MATH381H - HW 5

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Section 2.5

Question 4

a) We create the following function ($f(n)$) to examine if there exists a one-to-one correspondence between the set of positive integers ($\mathbb{Z}^+$) and the set of integers not divisible by 3 (A), where $n \in \mathbb{Z}^+$.

$$f : \mathbb{Z}^+ \to A, f(n) = \begin{cases} \frac{3(n-1)}{4} + 1 & 4|(n-1) \\ \frac{3(n-2)}{4} + 2 & 4|(n-2) \\ -(\frac{3(n-3)}{4} + 1) & 4|(n-3) \\ -(3(\frac{n}{4} - 1) + 2) & 4|n \end{cases}$$

To test for one-to-one correspondence we first test if our function is one-to-one. We let $a, b \in \mathbb{Z}^+$ and WLOG we let a, b satisfy the conditions of the first equation of our function. We let $\frac{3(a-1)}{4} + 1 = \frac{3(b-1)}{4} + 1$, which implies that $a = b$. Notice that equation 1 of our function is one-to-one since $f(a) = f(b)$ implies that $a = b$, which also holds true for the other three equations of our function. Thus, the function is one-to-one. Now we look to prove that we have a surjective function. We let some $y = f(n)$ and WLOG we use the first equation of our function again.

$$y = \frac{3(n-1)}{4} + 1$$
$$n = \frac{4(y-1)}{3} + 1$$

We see that $n \in \mathbb{Z}^+$ and satisfies the condition given with the first equation of our function. Thus, we can say that the first equation is surjective, and we can use the same proof to show that all equations of our function are surjective. Therefore, our function is also a surjective function. Given that our function is both injective and surjective, we can say that we have created a bijective function or a function with one-to-one correspondence from the set of positive integers ($\mathbb{Z}^+$) to the set of integers not divisible by 3 (A). Thus, the set of integers not divisible by 3 can be said to be infinitely countable.

b) For b) we set up a system of recursion relations, which allow for the one-to-one correspondence to seen:

$$f : \mathbb{Z}^+ \to A$$
$$a_1 = 5, a_2 = -5$$
$$a_3 = 10, a_4 = -10$$
$$a_5 = 15, a_6 = -15$$
$$a_7 = 20, a_8 = -20$$
$$a_9 = 25, a_{10} = -25$$
$$a_{11} = 30, a_{12} = -30$$
$$a_n = \begin{cases} a_{n-12} + 35 & n \pmod 2 = 1 \\ -a_{n-1} & n \pmod 2 = 0 \end{cases}$$

Where $n \in \mathbb{Z}^+$ as given by the recursion relation. This "function" is then one-to-one because each n value is only assigned one image. Furthermore, we cover all integers in the set we are interested in because it can be simplified to all integers divisible by 5 but not 35. Thus, we include all multiples of 5, but not those that are multiples of 35. Since our recursion relation displays a one-to-one correspondence between the set of positive integers and the set of integers divisible by 5 but not by 7, we have shown that the latter set is infinitely countable.

c) Each number in set A, which represents the set of real numbers with decimal representations consisting of all 1's, should be followed immediately by the negative number with the same value. Thus, we start by adding all real numbers and their negative counterparts of the form $1 + \frac{1^{a-1}}{10^a}$, where $a \in \mathbb{Z}^+$. The numbers $\{1, -1, 1.1, -1.1, 1.11, -1.11...\}$ will then be added. We next add the rational real number $\frac{10}{9}$ or 1 with infinitely repeating 1's. Next, we do the same with $10 + \frac{1^{a-1}}{10^a}$ and $\frac{100}{9}$, $100 + \frac{1^{a-1}}{10^a}$ and $\frac{1000}{9}$, and so on and so forth. We see that it is then possible to orderly list all the values in this set. Thus, this set is countably infinite.

d) We see that this set is not countable. Take 9 with infinite repeating 9's for example. This number cannot be written as a fraction or as the number 10 in this set (since 10 does not exist in this set), and is therefore an irrational number in the set. Thus, this set contains an infinite number of irrational numbers, and we know that the set of irrational numbers is not countable. Therefore, if this set is made up of the union of the countable set of rational real numbers (consisting of only 1's and 9's) and the uncountable set of irrational real numbers (consisting of only 1's and 9's) then this set must be uncountable by cardinality of union of sets rules.

