Analysis

Chapter 1

Real Numbers

1.1 Fields

Definition 1.1. A nonempty set F and two operations $+$ and $\cdot$ form a **field** if the following axioms (A 1) – (A 5), (M 1) – (M 5) and (D) are satisfied.

(A 1) $x + y \in F$ for any $x, y \in F$.

(A 2) $x + y = y + x$ for any $x, y \in F$.

(A 3) $(x + y) + z = x + (y + z)$ for any $x, y, z \in F$.

(A 4) There is an element $0 \in F$ such that $x + 0 = x$ for any $x \in F$.

(A 5) For each $x \in F$ there is an element $-x$ in F such that $x + (-x) = 0$.

(M 1) $x \cdot y \in F$ for any $x, y \in F$.

(M 1) $x \cdot y = y \cdot x$ for any $x, y \in F$.

(M 2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for any $x, y, z \in F$.

(M 3) There is an element $1 \in F \setminus \{0\}$ such that $x \cdot 1 = x$ for any $x \in F$.

(M 4) For each $x \in F \setminus \{0\}$ there is an element x^{-1} in F such that $x \cdot x^{-1} = 0$.

(D) $x \cdot (y + z) = x \cdot y + x \cdot z$ for any $x, y, z \in F$.

Theorem 1.2. Let F be a field. Then the following statements are true for any $x, y, z \in F$.

(a) If $x + y = x + z$, then $y = z$.

(b) If $x + y = x$, then $y = 0$.

(c) If $x + y = 0$, then $y = -x$.

(d) $-(-x) = x$.

Proof. Note that these statements are consequence of axioms (A 1) – (A 5).

(a) We have

$$\begin{aligned}
y &= 0 + y \\
&= (-x + x) + y \\
&= -x + (x + y) \\
&= -x + (x + z) \\
&= (-x + x) + z \\
&= 0 + z \\
&= z.
\end{aligned}$$

(b) Since $x + y = x = x + 0$, we have $y = 0$ by (a).

(c) Since $x + y = 0 = x + (-x)$, we have $y = -x$ by (a).

(d) Since $-x + x = 0$, we have $-(-x) = x$ by (c). □

Theorem 1.3. Let F be a field. Then the following statements are true for any $x \in F \setminus \{0\}$ and $y, z \in F$.

(a) If $x \cdot y = x \cdot z$, then $x = y$.

(b) If $x \cdot y = x$, then $y = 1$.

(c) If $x \cdot y = 1$, then $y = x^{-1}$.

(d) $(x^{-1})^{-1} = x$.

Proof. Note that these statements are consequence of axioms (M 1) – (M 5).

(a) We have

$$\begin{aligned}
y &= 1 \cdot y \\
&= (x^{-1} \cdot x) \cdot y \\
&= x^{-1} \cdot (x \cdot y) \\
&= x^{-1} \cdot (x \cdot z) \\
&= (x^{-1} \cdot x) \cdot z \\
&= 1 \cdot z \\
&= z.
\end{aligned}$$

(b) Since $x \cdot y = x = x \cdot 1$, we have $y = 1$ by (a).

(c) Since $x \cdot y = 1 = x \cdot x^{-1}$, we have $y = x^{-1}$ by (a).

(d) Since $x^{-1} + x = 1$, we have $(x^{-1})^{-1} = x$ by (c). □

Theorem 1.4. Let F be a field. Then the following statements are true for any $x, y \in F$.

(a) $0 \cdot x = 0$.

(b) $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$.

(c) $(-x) \cdot (-y) = x \cdot y$.

Proof.

(a) We have

$$0 \cdot x + 0 \cdot x = (0+0) \cdot x = 0 \cdot x,$$

implying $0 \cdot x = 0$.

(b) Since

$$(-x) \cdot y + x \cdot y = (-x + x) \cdot y = 0 \cdot y = 0,$$

we have $(-x) \cdot y = -(x \cdot y)$. One can prove $x \cdot (-y) = -(x \cdot y)$ similarly.

(c) We have

$$(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(x \cdot y)) = x \cdot y$$

by applying (b) twice. $\square$

1.2 Ordered Fields

Definition 1.5. An **ordered field** is a field on which relation $<$ is defined such that the following axioms (O 1) – (O 4) hold for any $x, y, z \in F$.

(O 1) One and only one of the statements $x = y$, $x < y$, $y < x$ is true.

(O 2) If $x < y$ and $y < z$, then $x < z$.

(O 3) If $x < y$, then $x + z < y + z$.

