Logic

Chapter 1

Propositional Logic

1.1 Basics of Propositional Logic

Definition 1.1. We choose an arbitrary countably infinite set A of **propositional atoms**, and we define **propositions** as follows.

1. Each propositional atom is a proposition.
2. If α is a proposition, then $\neg\alpha$ is a proposition.
3. If α and β are propositions, then $(\alpha \to \beta)$ is a proposition.

Definition 1.2. A **truth assignment** is a map $\tau : A \to \{0, 1\}$, and we further define

$$\tau(\neg\alpha) = \begin{cases} 0, & \text{if } \tau(\alpha) = 1 \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad \tau((\alpha \to \beta)) = \begin{cases} 0, & \text{if } \tau(\alpha) = 1 \text{ and } \tau(\beta) = 0 \\ 1, & \text{otherwise} \end{cases}$$

for each formula α and β. We say that τ **satisfies** formula α if $\tau(\alpha) = 1$.

1.2 The Proof System

Definition 1.3. The formulas of the forms (A1), (A2) and (A3) listed below are called **axioms**, where α, β, γ are formulas.

(A1) $\alpha \to (\beta \to \alpha)$.

(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

(A3) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$.

We denote the set of axioms by Λ.

Definition 1.4. Let Γ be a set of formulas and let α be a formula. We say that α can be **proved** from Γ if there exist formulas $\beta_1, \ldots, \beta_n$ with $\beta_n = \alpha$ such that

$$\beta_k \in \Gamma \cup \Lambda \quad \text{or} \quad \beta_j = (\beta_i \to \beta_k) \text{ for some } i, j \in \{1, 2, \ldots, k-1\}$$

holds for all $k \in \{1, \ldots, n\}$. We write $\Gamma \vdash \alpha$ if α can be proved from Γ, and if $\Gamma = \varnothing$, we write $\vdash \alpha$ for short.

Theorem 1.5 (Law of Identity). For any formula α, we have $\vdash \alpha \to \alpha$.

Proof. We have a proof of $\alpha \to \alpha$ as follows.

(1) $(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$. (A2)

(2) $\alpha \to ((\alpha \to \alpha) \to \alpha)$. (A1)

(3) $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$. (1, 2)

(4) $\alpha \to (\alpha \to \alpha)$. (A1)

(5) $\alpha \to \alpha$. (3, 4)

Thus, we can conclude that $\vdash \alpha \to \alpha$. $\square$

Theorem 1.6 (Duns Scotus Law). For any formula α and β, we have $\vdash \neg\alpha \to (\alpha \to \beta)$.

Proof. We have a proof of $\neg\alpha \to (\alpha \to \beta)$ as follows.

(1) $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta)) \to (\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta)))$. (A1)

(2) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$. (A3)

(3) $\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$. (1, 2)

(4) $(\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))) \to ((\neg\alpha \to (\neg\beta \to \neg\alpha)) \to (\neg\alpha \to (\alpha \to \beta)))$. (A2)

(5) $(\neg\alpha \to (\neg\beta \to \neg\alpha)) \to (\neg\alpha \to (\alpha \to \beta))$. (3, 4)

(6) $\neg\alpha \to (\neg\beta \to \neg\alpha)$. (A1)

(7) $\neg\alpha \to (\alpha \to \beta)$. (5, 6)

Thus, we can conclude that $\vdash \neg\alpha \to (\alpha \to \beta)$. $\square$

Theorem 1.7 (Modus Ponens). For any formula α and β, we have $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$.

Proof. We have a proof of $\alpha \to ((\alpha \to \beta) \to \beta)$ as follows.

