \cap L_2$ is also regular.

*Proof.* Straightforward since by De Morgan's law we have

$$L_1 \cap L_2 = \Sigma^* \setminus ((\Sigma^* \setminus L_1) \cup (\Sigma^* \setminus L_2)).$$

$\square$

## 1.3 Nondeterministic Finite State Automata

**Definition 1.11.** A **nondeterministic finite state automaton** (NFA) is a tuple

$$A = (Q, \Sigma, \delta, q_0, F),$$

where each component is as follows.

- $Q$ is a finite set of **states**.
- $\Sigma$ is a finite set of input symbols.
- $\delta \subseteq Q \times \Sigma \times Q$ is a relation, called the **transition relation**.
- $q_0 \in Q$ is called the **start state**.
- $F \subseteq Q$ is called the **accepting states**.

**Definition 1.12.** Let $A = (Q, \Sigma, \delta, q_0, F)$ be an NFA. For each string $w \in \Sigma^*$, we define $\delta_w \subseteq Q \times Q$ as follows, where $a \in \Sigma$ and $x \in \Sigma^*$.

- $\delta_\epsilon = \{(p, q) : p = q\}$.
- $\delta_a = \{(p, q) : (p, a, q) \in \delta\}$.
- $\delta_{xa} = \{(p, q) : (p, r) \in \delta_x \text{ and } (r, q) \in \delta_a \text{ for some } r \in Q\}$.

**Definition 1.13.** Let $A = (Q, \Sigma, \delta, q_0, F)$ be an NFA.

- We say that $A$ accepts a string $w \in \Sigma^*$ if there exists $q \in F$ such that $(q_0, q) \in \delta_w$.
- The **language** of $A$, denoted $L(A)$, is defined as the set of strings that are accepted by $A$.

**Theorem 1.14.** For every NFA $A$, there is a DFA $A'$ with $L(A') = L(A)$.

*Proof.* Let $A = (Q, \Sigma, \delta, q_0, F)$. We construct $A' = (\mathcal{P}(Q), \Sigma, \Delta, \{q_0\}, \Phi)$ as follows.

- $\Delta : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q)$ is the function with

  $$\Delta_a(P) = \bigcup_{p \in P} \{q \in Q : (p, q) \in \delta_a\}$$

  for any $P \subseteq Q$ and $a \in \Sigma$.
- $\Phi = \{P \subseteq Q : P \cap F \neq \varnothing\}$.

Now we prove that for any $w \in \Sigma^*$, for any $q \in Q$ and for any $P \subseteq Q$, we have $q \in \Delta_w(P)$ if and only if $(p, q) \in \delta_w$ for some $p \in P$. For the induction basis, let $w = \epsilon$, and we have

$$q \in \Delta_\epsilon(P) \quad \Leftrightarrow \quad q \in P \quad \Leftrightarrow \quad (p, q) \in \delta_\epsilon \text{ for some } p \in P.$$

For the induction step, let $w = xa$, where $x$ is any string and $a$ is any symbol. Note that by the construction of $\Delta$, we have $q \in \Delta_a(P)$ if and only if $(p, q) \in \delta_a$ for some $p \in P$. Thus, we can conclude that

$$\begin{aligned}
q \in \Delta_{xa}(P) \quad &\Leftrightarrow \quad q \in \Delta_a(\Delta_x(P)) \\
&\Leftrightarrow \quad (r, q) \in \delta_a \text{ for some } r \in \Delta_x(P) \\
&\phantom{\Leftrightarrow} \quad \text{and } (p, r) \in \delta_x \text{ for some } p \in P \\
&\Leftrightarrow \quad (p, q) \in \delta_{xa} \text{ for some } p \in P.
\end{aligned}$$