(O 4) If $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Definition 1.6. Let F be an ordered field. The relations $>$, $\leq$ and $\geq$ are defined as follows for any $x, y \in F$.

$$\begin{aligned} x > y \quad &\Leftrightarrow \quad y < x \\ x \leq y \quad &\Leftrightarrow \quad x < y \text{ or } x = y \\ x \geq y \quad &\Leftrightarrow \quad x > y \text{ or } x = y. \end{aligned}$$

Definition 1.7. Let F be an ordered field and let $S \subseteq F$.

- An **upper bound** of S is an element x in F such that $x \geq y$ for any $y \in S$. We say that S is **bounded above** if S has an upper bound.
- A **lower bound** of S is an element x in F such that $x \leq y$ for any $y \in S$. We say that S is **bounded below** if S has a lower bound.

Definition 1.8. Let F be an ordered field and let $S \subseteq F$.

- An element of S is called the **maximum** of S, denoted by $\max(S)$, if it is an upper bound of S.
- An element of S is called the **minimum** of S, denoted by $\min(S)$, if it is a lower bound of S.
- The minimum of the set of upper bounds of S is called the **supremum** of S, denoted by $\sup(S)$.
- The maximum of the set of lower bounds of S is called the **infimum** of S, denoted by $\inf(S)$.

1.3 The Real Field

Definition 1.9. $\mathbb{R}$ is an ordered field such that every nonempty subset S of $\mathbb{R}$ that is bounded above has a supremum. The elements of $\mathbb{R}$ are called the **real numbers**.

Theorem 1.10 (Archimedean Property). For any $x, y \in \mathbb{R}$ with $x > 0$, there is a positive integer n such that

$$n \cdot x > y.$$

Proof. Let

$$S = \{nx : n \text{ is a positive integer}\}.$$

Suppose that y is an upper bound of S. It follows that S has a supremum z. Note that $z - x$ is not an upper bound of S since $z - x < z$. Thus, $z - x < mx$ for some positive integer m, implying $z < (m + 1)x$, contradiction to the fact that z is an upper bound of S. Hence, y is not an upper bound of S, completing the proof. $\square$

Chapter 2

Basic Topology

2.1 Metric Spaces

Definition 2.1. A set X with a function $d : X \times X \to \mathbb{R}$ is a **metric space** if the following statements hold for any $x, y, z \in X$.

(a) $d(x, y) \geq 0$.

(b) $d(x, y) = 0$ if and only if $x = y$.

(c) $d(x, y) = d(y, x)$.

(d) $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 2.2. Let (X, d) be a metric space. Let $r > 0$ be a real number and let $x_0 \in X$. The **open ball** of radius r centered at x_0, denoted by $B_r(x_0)$, is defined by

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}.$$

Definition 2.3. Let (X, d) be a metric space and let $S \subseteq X$.

- S is **open** if for any $x \in S$, there is a real number $r > 0$ such that $B_r(x) \subseteq S$.
- S is **closed** if $X \setminus S$ is open.

Theorem 2.4. Let (X, d) be a metric space.

(a) X and $\varnothing$ are open.

(b) If S_1, S_2 are open subsets of X, then $S_1 \cap S_2$ is open.

(c) If $\{S_i : i \in I\}$ is a collection of open subsets of X, then

$$\bigcup_{i \in I} S_i$$

is open.

Definition 2.5. Let (X, d) be a metric space and let $S \subseteq X$.

- A point $x \in X$ is a **limit point** of S if there exists $y \in S \setminus \{x\}$ with $d(x, y) < \epsilon$ for any $\epsilon > 0$.
- A point $x \in X$ is an **isolated point** of S if x belongs to S and is not a limit point of S.

2.2 Compact Sets

Definition 2.6. Let (X, d) be a metric space and let $S \subseteq X$.

- A **cover** of S is a collection of subsets of X whose union contains S. An **open cover** of S is a cover of S whose elements are all open.
- We say that S is **compact** if every open cover Ω of S contains a finite cover Φ of S.

Theorem 2.7. Let (X, d) be a metric space and let $R \subseteq S \subseteq X$. If S is compact and R is closed, then R is compact.

Proof. Suppose that R has an open cover Ω. Then $\Omega' = \Omega \cup \{X \setminus R\}$ is an open cover of S since $X \setminus R$ is open. Let $\Phi' \subseteq \Omega'$ be a finite cover of S, and let $\Phi = \Phi' \setminus \{X \setminus R\}$. Then Φ is a finite open cover of R with $\Phi \subseteq \Omega$. Thus, R is compact. $\square$

Theorem 2.8 (Nested Interval Theorem). Let $\langle I_n \rangle$ be a sequence of rectangles in $\mathbb{R}^k$ such that $I_{n+1} \subseteq I_n$, then the intersection of $\{I_n : n \in \mathbb{N}\}$ is nonempty.

