# RESEARCH PLAN

# Dynamics of Dominated Splitting in Banach Spaces and its Feedback to Parabolic Partial Differential Equations

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## 0 Summary

Understanding the dynamics of dominated splitting in infinite dimensions is of great importance for the study of the long time behaviour of systems generated by certain models in biology, physics and chemistry. Here, we say that a dynamical system admits a *dominated splitting* (or*exponential separation* for the continuous time case) if the tangent vector bundle can be continuously decomposed into two complementary subbundles such that one direction converges exponentially fast to the other under the iteration of the tangent map. In particular, a hyperbolic splitting is a dominated splitting. In this project, we aim to study both geometrical and statistical properties of dominated splitting in Banach spaces and their application to some parabolic partial differential equations. More concretely, we shall,

1. Investigate the dynamics of dominated splitting in Banach spaces;
2. Investigate the ergodic theory and complexity for $C^1$-differentiable dynamical systems with dominated splitting in Banach spaces;
3. Investigate the lifting properties of skew-product semiflows generated by almost periodic scalar parabolic equations on the circle;
4. Investigate the dynamics of reaction-diffusion equations on thin domains.

In order to achieve these goals, the PI has already done much preparation work (Section 9) in this direction and also has already established an international collaboration network. Around three months per year of visiting abroad is scheduled in the research plan. Solving these problems, will establish an internationally recognized research group in this area in the department of Mathematics and Statistic at University of Helsinki.

# 1 Background

In biology, physics and chemistry, repetitive phenomena always capture the attention of people. Their mathematical fundamentals thus attract lots of experts and professionals. In these areas, models describing phenomena or potential laws of nature are usually formulated in terms of partial differential equations and functional differential equations. Therefore the underlying state spaces are infinite dimensional. The mathematical formulation of natural phenomena always involves simplifications of the physical laws. The question then is what features specific to a given model correspond to actual phenomena. However, those properties that are robust or even stable under perturbations of the model are likely to have real significance. A typical example is the well-known Smale's horseshoe [Smale67] (local hyperbolic dynamics), revealing complexity of phenomena. $C^2$-smooth Systems with positive entropy were proved to exhibit this characteristic, see [Katok80, LY11, LY12]. Compared to instability of orbits, the property of hyperbolicity is stable in the sense that any perturbation of the system is conjugate to the initial one, meaning that their relevant dynamical behaviours are actually qualitatively the same. Nevertheless, it was soon realised that some non-hyperbolic dynamical systems (see [Shub71, Lorenz63], for example) also exhibit robust features. After several decades of efforts and research, experts realised that at least for finite dimensional dynamics, robust phenomena reflect robust structures of the tangent map, that is, dominated splitting (see [Sambarino14] and the references therein), a notion which was first introduced independently by Liao, Mañé and Pliss. Here, we say that a dynamical system admits a *dominated splitting* (also called *exponential separation* in the continuous time case) if the tangent vector bundle can be continuously decomposed into two complementary subbundles such that one direction converges exponentially fast to the other under the iteration of the tangent map.

The domination property is so important that its dynamics have been thoroughly studied in finite dimensions for diffeomorphisms and flows, see [BDV05, PS09, Crovisier10] and references therein. However, related research about the dynamics of dominated splitting in infinite dimension is rare, except some work has been done by us [2]. It is the purpose of this project to study the geometrical and statistical properties of dominated splitting in Banach spaces and to apply them to certain specific partial differential equation (PDE) models.

# 2 Identification of the main research problems

## 2.1 Dynamics of dominated splitting

Let us start with the dynamical setting for certain classes of PDEs arising from application, for example, reaction-diffusion equations and some parabolic PDEs. We consider a separable Banach space $\mathbb{B}$ with norm $\|\cdot\|$ and a continuous map or semiflow on $\mathbb{B}$, i.e. a continuous mapping $F : T \times \mathbb{B} \to \mathbb{B}$, where $T = \mathbb{N}$ or $\mathbb{R}^+ \cup \{0\}$, with properties

$$F(t+s, x) = F(t, F(s, x)) \quad t, s \geq 0 \text{ and } F(0, x) = x.$$

For simplicity, we use $f^t$ to denote its time-$t$ map, i.e. $f^t(x) = F(t, x)$ and write $f = f^1$. Since we are interested in global behaviours of models, we focus on a certain compact invariant subset $\Lambda$. To apply geometric and differentiable techniques, assume further the basic smoothness condition $f^t \in C^1$.

