CHAPTER 1

Week 2 lecture notes

1.1 Recap from previous week

From last week

- The n-dimensional euclidean space $\mathbb{R}^n$
- functions where
$$f : \mathbb{R}^n \to \mathbb{R}^m$$

1.2 This week

- enrich algebraic manipulation of vectors where $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$
- preview of calculus with vectors

1.3 The dot product

For two vectors $\underline{v}$ and $\underline{u}$, their dot product is

$$\underline{u} \cdot \underline{v} = \sum_{k=1}^{n} u_k v_k$$

Note the similarity with finding the *norm* of a vector

$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}$$
$$\underline{v} \cdot \underline{v} = \|\underline{v}\|^2$$

Note some intuitive rules for the dot product, and their geometric meaning

$$\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$$
$$\underline{v} \cdot \underline{0} = 0$$

IMPORTANT - Commit to memory!!
For vectors $\underline{v}$ and $\underline{u}$ separated by angle θ

$$\underline{u} \cdot \underline{v} = \|\underline{u}\|\|\underline{v}\|cos\theta$$

Hence the angle θ is

$$arccos\frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\|\|\underline{v}\|}$$

For example, for vector $\underline{v} = \begin{pmatrix} 2 \\ 6 \\ 0 \\ -1 \\ 5 \end{pmatrix}$, find a vector $\underline{u}$ where all the elements are non-zero, which is orthogonal to v

one such vector is

$$\underline{u} = \begin{pmatrix} -6 \\ 2 \\ 343738943948390 \\ 3 \\ 1 \end{pmatrix} :)$$

1.3.1 The Cross Product

Note that unlike the dot product which works in *all dimensions*, the cross product only works in

- $\mathbb{R}^3$
- $\mathbb{R}^2$, if you cheat :)

The dot product of two vectors $\underline{u}$ and $\underline{v}$ is

$$\underline{u} \times \underline{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

Note the anti-commutative property of the cross product

$$\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

IMPORTANT - Commit to memory!!
Note the geometric understanding of the cross product, where the length of the cross product contains information about the length of $\underline{u}$ and $\underline{v}$ and the angle between them θ

$$\|\underline{u} \times \underline{v}\| = \|\underline{u}\|\|\underline{v}\|sin\theta$$

The length of $\underline{u} \times \underline{v}$ also gives the *area of the parallelogram* with sides $\underline{u}$ and $\underline{v}$

Note that instead of detecting orthogonality like the dot product, the cross product detects parallel vectors

Also note the *mutual orthogonality* of the cross product $\underline{u} \times \underline{v}$ to $\underline{u}$ and $\underline{v}$

1.3.2 The Scalar Triple Product

Don't need to commit to memory!!
The scalar triple product takes as its arguments three vectors in $\mathbb{R}^3$: $\underline{u}$, $\underline{v}$, $\underline{w}$, And returns a scalar

$$\underline{u} \cdot (\underline{v} \times \underline{w})$$

However, know the geometric interpretation of the scalar triple product, which contains information about the 3-d volume spanned by $\underline{u}, \underline{v}, \underline{w}$, in the shape of parallelopiped

Also note the anti-symmetrical properties of the scalar triple product

$$\begin{aligned}
&\underline{u} \cdot (\underline{v} \times \underline{w}) \\
=&\underline{v} \cdot (\underline{w} \times \underline{u}) \\
=&\underline{w} \cdot (\underline{u} \times \underline{v}) \\
=&-\underline{u} \cdot (\underline{w} \times \underline{v}) \\
=&-\underline{w} \cdot (\underline{v} \times \underline{u}) \\
=&-\underline{v} \cdot (\underline{u} \times \underline{w})
\end{aligned}$$

A note on PrepQuiz, Practice problems, and the Friday Quiz

- PrepQuiz is for conceptual understanding. Problems are different from those found in the Friday quiz
- Practice problems found on canvas are more similar to the problems expected on Friday

1.3.3 Calculus with parameterized curves

For a curve

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

The derivative is trivial

$$\gamma'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

If the position vector is denoted by $\gamma(t)$, then

$$\begin{aligned}
\text{velocity } \underline{v} &= \gamma' \\
\text{speed } s &= \|\underline{v}\| \\
\text{acceleration } \underline{a} &= \gamma''
\end{aligned}$$

Ability to decompose acceleration into its orthogonal components is NOT needed for 1410
The acceleration vector $\underline{a}$ can be decomposed into

$$\underline{a} = \underline{a}_{TAN} + \underline{a}_{NORMAL}$$

Note the unit tangent vector, especially note that all derivatives are w.r.t parameter t

$$\underline{\hat{T}} = \frac{\gamma'}{\|\gamma'\|}$$

and the unit normal vector

$$\begin{aligned}\underline{\hat{N}} &= \frac{\frac{d}{dt}\underline{\hat{T}}}{\|\frac{d}{dt}\underline{\hat{T}}\|} \\ &= \frac{\underline{T}'}{\|\underline{T}\|}\end{aligned}$$

1.3.4 Rules for derivatives of vectors

For two vectors $\underline{u}$ and $\underline{v}$,

$$\begin{aligned}(\underline{u}\cdot\underline{v})' &= \underline{u}'\cdot\underline{v}+\underline{u}\cdot\underline{v}' \\ (\underline{u}\times\underline{v})' &= \underline{u}'\times\underline{v}+\underline{u}\times\underline{v}'\end{aligned}$$

1.3.5 Arclength of a curve

For a curve

$$\gamma : [a.b] \to \mathbb{R}^n$$

As covered in 1400, don't memorize complicated formulae for volume or work done! Write down

$$l = \int dl$$

where dl can be understood as a velocity vector, hence the arclength element dl is the "speed" of moving along curve $\times$ the "time elapsed"

$$dl = \|\gamma'\| dt$$

hence

$$l = \int dl = \int_a^b \|\gamma'\| dt$$

CHAPTER 2

Week 3 Lecture

2.1 Matrix Algebra

2.1.1 Matrix-Vector Multiplication

Think about a matrix-vector multiplication as a weighted sum of the individual column elements based on the vector

Find the product of matrix $\begin{pmatrix} 1 & 7 & 2 & 0 \\ -1 & 5 & 0 & 3 \\ 0 & -4 & 2 & 1 \end{pmatrix}$ and vector $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} =$

$1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$

For two vectors $\underline{u}$ and $\underline{v} \in \mathbb{R}^n$,

$$\underline{u}^T \underline{v} = \underline{u} \cdot \underline{v}$$

Fact: For two matrices A, B

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

Proof of the inverse The inverse has to satisfy the condition

$$AA^{-1} = I$$

Hence for $B^{-1}A^{-1}$

$$(B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Proof of the transpose
Recall from lecture video that $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki}$$

$$\left(A^T B^T\right)_{ij} = \left(A^T_{ik} B^T_{kj}\right) = \sum_{k=1}^{n} A_{jk}B_{ki}$$

2.1.2 Gaussian elimination / Row reduction

Recall from last week that to find the line of intersection of two planes, we used the point and tangent vector method

Alternatively, for two planes

$$x - 2y + 5z = 10$$
$$6x - 2y + 3z = -1$$

We can solve for x and y with z as a parameter, which would give the parameterized equation of the line at the intersection of the two planes

$$\begin{pmatrix} 1 & -1 & 510 & \\ 6 & -2 & 1 & -1 \end{pmatrix}$$

2.1.3 Block-diagonal matrices

Consider the matrix block diagonal matrix A

$$\begin{pmatrix} 3 & 4 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 6 & -3 \end{pmatrix}$$

A^{-1} is also a block diagonal matrix, where each "sub-block" is the inverse of the original block

$$\begin{pmatrix} 3 & -4 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 2 & -\frac{5}{3} \end{pmatrix}$$

2.1.4 Finding the inverse via row reduction

To find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

We row reduce the following matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right)$$

CHAPTER 3

Week 4 Lecture

3.1 Coordinates & Bases

For two vectors $\underline{u}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\underline{u}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, and a vector $\underline{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$\underline{v}$ can be written in the $(\underline{u}_1, \underline{u}_2)$ basis as

$$\underline{v} = c_1\underline{u}_1 + c_2\underline{u}_2$$

The equation can be rewritten as

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

3.2 Orthogonality and orthonormativity

Definition: An orthogonal matrix Q is an matrix whose columns are an **orthonormal** basis

For example,

$$Q = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Or the Permutation matrix

$$P = \begin{bmatrix} \text{all 0's, except for} \\ \text{a unique 1 in each column / row} \end{bmatrix}$$

such as

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Methods for computing the determinant

1. Minor expansion
2. Row reduction!
3. Block-diagonal matrices

Specifically regarding block-diagonal matrices, we can find the determinant by taking the product of the respective block matrices along the diagonal

$$A = \begin{pmatrix} B_1 & \dots & \dots & 0 \\ \vdots & B_2 & & \\ \vdots & & \ddots & \\ & & & B_n \end{pmatrix}$$

$$Det(A) = \Pi_{i=1}^{n} Det B_i$$

Prof-G's improv proof on generalizing block diagonal matrices

For a matrix comprised of $2-by-2$ blocks

$$W = \left[\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right]$$

We want to proof that

$$Det(W) = (Det(A))(Det(C))$$

$$\left[\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right] = \left[\begin{array}{c|c} A & 0 \\ \hline B & I \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline B & C \end{array}\right]$$

Hence

$$Det\left(\left[\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right]\right) = Det \left[\begin{array}{c|c} A & 0 \\ \hline B & I \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline B & C \end{array}\right]$$

CHAPTER 4

Week 5 Lectures Week of 25th Sept

4.1 Multivariate functions

Multivariate functions map inputs in $ndimensions$ to output in m dimensions

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Such functions include both implicit and parameterized functions

- Parameterized equations such as a parameterized curve $r : \mathbb{R}^1 \rightarrow \mathbb{R}^3$
- Implicit equations such as a parameterized curve $x^2 + y^2 = 4$

However, MATH-1410 will work with the general case, where any multivariate function can be thought of as

$$f ; \mathbb{R}^n \rightarrow \mathbb{R}^m$$

4.1.1 The partial derivative

Definition: The partial derivative wrt a variable

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$$

The partial derivatives are

$$\frac{\delta f_1}{\delta x}, \frac{\delta f_2}{\delta x}, \frac{\delta f_1}{\delta y}, \frac{\delta f_2}{\delta y}$$

4.1.2 The Derivative

The derivative $[Df]$ can be thought of as a data structure, where the element

$$[Df]_{j,i}$$

Corresponds to the partial derivative of the $j - th$ output and $i - th$ input
For example, for

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 y - 2xy \\ x^3 + 3y^2 \end{pmatrix}$$

The derivative is

$$[Df] = \begin{bmatrix} 2xy - 2y & x^2 - 2x \\ 3x^2 & 6y \end{bmatrix}$$

Given a point, we can evaluate the derivative at that point. For example, at point $x = 1, y = 2$, we get

$$[Df]_{\substack{x=1\\y=2}} = \begin{bmatrix} 0 & -1 \\ 3 & 12 \end{bmatrix}$$

Given the derivative evaluated at a point, we can directly read off the sensitivity of the $j_t h$ output to the $i - th$ input.
In this case, the $2nd$ output is 12 times as sensitive as the $1st$ output to the $2nd$ input

4.2 The big lift!!

IMPORTANT: the derivative is a linear transformation that takes *vectors of rates of change of inputs* to *vectors of rates of change of outputs*

$$\underline{l} = [Df]_{\underline{a}}\underline{h}$$

4.2.1 Intuitive understanding of matrix as a linear transformation

Consider for a function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

If the function as derivative $[Df]_{\underline{a}}$, the derivative is of the form

$$\left[\begin{array}{c|c|c|c} \frac{\delta}{\delta x_1} & \frac{\delta}{\delta x_2} & \cdots & \frac{\delta}{\delta x_n} \end{array}\right]$$

The $i - th$ column of $[Df]_{\underline{a}}$ describes how $f(\underline{a})$ changes with a unit change to $\underline{a}$ in the $i - th$ dimension

In other words, the $i - th$ column describes a parametric curve of $f(\underline{a})$ in $m - th$ dimensional space, i.e.