(1) $(\alpha \to \beta) \to (\alpha \to \beta)$. (Theorem 1.5)

(2) $((\alpha \to \beta) \to (\alpha \to \beta)) \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))$. (A2)

(3) $((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)$. (1, 2)

(4) $(((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta)))$. (A1)

(5) $\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))$. (3, 4)

(6) $(\alpha \to (((\alpha \to \beta) \to \alpha) \to ((\alpha \to \beta) \to \beta))) \to ((\alpha \to ((\alpha \to \beta) \to \alpha)) \to (\alpha \to ((\alpha \to \beta) \to \beta)))$. (A2)

(7) $(\alpha \to ((\alpha \to \beta) \to \alpha)) \to (\alpha \to ((\alpha \to \beta) \to \beta))$. (5, 6)

(8) $\alpha \to ((\alpha \to \beta) \to \alpha)$. (A1)

(9) $\alpha \to ((\alpha \to \beta) \to \beta)$. (7, 8)

Thus, we can conclude that $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$. $\square$

Theorem 1.8 (Hypothetical Syllogism). For any formulas α, β and γ, we have $\vdash (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

Proof. We have a proof of $(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ as follows.

(1) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. (A2)

(2) $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))))$. (A1)

(3) $(\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$. (1, 2)

(4) $((\beta \to \gamma) \to ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))) \to (((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))))$. (A2)

(5) $((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))) \to ((\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$. (3, 4)

(6) $(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$. (A1)

(7) $(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. (5, 6)

Thus, we can conclude that $\vdash (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. $\square$

Theorem 1.9 (Clavius's Law). For any formula α, we have $\vdash (\neg\alpha \to \alpha) \to \alpha$.

Proof. We have a proof of $(\neg\alpha \to \alpha) \to \alpha$ as follows.

(1) $(\neg\alpha \to (\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha)))$. (A2)

(2) $\neg\alpha \to (\alpha \to \neg(\neg\alpha \to \alpha))$. (Theorem 1.6)

(3) $(\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))$. (1, 2)

(4) $(\neg\alpha \to \neg(\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)$. (A3)

(5) $((\neg\alpha \to \neg(\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)) \to (((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)))$. (Theorem 1.8)

(6) $((\neg\alpha \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))) \to ((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha))$. (4, 5)

(7) $(\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)$. (3, 6)

(8) $((\neg\alpha \to \alpha) \to ((\neg\alpha \to \alpha) \to \alpha)) \to (((\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha))$. (A2)

(9) $((\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)) \to ((\neg\alpha \to \alpha) \to \alpha)$ (7, 8)

(10) $(\neg\alpha \to \alpha) \to (\neg\alpha \to \alpha)$. (Theorem 1.5)

(11) $(\neg\alpha \to \alpha) \to \alpha$. (9, 10)

Thus, we can conclude that $\vdash (\neg\alpha \to \alpha) \to \alpha$. □

Theorem 1.10 (Elimination of Double Negation). For any formula α, we have $\vdash \neg\neg\alpha \to \alpha$.

Proof. We have a proof of $\neg\neg\alpha \to \alpha$ as follows.

(1) $((\neg\alpha \to \alpha) \to \alpha) \to ((\neg\neg\alpha \to (\neg\alpha \to \alpha)) \to (\neg\neg\alpha \to \alpha))$. (Theorem 1.8)

(2) $(\neg\alpha \to \alpha) \to \alpha$. (Theorem 1.9)

(3) $(\neg\neg\alpha \to (\neg\alpha \to \alpha)) \to (\neg\neg\alpha \to \alpha)$. (1, 2)

(4) $\neg\neg\alpha \to (\neg\alpha \to \alpha)$. (Theorem 1.6)

(5) $\neg\neg\alpha \to \alpha$. (3, 4)

Thus, we can conclude that $\vdash \neg\neg\alpha \to \alpha$. □

Theorem 1.11 (Introduction of Double Negation). For any formula α, we have $\vdash \alpha \to \neg\neg\alpha$.

Proof. We have a proof of $\alpha \to \neg\neg\alpha$ as follows.

(1) $(\neg\neg\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)$. (A3)

(2) $\neg\neg\neg\alpha \to \neg\alpha$. (Theorem 1.10)

(3) $\alpha \to \neg\neg\alpha$. (1, 2)

Thus, we can conclude that $\vdash \alpha \to \neg\neg\alpha$. □

Theorem 1.12 (Law of Contraposition). For any formulas α and β, we have $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$.

Proof. We have a proof of $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ as follows.

