Set Theory

Chapter 1

Axioms and Operations

1.1 Basic Axioms

For sets x and y, we write $x \in y$ to say that x is an element of y, and we write $x = y$ to say that x and y are equal. Furthermore, we define

$$\begin{aligned} x \notin y \quad &\Leftrightarrow \quad \neg(x \in y) \\ x \neq y \quad &\Leftrightarrow \quad \neg(x = y). \end{aligned}$$

Axiom I (Extensionality). Two sets are equal if they have exactly the same elements. Formally,

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

Definition 1.1. Let x and y be sets. We say that x is a **subset** of y, denoted by $x \subseteq y$, if every element of x belongs to y. Formally,

$$x \subseteq y \quad \Leftrightarrow \quad \forall z (z \in x \rightarrow z \in y).$$

Furthermore, x is a **proper subset** of y, denoted by $x \subsetneq y$, if both $x \subseteq y$ and $x \neq y$ hold.

Definition 1.2. The **empty set**, denoted by $\varnothing$, is the set that has no elements.

Axiom II (Pairing). For any two sets x and y, there is a set that consists of exactly x and y. Formally,

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y)).$$

Definition 1.3. The **pair set** of two sets x and y, denoted by $\{x, y\}$, is the set that consists of exactly x and y.

Axiom III (Power Set). For any set x, there is a set whose memebers are exactly the subsets of x. Formally,

$$\forall x \exists y \forall z (z \subseteq x \leftrightarrow z \in y).$$

Definition 1.4. The **power set** of a set x, denoted by $\mathcal{P}(x)$, is the set that consists of exactly the subsets of x.

Axiom IV (Separation Scheme). Let $\phi(z)$ be a formula. For any set x, there exists a set y such that for any set z, we have $z \in y$ if and only if both $z \in x$ and $\phi(z)$ hold. Formally,

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \phi(z)).$$

Definition 1.5. Let x, y be sets and let $\phi(z)$ be a formula. If for any set z, we have $z \in y$ if and only if $z \in x$ and $\phi(z)$, then we write

$$y = \{z \in x : \phi(z)\}.$$

Definition 1.6. For sets x and y, we define

$$x \setminus y = \{z \in x : z \notin y\}.$$

Theorem 1.7. There is no set to which every set belongs. Formally,

$$\forall x \exists y (y \notin x).$$

Proof. Let x be a set and let $y = \{z \in x : z \notin z\}$. Then

$$y \in y \quad \Leftrightarrow \quad y \in x \wedge y \notin y.$$

If $y \in x$, then

$$y \in y \quad \Leftrightarrow \quad y \notin y,$$

contradiction. Thus $y \notin x$, which completes the proof. □

Axiom V (Union). For any set x, there exists a set whose elements are exactly the elements of the elements of x. Formally,

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w)).$$

Definition 1.8. Let x be a set.

- We define the **union** of x, denoted by $\bigcup x$, to be the set that consists of the sets that belongs to at least one element of x. Formally, for any set z we have

$$z \in \bigcup x \quad \Leftrightarrow \quad \exists w (w \in x \wedge z \in w).$$

- If x is nonempty, we define the **intersection** of x, denoted by $\bigcap x$, to be the set that consists of the sets that belongs to all elements of x. Formally, for any set z we have

$$z \in \bigcap x \quad \Leftrightarrow \quad \forall w (w \in x \rightarrow z \in w).$$

For sets u and v, we define

$$u \cup v = \bigcup \{u, v\} \qquad \text{and} \qquad u \cap v = \bigcap \{u, v\}.$$

Chapter 2

Relations and Functions

2.1 Ordered Pairs

Definition 2.1. For sets x and y, we define

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

Lemma 2.2. Let x, y, y' be sets. If $\{x, y\} = \{x, y'\}$, then $y = y'$.

Proof. Suppose that $y \neq y'$. Since $y \in \{x, y\} = \{x, y'\}$ and $y \neq y'$, we have $y = x$. Then we have $y' \in \{x, y'\} = \{x, y\} = \{x\}$, implying $y' = x = y$, contradiction. Thus, $y = y'$. □

Theorem 2.3. For sets x, x', y, y', we have

$$\langle x, y\rangle = \langle x', y'\rangle$$

if and only if $x = x'$ and $y = y'$.

Proof. ($\Leftarrow$) Straightforward. ($\Rightarrow$) Suppose that $x \neq x'$. Since

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\},$$

either $\{x\} = \{x', y'\}$ or $\{x\} = \{x'\}$ holds. For both cases we all have $x' \in \{x\}$, implying $x' = x$, contradiction. Hence we have $x = x'$, and it follows that $\{x\} = \{x'\}$, implying $\{x, y\} = \{x', y'\}$, and thus $y = y'$. □

Lemma 2.4. If $x, y \in C$, then $\langle x, y\rangle \in \mathcal{P}(\mathcal{P}(C))$.

Proof. Since $\{x\}$ and $\{y\}$ are subsets of C, we have $\{x\}, \{x, y\} \in \mathcal{P}(C)$. It follows that $\{\{x\}, \{x, y\}\}$ is a subset of $\mathcal{P}(C)$, implying

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(C)).$$ □

Theorem 2.5. For any sets A and B, there is a set whose members are exactly the pairs (x, y) with $x \in A$ and $y \in B$.

