Linear Algebra

Chapter 1

Vector Spaces

1.1 Fields

Definition 1.1. A **field** is a set F with two operations, called **addition** (denoted by $+$) and **multiplication** (denoted by $\cdot$), which satisfy the following axioms.

(A 1) If $a \in F$ and $b \in F$, then $a + b \in F$.

(A 2) $a + b = b + a$ for all $a, b \in F$.

(A 3) $(a + b) + c = a + (b + c)$ for all $a, b, c \in F$.

(A 4) There is an element 0_F in F such that $0_F + a = a$ for all $a \in F$.

(A 5) For each $a \in F$ there is an element $-a$ in F such that $a + (-a) = 0_F$.

(M 1) If $a \in F$ and $b \in F$, then $a \cdot b \in F$.

(M 2) $a \cdot b = b \cdot a$ for all $a, b \in F$.

(M 3) $(a \cdot b) + c = a + (b \cdot c)$ for all $a, b, c \in F$.

(M 4) There is an element 1_F in $F \setminus \{0_F\}$ such that $1_F \cdot a = a$ for all $a \in F$.

(M 5) For each $a \in F \setminus \{0_F\}$ there is an element a^{-1} in F such that $a \cdot a^{-1} = 1_F$.

(D) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$.

Example. The set $\mathbb{Q}$ of rational numbers, the set $\mathbb{R}$ of real numbers, and the set $\mathbb{C}$ of complex numbers are fields.

Proposition 1.2. Let F be a field with $a, b, c \in F$.

(a) If $a + b = a + c$, then $b = c$.

(b) If $a + b = a$, then $b = 0_F$.

(c) If $a + b = 0_F$, then $b = -a$.

(d) $-(-a) = a$.

Proof.

(a) It can be proved by

$$
\begin{aligned}
b &= 0_F + b \\
&= (-a + a) + b \\
&= -a + (a + b) \\
&= -a + (a + c) \\
&= (-a + a) + c \\
&= 0_F + c \\
&= c.
\end{aligned}
$$

(b) By applying (a), it follows from $a + b = a + 0_F$ that $b = 0_F$.

(c) By applying (a), it follows from $a + b = a + (-a)$ that $b = -a$.

(d) Since $-a + a = 0_F$, we have $a = -(-a)$ by (c). □

Proposition 1.3. Let F be a field with $a, b, c \in F$ and $a \neq 0_F$.

(a) If $a \cdot b = a \cdot c$, then $b = c$.

(b) If $a \cdot b = a$, then $b = 1_F$.

(c) If $a \cdot b = 1_F$, then $b = a^{-1}$.

(d) $(a^{-1})^{-1} = a$.

Proof. The proof is omitted since it is similar to that of Proposition 1.2. □

Proposition 1.4. Let F be a field with $a, b \in F$.

(a) $0_F \cdot a = 0_F$.

(b) $(-a) \cdot b = -(a \cdot b) = a \cdot (-b)$.

(c) $(-a) \cdot (-b) = a \cdot b$.

Proof.

(a) Since
$$
0_F \cdot a + 0_F \cdot a = (0_F + 0_F) \cdot a = 0_F \cdot a,
$$
we have $0_F \cdot a = 0_F$ by Proposition 1.2 (b).

(b) Since
$$
(-a) \cdot b + a \cdot b = (-a + a) \cdot b = 0_F \cdot b = 0_F,
$$
we have $(-a) \cdot b = -(a \cdot b)$ by Proposition 1.2 (c). The other half can be proved similarly.

(c) By applying (b) twice, we have
$$
(-a) \cdot (-b) = -(a \cdot (-b)) = -(-(a \cdot b)) = a \cdot b. \qquad \square
$$

1.2 Vector Spaces

Definition 1.5. A **vector space** over a field F is a set V with two operations, called **addition** (denoted by $+$) and **scalar multiplication** (denoted by $\cdot$), which satisfy the following axioms.

(V 1) If $x \in V$ and $y \in V$, then $x + y \in V$.

(V 2) $x + y = y + x$ for all $x, y \in V$.

(V 3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$.

(V 4) There is an element 0_V in V such that $0_V + x = x$ for all $x \in V$.

(V 5) For each $x \in V$ there is an element $-x$ such that $x + (-x) = 0_V$.

(V 6) If $a \in F$ and $x \in V$, then $a \cdot x \in V$.

(V 7) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$ for all $a, b \in F$ and $x \in V$.

(V 8) $1_F \cdot x = x$ for all $x \in V$.

(V 9) $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in F$ and $x, y \in V$.

(V 10) $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in F$ and $x \in V$.

Remark.

- For simplification, we usually write ax instead of $a \cdot x$.

- The elements 0_V is usually called the **additive identity** of V, and $-x$ is called the **additive inverse** of x in V.

- **Subtraction** can be defined using additive inverses.

Examples.

- A field is a vector space over itself, e.g., $\mathbb{R}$ is a vector space over $\mathbb{R}$.

- $\mathbb{C}$ is a vector space over $\mathbb{R}$.

- $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Examples. Let F be a field.

- The set of **n-tuples** with entries from F, denoted F^n, is a vector space over F.

- The set of all $m \times n$ **matrices** with entries from F, denoted $F^{m \times n}$, is a vector space over F.

- The set of **polynomials** with coefficients from F, denoted $\mathcal{P}(F)$, is a vector space over F.

Proposition 1.6. Let V be a vector space with $x, y, z \in F$.

(a) If $x + y = x + z$, then $y = z$.

(b) If $x + y = x$, then $y = 0_V$.

(c) If $x + y = 0_V$, then $y = -x$.

(d) $-(-x) = x$.

Proof. The proof is omitted since it is similar to that of Proposition 1.2. □

Proposition 1.7. Let V be a vector space over a field F with $x \in V$ and $a \in F$.

(a) $0_F \cdot x = 0_V$.

(b) $a \cdot 0_V = 0_V$.

(c) $(-a) \cdot x = -(a \cdot x) = a \cdot (-x)$.

Proof. The proof is omitted since it is similar to that of Proposition 1.4. □

1.3 Subspaces

Definition 1.8. Let V be a vector space over F. A **subspace** of V is a subset W of V such that W is a vector space over F, where addition and scalar multiplication are the same as those defined on V.

Theorem 1.9. Let V be a vector space over F and let $W \subseteq V$. Then W is a subspace of V if and only if $0_V \in W$ and $ax + y \in W$ for any $a \in F$ and $x, y \in W$.

Proof. ($\Rightarrow$) Straightforward. ($\Leftarrow$) It suffices to prove the closeness of addition and scalar multiplication, and the existence of additive inverses. For any $a \in F$ and $x, y \in W$, we have

$$a \cdot x = a \cdot x + 0_W \in W \quad \text{and} \quad x + y = 1_F \cdot x + y \in W.$$

Furthermore, we have

$$x + (-1_F) \cdot x = 1_F \cdot x + (-1_F) \cdot x = (1_F + (-1_F)) \cdot x = 0_F \cdot x = 0_V,$$

for any $x \in W$, which completes the proof. $\qquad\square$

Example. The set $\mathcal{P}_n(F)$ of polynomials in $\mathcal{P}(F)$ with degree less than or equal to n is a subspace of $\mathcal{P}(F)$.

Definition 1.10. Let V be a vector space and let S_1 and S_2 be subsets of V. Then the **sum** of S_1 and S_2 is

$$S_1 + S_2 = \{x + y : x \in S_1 \text{ and } y \in S_2\}.$$

Theorem 1.11. Let V be a vector space and let W_1 and W_2 be subspaces of V. Then $W_1 + W_2$ is the minimal subspace of V that contains $W_1 \cup W_2$.

Proof. First we prove that $W_1 + W_2$ is a subspace of V. We have $0_V = 0_V + 0_V \in W_1 + W_2$, and for any $a \in F$, $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$, we have

$$\begin{aligned}
a(x_1 + x_2) + (y_1 + y_2) &= ax_1 + ax_2 + y_1 + y_2 \\
&= (ax_1 + y_1) + (ax_2 + y_2) \\
&\in W_1 + W_2.
\end{aligned}$$

Thus, it follows from Theorem 1.9 that $W_1 + W_2$ is a subspace of V.

Now we prove the minimality. Suppose that W is a subspace of V that contains $W_1 \cup W_2$. For any $x_1 \in W_1$ and $x_2 \in W_2$, we have $x_1 + x_2 \in W$. Thus, $W_1 + W_2 \subseteq W$, completing the proof. $\qquad\square$

1.4 Spanning Sets

Definition 1.12. Let V be a vector space over F and let $S \subseteq V$. A vector $x \in V$ is called a **linear combination** of S if $x = 0_V$ or there exist scalars $a_1, \ldots, a_n \in F$ and vectors $x_1, \ldots, x_n \in S$ such that

$$x = \sum_{i=1}^{n} a_i x_i = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

The set of all linear combinations of S is called the **span** of S, denoted by $\mathrm{span}(S)$.

Theorem 1.13. Let V be a vector space over F and let $S \subseteq V$. Then the following statements are true.

(a) $\mathrm{span}(S)$ is a subspace of V.

(b) If U is a subspace of V such that $S \subseteq U$, then $\mathrm{span}(S) \subseteq U$.

Proof.

(a) Let $c \in F$ and $x, y \in \mathrm{span}(S)$. Then there exist scalars $a_1, \ldots, a_n \in F$ and vectors $x_1, \ldots, x_n \in S$ such that

$$x = a_1 x_1 + \cdots + a_n x_n.$$

Also, there exist scalars $b_1, \ldots, b_n \in F$ and vectors $y_1, \ldots, y_m \in S$ such that

$$y = b_1 y_1 + \cdots + b_n y_m.$$

Thus, we have

$$\begin{aligned}
cx + y &= c(x_1 + \cdots + x_n) + (y_1 + \cdots + y_m) \\
&= cx_1 + \cdots + cx_n + y_1 + \cdots + y_m \\
&\in \mathrm{span}(S).
\end{aligned}$$

Furthermore, $0_V \in \mathrm{span}(S)$. Hence, $\mathrm{span}(S)$ is a subspace of V by **??**.

(b) Let $x \in \mathrm{span}(S)$. Then there exist scalars $a_1, \ldots, a_n \in F$ and vectors $x_1, \ldots, x_n \in S$ such that
$$x = a_1 x_1 + \cdots + a_n x_n.$$

Since $S \subseteq U$, we have $x_1, \ldots, x_n \in U$, and it follows that $x = a_1 x_1 + \cdots + a_n x_n \in U$ due to the closeness of U. Thus, $\mathrm{span}(S) \subseteq U$. $\qquad\square$

Definition 1.14. Let V be a vector space and let $S \subseteq V$. If $\mathrm{span}(S) = V$, then S is called a **spanning set** of V, and we also say S **spans** V.

Example. $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ is a spanning set of $\mathbb{R}^3$ since for any $x, y, z \in \mathbb{R}$,

$$(x, y, z) = \frac{-x + y + z}{2} \cdot (0, 1, 1) + \frac{x - y + z}{2} \cdot (1, 0, 1) + \frac{x + y - z}{2} \cdot (1, 1, 0).$$

Proposition 1.15. Let V be a vector space and let $R, S \subseteq V$.

(a) $S \subseteq \mathrm{span}(S)$.

(b) If $R \subseteq S$, then $\mathrm{span}(R) \subseteq \mathrm{span}(S)$.

(c) $S = \mathrm{span}(S)$ if and only if S is a subspace of V.

(d) $\mathrm{span}(R \cup S) = \mathrm{span}(R) + \mathrm{span}(S)$.

Proof.

(a) Straightforward.

(b) It is true since a linear combination of a subset of S is also a linear combination of S.

(c) ($\Rightarrow$) Straightforward from Theorem 1.13 (a).

($\Leftarrow$) Note that any linear combination of S is in S due to closeness of addition and scalar multiplication in S. Thus, $\mathrm{span}(S) \subseteq S$, and it follows that $S = \mathrm{span}(S)$.

(d) Since $R \subseteq \mathrm{span}(R)$ and $S \subseteq \mathrm{span}(S)$, we have $R \cup S \subseteq \mathrm{span}(R) + \mathrm{span}(S)$. Thus, by Theorem 1.13, we have $\mathrm{span}(R \cup S) \subseteq \mathrm{span}(R) + \mathrm{span}(S)$. On the other side, since
$$\mathrm{span}(R) \subseteq \mathrm{span}(R \cup S) \quad \text{and} \quad \mathrm{span}(S) \subseteq \mathrm{span}(R \cup S),$$
we can conclude that $\mathrm{span}(R) \cup \mathrm{span}(S) \subseteq \mathrm{span}(R \cup S)$. Thus, $\mathrm{span}(R) + \mathrm{span}(S) \subseteq \mathrm{span}(R \cup S)$ by **??**. $\qquad\square$

1.5 Linearly Independent Sets

Definition 1.16. Let V be a vector space over a field F and let $S \subseteq V$.

- S is **linearly dependent** if there exist scalars $a_1, a_2, \ldots, a_n \in F \setminus \{0_F\}$ and distinct vectors $x_1, x_2, \ldots, x_n \in S$ for some positive integer n such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0_V.$$

- S is **linearly independent** if it is not linearly dependent.

Remark.

- Note that $\varnothing$ is linearly independent.

Theorem 1.17. Let V be a vector space over a field F and let $S \subseteq V$. Then the following statements are equivalent.

(a) S is linearly dependent.

(b) There exists $x \in S$ with $x \in \operatorname{span}(S \setminus \{x\})$.

(c) There exists $x \in S$ with $\operatorname{span}(S) = \operatorname{span}(S \setminus \{x\})$.

Proof.

(i) First we assume (a) and prove (b). Suppose that

$$a_0 x_0 + a_1 x_1 + \cdots + a_n x_n = 0_V,$$

where $a_0, a_1, \ldots, a_n$ are nonzero scalars and $x_0, x_1, \ldots, x_n$ are distinct vectors. Then

$$
\begin{aligned}
x_0 &= (-a_0)^{-1}(a_1 x_1 + \cdots + a_n x_n) \\
&= ((-a_0)^{-1} a_1) x_1 + \cdots + ((-a_0)^{-1} a_n) x_n \\
&\in \operatorname{span}(S \setminus \{x_0\}).
\end{aligned}
$$

(ii) Then we assume (b) and prove (c). Since

$$x \in \operatorname{span}(S \setminus \{x\}) \quad \text{and} \quad S \setminus \{x\} \subseteq \operatorname{span}(S \setminus \{x\}),$$

we have $S \subseteq \operatorname{span}(S \setminus \{x\})$. Thus, $\operatorname{span}(S) \subseteq \operatorname{span}(S \setminus \{x\})$ by Theorem 1.13, and we can conclude that $\operatorname{span}(S) = \operatorname{span}(S \setminus \{x\})$.

(iii) Then we assume (c) and prove (b). It is straightforward since $x \in S \subseteq \operatorname{span}(S) = \operatorname{span}(S \setminus \{x\})$.

(iv) Finally we assume (b) and prove (a). Without loss of generality, let $a_1, \ldots, a_n \in F$ be nonzero scalars and $x_1, \ldots, x_n \in S \setminus \{x\}$ be distinct vectors such that $x = a_1 x_1 + \cdots + a_n x_n$. Then we have

$$(-1_F)x + a_1 x_1 + \cdots + a_n x_n = 0_V,$$

which completes the proof. $\qquad \square$

Example. Let $S = \{(0,1,1),(1,0,1),(1,1,0)\}$ be a subset of $\mathbb{R}^3$. Suppose that $a_1, a_2, a_3 \in \mathbb{R}$ are scalars such that

$$a_1(0,1,1) + a_2(1,0,1) + a_3(1,1,0) = (0,0,0).$$

Then we have the following system of equations.

$$
\begin{aligned}
a_2 + a_3 &= 0 \\
a_1 \qquad + a_3 &= 0 \\
a_1 + a_2 \qquad &= 0
\end{aligned}
$$

Since the only solution to this system of equations is $a_1 = a_2 = a_3 = 0$, we can conclude that S is linearly independent by Definition 1.16.

Example. Let $S = \{(1,1,1),(0,1,1),(1,0,1),(1,1,0)\}$ be a subset of $\mathbb{R}^3$. We can conclude that S is linearly dependent since

$$(1,1,1) = \frac{1}{2} \cdot (0,1,1) + \frac{1}{2} \cdot (1,0,1) + \frac{1}{2} \cdot (1,1,0).$$

Proposition 1.18. Let V be a vector space and let R, S be subsets of V with $R \subseteq S$.

(a) If R is linearly dependent, then so is S.

(b) If S is linearly independent, then so is R.

Proof.

(a) Suppose that R is linearly dependent. Then by Definition 1.16 there exists $x \in R$ such that $x \in \mathrm{span}(R \setminus \{x\})$. Also, we have $R \setminus \{x\} \subseteq S \setminus \{x\}$ since $R \subseteq S$. Thus, $x \in \mathrm{span}\, S \setminus \{x\}$, and it follows that S is linearly dependent.

(b) Straightforward from (a). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

1.6 Bases and Dimension

Definition 1.19. A **basis** for a vector space V is a linearly independent subset of V that spans V.

Examples.

- $\varnothing$ is a basis for $\{0_V\}$.

- $\{e_1, \ldots, e_n\}$ is a basis for F^n, where e_i is the n-tuple whose i-th component is 1_F and the other components are all 0_F.

- $\{E_{ij} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ is a basis for $F^{m \times n}$, where E_{ij} is the matrix whose (i, j)-entry is 1_F and the other entries are all 0_F.

- $\{t^0, t^1, t^2, \ldots, t^n\}$ is a basis for $\mathcal{P}_n(F)$.

- $\{t^0, t^1, t^2, \ldots\}$ is a basis for $\mathcal{P}(F)$.

Proposition 1.20. Let V be a vector space. If there exists a finite set S that spans V, then there is a subset Q of S that is a finite basis of V.

Proof. The proof is by induction on $|S|$. For the induction basis, suppose that $|S| = 0$, i.e., $S = \varnothing$. Then the proposition holds since one can choose $Q = \varnothing$ as a basis for V.

Now assume the induction hypothesis that the proposition holds for $|S| = n$ with $n \geq 0$. If S is linearly independent, then we can choose $Q = S$ as a basis for V. Otherwise, there exists $x \in S$ with $\mathrm{span}(S \setminus \{x\}) = \mathrm{span}(S)$, i.e., $S \setminus \{x\}$ spans V. Thus, by induction hypothesis there is a subset Q of $S \setminus \{x\}$ that is a basis for V, which completes the proof. $\qquad\square$

Theorem 1.21 (Replacement Theorem). Let V be a vector space over a field F. Let S be a finite set that spans V, and let $Q \subseteq V$ be a finite linearly independent set. Then $|Q| \leq |S|$, and there exists $R \subseteq S \setminus Q$ such that both $|Q \cup R| = |S|$ and $\mathrm{span}(Q \cup R) = V$ hold.