The $f^t$-invariant set $\Lambda$ is said to admit a *dominated splitting* if one can decompose its tangent bundle into two invariant subbundles $T_\Lambda\mathbb{B} = E \oplus F$, such that with $C > 0$ and $\alpha > 0$,

$$\frac{\|Df_x^t u\|}{\|Df_x^t v\|} \leq Ce^{-\alpha t}\frac{\|u\|}{\|v\|}, \quad t > 0$$

hold for all $u \in E(x)\backslash\{0\}$, $v \in F(x)\backslash\{0\}$ and $x \in \Lambda$. In particular, *a hyperbolic splitting is a dominated splitting*. For infinite dimensional dynamical systems generated by evolutionary PDEs, it is known that spatial properties of solutions, such as the number of zeros, homotopy type and symmetry properties, have a profound effect on their dynamical properties. These spatial properties usually produce dominated splitting/exponential separation; refer to [BNV94, CLM95, HuPoSa07, Húska06, Húska08]. However, almost all research on their dynamics have been focused on the special case, namely hyperbolic splitting and its stable and unstable manifolds. Results for the dynamics of a general dominated splitting are largely unknown. **The central theme of this study is to investigate the topological and statistical properties of dominated splitting for mappings and semiflows of Banach spaces**.

Considering that infinite dimensional theory is likely to be richer and more complex than finite dimensional ones, there is no reason to reinvent everything from scratch. It is thus reasonable to start by extending some core parts of finite dimensional theory to infinite dimensions. This can be made more explicit by recalling the theory of dominated splitting in finite dimensions. One can find that the basic tool and idea used for dominated splitting in finite dimensions is the famous Plaque Family Theorem [HPS77].

Assuming that there exists a dominated splitting $T_\Lambda\mathbb{B} = E \oplus F$ for the tangent system $Df^t$, plaque family theorem asserts that the locally invariant plaque families $\mathcal{W}^E$, $\mathcal{W}^F$ tangent to subbundles $E$, $F$ respectively, always exist for original nonlinear dynamics $f^t$. *These plaque families are actually the local stable and local unstable manifolds if the splitting is hyperbolic.*

On the other hand, let us put things in perspective: to study the long time behaviours of a PDE via dynamical system methods, there are basically three different levels. They are (1) the infinitesimal level, with regards to distortion properties of tangent maps, (2) local results, with respect to dynamical properties in neighbourhoods of an orbit, and (3) global (nonlocal) results, which is usually the goal we want to achieve, such as, convergency, specific complexity etc. To investigate the global behaviours, the usual idea is to use some techniques (for example, nontrivial recurrency of measures, transversality conditions) to connect different neighbourhoods together so that the local results can be applied. Concerning local results, the plaque family theorem for Banach spaces, is therefore extremely important for further related research in the field of infinite dimensional dynamical systems.

In our work [2], a version of plaque family theorem was obtained for mappings of Banach spaces with the assumption of existence of $C^1$-bump function. The next step is to find a way to remove or at least weaken this assumption. Then we will focus on finding a version of plaque family theorem for semiflows of Banach spaces with dominated splitting. After this, we will investigate geometric and statistical conditions so that the plaque families along an orbit are actually invariant manifolds. Then the well-developed thoeries and methods for invariant manifolds can be applied to investigate the global properties.

