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Fourier regularization for a backward heat equation ☆

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Abstract

In this paper a simple and convenient new regularization method for solving backward heat equation—Fourier regularization method is given. Meanwhile, some quite sharp error estimates between the approximate solution and exact solution are provided. A numerical example also shows that the method works effectively.

Keywords: Backward heat equation; Ill-posed problem; Fourier regularization; Error estimate

1. Introduction

The backward heat conduction problem (BHCP) is also referred to as the final value problem [1]. The BHCP is a typical ill-posed problem [2,3]. In general no solution which satisfies the heat conduction equation with final data and the boundary conditions exists. Even if a solution exists, it will not be continuously dependent on the final data such that the numerical simulations are very difficult and some special regularization methods are required. In the context of approximation method for this problem, many approaches have been investigated. Such authors as Lattes and Lions [4], Showalter [5], Ames et al. [6], Miller [7] have approximated the BHCP by quasi-reversibility method. Tautenhahn and Schröter established an optimal error estimate

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for a special BHCP [8], Seidman established an optimal filtering method [9]. Mera et al. [10] and Jourhmane and Mera [11] used many numerical methods with regularization techniques to approximate the problem. A mollification method has been studied by Hào [12]. Recently, Liu used a group preserving scheme to solve the backward heat equation numerically [13]. Kirkup and Wadsworth used an operator-splitting method [14].

To the authors' knowledge, so far there are many papers on the backward heat equation, but theoretically the error estimates of most regularization methods in the literature are Hölder type, i.e., the approximate solution v and the exact solution u satisfy $\|u(\cdot,t)-v(\cdot,t)\| \leqslant 2E^{1-\frac{t}{T}}\delta^{\frac{t}{T}}$, where E is an a priori bound on $u(x,0)$, and δ is the noise level on final data $u(x,T)$. In this paper we consider the following one-dimensional backward heat equation in an unbounded region [12],

$$\begin{cases} u_t = u_{xx}, & -\infty < x < \infty,\ 0 \leqslant t < T, \\ u(x,T) = \varphi_T(x), & -\infty < x < \infty, \end{cases} \tag{1.1}$$

where we want to determine the temperature distribution $u(\cdot,t)$ for $0 \leqslant t < T$ from the data $\varphi_T(x)$. The major object of this paper is to provide a quite simple and convenient new regularization method—Fourier regularization method. Meanwhile, we overstep the Hölder continuity, some more faster convergence error estimates are given. Especially, the convergence of the approximate solution at $t=0$ is also proved. This is an improvement of known results [9].

Let $\hat{g}(\xi)$ denote the Fourier transform of $g(x) \in L(\mathbb{R})$ and define it by

$$\hat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x)\,dx, \tag{1.2}$$

and let $\|g\|_{H^s}$ denote the norm on the Sobolev space $H^s(\mathbb{R})$ and define it by

$$\|g\|_{H^s} := \left(\int_{-\infty}^{\infty} \left|\hat{g}(\xi)\right|^2 \left(1+\xi^2\right)^s d\xi \right)^{\frac{1}{2}}. \tag{1.3}$$

When $s=0$, $\|\cdot\|_{H^0} := \|\cdot\|$ denotes the $L^2(\mathbb{R})$-norm.

As a solution of problem (1.1) we understand a function $u(x,t)$ satisfying (1.1) in the classical sense and for every fixed $t \in [0,T]$, the function $u(\cdot,t) \in L^2(\mathbb{R})$. In this class of functions, if the solution of problem (1.1) exists, then it must be unique [15]. We assume $u(x,t)$ is the unique solution of (1.1). Using the Fourier transform technique to problem (1.1) with respect to the variable x, we can get the Fourier transform $\hat{u}(\xi,t)$ of the exact solution $u(x,t)$ of problem (1.1):

$$\hat{u}(\xi,t) = e^{\xi^2(T-t)}\hat{\varphi}_T(\xi), \tag{1.4}$$

or equivalently,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi^2(T-t)} \hat{\varphi}_T(\xi)\,d\xi. \tag{1.5}$$

