Analysis

Chapter 1

Real Numbers

1.1 Fields

Definition 1.1. A nonempty set F and two operations $+$ and $\cdot$ form a **field** if the following axioms (A 1) – (A 5), (M 1) – (M 5) and (D) are satisfied.

(A 1) $x + y \in F$ for any $x, y \in F$.

(A 2) $x + y = y + x$ for any $x, y \in F$.

(A 3) $(x + y) + z = x + (y + z)$ for any $x, y, z \in F$.

(A 4) There is an element $0 \in F$ such that $x + 0 = x$ for any $x \in F$.

(A 5) For each $x \in F$ there is an element $-x$ in F such that $x + (-x) = 0$.

(M 1) $x \cdot y \in F$ for any $x, y \in F$.

(M 1) $x \cdot y = y \cdot x$ for any $x, y \in F$.

(M 2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for any $x, y, z \in F$.

(M 3) There is an element $1 \in F \setminus \{0\}$ such that $x \cdot 1 = x$ for any $x \in F$.

(M 4) For each $x \in F \setminus \{0\}$ there is an element x^{-1} in F such that $x \cdot x^{-1} = 0$.

(D) $x \cdot (y + z) = x \cdot y + x \cdot z$ for any $x, y, z \in F$.

Theorem 1.2. Let F be a field. Then the following statements are true for any $x, y, z \in F$.

(a) If $x + y = x + z$, then $y = z$.

(b) If $x + y = x$, then $y = 0$.

(c) If $x + y = 0$, then $y = -x$.

(d) $-(-x) = x$.

Proof. Note that these statements are consequence of axioms (A 1) – (A 5).

(a) We have

$$\begin{aligned} y &= 0 + y \\ &= (-x + x) + y \\ &= -x + (x + y) \\ &= -x + (x + z) \\ &= (-x + x) + z \\ &= 0 + z \\ &= z. \end{aligned}$$

(b) Since $x + y = x = x + 0$, we have $y = 0$ by (a).

(c) Since $x + y = 0 = x + (-x)$, we have $y = -x$ by (a).

(d) Since $-x + x = 0$, we have $-(-x) = x$ by (c). $\square$

Theorem 1.3. Let F be a field. Then the following statements are true for any $x \in F \setminus \{0\}$ and $y, z \in F$.

(a) If $x \cdot y = x \cdot z$, then $x = y$.

(b) If $x \cdot y = x$, then $y = 1$.

(c) If $x \cdot y = 1$, then $y = x^{-1}$.

(d) $(x^{-1})^{-1} = x$.

Proof. Note that these statements are consequence of axioms (M 1) – (M 5).

(a) We have

$$\begin{aligned} y &= 1 \cdot y \\ &= (x^{-1} \cdot x) \cdot y \\ &= x^{-1} \cdot (x \cdot y) \\ &= x^{-1} \cdot (x \cdot z) \\ &= (x^{-1} \cdot x) \cdot z \\ &= 1 \cdot z \\ &= z. \end{aligned}$$

(b) Since $x \cdot y = x = x \cdot 1$, we have $y = 1$ by (a).

(c) Since $x \cdot y = 1 = x \cdot x^{-1}$, we have $y = x^{-1}$ by (a).

(d) Since $x^{-1} + x = 1$, we have $(x^{-1})^{-1} = x$ by (c). $\square$

Theorem 1.4. Let F be a field. Then the following statements are true for any $x, y \in F$.

(a) $0 \cdot x = 0$.

(b) $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$.

(c) $(-x) \cdot (-y) = x \cdot y$.

Proof.

(a) We have
$$0 \cdot x + 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x,$$
implying $0 \cdot x = 0$.

(b) Since
$$(-x) \cdot y + x \cdot y = (-x + x) \cdot y = 0 \cdot y = 0,$$
we have $(-x) \cdot y = -(x \cdot y)$. One can prove $x \cdot (-y) = -(x \cdot y)$ similarly.