We expect several difficulties. Some of them was encountered while we was working on [2]. Few of them are possibly quite natural. Thus, we will have to invent new ideas, techniques or even make necessary assumptions. For example, does every separable Banach space admit a $C^1$-bump function? Also, infinite dimensional operators can have an essential spectrum. As a result, some basic assumptions for tangent dynamics should be made so that one can distinguish expanding, neutral and contracting directions. For dynamical systems generated by PDEs, dynamics are only defined for positive time and usually cannot be extended to the whole line. Hence the analytic method to define invariant manifolds is invalid. Moreover, tangent maps $Df^t$ are not invertible. The dynamics for given initial points have only a collective description but no explicit description. Sometimes the dynamics defined by concrete PDEs can only be $C^1$ smooth and thus theories developed under the $C^{1+\alpha}$ setting, for instance, [LY11, LY12], do not apply.

## 2.2 Ergodic theory for $C^1$-differentiable dynamical systems with dominated splitting in Banach spaces

For dynamical systems defined by mappings or (semi-)flows, ergodic theory offers a description of their global dynamics in terms of averages and almost sure behaviours. In finite dimensions, there is a well-developed ergodic theory. Here, we are interested in the smooth ergodic theory for $C^1$-differentiable dynamical systems with dominated splitting in Banach spaces. In particular, we expect to **generalize the results of Katok** [Katok80] **to $C^1$-mappings and $C^1$-semiflows of Banach spaces with dominated splitting**. Further, for an infinite dimensional dynamical system with positive entropy, we are interested in the **estimation of its complexity**.

Katok's results assert that for a $C^2$-diffeomeorphism of a compact Riemannian manifold with an ergodic hyperbolic (nonzero Lyapunov exponents) measure and with positive metric entropy, the Smale's horseshoe present. In pioneer works [LY11, LY12] toward the smooth ergodic theory of infinite dimensional dyanmcial systems, Lian and Young recently generalized Katok's results to $C^2$-mappings and $C^2$-semiflows of Hilbert spaces. Both the $C^2$-smoothness condition and working on Hilbert spaces are very crucial for their work. The geometry of a Hilbert space is needed for establishing Lyapunov metrics and $C^2$-assumption is used in exploiting uniform hyperbolicity on noninvariant sets. Considering requirements from application fields, unfortunately, sometimes neither requirements can be satisfied. The state space of the dynamical system generated by a concrete PDE model is usually a Banach space and the differentiablity of the dynamics usually can only be $C^1$; refer to [Henry81]. However, as explained in Subsection 2.1, the dynamical systems generated by PDEs have their specifics: dominated splitting. Due to the existence of domination property, one can expect a lower bound for angles between two splitted subbundles, and thus one can relax Hilbert space assumption to Banach space. Meanwhile, due to the $C^1$-continuity of plaque family, we plan to relax $C^2$-smoothness assumption to $C^1$. Of course, we possibly will loss the absolute continuity of foliation due to insufficient differentiability. As a result, we will investigate new techniqual skills and tools so that we can get a view of the whole dynamics in our new setting.

Now we consider in general a $C^1$-differentiable dynamical system with positive metric entropy. By [BGKM02], Li-York chaos occur in this dynamics. We studied further in [4] the complexity of a general topological dynamical system with positive entropy, and proved that chaos actually occur in closure of stable set of a point for almost every point. In particular, for a diffeomorphism on a smooth Riemannian manifold with positive entropy, the lower bound of complexity is obtained in terms of the metric entropy and Lyapunov exponents. In this research, we will first generalize this result to a $C^1$-differentiable dynamical system in Banach space with positive entropy. Next we will consider such systems with the property of dominated splitting. Assume further there exists an ergodic measure with a positive finite metric entorpy and smooth conditional densities on unstable manifolds (a measure satisfying these conditions is usually called an SRB measure). Due to the existence of dominated splitting, one can expect that local invariant manifolds exist at almost every point. Then the geometry in the unstable direction by SRB measures is expected to establish relationship between the Lyapunov exponents and the metric entropy. More precisely, we will verify or at least find conditions (for example, assuming the hyperbolicity for the measure) so that the Pesin's entropy formula holds.

