3. Linear Temporal Logic

Huixing Fang

School of Information Engineering
Yangzhou University

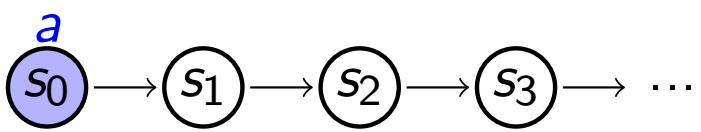
Outline

Definition 1 (Syntax of LTL)

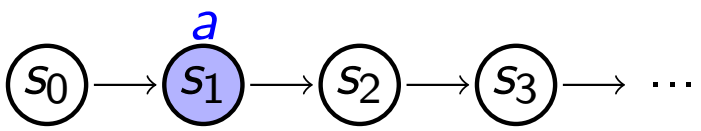
$$\varphi ::= \texttt{true} \mid a \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi_1 \mathbf{U} \varphi_2$$

where $a \in AP$. Precedence order: $\neg = \bigcirc \; > \mathbf{U} > \; \wedge$.

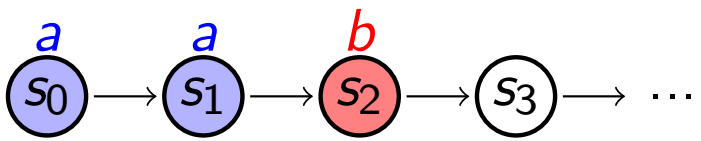
atomic prop. a



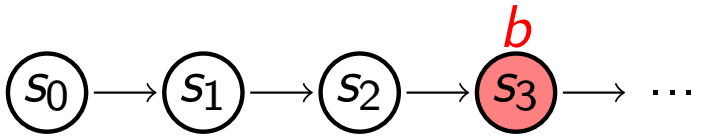
next operator $\bigcirc a$



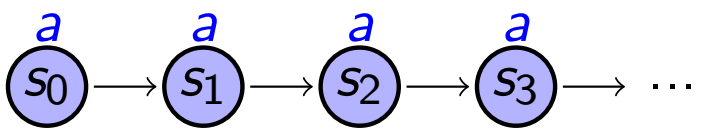
until operator $a \mathbf{U} b$



eventually $\Diamond b$



always $\Box a$



The until operator allows to derive modalities:

1. $\Diamond$("eventually")
   $\Diamond\varphi = \texttt{true}\,\mathbf{U}\,\varphi$
2. $\Box$ ("always")
   $\Box\varphi = \neg\Diamond\neg\varphi$

Example 2

mutual exclusion:	$\Box(\neg crit_1 \vee \neg crit_2)$	(1)
railroad-crossing:	$\Box(train\_is\_near \rightarrow gate\_is\_closed)$	(2)
progress property:	$\Box(request \rightarrow \Diamond response)$	(3)
traffic light:	$\Box(yellow \vee \bigcirc\neg red)$	(4)
infinitely often:	$\Box\Diamond\varphi$	(5)
eventually forever:	$\Diamond\Box\varphi$	(6)
unconditional fairness:	$\Box\Diamond crit_i$	(7)
strong fairness:	$\Box\Diamond wait_i \rightarrow \Box\Diamond crit_i$	(8)
weak fairness:	$\Diamond\Box wait_i \rightarrow \Box\Diamond crit_i$	(9)

Outline

Definition 3 (Semantics of LTL over Infinite Words)

The satisfaction relation between interpretation $\sigma = A_0A_1A_2... \in (2^{AP})^\omega$ and LTL formula is defined as follows:

1. $\sigma \models \texttt{true}$
2. $\sigma \models a$ iff $A_0 \models a$, i.e., $a \in A_0$
3. $\sigma \models \varphi_1 \wedge \varphi_2$ iff $\sigma \models \varphi_1$ and $\sigma \models \varphi_2$
4. $\sigma \models \neg\varphi$ iff $\sigma \not\models \varphi$
5. $\sigma \models \bigcirc\varphi$ iff $\mathit{suffix}(\sigma, 1) = A_1A_2A_3... \models \varphi$
6. $\sigma \models \varphi_1 \mathbf{U} \varphi_2$ iff there exists $j \geq 0$ such that $\mathit{suffix}(\sigma, j) = A_jA_{j+1}A_{j+2}... \models \varphi_2$ and $\mathit{suffix}(\sigma, i) = A_iA_{i+1}A_{i+2}... \models \varphi_1$, for $0 \leq i < j$

LT property of LTL formula φ: $\mathit{Words}(\varphi) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \varphi\}$.