Moreover, there holds

$$\hat{u}(\xi,0) = e^{\xi^2 T}\hat{\varphi}_T(\xi). \tag{1.6}$$

Denoting $\varphi_0(x) := u(x,0)$ and as usual, when we consider problem (1.1) in $L^2(\mathbb{R})$ for the variable x, we always assume there exists an a priori bound for $\varphi_0(x)$:

$$\|\varphi_0\| = \left\|u(\cdot,0)\right\| \leqslant E. \tag{1.7}$$

Due to (1.7), (1.6) and Parseval identity, we know

$$\|\varphi_0\|^2 = \int_{-\infty}^{\infty} \left|e^{\xi^2 T}\hat{\varphi}_T(\xi)\right|^2 d\xi < \infty. \tag{1.8}$$

Note that $e^{\xi^2 T} \to \infty$ as $|\xi| \to \infty$, (1.7) implies a rapid decay of $\hat{\varphi}_T(\xi)$ at high frequencies. But, in practice, the data at $t = T$ is often obtained on the basis of reading of physical instruments which is denoted by $\varphi_T^\delta(x)$. In such cases we cannot assume that it is given with absolute accuracy. Therefore such a decay of exact data is not likely to occur in the Fourier transform of the measured noisy data $\varphi_T^\delta(x)$. As measured data $\varphi_T^\delta(x)$ at $t = T$, its Fourier transform $\hat{\varphi}_T^\delta(\xi)$ is merely in $L^2(\mathbb{R})$ or $H^s(\mathbb{R})$, a small disturb in the data $\varphi_T(x)$ may cause a dramatically large errors in the solution $u(x,t)$ for $0 \leqslant t < T$. It is obvious that the severely ill-posedness of problem (1.1) is caused by disturb of high frequencies and many authors hope to recover the stability of problem (1.1) by filtering the high frequencies with suitable method. This idea has appeared earlier on in [16] and the author showed that in a subspace of $L^2(\mathbb{R})$ consisting of functions whose Fourier transforms have compact support, the backward heat equation (1.1) leads to a well-posed problem in the sense of Hadamard. He also provided a stable and convergent iteration scheme. Unfortunately, the result obtained in [16] is only a conditional stability and it cannot deal with any noise data. The essence of Fourier regularization method is just to eliminate all high frequencies from the solution, and instead consider (1.5) only for $|\xi| < \xi_{\max}$, where $\xi_{\max}$ is an appropriate positive constant. Recently, Fourier regularization method has been effectively applied to solve the sideways heat equation [17,18], a more general sideways parabolic equation [19] and numerical differentives [20]. This regularization method is rather simple and convenient for dealing with some ill-posed problems. However, as far as we know, there are not any results of Fourier method for treating backward heat equation until now. The present paper is devoted to establishing such a method for problem (1.1).

2. Fourier regularization and error estimates

As in the previous section, let $\varphi_T(x)$ and $\varphi_T^\delta(x)$ denote the exact and measured data at $t = T$, respectively, which satisfy

$$\left\|\varphi_T - \varphi_T^\delta\right\| \leqslant \delta. \tag{2.1}$$

For some $s \geqslant 0$ we assume there exists an a priori bound

$$\|\varphi_0\|_{H^s} \leqslant E. \tag{2.2}$$

We define a regularization approximate solution of problem (1.1) for noisy data $\varphi_T^\delta(x)$ which is called the Fourier regularized solution of problem (1.1) as follows:

$$u_{\delta,\xi_{\max}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi^2(T-t)} \hat{\varphi}_T^\delta(\xi)\chi_{\max}\, d\xi, \tag{2.3}$$

where $\chi_{\max}$ is the characteristic function of interval $[-\xi_{\max}, \xi_{\max}]$ and $\xi_{\max}$ is a constant which will be selected appropriately as regularization parameter.