## 2.3 Dynamics of Almost Periodic Scalar Parabolic Equations on $S^1$

Reaction-diffusion systems form an abstract framework to study reaction and diffusion phenomena in, among others, population dynamics and genetics, heat conduction, chemical kinetics and combustion theory, refer to [Britton86, VP09]. In this subsection, we are interested in the following scalar reaction-diffusion equation on the circle:

$$u_t = u_{xx} + f(t, x, u, u_x), \quad t > 0, x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

When the system is autonomous, its dynamics are relatively simple. For example, Poincaré-Bendixson phenomenon occurs [FM89] in this situation. Transversality of the stable and unstable manifolds of hyperbolic equilibria and periodic orbits also can be established [CR08]. Moreover, [JR10] showed that Morse-Smale property is generic with respect to the nonlinearity. When the system is time-periodic, dynamics become more complicated. For instance, chaotic behaviour can be exhibited [SF92]. However, we still know from [Tereščák94] that any $\omega$-limit set of the Poincaré map can be imbedded into a 2-dimensional manifold. For the general cases, the related studies are hard. It is well-known that the semiflow defined by an autonomous system satisfies the local flow property. However, this property is invalid in nonautomous case. As a result, the skew-product semiflow $\pi : \mathbb{R} \times \mathbb{B} \times H(f) \to \mathbb{B} \times H(f)$, defined by $\pi(t, U, g) = (u(t, x; U, g), g \cdot t)$, $g \in H(f)$, is used to investigate the dynamics of a nonautonomous system. Here, $\mathbb{B}$ is a suitable Banach space of functions and $H(f)$ is the closure of the set of time-shifts of $f$. Let $f$ and its derivatives be time-almost periodic. One can find out that the fiber spaces $\mathbb{B}(g \cdot t)$ do not come back to $\mathbb{B}(g)$ again, although they can approach to $\mathbb{B}(g)$ in the sense of the Hausdorff metric along a sequence of time moments. As a consequence, traditional dynamical methods cannot be applied. It is for this reason that only one partial result [SWZ15] was obtained so far for a general system.

In this work, we expect to **establish some order relationships for points in the same fiber**, using an adapted version of plaque family theorem for the skew-product semiflows whose fibers are Banach spaces and the non-increasing property of lap numbers. By virtue of the obtained results, we plan to **reduce the infinite dimensional skew-product semiflow generated by a general scalar reaction-diffusion system to finite-dimension one** in some sense. To be more precise, we first note that any variational system of the semiflow along a minimal set still admits an exponential separation by [CLM95]. Therefore, by plaque family theorem for Banach spaces proposed in Subsection 2.1, we expect to know the local fiber dynamics of the original skew-product semiflow. This characterization is much finer than the exponential dichotomy and its invariant manifolds. Note that it is the latter ones who played the key roles in the proofs of their main theorems in [SWZ15]. Moreover, their assumptions were made here so that they can study the special case: central direction is corresponding to one lap number only. By virtue of this observation, one naturally wishes a clearer investigation about the structure of the minimal set. We try to answer this question positively and will give a thorough investigation on the structure of any minimal set of a spatially-homogeneous system, with no assumptions made in [SWZ15].

## 2.4 Dynamics of Reaction-Diffusion Equations on Thin Domain

Based on the understanding of the dynamics of dominated splitting, we propose in this subsection a completely new method to study Hale and Raugel's problem in [HR92]. Let $\Omega \subset \mathbb{R}^{N_x+N_y}$ be a fixed smooth domain and consider the following reaction-diffusion equation

$$u_t = \Delta u + f(t, x, u), \qquad (E)_\varepsilon$$

in the thin domains $\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^{N_x} \times \mathbb{R}^{N_y} : (x, \frac{y}{\varepsilon}) \in \Omega\}$ with suitable Neumann boundary condition. Hale and Raugel ask: to what extent is it possible to approximate the model by means of an equation on a lower dimensional spatial domain? Is it possible to determine the approximant?

