Linear Algebra Done Right - Notes

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Chapter 1: Vector Spaces

1.1 Complex Numbers

Definition: a complex number is an **ordered pair** (a, b) where $a, b \in \mathbb{R}$. The set of all complex numbers is C

$$C = \{a + bi : a, b \in \mathbb{R}\}$$

Properties of complex numbers

- commutative

$$w + z = z + w \text{ and } wz = zw$$

- associativity

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \text{ and } (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

- identities

$$z + 0 = z \text{ and } z1 = z$$

- additive inverse

$$\forall z \in C \exists w \in C s.t. z + w = 0$$

- multiplicative inverse
- distributive

1.2 Vector spaces

Let F denote the set of all real and complex numbers. F^n denotes a list of n elements.

$$F^n = \{(x_1, \dots x_n) : x_j \in F \text{ for } j = 1, \dots n\}$$

Definition: A **vector space** is a set V along with addition on V and scalar multiplication on V such that the following properties hold

- commutativity
- associativity
- additive identity
- additive inverse
- multiplicative identity
- distributive properties

NOTE: Scalar multiplication in vector space depends on F. Therefore we say V is a vector space over F.

1.3 Properties of vector spaces

Proposition 1.2: A vector space has a unique additive identity.
Proposition 1.3: Each element in a vector space has a unique additive inverse
Proposition 1.4: $0v = 0$ for every $v \in V$
Proposition 1.5: $a_0 = 0$ for every $a \in F$
Proposition 1.6: $(-1)v = -v$ for every $v \in V$

1.4 Subspaces

Definition: A subset U of V is a subspace of V if U is also a vector space.

To proof U is a subspace, show

- additive identity (U contains 0)
- closed under addition
- closed under scalar multiplication

1.5 Sums and direct sums

For $U_1, \ldots U_m$ that are subspaces of V, the sum of $U_1, \ldots U_m$ denoted $U_1 + U_2 \ldots U_m$ is defined to be the set of all possible sums of $U_1, \ldots U_m$

$$U_1 + U_2 \ldots U_m = \{u_1 + \ldots u_m : u_1 \in U_1, \ldots u_m \in U_m\}$$

Definition: V is the direct sum of subspaces $U_1, U_2 \ldots U_m$ if each element of V can be written uniquely as a sum u_1

Proposition: If $U_1, \ldots U_n$ are subspaces of V, then $V = U_1 \oplus U_2 \ldots \oplus U_n$ if and only if

1. $V_1 = U_1 + \ldots U_n$
2. the only way to write 0 as a sum of $u_1 + \ldots u_n$ is by taking all u_j's equal to 0

Proposition: Suppose that U and W are subspaces of V, then $V = U \oplus W$ iff $V = U + W$ and $U \cap W = \{0\}$

Chapter 2: Finite dimensional vector spaces

2.1 Span and linear independence

Definition: a **linear combination** of a list $(v_1, \ldots v_m)$ of vectors in V is a vector of the form

$$a_1 v_1 + \ldots a_m v_m$$

where $a_1, \ldots a_m \in F$

Definition: The set of all linear combinations of $(v_1, \ldots v_m)$ is the **span** of $(v_1, \ldots v_m)$

$$span(v_1, \ldots, v_m) = \{a_1 v_1 + \ldots a_m v_m : a_1 \ldots a_m \in F\}$$

Definition: A list $(v_1, \ldots v_m)$ of vectors in V is called **linearly independent** if the only choice of $a_1, \ldots a_m \in F$ that makes $a_1 v_1 + \ldots + a_m v_m$ equal to 0 is $a_1 = \ldots = a_m = 0$

Removing a vector from a linearly independent list yields another linearly independent list.

Linear Independence Lemma: If $(v_1, \ldots, v_m)$ is linearly dependent in V and $v_1 \neq 0$, then there exists $j \in \{2, \ldots, m\}$ such that

1. $v_j \in span(v_1, \ldots v_{j-1})$
2. if the j-th term is removed from $(v_1, \ldots, v_m)$ the span of the remaining list equals $span(v_1, \ldots, v_m)$

2.2 Bases

Definition: A **basis** of V is a list of vectors in V that is linearly independent and spans V.

