1. Let A be the matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors for A.

To find the eigenvalues

$$\begin{aligned} det(A - \lambda I) &= 0 \\ det\left(\begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} - \lambda \end{bmatrix}\right) &= 0 \\ \left(\frac{1}{2} - \lambda\right)\left(\frac{2}{3} - \lambda\right) - \frac{1}{6} &= 0 \\ 6\lambda^2 - 7\lambda + 1 &= 0 \\ (6\lambda - 1)(\lambda - 1) &= 0 \end{aligned}$$

For $\lambda_1 = 1$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$a = b$$

$$v_1 = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for any } k$$

For $\lambda_2 = \frac{1}{6}$

$$\begin{aligned} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{1}{6} \begin{pmatrix} a \\ b \end{pmatrix} \\ \frac{1}{2}a + \frac{1}{2}b &= \frac{1}{6}a \\ \frac{1}{3}a + \frac{2}{3}b &= \frac{1}{6}b \\ b &= \frac{-2}{3}a \end{aligned}$$

$$v_2 = k \begin{pmatrix} 1 \\ \frac{-2}{3} \end{pmatrix} \text{ for any } k$$

(b) What is the determinant of A

$$det(A) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

(c) Let D be a diagonalized matrix similar to A,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

$$\begin{aligned} D &= P^{-1}AP \text{ where } P = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{2}{3} \end{bmatrix} \\ &= \left(-\frac{3}{5}\right) \begin{bmatrix} -\frac{2}{3} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{2}{3} \end{bmatrix} \\ &= \left(-\frac{3}{5}\right) \begin{bmatrix} -\frac{5}{3} & 0 \\ 0 & -\frac{5}{18} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \end{aligned}$$

An expression for A is

$$A = PDP^{-1}$$

An expression for A^n is

$$\begin{aligned} A^n &= \left(PDP^{-1}\right)^n \\ &= PDP^{-1} \cdot PDP^{-1} \dots PDP^{-1} \\ &= PD^nP^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{1}{6}\right)^n \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & -1 \\ -1 & 1 \end{bmatrix} \left(\frac{-3}{5}\right) \end{aligned}$$

(d) Trace of A is $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
(e) The transpose of A is

$$A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

The eigenvalues of A^T are the roots to

$$\begin{aligned} det\left(A^T - \lambda I\right) &= 0 \\ \left|\begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} - \lambda \end{bmatrix}\right| &= 0 \\ \left(\frac{1}{2} - \lambda\right)\left(\frac{2}{3} - \lambda\right) - \frac{1}{6} &= 0 \end{aligned}$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{6}$.

For $\lambda_1 = 1$,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

hence

$$\begin{aligned} \frac{1}{2}a + \frac{1}{3}b &= a \\ \frac{1}{2}a + \frac{2}{3}b &= b \\ b &= \frac{3}{4}a \end{aligned}$$

Therefore $v_1 = k \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix}$ for any k

For $\lambda_2 = \frac{1}{6}$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{6} \begin{pmatrix} a \\ b \end{pmatrix}$$

hence

$$\begin{aligned} \frac{1}{2}a + \frac{1}{3}b &= \frac{1}{6}a \\ \frac{1}{2}a + \frac{2}{3}b &= \frac{1}{6}b \\ b &= -a \end{aligned}$$

Therefore $v_2 = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for any k