Analysis

Chapter 1

Real Numbers

1.1 Ordered Fields

Definition 1.1. An **ordered field** is a set F on which addition $+ : F \times F \to F$, multiplication $\cdot : F \times F \to F$ and a binary relation $<$ are defined that satisfies the following axioms.

(A 1) $x + y = y + x$ for any $x, y \in F$.

(A 2) $(x + y) + z = x + (y + z)$ for any $x, y, z \in F$.

(A 3) There is an element 0 in F such that $x + 0 = x$ for any $x \in F$.

(A 4) For each $x \in F$ there is an element $-x$ in F such that $x + (-x) = 0$.

(M 1) $x \cdot y = y \cdot x$ for any $x, y \in F$.

(M 2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for any $x, y, z \in F$.

(M 3) There is an element 1 in $F \setminus \{0\}$ such that $x \cdot 1 = x$ for any $x \in F$.

(M 4) For each $x \in F \setminus \{0\}$ there is an element x^{-1} in F such that $x \cdot x^{-1} = 0$.

(D) $x \cdot (y + z) = x \cdot y + x \cdot z$ for any $x, y, z \in F$.

(O 1) Exactly one of the statements $x = y$, $x < y$, $y < x$ holds for any $x, y \in F$.

(O 2) $x < y$ and $y < z$ implies $x < z$ for any $x, y, z \in F$.

(O 3) $x < y$ implies $x + z < y + z$ for any $x, y, z \in F$.

(O 4) $0 < x$ and $0 < y$ implies $0 < xy$ for any $x, y \in F$.

Chapter 2

Basic Topology

2.1 Metric Spaces

Definition 2.1. A set X with a function $d : X \times X \to \mathbb{R}$ is a **metric space** if the following statements hold for any $x, y, z \in X$.

(a) $d(x, y) \geq 0$.

(b) $d(x, y) = 0$ if and only if $x = y$.

(c) $d(x, y) = d(y, x)$.

(d) $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 2.2. Let (X, d) be a metric space. Let $r > 0$ be a real number and let $x_0 \in X$. The **open ball** of radius r centered at x_0, denoted by $B_r(x_0)$, is defined by

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}.$$

Definition 2.3. Let (X, d) be a metric space and let $S \subseteq X$.

- S is **open** if for any $x \in S$, there is a real number $r > 0$ such that $B_r(x) \subseteq S$.
- S is **closed** if $X \setminus S$ is open.

Theorem 2.4. Let (X, d) be a metric space.

(a) X and $\varnothing$ are open.

(b) If S_1, S_2 are open subsets of X, then $S_1 \cap S_2$ is open.

(c) If $\{S_i : i \in I\}$ is a collection of open subsets of X, then

$$\bigcup_{i \in I} S_i$$

is open.

2.2 Compact Sets

Definition 2.5. Let (X, d) be a metric space and let $S \subseteq X$. An **open cover** of S is a collection $\{R_i : i \in I\}$ of open subsets of X such that

$$S \subseteq \bigcup_{i \in I} R_i.$$

Definition 2.6. Let (X, d) be a metric space and let $S \subseteq X$. We say that S is **compact** if for any open cover $\{R_i : i \in I\}$ of S there exist finitely many indices $i_1, \ldots, i_n \in I$ such that

$$S \subseteq \bigcup_{k=1}^{n} R_{i_k}.$$

Chapter 3

Sequences and Series

Definition 3.1. Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ **converges** to a point $x \in X$, denoted by

$$\lim_{n\to\infty} x_n = x,$$

if for any real number $\epsilon > 0$ there is a positive integer N such that $n \geq N$ implies $d(x_n, x) < \epsilon$.

- We say that $\{x_n\}$ is **convergent** if it converges to some point in X.
- We say that $\{x_n\}$ is **divergent** if it is not convergent.

Theorem 3.2. Let $\{x_n\}$ be a sequence in a metric space (X, d). If $\{x_n\}$ converges to both $x \in X$ and $x' \in X$, then $x = x'$.

Proof. For any $\epsilon > 0$, there exists a positive integer N such that

$$d(x_n, x) < \frac{\epsilon}{2} \quad \text{and} \quad d(x_n, x') < \frac{\epsilon}{2}$$

for each $n \geq N$, implying

$$d(x, x') \leq d(x_n, x) + d(x_n, x') < \epsilon. \qquad \square$$

Theorem 3.3. Let $\{a_n\}$ and $\{b_n\}$ be complex sequences with

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} b_n = M.$$

Let c be a complex number. Then the following statements are true.

(a) We have
$$\lim_{n\to\infty} (a_n + b_n) = L + M.$$

(b) We have
$$\lim_{n\to\infty} a_n b_n = LM.$$

(c) If $L \neq 0$ and $a_n \neq 0$ for each positive integer n, we have

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}.$$

Proof.

(a) For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{\epsilon}{2} \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2},$$

implying

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) Let $C > 0$ such that $|L| \leq C$ and $|b_n| \leq C$ for any positive integer n. For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{\epsilon}{2C} \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2C},$$

implying

$$\begin{aligned} |a_n b_n - LM| &= |(a_n - L)b_n + (b_n - M)L| \\ &\leq |a_n - L||b_n| + |b_n - M||L| \\ &< \frac{\epsilon(|b_n| + L)}{2C} \\ &\leq \epsilon. \end{aligned}$$

(c) For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{|L|^2\epsilon}{2} \quad \text{and} \quad |a_n - L| < \frac{|L|}{2}.$$

It follows that

$$|a_n| = |L + (a_n - L)| \geq |L| - |a_n - L| > \frac{|L|}{2},$$

implying

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \left|\frac{a_n - L}{a_n L}\right| = \frac{|a_n - L|}{|a_n||L|} < \frac{2|a_n - L|}{|L|^2} < \epsilon. \qquad \square$$

Chapter 4

Continuity

Definition 4.1. Let (X, d_X) and (Y, d_Y) be a metric spaces and let $S \subseteq X$. Let $f : S \to Y$ be a map. Then we say that $b \in Y$ is the **limit** of f at $a \in X$, denoted by

$$\lim_{x \to a} f(x) = b,$$

if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(x), b) < \epsilon$$

holds for any $x \in S$ with

$$0 < d_X(x, a) < \delta.$$

Chapter 5

Differentiation

Chapter 6

Integration