Logic

Chapter 1

Propositional Logic

1.1 The Language of Propositional Logic

Definition 1.1. Let V be a countably infinite set, whose elements are called **propositional variables**. We define the set of **formulas** as the minimal set of strings on the alphabet $V \cup \{\neg, \to, (,)\}$ satisfying the following properties.

(a) Each propositional variable in V is a formula on V.

(b) If α is a formula, then so is $\neg\alpha$.

(c) If α and β are formulas, then so is $(\alpha \to \beta)$.

1.2 Truth Assignment

Definition 1.2. A **truth assignment** is a function $\tau : V \to \{0, 1\}$, and it can be extended to have its domain the set of formulas as follows.

(a) $\tau(\neg\alpha) = 1 - \tau(\alpha)$ for any formula α.

(b) $\tau((\alpha \to \beta)) = 1 - \tau(\alpha)(1 - \tau(\beta))$ for any formulas α and β.

Definition 1.3. We say that a truth assignment τ **satisfies** a formula α if $\tau(\alpha) = 1$. Also, we say that τ **satisfies** a set Σ of formulas if it satisfies each formula in Σ.

Definition 1.4. Let Γ be a set of formulas and let α be a formula. We say that Γ **tautologically implies** α, denoted by $\Gamma \vDash \alpha$, if every truth assignment satisfying Γ also satisfies α.

1.3 Proof System

Definition 1.5. The collection Λ of **axioms** consists of the formulas listed below, where α, β, γ are formulas.

(A1) $\alpha \to (\beta \to \alpha)$.

(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

(A3) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$.

Definition 1.6. A **proof** of a formula α from a collection Γ of formulas is a sequence of formulas

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

satisfying the following properties.

(a) $\alpha_n = \alpha$.

(b) For $k \in \{1, 2, \ldots, n\}$, either $\alpha_k \in \Lambda \cup \Gamma$ or there exist $i, j \in \{1, 2, \ldots, k-1\}$ with $\alpha_j = \alpha_i \to \alpha_k$.

If there exists a proof of φ from Γ, we write $\Gamma \vdash \varphi$. If $\varnothing \vdash \varphi$, we write $\vdash \varphi$ for short.

Theorem 1.7 (Law of Identity). For any formula α, we have $\vdash \alpha \to \alpha$.

Proof. We have a proof of $\alpha \to \alpha$ as follows.

(1) $(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$. (A2)

(2) $\alpha \to ((\alpha \to \alpha) \to \alpha)$. (A1)

(3) $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$. (1, 2)

(4) $\alpha \to (\alpha \to \alpha)$. (A1)

(5) $\alpha \to \alpha$. (3, 4)

Thus, we can conclude that $\vdash \alpha \to \alpha$. $\square$

Theorem 1.8 (Duns Scotus Law). For any formula α and β, we have $\vdash \neg\alpha \to (\alpha \to \beta)$.

Proof. We have a proof of $\neg\alpha \to (\alpha \to \beta)$ as follows.

(1) $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta)) \to (\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta)))$. (A1)

(2) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$. (A3)

(3) $\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$. (1, 2)

(4) $(\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))) \to ((\neg\alpha \to (\neg\beta \to \neg\alpha)) \to (\neg\alpha \to (\alpha \to \beta)))$. (A2)

(5) $(\neg\alpha \to (\neg\beta \to \neg\alpha)) \to (\neg\alpha \to (\alpha \to \beta))$. (3, 4)

(6) $\neg\alpha \to (\neg\beta \to \neg\alpha)$. (A1)

(7) $\neg\alpha \to (\alpha \to \beta)$. (5, 6)

Thus, we can conclude that $\vdash \neg\alpha \to (\alpha \to \beta)$. $\square$

Theorem 1.9 (Modus Ponens). For any formula α and β, we have $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$.

Proof. We have a proof of $\alpha \to ((\alpha \to \beta) \to \beta)$ as follows.

(1) $(\alpha \to \beta) \to (\alpha \to \beta)$. (Theorem 1.7)

(2) $((\alpha \to \beta) \to (\alpha \to \beta)) \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))$. (A2)

(3) $((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)$. (1, 2)

(4) $(((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)))$. (A1)

(5) $\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))$. (3, 4)

(6) $(\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))) \to ((\alpha \to ((\alpha \to \beta) \to \alpha)) \to (\alpha \to ((\alpha \to \beta) \to \beta)))$. (A2)

(7) $(\alpha \to ((\alpha \to \beta) \to \alpha)) \to (\alpha \to ((\alpha \to \beta) \to \beta))$. (5, 6)

(8) $\alpha \to ((\alpha \to \beta) \to \alpha)$. (A1)

(9) $\alpha \to ((\alpha \to \beta) \to \beta)$. (7, 8)

Thus, we can conclude that $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$. $\square$

Theorem 1.10 (Hypothetical Syllogism). For any formulas α, β and γ, we have $\vdash (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

Proof. We have a proof of $(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ as follows.

