

# Analysis

<b>1</b>	<b>Real Numbers</b>	<b>2</b>
1.1	Fields . . . . .	2
1.2	Ordered Fields . . . . .	5
1.3	The Real Field . . . . .	6
<b>2</b>	<b>Basic Topology</b>	<b>7</b>
2.1	Metric Spaces . . . . .	7
2.2	Compact Sets . . . . .	9
<b>3</b>	<b>Sequences and Series</b>	<b>11</b>
3.1	Convergence . . . . .	11
3.2	Cauchy Sequences . . . . .	13
<b>4</b>	<b>Continuity</b>	<b>14</b>
4.1	Limits of Functions . . . . .	14
4.2	Continuous Functions . . . . .	15
4.3	Properties of Continuous Maps . . . . .	16
<b>5</b>	<b>Differentiation</b>	<b>17</b>
5.1	Derivatives . . . . .	17
5.2	The Mean Value Theorem . . . . .	19
<b>6</b>	<b>Integration</b>	<b>20</b>
6.1	Integrals . . . . .	20
<b>7</b>	<b>Sequences of Functions</b>	<b>21</b>
7.1	Pointwise Convergence . . . . .	21
7.2	Uniform Convergence . . . . .	22

# Chapter 1

## Real Numbers

### 1.1 Fields

**Definition 1.1.** A nonempty set  $F$  and two operations  $+$  and  $\cdot$  form a **field** if the following axioms (A 1) – (A 5), (M 1) – (M 5) and (D) are satisfied.

(A 1)  $x + y \in F$  for any  $x, y \in F$ .

(A 2)  $x + y = y + x$  for any  $x, y \in F$ .

(A 3)  $(x + y) + z = x + (y + z)$  for any  $x, y, z \in F$ .

(A 4) There is an element  $0 \in F$  such that  $x + 0 = x$  for any  $x \in F$ .

(A 5) For each  $x \in F$  there is an element  $-x$  in  $F$  such that  $x + (-x) = 0$ .

(M 1)  $x \cdot y \in F$  for any  $x, y \in F$ .

(M 1)  $x \cdot y = y \cdot x$  for any  $x, y \in F$ .

(M 2)  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$  for any  $x, y, z \in F$ .

(M 3) There is an element  $1 \in F \setminus \{0\}$  such that  $x \cdot 1 = x$  for any  $x \in F$ .

(M 4) For each  $x \in F \setminus \{0\}$  there is an element  $x^{-1}$  in  $F$  such that  $x \cdot x^{-1} = 1$ .

(D)  $x \cdot (y + z) = x \cdot y + x \cdot z$  for any  $x, y, z \in F$ .

**Theorem 1.2.** Let  $F$  be a field. Then the following statements are true for any  $x, y, z \in F$ .

(a) If  $x + y = x + z$ , then  $y = z$ .

(b) If  $x + y = x$ , then  $y = 0$ .

(c) If  $x + y = 0$ , then  $y = -x$ .

(d)  $-(-x) = x$ .

*Proof.* Note that these statements are consequence of axioms (A 1) – (A 5).

(a) We have

$$\begin{aligned}
y &= 0 + y \\
&= (-x + x) + y \\
&= -x + (x + y) \\
&= -x + (x + z) \\
&= (-x + x) + z \\
&= 0 + z \\
&= z.
\end{aligned}$$

(b) Since  $x + y = x = x + 0$ , we have  $y = 0$  by (a).

(c) Since  $x + y = 0 = x + (-x)$ , we have  $y = -x$  by (a).

(d) Since  $-x + x = 0$ , we have  $-(-x) = x$  by (c). □

**Theorem 1.3.** Let  $F$  be a field. Then the following statements are true for any  $x \in F \setminus \{0\}$  and  $y, z \in F$ .

(a) If  $x \cdot y = x \cdot z$ , then  $y = z$ .

(b) If  $x \cdot y = x$ , then  $y = 1$ .

(c) If  $x \cdot y = 1$ , then  $y = x^{-1}$ .

(d)  $(x^{-1})^{-1} = x$ .

*Proof.* Note that these statements are consequence of axioms (M 1) – (M 5).

(a) We have

$$\begin{aligned}
y &= 1 \cdot y \\
&= (x^{-1} \cdot x) \cdot y \\
&= x^{-1} \cdot (x \cdot y) \\
&= x^{-1} \cdot (x \cdot z) \\
&= (x^{-1} \cdot x) \cdot z \\
&= 1 \cdot z \\
&= z.
\end{aligned}$$

(b) Since  $x \cdot y = x = x \cdot 1$ , we have  $y = 1$  by (a).

(c) Since  $x \cdot y = 1 = x \cdot x^{-1}$ , we have  $y = x^{-1}$  by (a).

