# 1 Summary

1. For First Price sealed bid, perfect info,
   - Equilibrium bidding strategy in non-dominated strategies
     - highest bidder bids second highest v
     - second highest bidder bids $v - \delta$
     - all others bid less than their v, above 0
   - Expected revenue for seller
     - second highest v
2. For Second Price sealed bid, perfect info,
   - Equilibrium bidding strategy in weakly-dominated strategies
     - everyone bids their own v
   - Expected revenue for seller
     - second highest v
3. For First Price sealed bid, imperfect info,
   - Equilibrium bidding strategy in non-dominated strategies
     - expected second highest v conditional on having the highest v =

$$\frac{n-1}{n} v_i$$

   - Expected revenue for seller
     - expected second highest v =

$$\frac{n-1}{n+1}$$

4. For Second Price sealed bid, imperfect info,
   - Equilibrium bidding strategy in weakly-dominated strategies
     - bids their own v, $v_i$
   - Expected revenue for seller
     - expected second highest v =

$$\frac{n-1}{n+1}$$

5. For All Pay Auction, perfect info, common value
   - Equilibrium bidding strategy
     - no equilibrium in pure strategies
     - equilibrium in mixed strategies when everyone bids

$$b_i \sim U(0, v_{common})$$

   - Expected revenue for seller =

$$v_{commmon}$$

6. For All Pay Auction, perfect info, private value
   - Equilibrium bidding strategy
     - no equilibrium in pure strategies
     - equilibrium in mixed strategies when

$$\text{for i with larger v, } b_i \sim U(0, v_{secondlargest})$$
$$\text{for j with smaller v, } b_j = \begin{cases} 0 \text{ with 0.5 probability} \\ \sim U(0, v)\text{with 0.5 probability} \end{cases}$$

- Expected revenue for seller =

$$v_{secondhighest}$$

7. For All Pay Auction, imperfect info
   - Equilibrium bidding strategy

$$b_i = (n-1)\frac{v_i^n}{n}$$

   - Expected revenue for seller =

$$\frac{n-1}{n+1}$$

# 2 Bidding Strategy for First Price Sealed Bid

## 2.1 For 2 players

It can be shown that for 2 bidders with $v_1, v_2, \sim U(0,1)$, their equilibirum bidding strategy is uniquely $b_i = \frac{v_i}{2}$

Assuming that the $b_2$ is bidding with function $b_2 = B_2(v_2)$ such that $b_2 = kv_2, k <= 1$

$$\begin{aligned} u_1 &= P(kv_2 \leq b_1)(v_1 - b_1) \\ &= P(v_2 \leq \frac{b_1}{k})(v_1 - b_1) \\ &= (\frac{b_1}{k})(v_1 - b_1) \text{ since } v_2 \sim U(0,1), F_v(X) = X \end{aligned}$$

profits maximized when, $\frac{du_1}{db_1} = 0$

$$\frac{v_1 - b_1}{k} - \frac{b_1}{k} = 0$$
$$\frac{1}{k}(v_1 - 2b_1) = 0$$
$$\frac{v_1}{2} = b_1$$

this means that for any of opponent's k, bidding $b_i = \frac{v_i}{2}$ is a NE

## 2.2 Generalizing for $n$ players

For $n$ players each with $v_i \sim U(0,1)$, prove that the equilibrium bid is $b_i = \frac{n-1}{n}v_i$

### 2.2.1 Argument 1 - by profit max argument

$$\begin{aligned} u_1 &= P(v_1 - b_1) \text{ where P is the probability of winning} \\ &= P(\text{bid of n-1 others} < \ b_1\ )(v_1 - b_1) \text{ assuming everyone bidding with } b_j = kv_j, \\ &= P(kv < b_1)^{n-1}(v_1 - b_1) \\ &= (\frac{b_1}{k})^{n-1}(v_1 - b_1) \end{aligned}$$

To maximize profits, $\frac{du_1}{db_1} = 0$ when

$$\begin{aligned} \frac{1}{k^{n-1}}(v_1(n-1)b^{n-2} - nb^{n-1}) &= 0 \\ v_1(n-1)b^{n-2} - nb^{n-1} &= 0 \\ b^{n-2}(v_1(n-1) - nb) &= 0 \\ b &= \frac{n-1}{n}v \end{aligned}$$

