2.2

构造拉格朗日函数

$$L = \hat{U}(x,y) + \lambda(px+qy-I)$$

$$\begin{cases} \frac{\partial L}{\partial x} = \frac{\alpha}{x-x_0} + \lambda p = 0 \\ \frac{\partial L}{\partial y} = \frac{\beta}{y-y_0} + \lambda q = 0 \\ px+qy-I=0 \end{cases}$$

$x = -\frac{\alpha}{\lambda p} + x_0$ $\quad px = \frac{-\alpha}{\lambda} + px_0$

$y = -\frac{\beta}{\lambda q} + y_0$ $\quad qy = -\frac{\beta}{\lambda} + qy_0$

$px_0 + qy_0 - \frac{\alpha+\beta}{\lambda} = I$ $\quad px_0+qy_0 - \frac{1}{\lambda} = I$

$\frac{1}{\lambda} = \frac{px_0+qy_0+I}{\alpha+\beta}$ $\quad px = \frac{-p\alpha x_0 + q\alpha y_0 + \alpha I + \alpha px_0 + p\beta x_0}{\alpha+\beta}$

$= \alpha I + p\beta x_0 + q\alpha y_0$

同理可得 $\quad qy = \beta I - \beta p x_0 + \alpha q y_0$

3.3

$L = F(x) + \lambda(C - G(x))$ $\quad$ s.t. $\begin{cases} \sum_{j=1}^{n} x_j \le C \\ x_j \ge 0 \end{cases}$

$= \sum_{j=1}^{n}[\alpha_j x_j - \frac{1}{2}\beta_j x_j^2] + \lambda(C - \sum x_j)$

$$\begin{cases} \frac{\partial L}{\partial x} \le 0, & x \ge 0 \\ \frac{\partial L}{\partial \lambda} \ge 0, & \lambda \ge 0 \end{cases}$$

(i) 假设使用了 C' 资金

$\frac{\partial L}{\partial x_j} = \alpha_j - \beta_j x_j - \lambda \le 0$ $\quad$ ①

$\frac{\partial L}{\partial \lambda} = C' - \sum_{j=1}^{n} x_j \ge 0$ $\quad$ ②

由 ① 得 $\quad x_j \ge \frac{\alpha_j - \lambda}{\beta_j}$

将 x_j 代入 ② $\quad \sum_{j=1} x_j = C'$

$$\sum_{j=1}^{n}\frac{\alpha_j}{\beta_j} - \sum_{j=1}^{n}\frac{\lambda}{\beta_j} = C'$$

即 $H - \lambda K = C'$ $\quad \because C > H$

$\therefore C' < H < C$ $\quad$ 有资金未使用完

(ii)

$\because \alpha_j > \frac{H-C}{K}$

$x_j \ge \frac{\alpha_j - \lambda}{\beta_j} > \frac{H-C-\lambda K}{\beta_j K} = 0$

$\therefore x_j > 0$

(iii) 设 i 未得资金

$x_i = \frac{\alpha_i - \lambda}{\beta_i} = 0$ $\quad \alpha_i = \lambda$

$x_j = \frac{\alpha_j - \lambda}{\beta_j} > 0$

$\therefore \alpha_j - \alpha_i > 0$ $\quad \alpha_j > \alpha_i$

4.1

(1) $\sum_{g=1}^{G} Z_{fg} \le Z_f \qquad \sum_{c=1}^{C} x_{cg} \le X_g = \phi^g(Z_{1g}, Z_{2g}, \ldots Z_{Fg})$

$$L = W(U^1(x_{11}, \ldots x_{1G}), \ldots U^C(x_{C1}, \ldots x_{CG})) + \sum_g \lambda_f [Z_f - \sum_f Z_{fg}] + \sum_g \pi_g (X_g - \sum_c x_{cg})$$

最优一阶条件 $\frac{\partial L}{\partial x_{cg}} = \frac{\partial W}{\partial U_c} \cdot \frac{\partial U^c}{\partial x_{cg}} - \pi_g = 0$ ①

$\frac{\partial L}{\partial Z_{fg}} = -\lambda_f + \frac{\partial \phi^g}{\partial Z_{fg}} \pi_g$ ②

∴ 一阶条件不是 ① 新增 ②

(2) λ_f 代表要素 f 的影子价格，即 f 的边际价值

(3) ∵ 分散化

∴ 每个企业可独立决定要素投入量 Z_{fg}

企业投入要素直到边际产出价值 = 要素价格，

即 $\pi_g \cdot \frac{\partial \phi^g}{\partial Z_{fg}} = \lambda_f$

(4)

依题意 即证 $\sum_{c=1}^{C} I_c = \sum_{g=1}^{G} \pi_g X_g$

$\sum_{c=1}^{C} I_c = \sum_{c=1}^{C} \sum_{g=1}^{G} \pi_g x_{cg} = \sum_{g=1}^{G} \pi_g \sum_{c=1}^{C} x_{cg}$

最优配置时 $\sum_{c=1}^{C} x_{cg} = X_g$

∴ $\sum_{c=1}^{C} I_c = \sum_{g=1}^{C} \pi_g X_g$