$$\gamma : \mathbb{R}^1 \to \mathbb{R}^m$$

4.2.2 The derivative linear transformation as a weighted sum of columns

Recall that all matrix transformations are a weighted linear combination of the columns of the matrix

Applied to the derivative, when the derivative $[Df]_{\underline{a}}$ maps the rates of change of inputs

$$\underline{l} = [Df]_{\underline{a}}\underline{h}$$

,
The rates of change of outputs $\underline{l}$ is the column-wise combination of $[Df]_{\underline{a}}$, weighted by the elements of $\underline{h}$

4.2.3 An example with polar coordinates

For the function P that maps polar coordinates $\begin{pmatrix} r \\ \theta \end{pmatrix}$ to Euclidean coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} rcos\theta \\ rsin\theta \end{pmatrix}$$

The derivative is given by

$$[DP] = \begin{bmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{bmatrix}$$

Picking any point, such as $(1, \frac{\pi}{6}$,

$$[DP]_{1,\frac{\pi}{6}} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

4.2.4 Another example

For a parabola

$$z = x^2 + y^2$$

Note that z is a function of $xandy$, in the form

$$f : \mathbb{R}^2 \to \mathbb{R}$$

The derivative is

$$[Dz] = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

At point $(1, 2)$,

$$[Dz]_{\underline{a}} = \begin{bmatrix} 2 & 4 \end{bmatrix}$$

Writing the system in a parametric form, in the form

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$G \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \\ s^2 + t^2 \end{pmatrix}$$

The derivative $[DG]$ is

$$[DG] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2s & 2t \end{bmatrix}$$

4.2.5 Inferring information from a derivative

Given a derivative $[Df]_{\underline{a}}$, without knowing the underlying function f

$$[Df]_{\underline{a}} = \begin{bmatrix} 2 & 2 & 1 \\ -3 & 4 & 0 \\ 1 & 7 & 3 \\ 0 & 8 & 4 \end{bmatrix}$$

What can be inferred is

- the function f takes 3 inputs and returns 4 outputs
- the 2nd output is completely insensitive to the third input
- the 4th output is completely insensitive to the first input
- when all inputs are increasing at unit rate, all outputs are increasing

$$\underline{h} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, [Df]_{\underline{a}}\underline{h} = \begin{pmatrix} 5 \\ 1 \\ 11 \\ 12 \end{pmatrix}$$

4.2.6 Another example

For a function

$$\underline{y} = f(\underline{x}) = A\underline{x}$$

It can be shown that

$$[Df] = A$$

More importantly, if

$$\underline{y} = \begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix} \underline{x}$$

Then the inverse is

$$\underline{x} = \begin{bmatrix} \frac{2}{7} & \frac{5}{14} \\ \frac{1}{7} & -\frac{1}{14} \end{bmatrix} \underline{y}$$

Important:

$$\frac{\partial y_2}{\partial x_1} = 2, \quad \frac{\partial x_1}{\partial y_2} = \frac{5}{14}$$

4.2.7 A Taylor perspective

Recall from single variable calculus that for a function f, at very small values of h,

$$f(x+h) = f(x) + f'(x)h + \dots$$

In the multivariate case

$$f(\underline{x}+\underline{h}) = f(\underline{x}) + [Df]\underline{h} + \dots$$

Where

- $f(\underline{x})$ is the $0-th$ term
- $[Df]\underline{h}$ is the linear term

This **is** the definition of the derivative!! $\rightarrow$ it is the coefficient of the linear term in the Taylor series

Another example

For a function

$$S(A) = A^2$$

Where A is a $2by2$ matrix, the derivative $[DS]$ would have 4 inputs and 4 outputs

$$S(A+H) = (A+H)^2 = A^2 + AH + HA + H^2$$

$$\begin{aligned} S(A+H) &= S(A) + [DS]_A H \quad &+ \dots \\ &= A^2 + AH + HA \quad &+ H^2 \end{aligned}$$

Hence,

$$[DS]_A H = AH + HA$$

CHAPTER 5

Week 6 Lecture

5.1 Recap

Recall from last week that for a function of the form

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

The derivative evaluated at $\underline{a}$

$$[Df]_{\underline{a}}$$

is a **linear transformation**

This week's focus will be on the rules of differentiation.

5.2 Rules of differentiation

5.2.1 Linearity of derivatives

Rule 1: Differentiation is a **linear operator**

$$[D(f+g)] = [Df] + [Dg] \qquad \text{(addition)}$$

$$[D(cf)] = c\,[Df]\,, c \in \mathbb{R} \qquad \text{(scalar multiplication)}$$

Rule 2: The chain rule
Recall from single variable calculus that for two functions f, g, their composition $f \circ g$ has derivative

$$(f \circ g)'_a = f'_{g(a)} g'_a$$

In the multivariate case, for two functions f, g, their composition $f \circ g$ has derivative

$$[D(f \circ g)]_{\underline{a}} = [Df]_{g(\underline{a})}\,[Dg]_{\underline{a}}$$

Rule 3: The inverse rule
Recall that

$$e^{lnx} = x = ln(e^x)$$

Taking the composition

$$e \circ ln = ID, \text{ the identity function, where } ID(x) = x$$

Taking the derivative of the composition $(e \cdot ln)(x) = x$

$$e^{lnx} \cdot (lnx)' = 1$$

Replacing e^{lnx} with x

$$x \cdot (lnx)' = 1$$

Hence we can find the derivative of lnx

$$(lnx)' = \frac{1}{x}$$

In the multivariate case, for a function f and its inverse f^{-1}

$$f \circ f^{-1} = ID = f^{-1}f$$

Taking the derivative of both sides

$$\left[D(f \circ f^{-1}\right] = [D(ID)]$$

$$[Df]\left[D(f^{-1})\right] = I$$

Recall that if $BA = I, B = A^{-1}$, hence we can conclude that

$$\left[Df^{-1}\right] = [Df]^{-1}$$

Implications: we can compute the derivative of the inverse without knowing what the inverse is

Example:

$$P\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rcos\theta \\ rsin\theta \end{pmatrix}$$

$$P^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ arctan(\frac{y}{x}) \end{pmatrix}$$

We can find the derivative of $\left[D(P^{-1})\right]$ without taking the derivative of P^{-1}

We start by finding

$$[DP] = \begin{bmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{bmatrix}$$

Applying the inverse rule

$$\left[D(P^{-1})\right] = [DP]^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ \frac{sin\theta}{r} & \frac{cos\theta}{r} \end{bmatrix}$$

Note that the only place where the inverse is not defined is at the origin of the polar plane, where

$$r = 0$$

Note that r is also the determinant of $[DP]$

This leads us to the **inverse function theorem**

Inverse function theorem: A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is locally invertible at $F(\underline{a})$ **if**$[DF]_{\underline{a}}^{-1}$ exists (i.e., $Det\,[DF]_{\underline{a}} \neq 0$)

Note that the theorem goes if A then B, not A does **NOT** imply not B

- e.g. $f(x) = x^3$, the derivative at $x = 0$ is 0, but the inverse $x^{\frac{1}{3}}$ exists

Points worth taking note of

- For the single variable chain rule, the evaluation points are intuitive. For example,

$$\frac{d}{dx} sin(x^2) = 2xcos(x^2)$$

f' is evaluated at $g(a) = x^2$, NOT $a = x$. The same applies for the multivariate case

- In the single variable case, the derivative obtained by the chain rule is commutative

$$(f \circ g)'_a = f'_{g(a)} g'_a = g'_a f'_{g(a)}$$

The same does NOT hold for the multivariate case

$$[D(f \circ g)]_{\underline{a}} = [Df]_{g(\underline{a})} [Dg]_{\underline{a}} \neq [Dg]_{\underline{a}} [Df]_{g(\underline{a})}$$

As a fun example: demonstrating the linear property of the derivative using the chain rule

Give the single variable functions $u(x), v(x)$

Claim: $(u + v)' = u' + v'$

Consider: two functions, g, f such that

$$g : \mathbb{R}^1 \to \mathbb{R}^2, g(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

$$f : \mathbb{R}^2 \to \mathbb{R}^1, f \begin{pmatrix} u \\ v \end{pmatrix} = u + v$$

Taking the composition

$$(f \circ g)(x) = u(x) + v(x)$$

Taking the derivatives $[Df], [Dg]$

$$[Dg] = \begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$[Df] = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Applying the chain rule:

$$[D(f \circ g)] = [Df] [Dg] = u' + v'$$

As another fun example: demonstrating the single variable product rule using the chain rule
Give the single variable functions $u(x), v(x)$

Claim: $(uv)' = vu' + v'u$

Consider: two functions, g, f such that

$$g : \mathbb{R}^1 \to \mathbb{R}^2, g(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

$$f : \mathbb{R}^2 \to \mathbb{R}^1, f \begin{pmatrix} u \\ v \end{pmatrix} = uv$$

Taking the composition

$$(f \circ g)(x) = u(x)v(x)$$

Taking the derivatives $[Df]$, $[Dg]$

$$[Dg] = \begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$[Df] = \begin{bmatrix} v & u \end{bmatrix}$$

Applying the chain rule:

$$[D(f \circ g)] = [Df]\,[Dg] = vu' + v'u$$

The old school way of doing chain rules
For a function $z = z(x, y), x = x(t), y = y(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

We can arrive at the same derivative by setting up functions f, g correctly

$$g(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = z(x, y)$$

Taking the partial derivatives

$$[Df] = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$$

$$[Dg] = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$$

Applying the chain rule

$$[D(f \circ g)] = [Df]\,[Dg] = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

5.3 Another big lift! and purely optional - The implicit function theorem

Recall a simple example of implicit differentiation
Given

$$x^2 + y^2 = 1$$

Differentiating both sides implicitly

$$2xdx + 2ydy = 0$$

We can solve for $y = y(x)$ by finding

$$\frac{dy}{dx} = -\frac{2x}{2y}, \text{ solvable for } y \neq 0$$

This can be rewritten as

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Implicit function theorem: states that for

$$F(\underline{x},\underline{y}) = \underline{0}$$

Where there are m nonlinear equations and where $\underline{y}$ has m variables
We can locally solve for $\underline{y} = \underline{y}(\underline{x})$ if

$$Det\left[\frac{\partial F}{\partial \underline{y}}\right] \neq 0$$

In addition

$$\left[\frac{\partial \underline{y}}{\partial \underline{x}}\right]_{\underline{a}} = -\left[\frac{\partial F}{\partial \underline{y}}\right]^{-1}\left[\frac{\partial F}{\partial \underline{x}}\right]$$

5.4 Chain Rule examples

5.4.1 Example 1

For a function

$$f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^3 + 2uv \\ e^{uv} \\ sin(u^2+v) \end{pmatrix}$$

and another function

$$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xyz \\ x + 2y - 3z \end{pmatrix}$$

The composition $f \circ g$ and $g \circ f$ both exist

We can compute the derivatives

$$[Dg] = \begin{bmatrix} yz & xz & xy \\ 1 & 2 & -3 \end{bmatrix}$$

$$[Df] = \begin{bmatrix} 3u^2 + 2v & 2u \\ ve^{uv} & ue^{uv} \\ 2ucos(u^2+v) & cos(u^2+v) \end{bmatrix}$$

By the chain rule,

$$\begin{aligned} [D(f \circ g)] &= [Df]\,[Dg] \\ &= \begin{bmatrix} (3u^2+2v)(yz) + 2u & \dots & \dots \\ \vdots & \ddots & \dots \\ \vdots & \dots & \ddots \end{bmatrix} \end{aligned}$$

Note that the result of $[Df]\,[Dg]$ will likely have a mix of x, y, z and u, v, read the question carefully to determine if there's a need to convert to consistent variables

5.4.2 Example 2

Recall from single variable calculus that

$$(e^x)' = e^x$$

$$(a^x)' = a^x lna$$

What about something such as

$$((u(x))^x)' = ?????$$

$$(u(x)^{v(x)})' = ?????$$

For $u(x)^{v(x)}$, we can derive its derivative using the multivariate chain rule.