#### 2.2.2 Argument 2 - By Probability

Given that in equilibrium, players bid the expected value of the second highest v, assuming that they themselves have the highest v,

let $Y$ be the equilibrium bid $Y \sim max(v_2, v_3...v_n)$ *given that* $v_2, v_3...v_n < v_1$, find $E(Y|Y \leq v_1)$

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(max(v_2, v_3...v_n) \leq y) \\
&= P(v_2, v_3...v_n \leq y) \\
&= (P(v_2 \leq y))^{n-1} \\
&= y^{n-1} \text{ since } F_v(X) = X \text{ and all } v \sim U(0,1)
\end{aligned}$$

To find the CDF of Y conditional on $Y < v_1$,

$$F_Y(y|Y \leq v_1) = \frac{P((Y \leq y) \cap (Y \leq v_1))}{P(Y \leq v_1)}$$

Consider only case in which $y \leq v_1$, hence $(Y \leq y) \subseteq (Y \leq v_1)$

$$\frac{P((Y \leq y) \cap (Y \leq v_1))}{P(Y \leq v_1)} = \frac{P(Y \leq y)}{P(Y \leq v_1)} = \frac{y^{n-1}}{v_1^{n-1}}$$

The PDF of Y is thus

$$\begin{aligned}
f_Y(y) &= \frac{d}{dy}\frac{y^{n-1}}{v_1^{n-1}} \\
&= (n-1)\frac{y^{n-2}}{v^{n-1}}
\end{aligned}$$

The expected value of Y is

$$\begin{aligned}
E(Y|Y \leq v_1) &= \int_0^{v_1} y f_Y(y|y \leq v_1) dy \\
&= \frac{n-1}{v_1^{n-1}} \int_0^{v_1} y^{n-1} dy \\
&= \frac{n-1}{v^{n-1}} \left[\frac{y^n}{n}\right]_0^v \\
&= \frac{n-1}{v^{n-1}} \frac{v^n}{n} \\
&= \frac{n-1}{n} v
\end{aligned}$$

# 3 Expected Return for First Price Sealed Bid

## 3.1 Argument 1

Knowing that in equilibrium, each bidder bids $b_i = B_i(v_i)$ where $B_i$ is the equilibrium bidding function, the expected revenue is $E(B_i(X))$ where $X$ is the highest v.

### 3.1.1 Case with 2 players

For 2 bidders with $v_1,\ v_2$, each bidding $b_i = \frac{v_i}{2}$, the expected revenue is the half of the expected highest v.

$$\begin{aligned}
P(max(v_1,v_2)=x) &= P(v_1=x,v_2<x)+P(v_2=x,v_1<x)\\
&= 2P(v_1=x,v_2<x)\\
&= 2f_v(x)F_v(x)\\
&= 2x \text{ , since } v\sim U(0,1) \text{ and } F_v(x)=x
\end{aligned}$$

The expected value of $max(v_1,v_2)$ is

$$\begin{aligned}
E(f_{HV}(x))=\int_0^1 xf_{HV}(x)dx &= \int_0^1 2x^2dx\\
&= 2\left[\frac{x^3}{3}\right]_0^1\\
&= \frac{2}{3}
\end{aligned}$$

Since expected revenue is half of expected highest v,

$$expected\ revenue=\frac{3}{2}\times\frac{1}{2}=\frac{1}{3}$$

### 3.1.2 Generalization for n players

For $n$ bidders with $v_i\in(v_1,v_2,v_3...v_n)$, given that in equilibrium, each player bids with $b_i=\frac{n-1}{n}v_i$ as shown earlier, the expected revenue is $\frac{n-1}{n}v_{max}$ where $v_{max}=max\,(v_1,v_2,v_3...v_n)$ is the expected highest v

let $max\,(v_1,v_2,v_3...v_n)$ have PDF $f_{hv}(x)$, where

$$\begin{aligned}
f_{HV}(x) &= P(max\,(v_1,v_2,v_3...v_n)=x)\\
&= \sum_{i=1}^{n}P(v_i=x,\ (v_1,v_2...v_n)\,/v_i\le x)\\
&= nf_v(x)(F_v(x))^{n-1}\\
&= nx^{n-1}
\end{aligned}$$