(d) Since  $x^{-1} \cdot x = 1$ , we have  $(x^{-1})^{-1} = x$  by (c). □

**Theorem 1.4.** Let  $F$  be a field. Then the following statements are true for any  $x, y \in F$ .

(a)  $0 \cdot x = 0$ .

(b)  $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$ .

(c)  $(-x) \cdot (-y) = x \cdot y$ .

*Proof.*

(a) We have

$$0 \cdot x + 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x,$$

implying  $0 \cdot x = 0$ .

(b) Since

$$(-x) \cdot y + x \cdot y = (-x + x) \cdot y = 0 \cdot y = 0,$$

we have  $(-x) \cdot y = -(x \cdot y)$ . One can prove  $x \cdot (-y) = -(x \cdot y)$  similarly.

(c) We have

$$(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(x \cdot y)) = x \cdot y$$

by applying (b) twice. □

## 1.2 Ordered Fields

**Definition 1.5.** An **ordered field** is a field on which relation  $<$  is defined such that the following axioms (O 1) – (O 4) hold for any  $x, y, z \in F$ .

(O 1) One and only one of the statements  $x = y$ ,  $x < y$ ,  $y < x$  is true.

(O 2) If  $x < y$  and  $y < z$ , then  $x < z$ .

(O 3) If  $x < y$ , then  $x + z < y + z$ .

(O 4) If  $0 < x$  and  $0 < y$ , then  $0 < x \cdot y$ .

**Definition 1.6.** Let  $F$  be an ordered field. The relations  $>$ ,  $\leq$  and  $\geq$  are defined as follows for any  $x, y \in F$ .

$$\begin{aligned}x > y &\Leftrightarrow y < x \\x \leq y &\Leftrightarrow x < y \text{ or } x = y \\x \geq y &\Leftrightarrow x > y \text{ or } x = y.\end{aligned}$$

**Definition 1.7.** Let  $F$  be an ordered field and let  $S \subseteq F$ .

- An **upper bound** of  $S$  is an element  $x$  in  $F$  such that  $x \geq y$  for any  $y \in S$ . We say that  $S$  is **bounded above** if  $S$  has an upper bound.
- A **lower bound** of  $S$  is an element  $x$  in  $F$  such that  $x \leq y$  for any  $y \in S$ . We say that  $S$  is **bounded below** if  $S$  has a lower bound.

**Definition 1.8.** Let  $F$  be an ordered field and let  $S \subseteq F$ .

- An element of  $S$  is called the **maximum** of  $S$ , denoted by  $\max(S)$ , if it is an upper bound of  $S$ .
- An element of  $S$  is called the **minimum** of  $S$ , denoted by  $\min(S)$ , if it is a lower bound of  $S$ .
- The minimum of the set of upper bounds of  $S$  is called the **supremum** of  $S$ , denoted by  $\sup(S)$ .
- The maximum of the set of lower bounds of  $S$  is called the **infimum** of  $S$ , denoted by  $\inf(S)$ .

## 1.3 The Real Field

**Definition 1.9.**  $\mathbb{R}$  is an ordered field such that every nonempty subset  $S$  of  $\mathbb{R}$  that is bounded above has a supremum. The elements of  $\mathbb{R}$  are called the **real numbers**.

**Theorem 1.10 (Archimedean Property).** For any  $x, y \in \mathbb{R}$  with  $x > 0$ , there is a positive integer  $n$  such that

$$n \cdot x > y.$$

*Proof.* Let

$$S = \{nx : n \text{ is a positive integer}\}.$$

Suppose that  $y$  is an upper bound of  $S$ . It follows that  $S$  has a supremum  $z$ . Note that  $z - x$  is not an upper bound of  $S$  since  $z - x < z$ . Thus,  $z - x < mx$  for some positive integer  $m$ , implying  $z < (m + 1)x$ , contradiction to the fact that  $z$  is an upper bound of  $S$ . Hence,  $y$  is not an upper bound of  $S$ , completing the proof.  $\square$

# Chapter 2

## Basic Topology

### 2.1 Metric Spaces

**Definition 2.1.** A set  $X$  and a function  $d : X \times X \rightarrow \mathbb{R}$  form a **metric space** if the following properties hold for any  $x, y, z \in X$ .

1.  $d(x, y) \geq 0$ , and  $d(x, y) = 0$  holds if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$ .
3.  $d(x, y) \leq d(x, z) + d(z, y)$ .

**Remark.** We may use the underlying set  $X$  to represent the metric space  $(X, d)$ , and in this case, the distance function  $d$  is denoted by  $d_X$ .