We start by defining the correct functions

$$g(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

$$f\begin{pmatrix} u \\ v \end{pmatrix} = u^v$$

Note that the u and v in f and g stand for generic variables, and are not the same! (u, v are local to the function f, g)

We can compute the derivatives

$$[Dg] = \begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$\begin{aligned} [Df] &= \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \\ &= \begin{bmatrix} vu^{v-1} & u^v lnu \end{bmatrix} \end{aligned}$$

Applying the chain rule

$$\begin{aligned} [D(f \circ g)] &= [Df]\,[Dg] \\ &= vu^{v-1}u' + (u^v + lnu)v' \end{aligned}$$

Alternatively, using different variables
We start by defining the correct functions

$$g(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

$$f\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi^\eta$$

We can compute the derivatives

$$[Dg] = \begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$\begin{aligned} [Df] &= \begin{bmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \eta} \end{bmatrix} \\ &= \begin{bmatrix} \eta\xi^{\eta-1} & \xi^\eta ln\xi \end{bmatrix} \end{aligned}$$

Applying the chain rule

$$\begin{aligned} [D(f \circ g)] &= [Df]_{g(u,v)}\,[Dg] \\ &= \left. \begin{bmatrix} \eta\xi^{\eta-1}u' + (\xi^\eta + ln\xi)v' \end{bmatrix} \right|_{\xi=u,\eta=v} \\ &= vu^{v-1}u' + (u^v + lnu)v' \end{aligned}$$

5.5 Inverse Function Theorem examples

The inverse function theorem **is not**

$$\left[Df^{-1}\right] = [Df]^{-1}$$

While this statement is true, it is a result of **the chain rule**

The inverse function theorem deals with the existence of the inverse

5.5.1 Example 1

Given a function

$$f\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}2w+3vw\\e^{uv}\\sin(u^2+v)\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Can we express u, v, w as sum functions of x, y, z? (i.e. taking the inverse)

$$\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}f_1(x,y,z)\\f_2(x,y,z)\\f_3(x,y,z)\end{pmatrix}$$

The short answer is no!! but inverse function theorem lets us find the local inverse.

We start by taking the derivative

$$[Df]=\begin{bmatrix}0 & 3w & 2+3v\\ ve^{uv} & ue^{uv} & 0\\ 2ucos(u^2+v) & cos(u^2+v) & 0\end{bmatrix}$$

By the inverse function theorem,

$$\text{if } [Df]^{-1} \text{ exists locally, then } f^{-1} \text{ exists locally}$$

Note that the **inverse** is not necessarily true, i.e

$$\text{if } [Df]^{-1} \text{ does not exist locally, then we cannot conclude if } f^{-1} \text{ exists locally}$$

To find out if the inverse exists at $(0, 0, 0)$

$$[Df]_{\underline{0}}=\begin{bmatrix}0&0&2\\0&0&0\\0&1&0\end{bmatrix}$$

We can compute the determinant,

$$Det\,[Df]_{\underline{0}}=0$$

Hence, we **cannot conclude** whether the inverse exists at the origin (since the converse of the inverse of the Inverse Function Theorem is not necessarily true

5.6 Implicit Function Theorem

Given a function

$$F:\mathbb{R}^{n+m}\to\mathbb{R}^m$$

With m non-linear equations, where

$$F(\underline{x},\underline{y})=\underline{0}$$

and $\underline{x}$ has n variables and $\underline{y}$ has m variables,

We can solve locally for $\underline{y}=\underline{y}(\underline{x})$

5.6.1 Example 1

Given a system

$$2x_1^2-3x_2+y_1^2-4y_2^2=0$$
$$x_1^3-x_1^2+3y_1^2-y_2=0$$

Can we y_1 and y_2 as functions x_1, x_2? maybe via some nasty algebra?

Short answer is no!! and this system involves just polynomials. what if it has a bunch of square roots, logarithms and exponents

Instead, we can linearize the system and row reduce

CHAPTER 6

Week 8 Lecture

In week 7, we focused on **scalar fields**, or functions of the form

$$f : \mathbb{R}^n \to \mathbb{R}$$

It is a scalar field because it assigns a scalar to every point in the field

The derivative $[Df]$ is still a linear transformation that takes rates of change of inputs and gives rate of change of the output

$$[Df] = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

For scalar fields, we can find the gradient

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

6.1 Optimization

Recall from high school calculus that for a function $f : \mathbb{R} \to \mathbb{R}$, a is a critical point if it satisfies a few tests

Consider for a point near a, we can approximate such point via the Taylor Series

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2$$

The derivatives $f'(a)$ and $f''(a)$ determine what the function looks like near a.

- if $f'(a) = 0$, we can infer the shape of $f(a)$ about a via $f''(a)$
- if $f''(a) = 0$, we can infer the shape of $f(a)$ about a via $f'(a)$

In the multivariate case

$$f : \mathbb{R}^n \to \mathbb{R}$$

Locally at $\underline{a}$

$$f(\underline{a} + \underline{h}) = f(\underline{a}) + [Df]_{\underline{a}}\,\underline{h} + \frac{1}{2}\underline{h}^T \left[D^2 f\right]_{\underline{h}} + \dots$$

$\underline{a}$ is critical if $[Df]_{\underline{a}} = 0$ (i.e. all partial derivatives are 0) or undefined

The second order derivative In true multivariate case, the second order derivative is a tensor, which is beyond the scope of this course

For scalar functions, the second order derivative is a square, symmetrical matrix where

$$\left[D^2 f\right]_{ij} = \frac{\partial^2 f}{\partial x_i y_j}$$

For example

$$f = x^2 - 3xy + \frac{y^2}{2}$$

$$[Df] = \begin{bmatrix} 2x - 3y & -3x + y \end{bmatrix}$$

$$\left[D^2 f\right] = \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}$$

For critical points, this course will focus on $2d$ scalar fields

$$f : \mathbb{R}^2 \to \mathbb{R}$$

where critical points are either

- local minima
- local maxima
- saddle point
- degenerate

Algorithm for determining the nature of a critical point
For a critical point $\underline{a}$

- look at $Det\left[D^2 f\right]_{\underline{a}}$
  - if $det < 0$ then saddle point
  - if $det = 0$ then degenerate case, test fails
  - if $det > 0$ then compute the trace
    - if the trace > 0,
      - local minima
    - if the trace < 0,
      - local maxima
    - if the trace $= 0$?
      - this case will never show up, if trace is 0, determinant would have been 0, think about why!

6.2 Extension - Game theory

Consider a 2-player game, each with 2 strategies, with the following payout matrix

$$P = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$$

Suppose the players play at random, but with a probability distribution of their strategies

Let the two players be A and B

$$\underline{a} = \begin{pmatrix} x \\ 1-x \end{pmatrix}, \underline{b} = \begin{pmatrix} y \\ 1-y \end{pmatrix}$$

The average payout to A is a function of x, y

$$\begin{aligned} f(x,y) &= \underline{a}^T P \underline{b} \\ &= \underline{a}^T \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= \underline{a}^T \begin{pmatrix} -y + 2(1-y) \\ 3y - 2(1-y) \end{pmatrix} \\ &= \begin{pmatrix} x \\ 1-x \end{pmatrix} \begin{pmatrix} -y + 2(1-y) \\ 3y - 2(1-y) \end{pmatrix} \\ &= -8xy + 4x + 5y - 2 \end{aligned}$$

To find the critical point of P, we take the derivative

$$[Df] = \begin{bmatrix} -8y + 4 & -8x + 5 \end{bmatrix}$$

At critical point, all partial derivatives are 0, hence

$$x = \frac{5}{8}, y = \frac{1}{2}$$

To find the nature of the critical point

$$\left[D^2 f\right] = \begin{bmatrix} 0 & -8 \\ -8 & 0 \end{bmatrix}$$

Since $Det\left[D^2 f\right] < 0$, the critical point is a saddle point

6.3 Constrained Optimization

We want to maximize a function of the form

$$F : \mathbb{R}^n \to \mathbb{R}$$

subject to some constraints

Recall that in a simpler case, if we subject our system to the constraint that

$$x + y = 10$$

We can **parameterize** the system, by expressing y as a function of x, i.e. $y = 10 - x$, and solve for the optimization accordingly

In the real world, most optimization problems are **implicit**, and hence we need the **Lagrange**

6.3.1 The Lagrange equations

Suppose we want to **extremize** F subject to $G = c$

- $G = c$ is some constraint
- $G = c$ represents a level set

We need to solve for where the vectors of ∇F and ∇G are parallel, and the constraints are satisfied

$$\nabla F = \lambda \nabla G$$

$$G = c$$

where λ is the Lagrange Multiplier

Example
Suppose we want to solve
$$F = x^2 + y^2$$
Constrained to the implicit constraint
$$G = 3x + 2y = 6$$
Now we resist the urge to parameterize the system and solve for a single variable optimization, and turn instead to Lagrange

We begin by finding the gradients
$$\nabla F = \begin{pmatrix} 2x \\ 2y \end{pmatrix}, \nabla G = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
By the Lagrange Equation,
$$\nabla F = \lambda \nabla G$$
$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
We obtain 3 equations and 3 unknowns
$$\begin{aligned} 2x &= 3\lambda \\ 2y &= 2\lambda \\ 3x + 2y &= 6 \end{aligned}$$
Solving for x, y, we get
$$x = \frac{18}{13}, y = \frac{12}{13}$$
Can we tell whether this point is a local minima or maxima?

- Yes, but we need eigenvalues
- Out of scope for 1400

CHAPTER 7

Week 9 Lecture

7.1 Integration

Integration was tricky in single variable calculus because of all the methods and tricks needed

In this course, we will only use integration by substitution in full generality.

This will be a slow week! Everyone is back at the same place!

7.1.1 Integration in 1-D

Recall the two kinds of integrals in 1-D

- The definite integral
$$\int_{x=a}^{b} f dx$$
The definite integral is a scalar value
- The indefinite integral
$$\int f dx$$
The indefinite integral is a **class** of functions

The two objects are joined by the Fundamental Theorem of Integral Calculus

- From the indefinite integral, find the antiderivative, evaluate at the end points and take the difference
- You could think of the indefinite integral as a **means to an end**

In multivariable calculus, there is **no longer the indefinite integral**

- The indefinite integral has no multivariate analogue

There are certain 1-D interpretations of the derivative and integral that do not persist into higher dimensions. For example

- The derivative as *the slope*
- The integral as *the area*

Instead, we work with the intuition

- The derivative as a **linear transformation**
- The integral as a **mass**

7.1.2 The integral as a mass

Think of f as the density as a function of space. The mass element dM of a infinitesimal length dx is given by

$$dM = f(x)dx$$

The full mass is

$$M = \int dm$$

The only formula needed for calculating integrals is

$$M = \int dm$$

$$L = \int dl$$

$$A = \int da$$

Now in the multivariate case

For a multivariate density function on domain D, of the form

$$f : \mathbb{R}^2 \to \mathbb{R}$$

Where the density at x_i, y_i is given by $f(x_i, y_i)$

Consider the infinitesimal area element dA, its mass is given by

$$dM = fdA$$

$$M = \int fdA$$

Consider that the dimensions of dA is given by

- height dy
- width dx

Hence we get

$$M = \int fdA = \iint f(x, y)dxdy$$

The double integral is really

$$\lim_{\substack{\triangle x \to 0 \\ \triangle y \to 0}} \sum_j \sum_i f(x_i, y_j) \triangle x_i \triangle y_j$$

Likewise in 3-D, for a function $f(x, y, z)$

$$M = \int fdV = \iiint f(x, y, z)dxdydz$$

General intuition about integrals

If $f > 0$, what can you say about

$$\iint_D f(x,y)dxdy$$

We can conclude that

$$\iint_D f(x,y)dxdy > 0$$

7.2 Computing integrals and the Fubini theorem

Theorem: Some complicated stuff
Idea: For some function which depends on a bunch of variables

$$f(\underline{x})d\underline{x}$$

where

$$\underline{x} = (x_1, x_2, \dots x_n)$$

$$d\underline{x} = dx_1, dx_2, \dots dx_n$$

The integral over $\mathbb{R}^n$ with respect to the volume element $d\underline{x}$ is

$$\int_{\mathbb{R}^n} f(\underline{x})d\underline{x} = \int \left(\dots \left(\int \left(\int f dx_1 \right) dx_2 \right) \right) dx_n$$