(1) $(\beta \to \neg\neg\beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (Theorem 1.8)

(2) $\beta \to \neg\neg\beta$. (Theorem 1.11)

(3) $(\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$. (1, 2)

(4) $((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)))$. (A1)

(5) $(\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (3, 4)

(6) $(\alpha \to \beta) \to ((\neg\neg\alpha \to \alpha) \to (\neg\neg\alpha \to \beta))$. (Theorem 1.8)

(7) $((\alpha \to \beta) \to ((\neg\neg\alpha \to \alpha) \to (\neg\neg\alpha \to \beta))) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)))$. (A2)

(8) $((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \beta))$. (6, 7)

(9) $(\neg\neg\alpha \to \alpha) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \alpha))$. (A1)

(10) $\neg\neg\alpha \to \alpha$. (Theorem 1.10)

(11) $(\alpha \to \beta) \to (\neg\neg\alpha \to \alpha)$. (9, 10)

(12) $(\alpha \to \beta) \to (\neg\neg\alpha \to \beta)$. (8, 11)

(13) $((\alpha \to \beta) \to ((\neg\neg\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)))$. (A2)

(14) $((\alpha \to \beta) \to (\neg\neg\alpha \to \beta)) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta))$. (5, 13)

(15) $(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$. (12, 14)

(16) $((\neg\neg\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)) \to (((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to (\neg\beta \to \neg\alpha)))$. (Theorem 1.8)

(17) $(\neg\neg\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)$. (A3)

(18) $((\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)) \to ((\alpha \to \beta) \to (\neg\beta \to \neg\alpha))$. (16, 17)

(19) $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$. (15, 18)

Thus, we can conclude that $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$. $\square$

Theorem 1.13. For any formulas α and β, we have $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$.

Proof. We have a proof of $\alpha \to (\neg\beta \to \neg(\alpha \to \beta))$ as follows.

(1) $((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))$. (Theorem 1.12)

(2) $(((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))) \to (\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))))$. (A1)

(3) $\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta)))$. (1, 2)

(4) $(\alpha \to (((\alpha \to \beta) \to \beta) \to (\neg\beta \to (\neg(\alpha \to \beta))))) \to ((\alpha \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (\neg\beta \to (\neg(\alpha \to \beta)))))$. (A2)

(5) $(\alpha \to ((\alpha \to \beta) \to \beta)) \to (\alpha \to (\neg\beta \to (\neg(\alpha \to \beta))))$. (3, 4)

(6) $\alpha \to ((\alpha \to \beta) \to \beta)$. (Theorem 1.7)

(7) $\alpha \to (\neg\beta \to \neg(\alpha \to \beta))$. (5, 6)

Thus, we can conclude that $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$. □

Theorem 1.14 (Deduction Theorem). Let Γ be a set of formulas and let α and β be formulas. If $\Gamma \cup \{\alpha\} \vdash \beta$, then $\Gamma \vdash \alpha \to \beta$.

Proof. If $\beta \in \Lambda \cup \Gamma$, then we have $\Gamma \vdash \alpha \to \beta_k$ since $\vdash \beta_k \to (\alpha \to \beta_k)$. Furthermore, if $\beta = \alpha$, then we also have $\Gamma \vdash \alpha \to \beta$ since $\vdash \beta \to \beta$ by Theorem 1.5. Thus, one only needs to consider the case that $\beta \notin \Lambda \cup \Gamma \cup \{\alpha\}$.

Suppose that $(\beta_1, \beta_2, \ldots, \beta_n)$ is a proof of β from $\Gamma \cup \{\alpha\}$. For $1 \leq k \leq n$, we prove that $\Gamma \vdash \alpha \to \beta_k$ by induction on k. The induction basis holds for $k = 1$ since $\beta_1 \in \Lambda \cup \Gamma \cup \{\alpha\}$. For the induction step, let $k \geq 2$ and assume that $\Gamma \vdash \alpha \to \beta_\ell$ for $1 \leq \ell < k$. We have proved for the case that $\beta \in \Lambda \cup \Gamma \cup \{\alpha\}$, and thus we assume without loss of generality that there exist $1 \leq i < k$ and $1 \leq j < k$ such that $\beta_j = \beta_i \to \beta_k$. Note that $\Gamma \vdash \alpha \to \beta_i$ and $\Gamma \vdash \alpha \to (\beta_i \to \beta_k)$ hold by induction hypothesis. Therefore, since

$$\vdash (\alpha \to (\beta_i \to \beta_k)) \to ((\alpha \to \beta_i) \to (\alpha \to \beta_k)),$$

we can conclude that $\Gamma \vdash \alpha \to \beta_k$, which completes the proof. □