Proof. The proof is based on induction on $|Q|$. The theorem holds for $|Q| = 0$, i.e., $Q = \varnothing$, since we have $|\varnothing| \leq |S|$, $|\varnothing \cup S| = |S|$ and $\mathrm{span}(\varnothing \cup S) = V$.

Now suppose that the theorem is true for $|Q| = m$ with $m \geq 0$, and we prove that the theorem holds for $|Q| = m + 1$. Let $Q = \{x_1, \ldots, x_{m+1}\}$ and let $Q' = \{x_1, \ldots, x_m\}$. By induction hypothesis, there exists $R' = \{y_1, \ldots, y_k\} \subseteq S \setminus Q'$ such that $|Q'| + |R'| = |S|$ and $\mathrm{span}(Q' \cup R') = V$. Since $Q' \cup R'$ spans V, there exists $a_1, \ldots, a_m, b_1, \ldots, b_k \in F$ such that

$$x_{m+1} = \sum_{i=1}^{m} a_i x_i + \sum_{j=1}^{k} b_j y_j.$$

If $b_j = 0_F$ for all $j \in \{1, \ldots, k\}$, then $x_{m+1} \in \mathrm{span}(Q') = \mathrm{span}(Q \setminus \{x_{m+1}\})$, implying that Q is linearly dependent, contradiction. Thus, there must exist some $j \in \{1, \ldots, k\}$ such that $b_j \neq 0_F$.s Without loss of generality, suppose that $b_k \neq 0_F$ with $k \geq 1$. Also, let $R = \{y_1, \ldots, y_{k-1}\}$. Then $|Q \cup R| = (m+1) + (k-1) = |S|$, and we have $|Q| \leq |S|$. It follows that

$$Q' \cup R' \quad \subseteq \quad Q \cup R \cup \{y_k\} \quad \subseteq \quad \mathrm{span}(Q \cup R),$$

where the second inclusion holds because

$$y_k = (-b_k)^{-1} \left(\sum_{i=1}^{m} a_i x_i + (-1_F) x_{m+1} + \sum_{j=1}^{k-1} b_j y_j \right) \in \mathrm{span}(Q \cup R).$$

Then, we have

$$V = \mathrm{span}(Q' \cup R') \subseteq \mathrm{span}(Q \cup R) \subseteq V.$$

by Theorem 1.13. Thus, $\mathrm{span}(Q \cup R) = V$, which completes the proof. $\square$

Corollary 1.22. Let V be a vector space and Q be a linearly independent subset of V that is infinite. Then each spanning set of V is infinite.

Proof. Suppose that there is a finite set S that spans V. Let Q' be a subset of Q with $|Q'| = |S| + 1$. By Proposition 1.18, we can conclude that Q' is also linearly independent. Thus, we have $|Q'| \le |S|$ by replacement theorem (Theorem 1.21), contradiction. $\square$

Corollary 1.23. Let V be a vector space. If V has a finite basis, then each basis for V has the same size.

Proof. Let S be a finite basis for V and Q an arbitrary basis for V. Since $V = \mathrm{span}(S)$ and Q is linearly independent, it follows that Q is finite by Corollary 1.22, and thus we have $|Q| \le |S|$. Also, since $V = \mathrm{span}(Q)$ and S is linearly independent, we have $|S| \le |Q|$. Thus, $|Q| = |S|$. $\square$

Definition 1.24. Let V be a vector space.

- V is **finite-dimensional** if it has a finite basis. In this case, the number of vectors in each basis for V is called the **dimension** of V, denoted by $\dim(V)$.

- V is **infinite-dimensional** if it is not finite-dimensional.

Remark.

- If a vector space has a linearly independent subset that is infinite, we can conclude that it is infinite-dimensional by Corollary 1.22.

Examples. One can find the dimension of a vector space by any basis it admits.

- $\dim(\{0_V\}) = 0$.

- $\dim(F^n) = n$.

- $\dim(F^{m \times n}) = mn$.

- $\dim(\mathcal{P}_n(F)) = n + 1$.

- $\mathcal{P}(F)$ is infinite-dimensional.

Examples. Note that the dimension of a vector space depends on its field of scalars.

- Let $V = \mathbb{C}$ be a vector space over $\mathbb{R}$. Then we have $\dim(V) = 2$ since $\{1, i\}$ is a basis for V.

- Let $W = \mathbb{C}$ be a vector space over $\mathbb{C}$. Then we have $\dim(W) = 1$ since $\{1\}$ is a basis for V.

Proposition 1.25. Let V be a vector space. Then a subset of V of $n = \dim(V)$ vectors is linearly independent if and only if it is a spanning set of V.

Proof. ($\Rightarrow$) Suppose that Q is linearly independent with $|Q| = n$. By replacement theorem (Theorem 1.21), there exists $R \subseteq S \setminus Q$ such that $|Q \cup R| = |S|$ and $\mathrm{span}(Q \cup R) = V$. Since $|Q| = |S|$, we have $|R| = 0$, i.e., $R = \varnothing$. Thus, $\mathrm{span}(Q) = V$.

($\Leftarrow$) Suppose that S spans V with $|S| = n$. By Proposition 1.20, there is a subset Q of S that is a basis of V. Then we have $|Q| = n$, implying $Q = S$. Thus, S is a basis for V. $\square$

Proposition 1.26. Let V be a finite-dimensional vector space. Let $S = \{x_1, \ldots, x_n\}$ be a basis for V. Then for each $x \in V$, there exist a unique n-tuple $(a_1, \ldots, a_n) \in F^n$ with

$$x = a_1 x_1 + \cdots + a_n x_n.$$

Proof. Since $x \in \mathrm{span}(S)$, there exist scalars $a_1, \ldots, a_n \in F$ such that

$$x = a_1 x_1 + \cdots + a_n x_n.$$

Now we prove the uniqueness. Let $b_1, \ldots, b_n \in F$ be scalars with

$$x = b_1 x_1 + \cdots + b_n x_n.$$

Then we have
$$0_V = (a_1 - b_1)x_1 + \cdots + (a_n - b_n)x_n,$$

and it follows that $(a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n) = 0_{F^n}$ since S is linearly independent. Thus, $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$. $\square$

Proposition 1.27. Let V be a finite-dimensional vector space. Let V' be a subspace of V. Then the following statements are true.

(a) $\dim(V') \leq \dim(V)$.

(b) If $\dim(V') = \dim(V)$, then $V' = V$.

Proof. Let S and S' be bases for V and V', respectively.

(a) Since S' is linearly independent and $V = \mathrm{span}(S)$, we have $|S'| \leq |S|$ by replacement theorem (Theorem 1.21). Thus, $\dim(V') \leq \dim(V)$.

(b) Since S' is linearly independent and $|S'| = \dim(V)$, we have $\mathrm{span}(S') = V$ by Proposition 1.25. Thus, $V' = \mathrm{span}(S') = V$. $\square$

Example. Let W be the set of $n \times n$ diagonal matrices, which is a subspace of $F^{n \times n}$. Then one can verify that $\{E_{ii} : 1 \leq i \leq n\}$ is a basis for W, where E_{ij} is the matrix whose (i,j)-entry is 1_F and the other entries are 0_F. Thus, $\dim(W) = n$.

Chapter 2

Linear Transformations

2.1 Linear Transformations

Definition 2.1. Let V and W be vector spaces over a field F. A transformation $T : V \to W$ is said to be **linear** if

$$T(ax + y) = aT(x) + T(y)$$

holds for any scalar $a \in F$ and any vectors $x, y \in V$. The set of all linear transformations from V to W is denoted by $\mathcal{L}(V, W)$, and $\mathcal{L}(V)$ for short if $V = W$.

Proposition 2.2. Let V and W be vector spaces over a common field F. Let $T : V \to W$ be linear. Then we have the following properties.

(a) $T(0_V) = 0_W$.

(b) For nonnegative integer n,

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

hold for any $a_1, \ldots, a_n \in F$ and $x_1, \ldots, x_n \in V$.

Proof.

(a) Since

$$T(0_V) + T(0_V) = 1_F T(0_V) + T(0_V) = T(1_F 0_V + 0_V) = T(0_V),$$

we have $T(0_V) = 0_W$ by Proposition 1.6 (b).

(b) The proof is by induction on n. The induction basis with $n = 0$ is proved by

$$T\left(\sum_{i=1}^{0} a_i x_i\right) = T(0_V) = 0_W = \sum_{i=1}^{0} a_i T(x_i).$$

Now assume the induction hypothesis that the property holds for $n = k$. Then it follows that

$$T\left(\sum_{i=1}^{k+1} a_i x_i\right) = T\left(a_{k+1} x_{k+1} + \sum_{i=1}^{k} a_i x_i\right)$$

$$= a_{k+1} T(x_{k+1}) + T\left(\sum_{i=1}^{k} a_i x_i\right) \qquad \text{(linearity of } T)$$

$$= a_{k+1} T(x_{k+1}) + \sum_{i=1}^{k} a_i T(x_i) \qquad \text{(induction hypothesis)}$$

$$= \sum_{i=1}^{k+1} a_i T(x_i),$$

which completes the proof. $\qquad\square$

Theorem 2.3. If V and W are vector spaces over a field F, then $\mathcal{L}(V, W)$ is also a vector space over F.

Proof. For any $c \in F$ and $T_1, T_2 \in \mathcal{L}(V, W)$, since

$$
\begin{aligned}
(cT_1 + T_2)(ax + y) &= cT_1(ax + y) + T_2(ax + y) & \text{(linearity of } cT_1 + T_2) \\
&= c(aT_1(x) + T_1(y)) + (aT_2(x) + T_2(y)) & \text{(linearity of } T_1 \text{ and } T_2) \\
&= acT_1(x) + cT_1(y) + aT_2(x) + T_2(y) \\
&= a(cT_1(x) + T_2(x)) + (cT_1(y) + T_2(y)) \\
&= a(cT_1 + T_2)(x) + (cT_1 + T_2)(y) & \text{(linearity of } cT_1 + T_2)
\end{aligned}
$$

holds for each $a \in F$ and $x, y \in V$, we have $cT_1 + T_2 \in \mathcal{L}(V, W)$. Furthermore, $0_{\mathcal{F}(V,W)} \in \mathcal{L}(V, W)$. Thus, $\mathcal{L}(V, W)$ is a subspace of $\mathcal{F}(V, W)$. $\qquad\square$

Theorem 2.4. Let V and W be vector spaces and let $T : V \to W$ be linear. Then for any subset S of V, we have

$$T(\text{span}(S)) = \text{span}(T(S)).$$

Proof. If $y \in T(\text{span}(S))$, then there exist $a_i \in F$ and $x_i \in S$ for each $1 \le i \le n$ such that

$$y = T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i) \in \text{span}(T(S)).$$

Thus, $T(\text{span}(S)) \subseteq \text{span}(T(S))$.

On the other hand, if $y \in \text{span}(T(S))$, then there exist $a_i \in F$ and $x_i \in S$ for each $1 \le i \le n$ such that

$$y = \sum_{i=1}^{n} a_i T(x_i) = T\left(\sum_{i=1}^{n} a_i x_i\right) \in T(\text{span}(S)).$$

Thus, $\text{span}(T(S)) \subseteq T(\text{span}(S))$, which completes the proof. $\qquad\square$

2.2   Rank and Nullity

Definition 2.5. Let V and W be vector spaces. The **range** of a transformation $T : V \to W$, denoted by $\mathcal{R}(T)$, is defined by

$$\mathcal{R}(T) = \{y \in W : y = T(x) \text{ for some } x \in V\}.$$

Proposition 2.6. Let V and W be vector spaces over a field F. If $T : V \to W$ is linear, then $\mathcal{R}(T)$ is a subspace of W.

Proof. For each $a \in F$ and $y, y' \in \mathcal{R}(T)$, there exist $x, x' \in V$ such that $y = T(x)$ and $y' = T(x')$. Since

$$ay + y' = aT(x) + T(x') = T(ax + x'),$$

we have $ay + y' \in \mathcal{R}(T)$. Furthermore, $0_W = T(0_V) \in \mathcal{R}(T)$. Thus, $\mathcal{R}(T)$ is a subspace of W. $\square$

Definition 2.7. Let V and W be vector spaces. The **null space** of a transformation $T : V \to W$, denoted by $\mathcal{N}(T)$, is defined by

$$\mathcal{N}(T) = \{x \in V : T(x) = 0_W\}.$$

Proposition 2.8. Let V and W be vector spaces over a field F. If $T : V \to W$ is linear, then $\mathcal{N}(T)$ is a subspace of V.

Proof. For each $a \in F$ and $x, x' \in \mathcal{N}(T)$, we have

$$T(ax + x') = aT(x) + T(x') = a0_W + 0_W = 0_W.$$

Thus, $ax + x' \in \mathcal{N}(T)$. Furthermore, $0_V \in \mathcal{N}(T)$ since $T(0_V) = 0_W$. Thus, $\mathcal{N}(T)$ is a subspace of V. $\square$

Definition 2.9. Let X and Y be sets. Let $f : X \to Y$ be a function.

- f is **injective** if $T(x) = T(x')$ implies $x = x'$ for all $x, x' \in X$.

- f is **surjective** if there exists $x \in X$ with $T(x) = y$ for each $y \in Y$.

- f is **bijective** if f is injective and surjective.

Proposition 2.10. Let V and W be vector spaces and let $T : V \to W$ be linear. Let S be a subset of V. Then the following statements are true.

(a) T is injective if and only if $\mathcal{N}(T) = \{0_V\}$.

(b) If T is injective, then S is linearly dependent if and only of $T(S)$ is linearly dependent.

Proof.

(a) ($\Rightarrow$) We have $T(0_V) = 0_W$ since T is linear. If $T(x) = 0_W$, then $x = 0_V$ since T is injective. Thus, $\mathcal{N}(T) = \{0_V\}$.

($\Leftarrow$) Suppose that $x, y \in V$ be vectors with $T(x) = T(y)$. Since

$$T(x - y) = T(x) - T(y) = 0_W,$$

we have $x - y \in \mathcal{N}(T)$, and thus $x - y = 0_V$, implying $x = y$. Thus, T is injective.

(b) ($\Rightarrow$) If $x \in \text{span}(S \setminus \{x\})$ for some $x \in S$, then

$$
\begin{aligned}
T(x) \;&\in\; T(\text{span}(S \setminus \{x\})) \\
&=\; \text{span}(T(S \setminus \{x\})) && (T \text{ is linear}) \\
&=\; \text{span}(T(S) \setminus \{T(x)\}). && (T \text{ is injective})
\end{aligned}
$$

($\Leftarrow$) If $T(x) \in \text{span}(T(S) \setminus \{T(x)\})$ for some $x \in S$, then

$$
\begin{aligned}
T(x) \;&\in\; \text{span}(T(S) \setminus \{T(x)\}) \\
&=\; \text{span}(T(S \setminus \{x\})) && (T \text{ is injective}) \\
&=\; T(\text{span}(S \setminus \{x\})). && (T \text{ is linear})
\end{aligned}
$$

Thus, $x \in \text{span}(S \setminus \{x\})$ since T is injective. $\qquad\square$

Definition 2.11. Let V and W be vector spaces. Let $T : V \to W$ be linear.

- The **rank** of T, denoted by $\text{rank}(T)$, is the dimension of $\mathcal{R}(T)$.

- The **nullity** of T, denoted by $\text{nullity}(T)$, is the dimension of $\mathcal{N}(T)$.

Definition 2.12. Let $f : X \to Y$ be a function. Let D be a subset of X. Then the **restriction** of f to D is the function $f' : D \to Y$ with $f'(x) = f(x)$ for each $x \in D$.

Proposition 2.13. Let V and W be vector spaces and let $T : V \to W$ be linear. Let U be a subspace of V. Then the restriction of T to U is linear.

Proof. Let $T' : U \to W$ be the restriction of T to U. Then T' is linear since for each $a \in F$ and $x, y \in U$, we have

$$T'(ax + y) = T(ax + y) = aT(x) + T(y) = aT'(x) + T'(y). \qquad\square$$

Theorem 2.14 (Rank-nullity Theorem). Let V and W be finite-dimensional vector spaces over F. Let $T : V \to W$ be linear. Then we have

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Proof. Let S be a basis for V and Q a basis for $\mathcal{N}(T)$. By replacement theorem (Theorem 1.21), there is $R \subseteq S \setminus Q$ such that $Q \cup R$ is a basis for V.

We prove that $T(R)$ is a basis for $\mathcal{R}(T)$. First,

$$
\begin{aligned}
\mathcal{R}(T) &= T(\text{span}(Q \cup R)) \\
&= \text{span}(T(Q \cup R)) \\
&= \text{span}(T(Q) \cup T(R)) \\
&= \text{span}(T(R)). && (T(Q) = \{0_V\})
\end{aligned}
$$

Now we prove that $T(R)$ is linearly independent. Let T' be the restriction of T to R. Since R is linearly independent, it suffices to prove that T' is injective. Suppose that $T(x) = T(x')$ for some $x, x' \in R$. Then we have $T(x - x') = T(x) - T(x') = 0_W$, and thus $x - x' \in \mathcal{N}(T) = \text{span}(Q)$. It follows that x is a linear combination of $Q \cup \{x'\}$. If $x \neq x'$, then

$$x \in \text{span}(Q \cup \{x'\}) \subseteq \text{span}(Q \cup R \setminus \{x\}),$$

contradiction to the fact that $Q \cup R$ is linearly independent. Thus, T' is injective, implying $T(R)$ is linearly independent.

Note that $|T(R)| = |R|$ since T' is injective. Thus,

$$\text{nullity}(T) + \text{rank}(T) = |Q| + |T(R)| = |Q| + |R| = \dim(V). \qquad\square$$

2.3 Isomorphisms

Definition 2.15. Let X, Y, Z be sets. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the **composition** of f and g is the function $gf : X \to Z$ such that

$$(gf)(x) = g(f(x))$$

for all $x \in X$.

Definition 2.16. The **identity function** over a set X is a function $I_X : X \to X$ with $I_X(x) = x$ for all $x \in X$.

Definition 2.17. Let X and Y be sets. A function $f : X \to Y$ is said to be **invertible** if there exists a function $f^{-1} : Y \to X$, called the **inverse** of f, such that

$$f^{-1}f = I_X \quad \text{and} \quad ff^{-1} = I_Y.$$

Proposition 2.18. Let X and Y be sets. Let $f : X \to Y$ and $g : Y \to X$ be functions.