The expected value of $max\,(v_1,v_2,v_3...v_n)$ is thus

$$\begin{aligned}
E(f_{HV}(x))&=\int_0^1 xf_{HV}(x)dx\\
&=n\int_0^1 x^ndx\\
&=x\left[\frac{x^{n+1}}{n+1}\right]_0^1\\
&=\frac{n}{n+1}
\end{aligned}$$

Expected revenue is therefore

$$\frac{n-1}{n}\frac{n}{n+1}=\frac{n-1}{n+1}$$

## 3.2 Argument 2 - Prof Massi's argument

### 3.2.1 For 2 bidders

Knowing that in equilibrium, bidders bid their expected value of the second highest v assuming that they themselves have the highest v, Expected revenue = expected second highest v

For 2 bidders with $v_1$ $v_2 \sim U(0,1)$, let $Y$ be $min(v_1, v_2)$. Find E(Y)

The CDF of Y is

$$\begin{aligned} F_Y(y) &= P(min(v_1, v_2) \le y) \\ &= P(v_1 \le y, v_2 \le y) + P(v_1 \le y, v_2 > y) + P(v_2 \le y, v_1 > y) \\ &= y^2 + y(1-y) + y(1-y) \\ &= 2y - y^2 \end{aligned}$$

The CDF of Y can also be found by

$$\begin{aligned} F_Y(y) &= P(min(v_1, v_2) \le y) \\ &= 1 - P(min(v_1, v_2) \ge y) \\ &= 1 - P(v_1 \ge y, v_2 \ge y) \\ &= 1 - (1-y)^2 \\ &= 2y - y^2 \end{aligned}$$

The expected value of Y can thus be calculated from its PDF

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}(2y - y^2) \\ &= 2 - 2y \end{aligned}$$

$$\begin{aligned} E(Y) &= \int_0^1 y f_Y(y) dy = \int_0^1 2y - 2y^2 dy \\ &= \frac{1}{3} \end{aligned}$$

### 3.2.2 Generalisation for n players

For $n$ players with $v_i \in (v_1, v_2, v_3 ... v_n)\,, v_i \sim U(0,1)$, the expected revenue is the expected second highest v.

Hence assume $v_1 > v_i \in (v_2, v_3, ... v_n)$, let $Y$ be $max(v_2, v_3, ... v_n)$, find $E(Y|Y < v_1)$

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(max(v_2, v_3, ... v_n) \le y) \\ &= y^{n-1} \\ F_Y(y|Y < v_1) &= \frac{y^{n-1}}{v_1^{n-1}} \\ f_Y(y|Y < v_1) &= (n-1)\frac{y^{n-2}}{v_1^{n-1}} \end{aligned}$$

$$\begin{aligned} E(Y|Y < v_1) &= \int_0^{v_1} \frac{n-1}{v_1^{n-1}} y^{n-2} y dy \\ &= \frac{n-1}{v_1^{n-1}} \left[\frac{y^n}{n}\right]_0^{v_1} \\ &= \frac{n-1}{n} v_1 \end{aligned}$$

Since $v_1$ is the maximum v, find let $X$ be $max(v_1, v_2, v_3...v_n)$, find $E(X)$

$$\begin{aligned}
F_X(x) &= P(max(v_1, v_x, v_3...v_n) \leq x) \\
&= x^n \\
f_X(x) &= nx^{n-1} \\
E(X) &= \int_0^1 nx^n dx \\
&= n\left[\frac{x^{n+1}}{n+1}\right]_0^1 \\
&= \frac{n}{n+1}
\end{aligned}$$