(c) We have
$$(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(x \cdot y)) = x \cdot y$$
by applying (b) twice. □

1.2 Ordered Fields

Definition 1.5. An **ordered field** is a field on which relation $<$ is defined such that the following axioms (O 1) – (O 4) hold for any $x, y, z \in F$.

(O 1) One and only one of the statements $x = y$, $x < y$, $y < x$ is true.

(O 2) If $x < y$ and $y < z$, then $x < z$.

(O 3) If $x < y$, then $x + z < y + z$.

(O 4) If $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Definition 1.6. Let F be an ordered field. The relations $>$, $\leq$ and $\geq$ are defined as follows for any $x, y \in F$.

$$\begin{aligned} x > y \quad &\Leftrightarrow \quad y < x \\ x \leq y \quad &\Leftrightarrow \quad x < y \text{ or } x = y \\ x \geq y \quad &\Leftrightarrow \quad x > y \text{ or } x = y. \end{aligned}$$

Definition 1.7. Let F be an ordered field and let $S \subseteq F$.

- An **upper bound** of S is an element x in F such that $x \geq y$ for any $y \in S$. We say that S is **bounded above** if S has an upper bound.
- A **lower bound** of S is an element x in F such that $x \leq y$ for any $y \in S$. We say that S is **bounded below** if S has a lower bound.

Definition 1.8. Let F be an ordered field and let $S \subseteq F$.

- An element of S is called the **maximum** of S, denoted by $\max(S)$, if it is an upper bound of S.
- An element of S is called the **minimum** of S, denoted by $\min(S)$, if it is a lower bound of S.
- The minimum of the set of upper bounds of S is called the **supremum** of S, denoted by $\sup(S)$.
- The maximum of the set of lower bounds of S is called the **infimum** of S, denoted by $\inf(S)$.

1.3 The Real Field

Definition 1.9. $\mathbb{R}$ is an ordered field such that every nonempty subset S of $\mathbb{R}$ that is bounded above has a supremum. The elements of $\mathbb{R}$ are called the **real numbers**.

Theorem 1.10 (Archimedean Property). For any $x, y \in \mathbb{R}$ with $x > 0$, there is a positive integer n such that

$$n \cdot x > y.$$

Proof. Let

$$S = \{nx : n \text{ is a positive integer}\}.$$

Suppose that y is an upper bound of S. It follows that S has a supremum z. Note that $z - x$ is not an upper bound of S since $z - x < z$. Thus, $z - x < mx$ for some positive integer m, implying $z < (m+1)x$, contradiction to the fact that z is an upper bound of S. Hence, y is not an upper bound of S, completing the proof. $\square$

Chapter 2

Basic Topology

2.1 Metric Spaces

Definition 2.1. A set X and a function $d : X \times X \to \mathbb{R}$ form a **metric space** if the following properties hold for any $x, y, z \in X$.

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ holds if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq d(x, z) + d(z, y)$.

Remark. We may use the underlying set X to represent the metric space (X, d), and in this case, the distance function d is denoted by d_X.

Definition 2.2. Let X be a metric space. For any $\epsilon > 0$ and $x \in X$, we define the **open ball** of radius ϵ centered at x by

$$B_\epsilon(x) = \{y \in X : d_X(x, y) < \epsilon\}.$$

Definition 2.3. Let X be a metric space with $S \subseteq X$ and $x \in X$.

- We say that x is an **interior point** of S if $B_\epsilon(x) \subseteq S$ for some $\epsilon > 0$. If every point of S is an interior point of S, then S is said to be **open**.
- We say that x is an **limit point** of S if $(B_\epsilon(x) \setminus \{x\}) \cap S$ is not empty for all $\epsilon > 0$. If every limit point of S is a point of S, then S is said to be **close**.

Theorem 2.4. Let X be a metric space and $S \subseteq X$. Then S is open if and only if $X \setminus S$ is closed.