(a) If f is invertible, then f^{-1} is invertible.

(b) If f is invertible, then f^{-1} is linear.

(c) If f is invertible, then either $gf = I_X$ or $fg = I_Y$ implies $g = f^{-1}$.

(d) f is invertible if and only if f is bijective.

Proof.

(a) Straightforward from Definition 2.17.

(b) For $a \in F$ and $y, y' \in Y$, we have

$$
\begin{aligned}
f^{-1}(ay + y') &= f^{-1}(af(f^{-1}(y)) + f(f^{-1}(y'))) && (ff^{-1} = I_Y)\\
&= f^{-1}(f(af^{-1}(y) + f^{-1}(y'))) && \text{(linearity of } f)\\
&= af^{-1}(y) + f^{-1}(y'). && (f^{-1}f = I_X)
\end{aligned}
$$

Thus, f^{-1} is linear.

(c) If $gf = I_X$, then

$$g = gI_Y = g(ff^{-1}) = (gf)f^{-1} = I_X f^{-1} = f^{-1}.$$

If $fg = I_Y$, then

$$g = I_X g = (f^{-1}f)g = f^{-1}(fg) = f^{-1}I_Y = f^{-1}.$$

(d) ($\Rightarrow$) Suppose that f is invertible. Then f is injective since for each $x, x' \in X$ with $f(x) = f(x')$, we have

$$x = (f^{-1}f)(x) = f^{-1}(f(x)) = f^{-1}(f(x')) = (f^{-1}f)(x') = x'.$$

Also, f is surjective since for each $y \in Y$, we have

$$y = (ff^{-1})y = f(f^{-1}(y)).$$

($\Leftarrow$) If f is bijective, then for each $y \in Y$ there exists a unique element $x \in X$ with $f(x) = y$. Thus, there exists a function $g : Y \to X$ such that

$$g(f(x)) = x$$

for each $x \in X$. For any $y \in Y$, if $x \in X$ is the element such that $f(x) = y$, then we have

$$f(g(y)) = f(g(f(x))) = f(x) = y.$$

Thus, f is invertible since $gf = I_X$ and $fg = I_Y$. $\qquad\square$

Definition 2.19. Let V and W be vector spaces. An **isomorphism** from V onto W is a invertible linear transformation from V to W. If there is an isomorphism from V onto W, then V and W are said to be **isomorphic**, denoted by $V \cong W$.

Lemma 2.20. Let V and W be finite-dimensional vector spaces with $\dim(V) = \dim(W)$. Let $T : V \to W$ be linear. Then T is injective if and only if T is surjective.

Proof. ($\Rightarrow$) If T is injective, then $\mathcal{N}(T) = \{0_V\}$, implying $\mathrm{nullity}(T) = 0$. Then we have

$$\dim(\mathcal{R}(T)) = \mathrm{rank}(T) = \dim(V) - \mathrm{nullity}(T) = \dim(W) - 0 = \dim(W).$$

Since $\mathcal{R}(T)$ is a subspace of W with $\dim(\mathcal{R}(T)) = \dim(W)$, we can conclude that $\mathcal{R}(T) = W$ by Proposition 1.27.

($\Leftarrow$) If T is surjective, then $\mathcal{R}(T) = W$. Thus,

$$\mathrm{nullity}(T) = \dim(V) - \mathrm{rank}(T) = \dim(W) - \dim(W) = 0,$$

implying $\mathcal{N}(T) = \{0_V\}$. It follows that T is injective. $\qquad\square$

Lemma 2.21. Let V and W be finite-dimensional vector spaces over a field F. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a basis for V and let $y_1, y_2, \ldots, y_n$ be vectors in W. Then there exists a unique $T \in \mathcal{L}(V, W)$ with $T(x_i) = y_i$ for each $i \in \{1, \ldots, n\}$.

Proof. Let T be the transformation that satisfies

$$T(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n$$

for any $a_1, a_2, \ldots, a_n \in F$. It is obvious that $T(x_i) = y_i$ for each $i \in \{1, \ldots, n\}$, and T is linear since

$$
\begin{aligned}
T\left(c \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i \right) &= T\left(\sum_{i=1}^{n} (ca_i + b_i) x_i \right) \\
&= \sum_{i=1}^{n} (ca_i + b_i) y_i \\
&= c \sum_{i=1}^{n} a_i y_i + \sum_{i=1}^{n} b_i y_i \\
&= cT\left(\sum_{i=1}^{n} a_i x_i \right) + T\left(\sum_{i=1}^{n} b_i x_i \right)
\end{aligned}
$$

holds for any scalars $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c \in F$. To see the uniqueness, if $T' \in \mathcal{L}(V, W)$ satisfies $T'(x_i) = y_i$ for each $i \in \{1, \ldots, n\}$, then we have

$$\begin{aligned} T'(a_1 x_1 + \cdots + a_n x_n) &= a_1 T'(x_1) + \cdots + a_n T'(x_n) \\ &= a_1 T(x_1) + \cdots + a_n T(x_n) \\ &= T(a_1 x_1 + \cdots + a_n x_n). \end{aligned}$$

for any $a_1, \ldots, a_n \in F$. Thus, $T' = T$. $\qquad\qquad\square$

Theorem 2.22. Let V and W be finite-dimensional vector spaces over a field F. Then $V \cong W$ if and only if $\dim(V) = \dim(W)$.

Proof. ($\Rightarrow$) Let T be an isomorphism from V onto W. Since T is invertible, T is bijective. Then we have $\operatorname{rank}(T) = \dim(W)$ since $\mathcal{R}(T) = W$. Furthermore, since T is injective, we have $\operatorname{nullity}(T) = 0$, and it follows that $\operatorname{rank}(T) = \dim(V)$ by rank-nullity theorem (Theorem 2.14). Thus, $\dim(V) = \operatorname{rank}(T) = \dim(W)$.

($\Leftarrow$) Suppose that $S = \{x_1, x_2, \ldots, x_n\}$ is a basis for V and $R = \{y_1, y_2, \ldots, y_n\}$ is a basis for W. Then by Lemma 2.21 there exists $T \in \mathcal{L}(V, W)$ such that $T(x_i) = y_i$ for each $i \in \{1, \ldots, n\}$. Since R is a basis for W, for each $y \in W$ there exist scalars $a_1, \ldots, a_n \in F$ such that

$$y = \sum_{i=1}^{n} a_i y_i = \sum_{i=1}^{n} a_i T(x_i) = T\left(\sum_{i=1}^{n} a_i x_i\right).$$

It follows that T is surjective, and we can conclude that T is bijective by Lemma 2.20. Thus, T is an isomorphism from V onto W, implying $V \cong W$. $\qquad\square$

2.4 Coordinates and Matrix Representations

Definition 2.23. Let V be an finite-dimensional vector space over a field F with $\dim(V) = n$. An **ordered basis** for V is a finite sequence

$$\beta = (x_1, x_2, \ldots, x_n)$$

of vectors in V such that the set $S = \{x_1, x_2, \ldots, x_n\}$ is a basis for V.

Examples.

- The **standard ordered basis** for F^n is $(e_1, \ldots, e_n)$, where e_i is the n-tuple whose i-th component is 1_F and the other components are all 0_F.

- The **standard ordered basis** for $\mathcal{P}_n(F)$ is $(t^0, t^1, \ldots, t^n)$.

Definition 2.24. Let V be a finite-dimensional vector space over a field F. Let $\beta = (x_1, \ldots, x_n)$ be an ordered basis for V. Then we define $\phi_\beta : V \to F^n$ such that

$$\phi_\beta(x) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{for each } x \in V \text{ with } x = \sum_{i=1}^{n} a_i x_i,$$

where $a_1, a_2, \ldots, a_n \in F$. For each vector x in V, $\phi_\beta(x)$ is called the **coordinate** of x with respect to β, denoted by $[x]_\beta$.

Proposition 2.25. Let $\beta = (x_1, \ldots, x_n)$ be an ordered basis for a vector space V over F. Then ϕ_β is an isomorphism from V onto F^n.

Proof. ϕ_β is linear since

$$\phi_\beta \left(c \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i \right) = \phi_\beta \left(\sum_{i=1}^{n} (ca_i + b_i) x_i \right) = \begin{pmatrix} ca_1 + b_1 \\ \vdots \\ ca_n + b_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$= c \cdot \phi_\beta \left(\sum_{i=1}^{n} a_i x_i \right) + \phi_\beta \left(\sum_{i=1}^{n} b_i x_i \right)$$

holds for any $a_1, \ldots, a_n, b_1, \ldots, b_n, c \in F$. Also, ϕ_β is invertible since there exists $\phi_\beta^{-1} : F^n \to V$ with

$$\phi_\beta^{-1} \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \right) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

for any $a_1, a_2, \ldots, a_n \in F$. Thus, ϕ_β is an isomorphism. $\qquad \square$

Definition 2.26. Let V and W be finite-dimensional vector spaces over a field F. Let

$$\beta = (x_1, \ldots, x_n) \quad \text{and} \quad \gamma = (y_1, \ldots, y_m)$$

be ordered basis for V and W, respectively. Then we define $\Phi_\beta^\gamma : \mathcal{L}(V, W) \to F^{m \times n}$ by

$$\Phi_\beta^\gamma(T) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

for each $T \in \mathcal{L}(V, W)$, where

$$T(x_1) = a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m$$
$$T(x_2) = a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m$$
$$\vdots$$
$$T(x_n) = a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m$$

hold. For each linear $T : V \to W$, the matrix $\Phi_\beta^\gamma(T)$ is called the **matrix representation** of T with respect to β and γ, denoted by $[T]_\beta^\gamma$.

Proposition 2.27. Let $\beta = (x_1, \ldots, x_n)$ and $\gamma = (y_1, \ldots, y_m)$ be ordered bases for a vector spaces V and W over F, respectively. Then for any $T \in \mathcal{L}(V, W)$, we have

$$\left([T]_\beta^\gamma \right)_{ij} = \left([T(x_j)]_\gamma \right)_i$$

for any $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Proof. Let

$$[T]_\beta^\gamma = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Since $T(x_j) = a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m$, we have

$$[T(x_j)]_\gamma = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Thus,

$$\left([T(x_j)]_\gamma \right)_i = a_{ij}$$

holds, which completes the proof. $\square$

Theorem 2.28. Let β and γ be ordered bases for a vector spaces V and W over F, respectively. Then Φ_β^γ is an isomorphism from $\mathcal{L}(V, W)$ onto $F^{m \times n}$.

Proof. Let $\beta = (x_1, \ldots, x_n)$ and $\gamma = (y_1, \ldots, y_m)$. Note that Φ_β^γ is linear since for any $c \in F$ and $T_1, T_2 \in \mathcal{L}(V, W)$, we have

$$
\begin{aligned}
\left([cT_1 + T_2]_\beta^\gamma \right)_{ij} &= \left([(cT_1 + T_2)(x_j)]_\gamma \right)_i && \text{(Proposition 2.27)} \\
&= \left([cT_1(x_j) + T_2(x_j)]_\gamma \right)_i \\
&= \left(c[T_1(x_j)]_\gamma + [T_2(x_j)]_\gamma \right)_i && (\phi_\gamma \text{ is linear}) \\
&= c\left([T_1(x_j)]_\gamma \right)_i + \left([T_2(x_j)]_\gamma \right)_i \\
&= c\left([T_1]_\beta^\gamma \right)_{ij} + \left([T_2]_\beta^\gamma \right)_{ij} && \text{(Proposition 2.27)} \\
&= \left(c[T_1]_\beta^\gamma + [T_2]_\beta^\gamma \right)_{ij}
\end{aligned}
$$

for any $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. To prove that Φ_β^γ is invertible, let

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
$$

be an arbitrary matrix in $F^{m \times n}$. By Lemma 2.21, there exists a unique linear transformation $T : V \to W$ such that

$$
T(x_j) = \sum_{i=1}^n a_{ij} y_j
$$

for each $j \in \{1, \ldots, n\}$, and it follows that $[T]_\beta^\gamma = A$. Thus, there exists $(\Phi_\beta^\gamma)^{-1} : F^{m \times n} \to \mathcal{L}(V, W)$ such that $(\Phi_\beta^\gamma)^{-1}(A) = T$ with $[T]_\beta^\gamma = A$ for each $A \in F^{m \times n}$, which completes the proof. $\square$

Corollary 2.29. If V and W are finite-dimensional vector spaces over F with $\dim(V) = n$ and $\dim(W) = m$, then $\mathcal{L}(V, W)$ is finite-dimensional with $\dim(\mathcal{L}(V, W)) = mn$.

Proof. Straightforward from Theorem 2.22 and Theorem 2.28. $\square$

2.5 Matrix Multiplication

Definition 2.30. Let F be a field and let $A \in F^{\ell \times m}$ and $B \in F^{m \times n}$ be matrices. The **product** of A and B, denoted by AB, is a matrix in $F^{\ell \times n}$ that satisfies

$$(AB)_{ik} = \sum_{j=1}^{m} A_{ij} B_{jk}$$

for $i \in \{1, \ldots, \ell\}$ and $k \in \{1, \ldots, n\}$.

Proposition 2.31. Let U, V, W be vector spaces over F. If $T_1 : U \to V$ and $T_2 : V \to W$ are linear, then so is $T_2 T_1$.

Proof. For $a \in F$ and $x, y \in U$, we have

$$
\begin{aligned}
(T_2 T_1)(ax + y) &= T_2(T_1(ax + y)) && \text{(composition of } T_1 \text{ and } T_2) \\
&= T_2(a T_1(x) + T_1(y)) && \text{(linearity of } T_1) \\
&= a T_2(T_1(x)) + T_2(T_1(y)) && \text{(linearity of } T_2) \\
&= a(T_2 T_1)(x) + (T_2 T_1)(y). && \text{(composition of } T_1 \text{ and } T_2)
\end{aligned}
$$

Thus, $T_2 T_1$ is linear. $\square$

Theorem 2.32. Let U, V, W be finite-dimensional vector spaces with ordered bases

$$\alpha = (x_1, \ldots, x_n), \quad \beta = (y_1, \ldots, y_m), \quad \text{and} \quad \gamma = (z_1, \ldots, z_\ell),$$

respectively. If $T_1 : U \to V$ and $T_2 : V \to W$ are linear, then

$$[T_2 T_1]_\alpha^\gamma = [T_2]_\beta^\gamma [T_1]_\alpha^\beta.$$

Proof. Let $A = [T_2]_\beta^\gamma$, $B = [T_1]_\alpha^\beta$ and $C = [T_2 T_1]_\alpha^\gamma$. Then

$$T_2(y_j) = \sum_{i=1}^{\ell} A_{ij} z_i, \quad T_1(x_k) = \sum_{j=1}^{m} B_{jk} y_j, \quad \text{and} \quad (T_2 T_1)(x_k) = \sum_{i=1}^{\ell} C_{ik} z_i$$

hold for any $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$. Since for each $k \in \{1, \ldots, n\}$,

$$
\begin{aligned}
\sum_{i=1}^{\ell} C_{ik} z_i &= (T_2 T_1)(x_k) \\
&= T_2(T_1(x_k)) \\
&= T_2 \left(\sum_{j=1}^{m} B_{jk} y_j \right) \\
&= \sum_{j=1}^{m} B_{jk} T_2(y_j) \\
&= \sum_{j=1}^{m} B_{jk} \sum_{i=1}^{\ell} A_{ij} z_i \\
&= \sum_{i=1}^{\ell} \left(\sum_{j=1}^{m} A_{ij} B_{jk} \right) z_i,
\end{aligned}
$$

we have

$$C_{ik} = \sum_{j=1}^{m} A_{ij} B_{jk}$$

for each $i \in \{1, \ldots, \ell\}$ and $k \in \{1, \ldots, n\}$. Thus, $C = AB$. $\qquad\square$

Corollary 2.33. Let V and W be finite-dimensional vector spaces with ordered bases β and γ over a field F, respectively. If $T : V \to W$ is linear, then

$$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$$

for each $x \in V$.

Proof. Let $\alpha = (1_F)$ be an ordered basis for F. For each $x \in V$, let $\varphi : F \to V$ be the linear transformation with $\varphi(c) = cx$ for each $c \in F$. By Definition 2.26, we have

$$[\varphi]_\alpha^\beta = [\varphi(1_F)]_\beta \quad \text{and} \quad [T\varphi]_\alpha^\gamma = [(T\varphi)(1_F)]_\gamma.$$

Thus, it follows that

$$\begin{aligned}
[T(x)]_\gamma &= [T(\varphi(1_F))]_\gamma \\
&= [(T\varphi)(1_F)]_\gamma \\
&= [T\varphi]_\alpha^\gamma \\
&= [T]_\beta^\gamma [\varphi]_\alpha^\beta \qquad\qquad\qquad \text{(Theorem 2.32)} \\
&= [T]_\beta^\gamma [\varphi(1_F)]_\beta \\
&= [T]_\beta^\gamma [x]_\beta. \qquad\qquad\qquad\qquad\qquad\qquad\quad \square
\end{aligned}$$

2.6 Left-Multiplication Transformations

Definition 2.34. Let $A \in F^{m \times n}$ be a matrix. The **left-multiplication transformation** of A, denoted by L_A, is the transformation from F^n to F^m with

$$L_A(x) = Ax$$

for each $x \in F^n$.

Proposition 2.35. Let α, β and γ be standard ordered bases for F^n, F^m and F^ℓ, respectively. Then the following statements are true.

(a) L_A is linear for each $A \in F^{m \times n}$.

(b) $[L_A]_\alpha^\beta = A$ for each $A \in F^{m \times n}$.

(c) $L_{cA+B} = cL_A + L_B$ for each $c \in F$ and $A, B \in F^{m \times n}$.

(d) $L_{AB} = L_A L_B$ for each $A \in F^{\ell \times m}$ and $B \in F^{m \times n}$.

(e) $L_{I_n} = I_{F^n}$.

Proof.

(a) L_A is linear since for any $c \in F$ and $x, y \in F^n$,

$$
\begin{aligned}
\left[L_A(cx + y)\right]_i &= \left[A(cx + y)\right]_i \\
&= \sum_{j=1}^{n} A_{ij}\left[cx + y\right]_j \\
&= \sum_{j=1}^{n} A_{ij}(cx_j + y_j) \\
&= c\sum_{j=1}^{n} A_{ij}x_j + \sum_{j=1}^{n} A_{ij}y_j \\
&= c\left[Ax\right]_i + \left[Ay\right]_i \\
&= \left[cAx + Ay\right]_i \\
&= \left[cL_A(x) + L_A(y)\right]_i
\end{aligned}
$$

holds for each $i \in \{1, \dots, m\}$.