Most importantly: the order in which the integral is computed does not matter

Example For a function

$$D = f(x,y) = x^2 + yy$$

over the domain (rectangle) bound by $x = 1, x = 3, y = 0, y = 1$

The integral over D is given by

$$\int_D x^2 + y \; dA = \int_{y=0}^{1} \int_{x=1}^{3} x^2 + y \; dxdy = \int_{x=1}^{3} \int_{y=0}^{1} x^2 + y \; dydx$$

Evaluating the integral

$$\begin{aligned}
\int_D x^2 + y \; dA &= \int_{y=0}^{1} \int_{x=1}^{3} x^2 + y \; dxdy \\
&= \int_{y=0}^{1} \left(\frac{x^3}{3} + xy \right) \Bigg|_{x=1}^{3} dy \\
&= \int_{y=0}^{1} (9 + 3y) - (\frac{1}{3} + y) dy \\
&= y^2 + \frac{26}{3} y \Bigg|_{y=0}^{1} \\
&= \frac{29}{3}
\end{aligned}$$

Another example

Consider a density function

$$f = x^2 + y^2$$

Compute the mass of the domain D bound by

$$y = x^{\frac{1}{3}}, y = x^2$$

The set up of the double integral requires some thought

$$M = \int_{y=0}^{1} \int_{x=y^3}^{\sqrt{y}} x^2 + y^2 dxdy$$

Note that

- on the outermost integral, there can be no variables in the boundaries of integration because the integral is a **scalar** value
- on the inner integrals, think about fixing y, and track the end points of the row at a given y, the end points should be a function $x = f(y)$. Computing the inner integral gives a density function of y

We can also set up the double integral as

$$M = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} x^2 + y^2 dydx$$

Note that

- This time, to write down the limits of the inner integral, we fix x. Then, we keep track of the **column** at a given x, and we read off the upper and lower bounds of that column. Computing the integral would give us a density function with respect to x

7.3 A note not covered in lecture videos

By Fubini's theorem, if all limits are constant, and if the integrand is a product of a function of one variable, then we can "split up the integrals"

$$\int_c^d \int_a^b f(x)g(y)dxdy = \int_c^d g(y)dy \int_a^b f(x)dx$$

An example

$$\int_{z=2}^{5} \int_{y=-1}^{1} \int_{x=0}^{1} x^2y^3\sqrt{z} + x^4zdxdydz$$

This would take a while to compute, unless...

Note that the integral of $x^2y^3\sqrt{z}$ cancels out because $f(y) = y^3$ is an odd function, and the domain is symmetrical, hence $\int_{y=1}^{1} \int \int x^2y^3\sqrt{z}dxdzdy = 0$

Then, we can apply the above mentioned property, which states that the integral of a product is the product of integrals, provided all boundaries are constants

$$\begin{aligned}
\int_{z=2}^{5} \int_{y=-1}^{1} \int_{x=0}^{1} x^2y^3\sqrt{z} + x^4zdxdydz &= \int_{z=2}^{5} \int_{y=-1}^{1} \int_{x=0}^{1} dxdydz \\
&= \int_{z=2}^{5} zdz \int_{y=-1}^{1} dy \int_{x=0}^{1} x^4dx \\
&= \frac{21}{5}
\end{aligned}$$

7.4 A devilish example

Prove the following result

$$\int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} 1 + x \; dz \; dz = \frac{\pi}{3}$$

We start by looking at the domain

With respect to z axis, z is going from $z = 0$ to $z = \sqrt{1 - x^2 - y^2}$. Note that $z^2 = 1 - x^2 - y^2$ is the equation for a unit sphere.

Think of the first integral of collapsing 3-d mass along the z axis into the $x - y$ plane.

In the xyz space, we are collapsing the upper hemisphere

In the xy plane, for the y integral, we are collapsing the upper semicircle

In the x space, the domain goes from -1 to 1

For the whole integral, our domain is the quarter sphere, where z goes from 0 to 1, y goes from 0 to 1, and x goes from -1 to 1

We split up the integral using the linear property of integrals

$$\int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} 1 + x \; dz \; dz = \iiint_D 1 \; dz \; dz + \iiint_D x \; dz \; dz$$

The first integral is the integral of the volume element

$$\int\limits_{x=-1}^{1} \int\limits_{y=0}^{\sqrt{1-x^2}} \int\limits_{z=0}^{\sqrt{1-x^2-y^2}} 1 \; dz \; dz = \iiint\limits_D 1 dV = \frac{1}{4}\frac{4\pi}{3} = \frac{\pi}{3}$$

The second integral makes use of the fact that $f(x) = x$ is an odd function and the domain is symmetrical, hence it evaluates to 0

7.5 Another example

Recall that for f on domain D

The average is given by

$$\overline{f = \frac{\int_D f d\underline{x}}{\int_D 1 d\underline{x}} = VOL(D)}$$

7.6 Another example

Find the average of

$$f = xy^2$$

Step 1: write the formula!!

$$\overline{f = \frac{\int_D f d\underline{x}}{\int_D 1 d\underline{x}}}$$

Step 2: evaluate the denominator

$$\text{area of domain} = 2 \times 2 = 4$$

Step 3: evaluate the integral

$$\int_0^2 \int_0^2 xy^3 \; dx \; dy = \left.\frac{x^2}{2}\right|_0^2 \left.\frac{y^4}{4}\right|_0^2 = 2 \times 4 = 8$$

Step 4

$$\frac{8}{4} = 2$$

7.7 Another example with a weird domain

what if the domain is the square from $(0,0)$ to $(2,2)$, excluding the square from $(0,0)$ to $(1,1)$

step 1: compute the average on the domain $(0,0)$ to $(1,1)$

$$\overline{f} = \int_0^1 \int_0^1 xy^3 dx dy = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$$

step 2: compute the average on the restricted domain

$$\overline{f}_{restricted} = \frac{2 \times 4 - \frac{1}{8}}{4-1} = \frac{63}{24}$$

CHAPTER 8

Week 10 lectures

This week, we will think about two applications of multivariate integration

Physical mass The first intuition will require us to think of an integral as physical mass

The function f is a function of density. To find the mass between a and b, we integrate the mass element

$$M = \int dM$$

The mass element is given by

$$dM = f(x)dx$$

$$M = \int_{x=a}^{b} dx$$

Probability The function f is a probability density function on domain D

$$f \sim PDF$$

where

$$f(x) \geq 0, \int_D f = 1$$

To find the probability, we integrate the probability element

$$P = \int dP$$

For $A \in D$, the probability that $\underline{x}$ is in A is

$$P(\underline{x} \ in \ A) = \int_A DP = \int_A f(\underline{x})d\underline{x}$$

Normalized to

$$P(\underline{x} \ in \ D) = 1$$

In $\mathbb{R}^2$, the probability element in domain D is

$$dP = \frac{1}{AREA(D)}dxdy$$

Note that

$$P(x = c) = \int_{\{c\}} dP = \int_{x=c}^{c} f(x)dx = 0$$

8.1 Centroids & Centers of mass

For the centroid between f and g on domain D, whose area is A

In high school the formula to memorize would have been

$$\overline{x} = \frac{1}{A} \int_{x=a}^{b} x(f(x) - g(x))dx$$

$$\overline{y} = \frac{1}{2A} \int_{x=a}^{b} (f(x) - g(x))^2 dx$$

Instead, now with the language of double integrals, we can find $\overline{x}, \overline{y}$ using the average formula

$$\overline{x} = \frac{1}{A} \iint_D xdA$$

$$\overline{y} = \frac{1}{A} \iint_D ydA$$

Centroid

$$\overline{x} = \frac{\int_D xdV}{\int_D dV}$$

However, in calculating the centroid this way, we are assuming uniform density. If we want to find the center of mass, we need to use the **mass element**

$$\overline{x} = \frac{\int_D xdM}{\int_D dM}$$

where

$$dM = f(\underline{(}x)d\underline{x}$$

8.2 Expectation & Spread

For probability in $2D$, for a PDF of $f(x, y)$ over domain D such that $f(x, y) \geq 0$ and $\iint_D fdxdy = 1$

For two random variables X, Y

$$E(X) = \overline{x} = \int_D xdP$$

$$E(Y) = \overline{y} = \int_D ydP$$

The spread is measured by variance and standard deviation

$$V(X) = \int_D (x - E(X))^2 dP$$

$$\sigma(X) = \sqrt{V(X)}$$

Why doe the variance formula make sense? If we choose a good coordinate system (domain) such as $E(X) = 0$,

$$V = \int_D x^2 dP = \overline{x^2}$$

Now, it is easier to see why the variance is a measure of spread about the mean

8.3 Moment of inertia

For a domain $[-a, a]$ with uniform density, the moment of inertia is given by

$$\int_{-a}^{a} x^2 dM$$

In general, for a mass element dM at a distance r away from the axis, the integral element is the product of squared distance and the mass element

$$dI = r^2 dM$$

The moment of inertia is thus

$$I = \int dI = \int r^2 dM$$

Consider the symmetry between the two concepts

Physical	Probability
Variance $V(X)$ $\int x^2 dP$	Moment of Inertia I $\int r^2 dM$
Standard Deviation $\sqrt{V(X)}$	Radius of gyration $R_g = \sqrt{\frac{I}{M}}$

8.4 Things that we need to learn to compute this week

In the language of physics

- Mass
- Centroid
- Center of mass
- Moment of gyration
- Moments of inertia
- Radius of gyration

In the language of probability

- Probability
- Expectation
- Variance
- Standard Deviation

8.4.1 Example 1 - Probability

Given a probability density function

$$\rho = x^2 + y^2$$

On a domain

$$D = \{\|x\| \leq 2, \|y\| \leq 3\}$$

Note that total probability must be 1

$$\int\limits_{y=-3}^{3} \int\limits_{x=-2}^{2} C(x^2 + y^2) dx\ dy = 1$$

To find the constant C such that the integral is 1

$$\int_{y=-3}^{3}\int_{x=-2}^{2} C(x^2+y^2)dx\ dy = \ C\left(\frac{x^3}{3}y + x\frac{y^3}{3}\right)\Bigg|_{y=-3}^{3}\Bigg|_{x=-2}^{2}$$
$$= 104C$$

Hence the probability density

$$\rho = \frac{1}{104}\left(x^2+y^2\right)$$

Now we can compute the probability, for example, to find

$$\begin{aligned} P(\ X \geq 1\ ) &= \int_{\{x\geq 1\}} \rho\ dA \\ &= \int_{y=-3}^{3}\int_{x=1}^{2} \frac{1}{104}\left(x^2+y^2\right) dx\ dy \\ &= \ \frac{1}{104}\left(\frac{x^3y+xy^3}{3}\right)\Bigg|_{y=-3}^{3}\Bigg|_{x=1}^{2} \\ &= \frac{4}{13} \end{aligned}$$

Given the same probability density function, find $P(\ X \geq 1$ and $Y \leq -1\ )$

Given the same probability density function, find $P(\ X \geq 1$ or $Y \leq -1\ )$

8.4.2 Example - Moments of Inertia

The moment of inertia is calculated by integrating the moment of inertia element

$$I = \int dI$$

Where dI is given by

$$dI = r^2 dM$$

Where

- r is the radius to axis of rotation
- dM is the mass element, equals to ρdV

8.4.3 Example - More probability

For a given PDF

$$\rho = Ce^{-\alpha x}$$

on a domain

$$D = [0, \infty)$$

Integrating the PDF from 0 to infinity and setting total probability to 0, we solve for $C = \alpha$, hence the PDF is

$$\rho = \alpha e^{-\alpha x}$$

We can define a joint PDF for arbitrary number of variables by taking the product of ρ_i

$$\rho(x_1, x_2 \ldots x_n) = \alpha_1\alpha_2 \ldots \alpha_n e^{-\alpha_1 x - \alpha_2 x - \ldots \alpha_n x}$$

CHAPTER 9

Week 10 lectures

This week, we will think about two applications of multivariate integration

Physical mass The first intuition will require us to think of an integral as physical mass