The standard basis of F^n is

$$((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1))$$

Proposition: A list $(v_1, \ldots, v_n)$ of vectors in V is a basis if and only if every $v \in V$ can be written uniquely as

$$v = a_1 v_1 + \ldots + a_n v_n$$

Theorem: Every spanning list in a vector space can be reduced to a basis of the vector space.

Corollary: Every finite dimensional vector space has a basis.

Theorem: Every linearly independent list of vectors in a finite dimensional vector space can be extended to form a basis.

Theorem: If V is finite dimensional and U is a subspace of V, then there is a subspace W of V such that $V = U \oplus W$.

2.3 Dimension

Theorem: Any two bases of a finite-dimensional vector space have the same length.

Definition: The **dimension** of a finite dimensional vector space is the length of any basis of the vector space.

Proposition: If V is finite dimensional and U is a subspace of V

$$dim\ U \leq dim\ V$$

Proposition: If V is finite dimensional, every spanning list of vectors in V of length $dim\ V$ is a basis of V

Proposition: If V is finite dimensional, every independent list of vectors in V of length $dim\ V$ is a basis of V

Theorem: If U_1 and U_2 are subspace of a finite dimensional vector space, then

$$dim\,(U_1 + U_2) = dim\ U_1 + dim\ U_2 - dim\,(U_1 \cap U_2)$$

Proposition: Suppose V is finite dimensional and $U_1, \ldots, U_m$ are subspaces of V such that

$$V = U_1 + \ldots + U_m$$

$$dim\ V = dim\ U_1 + \ldots + dim\ U_m$$

Then

$$V = U_1 \oplus \ldots \oplus U_m$$

Chapter 3: Linear Maps

3.1 Definition of linear map

Definition: A **linear map** from V to W is a function $T : V \to W$ with the following properties

Additivity

$$T(u+v) = Tu + Tv \text{ for all } u, v \in F$$

Homogeneity

$$T(av) = a(Tv) \text{ for all } a \in F, v \in V$$

The set of all linear maps from V to W is denoted

$$\mathcal{L}(V, W)$$

Examples of linear maps

- zero
$$0 \in \mathcal{L}(V, W) \text{ where } 0v = 0$$
- identity
$$I \in \mathcal{L}(V, W) \text{ where } Iv = v$$
- differentiation i.e. $(f+g)' = f' + g'$ and $(af)' = af'$
$$T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R})) \text{ where } Tp = p'$$
- integration
$$T \in \mathcal{L}(P(\mathbb{R}), \mathbb{R}) \text{ where } Tp = \int_0^1 p(x)dx$$
- from F^n to F^m, e.g.
$$T \in (L)\left(F^3, F^2\right) \text{ where } T(x, y, z) = (a_{1,1}x + a_{1,2}y + a_{1,3}z, a_{2,1}x + a_{2,2}y + a_{2,3}z)$$

(L) constitute a vector space if we define addition and scalar multiplication.

Definition: **Addition of linear maps** is defined as

$$(S+T)v = Sv + Tv$$

For $(S+T), S, T \in \mathcal{L}(V, W)$.

Definition: **Scalar multiplication of linear maps** is defined as

$$(aT)v = a(Tv)$$

For $(aT), T \in \mathcal{L}(V, W), a \in F$.

Definition: **Product of linear maps** is defined as

$$(ST)(v) = S(Tv)$$

provided that

$$T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$$

Their product is then

$$ST \in \mathcal{L}(U, W)$$

I.e. for some pairs of linear maps where a useful product exists, their product is their composition.

Note that ST is only defined when T maps into the domain of S.

Properties

- Associativity
$$(T_1T_2)T_3 = T_1(T_2T_3)$$
- Associativity
$$TI = T \text{ and } IT = T \text{ for } T \in \mathcal{L}(V, W)$$
- Distributive
$$(S_1 + S_2)T = S_1T + S_2T$$
$$S(T_1 + T_2) = ST_1 + ST_2$$

3.2 Null spaces, ranges, injectivity and surjectivity

Definition: The **null space** of T for $T \in \mathcal{L}(V, W)$ is the subset of V consisting of those vectors that T maps to 0.

$$null\ T = \{v \in V : Tv = 0\}$$

Proposition: If $T \in \mathcal{L}(V, W)$ then $null\ T$ is a subspace of V.