(b) Let $T \in \mathcal{L}(V, W)$ be the transformation with $[T]_\alpha^\beta = A$. Then we have

$$T(x) = [T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha = Ax$$

for each $x \in F^n$ since α and β are standard ordered bases. Thus, $T = L_A$.

(c) It is proved by

$$[L_{cA+B}]_\alpha^\beta = cA + B = c[L_A]_\alpha^\beta + [L_B]_\alpha^\beta = [cL_A + L_B]_\alpha^\beta.$$

(d) It is proved by
$$[L_{AB}]_\alpha^\gamma = AB = [L_A]_\beta^\gamma [L_B]_\alpha^\beta = [L_A L_B]_\alpha^\gamma.$$

(e) Since
$$L_{I_n}(x) = I_n x = x = I_{F^n}(x)$$
holds for each $x \in F^n$, $L_{I_n} = I_{F^n}$. □

Lemma 2.36. Let U, V, W, X be vector spaces. Let
$$T_1, T_1' \in \mathcal{L}(U, V), \quad T_2, T_2' \in \mathcal{L}(V, W), \quad \text{and} \quad T_3 \in \mathcal{L}(W, X)$$
be linear transformations and let $c \in F$ be a scalar. Then the following statements are true.

(a) $T_1 I_U = T_1 = I_V T_1$.

(b) $T_2(T_1 + T_1') = T_2 T_1 + T_2 T_1'$.

(c) $(T_2 + T_2')T_1 = T_2 T_1 + T_2' T_1$.

(d) $c(T_2 T_1) = (cT_2)T_1 = T_2(cT_1)$.

(e) $T_3(T_2 T_1) = (T_3 T_2)T_1$.

Proof.

(a) Since
$$(T_1 I_U)(x) = T_1(I_U(x)) = T_1(x) = I_V(T_1(x)) = (I_V T_1)(x)$$
holds for each $x \in U$, we have $T_1 I_U = T_1 = I_V T_1$.

(b) Since

$$
\begin{aligned}
(T_2(T_1 + T_1'))(x) &= T_2((T_1 + T_1')(x)) && \text{(composition)} \\
&= T_2(T_1(x) + T_1'(x)) && \text{(addition)} \\
&= T_2(T_1(x)) + T_2(T_1'(x)) && \text{(linearity)} \\
&= (T_2 T_1)(x) + (T_2 T_1')(x) && \text{(composition)} \\
&= (T_2 T_1 + T_2 T_1')(x) && \text{(addition)}
\end{aligned}
$$

holds for each $x \in U$, we have $T_2(T_1 + T_1') = T_2 T_1 + T_2 T_1'$.

(c) Since

$$
\begin{aligned}
((T_2 + T_2')T_1)(x) &= (T_2 + T_2')(T_1(x)) && \text{(composition)} \\
&= T_2(T_1(x)) + T_2'(T_1(x)) && \text{(addition)} \\
&= (T_2 T_1)(x) + (T_2' T_1)(x) && \text{(composition)} \\
&= (T_2 T_1 + T_2' T_1)(x) && \text{(addition)}
\end{aligned}
$$

holds for each $x \in U$, we have $(T_2 + T_2')T_1 = T_2 T_1 + T_2' T_1$.

28

(d) Since

$$
\begin{array}{rcccc}
(c(T_2T_1))(x) & = & c(T_2T_1)(x) & = & cT_2(T_1(x)) \\
((cT_2)T_1)(x) & = & (cT_2)(T_1(x)) & = & cT_2(T_1(x)) \\
(T_2(cT_1))(x) & = & T_2(cT_1(x)) & = & cT_2(T_1(x))
\end{array}
$$

hold for each $x \in U$, we have $c(T_2T_1) = (cT_2)T_1 = T_2(cT_1)$.

(e) Since

$$
\begin{aligned}
(T_3(T_2T_1))(x) &= T_3((T_2T_1)(x)) & \text{(composition of } T_3 \text{ and } T_2T_1) \\
&= T_3(T_2(T_1(x))) & \text{(composition of } T_2 \text{ and } T_1) \\
&= (T_3T_2)(T_1(x)) & \text{(composition of } T_3 \text{ and } T_2) \\
&= ((T_3T_2)T_1)(x) & \text{(composition of } T_3T_2 \text{ and } T_1)
\end{aligned}
$$

holds for each $x \in U$, we have $T_3(T_2T_1) = (T_3T_2)T_1$. $\qquad\square$

Theorem 2.37. Let $A, A' \in F^{k \times \ell}$, $B, B' \in F^{\ell \times m}$ and $C \in F^{m \times n}$ be matrices and let $c \in F$ be a scalar. Then the following statements are true.

(a) $AI_\ell = A = I_k A$.

(b) $A(B + B') = AB + AB'$.

(c) $(A + A')B = AB + A'B$.

(d) $c(AB) = (cA)B = A(cB)$.

(e) $A(BC) = (AB)C$.

Proof. Straightforward from Lemma 2.36. $\qquad\square$

2.7 Invertible Matrices

Definition 2.38. A matrix $A \in F^{n \times n}$ is **invertible** if L_A is invertible. If A is invertible, then it has an **inverse**, denoted by A^{-1}, which is the matrix in $F^{n \times n}$ such that

$$L_{A^{-1}} = (L_A)^{-1}.$$

Proposition 2.39. The following statements are true for matrices $A, B \in F^{n \times n}$.

(a) If A is invertible, then $AA^{-1} = I_n = A^{-1}A$.

(b) If $AB = I_n$, then A and B are invertible. Furthermore, $A = B^{-1}$ and $B = A^{-1}$.

Proof.

(a) We have
$$L_{AA^{-1}} = L_A L_{A^{-1}} = L_A (L_A)^{-1} = I_{F^n} = L_{I_n}$$
and
$$L_{A^{-1}A} = L_{A^{-1}} L_A = (L_A)^{-1} L_A = I_{F^n} = L_{I_n},$$
implying $AA^{-1} = I_n = A^{-1}A$.

(b) Since AB is invertible, $L_{AB} = L_A L_B$ is injective and surjective. Thus, $L_A : F^n \to F^n$ is injective and $L_B : F^n \to F^n$ is surjective. It follows that L_A and L_B are bijective by Lemma 2.20, and thus are invertible, implying A and B are invertible. By Proposition 2.18 (c), we have $L_A = (L_B)^{-1}$ and $L_B = (L_A)^{-1}$. Thus, we have $A = B^{-1}$ and $B = A^{-1}$. □

2.8 Projections

Definition 2.40. Let $V_1, \ldots, V_k$ be subspaces of a vector space U. We say that U is the **direct sum** of $V_1, \ldots, V_k$, denoted

$$U = V_1 \oplus \cdots \oplus V_k,$$

if both of the following conditions hold.

(a) $U = V_1 + \cdots + V_k$.

(b) For each $i \in \{1, \ldots, k\}$, $V_i \cap (V_1 + \cdots + V_{i-1} + V_{i+1} + \cdots V_k) = \{0_U\}$.

Theorem 2.41. If $V_1, \ldots, V_k$ are subspaces of a finite-dimensional vector space U, then the following statements are equivalent.

(a) $W = V_1 \oplus \cdots \oplus V_k$.

(b) $W = V_1 + \cdots + V_k$. Moreover, if $x_1 + \cdots + x_k = 0_U$ with $x_i \in V_i$ for each $1 \le i \le k$, then $x_i = 0_U$ for each $1 \le i \le k$.

(c) Each vector $x \in U$ can be uniquely written as $x = x_1 + \cdots + x_k$, where $x_i \in V_i$ for $1 \le i \le k$.

(d) If β_i with $1 \le i \le k$ is an ordered basis for V_i, then $\beta_1 \cup \cdots \cup \beta_k$ is an ordered basis for U.

(e) For $1 \le i \le k$, there exists an ordered basis β_i for V_i, such that $\beta_1 \cup \cdots \cup \beta_k$ is an ordered basis for U.

Proof. First we assume (a) and prove (b). Clearly we have $W = V_1 + \cdots + V_k$ by Definition 2.40 (a). Now suppose that $x_1 + \cdots + x_k = 0_U$ with $x_i \in V_i$ for $1 \le i \le k$. For each $i \in \{1, \ldots, k\}$, we have

$$-x_i = x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_k,$$

implying

$$-x_i \in V_i \cap (V_1 + \cdots + V_{i-1} + V_{i+1} + \cdots V_k) = \{0_U\}.$$

Thus, $x_i = 0_U$, which proves (b).

Now we assume (b) and prove (c). For each vector $x \in U$, there exist vectors $x_i \in V_i$ for $1 \le i \le k$ such that $x = x_1 + \cdots + x_k$ since $V = V_1 + \cdots + V_k$. If there also exist $x_i' \in V_i$ for $1 \le i \le k$ such that $x = x_1' + \cdots + x_k'$, then we have

$$(x_1 - x_1') + \cdots + (x_k - x_k') = 0_U,$$

implying $x_i = x_i'$ for $1 \le i \le k$. Thus, (c) is proved.

Now we assume (c) and prove (d). Let $\beta = \beta_1 \cup \cdots \cup \beta_k$. Since $U = V_1 + \cdots + V_k$ by (c), it follows that $\text{span}(\beta) = U$. Thus, we only need to show that β is linearly independent. Suppose that for $1 \le i \le k$, the ordered basis β_i has m_i elements with $\beta_i = \{x_{ij} : 1 \le j \le m_i\}$. If there exist scalars $a_{ij} \in F$ for $1 \le i \le k$ and $1 \le j \le m_i$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{m_i} a_{ij} x_{ij} = 0_U,$$

then by (c) we have

$$\sum_{j=1}^{m_i} a_{ij} x_{ij} = 0_U$$

for each $i \in \{1, \ldots, k\}$. Since β_i is linearly independent, we can conclude that $a_{ij} = 0_F$ for any j with $1 \le j \le m_i$. Thus, β is linearly independent, which proves (d). Note that (e) follows from (d) immediately.

Finally, we assume (e) and prove (a). Since

$$\begin{aligned} U &= \operatorname{span}(\beta_1 \cup \cdots \cup \beta_k) \\ &= \operatorname{span}(\beta_1) + \cdots + \operatorname{span}(\beta_k) \\ &= V_1 + \cdots + V_k, \end{aligned}$$

it suffices to prove that $V_i \cap (V_1 + \cdots + V_{i-1} + V_{i+1} + \cdots + V_k) = \{0_U\}$ holds for all $1 \le i \le k$. Suppose that there is a nonzero vector $y \in U$ such that

$$y \in V_h \cap (V_1 + \cdots + V_{h-1} + V_{h+1} + \cdots + V_k)$$

for some $h \in \{1, \ldots, k\}$. For each $i \in \{1, \ldots, k\}$, let $\beta_i = \{x_{ij} : 1 \le j \le m_i\}$, where m_i is the number of vectors in β_i. Then we have

$$\sum_{j=1}^{m_h} -a_{hj} x_{hj} = y = \sum_{\substack{i \ne h \\ 1 \le i \le k}} \sum_{j=1}^{m_i} a_{ij} x_{ij}.$$

It follows that

$$\sum_{i=1}^{k} \sum_{j=1}^{m_i} a_{ij} x_{ij} = 0_U,$$

implying $a_{ij} = 0_F$ for each $i \in \{1, \ldots, k\}$ and for each $j \in \{1, \ldots, m_i\}$, contradiction. Hence (a) is proved, and thus the proof is completed. $\qquad \square$

Definition 2.42. Let V and W be subspaces of a vector space U with $U = V \oplus W$. Then the **projection** on V along W is a transformation $T : U \to U$ such that for any $x \in U$, $y \in V$ and $w \in W$ with $x = y + z$, we have $T(x) = y$.

Theorem 2.43. Let V and W be subspaces of a vector space U with $U = V \oplus W$. Let $T : U \to U$ be the projection on V along W. Then T is linear.

Proof. Let $a \in F$ and $x, x' \in U$. Furthermore, let

$$y = T(x), \quad z = x - T(x)$$

and

$$y' = T(x'), \quad z' = x' - T(x').$$

Then we have

$$\begin{aligned} T(ax + x') &= T(a(y + z) + (y' + z')) \\ &= T((ay + y') + (az + z')) \\ &= ay + y' \\ &= aT(x) + T(x'). \end{aligned} \qquad \square$$

Theorem 2.44. Let V and W be subspaces of a vector space U with $U = V \oplus W$. Let $T : U \to U$ be linear. Then T is the projectioon on V along W if and only if the following statements are satisfied.

(a) For any $y \in V$, $T(y) = y$.

(b) For any $z \in W$, $T(z) = 0_U$.

Proof. ($\Rightarrow$) Straightforward. ($\Leftarrow$) For any $x \in U$, let $y \in V$ and $w \in W$ be vectors with $x = y + z$. Then

$$T(x) = T(y + z) = T(y) + T(z) = y + 0_U = y.$$

Thus, T is the projection on V along W. $\qquad\square$

Chapter 3

Systems of Linear Equations

3.1 Elementary Matrices

Definition 3.1. Any one of the following three operations on matrices is called an **elementary row operation**.

(Type 1) Exchanging two different rows.

(Type 2) Multiplying a row by a nonzero scalar.

(Type 3) Adding a scalar multiple of a row to another row.

Similarly, any one of the following three operations on matrices is called an **elementary column operation**.

(Type 1) Exchanging two different columns.

(Type 2) Multiplying a column by a nonzero scalar.

(Type 3) Adding a scalar multiple of a column to another column.

Furthermore, an **elementary operation** is either an elementary row operation or an elementary column operation.

Definition 3.2. A matrix $X \in F^{n \times n}$ is **elementary** if it can be obtained from I_n by applying an elementary operation. We say that an elementary matrix is of type 1, 2, or 3 if its corresponding elementary operation is a type 1, 2, or 3 operation, respectively.

Proposition 3.3. Let $X \in F^{m \times m}$ and $Y \in F^{n \times n}$ be elementary matrices. Then the following statements hold for any matrix $A \in F^{m \times n}$.

(a) XA is the matrix obtained from A by applying the elementary row operation corresponding to X.

(b) AY is the matrix obtained from A by applying the elementary column operation corresponding to Y.

Proof. We will prove (a), and the proof of (b) is similar to that of (a) so that we omit it.

Let $\gamma = (e_1, e_2, \ldots, e_m)$ be the standard basis for F^m. Also, let

$$\text{row}(X) = (x_1, x_2, \ldots, x_m) \quad \text{and} \quad \text{col}(A) = (c_1, c_2, \ldots, c_n).$$

Then we have
$$(XA)_{ij} = \sum_{k=1}^{m} X_{ik}A_{kj} = \sum_{k=1}^{m} (x_i)_k (c_j)_k$$
for each $1 \le i \le m$ and $1 \le j \le n$.

First, suppose that X is of type 1, obtained from I_m by exchanging the p-th row and the q-th row. It follows that $x_p = e_q$, $x_q = e_p$, and $x_i = e_i$ for each $i \in \{1, \ldots, m\} \backslash \{p, q\}$. Thus,

$$(XA)_{pj} = \sum_{k=1}^{m} (e_q)_k (c_j)_k = (c_j)_q = A_{qj}$$

$$(XA)_{qj} = \sum_{k=1}^{m} (e_p)_k (c_j)_k = (c_j)_p = A_{pj}$$

$$(XA)_{ij} = \sum_{k=1}^{m} (e_i)_k (c_j)_k = (c_j)_i = A_{ij} \quad \text{for } i \in \{1, \ldots, m\} \setminus \{p, q\}$$

hold for any $j \in \{1, \ldots, n\}$, implying XA is the matrix obtained from A by exchanging the p-th row and the q-th row.

Secondly, suppose that X is of type 2, obtained from I_m by multiplying the p-th row by a scalar a. It follows that $x_p = ae_p$ and $x_i = e_i$ for $i \in \{1, \ldots, m\} \setminus \{p\}$. Thus,

$$(XA)_{pj} = \sum_{k=1}^{m} (ae_p)_k (c_j)_k = a(c_j)_p = aA_{pj}$$

$$(XA)_{ij} = \sum_{k=1}^{m} (e_i)_k (c_j)_k = (c_j)_i = A_{ij} \quad \text{for } i \in \{1, \ldots, m\} \setminus \{p\}$$

hold for any $j \in \{1, \ldots, n\}$, implying XA is the matrix obtained from A by multiplying the p-th row by a scalar a.

Finally, suppose that X is of type 3, obtained from I_m by adding the p-th row multiplied by a to the q-th row. It follows that $x_q = ae_p + e_q$ and $x_i = e_i$ for each $i \in \{1, \ldots, m\} \setminus \{q\}$. Thus,

$$(XA)_{qj} = \sum_{k=1}^{m} (ae_p + e_q)_k (c_j)_k = a(c_j)_p + (c_j)_q = aA_{pj} + A_{qj}$$

$$(XA)_{ij} = \sum_{k=1}^{m} (e_i)_k (c_j)_k = (c_j)_i = A_{ij} \quad \text{for } i \in \{1, \ldots, m\} \setminus \{q\}$$

hold for any $j \in \{1, \ldots, n\}$, implying XA is the matrix obtained from A by adding the p-th row multiplied by a to the q-th row. $\square$

Proposition 3.4. Let $X \in F^{n \times n}$ be an elementary matrix. Then X is invertible, and X^{-1} is also an elementary matrix.

Proof. There exists an elementary matrix $Y \in F^{n \times n}$ with $YX = I_n$ as follows.

- If X is of type 1 obtained from I_n by exchanging the p-th row and the q-th row, then Y is also of type 1 obtained from I_n by exchanging the p-th row and the q-th row.

- If X is of type 2 obtained from I_n by multiplying the p-th row by a scalar a, then Y is also of type 2 obtained from I_n by multiplying the p-th row by $(1/a)$.

- If X is of type 3 obtained from I_n by adding the p-th row multiplied by a scalar a to the q-th row, then Y is also of type 3 obtained from I_n by adding the p-th row multiplied by $(-a)$ to the q-th row.

Thus, by Proposition 2.39 (b) we can conclude that X is invertible and $Y = X^{-1}$, which completes the proof. $\qquad\square$

3.2 Rank and Nullity of Matrices

Definition 3.5. The **rank** and **nullity** of a matrix $A \in F^{m \times n}$, denoted by $\mathrm{rank}(A)$ and $\mathrm{nullity}(A)$, respectively, are defined by

$$\mathrm{rank}(A) = \mathrm{rank}(L_A)$$
$$\mathrm{nullity}(A) = \mathrm{nullity}(L_A).$$

Theorem 3.6. The following statements are true for any matrix $A \in F^{m \times n}$.

