Logic

Chapter 1

Propositional Logic

1.1 Formulas

In propositional logic, we reserve a countable set $\mathcal{V}$ of **propositional variables**.

Definition 1.1. We define **formulas** as follows.

1. Each propositional variable is a formula.
2. If α is a formula, then $\neg\alpha$ is a formula.
3. If α and β are formulas, then $(\alpha \to \beta)$ is a formula.

1.2 Truth Assignments

Definition 1.2. A **truth assignment** is a map $\tau : \mathcal{V} \to \{0, 1\}$, and we define the **truth value** $[\![\alpha]\!]_\tau$ of α under τ as follows.

1. For each propositional variable p, $[\![p]\!]_\tau = \tau(p)$.
2. For each formula α,
$$[\![\neg\alpha]\!]_\tau = \begin{cases} 0, & \text{if } [\![\alpha]\!]_\tau = 1 \\ 1, & \text{otherwise.} \end{cases}$$
3. For each formula α and β,
$$[\![(\alpha \to \beta)]\!]_\tau = \begin{cases} 0, & \text{if } [\![\alpha]\!]_\tau = 1 \text{ and } [\![\beta]\!]_\tau = 0 \\ 1, & \text{otherwise.} \end{cases}$$

1.3 Proofs

Definition 1.3. The formulas of the forms (A1), (A2) and (A3) listed below are called **axioms**, where α, β, γ are formulas.

(A1) $\alpha \to (\beta \to \alpha)$.

(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

(A3) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$.

We denote the set of axioms by Λ.

Definition 1.4. Let Γ be a set of formulas and let α be a formula. We say that α can be **proved** from Γ if there exist formulas $\beta_1, \ldots, \beta_n$ with $\beta_n = \alpha$ such that

$$\beta_k \in \Gamma \cup \Lambda \quad \text{or} \quad \beta_j = (\beta_i \to \beta_k) \text{ for some } i, j \in \{1, 2, \ldots, k-1\}$$

holds for all $k \in \{1, \ldots, n\}$. We write $\Gamma \vdash \alpha$ if α can be proved from Γ, and if $\Gamma = \varnothing$, we write $\vdash \alpha$ for short.

Theorem 1.5 (Law of Identity). For any formula α, we have $\vdash \alpha \to \alpha$.

Proof. We have a proof of $\alpha \to \alpha$ as follows.

(1) $(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$. (A2)

(2) $\alpha \to ((\alpha \to \alpha) \to \alpha)$. (A1)

(3) $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$. (1, 2)

(4) $\alpha \to (\alpha \to \alpha)$. (A1)

(5) $\alpha \to \alpha$. (3, 4)

Thus, we can conclude that $\vdash \alpha \to \alpha$. □

Theorem 1.6 (Duns Scotus Law). For any formula α and β, we have $\vdash \neg\alpha \to (\alpha \to \beta)$.

Proof. We have a proof of $\neg\alpha \to (\alpha \to \beta)$ as follows.

(1) $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta)) \to (\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta)))$. (A1)

(2) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$. (A3)

(3) $\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$. (1, 2)

(4) $(\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))) \to ((\neg\alpha \to (\neg\beta \to \neg\alpha)) \to (\neg\alpha \to (\alpha \to \beta)))$. (A2)

(5) $(\neg\alpha \to (\neg\beta \to \neg\alpha)) \to (\neg\alpha \to (\alpha \to \beta))$. (3, 4)

(6) $\neg\alpha \to (\neg\beta \to \neg\alpha)$. (A1)

(7) $\neg\alpha \to (\alpha \to \beta)$. (5, 6)

Thus, we can conclude that $\vdash \neg\alpha \to (\alpha \to \beta)$. $\square$

Theorem 1.7 (Modus Ponens). For any formula α and β, we have $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$.

Proof. We have a proof of $\alpha \to ((\alpha \to \beta) \to \beta)$ as follows.

(1) $(\alpha \to \beta) \to (\alpha \to \beta)$. (Theorem 1.5)

(2) $((\alpha \to \beta) \to (\alpha \to \beta)) \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))$. (A2)

(3) $((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)$. (1, 2)

(4) $(((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)))$. (A1)

(5) $\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))$. (3, 4)

(6) $(\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))) \to ((\alpha \to ((\alpha \to \beta) \to \alpha)) \to (\alpha \to ((\alpha \to \beta) \to \beta)))$. (A2)

(7) $(\alpha \to ((\alpha \to \beta) \to \alpha)) \to (\alpha \to ((\alpha \to \beta) \to \beta))$. (5, 6)

(8) $\alpha \to ((\alpha \to \beta) \to \alpha)$. (A1)

(9) $\alpha \to ((\alpha \to \beta) \to \beta)$. (7, 8)

Thus, we can conclude that $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$. $\square$

Theorem 1.8 (Hypothetical Syllogism). For any formulas α, β and γ, we have $\vdash (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

Proof. We have a proof of $(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ as follows.