Definition: A linear map $T : V \to W$ is injective if

$$\forall u, v \in V, Tu = Tv \implies u = v$$

Proposition: Let $T \in \mathcal{L}(V, W)$, T is injective if and only if

$$null\ T = \{0\}$$

We only need to check whether 0 is the only vector mapped to 0 to show injectivity.

Definition: For $T \in \mathcal{L}(V, W)$, the range of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$.

$$range\ T = \{Tv : v \in V\}$$

Proposition: If $T \in \mathcal{L}(V, W)$ then the range of T is a subspace of W.

Definition: A linear map $T : V \to W$ is surjective if its range equals W.

Rank-Nullity Theorem: If V is finite dimensional and $T \in \mathcal{L}(V, W)$ then *range* T is a finite dimensional subspace of W, and the sum of the dimension of the nullspace and the sum of the dimension of the range equals the dimension of the domain .

$$dim\ V = dim\ null\ T + dim\ range\ T$$

Corollary to Rank-Nullity 1: If V and W are finite dimensional and $dim\ V > dim\ W$ then no linear map from V to W is injective.

Proof for the corollary uses the fact that $dim\ null\ T > 0$, i.e. *null* T contains vectors other than 0.

Corollary to Rank-Nullity 2: If V and W are finite dimensional and $dim\ V < dim\ W$ then no linear map from V to W is surjective.

3.3 Matrix of a linear map

Definition: Let $T \in \mathcal{L}(V, W)$, and $(v_1, \dots, v_n)$ is a basis for V and $(w_1, \dots w_m)$ is a basis for W. For each $k \in [1, n]$,

$$Tv_k = a_{1,k}w_1 + \dots + a_{m,k}w_m$$

The $m \times n$ matrix formed by $a_{j,k}$ is the **matrix** of T

$$M\left(T, (v_1, \dots v_n), (w_1, \dots, w_m)\right)$$

Addition of matrices

$$M(T + S) = M(T) + M(S) \text{ for } T, S \in \mathcal{L}(V, W)$$

Scalar multiplication

$$M(cT) = cM(T) \text{ for } c \in F$$

3.3.1 Vector space of matrices

Definition: The set of all $m \times n$ matrices with entries in F constitutes a vector space, and the set is denoted
$$Mat(m, n, F)$$

Multiplying matrices Ideally we want
$$M(TS) + M(T)M(S)$$
In order to satisfy this, we define the product of matrices as such.
$$M(T) = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & & a_{m,n} \end{bmatrix}$$
$$M(S) = \begin{bmatrix} b_{1,1} & \dots & b_{1,p} \\ \vdots & & \vdots \\ b_{m,1} & & b_{n,p} \end{bmatrix}$$
Then $M(TS)$ is the $m \times p$ matrix whose entry in j row and k column is
$$\sum_{r=1}^{n} a_{j,r} b_{r,k}$$

3.4 Invertibility

Definition: A linear map $T \in \mathcal{L}(V, W)$ is invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that
$$ST = I \text{ for } I \in \mathcal{L}(V, V)$$
$$TS = I \text{ for } I \in \mathcal{L}(W, W)$$
S is the **inverse** of T.

If T is invertible, then it has a unique inverse T^{-1}

Proposition: A linear map is invertible if and only if it is injective and surjective.

Definition: Two vector spaces are **isomorphic** if there is an invertible linear map from one vector space onto another.

Theorem: Two finite-dimensional vector space are isomorphic if and only if they have the same dimension.

Proposition 3.19: Suppose that $(v_1, \dots, v_n)$ is a basis for V and $(w_1, \dots, w_m)$ is a basis of W, then M is an invertible linear map between $\mathcal{L}(V, W)$ and $Mat(m, n, F)$.

This is a little confusing, come back and make sense of this

This means there is a one-one correspondence between the matrix of a linear transformation from V to W and the set of all possible $m\ times n$ matrices with entries in F?

Consider the dimension of $Mat(m, n, F)$, the dimension can be found by choosing a basis. One of the possible bases is the set of $m \times n$ matrices with 0 in all entries except for a 1 in one entry. There are $m \times n = mn$ such matrices. Hence $dim(Mat(m, n, F)) = mn$.