Proof. ($\Rightarrow$) Suppose that x is a limit point of $X \setminus S$. Then $B_\epsilon(x) \setminus S \neq \varnothing$ for any $\epsilon > 0$, implying that x is not an interior point of S. Since S is open, we have $x \notin S$, i.e., $x \in X \setminus S$. Thus, $X \setminus S$ is closed.

($\Leftarrow$) Let $x \in S$. If x is a limit point of $X \setminus S$, then $x \in X \setminus S$ since $X \setminus S$ is closed, contradiction. Thus, x is not a limit point of $X \setminus S$, and there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq S$, implying that S is open. $\square$

Theorem 2.5. Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of open sets.

(a) $\bigcup_{\alpha \in A} S_\alpha$ is open.

(b) If A is nonempty and finite, then $\bigcap_{\alpha \in A} S_\alpha$ is open.

Proof.

(a) Suppose that $x \in \bigcup_{\alpha \in A} S_\alpha$. Then $x \in S_\alpha$ for some $\alpha \in A$. Since S_α is open, x is an interior point of S_α, and it follows that x is an interior point of $\bigcup_{\alpha \in A} S_\alpha$. Thus, $\bigcup_{\alpha \in A} S_\alpha$ is open.

(b) Suppose that $x \in \bigcap_{\alpha \in A} S_\alpha$. For each $\alpha \in A$, since S_α is open, we have $B_{\epsilon_\alpha}(x) \subseteq S_\alpha$ for some $\epsilon_\alpha > 0$. Since A is finite and nonempty, $\epsilon = \min(\{\epsilon_\alpha\}_{\alpha \in A})$ exists. It follows that $B_\epsilon(x) \subseteq \bigcap_{\alpha \in A} S_\alpha$, implying that x is an interior point of $\bigcap_{\alpha \in A} S_\alpha$. Thus, $\bigcap_{\alpha \in A} S_\alpha$ is open. □

Corollary 2.6. Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of closed sets.

(a) $\bigcap_{\alpha \in A} S_\alpha$ is closed.

(b) If A is nonempty and finite, then $\bigcup_{\alpha \in A} S_\alpha$ is closed.

Proof. Straightforward from Theorem 2.4 and Theorem 2.5. □

2.2 Compact Sets

Definition 2.7. Let (X, d) be a metric space and let $S \subseteq X$.

- A **cover** of S is a collection of subsets of X whose union contains S. An **open cover** of S is a cover of S whose elements are all open.
- We say that S is **compact** if every open cover Ω of S contains a finite cover Φ of S.

Theorem 2.8. Let (X, d) be a metric space and let $R \subseteq S \subseteq X$. If S is compact and R is closed, then R is compact.

Proof. Suppose that R has an open cover Ω. Then $\Omega' = \Omega \cup \{X \setminus R\}$ is an open cover of S since $X \setminus R$ is open. Let $\Phi' \subseteq \Omega'$ be a finite cover of S, and let $\Phi = \Phi' \setminus \{X \setminus R\}$. Then Φ is a finite open cover of R with $\Phi \subseteq \Omega$. Thus, R is compact. □

Theorem 2.9 (Nested Interval Theorem). Let $\langle I_n \rangle$ be a sequence of rectangles in $\mathbb{R}^k$ such that $I_{n+1} \subseteq I_n$, then the intersection of $\{I_n : n \in \mathbb{N}\}$ is nonempty.