2 Semantics

Review of execution, paths and traces

For transition system TS with labeling function $L : S \to 2^{AP}$,

1. execution: states + actions, $s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$
2. paths: sequences of states, $\pi = s_0\ s_1\ s_2\ \ldots$
3. traces: sequences of sets of atomic propositions

$$trace(\pi) = L(s_0)L(s_1)L(s_2)\ldots \in (2^{AP})^{\omega}$$

Semantics of $\Diamond$ and $\Box$ over Infinite Words

For $\sigma = A_0A_1A_2\ldots \in (2^{AP})^{\omega}$, and LTL formula φ

1. $\sigma \models \Diamond\varphi$ iff there exists $j \geq 0$ such that $A_jA_{j+1}A_{j+2}\ldots \models \varphi$
2. $\sigma \models \Box\varphi$ iff for all $j \geq 0$ we have $A_jA_{j+1}A_{j+2}\ldots \models \varphi$

Definition 4 (Semantics of LTL over Paths and States)

Let $TS = (S, Act, \rightarrow, S_0, AP, L)$ without terminal states, and let φ be an LTL formula over AP.

1. For infinite path fragment π of TS, the $\models$ relation is defined by

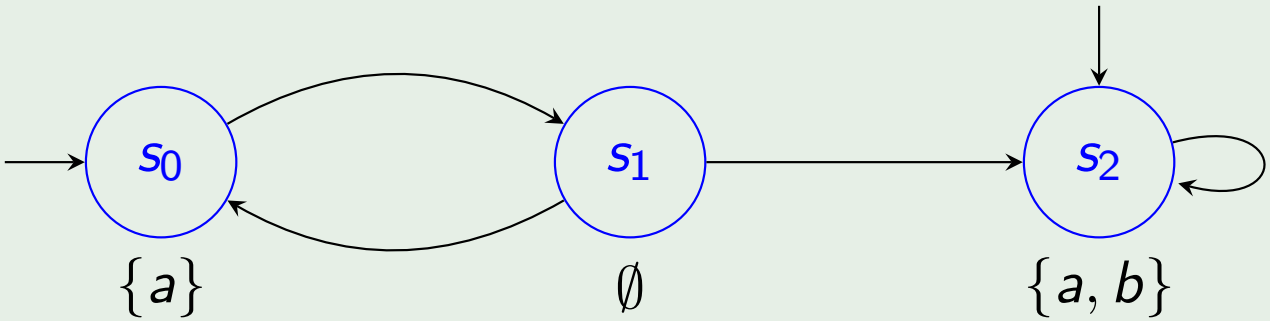
$$\pi = s_0 s_1 s_2 \ldots \models \varphi \quad \text{iff} \quad trace(\pi) \models \varphi$$
$$\text{iff} \quad trace(\pi) \in Words(\varphi) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \varphi\}$$

2. For state $s \in S$, the $\models$ relation is defined by

$$s \models \varphi \quad \text{iff} \quad \forall \pi \in Paths(s).\ \pi \models \varphi$$
$$\text{iff} \quad s \models Words(\varphi)$$
$$\text{iff} \quad Traces(s) \subseteq Words(\varphi)$$

- $Paths(s)$ = set of all maximal path fragments starting in state s
- $Traces(s) = \{trace(\pi) \mid \pi \in Paths(s)\}$

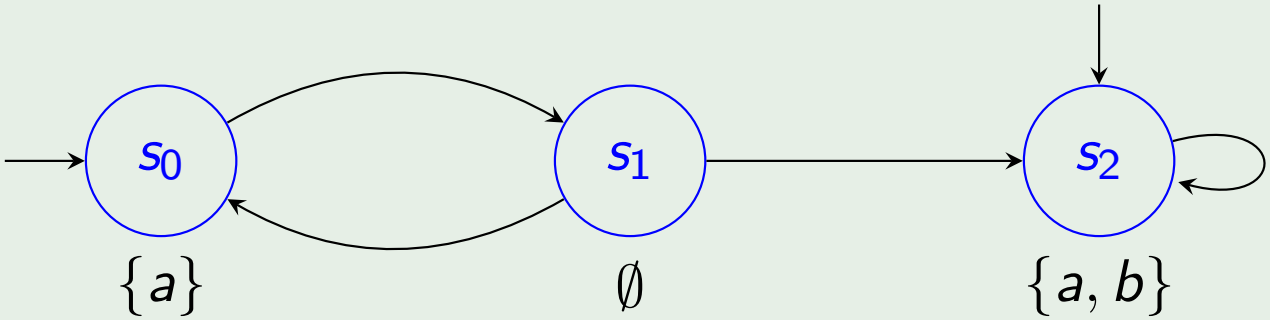
Example 5 (LTL-semantics over paths)



- $AP = \{a, b\}$
- $\pi = s_0 s_1 s_2 s_2 \ldots$
- $trace(\pi)$
  $= L(s_0)L(s_1)L(s_2)\ldots$
  $= \{a\}\emptyset\{a, b\}^\omega$