Question 10

1. We let A be the set of all real numbers, or simply equal to the set $\mathbb{R}$. We let B be the set of all real numbers not including 0, or simply equal to the set $\mathbb{R} - \{0\}$. This gives

$$A - B = (\mathbb{R}) - (\mathbb{R} - \{0\}) = \{0\}$$

In this case, $A - B$ is equal to the finite set $\{0\}$.

2. We let A be the set of all real numbers, or simply equal to the set $\mathbb{R}$. We let B be the set of all irrational numbers, or simply equal to the set $\mathbb{I}$. This gives

$$A - B = (\mathbb{R}) - (\mathbb{I}) = \mathbb{Q}$$

In this case, $A - B$ is equal to the infinitely countable set of rational numbers, $\mathbb{Q}$.

3. We let A be the set of all real numbers, or simply equal to the set $\mathbb{R}$. We let B be the set of all positive real numbers, or simply equal to the set $\mathbb{R}^+$. This gives

$$A - B = (\mathbb{R}) - (\mathbb{R}^+) = \mathbb{R}^- \cup \{0\}$$

In this case, $A - B$ is equal to the set of uncountable numbers described by $\mathbb{R}^- \cup \{0\}$.

Question 16

We let $B \subseteq A$, where A is a countable set. It follows by definition of countable sets that there exists a one-to-one correspondence between A and $\mathbb{Z}^+$. Furthermore we see that $\forall b \in B \implies b \in A$ by definition of a subset. Thus, if all elements of B exist in A, then there exists a one-to-one correspondence from B to $\mathbb{Z}^+$ as well. We can also say that given $B \subseteq A$, where A is a countable set, $|B| \leq |A|$ by definition of subset. Given $|B| \leq |A|$, there must exist a one-to-one function from B to A, which implies that there exists a one-to-one function from B to $\mathbb{Z}^+$ (since one exists for set A).

Question 18

Given that $|A| = |B|$ we can say that there exists a one-to-one and onto function $f : A \to B$. We now define a new function $g : P(A) \to P(B)$, where $g(C) = \{f(a), a \in C\}$ and $C \in P(A)$. Since $b = f(a) \in B$, it can be seen that $g(C) \subseteq B$, which gives that $g(C) \in P(B)$ by definition of subset and power set. Now we prove that g is both one-to-one and onto, to show equal cardinality. We assume that $g(C) = g(D)$, which gives that $\{f(a), a \in C\} = \{f(b), b \in D\}$. For this to be true:

$$|\{f(a), a \in C\}| = |\{f(b), b \in D\}|$$

and $\forall a \in C$ there must exist a corresponding $b \in D$ such that $f(a) = f(b)$, which implies that $a = b$ by definition of one-to-one function (f). Since all values in C are also in D and vice versa, we can say that $C = D$. Thus, since $g(C) = g(D) \implies C = D$, g is one-to-one. Furthermore, since the function f is onto, this implies that for every $c \in A$, there exists a $f(c) = d$. Thus, for all values in D, there will exist a pre-image in C, which can be written as $g(C) = D$. This shows that g is onto. Since g is both one-to-one and onto, we showed that $|P(A)| = |P(B)|$ with the defined function g.

Question 22

By definition of an onto function, we know that $|B| \leq |A|$ for there to exist such a function assuming A is finite. The reason is because every $b \in B$ has at least one pre-image $a \in A$. Thus, if A is finite, then it is countable and B is also finite, showing that B is also countable. Now assume A to be countably infinite. We let the function $g : \mathbb{Z}^+ \to A$ since A is countably infinite we know there exists an onto function, g (that is also one-to-one). Since f is onto as given by the problem we can say that $f : A \to B$. By definition of function composition and function composition properties: $f \circ g : \mathbb{Z}^+ \to B$, thus B is countably infinite by definition of infinitely countable sets.