**Definition 2.2.** Let  $X$  be a metric space. For any  $\epsilon > 0$  and  $x \in X$ , we define the **open ball** of radius  $\epsilon$  centered at  $x$  by

$$B_\epsilon(x) = \{y \in X : d_X(x, y) < \epsilon\}.$$

**Definition 2.3.** Let  $X$  be a metric space with  $S \subseteq X$  and  $x \in X$ .

- We say that  $x$  is an **interior point** of  $S$  if  $B_\epsilon(x) \subseteq S$  for some  $\epsilon > 0$ . If every point of  $S$  is an interior point of  $S$ , then  $S$  is said to be **open**.
- We say that  $x$  is a **limit point** of  $S$  if  $(B_\epsilon(x) \setminus \{x\}) \cap S$  is not empty for all  $\epsilon > 0$ . If every limit point of  $S$  is a point of  $S$ , then  $S$  is said to be **close**.

**Theorem 2.4.** Let  $X$  be a metric space and  $S \subseteq X$ . Then  $S$  is open if and only if  $X \setminus S$  is closed.

*Proof.* ( $\Rightarrow$ ) Suppose that  $x$  is a limit point of  $X \setminus S$ . Then  $B_\epsilon(x) \setminus S \neq \emptyset$  for any  $\epsilon > 0$ , implying that  $x$  is not an interior point of  $S$ . Since  $S$  is open, we have  $x \notin S$ , i.e.,  $x \in X \setminus S$ . Thus,  $X \setminus S$  is closed.

( $\Leftarrow$ ) Let  $x \in S$ . If  $x$  is a limit point of  $X \setminus S$ , then  $x \in X \setminus S$  since  $X \setminus S$  is closed, contradiction. Thus,  $x$  is not a limit point of  $X \setminus S$ , and there exists  $\epsilon > 0$  such that  $B_\epsilon(x) \subseteq S$ , implying that  $S$  is open.  $\square$

**Theorem 2.5.** Let  $\{S_\alpha\}_{\alpha \in A}$  be a collection of open sets.

- (a)  $\bigcup_{\alpha \in A} S_\alpha$  is open.

(b) If  $A$  is nonempty and finite, then  $\bigcap_{\alpha \in A} S_\alpha$  is open.

*Proof.*

- (a) Suppose that  $x \in \bigcup_{\alpha \in A} S_\alpha$ . Then  $x \in S_\alpha$  for some  $\alpha \in A$ . Since  $S_\alpha$  is open,  $x$  is an interior point of  $S_\alpha$ , and it follows that  $x$  is an interior point of  $\bigcup_{\alpha \in A} S_\alpha$ . Thus,  $\bigcup_{\alpha \in A} S_\alpha$  is open.
- (b) Suppose that  $x \in \bigcap_{\alpha \in A} S_\alpha$ . For each  $\alpha \in A$ , since  $S_\alpha$  is open, we have  $B_{\epsilon_\alpha}(x) \subseteq S_\alpha$  for some  $\epsilon_\alpha > 0$ . Since  $A$  is finite and nonempty,  $\epsilon = \min(\{\epsilon_\alpha\}_{\alpha \in A})$  exists. It follows that  $B_\epsilon(x) \subseteq \bigcap_{\alpha \in A} S_\alpha$ , implying that  $x$  is an interior point of  $\bigcap_{\alpha \in A} S_\alpha$ . Thus,  $\bigcap_{\alpha \in A} S_\alpha$  is open.  $\square$

**Corollary 2.6.** Let  $\{S_\alpha\}_{\alpha \in A}$  be a collection of closed sets.

- (a)  $\bigcap_{\alpha \in A} S_\alpha$  is closed.
- (b) If  $A$  is nonempty and finite, then  $\bigcup_{\alpha \in A} S_\alpha$  is closed.

*Proof.* Straightforward from Theorem 2.4 and Theorem 2.5.  $\square$



## 2.2 Compact Sets

**Definition 2.7.** Let  $(X, d)$  be a metric space and let  $S \subseteq X$ .

- A **cover** of  $S$  is a collection of subsets of  $X$  whose union contains  $S$ . An **open cover** of  $S$  is a cover of  $S$  whose elements are all open.
- We say that  $S$  is **compact** if every open cover  $\Omega$  of  $S$  contains a finite cover  $\Phi$  of  $S$ .

**Theorem 2.8.** Let  $(X, d)$  be a metric space and let  $R \subseteq S \subseteq X$ . If  $S$  is compact and  $R$  is closed, then  $R$  is compact.