To answer these questions, Hale and Raugel in [HR92] first considered an autonomous system on a curved squeezed domain. They proved the upper semicontinuity of the global attractors and topological equivalence of semiflows defined by system $(E)_\varepsilon$. Related results were then generalized to Lipschitz domains in [PR01] under certain natural conditions. This problem was also discussed on unbounded thin domain, see, [AP01, Elsken04]. Roughly speaking, related research sprung up after Hale and Raugel, up to 69 published articles so far. Most of the research is focused on autonomous and periodic systems. Methods they used, if we fit this problem into an abstract equation $v_t = -\tilde{A}_\varepsilon v + \hat{f}_\varepsilon(v)$, are to proceed from $\varepsilon$ to 0. By observing some properties of the limit system, we **propose to use a perturbation theory method to investigate Hale and Raugel's problem**. In a word, we will handle this question in an opposite way as others, to proceed from 0 to $\varepsilon$.

To show our idea, let $N_x = 1$. Assuming suitable smoothness conditions on the nonlinearity, one can make sure that every subsystem admits an exponential separation (see [CLM95]) along a fixed direction in $\mathbb{R}^{N_y}$. Further, the splittings change continuously with respect to these directions. Similar to finite dimensional flows (or skew-product flows), one expect to explore the dynamics of exponential separation for a semiflow (or a skew-product semiflow) in Banach spaces. Moreover, we expect to prove a persistence property for (skew-product) semiflows with explonential splitting in Banach spaces. Then these results will be used to study the dynamics of systems who are close to the limit system. Note that the obtained characterizations are much finer than former researchers can obtain from their key assumption of spectral convergence. The following question naturally arises. Based on this finer characterization, can we give a more clear investigation of Hale and Raugel's problem? We plan to work on this problem and try to give a positive answer.

The advantages of the perturbation theory method is that it can be adapted to a general thin domain. With this method, we can handle even a nonautomous systems (for example, $f$ and its derivatives are almost periodic), which is new in the field.

# 3 Objectives and methods

The proposed research will focus on geometric and statistical properties of dominated splitting in Banach spaces and their applications to reaction-diffusion equations. More precisely, the proposed research will provide a fundamental abstract theory on the dynamics of dominated splitting, including the plaque family theory and fake foliation as local results, geometric and statistical conditions and the generalized shadowing lemma as connecting tools for communication between dynamics in different neighborhoods. Based on these, we study the ergodic theory of $C^1$-differentiable dynamical systems with dominated splitting in Banach spaces and estimate their complexity. It is well-known that the study of dominated splitting and related examples are extremely important in finite dimensions to solve the Palis' conjecture. Reaction-diffusion equations, as a counterpart of examples that admit dominated splitting in finite dimensions, are hence very interesting for us. We introduce and show how the proposed tool, namely the plaque family theorem, is used to solve some interesting questions in the PDE field.

## 3.1 On the dynamics of dominated splitting in Banach spaces

The dynamics of dominated splitting for both mappings and semiflows in Banach spaces is the basis of the proposed research. It includes the plaque family theorem, fake foliation, geometric and statistical conditions for connecting dynamics in different neighborhoods. To get a version of the plaque family theorem, we will first use a $C^1$-bump functions to extend the local estimation about the nonlinear dynamics to the whole Banach space. As a consequece, we reduce the plaque family theorem into a version for a sequence of maps. Then we will construct a Banach space and equip it with a norm adapted to the domination property. Then the Hadamard-Perron method will be used to prove the existence of a plaque family. Next we will proceed with a similar argument or use fixed point theorem to prove $C^1$-regularity of the plaque family. To obtain a fake foliation result, we need to go further on the above results. We will use the idea from [HPS77] and consult with [BW10] for a construction of fake foliation. By permitting an acceptable angle between the tangent bundles and the constructed fake foliation, we will verify that the constructed result is indeed what we are looking for. Third, we will prove that the plaque family at a point turn into a stable or unstable manifold under certain uniform estimation. We will then check that at least for a hyperbolic ergodic measure on $\Lambda$ those conditions are satisfied at almost every point.

Besides, we will try, on the one hand, to find conditions for Banach spaces to admit a $C^1$-bump function. On the other hand, note that a separable Banach space admits an equivalent Gâteaux differentiable norm, and thus any separable Banach space has a Lipschitz continuous, Gâteaux differentiable bump function. We wish to prove a version of plaque family theorem with this norm. To overcome some difficulties in the semiflow cases, we will view it through composition of special section maps, which is defined by Young [LY12], and work along suitable sequences of Lyapunov coordinates. We will first assume a 1-1 relation for the dynamics in a neighborhood of $\Lambda$ and obtain some results, and then find alternative methods to weaken this assumption.

