习题 2.2

$\hat{U}(x,y) = \alpha \ln(x-x_0) + \beta \ln(y-y_0)$

s.t. $px+qy=I$

构建拉格朗日函数 $L(x,y,\lambda) = \alpha\ln(x-x_0)+\beta\ln(y-y_0)+\lambda[I-px-qy]$

$$\begin{cases} L_x = \dfrac{\partial L}{\partial x} = \dfrac{\alpha}{x-x_0} - p\lambda = 0 \\ L_y = \dfrac{\partial L}{\partial y} = \dfrac{\beta}{y-y_0} - q\lambda = 0 \\ L_\lambda = I - px - qy = 0 \end{cases}$$

$\therefore \lambda = \dfrac{\alpha}{p(x-x_0)} = \dfrac{\beta}{q(y-y_0)}$ $\quad \alpha qy - \alpha qy_0 = \beta px - \beta px_0$

$qy = I - px$ 代入 $\alpha(I-px) - \alpha qy_0 = \beta px - \beta px_0$ $\quad \alpha+\beta=1$

$\alpha I - \alpha qy_0 + \beta px_0 = \beta px + \alpha px = px$

$\therefore px = \alpha I - \alpha qy_0 + \beta px_0$

$px = I - qy$ 代入 $\alpha qy - \alpha qy_0 = \beta(I-qy) - \beta px_0$

$\alpha qy + \beta qy = \beta I - \beta px_0 + \alpha qy_0 = qy$

$\therefore qy = \beta I - \beta px_0 + \alpha qy_0$

习题 3.3

$\max \sum_{j=1}^{n} [\alpha_j x_j - \frac{1}{2} \beta_j x_j^2]$

(i) 假定 x 为 n 维向量

$L(x,\lambda) = \sum_{j=1}^{n} [\alpha_j x_j - \frac{1}{2}\beta_j x_j^2] + \lambda [c - \sum_{j=1}^{n} x_j]$

根据库恩-塔克定理，存在一个值 λ 满足

$L_{x_j}(\bar{x},\lambda) = \alpha_j - \beta_j x_j - \lambda \le 0 \quad x_j \geq 0$ 满足互补松弛条件

$L_\lambda(\bar{x},\lambda) = c - \sum_{j=1}^{n} x_j \geq 0 \quad \lambda \geq 0$ 满足互补松弛条件

假设在 $c > H$ 的条件下 $\lambda \neq 0$ 　则 $\lambda > 0$ 　$c = \sum_{j=1}^{n} x_j$

$c = \sum_{j=1}^{n} x_j > H = \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots$

① $x_j = 0 \quad \alpha_j - \beta_j x_j - \lambda < 0 \quad x_j = 0$

② $x_j > 0 \quad \alpha_j - \beta_j x_j = \lambda \quad x_j = \frac{\alpha_j - \lambda}{\beta_j}$

$\Big\}\ \sum_{j=1}^{n} x_j < \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots$

与 $c > H$ 矛盾.

∴ $c > H$ 的条件下 $\lambda \neq 0$ 不成立

∴ $\lambda = 0$

∴ $c - \sum_{j=1}^{n} x_j > 0$

∴ $\sum_{j=1}^{n} x_j < c$

总资金中有一部分未被使用

(ii) $\begin{cases} \alpha_j-\beta_j x_j-\lambda\le 0 & x_j\geqslant 0 \quad 满足互补松弛条件 \\ C-\sum_{j=1}^{n}x_j\geqslant 0 & \lambda\geqslant 0 \quad 满足互补松弛条件 \end{cases}$

① 当 $\lambda=0$ 时，则 $C>\sum_{j=1}^{n}x_j$

a、假设 $x_j>0$，则 $\alpha_j-\beta_j x_j-\lambda=0$ $\quad x_j=\frac{\alpha_j-\lambda}{\beta_j}=\frac{\alpha_j}{\beta_j}$ 项目j会得到资金

b、假设 $x_j=0$ $\quad x_j>\frac{\alpha_j-\lambda}{\beta_j}$ 则 $x_j>\frac{\alpha_j}{\beta_j}$

$\therefore \frac{\alpha_j}{\beta_j}<0$ 排除这种情况

② 当 $\lambda\ne 0$ 即 $\lambda>0$ 时，则 $C=\sum_{j=1}^{n}x_j$

a: 假设 $x_j>0$，则 $\alpha_j-\beta_j x_j-\lambda=0$ $\quad x_j=\frac{\alpha_j-\lambda}{\beta_j}$ 项目j会得到资金

b、假设 $x_j=0$ $\quad x_j>\frac{\alpha_j-\lambda}{\beta_j}$ $\quad \therefore \lambda>\alpha_j$

$\because \sum_{j=1}^{n}x_j=C$ $\quad x_j\geqslant\frac{\alpha_j-\lambda}{\beta_j}$