The main conclusion of this paper is:

Theorem 2.1. *Let $u(x,t)$ given by* (1.5) *and $u_{\delta,\xi_{\max}}(x,t)$ given by* (2.3) *be the exact solution and Fourier regularization solution of problem* (1.1)*, respectively, on the interval $t\in[0,T]$. Suppose conditions* (2.1) *and* (2.2) *hold. Then if we select*

$$\xi_{\max}=\left(\ln\left(\left(\frac{E}{\delta}\right)^{\frac{1}{T}}\left(\ln\frac{E}{\delta}\right)^{-\frac{s}{2T}}\right)\right)^{\frac{1}{2}}, \tag{2.4}$$

there holds the following logarithmic stability estimate:

$$\begin{aligned}&\left\|u(\cdot,t)-u_{\delta,\xi_{\max}}(\cdot,t)\right\|\\&\quad\leqslant E^{1-\frac{t}{T}}\delta^{\frac{t}{T}}\left(\ln\frac{E}{\delta}\right)^{-\frac{(T-t)s}{2T}}\left(1+\left(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln(\ln\frac{E}{\delta})^{-\frac{s}{2T}}}\right)^{\frac{s}{2}}\right).\end{aligned} \tag{2.5}$$

Proof. Due to Parseval formula and (1.5), (2.3), (1.6), (2.1), (2.2), we know

$$\begin{aligned}
&\left\|u(\cdot,t)-u_{\delta,\xi_{\max}}(\cdot,t)\right\|\\
&\quad=\left\|\hat u(\cdot,t)-\hat u_{\delta,\xi_{\max}}(\cdot,t)\right\|\\
&\quad=\left\|e^{\xi^2(T-t)}\hat\varphi_T(\xi)-e^{\xi^2(T-t)}\hat\varphi_T^{\delta}\chi_{\max}\right\|\\
&\quad\leqslant\left\|e^{\xi^2(T-t)}\hat\varphi_T(\xi)-e^{\xi^2(T-t)}\hat\varphi_T\chi_{\max}\right\|+\left\|e^{\xi^2(T-t)}\hat\varphi_T(\xi)\chi_{\max}-e^{\xi^2(T-t)}\hat\varphi_T^{\delta}\chi_{\max}\right\|\\
&\quad=\left(\int\limits_{|\xi|>\xi_{\max}}\left|e^{\xi^2(T-t)}\hat\varphi_T(\xi)\right|^2d\xi\right)^{\frac12}+\left(\int\limits_{|\xi|\leqslant\xi_{\max}}\left|e^{\xi^2(T-t)}\left(\hat\varphi_T^{\delta}(\xi)-\hat\varphi_T(\xi)\right)\right|^2d\xi\right)^{\frac12}\\
&\quad=\left(\int\limits_{|\xi|>\xi_{\max}}\left|e^{\xi^2(T-t)}e^{-\xi^2T}\hat\varphi_0(\xi)\right|^2d\xi\right)^{\frac12}+\left(\int\limits_{|\xi|\leqslant\xi_{\max}}\left|e^{\xi^2(T-t)}\left(\hat\varphi_T^{\delta}(\xi)-\hat\varphi_T(\xi)\right)\right|^2d\xi\right)^{\frac12}\\
&\quad=\left(\int\limits_{|\xi|>\xi_{\max}}\left|e^{-t\xi^2}\hat\varphi_0(\xi)\right|^2d\xi\right)^{\frac12}+\left(\int\limits_{|\xi|\leqslant\xi_{\max}}\left|e^{\xi^2(T-t)}\left(\hat\varphi_T^{\delta}(\xi)-\hat\varphi_T(\xi)\right)\right|^2d\xi\right)^{\frac12}\\
&\quad\leqslant\sup_{|\xi|>\xi_{\max}}\frac{e^{-t\xi^2}}{(1+\xi^2)^{\frac{s}{2}}}\left(\int\limits_{|\xi|>\xi_{\max}}\left|\hat\varphi_0(\xi)\right|^2\left(1+\xi^2\right)^s d\xi\right)^{\frac12}\\
&\qquad+\sup_{|\xi|\leqslant\xi_{\max}}e^{\xi^2(T-t)}\left(\int\limits_{|\xi|\leqslant\xi_{\max}}\left|\hat\varphi_T^{\delta}(\xi)-\hat\varphi_T(\xi)\right|^2d\xi\right)^{\frac12}\\
&\quad\leqslant\sup_{|\xi|>\xi_{\max}}\frac{e^{-t\xi^2}}{|\xi|^s}E+\sup_{|\xi|\leqslant\xi_{\max}}e^{\xi^2(T-t)}\delta\\
&\quad\leqslant\frac{e^{-t\ln((\frac{E}{\delta})^{\frac1T}(\ln\frac{E}{\delta})^{-\frac{s}{2T}})}}{(\ln((\frac{E}{\delta})^{\frac1T}(\ln\frac{E}{\delta})^{-\frac{s}{2T}}))^{\frac{s}{2}}}E+e^{(T-t)\ln((\frac{E}{\delta})^{\frac1T}(\ln\frac{E}{\delta})^{-\frac{s}{2T}})}\delta\\
&\quad=\left(\frac{E}{\delta}\right)^{-\frac{t}{T}}\left(\ln\frac{E}{\delta}\right)^{\frac{st}{2T}}E\left(\frac{1}{\frac1T\ln\frac{E}{\delta}+\ln(\ln\frac{E}{\delta})^{-\frac{s}{2T}}}\right)^{\frac{s}{2}}+\left(\frac{E}{\delta}\right)^{\frac{T-t}{T}}\left(\ln\frac{E}{\delta}\right)^{-\frac{(T-t)s}{2T}}\delta
\end{aligned}$$