Proposition: If V, W are finite dimensional, then

$$dim\ \mathcal{L}(V, W) = (dim\ V)(dim\ W)$$

Definition: An **operator** is a linear map from a vector space onto itself.

$$\mathcal{L}(V) = \mathcal{L}(V, V)$$

Theorem: Suppose V is finite-dimensional, if $T \in \mathcal{L}(V)$, then the following are equivalent

1. T is invertible
2. T is injective
3. T is surjective

Note: This only applies if V is finite-dimensional.

Chapter 4: Polynomials

4.1 Degree

Definition: A function $p : F \to F$ is called a **polynomial** with coefficients in F if there exists $a_0, \dots, a_m \in F$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m \text{ for all } z \in F$$

If p can be written in this form with $a_m \neq 0$, then p has degree m.

Note: If $a_0 = a_1 = \dots = a_m = 0$, then p has degree $-\infty$

Recall that $P(F)$ denotes the vector space of all polynomials with coefficients in F.

$P_m(F)$ denotes the subspace of $P(F)$ consisting of all polynomials with degree **at most** m.

Definition: A **root**, λ, of a polynomial is a number such that

$$p(\lambda) = 0$$

Proposition 4.1: Suppose $p \in P(F)$ is a polynomial with degree $m \geq 1$, and $\lambda \in F$, then λ is a root if and of if there is a polynomial $q \in P(F)$ with degree $(m-1)$ such that

$$p(z) = (z - \lambda)q(z) \text{ for all } z \in F$$

Corollary: Suppose $p \in P(F)$ is a polynomial with degree $m \geq 0$, then p has at most m distinct roots in F

Corollary: Suppose $a_0, \dots a_m \in F$. If

$$a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m = 0$$

for all $z \in F$, then $a_0 = \dots = a_m = 0$
i.e. If a polynomial is identically 0, then all coefficients must be 0

By the last corollary, $(1, z, \dots, z^m)$ is linearly independent in $P(F)$ (the only representation of 0 is trivial). This linear independence implies each polynomial can be be represented in one way.

Division Algorithm Lemma: suppose $p, q \in P(F)$ with $p \neq 0$, then there exists polynomials $s, r \in P(F)$ such that

$$q = sp + r$$

and $deg\ r < deg\ p$

Revisit proof for this lemma

4.2 Complex Coefficients

Fundamental Theorem of Algebra: Every nonconstant polynomial with complex coefficients has a root.

Corollary: If $p \in P(C)$ is a nonconstant polynomial, then p has a unique factorization (order irrelevant) of the form

$$p(z) = c(z - \lambda_1) \dots (z - \lambda_m)$$

where $c, \lambda_1, \dots, \lambda_m \in \mathbf{C}$

4.3 Complex Numbers

Definition: For $z = a + bi$, a is the **real part** of z, b is the **imaginary part** of z,

$$z = a + bi = Re\ z + (Im\ z)i$$

The **complex conjugate** of $z \in \mathbf{C}$ is $\bar{z}$

$$\bar{z} = Re\ z - (Im\ z)i$$

The **absolute value** of z is $|z|$

$$|z| = \sqrt{(Re\ z)^2 + (Im\ z)^2}$$

Properties of complex numbers

- additivity of real part
$$Re(w + z) = Re\ w + Re\ z$$
- additivity of imaginary part
$$Im(w + z) = Im\ w + Im\ z$$
- sum of z and $\bar{z}$
$$z + \bar{z} = 2Re\ z$$
- difference of z and $\bar{z}$
$$z - \bar{z} = 2(Im\ z)i$$
- product of z and $\bar{z}$
$$z\bar{z} = |z|^2$$
- additivity of complex conjugate
$$\overline{w + z} = \bar{w} + \bar{z}$$
- multiplicativity of complex conjugate
$$\overline{wz} = \bar{w}\bar{z}$$
- conjugate of conjugate
$$\bar{\bar{z}} = z$$
- multiplicativity of absolute value
$$|wz| = |w|\,|z|$$

4.4 Real Coefficients

Proposition: Suppose p is a polynomial with real coefficients, if $\lambda \in C$ is a root, then so is $\bar{\lambda}$

Proposition: Let $\alpha, \beta \in \mathbb{R}$, then there is a polynomial factorization of the form

$$x^2 + \alpha x + \beta = (x - \lambda_1)(x - \lambda_2)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ if and only if $\alpha^2 \geq 4\beta$

