2.2

$$\max.\ \hat{U}(x,y) = \alpha\ln(x-x_0) + \beta\ln(y-y_0) \qquad \alpha+\beta=1$$

$$\text{s.t.}\quad px + qy = I$$

$$\hat{L}(x,y,\lambda) = \alpha\ln(x-x_0) + \beta\ln(y-y_0) + \lambda[I - px - qy]$$

F.O.C.

$$\frac{\partial\hat{L}}{\partial x} = \frac{\alpha}{x-x_0} - \lambda p = 0 \qquad (1)$$

$$\frac{\partial\hat{L}}{\partial y} = \frac{\beta}{y-y_0} - \lambda q = 0 \qquad (2)$$

$$\frac{\partial\hat{L}}{\partial \lambda} = I - px - qy = 0 \qquad (3)$$

由(1) $\lambda(px - px_0) = \alpha \qquad (4)$

由(2) $\lambda(qy - qy_0) = \beta \qquad (5)$

(4)+(5) 再考虑(3) $\lambda(I - px_0 - qy_0) = \alpha+\beta = 1 \qquad (6)$

由(4)和(6) $\alpha(I - p x_0 - q y_0) = p x - p x_0$ (7)

由(5)和(6) $\beta(I - p x_0 - q y_0) = q y - q y_0$ (8).

对(7)和(8)化简得 $\begin{cases} p x = \alpha I + \beta p x_0 - \alpha q y_0 \\ q y = \beta I - \beta p x_0 + \alpha q y_0 \end{cases}$

3.3

$$\max_{\{x_j\}_{j=1,\dots n}} \sum_{j=1}^{n} \left[\alpha_j x_j - \frac{1}{2}\beta_j x_j^2\right]$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_j \le C$$

$$x_j \ge 0 \quad \forall j = 1, 2, \dots n$$

$$L(x_1, \dots x_n, \lambda) = \sum_{j=1}^{n}\left[\alpha_j x_j - \frac{1}{2}\beta_j x_j^2\right] + \lambda\left[C - \sum_{j=1}^{n} x_j\right]$$

KKT
$$\begin{cases} \dfrac{\partial L}{\partial x_j} \le 0, \quad x_j \ge 0 \quad \forall j \\ \dfrac{\partial L}{\partial \lambda} \ge 0, \quad \lambda \ge 0 \end{cases}$$

其中
$$\begin{cases} \dfrac{\partial L}{\partial x_j} = \alpha_j - \beta_j x_j + \lambda \\ \dfrac{\partial L}{\partial \lambda} = C - \sum_{j=1}^{n} x_j \end{cases}$$

(2) 证 $C > \sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} \Rightarrow C > \sum_{j=1}^{n} x_j$

反证：若 $C = \sum_{j=1}^{n} x_j$，$\lambda \geq 0$

由 $\lambda \geq 0$ 且 $\alpha_j - \beta_j x_j + \lambda \leq 0$

$$\alpha_j - \beta_j x_j \leq 0$$

$$x_j \geq \frac{\alpha_j}{\beta_j} \quad (\beta_j > 0 ?)$$

$$\sum_{j=1}^{n} x_j = C \geq \sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} \quad (\text{没矛盾?}).$$

(ii) $\alpha_j > \dfrac{\sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} - C}{\sum_{j=1}^{n} \frac{1}{\beta_j}} \Rightarrow x_j > 0 \ \forall j$

仅证：若 $\exists x_j = 0$

$\alpha_j - \beta_j x_j + \lambda = \alpha_j + \lambda \leq 0 \quad \alpha_j \leq 0$

对j其他 $x_i > 0 \quad \alpha_i - \beta_i x_i + \lambda = 0$

由 $\alpha_j - \beta_j x_j + \lambda \leq 0$

$$x_j \geq \frac{\alpha_j}{\beta_j} + \lambda \frac{1}{\beta_j}$$

$$C \geq \sum_{j=1}^{n} x_j \geq \sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} + \lambda \sum_{j=1}^{n} \frac{1}{\beta_j}$$

$$-\lambda = \frac{\displaystyle\sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} - C}{\displaystyle\sum_{j=1}^{n} \frac{1}{\beta_j}}$$ (沒推出矛盾?)

(iii) $\begin{cases} x_k = 0 & \alpha_k + \lambda \leq 0 \\ x_j > 0 & \alpha_j - \beta_j x_j + \lambda = 0 \end{cases}$

$$\alpha_k + \beta_j x_j - \alpha_j \leq 0 \quad (\beta_j > 0 ?)$$

$$\therefore \alpha_k \leq \alpha_j$$

4.1

$$X_g = \phi_g(Z_{1g}, \dots Z_{Fg})$$

$$\sum_{g=1}^{G} Z_{fg} \le Z_f \quad \forall f$$

$$\max.\ W = W(U_1, U_2, \dots U_C)$$

$$\text{s.t.} \quad U_c = U^c(X_{c1}, \dots X_{cG})$$

$$X_{1g} + \dots + X_{Cg} = X_g = \phi_g(Z_{1g}, \dots Z_{Fg}) \quad \forall g$$

$$\sum_{g=1}^{G} Z_{fg} \le Z_f \quad \forall f$$

$$L = W(U^1, \dots U^C) + \sum_g \pi_g [\phi_g(Z_{1g}, \dots Z_{Fg}) - \sum_c X_{cg}]$$

$$+ \sum_f \lambda_f (Z_f - \sum_g Z_{fg})$$

T.O.C.

$$\begin{cases} \dfrac{\partial L}{\partial x_{cg}} = 0 & \forall c,\ \forall g \quad ① \\ \dfrac{\partial L}{\partial z_{fg}} = 0 & \forall f,\ \forall g \quad ② \\ \dfrac{\partial L}{\partial \pi_g} = 0 & \forall g \quad ③ \\ \dfrac{\partial L}{\partial \lambda_f} = 0 & \forall f \quad ④ \end{cases}$$

(1) ①式没变，仍为 $\dfrac{\partial w}{\partial u^c}\dfrac{\partial u^c}{\partial x_{cg}} - \pi_g x_{cg} = 0$

故最优分配的一阶条件没变.

(2) 增加了②式 $\pi_g \dfrac{\partial \phi_g}{\partial f_g} - \lambda_f = 0$ 最优要素配置的新条件.

(3) π_g：商品g的产量 X_g 增加1单位带来的社会福利的边际增加量

λ_f：要素 f 增加1单位所能带来的社会福利边际增加量

(4) 对于专业化生产企业（生产g，市场价 p_g，租金 μ_f）

$$\max_{z_{fg}} \; p_g \phi_g(z_{1g}, \ldots z_{Fg}) - \sum_f \mu_f z_{fg}$$

$$\text{F.O.C.} \quad p_g \frac{\partial \phi_g}{\partial z_{fg}} - \mu_f = 0 \quad \forall f$$

只需令 $p_g = \pi_g$ $\lambda_f = \mu_f$ 即可.

(5) $$\sum_c I_c = \sum_c (\pi_1 X_{c1} + \cdots + \pi_G X_{cG})$$

$$= \sum_g \pi_g (X_{1g} + \cdots + X_{Cg}) = \text{总产出价值}$$