(a) $\mathcal{R}(L_A) = \mathrm{span}(\mathrm{col}(A))$.

(b) $\mathrm{rank}(A) = \dim(\mathrm{span}(\mathrm{col}(A)))$.

Proof.

(a) Let $\beta = (x_1, \ldots, x_n)$ and $\gamma = (y_1, \ldots, y_m)$ be the standard ordered basis for F^n and F^m, respectively. Then we have

$$Ax_i = [L_A(x_i)]_\gamma,$$

which is the ith column of $[L_A]_\beta^\gamma = A$. Thus, we have $L_A(\beta) = \mathrm{col}(A)$, and it follows that

$$\mathcal{R}(L_A) = L_A(F^n) = L_A(\mathrm{span}(\beta)) = \mathrm{span}(L_A(\beta)) = \mathrm{span}(\mathrm{col}(A)).$$

(b) By (a), we have

$$\mathrm{rank}(A) = \mathrm{rank}(L_A) = \dim(\mathcal{R}(L_A)) = \dim(\mathrm{span}(\mathrm{col}(A))). \qquad \square$$

Theorem 3.7. If $A \in F^{n \times n}$, then A is invertible if and only if $\mathrm{rank}(A) = n$.

Proof. ($\Rightarrow$) Suppose that A is invertible. It follows that $L_A : F^n \to F^n$ is also invertible, and thus is bijective. Therefore,

$$\mathrm{rank}(A) = \mathrm{rank}(L_A) = \dim(\mathcal{R}(L_A)) = \dim(F^n) = n.$$

($\Leftarrow$) Suppose that $\mathrm{rank}(A) = n$. Then we can conclude that $\mathcal{R}(L_A) = F^n$ since $\mathcal{R}(L_A)$ is a subspace of F^n with

$$\dim(\mathcal{R}(L_A)) = \mathrm{rank}(L_A) = \mathrm{rank}(A) = n = \dim(F^n).$$

Thus, L_A is surjective. It follows that L_A is bijective by Lemma 2.20, and thus L_A is invertible. Therefore, A is invertible. $\qquad \square$

Lemma 3.8. Let V and W be vector spaces and let $T : V \to W$ be linear. Let U be a subspace of V.

(a) $\dim(T(U)) \leq \dim(U)$.

(b) If T is injective, then $\dim(T(U)) = \dim(U)$.

Proof. Let S be a basis for U. Then we have $T(U) = T(\mathrm{span}(S)) = \mathrm{span}(T(S))$.

(a) Let Q be a basis for $T(U)$. By replacement theorem (Theorem 1.21),

$$\dim(T(U)) = |Q| \leq |T(S)| \leq |S| = \dim(U).$$

(b) If T is injective, then $T(S)$ is linearly independent. Thus, $T(S)$ is a basis for $T(U)$, implying
$$\dim(T(U)) = |T(S)| = |S| = \dim(U). \qquad \square$$

Theorem 3.9. The following statements hold for any matrix $A \in F^{m \times n}$.

(a) If $X \in F^{m \times m}$ is invertible, then $\operatorname{rank}(XA) = \operatorname{rank}(A)$.

(b) If $Y \in F^{n \times n}$ is invertible, then $\operatorname{rank}(AY) = \operatorname{rank}(A)$.

Proof.

(a) Since X is invertible, L_X is invertible, and thus is bijective. It follows that $\dim(L_X(U)) = \dim(U)$ for any subspace U of F^n since L_X is injective. Thus,

$$
\begin{aligned}
\operatorname{rank}(XA) &= \operatorname{rank}(L_{XA}) \\
&= \dim(L_X(L_A(F^n))) \\
&= \dim(L_A(F^n)) \\
&= \operatorname{rank}(L_A) \\
&= \operatorname{rank}(A).
\end{aligned}
$$

(b) Since Y is invertible, L_Y is invertible, and thus is bijective. It follows that $L_Y(F^n) = F^n$ since L_Y is surjective. Thus,

$$
\begin{aligned}
\operatorname{rank}(AY) &= \operatorname{rank}(L_{AY}) \\
&= \dim(L_A(L_Y(F^n))) \\
&= \dim(L_A(F^n)) \\
&= \operatorname{rank}(L_A) \\
&= \operatorname{rank}(A). \qquad \square
\end{aligned}
$$

Theorem 3.10. Let V and W be finite-dimensional vector spaces with bases β and γ, respectively. If $T : V \to W$ is linear, then

$$\operatorname{rank}(T) = \operatorname{rank}\left([T]_\beta^\gamma\right).$$

Proof. Let $A = [T]_\beta^\gamma$. Since $[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$ holds for any $x \in V$, we have

$$\phi_\gamma T = L_A \phi_\beta.$$

Thus, since ϕ_β and ϕ_γ are invertible, we have

$$\operatorname{rank}(T) = \operatorname{rank}(\phi_\gamma T) = \operatorname{rank}(L_A \phi_\beta) = \operatorname{rank}(L_A) = \operatorname{rank}(A). \qquad \square$$

Theorem 3.11. Let $A \in F^{m \times n}$ and let r be a nonnegative integer. Then $\operatorname{rank}(A) = r$ if and only if A can be transformed into

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

by performing a finite number of elementary operations.

Proof. ($\Leftarrow$) Since A can be transformed into D by a finite number of elementary operations, there exist elementary matrices $X_1, \ldots, X_p \in F^{m \times m}$ and $Y_1, \ldots, Y_q \in F^{n \times n}$ such that

$$X_p \cdots X_1 A Y_1 \cdots Y_q = D.$$

Since elementary matrices are invertible,

$$\operatorname{rank}(A) = \operatorname{rank}(X_p \cdots X_1 A Y_1 \cdots Y_q) = \operatorname{rank}(D) = r.$$

($\Rightarrow$) If A is the zero matrix, then we have $r = 0$, and thus the theorem holds in this case with $D = A$. Now suppose that A is not the zero matrix. The proof is by induction on $k = \min(m, n)$.

First, we show that A can be transformed into some matrix B by a finite number of elementary operations as follows, where $B_{11} = 1$, $B_{1j} = 0$ and $B_{i1} = 0$ for $2 \le i \le m$ and $2 \le j \le n$.

1. First, we turn the $(1,1)$-entry into a nonzero number by performing type 1 elementary operations.

   a. If the first row contains only zeros, perform a type 1 row operation by exchanging the first row and a nonzero row.

   b. If the $(1,1)$-entry is zero, perform a type 1 column operation by exchanging the first column and a column whose first entry is not zero.

2. Then we turn the $(1,1)$-entry into 1 by performing a type 2 operation.

3. Finally, we eliminate all nonzero entries in the first row and the first column except the $(1,1)$-entry by performing type 3 operations.

   a. For $2 \le i \le m$, if the $(i,1)$-entry is nonzero, perform a type 3 row operation by adding a multiple of the first row to the ith row such that the $(i,1)$-entry becomes zero.

   b. For $2 \le j \le n$, if the $(1,j)$-entry is nonzero, perform a type 3 column operation by adding a multiple of the first column to the jth column such that the $(1,j)$-entry becomes zero.

By Theorem 3.9, $\operatorname{rank}(B) = \operatorname{rank}(A) = r$ since B can be obtained from A by performing a finite number of elementary operations.

Now we prove the theorem by induction on $\min(m, n)$. For the induction basis, assume that $m = 1$ or $n = 1$ holds. Then $\operatorname{rank}(A) = 1$ since A is not the zero matrix, and thus the theorem holds with $D = B$.

Now assume that the theorem holds for $\min(m, n) = k$ with $k \ge 1$, and we prove that the theorem also holds for $\min(m, n) = k + 1$. Since $\min(m, n) \ge 2$, we have

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array} \right),$$

where B' is an $(m-1) \times (n-1)$ matrix. Note that $\operatorname{rank}(B') = \operatorname{rank}(B) - 1 = r - 1$. By induction hypothesis, B' can be transformed into

$$D' = \begin{pmatrix} I_{r-1} & O_1 \\ O_2 & O_3 \end{pmatrix}$$

by a finite number of elementary row and column operations. It follows that

$$D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{array} \right)$$

is obtained from B by performing these operations. Thus, A can be transformed into D by a finite number of elementary operations, which completes the proof. $\square$

Corollary 3.12. Let $A \in F^{m \times n}$ and let r be a nonnegative integer. Then $\operatorname{rank}(A) = r$ if and only if there exist invertible $X \in F^{m \times m}$ and $Y \in F^{n \times n}$ such that

$$XAY = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}.$$

Proof. ($\Leftarrow$) It is proved by

$$\operatorname{rank}(A) = \operatorname{rank}(XAY) = r.$$

($\Rightarrow$) By Theorem 3.11, there exist elementary matrices $X_1, \ldots, X_p \in F^{m \times m}$ and $Y_1, \ldots, Y_q \in F^{n \times n}$ such that

$$X_p \cdots X_1 A Y_1 \cdots Y_q = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}.$$

Thus, the theorem holds by assigning $X = X_p \cdots X_1$ and $Y = Y_1 \cdots Y_q$. $\square$

Theorem 3.13. For any $A \in F^{m \times n}$, $\operatorname{rank}(A^t) = \operatorname{rank}(A)$.

Proof. Let $r = \operatorname{rank}(A)$. By Corollary 3.12, there exist invertible matrices $X \in F^{m \times m}$ and $Y \in F^{n \times n}$ such that

$$XAY = D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_{3,} \end{pmatrix}$$

implying

$$Y^t A^t X^t = D^t.$$

Thus,

$$\operatorname{rank}(A^t) = \operatorname{rank}(Y^t A^t X^t) = \operatorname{rank}(D^t) = r. \qquad \square$$

Theorem 3.14.

(a) Let U, V, W be finite-dimensional vector spaces over F. For any linear transformations $T_1 : U \to V$ and $T_2 : V \to W$, we have

$$\operatorname{rank}(T_2 T_1) \le \operatorname{rank}(T_1) \quad \text{and} \quad \operatorname{rank}(T_2 T_1) \le \operatorname{rank}(T_2).$$

40

(b) For any matrices $A \in F^{\ell \times m}$ and $B \in F^{m \times n}$, we have

$$\mathrm{rank}(AB) \leq \mathrm{rank}(A) \quad \text{and} \quad \mathrm{rank}(AB) \leq \mathrm{rank}(B).$$

Proof.

(a) By Lemma 3.8, we have

$$\mathrm{rank}(T_2 T_1) = \dim(T_2(T_1(U))) \leq \dim(T_1(U)) = \mathrm{rank}(T_1).$$

Furthermore, since $T_1(U) \subseteq V$, we have $T_2(T_1(U)) \subseteq T_2(V)$. Thus,

$$\mathrm{rank}(T_2 T_1) = \dim(T_2(T_1(U))) \leq \dim(T_2(V)) = \mathrm{rank}(T_2).$$

(b) By (a), we can conclude that

$$\mathrm{rank}(AB) = \mathrm{rank}(L_{AB}) = \mathrm{rank}(L_A L_B) \leq \mathrm{rank}(L_A) = \mathrm{rank}(A)$$
$$\mathrm{rank}(AB) = \mathrm{rank}(L_{AB}) = \mathrm{rank}(L_A L_B) \leq \mathrm{rank}(L_B) = \mathrm{rank}(B). \qquad \square$$

3.3 Matrix Inverses

Theorem 3.15. Every invertible matrix is a product of elementary matrices.

Proof. Suppose A is an invertible $n \times n$ matrix. Since $\mathrm{rank}(A) = n$, there exist elementary matrices $X_1, \ldots, X_p, Y_1, \ldots, Y_q \in F^{n \times n}$ such that

$$X_p \cdots X_1 A Y_1 \cdots Y_q = I_n,$$

implying

$$A = X_1^{-1} \cdots X_p^{-1} Y_q^{-1} \cdots Y_1^{-1}.$$

Since the inverses of elementary matrices are elementary matrices, we can conclude that A is a product of elementary matrices. $\square$

3.4 Systems of Linear Equations

Definition 3.16. The system E of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}
$$

where a_{ij} and b_i are scalars in a field F and $x_1, x_2, \ldots, x_n$ are n variables that take values in F, is called a system of m **linear equations** in n unknowns over the field F. Furthremore, it can be rewritten as a matrix equation

$$
E : Ax = b
$$

with

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}, \quad
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad
b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},
$$

and the matrices

$$
A \in F^{m \times n} \quad \text{and} \quad (A \mid b) \in F^{m \times (n+1)}
$$

are called the **coefficient matrix** and the **augmented matrix** of E, respectively.

Definition 3.17. For any system $E : Ax = b$ of linear equations with $A \in F^{m \times n}$, the **solution set** of E, denoted by $S(E)$, is defined by

$$
S(E) = \{s \in F^n : As = b\}.
$$

Each element of $S(E)$ is called a **solution** to E.

Theorem 3.18. If $E : Ax = b$ is a system of linear equations, then $S(E)$ is nonempty if and only if $\operatorname{rank}(A) = \operatorname{rank}(A \mid b)$.

Proof. It is proved by

$$
\begin{aligned}
S(E) \neq \varnothing \iff & Ax = b \text{ for some } x \in F^n \\
\iff & b \in \mathcal{R}(L_A) \\
\iff & b \in \operatorname{span}(\operatorname{col}(A)) \\
\iff & \operatorname{span}(\operatorname{col}(A)) = \operatorname{span}(\operatorname{col}(A \mid b)) \\
\iff & \operatorname{rank}(A) = \operatorname{rank}(A \mid b). \qquad \square
\end{aligned}
$$

Definition 3.19. A system $E : Ax = b$ of linear equations with $A \in F^{m \times n}$ is said to be **homogeneous** if $b = 0_{F^m}$.

Proposition 3.20. The following statements are true for any homogeneous system $E : Ax = 0_{F^m}$ of linear equations with $A \in F^{m \times n}$.

(a) $S(E) = \mathcal{N}(L_A)$.

(b) $S(E)$ is a subspace of A with $\dim(S(E)) = \text{nullity}(A)$.

Proof. Straightforward. $\qquad\square$

Definition 3.21. For any system

$$E : Ax = b$$

of linear equations with $A \in F^{m \times n}$, the system

$$E_H : Ax = 0_{F^m}$$

of linear equations is called the **homogeneous system** corresponding to E.

Proposition 3.22. For any system $E : Ax = b$ of linear equations with $A \in F^{m \times n}$,

$$S(E) = \{s\} + S(E_H)$$

holds for any solution $s \in S(E)$.

Proof. For any $r \in F^n$, we have

$$
\begin{aligned}
r \in S(E) \iff & Ar = b \\
\iff & A(r - s) = 0_{F^m} \\
\iff & r - s \in S(E_H) \\
\iff & r \in \{s\} + S(E_H).
\end{aligned}
$$
$\qquad\square$

Theorem 3.23. Let $E : Ax = b$ be a system of linear equations with $A \in F^{n \times n}$. Then A is invertible if and only if E has exactly one solution.

Proof. ($\Rightarrow$) Suppose that $s \in F^n$ is a solution to E. Then we have $As = b$, implying $s = A^{-1}b$. Thus, $S(E) = \{A^{-1}b\}$.

($\Leftarrow$) Let $s \in F^n$ be the unique solution to E. Since $S(E) = \{s\} + S(E_H)$, we can conclude that $S(E_H) = \{0_{F^n}\}$, implying

$$\text{rank}(A) = n - \text{nullity}(A) = n - \dim(S(E_H)) = n - 0 = n.$$

Thus, A is invertible. $\qquad\square$

Theorem 3.24. Let $E : Ax = b$ and $E' : A'x = b'$ be systems of linear equations with $A, A' \in F^{m \times n}$. If there is an invertible matrix $X \in F^{m \times m}$ with

$$X(A \mid b) = (A' \mid b'),$$

then $S(E) = S(E')$.

Proof. For any $s \in F^n$, we have

$$
\begin{aligned}
s \in S(E) \iff & As = b \\
\iff & X(As) = Xb \\
\iff & A's = b' \\
\iff & s \in S(E').
\end{aligned}
$$
$\qquad\square$

Definition 3.25. A matrix is said to be in **reduced row echelon form** if it satisfies the following conditions.

(a) Any nonzero rows are above rows with all zeros.

(b) The first nonzero entry in each row is 1_F and it occurs to the right of the the first nonzero entry above it.

(c) The first nonzero entry in each row is the only nonzero entry in its column.

Theorem 3.26. Any matrix can be transformed into a matrix in reduced row echelon form by a finite number of elementary row operations.

Proof. One can repeat the following steps until all rows are processed or all nonzero columns are processed. At first, all rows and all columns has not been processed.

1. Find i such that the ith row is the first row that has not been processed, and find j such that the jth column is the first nonzero column that has not been processed.

2. If (i, j)-entry is zero, perform a type 1 row operation such that the (i, j)-entry becomes nonzero.

3. Perform a type 2 row operation to turn the (i, j)-entry into 1_F.

4. Perform type 3 row operations such that the (i, j)-entry becomes the only nonzero entry in the jth column.

5. Mark the ith row and the jth column as processed.

After the process above, any matrix should be transformed into a matrix in reduced row echelon form. □

Remark. The algorithm in the proof above is called **Gaussian-Jordan elimination**.

Chapter 4

Determinants

4.1 Characterization of the Determinant

Definition 4.1. A function $\delta : F^{n \times n} \to F$ is **n-linear** if

$$\delta(A) = k\delta(B) + \delta(C)$$

holds for any matrices $A, B, C \in F^{n \times n}$ satisfying the following properties for any $i \in \{1, \ldots, n\}$ and for any $k \in F$.

- The jth rows of A, B and C are identical for each $j \in \{1, \ldots, n\} \setminus \{i\}$.

- The ith row of A is the sum of the ith row of B multiplied by k and the ith row of C.

Definition 4.2. An n-linear function $\delta : F^{n \times n} \to F$ is **alternating** if

$$\delta(A) = 0_F$$

holds for any matrix $A \in F^{n \times n}$ that has two identical rows.

Proposition 4.3. Let $\delta : F^{n \times n} \to F$ be an alternating n-linear function and let $A \in F^{n \times n}$. Then the following statements are true.

(a) If $E_1 \in F^{n \times n}$ is an elementary matrix of type 1, then $\delta(E_1 A) = -\delta(A)$.

(b) If $E_2 \in F^{n \times n}$ is an elementary matrix of type 2 obtained by multiplying one row of I_n by scalar $k \in F$, then $\delta(E_2 A) = k\delta(A)$.