Finally we prove that $L(A') = L(A)$, which is given by

$$\begin{aligned}
w \in L(A') \quad &\Leftrightarrow \quad \Delta_w(\{q_0\}) \in \Phi \\
&\Leftrightarrow \quad \Delta_w(\{q_0\}) \cap F \neq \varnothing \\
&\Leftrightarrow \quad q \in \Delta_w(\{q_0\}) \text{ for some } q \in F \\
&\Leftrightarrow \quad (p, q) \in \delta_w \text{ for some } q \in F \text{ and } p \in \{q_0\} \\
&\Leftrightarrow \quad (q_0, q) \in \delta_w \text{ for some } q \in F \\
&\Leftrightarrow \quad w \in L(A).
\end{aligned}$$

□

**Theorem 1.15.** If $L_1$ and $L_2$ are regular languages over $\Sigma$, then $L_1L_2$ is also regular.

*Proof.* Let $A_1 = (Q_1, \Sigma, \delta^{(1)}, q_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta^{(2)}, q_2, F_2)$ be NFAs such that $L_1 = L(A_1)$ and $L_2 = L(A_2)$. We construct an NFA

$$A = (Q_1 \cup Q_2, \Sigma, \delta, q_1, F)$$

as follows.

- $\delta = \delta^{(1)} \cup \delta^{(2)} \cup \{(p, a, q) \in Q \times \Sigma \times Q : p \in F_1 \text{ and } (q_2, a, q) \in \delta^{(2)}\}$.
- If $q_2 \in F_2$, let $F = F_1 \cup F_2$. Otherwise, let $F = F_2$.

It can be shown that $L(A) = L(A_1)L(A_2)$, and thus $L_1L_2$ is regular. □

**Theorem 1.16.** If $L$ is a regular language over $\Sigma$, then $L^*$ is also regular.

*Proof.* Let $A = (Q, \Sigma, \delta, q_0, F)$ be an NFA with $L = L(A)$. We construct an NFA

$$A' = (Q \cup \{q_0'\}, \Sigma, \delta', q_0', F \cup \{q_0'\})$$

with

$$\delta' = \delta \cup \{(p, a, q) \in Q \times \Sigma \times Q : p \in F \cup \{q_0'\} \text{ and } (q_0, a, q) \in \delta\}.$$

It can be shown that $L(A') = (L(A))^*$, and thus $L^*$ is regular. □

## 1.4 Regular Expressions

**Definition 1.17.** Let $\Sigma$ be an alphabet. A **regular expression** over $\Sigma$ is a string in the minimal language over $\Sigma \cup \{\varnothing, \epsilon, ^*, +, (, )\}$ that satisfies the following conditions.

1. $\varnothing$ is a regular expression.
2. $\epsilon$ is a regular expression.
3. If $a \in \Sigma$, then $a$ is a regular expression.
4. If $e_1$ and $e_2$ are regular expressions, then so is $(e_1 e_2)$.
5. If $e_1$ and $e_2$ are regular expressions, then so is $(e_1 + e_2)$.
6. If $e$ is a regular expression, then so is $(e)^*$.

**Definition 1.18.** A regular expression $e$ over an alphabet $\Sigma$ defines a language $L(e)$ as follows.

1. $L(\varnothing) = \varnothing$.
2. $L(\epsilon) = \{\epsilon\}$.
3. $L(a) = \{a\}$ for each $a \in \Sigma$.
4. $L((e_1 e_2)) = L(e_1)L(e_2)$ for each regular expressions $e_1$ and $e_2$.
5. $L((e_1 + e_2)) = L(e_1) \cup L(e_2)$ for each regular expressions $e_1$ and $e_2$.
6. $L((e)^*) = L(e^*)$ for each regular expression $e$.

**Remark.** From now on, we may omit parentheses if there is no ambiguity.

**Lemma 1.19.** If $e$ is a regular expression over an alphabet $\Sigma$, then $L(e)$ is regular.

*Proof.* It can be easily shown that $\varnothing$ and $\{\epsilon\}$ are regular. Moreover, $\{a\}$ is regular for each $a \in \Sigma$. Thus, by Theorem 1.9, Theorem 1.15 and Theorem 1.16, we can conclude that for all regular expressions $e$, $L(e)$ is regular. □

**Lemma 1.20.** If $L$ is a regular language over an alphabet $\Sigma$, then there is a regular expression $e$ over $\Sigma$ such that $L(e) = L$.