(1) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. (A2)

(2) $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))))$. (A1)

(3) $(\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$. (1, 2)

(4) $((\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))) \to (((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))))$. (A2)

(5) $((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$. (3, 4)

(6) $(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$. (A1)

(7) $(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. (5, 6)

Thus, we can conclude that $\vdash (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. $\square$

Theorem 1.11 (Clavius's Law). For any formula α, we have $\vdash (\neg\alpha \to \alpha) \to \alpha$.

Proof. We have a proof of $(\neg\alpha \to \alpha) \to \alpha$ as follows.

(1) $(\neg\alpha \to (\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha)))$. (A2)

(2) $\neg\alpha \to (\alpha \to \neg(\neg\alpha \to \alpha))$. (Theorem 1.8)

(3) $(\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))$. (1, 2)

(4) $(\neg\alpha \to \neg(\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)$. (A3)

(5) $((\neg\alpha \to \neg(\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)) \to (((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)))$. (Theorem 1.10)

(6) $((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha))$. (4, 5)

(7) $(\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)$. (3, 6)

(8) $((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)) \to (((\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha))$. (A2)

(9) $((\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)$ (7, 8)

(10) $(\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)$. (Theorem 1.7)

(11) $(\neg\alpha \to \alpha) \to \alpha$. (9, 10)

Thus, we can conclude that $\vdash (\neg\alpha \to \alpha) \to \alpha$. □

Theorem 1.12 (Elimination of Double Negation). For any formula α, we have $\vdash \neg\neg\alpha \to \alpha$.

Proof. We have a proof of $\neg\neg\alpha \to \alpha$ as follows.

(1) $((\neg\alpha \to \alpha) \to \alpha) \to ((\neg\neg\alpha \to (\neg\alpha \to \alpha)) \to (\neg\neg\alpha \to \alpha))$. (Theorem 1.10)

(2) $(\neg\alpha \to \alpha) \to \alpha$. (Theorem 1.11)

(3) $(\neg\neg\alpha \to (\neg\alpha \to \alpha)) \to (\neg\neg\alpha \to \alpha)$. (1, 2)

(4) $\neg\neg\alpha \to (\neg\alpha \to \alpha)$. (Theorem 1.8)

(5) $\neg\neg\alpha \to \alpha$. (3, 4)

Thus, we can conclude that $\vdash \neg\neg\alpha \to \alpha$. □

Theorem 1.13 (Introduction of Double Negation). For any formula α, we have $\vdash \alpha \to \neg\neg\alpha$.

Proof. We have a proof of $\alpha \to \neg\neg\alpha$ as follows.

(1) $(\neg\neg\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)$. (A3)

(2) $\neg\neg\neg\alpha \to \neg\alpha$. (Theorem 1.12)

(3) $\alpha \to \neg\neg\alpha$. (1, 2)

Thus, we can conclude that $\vdash \alpha \to \neg\neg\alpha$. □

Theorem 1.14 (Law of Contraposition). For any formulas α and β, we have $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$.

Proof. We have a proof of $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ as follows.

(1) $(\beta \to \neg\neg\beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (Theorem 1.10)

(2) $\beta \to \neg\neg\beta$. (Theorem 1.13)

(3) $(\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$. (1, 2)

(4) $((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)))$. (A1)

(5) $(\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (3, 4)

(6) $(\alpha \to \beta) \to ((\neg\neg\alpha \to \alpha) \to (\neg\neg\alpha \to \beta))$. (Theorem 1.10)

(7) $((\alpha \to \beta) \to ((\neg\neg\alpha \to \alpha) \to (\neg\neg\alpha \to \beta))) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)))$. (A2)

(8) $((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \beta))$. (6, 7)

(9) $(\neg\neg\alpha \to \alpha) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha))$. (A1)

(10) $\neg\neg\alpha \to \alpha$. (Theorem 1.12)

(11) $(\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)$. (9, 10)

(12) $(\alpha \to \beta) \to (\neg\neg\alpha \to \beta)$. (8, 11)

(13) $((\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)))$. (A2)

(14) $((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (5, 13)

(15) $(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$. (12, 14)

(16) $((\neg\neg\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to (\neg\beta \to \neg\alpha)))$. (Theorem 1.10)

(17) $(\neg\neg\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)$. (A3)

(18) $((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to (\neg\beta \to \neg\alpha))$. (16, 17)

(19) $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$. (15, 18)

Thus, we can conclude that $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$. $\square$

Theorem 1.15. For any formulas α and β, we have $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$.

Proof. We have a proof of $\alpha \to (\neg\beta \to \neg(\alpha \to \beta))$ as follows.