1.3 Completeness

Lemma 1.15. Let τ be a truth assignment. For each formula ϕ, we define

$$\phi^{(\tau)} = \begin{cases} \phi, & \text{if } \tau(\phi) = 1 \\ \neg\phi, & \text{if } \tau(\phi) = 0. \end{cases}$$

Then for any formula α that consists of only the propositional variables $p_1, \dots, p_k$, we have

$$\left\{p_1^{(\tau)}, \dots, p_k^{(\tau)}\right\} \vdash \alpha^{(\tau)}.$$

Proof. Let

$$\Pi = \left\{p_1^{(\tau)}, \dots, p_k^{(\tau)}\right\}.$$

The proof is by induction. If α is atomic, i.e., $\alpha = p_i$ for some $i \in \{1, \dots, k\}$, then we have $\alpha^{(\tau)} \in \Pi$, and thus $\Pi \vdash \alpha^{(\tau)}$.

Now suppose that $\Pi \vdash \alpha^{(\tau)}$, and we prove that $\Pi \vdash \beta^{(\tau)}$ with $\beta = \neg\alpha$.

Case 1. If $\tau(\alpha) = 0$, then $\tau(\beta) = 1$, and it follows that $\alpha^{(\tau)} = \neg\alpha = \beta^{(\tau)}$, implying $\Pi \vdash \beta^{(\tau)}$.

Case 2. If $\tau(\alpha) = 1$, then $\tau(\beta) = 0$, and it follows that $\alpha^{(\tau)} = \alpha$ and $\beta^{(\tau)} = \neg\neg\alpha$. Since $\Pi \vdash \alpha$ and $\vdash \alpha \to \neg\neg\alpha$, we have $\Pi \vdash \neg\neg\alpha$, implying $\Pi \vdash \beta^{(\tau)}$.

Now suppose that $\Pi \vdash \alpha^{(\tau)}$ and $\Pi \vdash \beta^{(\tau)}$, and we prove that $\Pi \vdash \gamma^{(\tau)}$ with $\gamma = \alpha \to \beta$.

Case 1. If $\tau(\alpha) = 0$, then $\tau(\gamma) = 1$, and it follows that $\alpha^{(\tau)} = \neg\alpha$ and $\gamma^{(\tau)} = \alpha \to \beta$. Since $\Pi \vdash \neg\alpha$, and $\vdash \neg\alpha \to (\alpha \to \beta)$, we have $\Pi \vdash \alpha \to \beta$, implying $\Pi \vdash \gamma^{(\tau)}$.

Case 2. If $\tau(\beta) = 1$, then $\tau(\gamma) = 1$, and it follows that $\beta^{(\tau)} = \beta$ and $\gamma^{(\tau)} = \alpha \to \beta$. Since $\Pi \vdash \beta$ and $\beta \vdash (\alpha \to \beta)$, we have $\Pi \vdash \alpha \to \beta$, implying $\Pi \vdash \gamma^{(\tau)}$.

Case 3. If $\tau(\alpha) = 1$ and $\tau(\beta) = 0$, then $\tau(\gamma) = 0$, and it follows that $\alpha^{(\tau)} = \alpha$, $\beta^{(\tau)} = \neg\beta$, and $\gamma^{(\tau)} = \neg(\alpha \to \beta)$. Since $\Pi \vdash \alpha$, $\Pi \vdash \neg\beta$ and $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$, we have $\Pi \vdash \neg(\alpha \to \beta)$, implying $\Pi \vdash \gamma^{(\tau)}$. $\square$

Theorem 1.16 (Completness Theorem). For each formula α, $\vDash \alpha$ implies $\vdash \alpha$.

Proof. By Lemma 1.15,

$$p_1^{(\tau)}, p_2^{(\tau)}, \dots, p_k^{(\tau)} \vdash \alpha$$

holds for any truth assignment τ, where $p_1, p_2, \dots, p_k$ are the propositional variables that appears in α.