Theorem: If $p \in P(\mathbb{R})$ is a nonconstant polynomial, then p has a unique factorization of the form

$$p(x) = c(x - \lambda_1) \dots (x - \lambda_m)(x_2 + \alpha_1 x + \beta_1) \dots (x^2 + \alpha_M x + \beta_M)$$

where $c, \lambda_1, \dots, \lambda_m \in \mathbb{R}$ and $(\alpha_1, \beta_1), \dots (\alpha_M, \beta_M) \in \mathbb{R}^2$ with $\alpha_j^2 < 4\beta_j$

Chapter 5: Eigenvalues and eigenvectors

Eigenvalues and eigenvectors are concerned with linear maps from a vector space to itself. This constitutes the deepest and most important part of linear algebra.

5.1 Invariant subspaces

For some operator $T \in \mathcal{L}(V)$, assuming V can be direct sum decomposed into

$$V = U_1 \oplus \ldots \oplus U_m$$

the behavior of T can be understood by considering the behavior of $T|_{U_j}$, i.e. T restricted to the domain of U_j. However, $T|_{U_j}$ might not be an operator (i.e. might not map U_j to U_j. This is the motivating example for studying invariant subspaces.

Definition: For $T \in \mathcal{L}(V)$ and U a subspace of V, U is **invariant** under T if $u \in U$ implies $Tu \in U$ i.e.

$$u \in U \implies Tu \in U$$

i.e. U is invariant under T if

$$T|_U \in \mathcal{L}(U)$$

Examples of invariant subspaces

- null space

$$T \in \mathcal{L}(V) \implies \textit{null } T \text{ invariant under } T$$

- range

$$T \in \mathcal{L}(V) \implies \textit{range } T \text{ invariant under } T$$

5.1.1 One dimensional invariant subspaces

Subspaces of V of dimension 1 can be found by taking **any** $u \in V$, and finding the set of all scalar multiples of u

$$U = \{au : a \in F\}$$

Every one dimensional subspace of V has the same form.

If U is invariant under some $T \in \mathcal{L}(V)$, then by definition

$$Tu \in U \implies Tu = \lambda u \text{ for some } \lambda \in F$$

Definition: A scalar $\lambda \in F$ is an **eigenvalue** of $T \in \mathcal{L}(V)$ if there exists a nonzero vector $u \in V$ such that

$$Tu = \lambda u \text{ or } (T - \lambda I)u = 0$$

Corollary: T has a one dimensional invariant subspace if and only if T has an eigenvalue.

Note that

$$\begin{aligned}
&Tu = \lambda u \\
&\iff (T - \lambda I)u = 0 \\
&\iff (T - \lambda I) \text{ is not injective} \\
&\iff (T - \lambda I) \text{ is not invertible} \\
&\iff (T - \lambda I) \text{ is not surjective}
\end{aligned}$$

Definition: Suppose $T \in \lambda L(V)$ and $\lambda \in F$ is an eigenvalue of T, a vector $u \in V$ is called an **eigenvector** of T if

$$Tu = \lambda u$$

Corollary: Since $(T - \lambda I)u = 0$, the set of eigenvectors of T corresponding to λ is exactly

$$null(T - \lambda I)$$

Theorem: Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding nonzero eigenvectors. Then $(v_1, \ldots, v_m)$ is linearly independent.
Corollary: Each operator on V has at most $dim\ V$ distinct eigenvalues.

5.2 Polynomials applied to operators

Unlike linear maps, operators can be raised to powers. If $T \in \mathcal{L}(V)$ then $T^2 = TT \in \mathcal{L}(V)$.

$$T^m = T \ldots T$$

If T is invertible, then inverse of T is T^{-1}, and

$$T^{-m} = \left(T^{-1}\right)^m$$

Since T is an operator

$$T^m T^n = T^{m+n} \text{ and } (T^m)^n = T^{mn}$$

We can take the polynomial of an operator,

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \ldots + a_m T^m$$

If p and q are polynomials with coefficients in F, then

$$(pq)T = p(T)q(T)$$

5.3 Upper triangular matrices

Theorem: Every operator on a finite dimensional, non-zero complex vector space has an eigenvalue.

Since operators map a vector space onto itself, we only need to consider one basis, and matrices of operators will always be **square arrays**.