$$= \left(\frac{E}{\delta}\right)^{-\frac{t}{T}}\left(\ln\frac{E}{\delta}\right)^{\frac{st}{2T}} E\left(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln(\ln\frac{E}{\delta})^{-\frac{s}{2T}}}\right)^{\frac{s}{2}}\left(\ln\frac{E}{\delta}\right)^{-\frac{s}{2}}$$
$$+ E^{1-\frac{t}{T}}\delta^{\frac{t}{T}}\left(\ln\frac{E}{\delta}\right)^{-\frac{(T-t)s}{2T}}$$
$$= E^{1-\frac{t}{T}}\delta^{\frac{t}{T}}\left(\ln\frac{E}{\delta}\right)^{-\frac{(T-t)s}{2T}}\left(\left(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln(\ln\frac{E}{\delta})^{-\frac{s}{2T}}}\right)^{\frac{s}{2}}+1\right).$$

Note that $\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln(\ln\frac{E}{\delta})^{-\frac{s}{2T}}}$ is bounded for $\delta \to 0$, Therefore, the proof of estimate (2.5) is completed. □

Remark 2.1. When $s = 0$, the estimate (2.5) becomes

$$\left\|u(\cdot,t) - u_{\delta,\xi_{\max}}(\cdot,t)\right\| \leqslant 2E^{1-\frac{t}{T}}\delta^{\frac{t}{T}}. \tag{2.6}$$

From [8], we know this is an order optimal stability estimate in $L^2(\mathbb{R})$. However, from (2.6) we know when $t \to 0^+$, the accuracy of the regularized solution becomes progressively lower. At $t = 0$, it merely implies that the error is bounded by $2E$, i.e., the convergence of the regularization solution at $t = 0$ is not obtained theoretically. This defect is remedied by (2.5). In fact, for $t = 0$, (2.5) becomes

$$\left\|u(\cdot,0) - u_{\delta,\xi_{\max}}(\cdot,0)\right\| \leqslant E\left(\ln\frac{E}{\delta}\right)^{-\frac{s}{2}}\left(1+\left(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln(\ln\frac{E}{\delta})^{-\frac{s}{2T}}}\right)^{\frac{s}{2}}\right) \to 0$$

as $\delta \to 0^+$ and $s > 0$.