Proof. For each positive integer n, let

$$I_n = [a_n^{(1)}, b_n^{(1)}] \times \cdots \times [a_n^{(k)}, b_n^{(k)}].$$

For each $i \in \{1, \ldots, k\}$, we have

$$a_n^{(i)} \leq a_{n+m}^{(i)} \leq b_{n+m}^{(i)} \leq b_m^{(i)}$$

for any $n, m \in \mathbb{N}$. Thus, $\{a_n^{(i)} : n \in \mathbb{N}\}$ is bounded above, implying the existence of

$$x_i = \sup(\{a_n^{(i)} : n \in \mathbb{N}\}).$$

We have

$$a_n^{(i)} \leq x_i \leq b_m^{(i)}$$

for any $n, m \in \mathbb{N}$. Thus,

$$x = (x_1, \ldots, x_n) \in \bigcap_{n \geq 1} I_n,$$

completing the proof. □

Theorem 2.10. Every k-cell in $\mathbb{R}^k$ is compact.

Proof. Let $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$. We have

$$\|x - x'\| \leq \sqrt{\textstyle\sum_{i=1}^{k} (a_i - b_i)^2}$$

for any $x, x' \in I$. Assume that there is an open cover $\mathcal{O}$ of I that contains no finite subcover of I. Let $c_i = (a_i + b_i)/2$ for all $i \in \{1, \ldots, n\}$, and let

$$\mathcal{C} = \{I^{(1)} \times \cdots \times I^{(k)} : I^{(i)} \in \{[a_i, c_i], [c_i, b_i]\} \text{ for } 1 \leq i \leq k\}$$

be a collection of 2^k k-cells whose union is I. Then there must be a k-cell $I' \in \mathcal{C}$ cannot be covered by any finite subset of $\mathcal{O}$, or I could be covered by that set, contradtion.

Thus, if I is not compact, then we can construct a sequence $\langle I_n \rangle$ of k-cells which are not covered by any finite subset of $\mathcal{O}$ such that $I_1 = I$, $I_{n+1} \subseteq I_n$ for any integer $n \geq 1$, and

$$\|x - x'\| \leq \frac{\sqrt{\sum_{i=1}^{k}(a_i - b_i)^2}}{2^{n-1}}$$

holds for any $x, x' \in I_n$. It follows that there is a point $y \in \bigcap\{I_n\}$, and we have $y \in S$ for some $S \in \mathcal{O}$. Since S is open, we have $B_r(y) \subseteq S$ for some $r > 0$. Let N be a positive integer such that

$$2^N > \frac{\sqrt{\sum_{i=1}^{k}(a_i - b_i)^2}}{r/2}.$$

Then for any $x \in I_N$,

$$\|x - y\| \leq \frac{\sqrt{\sum_{i=1}^{k}(a_i - b_i)^2}}{2^{N-1}} < r,$$

implying $x \in B_r(y) \subseteq S$. It follows that $I_N \subseteq S$, and $\{S\}$ is a finite subset of $\mathcal{O}$, contradtion. Thus, I is compact. □

Theorem 2.11 (Heine–Borel Theorem). Let $S \subseteq \mathbb{R}^k$. Then S is compact if and only if S is closed and bounded.

Proof. ($\Leftarrow$) If S is closed and bounded, then there is a k-cell I with $S \subseteq I$. Since I is compact, and S is closed, we conclude that S is compact.

($\Rightarrow$) Suppose that S is compact. Then S is closed. Since $\mathcal{O} = \{B_r(0_{\mathbb{R}^k}) : r \in \mathbb{N}\}$ is an open cover of S, there is $\mathcal{O}' \subseteq \mathcal{O}$ such that $S \subseteq \bigcup \mathcal{O}'$. It can be shown that $\bigcup \mathcal{O}'$ is bounded, and thus S is bounded. □

Chapter 3

Sequences and Series

Definition 3.1. Let X be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ **converges** to a point $x \in X$, denoted by

$$\lim_{n\to\infty} x_n = x,$$

if for any real number $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d_X(x_n, x) < \epsilon$$

holds for all $n \in \mathbb{N}$ with $n \geq n_0$.

- We say that $(x_n)_{n\in\mathbb{N}}$ is **convergent** if it converges to some point in X.
- We say that $(x_n)_{n\in\mathbb{N}}$ is **divergent** if it is not convergent.