- $\pi \models a$, $\pi \not\models b$
- $\pi \models \bigcirc(\neg a \wedge \neg b)$
- $\pi \models \bigcirc\bigcirc(a \wedge b)$
- $\pi \models (\neg b)\mathbf{U}(a \wedge b)$

Example 6 (LTL-semantics over paths)



- $AP = \{a, b\}$
- $\pi = s_0 s_1 s_0 s_1 \ldots$
- $trace(\pi)$
  $= L(s_0)L(s_1)L(s_0)L(s_1)\ldots$
  $= \{a\}\emptyset\{a\}\emptyset$

- $\pi \models a \mathbf{U} b$, (?)
- $\pi \models \Diamond b \rightarrow (a \mathbf{U} b)$, (?)
- $\pi \models \bigcirc\bigcirc\neg b$, (?)
- $\pi \models \Box a$, (?)
- $\pi \models \Box\Diamond a$, (?)
- $\pi \models \Diamond\Box a$, (?)

Definition 7 (Interpretation of LTL formulas over TS)

Let $TS = (S, Act, \rightarrow, S_0, AP, L)$ without terminal states, and let φ be an LTL formula over AP.

$$\begin{aligned} TS \models \varphi &\text{ iff } s_0 \models \varphi \text{ for all } s_0 \in S_0 \\ &\text{iff } trace(\pi) \models \varphi \text{ for all } \pi \in Paths(TS) \\ &\text{iff } Traces(TS) \subseteq Words(\varphi) \\ &\text{iff } TS \models Words(\varphi) \end{aligned}$$

Review-1: An LT property over AP is a language E of infinite words over the alphabet $\Sigma = 2^{AP}$, i.e., $E \subseteq (2^{AP})^{\omega}$.

Review-2: Satisfaction relation $\models$ for TS and LT property E, $TS \models E$ iff $Traces(TS) \subseteq E$.

Outline

3 Specifying Properties

LTL-formulas for MUTEX protocols, $AP = \{wait_1, crit_1, wait_2, crit_2\}$

1. the mutual exclusion property

$$\varphi_m = \Box(\neg crit_1 \vee \neg crit_2)$$

2. every process enters the critical section infinitely often

$$\varphi_\ell = \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

3. every waiting process finally enters its critical section

$$\varphi_f = \Box(wait_1 \rightarrow \Diamond crit_1) \wedge \Box(wait_2 \rightarrow \Diamond crit_2)$$

Outline

4 Equivalence of LTL Formulae

Definition 8 (Equivalence of LTL formulas)

LTL formulae φ_1, φ_2. $\varphi_1 \equiv \varphi_2$ iff $\mathit{Words}(\varphi_1) = \mathit{Words}(\varphi_2)$ iff for all transition systems $\mathcal{T}$, $\mathcal{T} \models \varphi_1 \Leftrightarrow \mathcal{T} \models \varphi_2$.

Duality Rule: $\neg\bigcirc\varphi \equiv \bigcirc\neg\varphi$

Proof:

$$
\begin{aligned}
& A_0A_1\ldots \models \neg\bigcirc\varphi \\
\text{iff}\quad & A_0A_1\ldots \not\models \bigcirc\varphi \\
\text{iff}\quad & A_1A_2\ldots \not\models \varphi \\
\text{iff}\quad & A_1A_2\ldots \models \neg\varphi \\
\text{iff}\quad & A_0A_1A_2\ldots \models \bigcirc\neg\varphi
\end{aligned}
$$

4 Equivalence of LTL Formulae

The expansion laws describe the temporal modalities $\mathbf{U}$, $\Diamond$, and $\Box$ by means of a recursive equivalence.

1. until: $\boxed{\varphi \mathbf{U} \psi} \equiv \psi \vee (\varphi \wedge \bigcirc \boxed{\varphi \mathbf{U} \psi})$ least fixed point
2. eventually: $\boxed{\Diamond \psi} \equiv \psi \vee \bigcirc \boxed{\Diamond \psi}$ least fixed point
3. always: $\boxed{\Box \psi} \equiv \psi \wedge \bigcirc \boxed{\Box \psi}$ greatest fixed point

Expansion laws are fixed point equations

4 Equivalence of LTL Formulae

Until is the Least Solution of the Expansion Law (Lemma 5.18)

For LTL formuae φ and ψ, $\mathit{Words}(\varphi \mathbf{U} \psi)$ is the least LT property $P \subseteq (2^{AP})^\omega$ such that:

$$\mathit{Words}(\psi) \cup \{A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi) \mid A_1 A_2 \ldots \in P\} \subseteq P \qquad (*)$$