Moreover, comparing (2.5) with the result obtained in [12] we know estimate (2.5) is sharp and the best known estimate.

Remark 2.2. Note that $\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln(\ln\frac{E}{\delta})^{-\frac{s}{2T}}} \to T$, when $\delta \to 0^+$. Hence estimate (2.5) also can be rewritten as

$$\left\|u(\cdot,t) - u_{\delta,\xi_{\max}}(\cdot,t)\right\| \leqslant E^{1-\frac{t}{T}}\delta^{\frac{t}{T}}\left(\ln\frac{E}{\delta}\right)^{-\frac{(T-t)s}{2T}}\left(1+T^{\frac{s}{2}}+o(1)\right), \quad \text{for } \delta \to 0. \tag{2.7}$$

It is easy to see that the accuracy of the estimate increases with deceasing of T. This accords with the computation practice by another method given in [10].

Remark 2.3. In general, the a priori bound E is often unknown exactly in practice, therefore we do not have the exact a priori bound E. However, if we select

$$\xi^*_{\max} = \left(\ln\left(\left(\frac{1}{\delta}\right)^{\frac{1}{T}}\left(\ln\frac{1}{\delta}\right)^{-\frac{s}{2T}}\right)\right)^{\frac{1}{2}}, \tag{2.8}$$

we also have the estimate

$$\left\|u(\cdot,t) - u_{\delta,\xi^*_{\max}}(\cdot,t)\right\| \leqslant \delta^{\frac{t}{T}}\left(\ln\frac{1}{\delta}\right)^{-\frac{(T-t)s}{2T}}\left(1+E\left(\frac{\ln\frac{1}{\delta}}{\frac{1}{T}\ln\frac{1}{\delta}+\ln(\ln\frac{1}{\delta})^{-\frac{s}{2T}}}\right)^{\frac{s}{2}}\right), \tag{2.9}$$

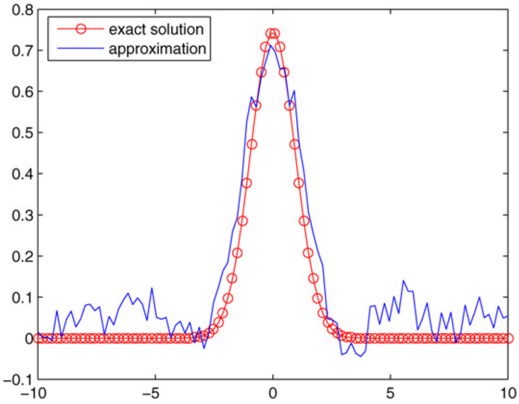

Fig. 1. $T = 1.00$, $t = 0.20$, $\xi_{\max} = 1.5396$.

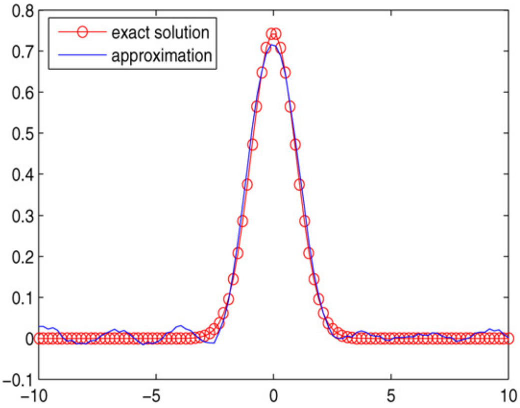

Fig. 2. $T = 1.00$, $t = 0.20$, $\xi_{\max} = 2.0883$.

or equivalently

$$\left\| u(\cdot,t) - u_{\delta,\xi^*_{\max}}(\cdot,t) \right\| \leqslant \delta^{\frac{t}{T}} \left(\ln \frac{1}{\delta} \right)^{-\frac{(T-t)s}{2T}} \left(1 + ET^{\frac{s}{2}} + o(1)\right), \quad \text{for } \delta \to 0, \tag{2.10}$$

where E is only a bounded positive constant and it is not necessary to know exactly in (2.9) and (2.10). This choice is helpful in our realistic computation.