*Proof.* Suppose that  $R$  has an open cover  $\Omega$ . Then  $\Omega' = \Omega \cup \{X \setminus R\}$  is an open cover of  $S$  since  $X \setminus R$  is open. Let  $\Phi' \subseteq \Omega'$  be a finite cover of  $S$ , and let  $\Phi = \Phi' \setminus \{X \setminus R\}$ . Then  $\Phi$  is a finite open cover of  $R$  with  $\Phi \subseteq \Omega$ . Thus,  $R$  is compact.  $\square$

**Theorem 2.9 (Nested Interval Theorem).** Let  $\langle I_n \rangle$  be a sequence of rectangles in  $\mathbb{R}^k$  such that  $I_{n+1} \subseteq I_n$ , then the intersection of  $\{I_n : n \in \mathbb{N}\}$  is nonempty.

*Proof.* For each positive integer  $n$ , let

$$I_n = [a_n^{(1)}, b_n^{(1)}] \times \cdots \times [a_n^{(k)}, b_n^{(k)}].$$

For each  $i \in \{1, \dots, k\}$ , we have

$$a_n^{(i)} \leq a_{n+m}^{(i)} \leq b_{n+m}^{(i)} \leq b_m^{(i)}$$

for any  $n, m \in \mathbb{N}$ . Thus,  $\{a_n^{(i)} : n \in \mathbb{N}\}$  is bounded above, implying the existence of

$$x_i = \sup(\{a_n^{(i)} : n \in \mathbb{N}\}).$$

We have

$$a_n^{(i)} \leq x_i \leq b_m^{(i)}$$

for any  $n, m \in \mathbb{N}$ . Thus,

$$x = (x_1, \dots, x_k) \in \bigcap_{n \geq 1} I_n,$$

completing the proof.  $\square$

**Theorem 2.10.** Every  $k$ -cell in  $\mathbb{R}^k$  is compact.

*Proof.* Let  $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ . We have

$$\|x - x'\| \leq \sqrt{\sum_{i=1}^k (a_i - b_i)^2}$$

for any  $x, x' \in I$ . Assume that there is an open cover  $\mathcal{O}$  of  $I$  that contains no finite subcover of  $I$ . Let  $c_i = (a_i + b_i)/2$  for all  $i \in \{1, \dots, k\}$ , and let

$$\mathcal{C} = \{I^{(1)} \times \cdots \times I^{(k)} : I^{(i)} \in \{[a_i, c_i], [c_i, b_i]\} \text{ for } 1 \leq i \leq k\}$$

be a collection of  $2^k$   $k$ -cells whose union is  $I$ . Then there must be a  $k$ -cell  $I' \in \mathcal{C}$  cannot be covered by any finite subset of  $\mathcal{O}$ , or  $I$  could be covered by that set, contradiction.

Thus, if  $I$  is not compact, then we can construct a sequence  $\langle I_n \rangle$  of  $k$ -cells which are not covered by any finite subset of  $\mathcal{O}$  such that  $I_1 = I$ ,  $I_{n+1} \subseteq I_n$  for any integer  $n \geq 1$ , and

$$\|x - x'\| \leq \frac{\sqrt{\sum_{i=1}^k (a_i - b_i)^2}}{2^{n-1}}$$

holds for any  $x, x' \in I_n$ . It follows that there is a point  $y \in \bigcap \{I_n\}$ , and we have  $y \in S$  for some  $S \in \mathcal{O}$ . Since  $S$  is open, we have  $B_r(y) \subseteq S$  for some  $r > 0$ . Let  $N$  be a positive integer such that

$$2^N > \frac{\sqrt{\sum_{i=1}^k (a_i - b_i)^2}}{r/2}.$$

Then for any  $x \in I_N$ ,

$$\|x - y\| \leq \frac{\sqrt{\sum_{i=1}^k (a_i - b_i)^2}}{2^{N-1}} < r,$$

implying  $x \in B_r(y) \subseteq S$ . It follows that  $I_N \subseteq S$ , and  $\{S\}$  is a finite subset of  $\mathcal{O}$ , contradiction. Thus,  $I$  is compact.  $\square$

**Theorem 2.11 (Heine–Borel Theorem).** Let  $S \subseteq \mathbb{R}^k$ . Then  $S$  is compact if and only if  $S$  is closed and bounded.

*Proof.* ( $\Leftarrow$ ) If  $S$  is closed and bounded, then there is a  $k$ -cell  $I$  with  $S \subseteq I$ . Since  $I$  is compact, and  $S$  is closed, we conclude that  $S$  is compact.

( $\Rightarrow$ ) Suppose that  $S$  is compact. Then  $S$  is closed. Since  $\mathcal{O} = \{B_r(0_{\mathbb{R}^k}) : r \in \mathbb{N}\}$  is an open cover of  $S$ , there is  $\mathcal{O}' \subseteq \mathcal{O}$  such that  $S \subseteq \bigcup \mathcal{O}'$ . It can be shown that  $\bigcup \mathcal{O}'$  is bounded, and thus  $S$  is bounded.  $\square$

# Chapter 3

## Sequences and Series

### 3.1 Convergence

Let  $\mathbb{N} = \{1, 2, 3, \dots\}$  denote the set of positive integers.