Moreover, $\mathit{Words}(\varphi \mathbf{U} \psi)$ agrees with the set

$$\mathit{Words}(\psi) \cup \{A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi) \mid A_1 A_2 \ldots \in \mathit{Words}(\varphi \mathbf{U} \psi)\}$$

The formulation "least LT property satisfying condition (*)" means that the following conditions hold:

1. $P = \mathit{Words}(\varphi \mathbf{U} \psi)$ satisfies $(*)$
2. $\mathit{Words}(\varphi \mathbf{U} \psi) \subseteq P$ for all LT properties P satisfying $(*)$

Outline

5 Weak Until, Release, and Positive Normal Form

The weak until operator $\mathbf{W}$:

$$\varphi \mathbf{W} \psi = (\varphi \mathbf{U} \psi) \vee \Box \varphi$$

Deriving "always" and "until" from "weak until":

$$\Box \varphi \equiv \varphi \mathbf{W} \mathit{false}$$
$$\varphi \mathbf{U} \psi \equiv (\varphi \mathbf{W} \psi) \wedge \Diamond \psi$$

Duality of $\mathbf{U}$ and $\mathbf{W}$:

$$\neg(\varphi \mathbf{U} \psi) \equiv (\neg\psi) \mathbf{W} (\neg\varphi \wedge \neg\psi)$$
$$\neg(\varphi \mathbf{W} \psi) \equiv (\neg\psi) \mathbf{U} (\neg\varphi \wedge \neg\psi)$$

5 Weak Until, Release, and Positive Normal Form

Expansion laws for **U** and **W**:

$$\varphi \mathbf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathbf{U} \psi))$$
$$\varphi \mathbf{W} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathbf{W} \psi))$$

Weak-Until is the Greatest Solution of the Expansion Law (Lemma 5.19)

- $\mathit{Words}(\varphi \mathbf{U} \psi)$ is the smallest LT-property P such that

$$\mathit{Words}(\psi) \cup \{A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi) \mid A_1 A_2 \ldots \in P\} \subseteq P$$

- $\mathit{Words}(\varphi \mathbf{W} \psi)$ is the largest LT-property P such that

$$P \subseteq \mathit{Words}(\psi) \cup \{A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi) \mid A_1 A_2 \ldots \in P\}$$

5 Weak Until, Release, and Positive Normal Form

Positive Normal Form for LTL (Weak-until PNF)

For $a \in AP$, the set of LTL formulae in **weak-until positive normal form (weak-until PNF)** is given by:

$$\varphi ::= \texttt{true} \mid \texttt{false} \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \bigcirc \varphi \mid \varphi_1 \mathbf{U} \varphi_2 \mid \varphi_1 \mathbf{W} \varphi_2$$

PNF also sometimes called negation normal form (NNF)

$\Diamond$ and $\Box$ can be derived:

$$\Diamond \varphi = \texttt{true} \mathbf{U} \varphi$$
$$\Box \varphi = \varphi \mathbf{W} \texttt{false}$$

Each LTL formula can be transformed into an equivalent LTL formula in PNF

5 Weak Until, Release, and Positive Normal Form

Each LTL formula can be transformed into an equivalent LTL formula in PNF by using the following transformations:

$$
\begin{aligned}
\neg\texttt{true} &\rightsquigarrow \texttt{false} \\
\neg\texttt{false} &\rightsquigarrow \texttt{true} \\
\neg\neg\varphi &\rightsquigarrow \varphi \\
\neg(\varphi \wedge \psi) &\rightsquigarrow \neg\varphi \vee \neg\psi \\
\neg\bigcirc\varphi &\rightsquigarrow \bigcirc\neg\varphi \\
\neg(\varphi \mathbf{U} \psi) &\rightsquigarrow (\varphi \wedge \neg\psi)\mathbf{W}(\neg\varphi \wedge \neg\psi) \\
&\vdots
\end{aligned}
$$

Outline

6 Fairness in LTL

Fairness Constraints

1. Unconditional fairness: e.g.,"Every process gets its turn infinitely often."
2. Strong fairness: e.g., "Every process that is enabled infinitely often gets its turn infinitely often."
3. Weak fairness: e.g., "Every process that is continuously enabled from a certain time instant on gets its turn infinitely often."

6 Fairness in LTL

Definition 9 (Unconditional, Strong, and Weak Fairness)

For transition system $TS = (S, Act, \rightarrow, I, AP, L)$ without terminal states, $A \subseteq Act$, and infinite execution fragment $\rho = s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} \ldots$ of TS:

1. ρ is unconditionally A-fair whenever $\overset{\infty}{\exists}\, j.\ \alpha_j \in A$.
2. ρ is strongly A-fair whenever $(\overset{\infty}{\exists}\, j.\ Act(s_j) \cap A \neq \emptyset) \Rightarrow (\overset{\infty}{\exists}\, j.\ \alpha_j \in A)$.
3. ρ is weakly A-fair whenever $(\overset{\infty}{\forall}\, j.\ Act(s_j) \cap A \neq \emptyset) \Rightarrow (\overset{\infty}{\exists}\, j.\ \alpha_j \in A)$.