Hence the expected revenue for seller is

$$\frac{n-1}{n}\frac{n}{n+1} = \frac{n-1}{n+1}$$

Alternatively, the PDF of the second highest v can be found directly. Let the second highest v have PDF $f_{2nd\ highest}(x)$.
taken from http://econ.ucsb.edu/ tedb/Courses/GameTheory/aucnotes.pdf , although it is technically wrong to claim that

$$P(X = x) = f_X(x)$$

$$\begin{aligned}
f_{2nd\ highest}(x) &= P(second\ highest\ v = x) \\
&= P(\text{one person has } v = x, \text{ one person has } v > x, \, n-2 \text{ others have } v \leq x)
\end{aligned}$$

probability of one person having $v = x$ is

$$P(v_i = x) = f_V(x) = 1$$

probability of one person having $v \geq x$ is

$$\begin{aligned}
P(v_j \geq x) &= 1 - P(x_j \leq x) \\
&= 1 - F_V(x)
\end{aligned}$$

probability of n-2 having $v \leq x$ is

$$x^{n-2}$$

number of ways to choose 1 person to have $v = x$ and 1 to have $v > x$ is $n(n-1)$

$$\begin{aligned}
f_{2nd\ highest}(x) &= P(\text{one person has } v = x, \text{ one person has } v > x, \, n-2 \text{ others have } v \leq x) \\
&= n(n-1)(1 - F_V(x))x^{n-2} \\
&= n(n-1)(1-x)x^{n-2}
\end{aligned}$$

the expected second highest v is thus

$$
\begin{aligned}
E(f_{2nd\ highest}(x)) &= \int_0^1 x f_{2nd\ highest}(x)dx \\
&= n(n-1)\int_0^1 (1-x)x^{n-1}dx \\
&= n(n-1)\int_0^1 x^{n-1} - xndx \\
&= n(n-1)\left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_0^1 \\
&= n(n-1)\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= n-1-\frac{n(n-1)}{n+1} \\
&= \frac{(n-1)(n+1)-n(n-1)}{n+1} \\
&= \frac{n-1}{n+1}
\end{aligned}
$$

# 4 All Pay Auctions, imperfect information

## 4.1 Case for 2 players

for 2 players with $v_1, v_2 \sim U(0,1)$, find their equilibrium bid function $B$. Assume B is a function that is monotonic increasing, and is hence invertible and differentiable.
Suppose bidder 1 is bidding $b_1 = B(x)$

$$
\begin{aligned}
u_1 &= P(v_1 - b_1) + (1-P)(-b_1), \text{ where P is probability of winning with } b_1 \\
&= Pv_1 - Pb_1 - b_1 + Pb_1 \\
&= Pv_1 - b_1 \\
&= P(B(x) > B(v_2))(v_1) - B(x) \\
&== x(v_1) - B(x)
\end{aligned}
$$

B is the NE bid function if x = v satisfies the profit-max equation $\frac{du_1}{dx} = 0$,

$$
\begin{aligned}
v_1 - B'(x) &= 0 \\
v_1 - B'(v_1) &= 0 \\
v_1 &= B'(v_1) \\
B(v_1) &= \int v_1 dv \\
&= \frac{v^2}{2} + C
\end{aligned}
$$

with the condition that $B(0) = 0$, $C = 0$, hence the eqm bid function is

$$
B(v_i) = \frac{v_i^2}{2}
$$

The expected revenue for seller is

$$
\begin{aligned}
E(R) &= E\left(\frac{v_1^2}{2} + \frac{v_2^2}{2}\right) \\
&= \int_0^1 v^2 dv = \frac{1}{3}
\end{aligned}
$$

# 5 Generalising for n players

for n players with $v_i \in (v_1, v_2, v_3...v_n),\ v_i \sim U(0,1)$, proof is similar to 2 player case above but expressed slightly differently.