*Proof.* Since $L$ is regular, there exists a DFA $A = (Q, \Sigma, \delta, q_0, F)$ with $L(A) = L$. Suppose that $Q = \{p_1, p_2, \ldots, p_n\}$ with $p_1 = q_0$. For any $i, j \in \{1, \ldots, n\}$ and for any $k \in \{0, \ldots, n\}$, let $L_{ij}^{(k)}$ denote the language of strings $w$ such that

- $\delta_w(p_i) = p_j$, and
- for each string $x$ with $\epsilon \sqsubset x \sqsubset w$, we have $\delta_x(p_i) = p_\ell$ for some $\ell \in \{1, \ldots, k\}$.

We are going to prove that for all $i, j \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, n\}$, there exists a regular expression $e_{ij}^{(k)}$ such that

$$L\big(e_{ij}^{(k)}\big) = L_{ij}^{(k)}.$$

The proof is by induction on $k$. For the induction basis, let $k = 0$. Let $\Pi_{ij} \subseteq \Sigma$ denote the set of symbols $a$ with $\delta_a(p_i) = p_j$. If $i \neq j$, we have

$$L_{ij}^{(0)} = \bigcup_{a \in \Pi_{ij}} \{a\},$$

and thus we can construct $e_{ij}^{(0)}$ by

$$e_{ij}^{(0)} = \sum_{a \in \Pi_{ij}} a.$$

(If $\Pi_{ij} = \varnothing$, then the summation is defined as $\varnothing$.) If $i = j$, we have

$$L_{ii}^{(0)} = \{\epsilon\} \cup \bigcup_{a \in \Pi_{ii}} \{a\},$$

and thus we can construct $e_{ii}^{(0)}$ by

$$e_{ii}^{(0)} = \epsilon + \sum_{a \in \Pi_{ii}} a.$$

Now for the induction step, let $k \geq 1$. Suppose that $w \in L_{ij}^{(k)}$. If there is no string $x$ with $\epsilon \sqsubset x \sqsubset w$ such that $\delta_x(p_i) = p_k$, then we have

$$w \in L_{ij}^{(k-1)}.$$

Otherwise, let $x_0, x_1, \ldots, x_\ell$ be all strings with $\epsilon \sqsubset x_0 \sqsubset x_1 \sqsubset \cdots \sqsubset x_\ell \sqsubset w$ such that

$$\delta_{x_0}(p_i) = \delta_{x_1}(p_i) = \cdots = \delta_{x_\ell}(p_i) = p_k.$$

Let $u_0, u_1, \ldots, u_{\ell+1}$ be strings such that

$$w = u_0 u_1 \cdots u_{\ell+1},$$

and $x_h = u_0 u_1 \cdots u_h$ for each $h \in \{0, \ldots, \ell\}$. Since $u_0 \in L_{ik}^{(k-1)}$, $u_{\ell+1} \in L_{kj}^{(k-1)}$, and $u_h \in L_{kk}^{(k-1)}$ for each $h \in \{1, \ldots, \ell\}$, it follows that

$$w \in L_{ik}^{(k-1)} \left(L_{kk}^{(k-1)}\right)^* L_{kj}^{(k-1)}.$$

As a result, we have

$$L_{ij}^{(k)} \subseteq L_{ij}^{(k-1)} \cup L_{ik}^{(k-1)} \left(L_{kk}^{(k-1)}\right)^* L_{kj}^{(k-1)},$$

implying

$$L_{ij}^{(k)} = L_{ij}^{(k-1)} \cup L_{ik}^{(k-1)} \left(L_{kk}^{(k-1)}\right)^* L_{kj}^{(k-1)}.$$