(c) If $E_3 \in F^{n \times n}$ is an elementary matrix of type 3, then $\delta(E_3 A) = \delta(A)$.

Proof. Let $\text{row}(A) = (x_1, \ldots, x_n)$.

(a) Let E_1 be obtained from I_n by interchanging the pth row and the qth row with

$p < q$. Then we have

$$
0_F = \delta \begin{pmatrix} x_1 \\ \vdots \\ x_p + x_q \\ \vdots \\ x_p + x_q \\ \vdots \\ x_n \end{pmatrix} = \delta \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_p + x_q \\ \vdots \\ x_n \end{pmatrix} + \delta \begin{pmatrix} x_1 \\ \vdots \\ x_q \\ \vdots \\ x_p + x_q \\ \vdots \\ x_n \end{pmatrix}
$$

$$
= \delta \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_p \\ \vdots \\ x_n \end{pmatrix} + \delta \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_q \\ \vdots \\ x_n \end{pmatrix} + \delta \begin{pmatrix} x_1 \\ \vdots \\ x_q \\ \vdots \\ x_p \\ \vdots \\ x_n \end{pmatrix} + \delta \begin{pmatrix} x_1 \\ \vdots \\ x_q \\ \vdots \\ x_q \\ \vdots \\ x_n \end{pmatrix}
$$

$$
= 0_F + \delta(A) + \delta(E_1 A) + 0_F.
$$

Thus, $\delta(E_1 A) = -\delta(A)$.

(b) Let E_2 be obtained from I_n by multiplying the pth row by some scalar k. Then we have

$$
\delta(E_2 A) = \delta \begin{pmatrix} x_1 \\ \vdots \\ k x_p \\ \vdots \\ x_n \end{pmatrix} = k \delta \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_n \end{pmatrix} = k \delta(A).
$$

(c) Let E_3 be obtained from I_n by adding the pth row multiplied by some scalar k to the qth row. If $p < q$, then we have

$$
\delta(E_3 A) = \delta \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ k x_p + x_q \\ \vdots \\ x_n \end{pmatrix} = k \delta \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_p \\ \vdots \\ x_n \end{pmatrix} + \delta \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_q \\ \vdots \\ x_n \end{pmatrix} = k 0_F + \delta(A) = \delta(A).
$$

The case that $q < p$ can be proved similarly. $\qquad\square$

Theorem 4.4. Let $\delta : F^{n \times n} \to F$ be an alternating n-linear function and let $A \in F^{n \times n}$. If $\mathrm{rank}(A) < n$, then $\delta(A) = 0_F$.

Proof. Since

$$
\dim(\mathrm{span}(\mathrm{row}(A))) = \mathrm{rank}(A^t) = \mathrm{rank}(A) < n,
$$

the rows of A is not a spanning set of F^n, and thus is linearly dependent, implying that there exists a row which is a linear combination of the other rows.

Therefore, A can be transformed into a matrix B that has two identical rows by a finite number of elementary row operations. It follows that

$$\delta(A) = \delta(E_p \cdots E_1 A) = \delta(B) = 0_F,$$

where $E_1, \ldots, E_p \in F^{n \times n}$ are elementary matrices. $\qquad\square$

Theorem 4.5. Let $\delta : F^{n \times n} \to F$ be an alternating n-linear function such that $\delta(I_n) = 1_F$. Then for any $A, B \in F^{m \times n}$, we have

$$\delta(AB) = \delta(A)\delta(B).$$

Proof. First, suppose that $\operatorname{rank}(A) < n$. Then we have $\operatorname{rank}(AB) < n$. Thus,

$$\delta(AB) = 0_F = \delta(A)\delta(B).$$

Now suppose that $\operatorname{rank}(A) = n$. That is, A is invertible, and thus $A = E_k \cdots E_1$ for some elementary matrices $E_1, \ldots, E_k \in F^{n \times n}$. Then we have

$$
\begin{aligned}
\delta(AB) &= \delta(E_k \cdots E_1 B) \\
&= \delta(E_k) \cdots \delta(E_1)\delta(B) \\
&= \delta(E_k) \cdots \delta(E_1)\delta(I_n)\delta(B) && (\delta(I_n) = 1_F) \\
&= \delta(E_k \cdots E_1 I_n)\delta(B) \\
&= \delta(A)\delta(B). && \square
\end{aligned}
$$

Theorem 4.6. There exists a unique alternating n-linear function $\delta : F^{n \times n} \to F$ with $\delta(I_n) = 1_F$.

Proof. Suppose that $\delta, \delta' : F^{n \times n} \to F$ are alternating n-linear functions with $\delta(I_n) = 1_F = \delta'(I_n)$. We prove that $\delta(A) = \delta(A')$ for any $A \in F^{n \times n}$. If $\operatorname{rank}(A) < n$, then

$$\delta(A) = 0_F = \delta'(A).$$

If $\operatorname{rank}(A) = n$, then A is invertible, and thus

$$A = E_p \cdots E_1$$

for some elementary matrices $E_1, \ldots, E_p \in F^{n \times n}$. It follows that

$$
\begin{aligned}
\delta(A) &= \delta(E_p \cdots E_1 I_n) \\
&= \delta(E_p) \cdots \delta(E_1)\delta(I_n) \\
&= \delta'(E_p) \cdots \delta'(E_1)\delta(I_n) \\
&= \delta'(E_p \cdots E_1 I_n) \\
&= \delta'(A). && \square
\end{aligned}
$$

Definition 4.7. The **determinant** of $A \in F^{n \times n}$ is

$$\det(A) = \delta(A),$$

where $\delta : F^{n \times n} \to F$ is the alternating n-linear function with $\delta(I_n) = 1_F$.

4.2 Permutations

Definition 4.8.

- A function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a **permutation** over $\{1, 2, \ldots, n\}$ if σ is bijective. The set of all permutations over $\{1, 2, \ldots, n\}$ is denoted by S_n.

- An inversion of $\sigma \in S_n$ is a pair (i, j) with $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$. The number of inversions of σ is denoted by $\rho(\sigma)$.

- The **sign** of $\sigma \in S_n$ is defined by

$$\mathrm{sgn}(\sigma) = (-1)^{\rho(\sigma)}.$$

Theorem 4.9. For any matrix $A \in F^{n \times n}$,

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}.$$

Proof. Let $\delta : F^{n \times n} \to F$ be the function

$$\delta(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}.$$

We prove that δ is an alternating n-linear function with $\delta(I_n) = 1_F$.

First, we show that δ is n-linear. Suppose that $A, B, C \in F^{n \times n}$ are matrices satisfying the following properties for any $p \in \{1, \ldots, n\}$ and for any $k \in F$.

- The ith rows of A, B and C are identical for each $i \in \{1, \ldots, n\} \setminus \{p\}$.

- The pth row of A is the sum of the pth row of B multiplied by k and the pth row of C.

Then we have

$$
\begin{aligned}
\delta(A) &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{p,\sigma(p)} \prod_{\substack{1 \le i \le n \\ i \ne p}} A_{i,\sigma(i)} \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(k B_{p,\sigma(p)} + C_{p,\sigma(p)}) \prod_{\substack{1 \le i \le n \\ i \ne p}} A_{i,\sigma(i)} \\
&= k \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) B_{p,\sigma(p)} \prod_{\substack{1 \le i \le n \\ i \ne p}} A_{i,\sigma(i)} + \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) C_{p,\sigma(p)} \prod_{\substack{1 \le i \le n \\ i \ne p}} A_{i,\sigma(i)} \\
&= k \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) B_{p,\sigma(p)} \prod_{\substack{1 \le i \le n \\ i \ne p}} B_{i,\sigma(i)} + \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) C_{p,\sigma(p)} \prod_{\substack{1 \le i \le n \\ i \ne p}} C_{i,\sigma(i)} \\
&= k\delta(B) + \delta(C).
\end{aligned}
$$

Now we show that δ is alternating. Suppose that $D \in F^{n \times n}$ is a matrix whose pth row and qth row are identical with $p \ne q$. For each $\sigma \in S_n$, let $\sigma' \in S_n$ be the permutation that satisfies the following properties.

- $\sigma'(p) = \sigma(q)$ and $\sigma'(q) = \sigma(p)$.

- $\sigma'(i) = \sigma(i)$ for each $i \in \{1, \ldots, n\} \setminus \{p, q\}$.

Then we have

$$
\begin{aligned}
\delta(D) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{1 \le i \le n} D_{i,\sigma(i)} \\
&= \sum_{\substack{\sigma \in S_n \\ \sigma(p) < \sigma(q)}} \text{sgn}(\sigma) \prod_{1 \le i \le n} D_{i,\sigma(i)} + \sum_{\substack{\sigma \in S_n \\ \sigma(p) > \sigma(q)}} \text{sgn}(\sigma) \prod_{1 \le i \le n} D_{i,\sigma(i)} \\
&= \sum_{\substack{\sigma \in S_n \\ \sigma(p) < \sigma(q)}} \text{sgn}(\sigma) \prod_{1 \le i \le n} D_{i,\sigma(i)} + \sum_{\substack{\sigma \in S_n \\ \sigma(p) < \sigma(q)}} \text{sgn}(\sigma') \prod_{1 \le i \le n} D_{i,\sigma'(i)} \\
&= \sum_{\substack{\sigma \in S_n \\ \sigma(p) < \sigma(q)}} (\text{sgn}(\sigma) + \text{sgn}(\sigma')) \prod_{1 \le i \le n} D_{i,\sigma(i)} \\
&= 0_F.
\end{aligned}
$$

Finally, we have

$$
\delta(I_n) = \text{sgn}(\sigma_0) = 1_F,
$$

where σ_0 is the identity permutation. Therefore, δ is an alternating n-linear function with $\delta(I_n) = 1_F$, and by Theorem 4.6 we can conclude that it is exactly the determinant function. $\qquad\square$

4.3   Properties of Determinants

Theorem 4.10. For any $A, B \in F^{n \times n}$, we have $\det(AB) = \det(A)\det(B)$.

Proof. Striaghtforward from Theorem 4.5. $\qquad\square$

Theorem 4.11. If $A \in F^{n \times n}$ is invertible, then $\det(A^{-1}) = (\det(A))^{-1}$.

Proof. It follows by

$$\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I_n) = 1_F. \qquad\square$$

Definition 4.12. Let $A, B \in F^{n \times n}$. We say that A and B are **similar**, denoted

$$A \sim B,$$

if there is an invertible matrix $Q \in F^{n \times n}$ such that $B = QAQ^{-1}$.

Theorem 4.13. For any $A, B \in F^{n \times n}$, if $A \sim B$, then $\det(A) = \det(B)$.

Proof. Suppose that Q is invertible such that $B = QAQ^{-1}$. Then

$$\begin{aligned}
\det(B) &= \det(QAQ^{-1}) \\
&= \det(Q) \cdot \det(A) \cdot \det(Q^{-1}) \\
&= \det(Q) \cdot \det(Q^{-1}) \cdot \det(A) \\
&= \det(I_n) \cdot \det(A) \\
&= \det(A). \qquad\square
\end{aligned}$$

Definition 4.14. Let $n \geq 2$. For any $A \in F^{n \times n}$ and for any $i, j \in \{1, \ldots, n\}$, let $\tilde{A}_{ij}$ denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and the jth column.

Theorem 4.15 (Laplace Expansion). Let $n \geq 2$. For any $A \in F^{n \times n}$ and $i \in \{1, \ldots, n\}$, we have

$$\det(A) = \sum_{j=1}^{n} A_{ij}(-1)^{i+j} \det(\tilde{A}_{ij}).$$

Proof. For $j \in \{1, \ldots, n\}$, let $B^{(j)}$ be the matrix obtained from A by replacing its ith row with e_j. Note that we can turn $B^{(j)}$ into a matrix

$$C^{(j)} = \begin{pmatrix} 1 & O \\ X & \tilde{A}_{ij} \end{pmatrix}$$

by $i - 1$ row swaps and $j - 1$ column swaps, where X is an $(n-1) \times 1$ matrix, and O is the $1 \times (n-1)$ zero matrix. Thus, we have

$$\det(B^{(j)}) = (-1)^{(i-1)+(j-1)} \det(C^j) = (-1)^{i+j} \det(\tilde{A}_{ij}).$$

Since $\det(\cdot)$ is n-linear, we have

$$\det(A) = \sum_{j=1}^{n} A_{ij} \det(B^{(j)}) = \sum_{j=1}^{n} A_{ij}(-1)^{i+j} \det(\tilde{A}_{ij}). \qquad\square$$

Chapter 5

Diagonalization

5.1 Eigenvalues and Eigenvectors

Definition 5.1. Let $T : V \to V$ be a linear operator on a vector space V over a field F. If

$$T(x) = \lambda x$$

holds for some scalar $\lambda \in F$ and some vector $x \in V \setminus \{0_V\}$, then (λ, x) is called an **eigenpair** of T, with λ and x called an **eigenvalue** and an **eigenvector** of T, respectively.

Definition 5.2. Let V be a finite-dimensional vector space over a field F. Let $T \in \mathcal{L}(V)$. An **eigenbasis** of V for T is an ordered basis of V in which every vector is an eigenvector of T.

Theorem 5.3. Let V be a vector space over a field F and let $T : V \to V$ be linear. Let $\beta = (x_1, x_2, \ldots, x_n)$ be an ordered basis for T. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$ be scalars. Then

$$[T]_\beta^\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

if and only if $T(x_i) = \lambda_i x_i$ for each $i \in \{1, 2, \ldots, n\}$.

Proof. ($\Rightarrow$) For each $i \in \{1, 2, \ldots, n\}$, we have

$$[T(x_i)]_\beta = \lambda_i e_i = [\lambda_i x_i]_\beta.$$

Thus, $T(x_i) = \lambda_i x_i$. ($\Leftarrow$) For each $i \in \{1, 2, \ldots, n\}$, we have $[T(x_i)]_\beta = [\lambda_i x_i]_\beta = \lambda_i e_i$, and it follows that

$$[T]_\beta^\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \qquad \square$$

Corollary 5.4. Let V be a finite-dimensional vector space and let $T : V \to V$ be linear. Let β be an ordered basis of T. Then $[T]_\beta^\beta$ is diagonal if and only if β is an eigenbasis of V for T.

Proof. Straightforward from Theorem 5.3. □

Definition 5.5. Let $A \in F^{n \times n}$. If

$$Ax = \lambda x$$

holds for some scalar $\lambda \in F$ and some vector $x \in V \setminus \{0_V\}$, then (λ, x) is called an **eigenpair** of A, with λ and x called an **eigenvalue** and an **eigenvector** of A, respectively.

Theorem 5.6. Let V be a finite-dimensional vector space with an ordered basis β. Let $\lambda \in F$ be a scalar and $x \in V$ be a vector. Then (λ, x) is an eigenpair of T if and only if $(\lambda, [x]_\beta)$ is an eigenpair of $[T]_\beta^\beta$.

Proof. ($\Rightarrow$) Suppose that $T(x) = \lambda x$. Then we have

$$[T]_\beta^\beta [x]_\beta = [T(x)]_\beta = [\lambda x]_\beta = \lambda [x]_\beta.$$

($\Leftarrow$) Since

$$[T(x)]_\beta = [T]_\beta^\beta [x]_\beta = \lambda [x]_\beta = [\lambda x]_\beta,$$

we can conclude that $T(x) = \lambda x$. □

5.2 Characteristic Polynomials and Eigenspaces

Theorem 5.7. Let $A \in F^{n \times n}$ be a matrix and let $\lambda \in F$ be a scalar. Then λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0_F$.

Proof. The proof is as follows.

$$
\begin{aligned}
\lambda \text{ is an eigenvalue of } A \quad &\Leftrightarrow \quad Ax = \lambda x \text{ for some } x \in F^n \setminus \{0_{F^n}\} \\
&\Leftrightarrow \quad (A - \lambda I_n)x \text{ for some } x \in F^n \setminus \{0_{F^n}\} \\
&\Leftrightarrow \quad (A - \lambda I_n) \text{ is not invertible} \\
&\Leftrightarrow \quad \det(A - \lambda I_n) = 0_F. \qquad \square
\end{aligned}
$$

Definition 5.8. The **characteristic polynomial** of $A \in F^{n \times n}$ is the polynomial $f_A \in \mathcal{P}(F)$ with

$$
f_A(t) = \det(A - t I_n).
$$

Theorem 5.9. Let $A, B \in F^{n \times n}$. If $A \sim B$, then $f_A = f_B$.

Proof. To be completed. $\qquad \square$

Definition 5.10. Let V be a finite-dimensional vector space over F with $\dim(V) = n$. For all $T \in \mathcal{L}(V)$, we define the **characteristic polynomial** of T as

$$
f_T = f_A,
$$

where $A = [T]_\beta^\beta$ for some arbitrary ordered basis β of V.

Theorem 5.11. Let V be a vector space over a field F and let $T \in \mathcal{L}(V)$. For any $\lambda \in F$ and $x \in V \setminus \{0_V\}$, (λ, x) is an eigenpair of T if and only if $x \in \mathcal{N}(T - \lambda I_V)$.

Proof. The proof is as follows.

$$
\begin{aligned}
(\lambda, x) \text{ is an eigenpair of } T \quad &\Leftrightarrow \quad T(x) = \lambda x \\
&\Leftrightarrow \quad T(x) = (\lambda I_V)(x) \\
&\Leftrightarrow \quad (T - \lambda I_V)(x) = 0_V \\
&\Leftrightarrow \quad x \in \mathcal{N}(T - \lambda I_V). \qquad \square
\end{aligned}
$$

Definition 5.12. Let V be a vector space over a field F. Let $T : V \to V$ be linear and let λ be an eigenvalue of T. Then

$$
E_T(\lambda) = \mathcal{N}(T - \lambda I_V).
$$

is called the **eigenspace** of T with respect to λ.