Now suppose that τ is a truth assignment and $p_1, \dots, p_j$ are propositional variables such that

$$p_1^{(\tau)}, \dots, p_{j-1}^{(\tau)}, p_j \vdash \alpha \quad \text{and} \quad p_1^{(\tau)}, \dots, p_{j-1}^{(\tau)}, \neg p_j \vdash \alpha.$$

Then we have

$$p_1^{(\tau)}, \dots, p_{j-1}^{(\tau)} \vdash p_j \to \alpha \quad \text{and} \quad p_1^{(\tau)}, \dots, p_{j-1}^{(\tau)} \vdash \neg p_j \to \alpha.$$

Since $\vdash (p_j \to \alpha) \to ((\neg p_j \to \alpha) \to \alpha)$, it follows that

$$p_1^{(\tau)}, \dots, p_{j-1}^{(\tau)} \vdash \alpha.$$

This process can be performed continually such that all the premises are eliminated. Thus, we conclude that $\vdash \alpha$. $\square$

Chapter 2

Predicate Logic

2.1 The Language of Predicate Logic

In this chapter, we reserve a countable set $\mathcal{V}$, whose elements are called **variables**.

Definition 2.1. A **vocabulary** is a pair

$$\mathcal{L} = (\mathcal{P}, \mathcal{F}),$$

where $\mathcal{P}$ is the set of **predicate symbols** and $\mathcal{F}$ is the set of **function symbols**. Each predicate symbol and each function symbol comes with an arity, the number of argument it expects.

Definition 2.2. We define **terms** as follows.

1. Each variable is a term.
2. If f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

Definition 2.3. We define **formulas** as follows.

1. If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is a formula.
2. If α is a formula, then $\neg\alpha$ is a formula.
3. If α and β are formulas, then $(\alpha \to \beta)$ is a formula.
4. If α is a formula and x is a variable, then $\forall x \alpha$ is a formula.

2.2 Models

Definition 2.4. A **model** of vocabulary $\mathcal{L} = (\mathcal{P}, \mathcal{F})$ is a triple

$$\mathcal{M} = \left(M, \left(P^{\mathcal{M}}\right)_{P \in \mathcal{P}}, \left(f^{\mathcal{M}}\right)_{f \in \mathcal{F}}\right),$$

where each component is as follows.

- M is a nonempty set called **universe**.
- To each n-ary relation symbol P an n-ary relation $P^{\mathcal{M}} \subseteq M^n$ is assigned.
- To each n-ary function symbol f an n-ary function $f^{\mathcal{M}} : M^n \to M$ is assigned.

Definition 2.5. Let $\mathcal{M}$ be a model of vocabulary $\mathcal{L} = (\mathcal{P}, \mathcal{F})$. An **object assignment** is a function σ that maps each variable to an element in M.

It can be extended to have its domain the set of terms such that for any n-ary function symbol f and any terms $t_1, \ldots, t_n$, we have

$$\sigma(f(t_1, \ldots, t_n)) = f^{\mathcal{M}}(\sigma(t_1), \sigma(t_2), \ldots, \sigma(t_n)).$$

Definition 2.6. For any model $\mathcal{M}$ and any object assignment σ, we define the satisfaction relation $(\mathcal{M}, \sigma) \vDash \phi$ for each formula ϕ as follows.

- $(\mathcal{M}, \sigma) \vDash P(t_1, \ldots, t_n)$ means $(t_1, \ldots, t_n) \in P^{\mathcal{M}}$.
- $(\mathcal{M}, \sigma) \vDash \neg\alpha$ holds if and only if $(\mathcal{M}, \sigma) \nvDash \alpha$.
- $(\mathcal{M}, \sigma) \vDash (\alpha \to \beta)$ holds if and only if either $(\mathcal{M}, \sigma) \nvDash \alpha$ or $(\mathcal{M}, \sigma) \vDash \beta$.
- $(\mathcal{M}, \sigma) \vDash \forall x \alpha$ holds if and only if $(\mathcal{M}, \sigma[x \mapsto c]) \vDash \alpha$ for all $c \in M$.