(1) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. (A2)

(2) $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))))$. (A1)

(3) $(\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$. (1, 2)

(4) $((\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))) \to (((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))))$. (A2)

(5) $((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$. (3, 4)

(6) $(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$. (A1)

(7) $(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. (5, 6)

Thus, we can conclude that $\vdash (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. $\square$

Theorem 1.9 (Clavius's Law). For any formula α, we have $\vdash (\neg\alpha \to \alpha) \to \alpha$.

Proof. We have a proof of $(\neg\alpha \to \alpha) \to \alpha$ as follows.

(1) $(\neg\alpha \to (\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha)))$. (A2)

(2) $\neg\alpha \to (\alpha \to \neg(\neg\alpha \to \alpha))$. (Theorem 1.6)

(3) $(\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))$. (1, 2)

(4) $(\neg\alpha \to \neg(\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)$. (A3)

(5) $((\neg\alpha \to \neg(\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)) \to (((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)))$. (Theorem 1.8)

(6) $((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha))$. (4, 5)

(7) $(\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)$. (3, 6)

(8) $((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)) \to (((\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha))$. (A2)

(9) $((\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)$ (7, 8)

(10) $(\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)$. (Theorem 1.5)

(11) $(\neg\alpha \to \alpha) \to \alpha$. (9, 10)

Thus, we can conclude that $\vdash (\neg\alpha \to \alpha) \to \alpha$. $\square$

Theorem 1.10 (Elimination of Double Negation). For any formula α, we have $\vdash \neg\neg\alpha \to \alpha$.

Proof. We have a proof of $\neg\neg\alpha \to \alpha$ as follows.

(1) $((\neg\alpha \to \alpha) \to \alpha) \to ((\neg\neg\alpha \to (\neg\alpha \to \alpha)) \to (\neg\neg\alpha \to \alpha))$. (Theorem 1.8)

(2) $(\neg\alpha \to \alpha) \to \alpha$. (Theorem 1.9)

(3) $(\neg\neg\alpha \to (\neg\alpha \to \alpha)) \to (\neg\neg\alpha \to \alpha)$. (1, 2)

(4) $\neg\neg\alpha \to (\neg\alpha \to \alpha)$. (Theorem 1.6)

(5) $\neg\neg\alpha \to \alpha$. (3, 4)

Thus, we can conclude that $\vdash \neg\neg\alpha \to \alpha$. $\square$

Theorem 1.11 (Introduction of Double Negation). For any formula α, we have $\vdash \alpha \to \neg\neg\alpha$.

Proof. We have a proof of $\alpha \to \neg\neg\alpha$ as follows.

(1) $(\neg\neg\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)$. (A3)

(2) $\neg\neg\neg\alpha \to \neg\alpha$. (Theorem 1.10)

(3) $\alpha \to \neg\neg\alpha$. (1, 2)

Thus, we can conclude that $\vdash \alpha \to \neg\neg\alpha$. $\square$

Theorem 1.12 (Law of Contraposition). For any formulas α and β, we have $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$.

Proof. We have a proof of $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ as follows.

(1) $(\beta \to \neg\neg\beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (Theorem 1.8)

(2) $\beta \to \neg\neg\beta$. (Theorem 1.11)

(3) $(\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$. (1, 2)

(4) $((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)))$. (A1)

(5) $(\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (3, 4)

(6) $(\alpha \to \beta) \to ((\neg\neg\alpha \to \alpha) \to (\neg\neg\alpha \to \beta))$. (Theorem 1.8)

(7) $((\alpha \to \beta) \to ((\neg\neg\alpha \to \alpha) \to (\neg\neg\alpha \to \beta))) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)))$. (A2)

(8) $((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \beta))$. (6, 7)

(9) $(\neg\neg\alpha \to \alpha) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha))$. (A1)

(10) $\neg\neg\alpha \to \alpha$. (Theorem 1.10)

(11) $(\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)$. (9, 10)

(12) $(\alpha \to \beta) \to (\neg\neg\alpha \to \beta)$. (8, 11)

(13) $((\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)))$. (A2)

(14) $((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (5, 13)

(15) $(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$. (12, 14)

(16) $((\neg\neg\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to (\neg\beta \to \neg\alpha)))$. (Theorem 1.8)

(17) $(\neg\neg\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)$. (A3)

(18) $((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to (\neg\beta \to \neg\alpha))$. (16, 17)

(19) $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$. (15, 18)

Thus, we can conclude that $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$. □

Theorem 1.13. For any formulas α and β, we have $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$.