Theorem 3.2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X. If $(x_n)_{n\in\mathbb{N}}$ converges to both $x \in X$ and $x' \in X$, then $x = x'$.

Proof. For any $\epsilon > 0$, there exists a positive integer N such that

$$d_X(x_n, x) < \frac{\epsilon}{2} \quad \text{and} \quad d_X(x_n, x') < \frac{\epsilon}{2}$$

hold for any integer $n \geq N$. It follows that

$$d_X(x, x') \leq d_X(x_n, x) + d_X(x_n, x') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

holds for any integer $n \geq N$. Thus, $x = x'$. $\square$

Theorem 3.3. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be real sequences with

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} b_n = M.$$

Then the following statements are true.

(a) $\lim_{n\to\infty}(a_n + b_n) = L + M$, and $\lim_{n\to\infty}(a_n - b_n) = L - M$.

(b) $\lim_{n\to\infty} a_n b_n = LM$.

(c) If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n^{-1} = L^{-1}$.

Proof.

(a) For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{\epsilon}{2} \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2},$$

implying

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) Let $C > 0$ such that $|L| \leq C$ and $|b_n| \leq C$ for any positive integer n. For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{\epsilon}{2C} \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2C},$$

implying

$$\begin{aligned} |a_n b_n - LM| &= |(a_n - L)b_n + (b_n - M)L| \\ &\leq |a_n - L||b_n| + |b_n - M||L| \\ &< \frac{\epsilon(|b_n| + L)}{2C} \\ &\leq \epsilon. \end{aligned}$$

(c) For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{|L|^2\epsilon}{2} \quad \text{and} \quad |a_n - L| < \frac{|L|}{2}.$$

It follows that

$$|a_n| = |L + (a_n - L)| \geq |L| - |a_n - L| > \frac{|L|}{2},$$

implying

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \left|\frac{a_n - L}{a_n L}\right| = \frac{|a_n - L|}{|a_n||L|} < \frac{2|a_n - L|}{|L|^2} < \epsilon.$$

□

Definition 3.4. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

- We say that $(a_n)_{n\in\mathbb{N}}$ is **increasing** (resp., **strictly increasing**) if $a_n \leq a_{n+1}$ (resp., $a_n < a_{n+1}$) holds for all $n \in \mathbb{N}$.
- We say that $(a_n)_{n\in\mathbb{N}}$ is **decreasing** (resp., **strictly decreasing**) if $a_n \geq a_{n+1}$ (resp., $a_n > a_{n+1}$) holds for all $n \in \mathbb{N}$.

Theorem 3.5. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If $(a_n)_{n\in\mathbb{N}}$ is increasing and its range is bounded above, then $(a_n)_{n\in\mathbb{N}}$ converges.

Proof. Let $L = \sup(\{a_n\}_{n\in\mathbb{N}})$. For any $\epsilon > 0$, since $L - \epsilon$ is not an upper bound of $\{a_n\}_{n\in\mathbb{N}}$, there exists $n_0 \in \mathbb{N}$ with $a_{n_0} > L - \epsilon$. Since $(a_n)_{n\in\mathbb{N}}$ is increasing, for any integer $n \geq n_0$ we have

$$L - \epsilon < a_{n_0} \leq a_n \leq L,$$

implying $|a_n - L| < \epsilon$. Thus, $(a_n)_{n\in\mathbb{N}}$ converges to L. □

Definition 3.6. Let X be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that

$$d_X(x_n, x_m) < \epsilon$$

holds for any $n, m \in \mathbb{N}$ with $n \geq n_0$ and $m \geq n_0$.