Therefore, we can construct $e_{ij}^{(k)}$ by

$$e_{ij}^{(k)} = e_{ij}^{(k-1)} + e_{ik}^{(k-1)} \left(e_{kk}^{(k-1)}\right)^* e_{kj}^{(k-1)}.$$

Now we can construct the regular expression $e$ with $L(e) = L$ as follows. Let $\Phi$ be the set of integers $j \in \{1, \ldots, n\}$ such that $p_j \in F$. Since

$$L = \bigcup_{j \in \Phi} L_{1j}^{(n)},$$

we can construct $e$ by

$$e = \sum_{j \in \Phi} e_{1j}^{(n)},$$

which completes the proof. □

**Theorem 1.21.** Let $\Sigma$ be an alphabet. A language $L$ over $\Sigma$ is regular if and only if there is a regular expression $e$ over $\Sigma$ such that $L(e) = L$.

*Proof.* Straightforward by Lemma 1.19 and Lemma 1.20. □

# Chapter 2

# Context-Free Languages

## 2.1 Context-Free Grammars

**Definition 2.1.** A **context-free grammar (CFG)** is a 4-tuple

$$G = (V, \Sigma, R, S),$$

where each component is as follows.

- $V$ is a finite set of **variables**.
- $\Sigma$ is a finite set of **terminals** with $V \cap \Sigma = \varnothing$.
- $R$ is a finite set of **rules**, where each rule is a pair $(A, \gamma)$ with $A \in V$ and $\gamma \in (V \cup \Sigma)^*$.
- $S$ is a special variable in $V$, called the **start variable**.

**Definition 2.2.** Let $G = (V, \Sigma, R, S)$ be a CFG.

- For any $A \in V$ and for any $\alpha, \beta, \gamma \in (V \cup \Sigma)^*$, we say that $\alpha A \beta$ **yields** $\alpha\gamma\beta$ under $G$, denoted by
  $$\alpha A \beta \underset{G}{\Rightarrow} \alpha\gamma\beta,$$
  if there is a rule $(A, \gamma) \in R$.
- For any $\alpha, \beta \in (V \cup \Sigma)^*$, we say that $\alpha$ **derives** $\beta$ under $G$, denoted by
  $$\alpha \overset{*}{\underset{G}{\Rightarrow}} \beta,$$
  if $\alpha = \beta$, or there exists $\gamma \in (V \cup \Sigma)^*$ such that $\alpha \overset{*}{\underset{G}{\Rightarrow}} \gamma$ and $\gamma \underset{G}{\Rightarrow} \beta$.

**Definition 2.3.** Let $G = (V, \Sigma, R, S)$ be a CFG. For any string $w \in \Sigma^*$, if

$$S \overset{*}{\underset{G}{\Rightarrow}} w,$$

then we say that $G$ **accepts** $w$. The set of string accepted by $G$ is called the **language** of $G$, denoted by $L(G)$. A language $L$ is **context-free** if $L = L(G)$ for some CFG $G$.

## 2.2 Pushdown Automata

**Definition 2.4.** A **pushdown automaton (PDA)** is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, s, \perp, F),$$

where each component is as follows.

- $Q$ is a finite set of **states**.
- $\Sigma$ is a finite alphabet, called the **input alphabet**.
- $\Gamma$ is a finite alphabet, called the **stack alphabet**.
- $\delta \subseteq Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \times Q \times \Gamma^*$ is the **transition relation**.
- $s \in Q$ is the **initial state**.
- $\perp \in \Gamma$ is the **initial stack symbol**.
- $F \subseteq Q$ is the set of **final states**.

**Definition 2.5.** A **configuration** of a PDA $M = (Q, \Sigma, \Gamma, \delta, s, \perp, F)$ is a triple

$$(q, w, \gamma),$$

where $q \in Q$ is the current state, $w \in \Sigma^*$ is the unprocessed input, and $\gamma \in \Gamma^*$ is the current stack. The **initial configuration** of $M$ on input string $w$ is $(s, w, \perp)$.