$\overset{\infty}{\exists}\, j$: there are infinitely many j.
$\overset{\infty}{\forall}\, j$: for nearly all j, for all, except for finitely many j.
The variable j ranges over the natural numbers.

For state s, let $Act(s)$ denote the set of actions that are executable in state s,

$$Act(s) = \{\alpha \in Act \mid \exists s' \in S.\ s \xrightarrow{\alpha} s'\}$$

Definition 10 (Fairness Assumption)

A fairness assumption for Act is a triple

$$\mathcal{F} = (\mathcal{F}_{ucond}, \mathcal{F}_{strong}, \mathcal{F}_{weak})$$

with $\mathcal{F}_{ucond}, \mathcal{F}_{strong}, \mathcal{F}_{weak} \subseteq 2^{Act}$. Execution ρ is **$\mathcal{F}$-fair** if

1. it is unconditionally A-fair for **all** $A \in \mathcal{F}_{ucond}$,
2. it is strongly A-fair for **all** $A \in \mathcal{F}_{strong}$, and
3. it is weakly A-fair for **all** $A \in \mathcal{F}_{weak}$.

Remark: A is a set of actions.

1. $FairPaths_{\mathcal{F}}(s)$: the set of $\mathcal{F}$-paths of s (i.e., infinite $\mathcal{F}$-fair path fragments that start in state s).
2. $FairPaths_{\mathcal{F}}(TS)$: set of $\mathcal{F}$-fair paths that start in some initial state of TS.
3. $FairTraces_{\mathcal{F}}(s) = trace(FairPaths_{\mathcal{F}}(s))$

6 Fairness in LTL

Definition 11 (Fair Satisfaction Relation for LT Properties)

Let P be an LT property over AP and F a fairness assumption over Act. Transition system $TS = (S, Act, \rightarrow, I, AP, L)$ fairly satisfies P, notation $TS \models_{\mathcal{F}} P$, iff $FairTraces_{\mathcal{F}}(TS) \subseteq P$.

In case a transition system has traces that are not $\mathcal{F}$-fair, then in general we are confronted with a situation

$$TS \models_{\mathcal{F}} P \qquad \text{whereas} \qquad TS \not\models P$$

By restricting the validity of a property to the set of fair paths, the verification can be restricted to "realistic" executions.

Definition 12 (LTL Fairness Constraints and Assumptions)

Let Φ and Ψ be propositional logic formulae over AP.

1. An unconditional LTL fairness constraint is an LTL formula of the form
$$ufair = \Box\Diamond\Psi.$$
2. A strong LTL fairness condition is an LTL formula of the form
$$sfair = \Box\Diamond\Phi \rightarrow \Box\Diamond\Psi.$$
3. A weak LTL fairness constraint is an LTL formula of the form
$$wfair = \Diamond\Box\Phi \rightarrow \Box\Diamond\Psi.$$

An LTL fairness assumption is a conjunction of LTL fairness constraints (of any arbitrary type).

6 Fairness in LTL

Notations:

1. LTL fairness assumptions are a conjunction of unconditional, strong, and weak fairness assumptions: $fair = ufair \wedge sfair \wedge wfair$.
2. Set of all fair paths starting in s, $FairPaths(s) = \{\pi \in Paths(s) \mid \pi \models fair\}$
3. Set of all traces induced by fair paths starting in s, $FairTraces(s) = \{trace(\pi) \mid \pi \in FairPaths(s)\}$

Definition 13 (Satisfaction Relation for LTL with Fairness)

For state s in transition system TS (over AP) without terminal states, LTL formula φ, and LTL fairness assumption $fair$ let

$$s \models_{fair} \varphi \quad \text{iff} \quad \forall \pi \in FairPaths(s).\ \pi \models \varphi \quad \text{and}$$
$$TS \models_{fair} \varphi \quad \text{iff} \quad \forall s_0 \in I.s_0 \models_{fair} \varphi.$$

TS satisfies φ under $fair$ if φ holds for all **fair paths** that originate from some initial state.