Let $T \in \mathcal{L}(V)$, suppose $(v_1, \ldots, v_n)$ is a basis for V, then

$$Tv_k = a_{1,k} v_1 + \ldots + a_{n,k} v_n$$

and the matrix of T with respect to $(v_1, \ldots, v_n)$ is

$$M(T, (v_1, \ldots, v_n)) = M(T) = \begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{bmatrix}$$

A central goal of linear algebra is to show that given an operator T, there exists a basis with respect to which T has a reasonable simple matrix.

If V is a complex vector space and $T \in \mathcal{L}(V)$, then there is a basis of V with respect to which $M(T)$ has the form

$$\begin{bmatrix} \lambda & & \\ 0 & * & \\ \vdots & & \\ 0 & & \end{bmatrix}$$

This is because since T has an eigenvalue λ and a corresponding eigenvector v, v can be extended to a basis of V.

From definition of eigenvectors

$$Tv = \lambda v$$

From definition of the matrix representation of T

$$Tv = a_{1,1}v + \ldots + a_{n,1}v_n$$

Proposition: Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V, then

1. the matrix of T with respect to $(v_1, \ldots, v_n)$ is upper triangular
2. $Tv_k \in span(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$
3. $span(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$

Notes for the above equivalence:

(1) and (2): If $M(T)$ is upper triangular, since we know that the k-th column of M acts on the k-th basis, i.e.

$$Tv_k = a_{1,k}v_1 + a_{2,k}v_2 + \ldots + a_{n,k}v_n$$

If $M(T)$ is upper triangular, then only $a_{1,k}, \ldots, a_{k,k}$ are non-zero. Hence

$$\begin{aligned} Tv_k &= a_{1,k}v_1 + a_{2,k}v_2 + \ldots + a_{k,k}v_k + 0 \times v_{k+1} + \ldots + 0 \times v_n \\ &= a_{1,k}v_1 + a_{2,k}v_2 + \ldots + a_{k,k}v_k \end{aligned}$$

Hence $Tv_k \in span(v_1, \ldots, v_k)$

(2) and (3): Recall that U is invariant under T if $u \in U$ implies $Tu \in U$,

$$\forall v \in span(v_1, \ldots, v_k), Tv \in span(v_1, \ldots, v_k)$$

Note: The lines above do not constitute a proof. Refer to the book for the full proof.

Theorem: Suppose V is a complex vector space and $T \in \mathcal{L}(V)$, then T has an upper triangular matrix with respect to some basis of V.

Note: This theorem does not apply to real vector spaces. The first vector in a basis for V with respect to which T has an upper triangular matrix must be an eigenvector.

The theorem above only guarantees the existence of an eigenvector for a non-zero complex vector space.

Proposition 5.16: Suppose $T \in \mathcal{L}(V)$ has an upper triangular matrix with respect to some basis of V, then T is invertible if and only if **all** entries on the diagonal of the upper triangular matrix are non-zero.

It would be good if we could compute the eigenvalues of an operator from its matrix exactly. However, no such method exists.

If we could find a basis with which the operator is upper triangular, then the computation of eigenvalues is trivial.

Proposition 5.18: Suppose $T \in \mathcal{L}(V)$ has an upper triangular matrix with respect to some basis of V, then the eigenvalues of T consists precisely of the entries on the diagonal of the upper triangular matrix.

Let $(v_1, \dots, v_n)$ be a basis for V with respect to which T has an upper triangular matrix

$$M(T, (v_1, \dots, v_n)) = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Let $\lambda \in F$, then

$$M(T - \lambda I, (v_1, \dots, v_n)) = \begin{bmatrix} \lambda_1 - \lambda & & & * \\ & \lambda_2 - \lambda & & \\ & & \ddots & \\ 0 & & & \lambda_n - \lambda \end{bmatrix}$$

By definition of eigenvalues and eigenvectors, λ is a eigenvalue of T if there exists some $u \in V$ such that

$$Tu = \lambda u \text{ or } (T - \lambda I)u = 0 \text{ or } T - \lambda I \text{ not invertible}$$

$T - \lambda I$ is not invertible if and only if λ equals one of the λ_j .

5.4 Diagonal matrices

Definition: A **diagonal matrix** is a square matrix with 0 everywhere except possibly along the diagonal.