Proof. We have a proof of $\alpha \to (\neg\beta \to \neg(\alpha \to \beta))$ as follows.

(1) $((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))$. (Theorem 1.12)

(2) $(((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))) \to (\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))))$. (A1)

(3) $\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta)))$. (1, 2)

(4) $(\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to (\neg(\alpha \to \beta))))) \to ((\alpha \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (\neg\beta \to (\neg(\alpha \to \beta)))))$. (A2)

(5) $(\alpha \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (\neg\beta \to (\neg(\alpha \to \beta))))$. (3, 4)

(6) $\alpha \to ((\alpha \to \beta) \to \beta)$. (Theorem 1.7)

(7) $\alpha \to (\neg\beta \to \neg(\alpha \to \beta))$. (5, 6)

Thus, we can conclude that $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$. □

Theorem 1.14 (Deduction Theorem). Let Γ be a set of formulas and let α and β be formulas. If $\Gamma \cup \{\alpha\} \vdash \beta$, then $\Gamma \vdash \alpha \to \beta$.

Proof. If $\beta \in \Lambda \cup \Gamma$, then we have $\Gamma \vdash \alpha \to \beta_k$ since $\vdash \beta_k \to (\alpha \to \beta_k)$. Furthermore, if $\beta = \alpha$, then we also have $\Gamma \vdash \alpha \to \beta$ since $\vdash \beta \to \beta$ by Theorem 1.5. Thus, one only needs to consider the case that $\beta \notin \Lambda \cup \Gamma \cup \{\alpha\}$.

Suppose that $(\beta_1, \beta_2, \ldots, \beta_n)$ is a proof of β from $\Gamma \cup \{\alpha\}$. For $1 \leq k \leq n$, we prove that $\Gamma \vdash \alpha \to \beta_k$ by induction on k. The induction basis holds for $k = 1$ since $\beta_1 \in \Lambda \cup \Gamma \cup \{\alpha\}$. For the induction step, let $k \geq 2$ and assume that $\Gamma \vdash \alpha \to \beta_\ell$ for $1 \leq \ell < k$. We have proved for the case that $\beta \in \Lambda \cup \Gamma \cup \{\alpha\}$, and thus we assume without loss of generality that there exist $1 \leq i < k$ and $1 \leq j < k$ such that $\beta_j = \beta_i \to \beta_k$. Note that $\Gamma \vdash \alpha \to \beta_i$ and $\Gamma \vdash \alpha \to (\beta_i \to \beta_k)$ hold by induction hypothesis. Therefore, since

$$\vdash (\alpha \to (\beta_i \to \beta_k)) \to ((\alpha \to \beta_i) \to (\alpha \to \beta_k)),$$

we can conclude that $\Gamma \vdash \alpha \to \beta_k$, which completes the proof. □

1.4 Completeness

Lemma 1.15. Let τ be a truth assignment. For each formula ϕ, we define

$$\phi^{(\tau)} = \begin{cases} \phi, & \text{if } \tau(\phi) = 1 \\ \neg\phi, & \text{if } \tau(\phi) = 0. \end{cases}$$

Then for any formula α that consists of only the propositional variables $p_1, \ldots, p_k$, we have

$$\left\{p_1^{(\tau)}, \ldots, p_k^{(\tau)}\right\} \vdash \alpha^{(\tau)}.$$

Proof. Let

$$\Pi = \left\{p_1^{(\tau)}, \ldots, p_k^{(\tau)}\right\}.$$

The proof is by induction. If α is atomic, i.e., $\alpha = p_i$ for some $i \in \{1, \ldots, k\}$, then we have $\alpha^{(\tau)} \in \Pi$, and thus $\Pi \vdash \alpha^{(\tau)}$.

Now suppose that $\Pi \vdash \alpha^{(\tau)}$, and we prove that $\Pi \vdash \beta^{(\tau)}$ with $\beta = \neg\alpha$.

Case 1. If $\tau(\alpha) = 0$, then $\tau(\beta) = 1$, and it follows that $\alpha^{(\tau)} = \neg\alpha = \beta^{(\tau)}$, implying $\Pi \vdash \beta^{(\tau)}$.

Case 2. If $\tau(\alpha) = 1$, then $\tau(\beta) = 0$, and it follows that $\alpha^{(\tau)} = \alpha$ and $\beta^{(\tau)} = \neg\neg\alpha$. Since $\Pi \vdash \alpha$ and $\vdash \alpha \to \neg\neg\alpha$, we have $\Pi \vdash \neg\neg\alpha$, implying $\Pi \vdash \beta^{(\tau)}$.