3. A numerical example

It is easy to verify that the function

$$u(x,t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}} \tag{3.1}$$

is the unique solution of the initial problem

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = e^{-x^2}, & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

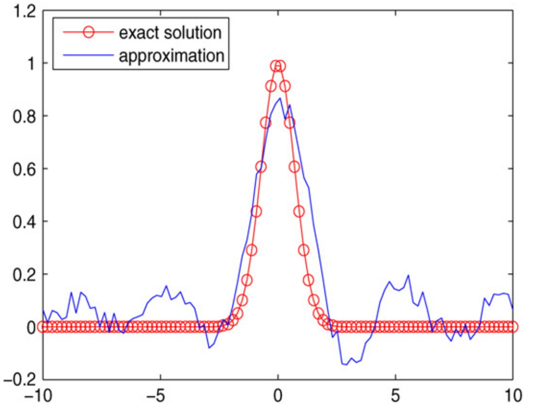

Fig. 3. $T = 1.00$, $t = 0$, $\xi_{\max} = 1.5206$.

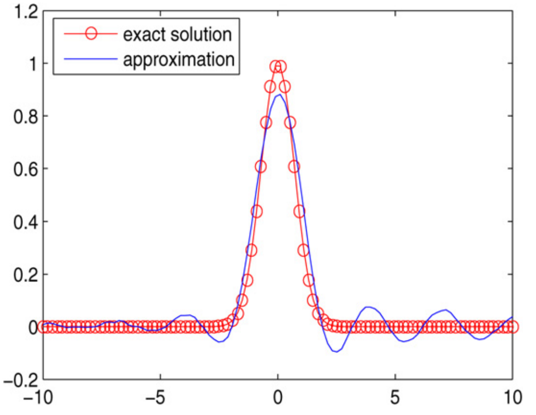

Fig. 4. $T = 1.00$, $t = 0$, $\xi_{\max} = 2.0705$.

Hence, $u(x,t)$ given by (3.1) is also the solution of the following backward heat equation for $0 \leqslant t < T$:

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ 0 \leqslant t < T, \\ u|_{t=T} = \frac{1}{\sqrt{1+4T}} e^{-\frac{x^2}{1+4T}}, & x \in \mathbb{R}. \end{cases} \tag{3.3}$$

Now we will focus on our numerical experiment to verify the theoretical results.

When the input data contain noises, we use the *rand* function given in Matlab to generate the noisy data

$$\left(\varphi_T^\delta\right)_i = (\varphi_T)_i + \epsilon\, rand(\varphi_T)_i$$

where $(\varphi_T)_i$ is the exact data and $rand(\varphi_T)_i$ is a random number in $[-1, 1]$, (φ_T) denotes the vector whose elements are $(\varphi_T)_i$ $(i = 1, 2, \dots, N_x)$, the magnitude ϵ indicates the noise level of the measurement data.

The following tests are done in the interval $x \in [-10, 10]$.

The regularization parameter $\xi_{\max}$ is computed by formula (2.8). Figures 1, 2 are based on $T = 1.00$, $t = 0.20$ with different noise levels. The noise levels are 6×10^{-2} and 6×10^{-3},

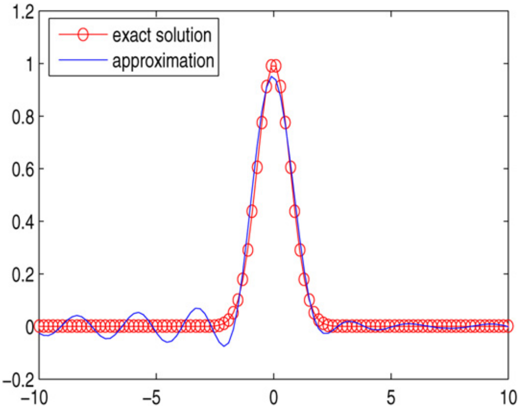

Fig. 5. $T = 1.00$, $t = 0$, $\xi_{max} = 2.5437$.

respectively. Figures 3–5 are based on $T = 1.00$, $t = 0$ with different noise levels 6×10^{-2}, 6×10^{-3}, and 6×10^{-4}, respectively.