### 3.2 Ergodic theory for $C^1$-differentiable dynamical systems with dominate splitting in Banach spaces

To generalize Katok's results to $C^1$-mappings and $C^1$-semiflows, we will first prove the existence of local stable and unstable manifolds at almost every point. This can be done for an ergodic hyperbolic measure via standard methods (see [ABC11], for example) in the differentiable dynamical systems, admitting the proposed plaque family theorems. Then we will prove Liao-Gan's shadowing lemma [Gan02] for Banach spaces. A discrete version of this lemma has been obatined in our work [2]. By virtue of this lemma (we also need a continuous time version), we expect to obtain hyperbolic periodic points and then a transversal homoclinic orbit. In the end, we try to prove the existence of Smale's horseshoe and inverstigate its complexity via the increasing rate of number of periodic orbits as the period increases. We first try to handle the difficulties appearing in the semiflow cases in a similar way in the former subsection. Otherwise, we try to invent new skills for specific difficulties.

To obtain an estimation for the lower bound of the chaos appearing in a $C^1$-differentiable dynamical system in Banach spaces with finite positive entropy, we will follow a similar outline in the proof, as we showed in [4]. Concretely, we will need to first check the geometric meaning of Lyapunov exponents obtained from the multiple ergodic theorem for Banach spaces. Then we will investigate a relation between Bowen dimension entropy of any non-compact set in terms of its Hausdorff dimension and Lyapunov exponent. Next, as we have done in [4], we will work on generic points of the measure and refine the above results. Then [4, Theorem 4.9] will be applied to prove the desired results. Now we assume further the domination property for the dynamics and assume the existence of an SRB measure. Our aim is to prove Pesin's entropy formula for a $C^1$-differentiable dynamical system with dominated splitting. First, we will verify the existence of unstable manifolds $W^u_{\delta(x)}$ at almost every point $x$, by using the proposed plaque family theorem in Subsection 2.1. Second, we focus on finding a partition $\mathcal{P}$ so that almost surely: (1) $\mathcal{P}(x) \subset W^u_{\delta(x)}$ and (2) $\mathcal{P}(x)$ has a nonempty interior hold. By the absolutely continuity condition of the SRB measure, we have $\mu_x^{\mathcal{P}} \ll l^u_{W_{\delta(x)}}(x)$ where $l$ is the Lebesgue measure. We will then consult with [BY15] for a geometry in Banach spaces. In the end, one will try to use Ledrappier and Young's [LeYo85, Section 6] ideas to prove the Pesin's entropy formula.

### 3.3 Dynamics of almost periodic scalar parabolic equations on $S^1$

Fix a minimal set of the skew-product semiflow generated by an almost periodic scalar reaction-diffusion equation. To investigate its structure, the key idea is to find a partially hyperbolic structure for its variational system over this set. Meanwhile, we need to make sure that the central direction corresponds to one lap number only and the other two directions are characterized by the rest lap numbers. Then one can proceed to nonlinear dynamics and use the non-increasing properties of lap numbers to establish an order for points in the same fiber of the minimal sets. Combined with rotation invariance property of solutions, one can expect to prove the following dichotomies for a minimal set of a spatially-homogeneous system: the minimal set is either an almost 1-cover of its base and the dynamics are topologically conjugate to a minimal flow in a 1-dimensional skew-product semiflow, or it can be residually embedded into an almost automorphic forced circle-flow.

In detail, we will use Floquet subbundles which were established in [CLM95]. Since they are exponentially separated, the proposed plaque family theorem in Subsection 2.1 will assert the existence of local invariant plaque families. Now we will use ideas from our work [2], to investigate the desired partially hyperbolic structure for this minimal set. In general, the dimension of the central direction is determined by the number of bundles whose Lyapunov spectrum can cover zero. So we will next use the rotation invariance of solutions which was already observed in [SWZ15], to check that a zero Lyapunov exponent actually exists. As a consequence, the central direction corresponds to one lap number only. Then by virtue of these, we will try to prove the other two plaque families are actually local stable and local unstable manifolds. Next, non-increasing properties of lap numbers and rotation invariance property of solutions will be used to prove the above mentioned results.