(1) $((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))$. (Theorem 1.14)

(2) $(((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))) \to (\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))))$. (A1)

(3) $\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta)))$. (1, 2)

(4) $(\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to (\neg(\alpha \to \beta))))) \to ((\alpha \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (\neg\beta \to (\neg(\alpha \to \beta)))))$. (A2)

(5) $(\alpha \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (\neg\beta \to (\neg(\alpha \to \beta))))$. (3, 4)

(6) $\alpha \to ((\alpha \to \beta) \to \beta)$. (Theorem 1.9)

(7) $\alpha \to (\neg\beta \to \neg(\alpha \to \beta))$. (5, 6)

Thus, we can conclude that $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$. □

Theorem 1.16 (Deduction Theorem). Let Γ be a set of formulas and let α and β be formulas. If $\Gamma \cup \{\alpha\} \vdash \beta$, then $\Gamma \vdash \alpha \to \beta$.

Proof. If $\beta \in \Lambda \cup \Gamma$, then we have $\Gamma \vdash \alpha \to \beta_k$ since $\vdash \beta_k \to (\alpha \to \beta_k)$. Furthermore, if $\beta = \alpha$, then we also have $\Gamma \vdash \alpha \to \beta$ since $\vdash \beta \to \beta$ by Theorem 1.7. Thus, one only needs to consider the case that $\beta \notin \Lambda \cup \Gamma \cup \{\alpha\}$.

Suppose that $(\beta_1, \beta_2, \ldots, \beta_n)$ is a proof of β from $\Gamma \cup \{\alpha\}$. For $1 \leq k \leq n$, we prove that $\Gamma \vdash \alpha \to \beta_k$ by induction on k. The induction basis holds for $k = 1$ since $\beta_1 \in \Lambda \cup \Gamma \cup \{\alpha\}$. For the induction step, let $k \geq 2$ and assume that $\Gamma \vdash \alpha \to \beta_\ell$ for $1 \leq \ell < k$. We have proved for the case that $\beta \in \Lambda \cup \Gamma \cup \{\alpha\}$, and thus we assume without loss of generality that there exist $1 \leq i < k$ and $1 \leq j < k$ such that $\beta_j = \beta_i \to \beta_k$. Note that $\Gamma \vdash \alpha \to \beta_i$ and $\Gamma \vdash \alpha \to (\beta_i \to \beta_k)$ hold by induction hypothesis. Therefore, since

$$\vdash (\alpha \to (\beta_i \to \beta_k)) \to ((\alpha \to \beta_i) \to (\alpha \to \beta_k)),$$

we can conclude that $\Gamma \vdash \alpha \to \beta_k$, which completes the proof. □

1.4 Soundness and Completeness

Theorem 1.17. Let α be a formula which consists of only the propositional variables $p_1, \ldots, p_k$ and let τ be a truth assignment. Let $p_1^*, \ldots, p_k^*$ be formulas such that for each $i \in \{1, \ldots, k\}$,

$$p_i^* = \begin{cases} p_i, & \text{if } \tau(p_i) = 1 \\ \neg p_i, & \text{if } \tau(p_i) = 0. \end{cases}$$

Furthermore, let α^* be the formula defined by

$$\alpha^* = \begin{cases} \alpha, & \text{if } \tau(\alpha) = 1 \\ \neg \alpha, & \text{if } \tau(\alpha) = 0. \end{cases}$$

Then we have

$$\{p_1^*, \ldots, p_k^*\} \vdash \alpha^*.$$

Proof. The proof is by induction on the complexity of α. It is straightforward that the theorem holds when $\alpha = p_i$ for some $i \in \{1, \ldots, k\}$.

Now suppose that $\{p_1^*, \ldots, p_k^*\} \vdash \alpha^*$, and we prove that

$$\{p_1^*, \ldots, p_k^*\} \vdash \beta^*$$

with $\beta = \neg\alpha$. If $\tau(\alpha) = 0$, then $\tau(\beta) = 1$, and we have $\alpha^* = \neg\alpha = \beta^*$. Thus, $\{p_1^*, \ldots, p_k^*\} \vdash \beta^*$. If $\tau(\alpha) = 1$, then $\tau(\beta) = 0$, and we have $\alpha^* = \alpha$ and $\beta^* = \neg\neg\alpha$. Since

$$\{p_1^*, \ldots, p_k^*\} \vdash \alpha \quad \text{and} \quad \vdash \alpha \to \neg\neg\alpha,$$

we have $\{p_1^*, \ldots, p_k^*\} \vdash \beta^*$.