6 Fairness in LTL

Theorem 14 (Reduction of $\models_{fair}$ to $\models$)

For transition system TS without terminal states, LTL formula φ, and LTL fairness assumption fair:

$$TS \models_{fair} \varphi \quad \textit{iff} \quad TS \models fair \rightarrow \Phi.$$

Outline

7 Automata-Based LTL Model Checking

For transition system TS and LTL formula φ, let $\mathcal{A}$ be an NBA (Nondeterministic Büchi Automaton) with $\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\neg\varphi)$:

$$
\begin{aligned}
TS \models \varphi \quad & \text{iff} \quad \mathit{Traces}(TS) \subseteq \mathit{Words}(\varphi) \\
& \text{iff} \quad \mathit{Traces}(TS) \cap ((2^{AP})^\omega \setminus \mathit{Words}(\varphi)) = \emptyset \\
& \text{iff} \quad \mathit{Traces}(TS) \cap \mathit{Words}(\neg\varphi) = \emptyset \\
& \text{iff} \quad \mathit{Traces}(TS) \cap \mathcal{L}_\omega(\mathcal{A}) = \emptyset
\end{aligned}
$$

7 Automata-Based LTL Model Checking

Algorithm 1: Automaton-based LTL model checking

Input: finite transition system TS and LTL formula φ (both over AP)
Output: "yes" if $TS \models \varphi$; otherwise, "no" plus a counterexample

1 Construct an NBA $A_{\neg\varphi}$ such that $\mathcal{L}_\omega(A_{\neg\varphi}) = \mathit{Words}(\neg\varphi)$
2 Construct the product transition system $TS \otimes A_{\neg\varphi}$
3 **if** $\exists \pi \in \mathit{Pahts}(TS \otimes A_{\neg\varphi})$ *satisfying the accepting condition of* A **then**
4 | return "no" and an expressive prefix of π
5 **else**
6 | return "yes"
7 **end**

7 Automata-Based LTL Model Checking

$$
\begin{aligned}
& TS \models \text{LTL-formula } \varphi \\
\text{iff} \quad & \mathit{Traces}(TS) \cap \mathcal{L}_\omega(A_{\neg\varphi}) = \emptyset \\
\text{iff} \quad & \text{there is NO path } \langle s_0, q_0\rangle\langle s_1, q_1\rangle\langle s_2, q_2\rangle\ldots \\
& \text{in } TS \otimes A_{\neg\varphi} \text{ s.t. } q_i \in F \text{ for infinitely many } i \in \mathbb{N} \\
\text{iff} \quad & TS \otimes A_{\neg\varphi} \models \Diamond\Box\neg F
\end{aligned}
$$

Definition 15 (Nondeterministic Büchi Automaton (NBA))

A Büchi automaton (NBA) $\mathcal{A}$ is a tuple $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ where

- Q is a finite set of states,
- Σ is an alphabet,
- $\delta : Q \times \Sigma \to 2^Q$ is a transition function,
- $Q_0 \subseteq Q$ is a set of initial states, and
- $F \subseteq Q$ is a set of accept states, called the acceptance set.

- A **run** for $\sigma = A_0 A_1 A_2 \ldots \in \Sigma^\omega$ denotes an infinite sequence $q_0 q_1 q_2 \ldots$ of states in $\mathcal{A}$ such that $q_0 \in Q_0$ and $q_i \xrightarrow{A_i} q_{i+1}$ for $i \geq 0$.
- Run $q_0 q_1 q_2 \ldots$ is **accepting** if $q_i \in F$ for infinitely many indices $i \in \mathbb{N}$.
- The **accepted language** of $\mathcal{A}$ is
  $\mathcal{L}_\omega(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{there exists an accepting run for } \sigma \text{ in } \mathcal{A}\}$

Definition 16 (Nonblocking NBA)

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. $\mathcal{A}$ is called **nonblocking** if $\delta(q, a) \neq \emptyset$ for all states q and all symbols $a \in \Sigma$.

Remark: For each NBA $\mathcal{A}$ there exists a nonblocking NBA $trap(\mathcal{A})$ with $|trap(\mathcal{A})| = O(|\mathcal{A}|)$ and $\mathcal{A} \equiv trap(\mathcal{A})$.

In nonblocking NBA $trap(\mathcal{A})$,

$$\delta'(q, A) = \begin{cases} \delta(q, A) & \text{if } q \in Q \text{ and } \delta(q, A) \neq \emptyset \\ \{q_{trap}\} & \text{otherwise} \end{cases}$$

$trap(\mathcal{A})$ is obtained from $\mathcal{A}$ by inserting a nonaccept trapping state q_{trap} equipped with a self-loop for each symbol of Σ.