The function f is a function of density. To find the mass between a and b, we integrate the mass element

$$M = \int dM$$

The mass element is given by

$$dM = f(x)dx$$

$$M = \int_{x=a}^{b} dx$$

Probability The function f is a probability density function on domain D

$$f \sim PDF$$

where

$$f(x) \geq 0, \int_D f = 1$$

To find the probability, we integrate the probability element

$$P = \int dP$$

For $A \in D$, the probability that $\underline{x}$ is in A is

$$P(\underline{x}\ in\ A) = \int_A DP = \int_A f(\underline{x})d\underline{x}$$

Normalized to

$$P(\underline{x}\ in\ D) = 1$$

In $\mathbb{R}^2$, the probability element in domain D is

$$dP = \frac{1}{AREA(D)}dxdy$$

Note that

$$P(x = c) = \int_{\{c\}} dP = \int_{x=c}^{c} f(x)dx = 0$$

9.1 Centroids & Centers of mass

For the centroid between f and g on domain D, whose area is A

In high school the formula to memorize would have been

$$\overline{x} = \frac{1}{A}\int_{x=a}^{b} x(f(x) - g(x))dx$$

$$\overline{y} = \frac{1}{2A}\int_{x=a}^{b} (f(x) - g(x))^2 dx$$

Instead, now with the language of double integrals, we can find $\overline{x}, \overline{y}$ using the average formula

$$\overline{x} = \frac{1}{A}\iint_D xdA$$

$$\overline{y} = \frac{1}{A}\iint_D ydA$$

Centroid

$$\overline{x} = \frac{\int_D xdV}{\int_D dV}$$

However, in calculating the centroid this way, we are assuming uniform density. If we want to find the center of mass, we need to use the **mass element**

$$\overline{x} = \frac{\int_D xdM}{\int_D dM}$$

where

$$dM = f(\underline{(}x)d\underline{x}$$

9.2 Expectation & Spread

For probability in $2D$, for a PDF of $f(x, y)$ over domain D such that $f(x, y) \geq 0$ and $\iint_D fdxdy = 1$

For two random variables X, Y

$$E(X) = \overline{x} = \int_D xdP$$

$$E(Y) = \overline{y} = \int_D ydP$$

The spread is measured by variance and standard deviation

$$V(X) = \int_D (x - E(X))^2 dP$$

$$\sigma(X) = \sqrt{V(X)}$$

Why doe the variance formula make sense? If we choose a good coordinate system (domain) such as $E(X) = 0$,

$$V = \int_D x^2 dP = \overline{x^2}$$

Now, it is easier to see why the variance is a measure of spread about the mean

9.3 Moment of inertia

For a domain $[-a, a]$ with uniform density, the moment of inertia is given by

$$\int_{-a}^{a} x^2 dM$$

In general, for a mass element dM at a distance r away from the axis, the integral element is the product of squared distance and the mass element

$$dI = r^2 dM$$

The moment of inertia is thus

$$I = \int dI = \int r^2 dM$$

Consider the symmetry between the two concepts

Physical	Probability
Variance $V(X)$ $\int x^2 dP$	Moment of Inertia I $\int r^2 dM$
Standard Deviation $\sqrt{V(X)}$	Radius of gyration $R_g = \sqrt{\frac{I}{M}}$

9.4 Things that we need to learn to compute this week

In the language of physics

- Mass
- Centroid
- Center of mass
- Moment of gyration
- Moments of inertia
- Radius of gyration

In the language of probability

- Probability
- Expectation
- Variance
- Standard Deviation

9.4.1 Example 1 - Probability

Given a probability density function

$$\rho = x^2 + y^2$$

On a domain

$$D = \{\|x\| \leq 2, \|y\| \leq 3\}$$

Note that total probability must be 1

$$\int\limits_{y=-3}^{3} \int\limits_{x=-2}^{2} C(x^2 + y^2) dx\ dy = 1$$

To find the constant C such that the integral is 1

$$\int_{y=-3}^{3}\int_{x=-2}^{2} C(x^2+y^2)dx\ dy = C\left(\frac{x^3}{3}y + x\frac{y^3}{3}\right)\Bigg|_{y=-3}^{3}\Bigg|_{x=-2}^{2}$$
$$= 104C$$

Hence the probability density

$$\rho = \frac{1}{104}\left(x^2+y^2\right)$$

Now we can compute the probability, for example, to find

$$\begin{aligned} P(\ X \geq 1\ ) &= \int_{\{x\geq 1\}} \rho\ dA \\ &= \int_{y=-3}^{3}\int_{x=1}^{2} \frac{1}{104}\left(x^2+y^2\right) dx\ dy \\ &= \frac{1}{104}\left(\frac{x^3y+xy^3}{3}\right)\Bigg|_{y=-3}^{3}\Bigg|_{x=1}^{2} \\ &= \frac{4}{13} \end{aligned}$$

Given the same probability density function, find $P(\ X \geq 1$ and $Y \leq -1\ )$

Given the same probability density function, find $P(\ X \geq 1$ or $Y \leq -1\ )$

9.4.2 Example - Moments of Inertia

The moment of inertia is calculated by integrating the moment of inertia element

$$I = \int dI$$

Where dI is given by

$$dI = r^2 dM$$

Where

- r is the radius to axis of rotation
- dM is the mass element, equals to ρdV

9.4.3 Example - More probability

For a given PDF

$$\rho = Ce^{-\alpha x}$$

on a domain

$$D = [0, \infty)$$

Integrating the PDF from 0 to infinity and setting total probability to 0, we solve for $C = \alpha$, hence the PDF is

$$\rho = \alpha e^{-\alpha x}$$

We can define a joint PDF for arbitrary number of variables by taking the product of ρ_i

$$\rho(x_1, x_2 \ldots x_n) = \alpha_1\alpha_2\ldots\alpha_n e^{-\alpha_1 x-\alpha_2 x-\ldots\alpha_n x}$$

CHAPTER 10

Week 11 lectures

10.1 Changing coordinates

10.1.1 Polar coordinates

Consider a circle centered at origin with radius R, to compute the area using $x - y$ coordinates, we can evaluate

$$A = \int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dydx$$

We could instead convert to polar coordinates, where

$$dA = r\ dr\ d\theta$$

$$\begin{aligned} A &= \int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dydx \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^{R} dr\ d\theta \\ &= r|_0^R\ \theta|_0^{2\pi} \\ &= 2\pi R \end{aligned}$$

10.1.2 Cylindrical coordinates

$$dV = r\ drd\theta dz$$

$$\begin{aligned} x &= rcos\theta \\ y &= rsin\theta \\ z &= z \end{aligned}$$

Considering finding the moment of inertia of a cylinder of height h and radius R about the z axis

$$\begin{aligned} I &= \int dI \\ &= \int r^2 dM \\ &= \int_{z=0}^{h} \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r^2 \left(r dr d\theta dz\right) \\ &= \frac{2\pi h R^4}{4} \end{aligned}$$

10.1.3 Spherical coordinates

The spherical coordinates are

$$\begin{aligned} x &= \rho cos\theta sin\phi \\ y &= \rho sin\theta sin\phi \\ z &= \rho cos\phi \end{aligned}$$

Where

- $\rho : 0 \leq \rho \leq R$, radius
- $\theta : 0 \leq \theta \leq 2\pi$, azimuth
- $\phi : 0 \leq \phi \pi R$, inclination (angle from north pole)

What is the volume element in spherical coordinates? is it perhaps

$$V = \int dV = \int_{\phi=0}^{\pi} \int_{\theta}^{2\pi} \int_{\rho=0}^{R} d\rho d\theta d\phi$$

NO!! The volume element is

$$dV = \rho^2 sin\phi \ d\rho \ d\phi \ d\theta$$

We can try computing the volume of a sphere to test it out

$$\begin{aligned} V &= \int dV \\ &= \int_{\phi=0}^{\pi} \int_{\theta}^{2\pi} \int_{\rho=0}^{R} \rho^2 sin\phi d\rho d\theta d\phi \\ &= \frac{R^3}{3} 2\pi \times 2 \\ &= \frac{4}{3}\pi R^3 \end{aligned}$$

10.1.4 Finding the area element in polar coordinates

We want to show that

$$dA = r \ dr \ d\theta$$

Recall linear transformations from earlier in the semester, and the relationship between the determinant and the scaling of n-volume

Consider the transformation into polar coordinates

$$P \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rcos\theta \\ rsin\theta \end{pmatrix}$$

Recall from the chapter on differentiation that

$$[DP] = \begin{bmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{bmatrix}$$

Hence,

$$det\,[DP] = r\left(cos^2\theta + sin^2\theta\right) = r$$

And

$$dxdy = r\ dr\ d\theta$$

10.1.5 Finding the volume element in spherical coordinates

The transformation is as such

$$S\begin{pmatrix} \rho \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\ cos\theta\ sin\phi \\ \rho\ sin\theta\ sin\phi \\ \rho\ cos\phi \end{pmatrix}$$

The derivative is

$$[DS] = \begin{bmatrix} cos\theta\ sin\phi & -\rho\ sin\theta\ sin\phi & \rho\ cos\theta\ cos\phi \\ sin\theta\ sin\phi & \rho\ cos\theta\ sin\phi & \rho\ sin\theta\ cos\phi \\ cos\phi & 0 & -\rho\ sin\phi \end{bmatrix}$$

And

$$det\,[DS] = \rho^2 sin\phi$$

10.2 Change of variables theorem

10.2.1 Recall U-SUBS!!!

Recall from single variable that we can integrate by substitution via

$$u = F(x)$$
$$du = F'(x)dx$$

What we really were doing was

1. writing down a non-linear coordinate transformation
2. finding the new length element

Now we generalize into higher dimensions. Instead of calling it u-subs, we call it the **change of variables theorem**

10.2.2 Change of variables theorem

If

$$\underline{u} = F(\underline{x})$$

Then

$$d\underline{u} = |Det\,[DF]|\,d\underline{x}$$

Expressed in words, in arbitrary dimensions, the n-volume element in a new coordinate system is related to the volume element in the original coordinate system by the absolute value of the determinant of the transformation

This is a good point in the semester to look back and have our little minds blown by how far we've come. We can relate volume elements before and after non-linear transformations using little more than determinants and derivatives.

In old school calculus textbooks that do not formally introduce the derivative as a linear transformation, this theorem would be a lot more verbose (involving a bunch of partial derivatives

10.2.3 Examples with coordinate systems

Consider an inverted cone (with a flat, circular base) symmetrical about the z-axis, with height h , base radius R, with its tip at the origin. How would we describe this shape? Which coordinate system would we use?

Spherical coordinates We can observe that

- $0 \leq \theta \leq 2\pi$
- $0 \leq \phi \leq arctan\left(\frac{R}{h}\right)$
- $0 \leq \rho \leq$???????????

Cylindrical coordinates?

- $0 \leq z \leq h$
- $0 \leq \theta \leq 2\pi$
- $0 \leq r \leq z\frac{R}{h}$

10.2.4 Example, 9 Nov

For a cube centered at origin with length L, with density $f = \frac{1}{(x^2+y^2+z^2)^\alpha}$ for some constant $\alpha > 0$, does the mass converge?

$$M = \int dM = \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(x^2 + y^2 + z^2\right)^{-\alpha} dxdydz = ?????????????????$$

Trying spherical coordinates

$$\begin{aligned} M &= \int dM \\ &= \iiint \rho^{-2\alpha} \rho^2 sin\phi d\rho \; d\phi \; d\theta \end{aligned}$$

To determine if mass is finite, we only need to care about the asymptotic behavior, i.e. whether the mass of the infinitesimal sphere near the origin is finite
Hence, we can define the limits of integration as such for some ϵ

$$\begin{aligned} M &= \int dM \\ &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{\epsilon} \rho^{-2\alpha} \rho^2 sin\phi d\rho \; d\phi \; d \\ &= \left. \frac{4\pi\rho^{2-2\alpha+1}}{3-2\alpha} \right|_{\rho=0}^{\epsilon} \end{aligned}$$

Where is the badness here?