We define the single-step relation $\vdash_M$ such that for any $p, q \in Q$, $a \in \Sigma$, $h \in \Gamma$, $w \in \Sigma^*$ and $\beta, \gamma \in \Gamma^*$,

$$(p, aw, h\gamma) \vdash_M (q, w, \beta\gamma)$$

holds if and only if $(p, a, h, q, \beta) \in \delta$. The reflexive transitive closure of $\vdash_M$ is denoted by $\vdash_M^*$.

**Definition 2.6.** Let $M = (Q, \Sigma, \Gamma, \delta, s, \perp, F)$ be a PDA. A string $w \in \Sigma^*$ is **accepted** by $M$ if

$$(s, w, \perp) \vdash_M^* (q, \epsilon, \gamma)$$

for some $q \in F$ and $\gamma \in \Gamma^*$. The **language** $L(M)$ accepted by $M$ is defined as the collection of strings that are accepted by $M$.

**Theorem 2.7.** If $L$ is context-free, then there is a PDA $M$ that accepts $L$.

# Chapter 3

# Decidability

## 3.1 Turing Machines

**Definition 3.1.** A **Turing machine** is an 8-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, q_{\text{acc}}, q_{\text{rej}}),$$

where each component is as follows.

- $Q$ is the finite set of **states**.
- $\Sigma$ is the finite set of **input symbols**.
- $\Gamma$ is the finite set of **tape symbols** with $\Sigma \subseteq \Gamma$.
- $\delta : (Q \setminus \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \to Q \times \Gamma \times \{-1, 0, +1\}$ is the **transition function**.
- $q_0 \in Q$ is the **initial state**.
- $\sqcup \in \Gamma \setminus \Sigma$ is a special symbol, called the **blank symbol**.
- $q_{\text{acc}}$ and $q_{\text{rej}}$ are distinct states in $Q$, called the **accepting state** and the **rejecting state**, respectively.

**Definition 3.2.** Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, q_{\text{acc}}, q_{\text{rej}})$ be a Turing machine.

- A **configuration** of $M$ is a triple in $Q \times \{1, 2, \dots\} \times \Gamma^*$.
- We define a binary relation $\vdash_M$ over $Q \times \{1, 2, \dots\} \times \Gamma^*$ such that for any $p, q \in Q$, $i, j \in \{1, 2, \dots\}$ and $u, v \in \Gamma^*$,

  $$(p, i, u) \underset{M}{\vdash} (q, j, v)$$

  if and only if

  $$\begin{aligned} u^{(1)} \cdots u^{(i-1)} u^{(i+1)} \cdots u^{(n)} \sqcup \sqcup \cdots &= v^{(1)} \cdots v^{(i-1)} v^{(i+1)} \cdots v^{(m)} \sqcup \sqcup \cdots \\ \delta(p, u^{(i)}) &= (q, v^{(i)}, j - i). \end{aligned}$$

  Let $\vdash_M^{(n)}$ denote the $n$th power of $\vdash_M$, and let $\vdash_M^* = \bigcup_{n \in \mathbb{N}} \vdash_M^{(n)}$.

**Remark.** $\vdash_M$ is a partial function.

**Definition 3.3.** Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, q_{\text{acc}}, q_{\text{rej}})$ be a Turing machine. Let $u \in \Sigma^*$.

- We say that $M$ **accepts** $u$ if $(q_0, 1, w) \vdash_M^* (q_{\text{acc}}, j, v)$ for some $j \in \{1, 2, \dots\}$ and $v \in \Gamma^*$.
- We say that $M$ **rejects** $u$ if $(q_0, 1, w) \vdash_M^* (q_{\text{rej}}, j, v)$ for some $j \in \{1, 2, \dots\}$ and $v \in \Gamma^*$.
- We say that $M$ **halts** on input $u$ if $M$ either accepts or rejects $u$.

If $M$ halts on $u$, then we have the following definitions.