As shown earlier the expected payoffs are

$$u_i = Pv_i - b_i$$

expressing payoffs in terms of $b_i$, a function $V$ can be defined as the inverse of equilibrium bidding function $B$, that is $V - B^{-1}$

$$b_i = B(v_i)$$
$$v_i = V(b_i)$$

hence to express $u_i$ in terms of $b_i$,

$$u_i = P(b_i > max(b_1, b_2, b_3...b_n)/b_i)(v_i) - b_i$$

since everyone bidding with $B$ in equilibrium, and $B$ is monotonically increasing,

$$\begin{aligned} P(b_i > max(b_1, b_2, b_3....b_n)/b_i) &= (P(b_i > b_j))^{n-1} \\ &= (P(V(b_i) > v))^{n-1} \\ &= (F_V(V(b_i)))^{n-1} \qquad = (V(b_i))^{n-1} \end{aligned}$$

$u_i$ can thus be expressed as

$$u_i = (V(b_i))^{n-1}\, v_i - b_i$$

since the bidder chooses $b_i$ to maximize $u_i$,

$$\begin{aligned} \frac{du_i}{db_i} &= v_i(n-1)\,(V(b_i))^{n-2}\, V'(b_i) - 1 \\ &= (n-1)(v_i)^{n-1}V'(b_i) - 1, \text{ since } v_i = V(b_i) \end{aligned}$$

since $V^{-1} = B$, $V'(b_i) = \frac{1}{B'(v_i)}$, hence

$$\frac{du_i}{db_i} = \frac{(n-1)v_i^{n-1}}{B'(v_i)} - 1$$

profit maximized when

$$\begin{aligned} \frac{(n-1)v_i^{n-1}}{B'(v_i)} &= 1 \\ (n-1)v_i^{n-1} &= B'(v_i) \\ B(v_i) &= \int (n-1)v_i^{n-1} dv \\ &= \frac{(n-1)v_i^n}{n} + C. \end{aligned}$$

Given additional condition that $B(0) = 0$, hence $C = 0$ and

$$b_i = B(v_i) = \frac{(n-1)v_i^n}{n}$$

the expected revenue is thus

$$\begin{aligned} E\left(\sum_{i=1}^{i=n} \frac{(n-1)v_i^n}{n}\right) &= \int_0^1 (n-1)v^n dv \\ &= (n-1)\left[\frac{v^{n+1}}{n+1}\right]_0^1 \\ &= \frac{n-1}{n+1} \end{aligned}$$

## 6 Generalizing for n players, with different uniform distribution

for n players each with $v_i \sim U(0, a)$, to show that normalizing works

$$\begin{aligned} u_1 &= Pv_1 - b_1 \\ &= (P(b_1 > b_i))^{n-1}(v_1) - b_1 \end{aligned}$$

Given equilibrium bidding function and its inverse

$$b_i = B(v_i)$$

$$v_i = V(b_i)$$

$$\begin{aligned} u_1 &= (P(B(v_i) < b_1))^{n-1} v_1 - b_1 \\ &= (P(v_i < V(b_1)))^{n-1} v_1 - b_1 \end{aligned}$$

since the CDF of $v$ is now

$$F_V(x) = P(v \leq x) = \frac{x}{a}$$

$$\begin{aligned} u_1 &= \left[\frac{V(b_1)}{a}\right]^{n-1} v_1 - b_1 \\ \frac{du_1}{db_1} &= (n-1)\left[\frac{V(b_1)}{a}\right]^{n-2} \frac{V'(b_1)}{a} v_1 - 1 \\ &= (n-1)\left(\frac{1}{a}\right)^{n-1} v_1^{n-1} \frac{1}{B'(v_1)} - 1 \end{aligned}$$

$u_1$ max when

$$B'(v_1) = (n-1)\left(\frac{1}{a}\right)^{n-1} v_1^{n-1}$$

$$B(v_1) = \int (n-1)\left(\frac{1}{a}\right)^{n-1} v_1^{n-1} dv$$

$$B(v_1) = (n-1)\left(\frac{1}{a}\right)^{n-1} \frac{v_1^n}{n}$$

hence the expected revenue is

$$E(R) = \int_0^a B(v_1) \times n \times f(v) dv$$

since f(v) is the pdf of v

$$f(v) = \frac{1}{a}$$

$$\begin{aligned} E(R) &= \int_0^a B(v_1) \times n \times f(v) dv \\ &= \int_0^a (n-1)\left[\frac{1}{a}\right]^n v_1^n dv \\ &= (n-1)\left(\frac{1}{a}\right)^n \left[\frac{v^{n+1}}{n+1}\right]_0^a \\ &= \frac{n-1}{n+1} a \end{aligned}$$