Chapter 4

Continuity

4.1 Limits of Functions

Definition 4.1. Let X and Y be a metric spaces and let $f : D \to Y$ be a map with $D \subseteq X$. Let $a \in X$ be a limit point and $b \in Y$. Then we say that b is the **limit** of f at a, denoted

$$\lim_{x \to a} f(x) = b,$$

if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$,

$$0 < d_X(x, a) < \delta \quad \Rightarrow \quad d_Y(f(x), b) < \epsilon.$$

4.2 Continuous Functions

Definition 4.2. Let X and Y be a metric spaces and let $f : D \to Y$ be a map with $D \subseteq X$. We say that f is **continuous** at $a \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(x), f(a)) < \epsilon$$

holds for any $x \in D$ with

$$d_X(x, a) < \delta.$$

Also, we say that f is **continuous** on D if f is continuous at every point of D.

Theorem 4.3. Let X and Y be metric spaces. Let $f : X \to Y$ be a map. Then f is continuous if and only if $f^{-1}(E)$ is open for any open set E in Y.

Proof. To be completed. □

4.3 Properties of Continuous Maps

Theorem 4.4. Let X and Y be metric spaces, and let $f : X \to Y$ be a continuous map. If $K \subseteq X$ is compact, then $f(K)$ is compact.

Proof. For any open cover $\{V_\alpha\}_{\alpha \in A}$ of $f(K)$, we have

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha).$$

Since f is continuous, $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$ is an open cover of K. Due to compactness of K, there exist $\alpha_1, \ldots, \alpha_m \in A$ such that

$$K \subseteq \bigcup_{i=1}^{m} f^{-1}(V_{\alpha_i}),$$

and we have

$$f(K) \subseteq f\left(\bigcup_{i=1}^{m} f^{-1}(V_{\alpha_i})\right) = f\left(f^{-1}\left(\bigcup_{i=1}^{m} V_{\alpha_i}\right)\right) = \bigcup_{i=1}^{m} V_{\alpha_i}.$$

Thus, $f(K)$ is compact. □

Theorem 4.5. Let X be a metric space and let $f : X \to \mathbb{R}$ be a continuous map. If $K \subseteq X$ is compact, then $\max(f(K))$ and $\min(f(K))$ exist.

Proof. Since f is continuous and K is compact, $f(K)$ is a compact subset of $\mathbb{R}$. Thus, $f(K)$ has maximum and minimum. □

Theorem 4.6 (Intermediate Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous and let $c \in \mathbb{R}$. If $f(a) < c < f(b)$, then $f(x) = c$ for some $x \in (a, b)$.

Proof. To be completed. □

Chapter 5

Differentiation

5.1 Derivatives

Definition 5.1. Let $f : D \to \mathbb{R}$ be a map with $D \subseteq \mathbb{R}$. For each $a \in \mathbb{R}$ such that $(a - \delta, a + \delta) \subseteq D$ for some $\delta > 0$, we define the **derivative** of f at a by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

We say that f is **differentiable** at a if $f'(a)$ exists.

Theorem 5.2. Let $f : D \to \mathbb{R}$ be a map with $D \subseteq \mathbb{R}$. For each $a \in \mathbb{R}$ such that $(a - \delta, a + \delta) \subseteq D$ for some $\delta > 0$, if f is differentiable at a, then f is continuous at a.

Proof. We have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{h\to 0} f(a+h) \\
&= \lim_{h\to 0} \left(f(a) + \frac{f(a+h) - f(a)}{h} \cdot h \right) \\
&= f(a) + f'(a) \cdot 0 \\
&= f(a).
\end{aligned}$$

$\square$

Theorem 5.3. Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be maps with $D \subseteq \mathbb{R}$ and $E \subseteq \mathbb{R}$. If both f and g are differentiable at $a \in \mathbb{R}$, then the following statements are true.

(a) $f + g$ is differentiable at a, and $(f + g)'(a) = f'(a) + g'(a)$.

(b) $f \cdot g$ is differentiable at a, and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

(c) If $g(a) \neq 0$, then $1/g$ is differentiable at a, and $(1/g)'(a) = -g'(a)/(g(a))^2$.

Proof.