- The **running time** of $M$ on input $u$ is the integer $t$ such that

$$(q_0, 1, u) \overset{(t)}{\underset{M}{\vdash}} (q_{\text{acc}}, j, v) \qquad \text{or} \qquad (q_0, 1, u) \overset{(t)}{\underset{M}{\vdash}} (q_{\text{rej}}, j, v),$$

  where $j \in \{1, 2, \dots\}$ and $v \in \Sigma^*$.

- The **accessed space** of $M$ on input $u$ is the maximum integer $s$ such that

$$(q_0, 1, u) \overset{*}{\underset{M}{\vdash}} (q, s, v),$$

  where $q \in Q$ and $v \in \Sigma^*$.

**Definition 3.4.** Let $L$ be a language over $\Sigma$. Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, q_{\text{acc}}, q_{\text{rej}})$ be a Turing machine.

- We say that $M$ **recognizes** $L$ if for each $w \in \Sigma^*$,

$$w \in L \quad \Leftrightarrow \quad M \text{ accepts } w.$$

  A language is **recursively enumerable** if it is recognized by some Turing machine. The collection of recursively enumerable languages is denoted by **RE**.

- We say that $M$ **decides** $L$ if for each $w \in \Sigma^*$,

$$\begin{aligned} w \in L \quad &\Rightarrow \quad M \text{ accepts } w \\ w \notin L \quad &\Rightarrow \quad M \text{ rejects } w. \end{aligned}$$

  A language is **recursive** if it is decided by some Turing machine. The collection of recursive languages is denoted by **R**.

**Remark.** If $M$ decides $L$, then $M$ recognizes $L$. Thus, $\mathbf{R} \subseteq \mathbf{RE}$.

## 3.2 Variants of Turing Machines

**Definition 3.5.** A ***k*-tape Turing machine** is

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, q_{\mathrm{acc}}, q_{\mathrm{rej}}),$$

where

$$\delta : (Q \setminus \{q_{\mathrm{acc}}, q_{\mathrm{rej}}\}) \times \Gamma^k \to Q \times \Gamma^k \times \{-1, 0, +1\}^k$$

is the **transition function** and other components are the same as those in the definition of Turing machine.

- A **configuration** of $M$ is a triple in $Q \times \{1, 2, \dots\}^k \times (\Gamma^*)^k$, and we define the binary relation $\vdash_M$ over the configurations of $M$ such that for any $p, q \in Q$, $i_1, \dots, i_k, j_1, \dots, j_k \in \{1, 2, \dots\}$ and $u_1, \dots, u_k, v_1, \dots, v_k \in \Gamma^*$,

  $$(p, (i_1, \dots, i_k), (u_1, \dots, u_k)) \quad \underset{M}{\vdash} \quad (q, (j_1, \dots, j_k), (v_1, \dots, v_k))$$

  if and only if

  $$u_\kappa^{(1)} \cdots u_\kappa^{(i_\kappa - 1)} u_\kappa^{(i_\kappa + 1)} \cdots u_\kappa^{(n_\kappa)} \sqcup \sqcup \cdots \quad = \quad v_\kappa^{(1)} \cdots v_\kappa^{(i_\kappa - 1)} v_\kappa^{(i_\kappa + 1)} \cdots v_\kappa^{(m_\kappa)} \sqcup \sqcup \cdots$$

  for all $\kappa \in \{1, \dots, k\}$ and

  $$\delta(p, (u_1^{(i_1)}, \dots, u_k^{(i_k)})) \quad = \quad (q, (v_1^{(i_1)}, \dots, v_k^{(i_k)}), (j_1 - i_1, \dots, j_k - i_k)).$$

- We say that $M$ **accepts** (resp., **rejects**) $w \in \Sigma^*$ if

  $$(q_0, (1, 1, \dots, 1), (w, \epsilon, \dots, \epsilon)) \quad \overset{*}{\underset{M}{\vdash}} \quad (q_{\mathrm{acc}}, (j_1, j_2, \dots, j_k), (v_1, v_2, \dots, v_k))$$
  $$\Big(\text{resp., } (q_0, (1, 1, \dots, 1), (w, \epsilon, \dots, \epsilon)) \quad \overset{*}{\underset{M}{\vdash}} \quad (q_{\mathrm{rej}}, (j_1, j_2, \dots, j_k), (v_1, v_2, \dots, v_k))\Big)$$

  for some $j_1, j_2, \dots, j_k \in \{1, 2, \dots\}$ and $v_1, v_2, \dots, v_k \in \Gamma^*$. If $M$ either accepts or rejects $w$, then we say that $M$ **halts** on $w$.