5.3 Diagonalizability

5.4 Cayley-Hamilton Theorem

Definition 5.13. Let V be a vector space and let $T \in \mathcal{L}(V)$. A subspace W of V is a **T-invariant subspace** of V if
$$T(W) \subseteq W.$$

Theorem 5.14. Let V be a finite-dimensional vector space and let $T : V \to V$ be linear. Let W be a T-invariant subspace of V and define $T' : W \to W$ as the transformation such that $T'(x) = T(x)$ for any $x \in W$. Then we have
$$f_{T'}(t) \mid f_T(t).$$

Proof. Let $\gamma = (x_1, \ldots, x_k)$ be an ordered basis of W. By replacement theorem (Theorem 1.21), there is an ordered basis $\beta = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)$ of V. It can be shown that
$$[T]_\beta^\beta = \begin{pmatrix} [T']_\gamma^\gamma & X \\ O & Y \end{pmatrix}$$
for some $X \in F^{k \times (n-k)}$ and $Y \in F^{(n-k) \times (n-k)}$. Thus, we have
$$\begin{aligned} f_T(t) &= \det([T]_\beta^\beta - tI_n) \\ &= \det \begin{pmatrix} [T']_\gamma^\gamma - tI_k & X \\ O & Y - tI_{n-k} \end{pmatrix} \\ &= \det([T']_\gamma^\gamma - tI_k) \cdot \det(Y - tI_{n-k}) \\ &= f_{T'}(t) \cdot \det(Y - tI_{n-k}). \end{aligned}$$ $\square$

Definition 5.15. Let V be a vector space and let $T \in \mathcal{L}(V)$. The **T-cyclic subspace** of V generated by $x \in V$ is defined as
$$C_T(x) = \operatorname{span}\left(\bigcup_{i=0}^\infty \{T^i(x)\} \right).$$

Theorem 5.16. Let V be a vector space and let $T \in \mathcal{L}(V)$. Then the following statements hold for any $x \in V$.

(a) $C_T(x)$ is a T-invariant subspace of V.

(b) If W is a T-invariant subspace of V with $x \in W$, then $C_T(x) \subseteq W$.

Proof.

(a) Suppose that $y \in C_T(x)$ with
$$y = \sum_{i=0}^k a_i T^i(x).$$

Then we have
$$T(y) = T \left(\sum_{i=0}^k a_i T^i(x) \right) = \sum_{i=0}^k a_i T^{i+1}(x) \in C_T(x).$$

It follows that $T(C_T(x)) \subseteq C_T(x)$, and thus $C_T(x)$ is T-invariant.

(b) Since $x \in U$ and $T(U) \subseteq U$, we can conclude that $T^i(x) \in U$ holds for any nonnegative integer i. Thus, we have

$$\bigcup_{i=0}^{\infty} \{T^i(x)\} \subseteq U,$$

implying

$$C_T(x) = \operatorname{span}\left(\bigcup_{i=0}^{\infty} \{T^i(x)\}\right) \subseteq U. \qquad \square$$

Chapter 6

Inner Product Spaces

6.1 Inner Products and Norms

Definition 6.1. Let V be a vector space over a field $F \in \{\mathbb{R}, \mathbb{C}\}$. A function

$$\langle \cdot \mid \cdot \rangle : V \times V \to F$$

is called an **inner product** on V if it satisfies the following properties for all $x, x', y \in V$.

(a) $\langle ax + x' \mid y \rangle = a \langle x \mid y \rangle + \langle x' \mid y \rangle$.

(b) $\langle x \mid y \rangle = \overline{\langle y \mid x \rangle}$.

(c) $\langle x \mid x \rangle \in \mathbb{R}^+$ for any $x \in V \setminus \{0_V\}$.

A vector space equipped with an inner product is called an **inner product space**.

Proposition 6.2. Let V be an inner product space over a field $F \in \{\mathbb{R}, \mathbb{C}\}$. Then the following statements are true for $x, y, y' \in V$ and $a \in F$.

(a) $\langle x \mid ay + y' \rangle = \overline{a} \langle x \mid y \rangle + \langle x \mid y' \rangle$.

(b) $\langle x \mid 0_V \rangle = 0_F = \langle 0_V \mid x \rangle$.

(c) $\langle x \mid x \rangle = 0_F$ if and only if $x = 0_V$.

(d) If $\langle x \mid y \rangle = \langle x \mid y' \rangle$ holds for all $x \in V$, then $y = y'$.

Proof.

(a) It is proved by

$$\langle x \mid ay + y' \rangle = \overline{\langle ay + y' \mid x \rangle} = \overline{a \langle y \mid x \rangle + \langle y' \mid x \rangle} = \overline{a} \langle x \mid y \rangle + \langle x \mid y' \rangle.$$

(b) By

$$\langle x \mid x \rangle = \langle x \mid 1_F x + 0_V \rangle = \overline{1_F} \langle x \mid x \rangle + \langle x \mid 0_V \rangle = \langle x \mid x \rangle + \langle x \mid 0_V \rangle$$

and

$$\langle x \mid x \rangle = \langle 1_F x + 0_V \mid x \rangle = 1_F \langle x \mid x \rangle + \langle 0_V \mid x \rangle = \langle x \mid x \rangle + \langle 0_V \mid x \rangle,$$

we have $\langle x \mid 0_V \rangle = 0_F = \langle 0_V \mid x \rangle$.

(c) ($\Leftarrow$) If $x = 0_V$, then $\langle x \mid x \rangle = 0_F$ by (b).

($\Rightarrow$) If $\langle x \mid x \rangle = 0_F$, then $x = 0_V$ by Definition 6.1 (c).

(d) Note that
$$\langle x \mid y - y' \rangle = \langle x \mid y \rangle + \overline{(-1_F)} \langle x \mid y' \rangle = 0_F$$
holds for all $x \in V$. Since $\langle y - y' \mid y - y' \rangle = 0_F$, we have $y - y' = 0_V$, and thus $y = y'$. $\qquad \square$

Definition 6.3. Let V be an inner product space over a field F.

- For $x, y \in V$, we say that x and y are **orthogonal** if
$$\langle x \mid y \rangle = 0_F.$$

- A subset S of V is **orthogonal** if any two distinct vectors in S are orthogonal.

Theorem 6.4. Let V be an inner product space over a field F. Let S be an orthogonal subset of $V \setminus \{0_V\}$ and let $x_1, \ldots, x_n$ be distinct vectors in S. Then for $y \in V$, if
$$y = \sum_{i=1}^{n} a_i x_i$$
for some $a_1, \ldots, a_n \in F$, then
$$a_i = \frac{\langle y \mid x_i \rangle}{\langle x_i \mid x_i \rangle}$$
for each $i \in \{1, \ldots, n\}$.

Proof. For each $i \in \{1, \ldots, n\}$, we have
$$\langle y \mid x_i \rangle = \left\langle \sum_{j=1}^{n} a_j x_j \;\middle|\; x_i \right\rangle = \sum_{j=1}^{n} a_j \langle x_j \mid x_i \rangle = a_i \langle x_i \mid x_i \rangle,$$
implying
$$a_i = \frac{\langle y \mid x_i \rangle}{\langle x_i \mid x_i \rangle}. \qquad \square$$

Corollary 6.5. Let V be an inner product space over a field F. If S is an orthogonal subset of $V \setminus \{0_V\}$, then S is linearly independent.

Proof. Suppose that there exist scalars $a_1, \ldots, a_n \in F$ and distinct vectors $x_1, \ldots, x_n \in S$ such that
$$\sum_{i=1}^{n} a_i x_i = 0_V.$$
Then we have
$$a_i = \frac{\langle 0_V \mid x_i \rangle}{\langle x_i \mid x_i \rangle} = 0_F$$
for each $i \in \{1, \ldots, n\}$. Thus, S is linearly independent. $\qquad \square$

Theorem 6.6 (Gram-Schmidt Process). Let V be a finite-dimensional inner product space over a field F. Let $R = \{x_1, \ldots, x_n\}$ be a linearly independent subset of V. Then the set $S = \{y_1, \ldots, y_n\}$ with

$$y_i = x_i - \sum_{j=1}^{i-1} \frac{\langle x_i \mid y_j \rangle}{\langle y_j \mid y_j \rangle} y_j$$

for $1 \leq i \leq n$ is an orthogonal set of nonzero vectors satisfying $\mathrm{span}(S) = \mathrm{span}(R)$.

Proof. The proof is by induction on n. The theorem holds for $n = 0$. To show the induction step, let $n \geq 1$. By the induction hypothesis, $\langle y_j \mid y_i \rangle = 0_F$ for distinct $i, j \in \{1, \ldots, n-1\}$. Then since for $i \in \{1, \ldots, n-1\}$, we have

$$
\begin{aligned}
\langle y_n \mid y_i \rangle &= \left\langle x_n - \sum_{j=1}^{n-1} \frac{\langle x_n \mid y_j \rangle}{\langle y_j \mid y_j \rangle} y_j \,\middle|\, y_i \right\rangle \\
&= \langle x_n \mid y_i \rangle - \sum_{j=1}^{n-1} \frac{\langle x_n \mid y_j \rangle}{\langle y_j \mid y_j \rangle} \langle y_j \mid y_i \rangle \\
&= \langle x_n \mid y_i \rangle - \frac{\langle x_n \mid y_i \rangle}{\langle y_i \mid y_i \rangle} \langle y_i \mid y_i \rangle \\
&= 0_F,
\end{aligned}
$$

we can conclude that S is orthogonal. Furthermore, if $y_n = 0_V$, then

$$x_n \in \mathrm{span}(\{y_1, \ldots, y_{n-1}\}) = \mathrm{span}(\{x_1, \ldots, x_{n-1}\})$$

because

$$x_n = y_n + \sum_{j=1}^{n-1} \frac{\langle x_n \mid y_j \rangle}{\langle y_j \mid y_j \rangle} y_j,$$

contradiction to the fact that R is linearly independent. Thus, $y_n \neq 0_V$, implying $0_V \notin S$. It follows that S is linearly independent by Corollary 6.5. Therefore, since $|S| = \dim(\mathrm{span}(R))$, we have $\mathrm{span}(S) = \mathrm{span}(R)$. $\qquad\square$

Definition 6.7. Let V be an inner product space. For each vector $x \in S$, the **norm** of x is a nonnegative real number, defined as

$$\|x\| = \sqrt{\langle x \mid x \rangle}.$$

Proposition 6.8. Let V be an inner product space over a field $F \in \{\mathbb{R}, \mathbb{C}\}$. Then the following statements are true for any vectors $x, y \in V$ and any scalar $a \in F$.

(a) $\|ax\| = |a| \cdot \|x\|$.

(b) $\|x\| = 0_F$ if and only if $x = 0_V$.

Proof.

(a) We have

$$\|ax\| = \sqrt{\langle ax \mid ax \rangle} = \sqrt{a\bar{a}\langle x \mid x \rangle} = \sqrt{|a|^2 \langle x \mid x \rangle} = |a| \cdot \|x\|.$$

(b) We have
$$\|x\| = 0_F \quad \Leftrightarrow \quad \langle x \mid x \rangle = 0_F \quad \Leftrightarrow \quad x = 0_V.$$

$\square$

Theorem 6.9 (Pythagorean Theorem). Let V be an inner product space over a field $F \in \{\mathbb{R}, \mathbb{C}\}$. Then for any vectors $x, y \in V$ with $\langle x \mid y \rangle = 0_F$, we have
$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Proof. We have
$$\begin{aligned}
\|x + y\|^2 &= \langle x + y \mid x + y \rangle \\
&= \langle x \mid x + y \rangle + \langle y \mid x + y \rangle \\
&= \langle x \mid x \rangle + \langle x \mid y \rangle + \langle y \mid x \rangle + \langle y \mid y \rangle \\
&= \langle x \mid x \rangle + 0_F + 0_F + \langle y \mid y \rangle \\
&= \|x\|^2 + \|y\|^2.
\end{aligned}$$
$\square$

Definition 6.10. Let V be an inner product space over a field $F \in \{\mathbb{R}, \mathbb{C}\}$. We say that a subset S of V is **orthonormal** if S is orthogonal and $\|x\| = 1_F$ for each $x \in S$.

Theorem 6.11. Let V be an inner product space over a field $F \in \{\mathbb{R}, \mathbb{C}\}$. Let S be an orthonormal subset of V and let $x_1, \ldots, x_n$ be distinct vectors in S. Then for $y \in V$, if
$$y = \sum_{i=1}^{n} a_i x_i$$
for some $a_1, \ldots, a_n \in F$, then
$$a_i = \langle y \mid x_i \rangle$$
for each $i \in \{1, \ldots, n\}$.

Proof. Since S is orthonormal, we have $0_V \notin S$. It follows that
$$a_i = \frac{\langle y \mid x_i \rangle}{\langle x_i \mid x_i \rangle} = \frac{\langle y \mid x_i \rangle}{1_F} = \langle y \mid x_i \rangle$$
for each $i \in \{1, \ldots, n\}$ by Theorem 6.4.
$\square$

Definition 6.12. Let V be an inner product space over F and let S be a subspace of V. The **orthogonal complement** of S, denoted $S^\perp$, is the set of vectors that are orthogonal to every vector in S, i.e.,
$$S^\perp = \{x \in V : \langle x \mid y \rangle = 0_F \text{ for all } y \in S\}.$$

Theorem 6.13. Let V be an inner product space over F. For any subset S of V, $S^\perp$ is a subspace of V.

Proof. We have $0_V \in S^\perp$ since $\langle 0_V \mid z \rangle = 0_F$ for any $z \in S$. For any $a \in F$ and $x, y \in S^\perp$, we have
$$\begin{aligned}
\langle ax + y \mid z \rangle &= a \langle x \mid z \rangle + \langle y \mid z \rangle \\
&= a0_F + 0_F \\
&= 0_F
\end{aligned}$$
for any $z \in S$, implying $ax + y \in S^\perp$. Thus, $S^\perp$ is a subspace of V by **??**.
$\square$

Theorem 6.14. Let V be a finite-dimensional inner product space over F. If W is a subspace of V, then $W \oplus W^\perp = V$.

Proof. Let $R = \{y_1, \ldots, y_k\}$ be an orthonormal basis of W. We have $W + W^\perp \subseteq V$ since W and $W^\perp$ are subspaces of V. To prove $V \subseteq W + W^\perp$, suppose that $x \in V$, and let

$$y = \sum_{i=1}^{k} \langle x \mid y_i \rangle \, y_i$$

be a vector in W. Then $x - y \in V^\perp$ since

$$\langle x - y \mid y_j \rangle = \left\langle x - \sum_{i=1}^{k} \langle x \mid y_i \rangle \, y_i \,\middle|\, y_j \right\rangle$$

$$= \langle x \mid y_j \rangle - \sum_{i=1}^{k} \langle x \mid y_i \rangle \langle y_i \mid y_j \rangle$$

$$= \langle x \mid y_j \rangle - \langle x \mid y_j \rangle$$

$$= 0_F.$$

Thus,

$$x = y + (x - y) \in V + V^\perp,$$

implying $V \subseteq W + W^\perp$, and thus $W + W^\perp = V$.

Furthermore, for any $x \in W \cap W^\perp$, we have $\langle x \mid x \rangle = 0_F$, implying $x = 0_V$. Thus, we have $W \cap W^\perp = \{0_V\}$, which implies $W \oplus W^\perp = V$. $\qquad\square$

6.2 The Adjoint of a Linear Operator

Theorem 6.15. Let V be a finite-dimensional inner product space over $F \in \{\mathbb{R}, \mathbb{C}\}$. Let $f : V \to F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that

$$f(x) = \langle x \mid y \rangle$$

for all $x \in V$.

Proof. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an orthonormal basis for V. Then we have

$$f(x) = f\left(\sum_{i=1}^{n} \langle x \mid x_i \rangle \cdot x_i\right)$$

$$= \sum_{i=1}^{n} \langle x \mid x_i \rangle \cdot f(x_i)$$

$$= \left\langle x \,\middle|\, \sum_{i=1}^{n} \overline{f(x_i)} \cdot x_i \right\rangle.$$

Thus, there exists

$$y = \sum_{i=1}^{n} \overline{f(x_i)} \cdot x_i$$

such that $f(x) = \langle x \mid y \rangle$ for all $x \in V$.

Furthermore, if there exists $y' \in V$ such that $f(x) = \langle x \mid y' \rangle$ for all $x \in V$, then we have $y' = y$ by Proposition 6.2 (d), which completes the proof. $\square$

Theorem 6.16. Let V be a finite-dimensional inner product space over $F \in \{\mathbb{R}, \mathbb{C}\}$. For any linear operator $T : V \to V$, there exists a unique operator $T' : V \to V$ such that

$$\langle T(x) \mid y \rangle = \langle x \mid T'(y) \rangle$$

for all $x, y \in V$. Also, T' is linear.

Proof. Suppose that $y \in V$ is an arbitrary vector. Let $f : V \to F$ be a function such that $f(x) = \langle T(x) \mid y \rangle$ for each $x \in V$. Then f is linear since

$$f(ax_1 + x_2) = \langle T(ax_1 + x_2) \mid y \rangle$$
$$= \langle aT(x_1) + T(x_2) \mid y \rangle$$
$$= a \langle T(x_1) \mid y \rangle + \langle T(x_2) \mid y \rangle$$
$$= af(x_1) + f(x_2)$$

holds for each $a \in F$ and for each $x_1, x_2 \in V$. Since f is linear, there exists a vector $y' \in V$ such that $f(x) = \langle x \mid y' \rangle$ by Theorem 6.15. Thus, we can define $T' : V \to V$ as the functoin with $T'(y) = y'$, implying $\langle T(x) \mid y \rangle = \langle x \mid T'(y) \rangle$ for each $x, y \in V$.

Now we prove that T' is linear. For any $a \in F$ and $x, y_1, y_2 \in V$, we have

$$\langle x \mid T'(ay_1 + y_2) \rangle = \langle T(x) \mid ay_1 + y_2 \rangle$$
$$= \overline{a} \langle T(x) \mid y_1 \rangle + \langle T(x) \mid y_2 \rangle$$
$$= \overline{a} \langle x \mid T'(y_1) \rangle + \langle x \mid T'(y_2) \rangle$$
$$= \langle x \mid aT'(y_1) + T'(y_2) \rangle.$$

Thus, we can conclude that $T'(ay_1+y_2) = aT'(y_1)+T'(y_2)$ for any $a \in F$ and $y_1, y_2 \in V$ by Proposition 6.2 (d).

To show that T' is unique, suppose that $T'' : V \to V$ is linear and satisfies $\langle T(x) \mid y \rangle = \langle x \mid T''(y) \rangle$ for any $x, y \in V$. Then we have

$$\langle x \mid T''(y) \rangle = \langle T(x) \mid y \rangle = \langle x \mid T'(y) \rangle ,$$

implying $T''(y) = T'(y)$ for any $y \in V$ by Proposition 6.2 (d). Thus, $T'' = T'$. $\square$

Definition 6.17. Let V be a finite-dimensional inner product space over $F \in \{\mathbb{R}, \mathbb{C}\}$ and let $T : V \to V$ be linear. The **adjoint** of T, denoted T^*, is the linear operator satisfying

$$\langle T(x) \mid y \rangle = \langle x \mid T^*(y) \rangle$$

for all $x, y \in V$.