**Definition 3.1.** Let  $\langle x_n \rangle_{n \in \mathbb{N}}$  be a sequence in a metric space  $X$ . We say that  $\langle x_n \rangle_{n \in \mathbb{N}}$  **converges** to a point  $x \in X$ , denoted by

$$\lim_{n \rightarrow \infty} x_n = x,$$

if for any  $\epsilon > 0$  there exists some  $n_0 \in \mathbb{N}$  such that

$$n \geq n_0 \quad \Rightarrow \quad d_X(x_n, x) < \epsilon$$

holds for all  $n \in \mathbb{N}$ .

- We say that  $\langle x_n \rangle_{n \in \mathbb{N}}$  is **convergent** if it converges to a point in  $X$ .
- We say that  $\langle x_n \rangle_{n \in \mathbb{N}}$  is **divergent** if it is not convergent.

**Theorem 3.2.** Let  $\langle x_n \rangle_{n \in \mathbb{N}}$  be a sequence in a metric space  $X$ . If  $\langle x_n \rangle_{n \in \mathbb{N}}$  converges to both  $x \in X$  and  $x' \in X$ , then  $x = x'$ .

*Proof.* For any  $\epsilon > 0$ , there exists some  $n_0 \in \mathbb{N}$  such that

$$d_X(x_n, x) < \frac{\epsilon}{2} \quad \text{and} \quad d_X(x_n, x') < \frac{\epsilon}{2}$$

hold for all  $n \in \mathbb{N}$  with  $n \geq n_0$ . Then we have

$$d_X(x, x') \leq d_X(x_n, x) + d_X(x_n, x') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all  $n \in \mathbb{N}$  with  $n \geq n_0$ . Thus  $d_X(x, x') = 0$ , implying  $x = x'$ . □

**Theorem 3.3.** Let  $\langle a_n \rangle_{n \in \mathbb{N}}$  and  $\langle b_n \rangle_{n \in \mathbb{N}}$  be sequence in  $\mathbb{R}$  with

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = M.$$

Then the following statements are true.

(a) We have

$$\lim_{n \rightarrow \infty} (a_n + b_n) = L + M \quad \text{and} \quad \lim_{n \rightarrow \infty} (a_n - b_n) = L - M.$$

(b) We have

$$\lim_{n \rightarrow \infty} a_n b_n = LM.$$

(c) If  $L \neq 0$  and  $a_n \neq 0$  for all  $n \in \mathbb{N}$ , then we have

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{L}.$$

*Proof.*

(a) For any  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$|a_n - L| < \frac{\epsilon}{2} \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2}$$

holds for all  $n \in \mathbb{N}$  with  $n \geq n_0$ . Then

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and

$$|(a_n - b_n) - (L - M)| \leq |a_n - L| + |-b_n + M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

hold for all  $n \in \mathbb{N}$  with  $n \geq n_0$ , completing the proof.

(b) Choose  $C > 0$  such that  $|L| \leq C$  and  $|b_n| \leq C$  for all  $n \in \mathbb{N}$ . For any  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$|a_n - L| < \frac{\epsilon}{2C} \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2C}$$

hold for all  $n \in \mathbb{N}$  with  $n \geq n_0$ . Then

$$\begin{aligned} |a_n b_n - LM| &= |(a_n - L)b_n + (b_n - M)L| \\ &\leq |a_n - L| \cdot |b_n| + |b_n - M| \cdot |L| \\ &\leq |a_n - L| \cdot C + |b_n - M| \cdot C \\ &< \epsilon \end{aligned}$$

holds for all  $n \in \mathbb{N}$  with  $n \geq n_0$ , completing the proof.

(c) For any  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$|a_n - L| < \min \left( \left\{ \frac{|L|^2 \epsilon}{2}, \frac{|L|}{2} \right\} \right)$$

hold for all  $n \in \mathbb{N}$  with  $n \geq n_0$ . Then for all  $n \in \mathbb{N}$  with  $n \geq n_0$  we have

$$|a_n| = |L + (a_n - L)| \geq |L| - |a_n - L| > \frac{|L|}{2},$$

implying

$$\left| \frac{1}{a_n} - \frac{1}{L} \right| = \left| \frac{a_n - L}{a_n L} \right| = \frac{|a_n - L|}{|a_n| \cdot |L|} < \frac{|L|^2 \epsilon / 2}{|L|/2 \cdot |L|} = \epsilon,$$

completing the proof. □

## 3.2 Cauchy Sequences

**Definition 3.4.** Let  $\langle x_n \rangle_{n \in \mathbb{N}}$  be a sequence in a metric space  $X$ . We say that  $\langle x_n \rangle_{n \in \mathbb{N}}$  is a **Cauchy sequence**, if for any  $\epsilon > 0$  there exists some  $n_0 \in \mathbb{N}$  such that

$$n \geq n_0 \quad \text{and} \quad m \geq n_0 \quad \Rightarrow \quad d_X(x_n, x_m) < \epsilon$$

holds for all  $n, m \in \mathbb{N}$ .