$\sum_{j=1}^{n}\frac{\alpha_j-\lambda}{\beta_j}=\sum_{j=1}^{n}\frac{\alpha_j}{\beta_j}-\lambda\sum_{j=1}^{n}\frac{1}{\beta_j}=H-\lambda K$

$\therefore C\geqslant H-\lambda K$ $\quad \lambda\geqslant\frac{H-C}{K}$

$\because a_j>(H-C)/K$

$\therefore a_j\geqslant\lambda$ 但与 $\lambda>a_j$ 互相矛盾

$\therefore x_j=0$ 不成立

$\therefore x_j>0$ 每个项目都会得到一些资金

(iii) $\alpha_j - \beta_j x_j - \lambda \leq 0$ $x_j \geq 0$ 满足互补松弛条件

当 $x_j = 0$ 时 $\alpha_j - \beta_j x_j - \lambda < 0$ $\alpha_j < \lambda$

即当项目未得到资金时，$\alpha_j < \lambda$

当 $x_j > 0$ 时 $\alpha_j - \beta_j x_j - \lambda = 0$ $\alpha_j = \lambda + \beta_j x_j > \lambda$

∴ 未得到资金的项目对应的 $\alpha$ 一定比任何获得资金的项目小

习题4.1

(1) $Z_f \geq \sum_{g=1}^{G} z_{fg}$

$X_g = \Phi(z_{1g}, z_{2g} \cdots z_{Fg}) \geq \sum_c x_{cg}$

构建拉格朗日函数

$$L = W(U^1(x_{11}, \cdots x_{1G}) \cdots U^c(x_{c1}, \cdots x_{cG})) + \sum_g \pi_g(\Phi^g(z_{1g} \cdots z_{Fg}) - \sum_c x_{cg})$$

$$+ \sum_f \pi_f(Z_f - \sum_{g=1}^{G} z_{fg})$$

$$\frac{\partial L}{\partial x_{cg}} = \frac{\partial W}{\partial U^c} \frac{\partial U^c}{\partial x_{cg}} - \pi_g = 0 \quad (1)$$

$$\frac{\partial L}{\partial z_{fg}} = \pi_g \cdot \frac{\partial \Phi^g}{\partial z_{fg}} - \pi_f = 0 \quad (2)$$

式(1)和原来的一阶条件一样，但增加了最优要素配置的新条件(2)

(2) $\pi_g = \frac{\partial W}{\partial U^c} \cdot \frac{\partial U^c}{\partial x_{cg}}$ $\pi_g$是商品g的影子价格，是边际社会价值

指消费者增加一单位商品g，增加的社会福利

$\pi_f = \pi_g \cdot \frac{\partial \Phi^g}{\partial z_{fg}}$ $\pi_f$是投入要素f的影子价格，是边际成本，

指多投入一单位生产要素f增加的社会福利

(3) 假设每个企业都只生产一种产品，即假设企业g只生产商品g

$$X_g = \Phi^g(z_{1g}, z_{2g} \cdots z_{Fg})$$

$$\sum_{g=1}^{G} z_{fg} \le Z_f$$

$$\frac{\partial L}{\partial z_{fg}} = \pi_g \frac{\partial \Phi^g}{\partial z_{fg}} - \pi_f = 0$$

在完全竞争市场条件下，企业是价格接受者

只生产一种商品，根据 $\pi_g$ 和 $\pi_f$ 也可达到最优配置

(4). 消费者总收入：$I_c = \sum_{f=1}^{F} \pi_f Z_f$ 　　总产出价值 $\sum_{g=1}^{G} \pi_g X_g$

$$\frac{\partial L}{\partial z_{fg}} = 0 \qquad \therefore \pi_f = \pi_g \frac{\partial \Phi^g}{\partial z_{gf}}$$

$$\sum_{f=1}^{F} \pi_f z_{fg} = \sum_{f=1}^{F} \left(\pi_g \frac{\partial \Phi^g}{\partial z_{gf}}\right) z_{fg}$$ 各要素支出之和

$$\pi_g X_g = \pi_g \cdot \Phi(z_{g1} \cdots z_{gf})$$ 总收入

总收入 = 总成本 　$\pi_g X_g = \sum_{f=1}^{F} \pi_f z_{fg}$

$$I_c = \sum_{f=1}^{F} \pi_f Z_f = \sum_{f=1}^{F} \pi_f \left(\sum_{g=1}^{G} z_{gf}\right) = \sum_{g=1}^{G} \underbrace{\sum_{f=1}^{F} \pi_f z_{gf}}_{\pi_g X_g}$$

$$\therefore I_c = \sum_{g=1}^{G} \pi_g X_g = 总产出价值.$$

∴ 分配给消费者的收入 $I_c$ 加起来正好等于总产出价值.