Now suppose that $\{p_1^*, \ldots, p_k^*\} \vdash \alpha^*$ and $\{p_1^*, \ldots, p_k^*\} \vdash \beta^*$, and we prove that

$$\{p_1^*, \ldots, p_k^*\} \vdash \gamma^*$$

with $\gamma = \alpha \to \beta$. If $\tau(\alpha) = 0$, then $\tau(\gamma) = 1$, and we have $\alpha^* = \neg\alpha$ and $\gamma^* = \alpha \to \beta$. Since $\{p_1^*, \ldots, p_k^*\} \vdash \neg\alpha$

$$\{p_1^*, \ldots, p_k^*\} \vdash \neg\alpha \quad \text{and} \quad \vdash \neg\alpha \to (\alpha \to \beta)$$

we have $\{p_1^*, \ldots, p_k^*\} \vdash \gamma^*$. If $\tau(\beta) = 1$, then $\tau(\gamma) = 1$, and we have $\beta^* = \beta$ and $\gamma^* = \alpha \to \beta$. Since

$$\{p_1^*, \ldots, p_k^*\} \vdash \beta \quad \text{and} \quad \vdash \beta \to (\alpha \to \beta)$$

we have $\{p_1^*, \ldots, p_k^*\} \vdash \gamma^*$. If $\tau(\alpha) = 1$ and $\tau(\beta) = 0$, then $\tau(\gamma) = 0$, and we have $\alpha^* = \alpha$, $\beta^* = \neg\beta$ and $\gamma^* = \neg(\alpha \to \beta)$. Since

$$\{p_1^*, \ldots, p_k^*\} \vdash \alpha, \quad \{p_1^*, \ldots, p_k^*\} \vdash \neg\beta, \quad \text{and} \quad \vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta)),$$

we have $\{p_1^*, \ldots, p_k^*\} \vdash \gamma^*$, completing the proof. $\square$

Chapter 2

Predicate Logic

2.1 The Language of Predicate Logic

In this chapter, we reserve a countable set $\mathcal{V}$, in which each element is called a **variable**.

Definition 2.1. A **vocabulary** is a pair

$$\mathcal{L} = (\mathcal{P}, \mathcal{F}),$$

where $\mathcal{P}$ is the set of **predicate symbols** and $\mathcal{F}$ is the set of **function symbols**. Each predicate symbol and each function symbol comes with an arity, the number of argument it expects.

Definition 2.2. We define **terms** as follows.

- Each variable is a term.
- If f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

Definition 2.3. We define **formulas** as follows.

- If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is a formula.
- If α is a formula, then $\neg\alpha$ is a formula.
- If α and β are formulas, then $(\alpha \to \beta)$ is a formula.
- If α is a formula and x is a variable, then $\forall x \alpha$ is a formula.

2.2 Models

Definition 2.4. A **model** of vocabulary $\mathcal{L} = (\mathcal{P}, \mathcal{F})$ is a triple

$$\mathcal{M} = \left(A, \left(P^{\mathcal{M}}\right)_{P\in\mathcal{P}}, \left(f^{M}\right)_{f\in\mathcal{F}}\right),$$

where each component is as follows.

- A is a nonempty set called **universe**.
- To each n-ary relation symbol P an n-ary relation $P^{\mathcal{M}} \subseteq A^n$ is assigned.
- To each n-ary function symbol f an n-ary function $f^{\mathcal{M}} : A^n \to A$ is assigned.

Definition 2.5. Let $\mathcal{M}$ be a model of vocabulary $\mathcal{L} = (\mathcal{P}, \mathcal{F})$. An **object assignment** is a function σ that maps each variable to an element in A.

It can be extended to have its domain the set of terms such that for any n-ary function symbol f and any terms $t_1, \ldots, t_n$, we have

$$\sigma(f(t_1, \ldots, t_n)) = f^{M}(\sigma(t_1), \sigma(t_2), \ldots, \sigma(t_n)).$$

Definition 2.6. For any model $\mathcal{M}$ and any object assignment σ, we define the satisfaction relation $(\mathcal{M}, \sigma) \vDash \phi$ for each formula ϕ as follows.

- $(\mathcal{M}, \sigma) \vDash P(t_1, \ldots, t_n)$ holds if and only if $(t_1, \ldots, t_n) \in P^{\mathcal{M}}$ holds.
- $(\mathcal{M}, \sigma) \vDash \neg\alpha$ holds if and only if $(\mathcal{M}, \sigma) \vDash \alpha$ does not hold.
- $(\mathcal{M}, \sigma) \vDash (\alpha \to \beta)$ holds if and only if $(\mathcal{M}, \sigma) \vDash \alpha$ does not hold or $(\mathcal{M}, \sigma) \vDash \beta$ holds.
- $(\mathcal{M}, \sigma) \vDash \forall x\alpha$ holds if and only if $(\mathcal{M}, \sigma[x \mapsto c]) \vDash \alpha$ holds for all $c \in A$.