- If $M$ halts on $w \in \Sigma^*$, then the **running time** of $M$ on input $w$ is the integer $t$ with

  $$(q_0, (1, 1, \dots, 1), (w, \epsilon, \dots, \epsilon)) \quad \overset{(t)}{\underset{M}{\vdash}} \quad (q_{\mathrm{acc}}, (j_1, j_2, \dots, j_k), (v_1, v_2, \dots, v_k))$$

  or

  $$(q_0, (1, 1, \dots, 1), (w, \epsilon, \dots, \epsilon)) \quad \overset{(t)}{\underset{M}{\vdash}} \quad (q_{\mathrm{rej}}, (j_1, j_2, \dots, j_k), (v_1, v_2, \dots, v_k)),$$

  and the **accessed space** of $M$ on input $w$ is the maximum sum of integers $s_1, \dots, s_k$ with

  $$(q_0, (1, 1, \dots, 1), (w, \epsilon, \dots, \epsilon)) \overset{*}{\underset{M}{\vdash}} (q, (j_1, j_2, \dots, j_k), (v_1, v_2, \dots, v_k)),$$

  where $q \in Q$, $j_1, j_2, \dots, j_k \in \{1, 2, \dots\}$ and $v_1, v_2, \dots, v_k \in \Gamma^*$.

**Theorem 3.6.** Every $k$-tape Turing machine has an equivalent 1-tape Turing machine.

*Proof.* Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, q_{\text{acc}}, q_{\text{rej}})$ be a $k$-tape Turing machine. We show how to convert $M$ to an equivalent 1-tape Turing machine $M' = (Q', \Sigma, \Gamma', \delta', q_0, \sqcup, q_{\text{acc}}, q_{\text{rej}})$, where we use

$$\Gamma' = \Gamma \times \{\square, \dot{\square}\} \cup \{\#\}$$

as the tape symbols. On input $w \in \Sigma^*$, the machine $M'$ performs the following steps:

1. Format the tape by turning its content from
$$w_1 \cdots w_n$$
into
$$\# \dot{w}_1 \cdots w_n \sqcup \# \dot{\sqcup} \# \dot{\sqcup} \# \cdots \# \dot{\sqcup} \#$$
such that the tape of $M'$ can simulate the $k$ tapes of $M$.

2. Continue performing steps 3 – 4 until it halts.

3. Scan the tape from the first $\#$ to the last $\#$ to determine the $k$ symbols under the dots.

4. Update the tape according to the transition function $\delta$ of $M$. If there is any $\dot{\#}$, then replace it with a $\dot{\sqcup}$ and then insert a $\#$ after it.

We have an implementation as follows.

$$\delta'(q, a) = \begin{cases} ((\text{init}, 1, \dot{a}), \triangleright, +1), & \text{if } q = (\text{init}, 1, \triangleright) \\ ((\text{init}, 1, a), b, +1), & \text{if } q = (\text{init}, 1, b) \text{ with } b \in \Gamma \cup \dot{\Gamma} \text{ and } a \neq \sqcup \\ ((\text{init}, 1, \#), b, +1), & \text{if } q = (\text{init}, 1, b) \text{ with } b \in \Gamma \cup \dot{\Gamma} \text{ and } a = \sqcup \\ ((\text{init}, \kappa + 1, \dot{\sqcup}), \#, +1), & \text{if } q = (\text{init}, \kappa, \#) \text{ with } 1 \leq \kappa \leq k - 1 \\ ((\text{init}, \text{back}), \#, 0), & \text{if } q = (\text{init}, k, \#) \\ ((\text{init}, \kappa, \#), \dot{\sqcup}, +1), & \text{if } q = (\text{init}, \kappa, \dot{\sqcup}) \text{ with } 2 \leq \kappa \leq k \\ ((\text{init}, \text{back}), a, -1), & \text{if } q = (\text{init}, \text{back}) \text{ and } a \neq \triangleright \\ ((\text{scan}, p, \epsilon), \triangleright, +1), & \text{if } q = (\text{init}, \text{back}) \text{ and } a = \triangleright \\ \cdots \end{cases}$$