(a) We have

$$\begin{aligned}
(f+g)'(a) &= \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h} \right) \\
&= f'(a) + g'(a).
\end{aligned}$$

(b) We have

$$\begin{aligned}
(f \cdot g)'(a) &= \lim_{h \to 0} \frac{(f \cdot g)(a+h) - (f \cdot g)(a)}{h} \\
&= \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \cdot g(a+h) + f(a) \cdot \frac{g(a+h) - g(a)}{h} \right) \\
&= f'(a) \cdot g(a) + f(a) \cdot g'(a).
\end{aligned}$$

(c) We have

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{h \to 0} \frac{(1/g)(a+h) - (1/g)(a)}{h} \\
&= \lim_{h \to 0} \frac{g(a) - g(a+h)}{hg(a)g(a+h)} \\
&= \frac{-g'(a)}{(g(a))^2}.
\end{aligned}$$

□

Theorem 5.4 (Chain Rule). Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be maps with $D \subseteq \mathbb{R}$ and $E \subseteq \mathbb{R}$. If f is differentiable at $a \in \mathbb{R}$ and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof. To be completed. □

5.2 The Mean Value Theorem

Theorem 5.5. Let $a \in \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a map with $D \subseteq \mathbb{R}$. If f is differentiable at a and f has a local maximum at a, then $f'(a) = 0$.

Proof. Assume for contradiction that $f'(a) \neq 0$. Choose $\delta > 0$ such that $f(x) \leq f(a)$ and

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \frac{|f'(a)|}{2}$$

hold for all $x \in (a - \delta, a + \delta)$. If $f'(a) > 0$, then

$$f(x) - f(a) > \frac{f'(a)(x - a)}{2} > 0$$

for all $x \in (a, a + \delta)$, contradiction. If $f'(a) < 0$, then

$$f(x) - f(a) > \frac{f'(a)(x - a)}{2} > 0$$

for all $x \in (a - \delta, a)$, contradiction. Thus, $f'(a) = 0$. $\square$

Chapter 6

Integration

6.1 Integrals

Definition 6.1. Let $[a, b]$ be a given interval. A **partition** of $[a, b]$ is a finite set

$$P = \{x_0, x_1, \ldots, x_n\} \qquad \text{with} \qquad a \leq x_0 < x_1 < \cdots < x_{n-1} < x_n \leq b.$$

For every partition P of $[a, b]$, the **upper sum** of f with respect to P is defined by

$$U(f, P) = \sum_{i=1}^{n} \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \cdot (x_i - x_{i-1}),$$

and the **lower sum** of f with respect to P is defined by

$$L(f, P) = \sum_{i=1}^{n} \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \cdot (x_i - x_{i-1}).$$

Finally, we define the **upper integral** and the **lower integral** of f on $[a, b]$ by

$$\overline{\int_a^b} f(x)\, dx = \inf\{U(f, P) : P \text{ is a parition of } [a, b]\}$$

and

$$\underline{\int_a^b} f(x)\, dx = \sup\{L(f, P) : P \text{ is a parition of } [a, b]\},$$

respectively. If

$$\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx,$$

then we say that f is **integrable** on $[a, b]$, and this common value is denoted by

$$\int_a^b f(x)\, dx.$$

Chapter 7

Sequences of Functions

7.1 Pointwise Convergence

Definition 7.1. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued functions defined on $D \subseteq \mathbb{R}$. We say that $(f_n)_{n\in\mathbb{N}}$ **converges pointwise** to a function f defined on D if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in D$.

7.2 Uniform Convergence

Definition 7.2. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued functions defined on $D \subseteq \mathbb{R}$. We say that $(f_n)_{n\in\mathbb{N}}$ **converges uniformly** to a function f defined on D, if for each $\epsilon \in \mathbb{R}^+$ there exists $n_0 \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \epsilon$$

for all $x \in D$ and $n \in \mathbb{N}$.