**Definition 3.5.** Let  $\langle a_n \rangle_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ .

- We say that  $\langle a_n \rangle_{n \in \mathbb{N}}$  is **increasing** (resp., **strictly increasing**) if

$$a_n \leq a_{n+1} \quad (\text{resp., } a_n < a_{n+1})$$

holds for all  $n \in \mathbb{N}$ .

- We say that  $\langle a_n \rangle_{n \in \mathbb{N}}$  is **decreasing** (resp., **strictly decreasing**) if

$$a_n \geq a_{n+1} \quad (\text{resp., } a_n > a_{n+1})$$

holds for all  $n \in \mathbb{N}$ .

**Theorem 3.6.** Let  $\langle a_n \rangle_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . If  $\langle a_n \rangle_{n \in \mathbb{N}}$  is increasing and its range  $A = \{a_n : n \in \mathbb{N}\}$  is bounded above, then  $\langle a_n \rangle_{n \in \mathbb{N}}$  converges.

*Proof.* Let  $L = \sup(A)$ . For any  $\epsilon > 0$ , since  $L - \epsilon$  is not an upper bound of  $A$ , there exists  $n_0 \in \mathbb{N}$  with  $a_{n_0} > L - \epsilon$ . Since  $\langle a_n \rangle_{n \in \mathbb{N}}$  is increasing, for all  $n \in \mathbb{N}$  with  $n \geq n_0$  we have

$$L - \epsilon < a_{n_0} \leq a_n \leq L,$$

and thus  $|a_n - L| < \epsilon$ . It follows that  $\langle a_n \rangle_{n \in \mathbb{N}}$  converges to  $L$ . □

# Chapter 4

## Continuity

### 4.1 Limits of Functions

**Definition 4.1.** Let  $X$  and  $Y$  be metric spaces and let  $f : D \rightarrow Y$  be a map with  $D \subseteq X$ . Let  $a \in X$  be a limit point and  $b \in Y$ . Then we say that  $b$  is the **limit** of  $f$  at  $a$ , denoted

$$\lim_{x \rightarrow a} f(x) = b,$$

if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in D$ ,

$$0 < d_X(x, a) < \delta \quad \Rightarrow \quad d_Y(f(x), b) < \epsilon.$$

## 4.2 Continuous Functions

**Definition 4.2.** Let  $X$  and  $Y$  be metric spaces and let  $f : D \rightarrow Y$  be a map with  $D \subseteq X$ . We say that  $f$  is **continuous** at  $a \in X$  if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$d_Y(f(x), f(a)) < \epsilon$$

holds for any  $x \in D$  with

$$d_X(x, a) < \delta.$$

Also, we say that  $f$  is **continuous** on  $D$  if  $f$  is continuous at every point of  $D$ .

**Theorem 4.3.** Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$  be a map. Then  $f$  is continuous if and only if  $f^{-1}(E)$  is open for any open set  $E$  in  $Y$ .

*Proof.* To be completed. □

### 4.3 Properties of Continuous Maps

**Theorem 4.4.** Let  $X$  and  $Y$  be metric spaces, and let  $f : X \rightarrow Y$  be a continuous map. If  $K \subseteq X$  is compact, then  $f(K)$  is compact.

*Proof.* For any open cover  $\{V_\alpha\}_{\alpha \in A}$  of  $f(K)$ , we have

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha).$$

Since  $f$  is continuous,  $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$  is an open cover of  $K$ . Due to compactness of  $K$ , there exist  $\alpha_1, \dots, \alpha_m \in A$  such that

$$K \subseteq \bigcup_{i=1}^m f^{-1}(V_{\alpha_i}),$$

and we have

$$f(K) \subseteq f\left(\bigcup_{i=1}^m f^{-1}(V_{\alpha_i})\right) = f\left(f^{-1}\left(\bigcup_{i=1}^m V_{\alpha_i}\right)\right) = \bigcup_{i=1}^m V_{\alpha_i}.$$

Thus,  $f(K)$  is compact. □

**Theorem 4.5.** Let  $X$  be a metric space and let  $f : X \rightarrow \mathbb{R}$  be a continuous map. If  $K \subseteq X$  is compact, then  $\max(f(K))$  and  $\min(f(K))$  exist.