It can be shown that if $M$ runs on $w$ in time $t$, then $M'$ runs on $w$ in time $O(t^2)$. $\square$

**Definition 3.7.** A **nondeterministic Turing machine** is

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, q_{\text{acc}}, q_{\text{rej}}),$$

where

$$\delta \subseteq ((Q \setminus \{q_{\text{acc}}, q_{\text{reg}}\}) \times \Gamma) \times (Q \times \Gamma \times \{-1, 0, +1\})$$

is the **transition relation** and other components are the same as those in the definition of Turing machine.

- A **configuration** of $M$ is a triple in $Q \times \{1, 2, \dots\} \times \Gamma^*$, and we define the binary relation $\vdash_M$ over the configurations of $M$ such that for any $p, q \in Q$, $i, j \in \{1, 2, \dots\}$ and $u, v \in \Gamma^*$,

$$(p, i, u) \underset{M}{\vdash} (q, j, v)$$

  if and only if

$$\begin{aligned} u^{(1)} \cdots u^{(i-1)} u^{(i+1)} \cdots u^{(n)} \sqcup \sqcup \cdots &= v^{(1)} \cdots v^{(i-1)} v^{(i+1)} \cdots v^{(m)} \sqcup \sqcup \cdots \\ ((p, u^{(i)}), (q, v^{(i)}, j - i)) &\in \delta. \end{aligned}$$

- Let $u \in \Sigma^*$. We say that $M$ **diverges** on input $u$ (i.e., $M$ does not **halt** on $u$) if for any integer $t \geq 1$ there exist $q \in Q$, $j \in \{1, 2, \dots\}$ and $v \in \Gamma^*$ such that

$$(q_0, 1, u) \overset{(t)}{\underset{M}{\vdash}} (q, j, v).$$

  We say that $M$ **accepts** $u$ if

$$(q_0, 1, u) \overset{*}{\underset{M}{\vdash}} (q_{\text{acc}}, j, v)$$

  for some $j \in \{1, 2, \dots\}$ and $v \in \Gamma^*$. We say that $M$ **rejects** $u$ if $M$ neither accepts $u$ nor diverges on $u$.

- If $M$ halts on $u \in \Sigma^*$, then the **running time** of $M$ on input $u$ is the maximum integer $t$ with

$$(q_0, 1, u) \overset{(t)}{\underset{M}{\vdash}} (q_{\text{acc}}, j, v) \qquad \text{or} \qquad (q_0, 1, u) \overset{(t)}{\underset{M}{\vdash}} (q_{\text{rej}}, j, v),$$

  and the **accessed space** of $M$ on input $u$ is the maximum integer $s$ with

$$(q_0, 1, u) \overset{*}{\underset{M}{\vdash}} (q, s, v)$$

  where $q \in Q$, $j \in \{1, 2, \dots\}$ and $v \in \Gamma^*$.

# Chapter 4

# Time Complexity

## 4.1 P

**Definition 4.1.** We define
$$\mathbf{P} = \bigcup_{k \in \mathbb{N}} \mathbf{TIME}(n^k).$$

## 4.2 NP

**Definition 4.2.** We define

$$\mathbf{NP} = \bigcup_{k \in \mathbb{N}} \mathbf{NTIME}(n^k).$$