Question 28

Given that $\mathbb{Z}^+$ is countably infinite, we can write the Cartesian Product of $\mathbb{Z}^+$ in the form of an infinite counting table:

$$\begin{matrix} (0,0) & (0,1) & (0,2) & \cdots \\ (1,0) & (1,1) & (1,2) & \cdots \\ (2,0) & (2,1) & (2,2) & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

We can count this infinite table as (0,0), (0,1), (1,0) and so on, with each element of the table having an index value in $\mathbb{Z}^+$. Therefore, there exists a one-to-one correspondence between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Z}^+$.

Question 30

We list out the tuple (0,0,0) first and begin with tuples in the form (a,b,c) increasing in value with positive and negative integers. We know that a set containing these tuples is countable $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, which can be proved in a similar fashion as done in Exercise 28. Furthermore, we know that each of these tuples will contain 2 real roots. Thus, we can write a new set in the same order of the old set containing the tuples, just with the root expansion. Thus, we have written a countable set containing the real roots to the quadratic equation $ax^2 + bx + c = 0$ for integers a,b, and c.

Question 34

a) We show that f(x) is both injective and surjective to show that it is bijective.
INJECTIVE We let $a, b \in (0, 1)$. We let f(a) = f(b):

$$\frac{2a-1}{2a(1-a)} = \frac{2b-1}{2b(1-b)}$$
$$(b-a)(8ab - 2b - 2a + 4) = 0$$

This gives that $a = b$ or $a = \frac{b-2}{4b-1}$, but we see that when $b > 0.25$, $a < 0$, which contradicts our initial assumption that $a \in (0, 1)$, thus $a = b$. We showed that $f(a) = f(b) \implies a = b$, which means our function is injective. SURJECTIVE We find the inverse of function f. We can write:

$$y = \frac{2x-1}{2x-2x^2}$$
$$2yx^2 + (2-2y)x - 1 = 0$$
$$\delta = 4 - 8y + 4y^2 - 4(2y)(-1) = 4y^2 + 4 > 0$$
$$x = \frac{2 - 2y \pm \sqrt{\delta}}{4y}$$

Thus, since there exists an inverse to our equation (for $y \neq 0$) our function is surjective. Since the function is both surjective and injective, it follows that it must be a bijective function.

b) We can say that $|(0,1)| \leq |\mathbb{R}|$ by definition, since there exists a one-to-one function from (0,1) to $\mathbb{R}$ as shown in part a). We know that the given function is bijective, which means the inverse of the given function will also be one-to-one. Thus, by definition since there exists a one-to-one function from $\mathbb{R}$ to $(0,1)$, we can say that $|\mathbb{R}| \leq |(0,1)|$ by definition. Using the Schroder-Bernstein theorem we can then say that $|(0,1)| = |\mathbb{R}|$ since $|(0,1)| \leq |\mathbb{R}|$ and $|\mathbb{R}| \leq |(0,1)|$.

Question 38

Let A be the set of functions $f : \mathbb{Z}^+ \to \{0,1,2,3,4,5,6,7,8,9\}$. We then let $g : (0,1) \to A$ such that $g(x) = f$ where $f(n) = d_n$ for $0.d_1d_2d_3...d_n$. We now show that g is injective. Let $a, b \in (0,1)$, and assume that $g(a) = g(b) \implies f_a = f_b$. f_a and f_b must have the same number of digits after the decimal point, by the function definition. It follows then that a and b must be the same number for this to happen. Thus, $g(a) = g(b)$ implies that $a = b$, which means that g is injective. g is onto because every image $f \in A$ is created by a specific value x such that $x = 0.f(1)f(2)...f(n)$, thus g is surjective. Since g is both injective and surjective, we can say that there exists a one-to-one correspondence between $(0,1)$ and A. Furthermore, we know that $(0,1)$ is uncountable, as shown by answers to Exercise 34, which show equal cardinality between $(0,1)$ and uncountable set $\mathbb{R}$. It follows that A must also be uncountable for there to exist a one-to-one correspondence with $(0,1)$.