There are a few actually

- denominator cannot be zero, hence $\alpha \neq \frac{3}{2}$
- mass cannot be negative, hence $\alpha < \frac{3}{2}$
- integrating
$$\int x^n dx = \frac{x^{n+1}}{n+1}$$
only works for $n \neq -1$, hence $\alpha \neq \frac{3}{2}$

10.2.5 Another example in changing variables

We want to evaluate

$$\iint_D xy(x^2+y^2)dxdy$$

Where

$$D : 1 \leq xy \leq 4, 1 \leq x^2 - y^2 \leq 3$$

Recall that for

$$\underline{u} = F(\underline{x})$$

We have

$$d\underline{u} = |det\,[DF]|\,d\underline{x}$$

We change coordinates to hopefully make bounds of integration constant!
Let:

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} xy \\ x^2-y^2 \end{pmatrix}$$

$$[DF] = \begin{bmatrix} y & x \\ 2x & \\ -2y & \end{bmatrix}$$

$$Det\,[DF] = -2y^2 - 2x^2$$

Hence we get

$$\begin{aligned} dudv &= \left|-2y^2 - 2x^2\right| dxdy \\ &= 2(x^2+y^2)dxdy \\ dxdy &= \frac{dudv}{2(x^2+y^2)} \end{aligned}$$

The integral is thus

$$\begin{aligned} \iint_D \frac{xy(x^2+y^2)dudv}{2(x^2+y^2)} &= \int_{v=1}^{3}\int_{u=1}^{4} \frac{u}{2}dudv \\ &= v|_1^3\, \frac{u^2}{4}\bigg|_1^4 \end{aligned}$$

10.3 Addenda!

10.3.1 Surface area

For the surface area of a sphere of radius R, the surface area element is given by

$$d\sigma = R^2 sin\phi d\phi d\theta$$

This can be thought of as the volume element, except ρ is now fixed at R
We can verify by taking the integral

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} d\sigma = 4\pi R^2$$

10.3.2 Gaussian

In 1D, it is given by

$$\frac{1}{sqrt2\pi}e^{-\frac{x^2}{2}}$$

In 2D

$$\frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}$$

CHAPTER 11

Week 12 lectures - Fields

11.1 Fields

Recall from the whole semester that we have seen
scalar fields, which have the form

$$f : \mathbb{R}^n \to \mathbb{R}$$

vector fields, an example of which is the **gradient**

$$\nabla f$$

***smiley* fields**, where every point is a *smiley*

- smiley can be anything
- it can be a matrix, such as in

$$F : \mathbb{R}^n \to \mathbb{R}^m$$

- the derivative $[DF]$ is a matrix field

Moving on to think about integrals, given a scalar field $f : \mathbb{R}^n \to \mathbb{R}$, we can compute the integral over a domain, or we can compute a path integral
The path can be parameterized by $\gamma(t)$

$$\gamma : [a, b] \to \mathbb{R}^n$$

The path integral can be computed by integrating the arc length element, weighted by the scalar field

$$\int_\gamma f \ dl$$

$$= \int_\gamma f \ \ |\gamma'| \, dt$$

For example, given $r(t) = \begin{pmatrix} t \\ t^2 - 1 \end{pmatrix}$ and a scalar field $f = x^2 + y^2$ for $0 \leq t \leq 2$, we can compute that

$$dl = \left| \begin{pmatrix} 1 \\ 2t \end{pmatrix} \right| dt = \sqrt{1 + 4t^2} dt$$

The path integral is

$$\int_\gamma f \ dl = \int_{t=0}^{2} \left(t^2 + \left(t^2 - 1\right)^2 \right) \sqrt{1 + 4t^2} dt$$

Note: the path integral $\int_\gamma f \ dl$ is independent of the parameterization of γ

- this can be proved via the Change of Variables Theorem
- this makes sense intuitively!

11.2 Something new!! - the 1-form

Recall that for a scalar field f, we have the differential df

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

The question we must ask ourselves is

What is dx?

It is a **1-form**!! A 1-form on $\mathbb{R}^n$

- it takes in a vector and returns a scalar linearly

For example, in $\mathbb{R}^3$, for a vector $\underline{v}$

$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$\begin{aligned} dx\left(\underline{v}\right) &= v_x \\ dy\left(\underline{v}\right) &= v_y \\ dz\left(\underline{v}\right) &= v_z \end{aligned}$$

Why is this useful?: we can do math with it!

For example, we could define a 1-form α

$$\alpha = dx - 2dy + 5dz$$

$$\begin{aligned} \alpha\left(\underline{v}\right) &= dx(\underline{v}) - 2dy(\underline{v}) + 5dz(\underline{v}) \\ &= v_x - 2v_y + 5v_z \end{aligned}$$

We can effectively think of dx as the operator

$$dx \sim i\cdot \;\text{ OR }\; \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cdot$$

11.2.1 1-form fields!

A 1-form field has a 1-form at every point

$$\alpha = y\ dx + x^2\ dy - 3(x^2 + y^2)\ dz$$

This is **very different from a vector field**!!
Consider the scalar field

$$f = x^2 - 2xy + y^2$$

The gradient (vector field) is

$$\nabla f = (2x - 2y)\,\hat{i} + (2y - 2x)\hat{j}$$

- each point nudges in the direction of maximum increase

The differential (1-form field) is

$$df = (2x - 2y)\,dx + (2y - 2x)dy$$

- dx are measurement devices that measures small changes in x
- df can be thought of an operator, which is equivalent to $\nabla f\cdot$

11.2.2 Something new!!

Given a 1-form field α and path γ,

Define

$$\int_\gamma \alpha$$

Think!! What does it mean to take the dot product of the **velocity vector** along a path and the 1-form at that point?

$$\int_\gamma \alpha = \int_\gamma \left(\alpha_{\gamma(t)}\right)\left(\gamma'(t)\right) dt$$

Example
For a 1-form field

$$\alpha = y\ dx - 2x\ dy$$

and a path

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, 0 \le t \le 1$$

The velocity vector is

$$\gamma' = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

The path integral is

$$\begin{aligned}
\int_\gamma \alpha &= \int_\gamma y\ dx - 2x\ dy \\
&= \int_{t=0}^{1} \left(t^2\ dx - 2t\ dy\right) \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt \\
&= \int_{t=0}^{1} t^2 \cdot 1 - 2t \cdot 2t dt \\
&= \int_{t=0}^{1} -3t^2 dt
\end{aligned}$$

Why go through all the trouble?
This is useful!! The standard definite integral that we know and love

$$\int_{x=a}^{b} f(x) dx$$

is really the integral of the one form field

$$\alpha = f(x) dx$$

over a parameterized path

$$\gamma(t) = t, a \le t \le b$$

Hence, if we follow all the steps outlined above

$$\int_{x=a}^{b} f(x) dx = \int_{t=a}^{b} f(t) dt$$

More concretely....
Why integrate 1-form fields?

- Work
- Flux
- etc etc

11.3 The APEX

Consider a gradient (vector field), it has an associated gradient 1-form field.

Just as not every vector field is a gradient vector field, not very 1-form field is a gradient 1-form field

Independence of Path Theorem: integrating over a gradient 1-form field, the integral is independent of the path and only depends on the endpoints

For a gradient one form field df, over a path γ from a to b

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

Note the similarity of this idea to the fundamental theorem of integral calculus

11.4 Thursday Lecture - Recap

Recall that the biggest themes this week are

- 1-form fields in a form similar to
$$\alpha = y^2 \ dx - x \ dy$$
- integrating over a 1-form field along a parameterized path

11.4.1 Work, circulation & Flux

Consider the planar vector field denoting the force at every point in planar space

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$$

We can define the work 1-form

$$\alpha_{\vec{F}} = F_x \ dx + F_y \ dy$$

Instead of using the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ to denote the force in each component at every point, we use the 1-form dx and dy to measure the work done in each component.

We can also define the flux 1-form

$$\Phi_{\vec{F}} = F_x \ dy - F_y \ dx$$

Example
For a parameterized path γ along a circle of radius 2, counter-clockwise (positive)

Note that the integral over a 1-form field is independent of the parameterization, only the geometric path

Given a vector field in $\mathbb{R}^2$

$$\vec{F} = y\hat{\imath} - x^2\hat{\jmath}$$

We write down the work 1-form

$$\alpha_{\vec{F}} = ydx - x^2dy$$

One possible parameterization of γ is

$$\gamma(t) = \begin{pmatrix} 2cost \\ 2sint \end{pmatrix}, 0 \le t \le 2\pi$$

The work done is

$$\int_\gamma \alpha_{\vec{F}} = \int_\gamma ydx - x^2dy$$

Finding the velocity vector (to relate dx, dy to dt

$$\gamma' = \begin{pmatrix} -2sint \\ 2cost \end{pmatrix}$$

We can now express work done in terms of t

$$\begin{aligned} \int_\gamma \alpha_{\vec{F}} &= \int_\gamma ydx - x^2dy \\ &= \int_{t=0}^{2\pi} (2sint)(-2sint) - (4cos^2t)(2cost)dt \end{aligned}$$

Some trigonometric tricks to remember!!

- to integrate sin^2t, use double angle formula
- to integrate cos^3t, pull out a $cos^2t = (1-sin^2t)$, splitting one term into two terms can be integrated individually

11.5 Independence of path theorem

The main idea is that for some *special* 1-form fields, the integral is independent of the path

More formally, if the 1-form field is a gradient 1-form field, the integral is independent of the path and depends only on the end points

$$\int_\gamma df = f|_{\gamma(b)}^{\gamma(a)} = f(\gamma(b)) - f(\gamma(a))$$

BUT!!! two complications

- gradient 1-forms are rate, when is α a gradient?
- how do you find f?

11.5.1 Example

Given a 1-form field α

$$\int_\gamma \alpha = \int_\gamma \left(\frac{2x}{y} - 1\right) dx + \left(3y^2 - \frac{x^2}{y^2}\right) dy$$

and a path

$$\gamma(t) = \begin{pmatrix} 1 + arctan\left(t^2 - t\right) \\ t + 2cos3\pi t \end{pmatrix}$$

Prof-g's nugget of wisdom: the more complicated an integral looks, the more relaxed you should feel.
Step 1: look for a potential, f (i.e. df is gradient)
f is a potential if

$$\alpha = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

We check that

$$\frac{2x}{y} - 1 = \frac{\partial f}{\partial x}, f = \frac{x^2}{y} - x + C(y)$$

We take the partial w.r.t y, getting

$$\frac{\partial}{\partial y}\frac{x^2}{y} - x + C(y) = -\frac{x^2}{y^2} + C'(y)$$

We also check that

$$\frac{\partial f}{\partial y} = 3y^2 - \frac{x^2}{y^2} = -\frac{x^2}{y^2} + C'(y)$$

ANSATZ: We try to find f by writing down

$$f = \frac{x^2}{y} - x + y^3$$

ANSATZ is the German word for an unverified guess. but it is in German. so it is an intelligent guess. pronounce it like you are shouting at someone – Prof G
We verify

$$\frac{\partial}{\partial x}\frac{x^2}{y} - x + y^3 = \frac{2x}{y} - 1$$
$$\frac{\partial}{\partial y}\frac{x^2}{y} - x + y^3 = 3y^2 - \frac{x^2}{y^2}$$

Hence, our **ANSATZ** is no longer an **ANSATZ**

Now the problem is easy, we find the start and end points by

$$\gamma(0) = \begin{pmatrix} 1 + arctan(0) \\ 0 + 2cos0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\gamma(1) = \begin{pmatrix} 1 + arctan(0) \\ 1 + 2cos3\pi \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

By independence of path theorem

$$\begin{aligned}\int_\gamma df &= f(\gamma(b)) - f(\gamma(a)) \\ &= -1 - \frac{15}{2} \\ &= -17/2\end{aligned}$$

11.5.2 Another example

Given a 1-form field

$$\int_\gamma ye^{xy}dx + \left(xe^{xy} - ze^{-yz}\right)dy + \left(e^z - ye^{-yz}\right)dz$$

And a path

$$\gamma(t) = \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix}, 0 \le t \le 1$$

You can write down an unsubstantiated ANSATZ, as long as you show that it works!
Our logic in approaching is problem is
If ye^{xy} is $\frac{\partial f}{\partial x}$,

$$f = e^{xy} + C(y, z)$$

If $xe^{xy} - ze^{-yz}$ is $\frac{\partial f}{\partial y}$,

$$f = e^{xy} + e^{-yz} + C_2(x, z)$$

If $e^z - ye^{-yz}$ is $\frac{\partial f}{\partial z}$,

$$f = e^z + e^{-yz} + C_3(x, y)$$

Merging it all

$$f = e^{xy} + e^{-yz} + e^z$$

11.6 Parameterizing paths

Make sure you know how to parameterize paths

- straight line paths
  - for a line from $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 3 \\ 5 \\ -3 \end{pmatrix}$, we can parameterize it with

$$\gamma(t) = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3-(-1) \\ 5-2 \\ -3-0 \end{pmatrix}, 0 \leq t \leq 1$$

- circular paths

11.7 Prof-g's favorite 1-form

$$\alpha = xdy$$

Note that that this is not a gradient

$$f = xy, df = ydx + xdy \neq xdy$$

Is there a way to get from a point in the 3rd quadrant into the 1st quadrant without a movement being detected by α????