Definition 17 (Product of Transition System and NBA)

Let $TS = (S, Act, \rightarrow, I, AP, L)$ be a transition system without terminal states and $\mathcal{A} = (Q, 2^{AP}, \delta, Q_0, F)$ a nonblocking NBA. Then, $TS \otimes \mathcal{A}$ is the following transition system:

$$TS \otimes \mathcal{A} = (S \times Q, Act, \rightarrow', I', AP', L')$$

where $\rightarrow'$ is the smallest relation defined by the rule

$$\frac{s \xrightarrow{\alpha} t \wedge p \xrightarrow{L(t)} q}{\langle s, p \rangle \xrightarrow{\alpha}{}' \langle t, q \rangle}$$

and

- $I' = \{\langle s_0, q \rangle \mid s_o \in I \wedge \exists q_0 \in Q_0.\ q_0 \xrightarrow{L(s_0)} q\}$,
- $AP' = Q$ and $L' : S \times Q \rightarrow 2^Q$ is given by $L'(\langle s, q \rangle) = \{q\}$.

7.1 From LTL to NBA

Construction of an NBA $\mathcal{A}_\varphi$ satisfying $\mathcal{L}_\omega(\mathcal{A}_\varphi) = \mathit{Words}(\varphi)$ for the LTL formula φ

LTL formula φ

$\downarrow$

Generalized Nondeterministic Büchi automaton G_φ

$\downarrow$

Nondeterministic Büchi Automaton $\mathcal{A}_\varphi$

Definition 18 (Generalized NBA (GNBA))

A generalized NBA is a tuple $G = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ where Q, Σ, δ, Q_0 are defined as for an NBA, and $\mathcal{F}$ is a (possibly empty) subset of 2^Q.

The elements $F \in \mathcal{F}$ are called **acceptance sets**. The infinite run $q_0 q_1 q_2 \ldots \in Q^\omega$ is called **accepting** if

$$\forall F \in \mathcal{F}.\ (\overset{\infty}{\exists}\ j \in \mathbb{N}.\ q_j \in F)\ .$$

The accepted language of G is:

$$\mathcal{L}_\omega(G) = \{\sigma \in \Sigma^\omega \mid \text{there exists an accepting run for } \sigma \text{ in } G\}$$

Review: infinite words can be defined as functions $\sigma : \mathbb{N} \to \Sigma$ and the notation $\sigma = A_1 A_2 A_3 \ldots$ means that $\sigma(i) = A_i$ for all $i \in \mathbb{N}$.

7.1 From LTL to NBA

Let infinite word $\sigma = A_0A_1A_2... \in Words(\varphi)$, an infinite words $\bar{\sigma} = B_0B_1B_2...$ satisfies

$$\psi \in B_i \quad \text{iff} \quad A_iA_{i+1}A_{i+2}... \models \psi \qquad (\psi \in closure(\varphi))$$

Example 19

If $\varphi = a\mathbf{U}(\neg \wedge b)$, $\sigma = \{a\}\{a,b\}\{b\}...$ then

$$closure(\varphi) = \{a, b, \neg a, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), \varphi, \neg\varphi\}$$
$$B_0 = \{a, \neg b, \neg(\neg a \wedge b), \varphi\}$$
$$B_1 = \{a, b, \neg(\neg a \wedge b), \varphi\}$$
$$B_2 = \{\neg a, b, \neg a \wedge b, \varphi\}$$
$$...$$

The GNBA G_φ is constructed such that the sets Bi constitute its states

7.1 From LTL to NBA

Definition 20 (Closure of φ)

The closure of LTL formula φ is the set $\mathit{closure}(\varphi)$ consisting of all subformulae ψ of φ and their negation $\neg\psi$ (where ψ and $\neg\neg\psi$ are identified).

Definition 21 (Elementary Sets of Formulae)

$B \subseteq \mathit{closure}(\varphi)$ is elementary if it is consistent with respect to propositional logic, maximal, and locally consistent with respect to the until operator.

Properties of elementary sets of formulae

1. B is **consistent** with respect to propositional logic, for all $\varphi_1 \wedge \varphi_2, \psi \in \mathit{closure}(\varphi)$:
   - $\varphi_1 \wedge \varphi_2 \in B \Leftrightarrow \varphi_1 \in B$ and $\varphi_2 \in B$
   - $\psi \in B \Rightarrow \neg\psi \notin B$
   - $\mathtt{true} \in \mathit{closure}(\varphi) \Rightarrow \mathtt{true} \in B$.
2. B is **locally consistent** with respect to the until operator, for all $\varphi_1 \mathbf{U} \varphi_2 \in \mathit{closure}(\varphi)$:
   - $\varphi_2 \in B \Rightarrow \varphi_1 \mathbf{U} \varphi_2 \in B$
   - $\varphi_1 \mathbf{U} \varphi_2 \in B$ and $\varphi_2 \notin B \Rightarrow \varphi_1 \in B$.
3. B is **maximal**, for all $\psi \in \mathit{closure}(\varphi)$:
   - $\psi \notin B \Rightarrow \neg\psi \in B$.