Proof. For each positive integer n, let

$$I_n = [a_n^{(1)}, b_n^{(1)}] \times \cdots \times [a_n^{(k)}, b_n^{(k)}].$$

For each $i \in \{1, \ldots, k\}$, we have

$$a_n^{(i)} \le a_{n+m}^{(i)} \le b_{n+m}^{(i)} \le b_m^{(i)}$$

for any $n, m \in \mathbb{N}$. Thus, $\{a_n^{(i)} : n \in \mathbb{N}\}$ is bounded above, implying the existence of

$$x_i = \sup(\{a_n^{(i)} : n \in \mathbb{N}\}).$$

We have

$$a_n^{(i)} \le x_i \le b_m^{(i)}$$

for any $n, m \in \mathbb{N}$. Thus,

$$x = (x_1, \ldots, x_n) \in \bigcap_{n \ge 1} I_n,$$

completing the proof. $\square$

Theorem 2.9. Every k-cell in $\mathbb{R}^k$ is compact.

Proof. Let $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$. We have

$$\|x - x'\| \le \sqrt{\textstyle\sum_{i=1}^k (a_i - b_i)^2}$$

for any $x, x' \in I$. Assume that there is an open cover $\mathcal{O}$ of I that contains no finite subcover of I. Let $c_i = (a_i + b_i)/2$ for all $i \in \{1, \ldots, n\}$, and let

$$\mathcal{C} = \{I^{(1)} \times \cdots \times I^{(k)} : I^{(i)} \in \{[a_i, c_i], [c_i, b_i]\} \text{ for } 1 \le i \le k\}$$

be a collection of 2^k k-cells whose union is I. Then there must be a k-cell $I' \in \mathcal{C}$ cannot be covered by any finite subset of $\mathcal{O}$, or I could be covered by that set, contradtion.

Thus, if I is not compact, then we can construct a sequence $\langle I_n \rangle$ of k-cells which are not covered by any finite subset of $\mathcal{O}$ such that $I_1 = I$, $I_{n+1} \subseteq I_n$ for any integer $n \geq 1$, and

$$\|x - x'\| \leq \frac{\sqrt{\sum_{i=1}^{k} (a_i - b_i)^2}}{2^{n-1}}$$

holds for any $x, x' \in I_n$. It follows that there is a point $y \in \bigcap \{I_n\}$, and we have $y \in S$ for some $S \in \mathcal{O}$. Since S is open, we have $B_r(y) \subseteq S$ for some $r > 0$. Let N be a positive integer such that

$$2^N > \frac{\sqrt{\sum_{i=1}^{k} (a_i - b_i)^2}}{r/2}.$$

Then for any $x \in I_N$,

$$\|x - y\| \leq \frac{\sqrt{\sum_{i=1}^{k} (a_i - b_i)^2}}{2^{N-1}} < r,$$

implying $x \in B_r(y) \subseteq S$. It follows that $I_N \subseteq S$, and $\{S\}$ is a finite subset of $\mathcal{O}$, contradtion. Thus, I is compact. □

Theorem 2.10 (Heine–Borel Theorem). Let $S \subseteq \mathbb{R}^k$. Then S is compact if and only if S is closed and bounded.

Proof. ($\Leftarrow$) If S is closed and bounded, then there is a k-cell I with $S \subseteq I$. Since I is compact, and S is closed, we conclude that S is compact.

($\Rightarrow$) Suppose that S is compact. Then S is closed. Since $\mathcal{O} = \{B_r(0_{\mathbb{R}^k}) : r \in \mathbb{N}\}$ is an open cover of S, there is $\mathcal{O}' \subseteq \mathcal{O}$ such that $S \subseteq \bigcup \mathcal{O}'$. It can be shown that $\bigcup \mathcal{O}'$ is bounded, and thus S is bounded. □

Chapter 3

Sequences and Series

Definition 3.1. Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ **converges** to a point $x \in X$, denoted by

$$\lim_{n\to\infty} x_n = x,$$

if for any real number $\epsilon > 0$ there is a positive integer N such that $n \geq N$ implies $d(x_n, x) < \epsilon$.

- We say that $\{x_n\}$ is **convergent** if it converges to some point in X.
- We say that $\{x_n\}$ is **divergent** if it is not convergent.