*Proof.* Since  $f$  is continuous and  $K$  is compact,  $f(K)$  is a compact subset of  $\mathbb{R}$ . Thus,  $f(K)$  has maximum and minimum. □

**Theorem 4.6 (Intermediate Value Theorem).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and let  $c \in \mathbb{R}$ . If  $f(a) < c < f(b)$ , then  $f(x) = c$  for some  $x \in (a, b)$ .

*Proof.* To be completed. □



# Chapter 5

## Differentiation

### 5.1 Derivatives

**Definition 5.1.** Let  $f : D \rightarrow \mathbb{R}$  be a map with  $D \subseteq \mathbb{R}$ . For each  $a \in \mathbb{R}$  such that  $(a - \delta, a + \delta) \subseteq D$  for some  $\delta > 0$ , we define the **derivative** of  $f$  at  $a$  by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

We say that  $f$  is **differentiable** at  $a$  if  $f'(a)$  exists.

**Theorem 5.2.** Let  $f : D \rightarrow \mathbb{R}$  be a map with  $D \subseteq \mathbb{R}$ . For each  $a \in \mathbb{R}$  such that  $(a - \delta, a + \delta) \subseteq D$  for some  $\delta > 0$ , if  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

*Proof.* We have

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{h \rightarrow 0} f(a+h) \\ &= \lim_{h \rightarrow 0} \left( f(a) + \frac{f(a+h) - f(a)}{h} \cdot h \right) \\ &= f(a) + f'(a) \cdot 0 \\ &= f(a). \end{aligned} \quad \square$$

**Theorem 5.3.** Let  $f$  and  $g$  be real functions. If both  $f$  and  $g$  are differentiable at  $a \in \mathbb{R}$ , then the following statements are true.

- (a)  $f + g$  is differentiable at  $a$ , and  $(f + g)'(a) = f'(a) + g'(a)$ .
- (b)  $f \cdot g$  is differentiable at  $a$ , and  $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$ .
- (c) If  $g(a) \neq 0$ , then  $1/g$  is differentiable at  $a$ , and  $(1/g)'(a) = -g'(a)/(g(a))^2$ .

*Proof.*

- (a) We have

$$\begin{aligned} (f + g)'(a) &= \lim_{h \rightarrow 0} \frac{(f + g)(a+h) - (f + g)(a)}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h} \right) \\ &= f'(a) + g'(a). \end{aligned}$$

(b) We have

$$\begin{aligned}
(f \cdot g)'(a) &= \lim_{h \rightarrow 0} \frac{(f \cdot g)(a+h) - (f \cdot g)(a)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h \rightarrow 0} \left( \frac{f(a+h) - f(a)}{h} \cdot g(a+h) + f(a) \cdot \frac{g(a+h) - g(a)}{h} \right) \\
&= f'(a) \cdot g(a) + f(a) \cdot g'(a).
\end{aligned}$$

(c) We have

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{h \rightarrow 0} \frac{(1/g)(a+h) - (1/g)(a)}{h} \\
&= \lim_{h \rightarrow 0} \frac{g(a) - g(a+h)}{hg(a)g(a+h)} \\
&= \frac{-g'(a)}{(g(a))^2}.
\end{aligned}$$

□

**Theorem 5.4 (Chain Rule).** Let  $f$  and  $g$  be real functions. If  $f$  is differentiable at  $a \in \mathbb{R}$  and  $g$  is differentiable at  $f(a)$ , then  $g \circ f$  is differentiable at  $a$ , and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

*Proof.* Let

$$u(x, h) = \begin{cases} 0, & \text{if } h = 0 \\ \frac{f(x+h) - f(x)}{h} - f'(x), & \text{otherwise} \end{cases}$$

and

$$v(y, k) = \begin{cases} 0, & \text{if } k = 0 \\ \frac{g(y+k) - g(y)}{k} - g'(y), & \text{otherwise.} \end{cases}$$

Note that we have

$$\begin{aligned}
f(x+h) &= f(x) + h \cdot (f'(x) + u(x, h)) \\
g(y+k) &= g(y) + k \cdot (g'(y) + v(y, k)).
\end{aligned}$$

Then for every  $h \neq 0$  we have

$$\begin{aligned}
\frac{g(f(a+h)) - g(f(a))}{h} &= \frac{(f(a+h) - f(a)) \cdot (g'(f(a)) + v(f(a), f(a+h) - f(a)))}{h} \\
&= (f'(a) + u(a, h)) \cdot (g'(f(a)) + v(f(a), (f'(a) + u(a, h))h)).
\end{aligned}$$

Since

$$\lim_{h \rightarrow 0} u(x, h) = 0 \quad \text{and} \quad \lim_{k \rightarrow 0} v(y, k) = 0,$$

it follows that

$$(g \circ f)'(a) = \lim_{h \rightarrow 0} \frac{g(f(a+h)) - g(f(a))}{h} = f'(a) \cdot g'(f(a)).$$

□

## 5.2 The Mean Value Theorem

**Theorem 5.5.** Let  $f$  be a real function. If  $f$  has a local maximum at  $a$  and  $f$  is differentiable at  $a$ , then  $f'(a) = 0$ .