7.1 From LTL to NBA

Theorem 22 (GNBA for LTL Formula)

For any LTL formula φ (over AP) there exists a GNBA G_φ over the alphabet 2^AP such that

1. $Words(\varphi) = \mathcal{L}_\omega(G_\varphi)$.
2. G_φ *can be constructed in time and space* $2^{O(|\varphi|)}$.
3. *The number of accepting sets of* G_φ *is bounded above by* $O(|\varphi|)$.

7.1 From LTL to NBA

Let φ be an LTL formula over AP. Let $G_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$:

- Q : set of all elementary sets of formulae $B \subseteq \mathit{closure}(\varphi)$,
- $Q_0 = \{B \in Q \mid \varphi \in B\}$,
- $\mathcal{F} = \{F_{\varphi_1 \mathbf{U} \varphi_2} \mid \varphi_1 \mathbf{U} \varphi_2 \in \mathit{closure}(\varphi)\}$, where $F_{\varphi_1 \mathbf{U} \varphi_2} = \{B \in Q \mid \varphi_1 \mathbf{U} \varphi_2 \notin B \text{ or } \varphi_2 \in B\}$.

The transition relation $\delta : Q \times 2^{AP} \to 2^Q$ is given by:

- If $A \neq B \cap AP$, then $\delta(B, A) = \emptyset$
- If $A = B \cap AP$, then $\delta(B, A)$ is the set of all elementary sets of formulae B' satisfying
  1. for every $\bigcirc\psi \in \mathit{closure}(\varphi)$: $\bigcirc \in B \Leftrightarrow \psi \in B'$, and
  2. for every $\varphi_1 \mathbf{U} \varphi_2 \in \mathit{closure}(\varphi)$: $\varphi_1 \mathbf{U} \varphi_2 \in B \Leftrightarrow (\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \mathbf{U} \varphi_2 \in B'))$.

Theorem 23

From each GNBA G there exists an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(G) = \mathcal{L}_\omega(\mathcal{A})$.

Proof. Let $G = (Q, \Sigma, \delta, Q_0, \mathcal{F})$, with $\mathcal{F} = \{F_1, ..., F_k\}$ and

1. If $k = 1$ then G is an NBA
2. If $k \geq 2$ then NBA $\mathcal{A}$ results from k copies of G:
   $\mathcal{A} = (Q', \Sigma, \delta', Q_0', F')$ where:
   - $Q' = Q \times \{1, ..., k\}$,
   - $Q_0' = Q_0 \times \{1\} = \{\langle q_0, 1\rangle \mid q_0 \in Q_0\}$, and
   - $F' = F_1 \times \{1\} = \{\langle q_F, 1\rangle \mid q_F \in F_1\}$.

   The transition function δ' :

$$\delta'(\langle q, i\rangle, A) = \begin{cases} \{\langle q', i\rangle \mid q' \in \delta(q, A)\} & \text{if } q \notin F_i \\ \{\langle q', i+1\rangle \mid q' \in \delta(q, A)\} & \text{otherwise } (i = k \text{ back}) \end{cases}$$

$$size(\mathcal{A}) = O(Size(G) \cdot |\mathcal{F}|)$$

7.2 Complexity of LTL to NBA

For each LTL formula φ, there is an NBA $\mathcal{A}$ s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\varphi)$ and $\mathit{size}(\mathcal{A}) \leq 2^{cl(\varphi)} \cdot |\varphi| = 2^{O|\varphi|}$

1. From LTL formula φ to GNBA G of size $2^{cl(\varphi)}$
2. From GNBA G to NBA $\mathcal{A}$ of size $\mathit{size}(G) \cdot |\mathcal{F}|$
3. $|\mathcal{F}|$: number of acceptance sets in G, $|\mathcal{F}| \leq |\varphi|$

7.3 Complexity of LTL Model Checking

Theorem 24

The LTL model-checking problem is PSPACE-complete (PSPACE and PSPACE-hard).

1. **PTIME** (or briefly P) denotes the class of all decision problems that can be solved by a deterministic polytime algorithm
2. **NP** denotes the class of all decision problems that can be solved by a nondeterministic polytime algorithm.
3. **PSPACE** denotes the class of all decision problems that can be solved by a deterministic polyspace algorithm.
4. Decision problem P is **PSPACE-hard** if all problems in PSPACE are polynomially reducible to P.

Summary

1. LTL is a logic for formalizing path-based properties
2. LTL formulae can be transformed algorithmically into nondeterministic Büchi automata (NBA). This transformation can cause an exponential blowup.
3. The LTL model-checking problem can be solved by a nested depth-first search in the product of the given transition system and an NBA for the negated formula.
4. The time complexity of the automata-based model-checking algorithm for LTL is linear in the size of the transition system and exponential in the length of the formula