### 3.4 Dynamics of Reaction-Diffusion Equations on Thin Domain

First, we will first try to use the cone theory to prove a persistence property for dominated splittings of semiflows in Banach spaces. Intuitively, this result can be considered as a special case of the palque family theorem. Plaque family is a nonlinear perturbation of the tangent semiflow, while the persistence property we wish to have is only a linear perturbation of the tangent semiflow. Therefore, we certainly expect its existence. Based on this result and on observations on the limit system, we can obtain the domination property for the systems near the limit system. Hence, the systems $(E)_\varepsilon$ for small $\varepsilon$ are included. Now we will use the proposed plaque family theorem in Subsection 2.1 to variational systems of $(E)_\varepsilon$. As a result, the local dynamics of systems $(E)_\varepsilon$ will be made clear. Note that once the invariant sets (probably their global attractors) in $(E)_\varepsilon$ exist, they are continuous. Further, we plan to use the perturbation skills established in work [3], to construct a discrete Lyapunov function (In general, this is not true for a reaction-diffusion equation in high dimensions. And it is because of this, complexity of dynamics can be presented) for nearby systems. Then we try to use the methods of [SY98] and [3, 5] to study the liftying properties of those invariant sets. We expect to explore the covergence property from the persistence property of domination.

## 4 Results and time schedule

The results of the proposed research will be of intrinsic mathematical value both in pure and applied fields. Our results will be published in the best international journals.

### 4.1 Time schedule

2016-2017 *The dynamics of dominated splitting for mappings and semiflows in Banach spaces.* This is fundamental to the proposed research and will expose the geometrical and statistical properties of infinite dimensional dynamical systems with dominated splitting.

2017 *Ergodic theory for $C^1$-differentiable dynamical systems in Banach spaces with dominated splitting and their complexity.*

2018 *Lifting properties in skew-product semiflows generated by almost periodic scalar parabolic equations on the circle.* Find a partially hyperbolic structure for varational system along any minimal set and prove the above mentioned dichotomies for the structural of minimal sets.

2019 *Dynamics of reaction-diffusion equations on thin domain.* Investigate Hale and Raugel's problem from perturbation theory point of view. Show how the dynamics in a lower dimensional affect the high dimensional dynamics and determine the approximant.

Many of proposed topics of research are directly related to the previous work by myself and collaborators. With the methods and results already obtained, we certainly expect new theorems to be proven, even if there is still a lot of work to be done. However, there are some problems and difficulties which are hard to or even didn't estimate in advance. Thus, we will first focus on topics that are more familiar to us. Then we will move on to the more ambitious and hence more challenging parts of the research plan.

# 5 Researcher training and research career

I plan to lecture an advanced course on the topic of the research project at University of Helsinki. This will hopefully attract an MSc student to the subject, so that I could supervise his or her MSc thesis studies.

I was appointed as a teaching assistant of several undergraduate and graduate courses in USTC from 2005 to 2008, two advanced courses in University of Helsinki in 2014. I once was a research assistant of Prof. Huang Wen from 2007 to 2008. Besides, I gave one to two short lectures (around 8 hours in total) per year during the sandwich time of Biomathematic group in University of Helsinki. The contents are about cutting edge problems in mathematics and biomathematics. The pedagogical skills were praised by senior researcher Eva Kisdi (http://www.mv.helsinki.fi/home/kisdi/).