*Proof.* Assume for contradiction that  $f'(a) \neq 0$ . Choose  $\delta > 0$  such that  $f(x) \leq f(a)$  and

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \frac{|f'(a)|}{2}$$

hold for all  $x \in (a - \delta, a + \delta)$ . If  $f'(a) > 0$ , then

$$f(x) - f(a) > \frac{f'(a)(x - a)}{2} > 0$$

for all  $x \in (a, a + \delta)$ , contradiction. If  $f'(a) < 0$ , then

$$f(x) - f(a) > \frac{f'(a)(x - a)}{2} > 0$$

for all  $x \in (a - \delta, a)$ , contradiction. Thus,  $f'(a) = 0$ .  $\square$

**Theorem 5.6 (Rolle's Theorem).** Let  $f$  be a real function that is continuous on  $[a, b]$  and is differentiable on  $(a, b)$ . If  $f(a) = f(b)$ , then there exists  $c \in (a, b)$  such that  $f'(c) = 0$ .

*Proof.* Let  $c^+, c^- \in [a, b]$  such that  $f(c^-) \leq f(x) \leq f(c^+)$  for all  $x \in [a, b]$ . If  $\{c^+, c^-\} = \{a, b\}$ , then  $f$  is a constant function, and thus  $f'(c) = 0$  for all  $c \in (a, b)$ . Otherwise, let  $c \in \{c^+, c^-\} \setminus \{a, b\}$ , and we have  $f'(c) = 0$ .  $\square$

**Theorem 5.7 (Mean Value Theorem).** Let  $f$  be a real function that is continuous on  $[a, b]$  and is differentiable on  $(a, b)$ . Then there exists  $c \in (a, b)$  such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}.$$

*Proof.* Define  $g : [a, b] \rightarrow \mathbb{R}$  with

$$g(x) = f(x) - f(a) - \frac{f(a) - f(b)}{a - b} \cdot (x - a).$$

Note that  $g$  is continuous on  $[a, b]$  and is differentiable on  $(a, b)$ . Also we have  $g(a) = g(b) = 0$ . Thus, there exists  $c \in (a, b)$  with  $g'(c) = 0$ . Since

$$g'(x) = f'(x) - \frac{f(a) - f(b)}{a - b}$$

holds for all  $x \in (a, b)$ , we have

$$f'(c) = \frac{f(a) - f(b)}{a - b}. \quad \square$$

# Chapter 6

## Integration

### 6.1 Integrals

**Definition 6.1.** Let  $[a, b]$  be a given interval. A **partition** of  $[a, b]$  is a finite set

$$P = \{x_0, x_1, \dots, x_n\} \quad \text{with} \quad a \leq x_0 < x_1 < \dots < x_{n-1} < x_n \leq b.$$

For every partition  $P$  of  $[a, b]$ , the **upper sum** of  $f$  with respect to  $P$  is defined by

$$U(f, P) = \sum_{i=1}^n \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \cdot (x_i - x_{i-1}),$$

and the **lower sum** of  $f$  with respect to  $P$  is defined by

$$L(f, P) = \sum_{i=1}^n \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \cdot (x_i - x_{i-1}).$$

Finally, we define the **upper integral** and the **lower integral** of  $f$  on  $[a, b]$  by

$$\int_a^{\overline{b}} f(x) dx = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and

$$\int_a^{\underline{b}} f(x) dx = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\},$$

respectively. If

$$\int_a^{\overline{b}} f(x) dx = \int_a^{\underline{b}} f(x) dx,$$

then we say that  $f$  is **integrable** on  $[a, b]$ , and this common value is denoted by

$$\int_a^b f(x) dx.$$

# Chapter 7

## Sequences of Functions

### 7.1 Pointwise Convergence

**Definition 7.1.** Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of real-valued functions defined on  $D \subseteq \mathbb{R}$ . We say that  $(f_n)_{n \in \mathbb{N}}$  **converges pointwise** to a function  $f$  defined on  $D$  if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

for all  $x \in D$ .

## 7.2 Uniform Convergence

**Definition 7.2.** Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of real-valued functions defined on  $D \subseteq \mathbb{R}$ . We say that  $(f_n)_{n \in \mathbb{N}}$  **converges uniformly** to a function  $f$  defined on  $D$ , if for each  $\epsilon \in \mathbb{R}^+$  there exists  $n_0 \in \mathbb{N}$  such that

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \epsilon$$

for all  $x \in D$  and  $n \in \mathbb{N}$ .