Proof. Since $x, y \in A \cup B$, the set of pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$ can be given by

$$\{z \in \mathcal{P}(\mathcal{P}(A \cup B)) : z = \langle x, y\rangle \text{ for some } x \in A \text{ and } y \in B\}.$$ □

Definition 2.6. For any sets A and B, the **Cartesian product** of A and B, denoted by $A \times B$, is the set whose memebers are exactly the pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$.

2.2 Relations

Definition 2.7. A **relation** is a set of ordered pairs. For any relation R, the **domain** and the **range** of R, denoted by $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$, respectively, are defined as follows.

- $\mathrm{dom}(R)$ is the collection of sets x with $\langle x, y\rangle \in R$ for some y.
- $\mathrm{ran}(R)$ is the collection of sets y with $\langle x, y\rangle \in R$ for some x.

Definition 2.8. Let R and S be relations and let X be a set.

- The **inverse** of R, denoted by R^{-1}, is the set of all pairs $\langle y, x\rangle$ with $\langle x, y\rangle \in R$.
- The **restriction** of R to X, denoted by $R \upharpoonright X$, is the set of all pairs $\langle x, y\rangle \in R$ with $x \in X$.
- The **composition** of R and S, denoted by $R \circ S$, is the set of all pairs $\langle x, z\rangle$ with $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S$.

Definition 2.9. A **function** is a relation f such that for any set $x \in \mathrm{dom}(f)$, there exists a unique set y such that $\langle x, y\rangle \in f$. The unique set y with respect to x is called the **value** of f at x and is denoted $f(x)$.

- We say that f is a function from A to B, denoted by $f : A \to B$, if $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.
- f is **one-to-one** if for any $y \in \mathrm{ran}(f)$, there exists a unique set $x \in \mathrm{dom}(f)$ with $f(x) = y$.

Definition 2.10. For any sets A and B, the set of functions from A to B is denoted by B^A.

2.3 Equivalence Relations and Ordering Relations

Definition 2.11. Let A be a set. An **equivalence relation** on A is a relation $R \subseteq A \times A$ that satisfies the following three conditions.

- Reflexivity: $\langle x, x \rangle \in R$ for any $x \in A$.
- Symmetry: $\langle x, y \rangle \in R$ implies $\langle y, x \rangle \in R$ for any $x, y \in A$.
- Transitivity: $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ implies $\langle x, z \rangle \in R$ for any $x, y, z \in A$.

Chapter 3

Natural Numbers

3.1 Inductive Sets

Definition 3.1. The **successor** of a set x, denoted x^+, is defined by

$$x^+ = x \cup \{x\}.$$

Definition 3.2. A set A is **inductive** if $\varnothing \in A$, and for any $a \in A$, we have $a^+ \in A$.

Axiom VI (Infinity). There exists an inductive set.

Definition 3.3. A **natural number** is a set belonging to all inductive sets. The set of natural numbers is denoted by ω.

Theorem 3.4. ω is inductive.

Proof. First, $\varnothing \in \omega$ since $\varnothing$ belongs to all inductive sets by definition. For any set $x \in \omega$, x belongs to all inductive sets, implying that x^+ belongs to all inductive sets, and thus $x^+ \in \omega$. Thus, ω is inductive. $\square$

Definition 3.5. Let

$$0 = \varnothing, \quad 1 = \varnothing^+, \quad 2 = (\varnothing^+)^+, \quad 3 = ((\varnothing^+)^+)^+, \quad 4 = (((\varnothing^+)^+)^+)^+, \quad \ldots$$

denote the natural numbers.

3.2 Recursion

Theorem 3.6 (Recursion Theorem). Let A be a set. Let $a \in A$ and $G : A \to A$. Then there is a unique function $f : \omega \to A$ such that $f(0) = a$ and $f(n^+) = G(f(n))$ for all $n \in \omega$.

Proof. Let H be the set of functions $h \subseteq \omega \times X$ satisfying the following conditions.

1. If $0 \in \mathrm{dom}(h)$, then $h(0) = a$.
2. For any $n \in \omega$, if $n^+ \in \mathrm{dom}(h)$, then $n \in \mathrm{dom}(h)$ and $h(n^+) = G(h(n))$.

Let $f = \bigcup H$. The rest of this proof is to be completed. □

3.3 Arithmetic

Definition 3.7. For $n, m \in \omega$, we define

$$n + 0 = n \qquad \text{and} \qquad n + m^+ = (n + m)^+$$

for all $n, m \in \omega$.

Definition 3.8. For $n, m \in \omega$, we define

$$n \cdot 0 = 0 \qquad \text{and} \qquad n \cdot m^+ = n \cdot m + n$$

for all $n, m \in \omega$.

3.4 Ordering

Definition 3.9. We define binary relations $<$ and $\leq$ over the set ω of natural numbers such that

$$n < m \quad \Leftrightarrow \quad n \in m$$

and

$$n \leq m \quad \Leftrightarrow \quad n \in m \text{ or } n = m$$

for $n, m \in \omega$.