Yes, you move in a path parallel to the x axis ($dy = 0$), and then move along the y axis ($x = 0$), and finally along a path parallel to the axis again.

This is not why it is his favorite 1-form, it just helps with checking your understanding of 1-forms

CHAPTER 12

Week 13 lectures

Recall from before thanksgiving break our discussion of 1-forms. A 1-form is a operator that takes a vector and spits out a scalar.

An example of a 1-form field is

$$\alpha = xdx - 2ydy$$

1-form fields tend to have some nice geometric interpretation when integrated over a path γ

$$\int_\gamma \alpha$$

In particular, we touched briefly on the **independence of path** theorem.

Given some gradient 1-form df

$$\int_\gamma df = f|_{\gamma(start)}^{\gamma(end)}$$

12.0.1 Thinking more deeply about form fields

linear 1-forms fields have the form

$$\alpha = adx + bdy + cdz$$

for some constant a, b, c

The basis 1-forms are

$$dx, dy, dz$$

Geometric interpretation: oriented projected length (dz)
linear 2-forms fields have the form

$$\beta = dx \wedge dy - 2dy \wedge dz$$

The basis 2-forms are

$$dx \wedge dy = -dy \wedge dx$$
$$dy \wedge dz = -dz \wedge dy$$
$$dz \wedge dx = -dx \wedge dz$$

This field takes 2 vectors and spits out a scalar, based on some computation of the projected area
What does $dx \wedge dy$ mean??

$$(dx \wedge dy)(\underline{u}, \underline{v}) = Det \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$$

$(dx \wedge dy)$ tells us which components of u, v we will use to compute the determinant
Geometric interpretation: oriented projected area $(dx \wedge dy\ )$

12.0.2 Generalizing to fields

Recall for 1-forms, we covered

$$\text{Basis 1-forms}(dx, dy) \to \text{Linear 1-forms}(2dx + 5dy) \to \text{fields}(ydx + xdy)$$

For 2-forms

$$\text{Basis 2-forms}(dx \wedge dy) \to \text{Linear 2-forms}(2dx \wedge dy + 5dy \wedge dz) \to \text{fields}(x^2 dx \wedge dy + ydy \wedge dz)$$

12.0.3 Algebra of forms

Very generally, $\wedge \approx$ "wedge" "product"
Consider

$$\begin{aligned}&(dx + 2dy) \wedge (dy - 3dz)\\ =&dx \wedge dy - 3dx \wedge dz + 2dy \wedge dy - 6dy \wedge dz\end{aligned}$$

The rules

- $dx \wedge dx = 0$
- $dx \wedge dy = -dy \wedge dx$

12.0.4 Calculus of forms

The idea of forms is connected to the idea of implicit differentiation.

Consider "d" as an operator

$$d(\text{0-form field}) = \text{1-form field}$$

i.e.

$$f \to df, \text{ which is a gradient 1-form}$$

For example

$$\begin{aligned}f &= x^2 y - z\\ df &= 2xydx + x^2 dy - dz\end{aligned}$$

Rule:

$$d(f\alpha) = df \wedge \alpha$$

For example

$$\begin{aligned}&d(x^2 dx - xydy)\\ =&d(x^2 dx) - d(xy\ dy)\\ =&(2x\ dx) \wedge dx - (y\ dx + x\ dy) \wedge dy\\ =&0 - y\ dx \wedge dy - 0\\ =&-y\ dx \wedge dy\end{aligned}$$

12.0.5 Some strange pattern

A momentary side quest!!!

$$d(d\alpha) = 0$$

12.1 Green's Theorem

In the generic form
Integrating over some oriented boundary ∂D

$$\int_{\partial D} f dx + g dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

Work / Circulation
Given

$$\vec{F} = F_x \underline{i} + F_y \underline{j}$$

$$\int_{\partial D} F_x dx + F_y dy = \int_D \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dA$$

Where

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \nabla \times \vec{F} \text{ is known as the } \textbf{curl}$$

Flux / Divergence
Given

$$\vec{F} = F_x \underline{i} + F_y \underline{j}$$

$$\int_{\partial D} F_x dy - F_y dx = \int_D \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) dA$$

Where

$$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \nabla \cdot \vec{F} \text{ is known as the } \textbf{divergence}$$

12.2 The end of the story

$$\int_{\partial D} \alpha = \int_D d\alpha$$

Given a 1-form

$$\alpha = f dx + g dy$$

$$\begin{aligned}
d\alpha &= df \wedge dx + dg \wedge dy \\
&= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \right) \wedge dy \\
&= \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy \\
&= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy \\
&= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA, \text{provided boundary is oriented correctly}
\end{aligned}$$

12.3 Continued

Recall from the previous week the independence of path theorem

$$\int_\gamma df = f(\gamma(end)) - f(\gamma(start))$$

From this week, we have

$$\int_{\partial D} f dx + g dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

12.3.1 An example

Given

$$\alpha = \left(cosx + 3x^2 - 2y\right) dx + \left(x^3 + 4x - e^{2y}\right) dy$$

Integrated over a circle parameterized by

$$\gamma(t) = \begin{pmatrix} x_0 + Rcost \\ y_0 + Rsint \end{pmatrix}$$

It would be horrible to integrate this via usub. But it is much easier with Green's Theorem

12.3.2 More on Green's Theorem

Recall that for

$$\alpha = xdy$$

$$\begin{aligned} \int_{\partial D} xdy &= \iint 1 - 0dA \\ &= A \end{aligned}$$

Consider applying Green's Theorem to finding the centroid

$$\overline{X} = \frac{\iint xdA}{A} = \frac{???????}{\int_{\partial D} x\ dy}$$

$$\iint_D xdA = \int_{\partial D} fdx + gdy = \int_{\partial D} \frac{1}{2}x^2 dy$$

Hence

$$\begin{aligned} \overline{X} &= \frac{\iint xdA}{A} = \frac{\frac{1}{2}x^2 dy}{\int_{\partial D} x\ dy} \\ &= \frac{1}{2A}\int_{\partial D} x^2 dy \end{aligned}$$

12.3.3 Another example

Given 2 0-forms

$$f = 3x + 2y - z$$

$$g = 5y - 4z$$

We can compute the gradients

$$df = 3dx + 2dy - dz$$

$$dg = 5dy - 4dz$$

What are the gradients of f, g?

$$\nabla f = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \nabla g = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}$$

Differentiating the 1-forms gets us

$$d(df) = 0 = d(dg)$$

Taking the wedge products of df, dg,

$$\begin{aligned} df \wedge dg &= (3dx + 2dy - dz) \wedge (5dy - 4dz) \\ &= 15dx \wedge dy - 12dx \wedge dz - 8dy \wedge dz - 5dz \wedge dy \\ &= 15dx \wedge dy - 12dx \wedge dz - 3dy \wedge dz \end{aligned}$$

12.4 Flux 2-forms

A 3-D Vector field has associated with it a flux 2-form

$$\phi_{\vec{F}} = F_x dy \wedge dx + F_y dz \wedge dx + F_z dx \wedge dy$$

The wedge product of 2 scalar fields f and g gives the flux 2-form of a constant vector field, i.e. If

$$\phi_{\vec{F}} = 15dx \wedge dy - 12dx \wedge dz - 3dy \wedge dz$$

Then

$$\vec{F} = \begin{pmatrix} -3 \\ 12 \\ 15 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}$$

12.5 Curl, grad, divergence

In 3D, there are two ways to interpret a vector field

$$\vec{F} = F_x \underline{i} + F_y \underline{j} + F_x \underline{k}$$

This can be interpreted as

$$\alpha_{\vec{F}} \text{1-form}$$

$$d\alpha_{\vec{F}} \text{ is known as the curl, } \nabla \times \vec{F}\text{, measures rotation, infinitesimal spin}$$

This can also be interpreted as

$$\phi_{\vec{F}} \text{2-form}$$

$$d\phi_{\vec{F}} \text{ is known as the div, } \nabla \cdot \vec{F}\text{, measures expansion, infinitesimal flux}$$

12.5.1 Example

Given

$$\vec{F} = x\underline{i} + xz\underline{j} + yz\underline{k}$$

For divergence

$$\begin{aligned} Div(\vec{F}) = \nabla \cdot \vec{F} &= \frac{\partial}{\partial x} x + \frac{\partial}{\partial y}(-xz) + \frac{\partial}{\partial z}(yz) \\ &= 1 + y \end{aligned}$$

For curl

$$\nabla \times \vec{F} = (z + x)\,\underline{i} + (0)\,\underline{j} + (-z)\,\underline{k}$$

12.6 3 non-trivial observations

1. "d" is curl (differentiating a 1-form)

$$d\alpha_{\vec{F}} = \phi_{\nabla \times \vec{F}}$$

2. "d" is divergence (differentiating a 2-form)

$$d\phi_{\vec{F}} = \left(\nabla \times \vec{F}\right) dx \wedge dy \wedge dz$$

3. "d" is gradient (differentiating a 0-form)

$$df = \alpha_{\nabla f}$$

CHAPTER 13

Week 14

This week is the end.

Bulk of this week will be spent understanding

$$\int_{\partial D} \alpha = \int_D^{d\alpha}$$

Where

- α is any k-form field
- D is some domain of dimension $k+1$

This is **the** Stoke's Theorem, and this week's content covers a specific case of Stoke's Theorem.

This is the same idea as

- Fundamental theorem of integral calculus
- Independence of path (2-D)
- Green's Theorem (2-D)
- Gauss' Theorem (3-D)
- Classical Stoke's Theorem (3-D)

13.1 Relooking Green's Theorem

Recall from last week that Green's Theorem can be written in several different forms.

13.1.1 Work / Circulation

Given a vector field

$$\vec{F} = F_x \underline{i} + F_y \underline{j}$$

The work done by the vector field along the boundary can be computed by taking the double integral of some density over the internal area.

The density is related to the vector field. It is known as the curl.

Curl is a concept that only makes sense in 3-D. In the case of 2-D, we are looking at the component of curl about the $\underline{k}$ axis

$$\int_{\partial D} F_x dx + F_y dy = \iint_D \left(\nabla \times \vec{F}\right) \cdot \vec{k} dA = \iint_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} dA$$

13.1.2 Flux

The flux of the vector field over the boundary can be computed by some combination of the partial derivatives over the internal area.

The integrand is known as the divergence.

$$\int_{\partial D} F_x dy - F_y dx = \iint_D \left(\nabla \cdot \vec{F}\right) dA = \iint_D \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} dA$$

13.1.3 Going back to the general equation

Note that these two are both in the form

$$\int_{\partial D} \alpha = \int_D d\alpha$$

What is D and $d\alpha$???

In the 2-D cases mentioned above,

- D : 2-D domain
- $d\alpha$: 2-form in $\mathbb{R}^2$

Since in 2-D, there is only 1 2-form

$$d\alpha = f(x, y) dx \wedge dy$$

Hence, we **define by fiat** that in $\mathbb{R}^2$

$$\int_D f dx \wedge dy = \iint_D f dA$$

13.1.4 Lifting up by 1 dimension

Consider the following integral. What is it?

$$\int_S \beta$$

Where

- β is some 2-form field in 3-D
- S is some oriented surface in $\mathbb{R}^3$

Recall that given α, a 1-form field, we integrate it over a path, since 1-form operates on a single vector each.