# 6 Research environment and mobility

The research will be conducted in the Department of Mathematics and Statistics at University of Helsinki. The department has large and active research groups both in pure and applied math: mathematical physics and biomathematics. The leaders of these two groups, Prof. Antti Kupiainen and Prof. Mats Gyllenberg are internationally recognized scholar who have achieved important results in many areas in pure and applied math, including those related infinite dimensional dynamical systems generated by PDEs. Adjunct Prof. Mikko Stenlund is an excellent researcher on ergodic theory and dynamical systems, whose work obviously related to the present proposal. Ideas and thoughts are freely discussed among young researchers and several senior researchers: Antti Kemppainen, Konstantin Lzyurov, Stefan Geritz, Eva Kisdi and Ping Yan, who are my good friends. Seminars of these groups are usually held once a week. The CoE in Analysis and Dynamics has many short-term foreign visitors. Several mini-courses and lecture series are given every year on cutting-edge topics.

The implementation of the project requires international collaboration, and thus also a lot of mobility. I am scheduling around three months per year of visiting abroad to achieve my goals. I expect that I need to work with my collaborator Prof. Dawei Yang (Soochow University) around one month per year on the first topic of the proposal. Since I already have regular discussions with members of Prof. Lai-Sang Youngs group (Courant Institute of Mathematical Sciences), I am planing to visit her research group one month per year to study the smooth ergodic theory of infinite dimensional differentiable dynamical systems. I wish to initiate collaboration with Prof. Geneviève Raugel (Université Paris-Sud) on Hale-Raugel's problem. To learn further about Palis conjecture and to keep contact with the field of differentiable dynamical systems, I am planning to visit Prof. Sylvain Crovisier (Université Paris-Sud) at least two weeks per year. If it is possible, we will work on Palis conjecture in Banach spaces. I am also looking forward to working jointly with Prof. Huang Wen (University of Science and Technology of China) for around three weeks per year. We both have interest in the estimation of complexity of infinite dimensional differentiable dynamical systems with positive entropy and domination property.

# 7 Ethical issues

There are no ethical problems related to the project.

# 8 Budget

In the budget table, the material costs are mostly books to be purchased. The travel expenses are calculated so that they can cover two to three conferences and around three months of abroad visiting to three different places (France, USA and China) per year. Details of the expected cost per month are as follows. Flight fees: 4 flights, 2530 euro. Hotel cost: 104 days, 10400 euro. Per dim for the visiting of three different places, France, USA and China. Services fee mainly include, for example, conference fees and publication fees. Other cost is applied for the mobility allowance.

| Year | 2016 | 2017 | 2018 | 2019 | Total |
|---|---|---|---|---|---|
| Salaries | 11 865 | 36 663 | 37 763 | 25 930 | 112 221 |
| Indirect employee costs | 6 644 | 20 531 | 21 147 | 14 521 | 62 843 |
| Total overheads share | 17 954 | 55 478 | 57 143 | 39237 | 169 812 |
| Materials | 120 | 120 | 0 | 0 | 240 |
| Services | 400 | 1100 | 1100 | 700 | 3 300 |
| Travel expences | 2 925 | 8 874 | 8 874 | 5 949 | 26 622 |
| Other cost | 1 200 | 3 500 | 3 500 | 2 500 | 10 700 |
| Total cost | 41 108 | 126 266 | 129 527 | 88 837 | 385 738 |
| Academy fund. per. | 70% | 70% | 70% | 70% | 70% |
| Academy in euros | 28776 | 88456 | 90739 | 62116 | 270087 |
| of which 70% is applied from the Academy | | | | | 270 017 |

Table 1: Budget table

# 9 The PI's five most important work relevant to the topic of this proposal


[1 ] Fang, Chun (2015). On the dynamics of non-hyperbolic minimal sets for tridiagonal competitive-cooperative systems, manuscript, available upon request.

[2 ] Fang, Chun; Gyllenberg, Mats; Yang, Dawei (2015), Positive entropies and horseshoes in infinite dimensional dynamics via an approach of plaque family, preprint, available upon request.

[3 ] Fang, Chun; Gyllenberg, Mats; Wang, Yi (2014). Perturbation theory for tridiagonal competitive-cooperative systems, preprint, available upon request.

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Results in [1] will be reported in the Biomath seminar of University of Helsinki on 14.10.2015; Partial results of [3] was reported in the International conference on Advances on Fractals and Related topics on 14.12.2012, Hong Kong.

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