Question 40

We begin with a function $f : S \to P(S)$. Suppose T is a subset of S such that

$$T = \{s \in S | s \notin f(s)\}$$

We want to show that $T \notin f(S)$. We let $T = f(s)$. Thus, if $s \in T$, it follows by definition of T that $s \notin f(s) = T$. We have a contradiction so $T \neq f(s)$ for all s. It follows that f cannot be onto.

Section 4.1

Question 6

Given $a|c$ and $b|d$, we let $c = an$ and $d = bm$ where $n, m \in \mathbb{Z}$. Thus, $ab|cd$ simply becomes $ab|anbm$ and thus $\frac{abnm}{ab} = nm$. We know nm is an integer since both n and m are. Thus, $ab|cd$ given $a|c$ and $b|d$

Question 10

Given that $a|b$, we can let $b = an$ for $n \in \mathbb{Z}$. We let $a + b = a + an = (n+1)a$ equal the odd number $2k + 1$ for some $k \in \mathbb{Z}$. It follows that both $n + 1$ and a must be odd, since their product is the odd number $2k + 1$. The only way to multiply and get an odd number is for both factors to be odd. Thus, the integer a must be odd.

Question 14

a) 44 divided by 8
Quotient: 5
Remainder: 4

b) 777 divided by 21
Quotient: 37
Remainder: 0

c) -123 divided by 19
Quotient: -6
Remainder: -9

d) -1 divided by 23
Quotient: 0
Remainder: -1

e) -2002 divided by 87
Quotient: -23
Remainder: -1

f) 0 divided by 17
Quotient: 0
Remainder: 0

g) 1234567 divided by 1001
Quotient: 1233
Remainder: 334

h) -100 divided by 101
Quotient: 0
Remainder: -100

Question 20

This is false, we provide the counterexample where $a = 4$, $b = 5$, and $d = 9$. Thus, this gives

$$(a+b) \textbf{ div } d = a \textbf{ div } d + b \textbf{ div } d$$
$$(4+5) \textbf{ div } 9 = 4 \textbf{ div } 9 + 5 \textbf{ div } 9$$
$$1 = 0 + 01 \neq 0$$

Thus, our counterexample proves that this is a false proposition.

Question 24

We let $a = dq + r$ for $a, d, q, r \in \mathbb{Z}$. Simplifying

$$\begin{aligned} q &= \frac{a-r}{d},\ r < d \rightarrow \frac{r}{d} < 1 \text{ by definition of remainder} \\ q + \frac{r}{d} &= \frac{a}{d} \\ \lfloor q + \frac{r}{d} \rfloor &= \lfloor \frac{a}{d} \rfloor \\ q &= \lfloor \frac{a}{d} \rfloor \end{aligned}$$

We can rewrite out initial equation as $r = a - dq$ and plugging in $q = \lfloor \frac{a}{d} \rfloor$ we get that $r = a - d\lfloor \frac{a}{d} \rfloor$.

Question 30

a) $a = -3$, since $43 - 2(23) = -3$

b) $a = -12$ since $17 - 29 = -12$

c) $a = 94$ since $-11 + 5(21) = 94$

Question 38

a)

$$\begin{aligned} &(19^2 \textbf{ mod } 41) \textbf{ mod } 9 \\ &= 33 \textbf{ mod } 9 \\ &= 6 \end{aligned}$$

b)

$$\begin{aligned} &(32^3 \textbf{ mod } 13)^2 \textbf{ mod } 11 \\ &= 64 \textbf{ mod } 11 \\ &= 9 \end{aligned}$$

c)

$$\begin{aligned} &(7^3 \textbf{ mod } 23)^2 \textbf{ mod } 31 \\ &= 441 \textbf{ mod } 31 \\ &= 7 \end{aligned}$$

d)

$$\begin{aligned} &(21^2 \textbf{ mod } 15)^3 \textbf{ mod } 22 \\ &= 216 \textbf{ mod } 22 \\ &= 18 \end{aligned}$$