An operator $T \in \mathcal{L}(V)$ has a diagonal matrix with respect to some basis $(v_1, \dots, v_n)$ if and only if

$$\begin{gathered} Tv_1 = \lambda_1 v_1 \\ \dots \\ Tv_n = \lambda_n v_n \end{gathered}$$

i.e. An operator T has a diagonal matrix with respect to some basis V if and only if V has a basis consisting of eigenvectors of T.

Note: Not every operator has a diagonal matrix with respect to some basis. Even though every operator on a (finite) complex vector spaces has a eigenvector, there may not be enough linearly independent eigenvectors of T to form a basis for V.

Proposition 5.20: If $T \in \mathcal{L}(V)$ has $dim\ V$ distinct eigenvalues, then T has a diagonal matrix with respect to some basis V.

Note: the converse of this is not true. Operators with fewer eigenvalues than the dimension of the domain may also have diagonal matrices.

Proposition 5.21: Suppose $T \in \mathcal{L}(V)$, and $\lambda_1, \ldots, \lambda_m$, are distinct eigenvalues of T, then the following are equivalent

1. T has a diagonal matrix with respect to some basis of V
2. V has a basis consisting of eigenvectors of T
3. there exist one-dimensional subspaces $U_1, \ldots, U_n$ of V, each invariant under T such that
$$V = U_1 \oplus \ldots \oplus U_n$$
4. $V = null(T - \lambda_1 I) \oplus \ldots \oplus null(T - \lambda_m I)$
5. $dim\ V = dim\ null(T - \lambda_1) + \ldots + dim\ null(T - \lambda_m I)$

5.5 Invariant subspaces on real vector spaces

Theorem 5.24: Every operator on a finite dimensional, nonzero, real vector space has an invariant subspace of dimension 1 or 2.

Definition: Suppose U and W are subspaces of V such that
$$V = U \oplus W$$
Each vector $v \in V$ can be written as
$$v = u + w$$
where $u \in U$ and $w \in W$. We define a **projection** operator $P_{U,W} \in \mathcal{L}(V)$ such that
$$P_{U,W} v = u$$

Note: $P_{U,W}$ is the projection onto U with nullspace of W.
Properties of projection

- $$v = P_{U,W} v + P_{W,U} v$$
- $$P_{U,W}^2 = P_{U,W}$$
- range of P
$$range\ P_{U,W} = U$$
- kernel of P
$$null\ P_{U,W} = W$$

Theorem 5.26: Every operator on an odd-dimensional real vector space has an eigenvalue.

Chapter 6: Inner product spaces

6.1 Inner product

Definition: The 'length' of a vector is called the **norm** of x, denoted $\|x\|$.

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$$

Note that the norm is not linear on $\mathbf{R}^n$

Defintion: For $x, yin\mathbf{R}^n$, the **dot product** of x, y is

$$x \cdot y = x_1 y_1 + \ldots + x_n y_n$$

Note that

- dot product returns a number, not vector
- $x \cdot x \geq 0$ for all $x \in \mathbf{R}^n$
- $x \cdot x = 0$ if and only if $x = 0$

- if $y \in \mathbf{R}^n$ is fixed, then map from R^n to$\mathbf{R}$ is linear
- $x \cdot y = y \cdot x$

Definition: An **inner product** on V is a function that takes each ordered pair (u, v) of elements in V to a number $\langle u, v \rangle \in F$ with the following properties

- positivity
$$\langle v, v \rangle \geq 0 \text{ for all } v \in V$$
- definiteness
$$\langle v, v \rangle = 0 \text{ if and only if } v = 0$$
- additivity in first slot
$$\langle u = v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \text{ for all } u, v, w \in V$$
- homogeneity in first slot
$$\langle av, w \rangle = a \langle v, w \rangle$$
- conjugate symmetry
$$\langle av, w \rangle = \overline{\langle w, v \rangle}$$

6.2 Norm

6.3 Orthonormal bases

6.4 Orthogonal projections and minimization problems

6.5 Linear functions and adjoints

Chapter 7: Operators on inner product spaces

7.1 Self adjoint and normal operators

7.2 Spectral theorem

7.3 Normal operators on real inner product spaces

7.4 Positive operators

7.5 Isometries

7.6 Polar and singular value decompositions

Chapter 8: Operators on complex vector spaces

Chapter 9: Operators on real vector spaces

Chapter 10: Trace and determinant