Repeated games

1 twice repeated PD

strategy profile resemble

$$X,(X,X,X,X)$$

where X is strategy in first round, and (X,X,X,X) are strategies in second round for each of the 4 histories that could arise in first round, there are thus $2^5 = 32$ different strategies

let payoffs be

$$G > C > D > B$$

The payoffs are

$$U = \sum_{t=0}^{T-1} \delta^t g_{t+1}$$

2 One Shot deviation principle

a strategy profile is SGPE if and only if at **any subgame**, no player can improve her payoff by change her action from said profile for one round only, ie no profitable one shot deviation

- finding a SGPE thus entails finding the consequences of a one shot deviation at stage t knowing that at stage $t+1$ players will play the specified strategy profile
- deviation must not be profitable for all t

3 Grim trigger

the grim trigger strategy:

- cooperate and remain in cooperative state until there is a deviation
- remain in punishment stage and defect forever

standard	C	C	C	C	C	C	C	C	C	C
	C	C	C	C	C	C	C	C	C	C
with defection	C	C	C	C	D	D	D	D	D	D
	C	C	C	C	C	D	D	D	D	D

each stage thus falls into one of two categories

- cooperative state: no defect yet

$$V(Coop) = \sum_{i=0}^{\infty} C\delta^i = \frac{C}{1-\delta}$$

- punishment state: defect at least once before

$$V(Pun) = \sum_{i=0}^{\infty} D\delta^i = \frac{D}{1-\delta}$$

To test for SGPE

1. first test for profitability of deviation under cooperative state

$$V(Coop) = V(Coop)$$

$$V(Defect) = G + \delta V(Pun)$$

2. then test for profitability of deviation under punishment state

$$V(Coop) = B + \delta V(Pun)$$

$$V(Defect) = V(Pun)$$

Alternatively, the consequences of deviating under cooperative state could be expressed as

$$\sum \text{gain from first round + losses for all future rounds}$$

$$\sum (G-C) + \delta(D-C) + \delta^2(D-C) + ...\delta^{\infty}(D-C) \geq 0$$

$$(G-C) + \frac{\delta(D-C)}{1-\delta} \geq 0$$

$$(G-C) \geq \frac{\delta(C-D)}{1-\delta}$$

4 Tit for tat

The TFT strategy:

- start in mutual cooperation, then each player copies the move of the opponent in the previous round

standard	C	C	C	C	C	C	C	C	C	C
	C	C	C	C	C	C	C	C	C	C
with defection	C	C	C	C	D	C	D	C	D	C
	C	C	C	C	C	D	C	D	C	D

each stage thus falls into one of four categories, let $V(X, X)$ denote the state of a stage where X, X were the strategies for the previous round

- cooperative state: last round was C, C

$$V(C, C) = \sum_{i=0}^{\infty} C\delta^i = \frac{C}{1-\delta}$$

- punishment state: last round was D, D

$$V(D, D) = \sum_{i=0}^{\infty} D\delta^i = \frac{D}{1-\delta}$$

- alternating state 1: last round was C, D

$$V(C, D) = \frac{G + B\delta}{1-\delta^2}$$

- alternating state 1: last round was D, C

$$V(D, C) = \frac{B + G\delta}{1-\delta^2}$$

now test for profitability of deviation under each stage

- under state V(C, C)
  - TFT strategy is to play C, for a payoff of
    $$V(C,C)$$
  - deviation strategy is to play D, stage effectively becomes $V(C,D)$ (as if opponent played D in the previous round, payoffs are
    $$V(C,D)$$
    hence first condition for TFT sustaining cooperation as SGPE is
    $$V(C,C) \geq V(C,D) \tag{1}$$
- under state V(D, D)
  - TFT strategy is to play D, for a payoff of
    $$V(D,D)$$
  - deviation strategy is to play C, stage effectively becomes $V(D,C)$ (as if opponent played D in the previous round, payoffs are
    $$V(D,C)$$
    hence second condition for TFT sustaining cooperation as SGPE is
    $$V(D,D) \geq V(D,C) \tag{2}$$
- under state V(C, D)
  - TFT strategy is to play D, for a payoff of
    $$V(C,D)$$
  - deviation strategy is to play C, stage effectively becomes $V(C,C)$ (as if opponent played D in the previous round, payoffs are
    $$V(C,C)$$
    hence second condition for TFT sustaining cooperation as SGPE is
    $$V(C,D) \geq V(C,C) \tag{3}$$
- under state V(D, C)
  - TFT strategy is to play C, for a payoff of
    $$V(D,C)$$
  - deviation strategy is to play D, stage effectively becomes $V(D,D)$ (as if opponent played D in the previous round, payoffs are
    $$V(D,D)$$
    hence second condition for TFT sustaining cooperation as SGPE is
    $$V(D,C) \geq V(D,D) \tag{4}$$

hence TFT is a SGPE if and only if

$$V(C,C) = V(C,D)$$
$$V(D,D) = V(D,C)$$

5 Stick and carrorts

Stick and carrots strategy:

- define punishment state as playing minimax, which is the minimum payoff that one player can induce to the other
- more specifically, this entails P1 playing best response to P2, and P2 minimizing Π_1 wrt $p2$
- play C if there is no defection
- punish for t rounds and then return to cooperation, unless there is deviation in which case restart the clock

given the collusive pricing example,

- profit function is

$$\Pi_i = (p_i - 8)(44 - 2p_i + p_2)$$

- BR for price is

$$p_i = \frac{60 + p_2}{4}$$

- under collusion / cooperation,

$$p_1 = p_2 = 26,\ \Pi_1 = \Pi_2 = 324$$

- under NE

$$p_1 = p_2 = 20,\ \Pi_1 = \Pi_2 = 288$$

- under minimax,

$$p_1 = p_2 = 8,\ \Pi_1 = \Pi_2 = 0$$

hence game resembles

standard	28	28	28	28	28	28	28	28	28
	28	28	28	28	28	28	28	28	28
with defection	28	28	28	28	28	21.5	8	28	28
	28	28	28	28	28	28	8	28	28
with defection in punsihment state	28	28	28	28	28	21.5	8	8	28
	28	28	28	28	28	28	17	8	28

the value of cooperative state

$$V(C) = 324 + \delta V(C) = \frac{324}{1-\delta}$$

the value of punsihment state

$$V(D) = 0 + \delta V(C) = \delta \frac{324}{1-\delta}$$

to test for SGPE

- under cooperative state
  - SC strategy is to play 26

$$V(SC) = V(C) = 324 + \delta V(C)$$

  - deviation would be to play BR to 26, which is 21.5, giving payoff $\frac{729}{2}$ in the current round

$$V(dev) = \frac{729}{2} + \delta V(D) = \frac{729}{2} + \delta^2 V(C)$$

    hence deviation not profitable if

$$324 + \delta V(C) \geq \frac{729}{2} + \delta^2 V(C) \tag{5}$$

$$\delta V(C)(1-\delta) \geq \frac{729}{2} - 324$$

$$\delta \geq \frac{40.5}{324}$$

- under punishment state
  - SC strategy is to play 8
    $$V(SC) = V(D) = \delta V(C)$$
  - deviation strategy is to deviate from 8 and play BR to opponent's 8, playing 17, giving a payoff of 162
    $$V(dev) = 162 + \delta V(D)$$

    hence deviation not profitable if

$$V(D) \geq 162 + \delta V(D) \tag{6}$$
$$V(D)(1-\delta) \geq 162$$
$$\delta V(C)(1-\delta) \geq 162$$
$$\delta 324 \geq 162$$