Now suppose that $\Pi \vdash \alpha^{(\tau)}$ and $\Pi \vdash \beta^{(\tau)}$, and we prove that $\Pi \vdash \gamma^{(\tau)}$ with $\gamma = \alpha \to \beta$.

Case 1. If $\tau(\alpha) = 0$, then $\tau(\gamma) = 1$, and it follows that $\alpha^{(\tau)} = \neg\alpha$ and $\gamma^{(\tau)} = \alpha \to \beta$. Since $\Pi \vdash \neg\alpha$, and $\vdash \neg\alpha \to (\alpha \to \beta)$, we have $\Pi \vdash \alpha \to \beta$, implying $\Pi \vdash \gamma^{(\tau)}$.

Case 2. If $\tau(\beta) = 1$, then $\tau(\gamma) = 1$, and it follows that $\beta^{(\tau)} = \beta$ and $\gamma^{(\tau)} = \alpha \to \beta$. Since $\Pi \vdash \beta$ and $\beta \vdash (\alpha \to \beta)$, we have $\Pi \vdash \alpha \to \beta$, implying $\Pi \vdash \gamma^{(\tau)}$.

Case 3. If $\tau(\alpha) = 1$ and $\tau(\beta) = 0$, then $\tau(\gamma) = 0$, and it follows that $\alpha^{(\tau)} = \alpha$, $\beta^{(\tau)} = \neg\beta$, and $\gamma^{(\tau)} = \neg(\alpha \to \beta)$. Since $\Pi \vdash \alpha$, $\Pi \vdash \neg\beta$ and $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$, we have $\Pi \vdash \neg(\alpha \to \beta)$, implying $\Pi \vdash \gamma^{(\tau)}$. $\square$

Theorem 1.16 (Completness Theorem). For each formula α, $\vDash \alpha$ implies $\vdash \alpha$.

Proof. By Lemma 1.15,

$$p_1^{(\tau)}, p_2^{(\tau)}, \ldots, p_k^{(\tau)} \vdash \alpha$$

holds for any truth assignment τ, where $p_1, p_2, \ldots, p_k$ are the propositional variables that appears in α.

Now suppose that τ is a truth assignment and $p_1, \ldots, p_j$ are propositional variables such that

$$p_1^{(\tau)}, \ldots, p_{j-1}^{(\tau)}, p_j \vdash \alpha \quad \text{and} \quad p_1^{(\tau)}, \ldots, p_{j-1}^{(\tau)}, \neg p_j \vdash \alpha.$$

Then we have

$$p_1^{(\tau)}, \ldots, p_{j-1}^{(\tau)} \vdash p_j \to \alpha \quad \text{and} \quad p_1^{(\tau)}, \ldots, p_{j-1}^{(\tau)} \vdash \neg p_j \to \alpha.$$

Since $\vdash (p_j \to \alpha) \to ((\neg p_j \to \alpha) \to \alpha)$, it follows that

$$p_1^{(\tau)}, \ldots, p_{j-1}^{(\tau)} \vdash \alpha.$$

This process can be performed continually such that all the premises are eliminated. Thus, we conclude that $\vdash \alpha$. $\square$

Chapter 2

First-order Logic

2.1 Formulas

In first-order logic, we reserve a countable set $\mathcal{X}$ of **variables**, a countable set $\mathcal{R}$ of **relation symbols**, and a countable set $\mathcal{F}$ of **function symbols**. Each relation symbol and function symbol is associated with an arity.

Definition 2.1. We define **terms** as follows.

1. Each variable is a term.
2. If f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

Definition 2.2. We define **formulas** as follows.

1. If R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is a formula.
2. If α is a formula, then $\neg\alpha$ is a formula.
3. If α and β are formulas, then $(\alpha \to \beta)$ is a formula.
4. If α is a formula and x is a variable, then $\forall x \alpha$ is a formula.

2.2 Structures

Definition 2.3. A **structure** is a triple

$$\mathcal{A} = \Big(A, \big(R^{\mathcal{A}}\big)_{R\in\mathcal{R}}, \big(f^{\mathcal{A}}\big)_{f\in\mathcal{F}}\Big),$$

where each component is as follows.

- A is a nonempty set called **domain**.
- To each n-ary relation symbol R an n-ary relation $R^{\mathcal{A}} \subseteq A^n$ is assigned.
- To each n-ary function symbol f an n-ary function $f^{\mathcal{A}} : A^n \to A$ is assigned.