Theorem 3.2. Let $\{x_n\}$ be a sequence in a metric space (X, d). If $\{x_n\}$ converges to both $x \in X$ and $x' \in X$, then $x = x'$.

Proof. For any $\epsilon > 0$, there exists a positive integer N such that

$$d(x_n, x) < \frac{\epsilon}{2} \quad \text{and} \quad d(x_n, x') < \frac{\epsilon}{2}$$

for each $n \geq N$, implying

$$d(x, x') \leq d(x_n, x) + d(x_n, x') < \epsilon. \qquad \square$$

Theorem 3.3. Let $\{a_n\}$ and $\{b_n\}$ be complex sequences with

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} b_n = M.$$

Let c be a complex number. Then the following statements are true.

(a) We have
$$\lim_{n\to\infty} (a_n + b_n) = L + M.$$

(b) We have
$$\lim_{n\to\infty} a_n b_n = LM.$$

(c) If $L \neq 0$ and $a_n \neq 0$ for each positive integer n, we have
$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}.$$

Proof.

(a) For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{\epsilon}{2} \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2},$$

implying

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) Let $C > 0$ such that $|L| \leq C$ and $|b_n| \leq C$ for any positive integer n. For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{\epsilon}{2C} \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2C},$$

implying

$$\begin{aligned} |a_n b_n - LM| &= |(a_n - L)b_n + (b_n - M)L| \\ &\leq |a_n - L||b_n| + |b_n - M||L| \\ &< \frac{\epsilon(|b_n| + L)}{2C} \\ &\leq \epsilon. \end{aligned}$$

(c) For any $\epsilon > 0$, there exists a positive integer N such that for any $n \geq N$, we have

$$|a_n - L| < \frac{|L|^2\epsilon}{2} \quad \text{and} \quad |a_n - L| < \frac{|L|}{2}.$$

It follows that

$$|a_n| = |L + (a_n - L)| \geq |L| - |a_n - L| > \frac{|L|}{2},$$

implying

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \left|\frac{a_n - L}{a_n L}\right| = \frac{|a_n - L|}{|a_n||L|} < \frac{2|a_n - L|}{|L|^2} < \epsilon.$$

□

Definition 3.4. Let $\langle a_n \rangle$ be a sequence of real numbers.

- We say that $\langle a_n \rangle$ is **increasing** (resp., **strictly increasing**) if $a_n \leq a_{n+1}$ (resp. $a_n < a_{n+1}$) holds for all $n \in \mathbb{N}$.
- We say that $\langle a_n \rangle$ is **decreasing** (resp., **strictly decreasing**) if $a_n \geq a_{n+1}$ (resp. $a_n > a_{n+1}$) holds for all $n \in \mathbb{N}$.

Theorem 3.5. Let $\langle a_n \rangle$ be a sequence of real numbers. If $\langle a_n \rangle$ is increasing and bounded above, then $\langle a_n \rangle$ converges.

Proof. Let $L = \sup(\{a_n : n \in \mathbb{N}\})$. For any $\epsilon > 0$, since $L - \epsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$, there exists a positive integer N with $a_N > L - \epsilon$. Since $\langle a_n \rangle$ is increasing, we have

$$L - \epsilon < a_N < a_n \leq L$$

for all positive integer $n > N$, implying $|a_n - L| \leq \epsilon$ for all positive integer $n > N$. Thus, $\langle a_n \rangle$ converges to L. □

Definition 3.6. Let (X, d) be a metric space and let $\langle x_n \rangle$ be a sequence in X. We say that $\langle x_n \rangle$ is a **Cauchy sequence** if for any $\epsilon > 0$ there is a positive integer N, such that $n \geq N$ and $m \geq N$ implies $d(x_n, x_m) < \epsilon$.

Chapter 4

Continuity

4.1 Limits of Functions

Definition 4.1. Let (X, d_X) and (Y, d_Y) be a metric spaces. Let $f : D \to Y$ be a map with $D \subseteq X$. Then we say that $b \in Y$ is the **limit** of f at $a \in X$, denoted by

$$\lim_{x \to a} f(x) = b,$$

if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(x), b) < \epsilon$$

holds for any $x \in S$ with

$$0 < d_X(x, a) < \delta.$$

Definition 4.2. Let (X, d_x) and (Y, d_Y) be metric spaces. Let $f : D \to Y$ be a map with $D \subseteq X$. Then f is **continuous** at $a \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(x), f(a)) < \epsilon$$

holds for any $x \in D$ with

$$d_X(x, a) < \delta.$$

4.2 Continuity and Compactness

Chapter 5

Differentiation

Chapter 6

Integration