Now, we consider surface in $\mathbb{R}^3$

Let S be the parameterized surface in $\mathbb{R}^3$. Given an implicit surface

$$z = x^2 + y^2$$

$$S(s, t) = \begin{pmatrix} s \\ t \\ s^2 + t^2 \end{pmatrix}, s^2 + t^2 \leq R^2$$

Think of S as mapping the 2-D plane of s and t (within the specified domain) to a surface in 3-D

Consider the integral

$$\int_S dx \wedge dy + dy \wedge dz$$

Note that

- $dx \wedge dy$ measures oriented projected area in the x, y plane, $= \pi R^2$
- $dy \wedge dz$ measures oriented projected area in the y, z plane $= 0$, because the parabola has two oriented projected areas, oriented in opposite directions

The integral is thus

$$\begin{aligned}&\int_S dx \wedge dy + dy \wedge dz\\ =&\pi R^2 + 0\\ =&\pi R^2\end{aligned}$$

Now we return to the theorem and compute the integral again

Taking the derivative of S,

$$S(s,t) = \begin{pmatrix} s \\ t \\ s^2+t^2 \end{pmatrix}, [DS] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2s & 2t \end{bmatrix}$$

We now have another way to compute the integral over the surface S

$$\begin{aligned}&\int_S dx \wedge dy + dy \wedge dz\\ =&\iint_{s'} (dx \wedge dy + dy \wedge dz)\,[DS]\, dsdt \text{ where } s' \text{ is the circular domain in } s, t\\ =&\iint_{s'} (dx \wedge dy) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2s & 2t \end{bmatrix} + (dy \wedge dz) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2s & 2t \end{bmatrix} dsdt\\ =&\iint_{s'} Det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + Det \begin{bmatrix} 0 & 1 \\ 2s & 2t \end{bmatrix} dsdt\\ =&\iint_{s'} 1 - 2sdsdt\\ =&\iint_{s'} dsdt - \iint 2sdsdt\\ =&\pi R^2 - 0 \text{ since } 2s \text{ is an odd function integrated over a symmetric domain}\end{aligned}$$

Generalizing

$$\int_S \beta = \iint_{S'} \beta_{S(s,t)}\,[DS]\, dsdt = \iint_{S'} \beta_{S(s,t)} \frac{\partial S}{\partial s}\frac{\partial S}{\partial t} dsdt$$

The question is why????

One good applications is flux, which is given by the flux 2-form

$$\Phi_{\vec{F}} = F_x dy \wedge dx + F_y dz \wedge dx + F_z dx \wedge dy$$

For some curve S,

$$\int_S \Phi_{\vec{F}} = \text{Flux of } \vec{F} \text{ across } S$$

In the classical physics style, flux is calculated by taking some infinitesimal surface area element $d\sigma$, with an associated unit normal vector $\underline{\hat{n}}$

$$\begin{aligned}\text{Flux} &= \iint \left(\vec{F} \cdot \underline{\hat{n}}\right) d\sigma \\ &= \int_S \Phi_{\vec{F}}\end{aligned}$$

13.2 Green's Theorem in 3D

$$\begin{aligned}\text{"Gauss"} &: \text{"Div"} \\ \text{"Stokes"} &: \text{"Circulation?"}\end{aligned}$$

13.2.1 Gauss' Theorem

Guass: What is the flux of $\vec{F}$ across the boundary of some 3-D domain?

The boundary of D, ∂D is some **closed 2-D surface**

Theorem: The flux over the boundary is equal to the integral of the derivative of the flux 2-form over the interior of the domain.

$$\begin{aligned}\int_{\partial D} \Phi_{\vec{F}} &= \int_D d\Phi_{\vec{F}} \\ &= \iiint_D \left(\nabla \cdot \vec{F}\right) dV\end{aligned}$$

The classical physics version of the theorem is stated as

$$\iint_{\partial D} \left(\vec{F} \cdot \underline{\hat{n}}\right) d\sigma = \iiint D \left(\nabla \cdot \vec{F}\right) dV$$

13.2.2 Stokes' Theorem

What is the circulation of a vector field $\vec{F}$ in $\mathbb{R}^3$ over some oriented 2-D domain (some surface in 3-D) with a boundary ∂D ?

Recall that the derivative of any 1-form field gives a 2-form field, which is itself the flux 2-form of some function

Theorem: The circulation over the boundary is equal to the integral of the flux 2-form of the curl of $\vec{F}$ over the interior of the domain.

$$\int_{\partial D} \alpha_{\vec{F}} = \int_D d\alpha_{\vec{F}}$$

$$\int_{\partial D} \vec{F} \cdot d\underline{x} = \iint_D \left(\nabla \times \vec{F}\right) \cdot \underline{\hat{n}} d\sigma$$

The circulation of F on ∂D is equal to the flux of $\nabla \times \vec{F}$ across D

13.2.3 Dropping back down to Greens in 2-D

Recall that

$$\text{circulation on } \partial D = \text{Flux of } \nabla \times \vec{F} \cdot \hat{k} \to \text{Stokes}$$
$$\text{flux of } \vec{F} \text{ across } \partial D = \iint_D Div(\vec{F}) \to \text{Gauss}$$

13.2.4 An example

Given

$$\vec{F} = \left(x^2 + y^2\right)\hat{\imath} + 2e^z\hat{\jmath} + (x^2 + z^2)\hat{k}$$

Compute the flux of $\vec{F}$ across a cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$
By Gauss,

$$\begin{aligned}
\text{Flux} &= \int_0^1 \int_0^1 \int_0^1 \nabla \cdot \vec{F} dxdydz \\
&= \int_0^1 \int_0^1 \int_0^1 2x + 2zdxdydz \\
&= 1 + 1 \\
2
\end{aligned}$$

13.3 Thursday Lecture

Recall that by Stoke's Theorem, in $\mathbb{R}^n$

$$\int_{\partial D} \alpha = \int_D d\alpha$$

Where ∂D is some boundary and D is in k dimensions, and α is a k-form

n	k		
2	1	Green's	
3	1	Stokes	
3	2	Gauss	
1	0	FTIC	$(f(b) - f(a) = \int_{a,b} f - \int_{a,b} df)$
n	0	IoP	

13.3.1 Practice problem 1

Given a vector field

$$\vec{F} = y^2\hat{\imath} - x^2\hat{\jmath} + z\hat{k}$$

What is the flux of $\vec{F}$?

When we see **flux**, we ask ourselves, is the question asking for the **flux of** f or **flux of the curl**?

In this case, they are asking for the flux of f, so we know that we **don't need Stokes**

Can we use Gauss? Gauss calculates flux but it applies to a closed surface. In this case, we want the flux over an open top hemisphere. Hence no Gauss

What we can do is a direct integration

$$\Phi_{\vec{F}} = y^2 dy \wedge dz - x^2 dz \wedge dx + 2dx \wedge dy$$

This is doable, but it is not obvious / easy

$$S(s,t) = \begin{pmatrix} s \\ t \\ \sqrt{R^2 - s^2 - t^2} \end{pmatrix}$$

$$[DS] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ ???? & ??? \end{bmatrix}$$

THINK MARK THINK!!!
We go back to the flux two form

$$\Phi_{\vec{F}} = y^2 dy \wedge dz - x^2 dz \wedge dx + 2dx \wedge dy$$

Because of the 2-forms, we are working with the projected area onto each of the 3 planes

Consider the $y - z$ plane, the net flux is zero **by symmetry**

$$\int y^2 dy \wedge dz = 0$$

Likewise in the $x - z$ plane, the net flux is also zero **by symmetry**

$$\int -x^2 dx \wedge dz = 0$$

The only non-vanishing net flux is given by

$$\int 2dx \wedge dy = 2, \text{ the oriented projected area in the } x, y \text{ plane}$$

NOTE In general,

$$\int f(x,y) dx \wedge dy = 0$$

Or

$$\int f(x_i, x_j) dx_i \wedge dx_j = 0$$

WE ARE NOT DONE
We will try using Gauss again, because we are stubborn

We now make the domain a solid hemisphere, such that its boundary has two pieces

By Gauss, the net flux is

$$\int_{\text{spherical boundary}} \Phi - \int_{\text{circular boundary at base}} \Phi = \iiint_D \nabla \cdot \vec{F} dV$$

We then show that the divergence is 0

Verify!

Hence

$$\int_{\text{spherical boundary}} \Phi - \int_{\text{circular boundary at base}} \Phi = \iiint_D \nabla \cdot \vec{F} dV = 0$$

Now we find

$$\int_{\text{circular boundary at base}} \Phi = \int 2dx \wedge dy$$

Which is the same computation we did above

13.3.2 Another example

Given a vector field

$$\vec{F} = (y + xsinx^2)\hat{\imath} + \left(x^2 + e^{y^2-5y}\right)\hat{\jmath} + (x^2+y^2)\hat{k}$$

Find the flux of the curl of $\vec{F}$ over the surface defined by

$$z = cos^3\left(\frac{\pi}{2}\left(x^2+y^2\right)\right), x^2+y^2 \leq 1$$

By Stokes, the integral over the boundary of the surface of the work 1-form of f is equal to the integral over the interior of D of the derivative of the work 1-form

$$\int_{\partial S} \alpha_{\vec{F}} = \int_S d\alpha_{\vec{F}} = \int_S \Phi_{\nabla\times\vec{F}}$$

The right hand side is what the question asked for, the left hand side is easier to compute

Along the boundary of S,

$$z = cos^3\left(\frac{\pi}{2}(x^2+y^2)\right) = cos^3\frac{\pi}{2} = 0$$

The strategy
We compute the circulation over the boundary, by computing

$$\int_{\partial S} \alpha_{\vec{F}} \int_{t=0}^{2\pi} \left(sint + costsintcos^2t\right)(-sint) + (????)\,costdt$$

Where the boundary is parameterized as

$$\gamma(t) = \begin{pmatrix} cost \\ sint \end{pmatrix}$$

This is not going to work....
We try again
We use the independence of surface, as long as the boundary remains the same.

We use the surface

$$z = 0, x^2+y^2 \leq 1$$

We want to calculate

$$\int \Phi_{\nabla\times\vec{F}}$$

We write down the work 1-form

$$\alpha_{\vec{F}} = \left(y^2 + xsinx^2\right)dx + \left(x^2 + e^{y^2-5y}\right)dy + \left(x^2+y^2\right)dz$$

The derivative of work 1-form is

$$\begin{aligned} d\alpha_{\vec{F}} &= dy\wedge dx + 2xdx\wedge dy + 2xdx\wedge dz + 2ydy\wedge dz \\ &= (2x-1)dx\wedge dy + 2xdx\wedge dz + 2ydy\wedge dz \end{aligned}$$

The projected area of $2x$ in the $x-z$ plane is 0

The projected area of $2y$ in the $y-z$ plane is 0

Hence

$$\int_{\text{disc}} d\alpha_{\vec{F}} = \iint 2x - 1dx\wedge dy = 2\iint xdA - \iint dA = -\pi$$

The derivative of the work 1-form is equal to the flux of the curl!!!

$$d\alpha_{\vec{F}} = \Phi_{\nabla\times\vec{F}}$$

13.3.3 Another another example

For an object submerged in a fluid of constant uniform density ρ,
The pressure exerted by the fluid is **orthogonal** to the surface of the object

The net force in the x and y directions are 0

Buoyancy is related to the oriented projected area in the z direction, i.e. in the $x - y$ plane

Hence, buoyancy is a 2-form! Taking into account that buoyancy is also a function of depth, given by the z coordinate, we get

$$\beta = \rho z dx \wedge dy$$

The net buoyancy force is

$$\int_{\text{skin}} \beta = \int_{\partial D} \beta$$

By Gauss Theorem

$$\begin{aligned}
&\int_{\partial D} \beta \\
=&\int_D d\beta \\
=&\iiint_D \rho dz \wedge dx \wedge dy \\
=&\iiint_D \rho dV \\
=&\rho \times \text{Volume} \\
=&mass
\end{aligned}$$