From these figures, we can conclude that the regularization parameter rule given by (2.8) is valid and the numerical solution is stable at $t = 0$. This accords with our theoretical results.

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References

[1] A. Carasso, Error bounds in the final value problem for the heat equation, SIAM J. Math. Anal. 7 (1976) 195–199.
[2] M.M. Lavrentév, V.G. Romanov, S.P. Shishat'skiĭ, Ill-posed Problems of Mathematical Physics and Analysis, Amer. Math. Soc., Providence, RI, 1986.
[3] V. Isakov, Inverse Problems for Partial Differential Equations, Springer-Verlag, New York, 1998.
[4] R. Lattes, J.L. Lions, Methode de Quasi-Reversibility et Applications, Dunod, Paris, 1967 (English translation: R. Bellman, Elsevier, New York, 1969).
[5] R.E. Showalter, The final value problem for evolution equations, J. Math. Anal. Appl. 47 (1974) 563–572.
[6] K.A. Ames, W.C. Gordon, J.F. Epperson, S.F. Oppenhermer, A comparison of regularizations for an ill-posed problem, Math. Comput. 67 (1998) 1451–1471.
[7] K. Miller, Stabilized quasireversibility and other nearly best possible methods for non-well-posed problems, in: Symposium on Non-Well-Posed Problems and Logarithmic Convexity, in: Lecture Notes in Math., vol. 316, Springer-Verlag, Berlin, 1973, pp. 161–176.
[8] U. Tautenhahn, T. Schröter, On optimal regularization methods for the backward heat equation, Z. Anal. Anwendungen 15 (1996) 475–493.
[9] T.I. Seidman, Optimal filtering for the backward heat equation, SIAM J. Numer. Anal. 33 (1996) 162–170.
[10] N.S. Mera, L. Elliott, D.B. Ingham, D. Lesnic, An iterative boundary element method for solving the one-dimensional backward heat conduction problem, Internat. J. Heat Mass Transfer 44 (2001) 1937–1946.
[11] M. Jourhmane, N.S. Mera, An iterative algorithm for the backward heat conduction problem based on variable relaxation factors, Inverse Problems in Engineering 10 (2002) 293–308.
[12] D.N. Hào, A mollification method for ill-posed problems, Numer. Math. 68 (1994) 469–506.
[13] C.S. Liu, Group preserving scheme for backward heat conduction problems, Internat. J. Heat Mass Transfer 47 (2004) 2567–2576.
[14] S.M. Kirkup, M. Wadsworth, Solution of inverse diffusion problems by operator-splitting methods, Appl. Math. Modelling 26 (10) (2002) 1003–1018.

[15] L.C. Evans, Partial Differential Equations, Amer. Math. Soc., Providence, RI, 1998.
[16] W.L. Miranker, A well posed problem for the backward heat equation, Proc. Amer. Math. Soc. 12 (2) (1961) 243–274.
[17] L. Eldén, F. Berntsson, T. Regińska, Wavelet and Fourier method for solving the sideways heat equation, SIAM J. Sci. Comput. 21 (6) (2000) 2187–2205.
[18] C.L. Fu, X.T. Xiong, P. Fu, Fourier regularization method for solving the surface heat flux from interior observations, Math. Comput. Modelling 42 (2005) 489–498.
[19] C.L. Fu, Simplified Tikhonov and Fourier regularization methods on a general sideways parabolic equation, J. Comput. Appl. Math. 167 (2004) 449–463.
[20] Z. Qian, C.L. Fu, X.T. Xiong, T. Wei, Fourier truncation method for high order numerical derivatives, Appl. Math. Comput., in press, doi:10.1016/j.amc.2006.01.057.