Theorem 6.18. Let V be a finite-dimensional inner product space and let β be an ordered orthonormal basis of V. If $T : V \to V$ is linear, then

$$[T^*]_\beta^\beta = \left([T]_\beta^\beta\right)^*.$$

Proof. Suppose that $\dim(V) = n$ and $\beta = (x_1, x_2, \ldots, x_n)$. Let $A = [T^*]_\beta^\beta$ and $B = [T]_\beta^\beta$ be $n \times n$ matrices. Then for any $i, j \in \{1, \ldots, n\}$, we have

$$A_{ij} = \langle T^*(x_j) \mid x_i \rangle = \langle x_j \mid T(x_i) \rangle = \overline{\langle T(x_i) \mid x_j \rangle} = \overline{B_{ji}},$$

and thus $A = B^*$. $\square$

Theorem 6.19. Let V be a finite-dimensional inner product space over F. Then the following statements hold for any $a \in F$ and $T_1, T_2, T \in \mathcal{L}(V)$.

(a) $(a \cdot T_1 + T_2)^* = \bar{a} \cdot T_1^* + T_2^*$.

(b) $(T_1 T_2)^* = T_2^* T_1^*$.

(c) $(T^*)^* = T$.

(d) $I_V^* = I_V$.

Proof. To be completed. $\square$

Corollary 6.20. Let $F \in \{\mathbb{R}, \mathbb{C}\}$ be a field. Then the following statements are true for any $c \in F$ and $A, B \in F^{n \times n}$.

(a) $(cA + B)^* = \bar{c}A^* + B^*$.

(b) $(AB)^* = B^* A^*$.

(c) $(A^*)^* = A$.

(d) $I_n^* = I_n$.

Proof. Straightforward from Theorem 6.19. $\square$

Theorem 6.21. Let V be a inner product space over F and let $T : V \to V$ be linear. Then $\mathcal{R}(T^*)^\perp = \mathcal{N}(T)$.

Proof. The theorem is proved by

$$
\begin{aligned}
x \in \mathcal{R}(T^*)^\perp &\Leftrightarrow &\langle x \mid T^*(y) \rangle = 0_F \text{ for all } y \in V \\
&\Leftrightarrow &\langle T(x) \mid y \rangle = 0_F \text{ for all } y \in V \\
&\Leftrightarrow &T(x) = 0_V \\
&\Leftrightarrow &x \in \mathcal{N}(T).
\end{aligned}
$$
$\square$

Theorem 6.22. Let V be a finite-dimensional inner product space over F and let $T \in \mathcal{L}(V)$. Then $\overline{\lambda}$ is an eigenvalue of T^* if and only if λ is an eigenvalue of T.

Proof. The theorem is proved by

$$
\begin{aligned}
\mathcal{N}(T^* - \overline{\lambda}I_V) = \{0_V\} &\Leftrightarrow &\mathcal{R}(T^* - \overline{\lambda}I_V) = V \\
&\Leftrightarrow &\mathcal{R}(T^* - \overline{\lambda}I_V)^\perp = \{0_V\} \\
&\Leftrightarrow &\mathcal{N}(T - \lambda I_V) = \{0_V\}.
\end{aligned}
$$
$\square$

6.3 Normal and Self-Adjoint Operators

Definition 6.23. A polynomial f in $\mathcal{P}(F)$ **splits** if there are scalars $c, a_1, \ldots, a_n$ in F such that

$$f(t) = c \prod_{i=1}^{n} (t - a_i).$$

Theorem 6.24 (Schur's Theorem). Let V be a finite-dimensional inner product space over $F \in \{\mathbb{R}, \mathbb{C}\}$ and let $T : V \to V$ be linear. If f_T splits, then there is an orthonormal ordered basis β of V such that $[T]_\beta^\beta$ is upper triangular.

Proof. The proof is by induction on $n = \dim(V)$. The theorem holds trivially for $n = 1$. For $n \geq 2$, since f_T splits, T has an eigenvalue, and thus T^* has an eigenvalue by Theorem 6.22.

Suppose that (λ, x) is an eigenpair of T^* with $\|x\| = 1_F$. Let $W = \{x\}^\perp$. Then $\dim(W) = n - 1$, and we can conclude that W is T-invariant since

$$\langle x \mid T(y) \rangle = \langle T^*(x) \mid y \rangle = \langle \lambda x \mid y \rangle = \lambda \langle x \mid y \rangle = 0_F$$

for any $y \in W$. Define $T' : W \to W$ with $T'(y) = T(y)$ for each $y \in W$. It follows that $f_{T'}(t) \mid f_T(t)$, and thus $f_{T'}(t)$ splits. By induction hypothesis, there is an orthonormal ordered basis

$$\beta' = (x_1, \ldots, x_{n-1})$$

of W such that $A = [T']_{\beta'}^{\beta'}$ is upper triangular. We can conclude that

$$\beta = (x_1, \ldots, x_{n-1}, x)$$

is an orthonormal ordered basis of V, and it follows that $B = [T]_\beta^\beta$ is upper triangular since $B_{ij} = A_{ij}$ for all $i, j \in \{1, \ldots, n-1\}$, which completes the proof. $\square$

Definition 6.25. Let V be an inner product space over $F \in \{\mathbb{R}, \mathbb{C}\}$.

- We say that $T \in \mathcal{L}(V)$ is **normal** if $TT^* = T^*T$.

- We say that $A \in F^{n \times n}$ is **normal** if $AA^* = A^*A$.

Theorem 6.26. Let V be an inner product space over $F \in \{\mathbb{R}, \mathbb{C}\}$, and let $T : V \to V$ be normal. Then the following statements hold.

(a) $\|T(x)\| = \|T^*(x)\|$ for any $x \in V$.

(b) $T - cI_V$ is normal for any $c \in F$.

(c) If (λ, x) is an eigenpair of T, then $(\overline{\lambda}, x)$ is an eigenpair of T^*.

(d) If λ_1 and λ_2 are distinct eigenvalues of T, then for any $x \in E_T(\lambda_1)$ and $y \in E_T(\lambda_2)$ we have $\langle x \mid y \rangle = 0_F$.

Proof. To be completed. $\square$

Theorem 6.27. Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let $T : V \to V$ be linear. Then T is normal if and only if there is an orthonormal eigenbasis of V for T.

Proof. ($\Rightarrow$) It can be shown that $f_T(t)$ splits by fundamental theorem of algebra. Thus, by Schur's theorem (Theorem 6.24), there is an orthonormal ordered basis $\beta = (x_1, \ldots, x_n)$ such that $A = [T]_\beta^\beta$ is upper triangular. By induction on $j \in \{1, \ldots, n\}$, we show that (A_{jj}, x_j) is an eigenpair of T. The induction basis with $j = 1$ holds trivially since A is upper triangular, implying $T(x_1) = A_{11}x_1$. For $j \geq 2$, we have

$$T(x_j) = \sum_{i=1}^{j} A_{ij}x_i,$$

and since

$$A_{ij} = \langle T(x_j) \mid x_i \rangle = \langle x_j \mid T^*(x_i) \rangle = \left\langle x_j \mid \overline{A_{ii}}x_i \right\rangle = A_{ii} \langle x_j \mid x_i \rangle = 0_F$$

holds for any $i \in \{1, \ldots, j-1\}$, it follows that $T(x_j) = A_{jj}x_j$. Thus, $x_1, \ldots, x_n$ are eigenvectors of T, implying β is an orthonormal eigenbasis of V.

($\Leftarrow$) Suppose that β is an orthonormal eigenbasis of T. Then $[T]_\beta^\beta$ is diagonal, implying

$$[T^*]_\beta^\beta = \left([T]_\beta^\beta\right)^*$$

is diagonal. It follows that

$$[TT^*]_\beta^\beta = [T]_\beta^\beta[T^*]_\beta^\beta = [T^*]_\beta^\beta[T]_\beta^\beta = [T^*T]_\beta^\beta,$$

implying $TT^* = T^*T$. $\qquad\square$

Definition 6.28. Let V be a finite-dimensional inner product space over $F \in \{\mathbb{R}, \mathbb{C}\}$.

- We say that $T \in \mathcal{L}(V)$ is **self-adjoint** if $T^* = T$.

- We say that $A \in F^{n \times n}$ is **self-adjoint** if $A^* = A$.

Theorem 6.29. Let V be a finite-dimensional inner product space over $F \in \{\mathbb{R}, \mathbb{C}\}$ and let $T : V \to V$ be self-adjoint. Then the following statements hold.

(a) Every eigenvalue of T is real.

(b) $f_T(t)$ splits.

Proof.

(a) Suppose that (λ, x) is an eigenpair of T. Note that T is normal since T is self-adjoint. By Theorem 6.26, $(\overline{\lambda}, x)$ is an eigenpair of $T^* = T$. Thus, $\overline{\lambda} = \lambda$, implying λ is real.

(b) Define $g_T(t) \in \mathcal{P}(\mathbb{C})$ such that $g_T(t) = f_T(t)$. By the fundamental theorem of algebra, we have

$$g_T(t) = \prod_{i=1}^{n}(t - \lambda_i)$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. By (a), we can conclude that $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Thus, $f_T(t)$ splits even if $F = \mathbb{R}$. $\qquad\square$

Theorem 6.30. Let V be a finite-dimensional inner product space over $\mathbb{R}$ and let $T : V \to V$ be linear. Then T is self-adjoint if and only if there is an orthonormal eigenbasis of V for T.

Proof. ($\Rightarrow$) By Theorem 6.29 (b), $f_T(t)$ splits. Thus, by Schur's theorem (Theorem 6.24), there is an orthonormal ordered basis β such that $A = [T]_\beta^\beta$ is upper triangular. Moreover, $A^t = A$ since T is self-adjoint. Thus, A is diagonal, implying β is an orthonormal eigenbasis of V for T.

($\Leftarrow$) Suppose that β is an orthonormal eigenbasis of V for T. It follows that $A = [T]_\beta^\beta$ is diagonal, implying that A is self-adjoint. Thus, T is self-adjoint. $\square$

6.4 Unitary and Orthogonal Operators

Definition 6.31. Let V be an inner product space over F. Let $T : V \to V$ be linear.

- We say that T is **unitary** if $F = \mathbb{C}$ and $\|T(x)\| = \|x\|$ for any $x \in V$.

- We say that T is **orthogonal** if $F = \mathbb{R}$ and $\|T(x)\| = \|x\|$ for any $x \in V$.

Theorem 6.32. Let V be a finite-dimensional inner product space over F. Then the following statements are equivalent.

(a) $\|T(x)\| = \|x\|$ for any $x \in V$.

(b) $T^*T = I_V$.

(c) $\langle T(x) \mid T(y) \rangle = \langle x \mid y \rangle$ for any $x, y \in V$.

(d) If S is an orthonormal basis of V, so is $T(S)$.

(e) There is a subset S of V such that both S and $T(S)$ are orthonormal bases of V.

Proof. First we prove (b) from (a). Note that T^*T is self-adjoint and normal since $(T^*T)^* = T^*(T^*)^* = T^*T$. Thus, there exists an orthonormal basis $S = \{x_1, \ldots, x_n\}$ of V such that for any $i \in \{1, \ldots, n\}$, $T^*T(x_i) = \lambda_i x_i$ holds for some $\lambda_i \in F$. Since

$$\lambda_i = \lambda_i \langle x_i \mid x_i \rangle = \langle \lambda_i x_i \mid x_i \rangle = \langle T^*T(x_i) \mid x_i \rangle = \langle T(x_i) \mid T(x_i) \rangle = \langle x_i \mid x_i \rangle = 1_F$$

holds for each $i \in \{1, \ldots, n\}$, we have $T^*T(x) = x$ for any $x \in V$ by Lemma 2.21. Thus, $T^*T = I_V$.

Now we prove (c) from (b). The proof is given by

$$\langle T(x) \mid T(y) \rangle = \langle x \mid T^*T(y) \rangle = \langle x \mid y \rangle$$

for any $x, y \in V$.

Now we prove (d) from (c). Let $S = \{x_1, \ldots, x_n\}$ be an orthonormal basis of V. Then for any $i, j \in \{1, \ldots, n\}$ we have

$$\langle T(x_i) \mid T(x_j) \rangle = \langle x_i \mid x_j \rangle = [\![i = j]\!],$$

implying $T(S)$ is an orthonormal basis of V.

The proof of (e) from (d) is trivial. To prove (e) from (a), let $S = \{x_1, \ldots, x_n\}$. Then for any $x \in V$ with $x = a_1 x_1 + \cdots + a_n x_n$ for some $a_1, \ldots, a_n \in F$, we have

$$\langle T(x) \mid T(x) \rangle = \left\langle \sum_{i=1}^{n} a_i T(x_i) \, \middle| \, \sum_{j=1}^{n} a_j T(x_j) \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \langle T(x_i) \mid T(x_j) \rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \langle x_i \mid x_j \rangle$$

$$= \left\langle \sum_{i=1}^{n} a_i x_i \, \middle| \, \sum_{j=1}^{n} a_j x_j \right\rangle$$

$$= \langle x \mid x \rangle,$$

completing the proof. $\qquad \square$

Theorem 6.33. Let V be a finite-dimensional inner product space over F and let $T : V \to V$ be linear.

(a) Let $F = \mathbb{C}$. Then T is unitary if and only if V has an orthonormal eigenbasis for T and $|\lambda| = 1_F$ holds for each eigenvalue λ of T.

(b) Let $F = \mathbb{R}$. Then T is orthogonal and self-adjoint if and only if V has an orthonormal eigenbasis for T and $|\lambda| = 1_F$ holds for each eigenvalue λ of T.

Proof. We prove both statements simultaneously.

($\Rightarrow$) V has an orthonormal eigenbasis by Theorem 6.27 and Theorem 6.30. For each eigenpair (λ, x) of T, $(\overline{\lambda}, x)$ is an eigenpair of T^* by Theorem 6.26 (c), and we have
$$|\lambda|^2 x = \lambda \overline{\lambda} x = \lambda T^*(x) = T^*(\lambda x) = T^*(T(x)) = x$$
by Theorem 6.32, implying $|\lambda| = 1_F$.

($\Leftarrow$) Let $\beta = (x_1, \ldots, x_n)$ be an orthonormal eigenbasis of V for T. For each $i \in \{1, \ldots, n\}$, let λ_i be the corresponding eigenvalue of x_i for T. Since T is normal by Theorem 6.27 and Theorem 6.30, we have

$$T^*(T(x_i)) = T^*(\lambda_i x_i) = \lambda_i T^*(x_i) = \lambda_i \overline{\lambda_i} x_i = |\lambda_i|^2 x_i = x_i$$

for any $i \in \{1, \ldots, n\}$, implying $T^*T = I_V$. The proof is completed due to Theorem 6.32. $\square$

Definition 6.34. Let $Q \in F^{n \times n}$ with $F \in \{\mathbb{C}, \mathbb{R}\}$.

- We say that Q is **unitary** if $Q^*Q = I_n$.

- We say that Q is **orthogonal** if $Q^tQ = I_n$.

Definition 6.35. Let $A, B \in F^{n \times n}$ with $F \in \{\mathbb{C}, \mathbb{R}\}$.

- We say that A and B are **unitarily equivalent** if $B = QAQ^*$ for some unitary $Q \in F^{n \times n}$.

- We say that A and B are **orthogonally equivalent** if $B = QAQ^t$ for some orthogonal $Q \in F^{n \times n}$.

Theorem 6.36. Let $A \in F^{n \times n}$.

(a) If $F = \mathbb{C}$, then A is normal if and only if A is unitarily equivalent to a diagonal matrix in $F^{n \times n}$.

(b) If $F = \mathbb{R}$, then A is self-adjoint if and only if A is orthogonally equivalent to a diagonal matrix in $F^{n \times n}$.

Proof. We prove both statements simultaneously.

($\Rightarrow$) Let α be the standard ordered basis of F^n, and let $\beta = (x_1, \ldots, x_n)$ be an orthonormal eigenbasis of F^n for L_A. Then $B = [L_A]_\beta^\beta$ is diagonal. Define $Q = [I_{F^n}]_\beta^\alpha$, and we have

$$(Q^*Q)_{ij} = \sum_{k=1}^{n} \overline{Q_{ki}} Q_{kj} = \sum_{k=1}^{n} \overline{(x_i)_k}(x_j)_k = x_i^* x_j = [\![i = j]\!]$$

for any $i, j \in \{1, \ldots, n\}$, implying $Q^*Q = I_n$. Thus, A and B are unitarily equivalent since

$$B = [L_A]_\beta^\beta = [I_{F^n}]_\alpha^\beta [L_A]_\alpha^\alpha [I_{F^n}]_\beta^\alpha = Q^{-1}AQ = Q^*AQ.$$

($\Leftarrow$) Let Q be a unitary matrix such that $B = QAQ^*$ is diagonal. We have $A = Q^*BQ$. Thus, A is normal since

$$A^*A = (Q^*B^*Q)(Q^*BQ) = Q^*B^*BQ = Q^*BB^*Q = (Q^*BQ)(Q^*B^*Q) = AA^*.$$

If $F = \mathbb{R}$, then $B^* = B$, and it follows that A is self-adjoint since

$$A^* = Q^*B^*Q = Q^*BQ = A. \qquad \square$$

6.5 The Singular Value Decomposition

Definition 6.37. Let V be an inner product space over F.

- A self-adjoint operator $T : V \to V$ is said to be **positive semidefinite** and **positive definite** if $\langle T(x) \mid x \rangle$ is nonnegative and positive for any $x \in V$, respectively.

- A self-adjoint matrix $A \in F^{n \times n}$ is said to be **positive semidefinite** and **positive definite** if L_A is positive semidefinite and positive definite, respectively.

Theorem 6.38. Let V and W be finite-dimensional inner product spaces over F with $\dim(V) = n$ and $\dim(W) = m$. Let $T : V \to W$ be linear with $\operatorname{rank}(T) = r$. Then there exist positive real numbers $\sigma_1, \ldots, \sigma_r$ and orthonormal bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ of V and W, respectively, such that the following statements hold with $\sigma_k = 0_F$ for $k > r$, $x_k = 0_V$ for $k > n$, and $y_k = 0_W$ for $k > m$.

(a) $T(x_i) = \sigma_i y_i$ for any $i \in \{1, \ldots, n\}$.

(b) $T^*(y_j) = \sigma_j x_j$ for any $j \in \{1, \ldots, m\}$.

(c) (σ_i^2, x_i) is an eigenpair of T^*T for any $i \in \{1, \ldots, n\}$.

(d) (σ_j^2, y_j) is an eigenpair of TT^* for any $j \in \{1, \ldots, m\}$.

Proof. To be completed. $\qquad \square$