2026 考研数学零基础提前学课堂手迹版讲义

新浪微博：考研数学周洋鑫

零基础提前学（11）

【考点1】原函数　$F'(x)$　$f(x)$　　$F'(x)=f(x)$　　$[F(x)+c]'=f(x)$

【考点2】不定积分

1. 不定积分的概念

2. 不定积分的至高理解

若 $F'(x)=f(x)$ 则 $\int f(x)\,dx = F(x)+c$.

$F(x) \to x \to f(x)$

$\int f(x)\,dx = F(x)+c$.

$\int \dfrac{甲}{}\,d\underline{乙} = \underline{内}+c$　　$\dfrac{d内}{d乙}=甲$

【注】不定积分反问题——可秒

【例8.2】 判断下列不定积分是否正确.

（1）$\displaystyle\int \sin x\cos x\,dx = \dfrac{1}{2}\sin^2 x + C$;　✓

（2）$\displaystyle\int \dfrac{1}{\sqrt{1+x^2}}\,dx = \ln\left(x+\sqrt{1+x^2}\right)+C$.　✓

（3）$\displaystyle\int f'(ax)\,dx = f(ax)+C$.　✗

（1）$\left(\dfrac{1}{2}\sin^2 x\right)'_x = \dfrac{1}{2}\cdot 2\sin x\cdot\cos x = \sin x\cos x$

（2）$\left[\ln\left(x+\sqrt{x^2+1}\right)\right]' = \dfrac{1}{\sqrt{x^2+1}}$

（3）$\left[f(ax)\right]'_x = f'(ax)\cdot a$

求导 ✓

【例8.3】 已知 $\displaystyle\int f(x)\,dx = e^{\sin x}+C$ ，则 $f(x)=$ _____ .　反问题

解：$f(x) = \left[e^{\sin x}\right]'_x = e^{\sin x}\cdot\cos x$　✓

【例8.4】 利用不定积分定义求下列不定积分.（多动脑想！）

（1）$\int x^3 dx = $ _____$\dfrac{1}{4}x^4 + C$_____.

（2）$\int \dfrac{1}{x} dx = $ _____$\ln|x| + C$_____.

（3）$\int a^x dx = $ _____$\dfrac{1}{\ln a} a^x + C$_____.

（4）$\int e^x dx = $ _____$e^x + C$_____.

导数表 ✓

积分表 ✓

（5）$\int \sin x dx = $ _____$-\cos x + C.$_____.

（6）$\int \cos x dx = $ _____$\sin x + C.$_____.

（7）$\int \sec^2 x dx = $ _____$\tan x + C$_____.

（8）$\int \csc^2 x dx = $ _____$-\cot x + C.$_____.

（9）$\int \sec x \tan x dx = $ _____$\sec x + C$_____.

（10）$\int \csc x \cot x dx = $ _____$-\csc x + C$_____.

（11）$\int \dfrac{1}{\sqrt{1-x^2}} dx = $ _____$\arcsin x + C.$_____.

（12）$\int \dfrac{1}{1+x^2} dx = $ _____$\arctan x + C.$_____.

（13）$\int \dfrac{1}{\sqrt{x^2+a^2}} dx = $ _____$\ln\left(x + \sqrt{x^2+a^2}\right) + C$_____.

（14）$\int \dfrac{1}{\sqrt{x^2-a^2}} dx = $ _____$\ln\left|x + \sqrt{x^2-a^2}\right| + C.$_____.

【考点3】不定积分运算性质

（1）数乘：$\int kf(x)dx = k\int f(x)dx$

（2）加减法：$\int\left[f(x) \pm g(x)\right]dx = \int f(x)dx \pm \int g(x)dx$.

【思考】 $\int f'(x)dx = f(x) + C$ $f'(x)dx = df(x)$

$\int df(x) = f(x) + C$

$\int d \Rightarrow$ 加C

$d \int \Rightarrow$ 不加C.

【注】不定积分与微分的逆运算问题.

【考】不定积分的计算问题

1. 积分表（基础）
2. 凑微分法（后续方法的基础）
3. 第二类换元
4. 分部积分法
5. 有理分式积分（基础班再讲）

$\checkmark\displaystyle\int\frac{1}{2\sqrt{x}}dx=\sqrt{x}+c.$

$\checkmark\displaystyle\int\frac{1}{x^2}dx=\frac{1}{x}+c.$

【考点4】基本积分表（不定积分运算基础，必记！）

1. $\displaystyle\int x^\alpha dx=\frac{x^{\alpha+1}}{\alpha+1}+C$ （$\alpha\neq -1$，实常数） 　　2. $\displaystyle\int\frac{1}{x}dx=\ln|x|+C$

3. $\displaystyle\int a^x dx=\frac{1}{\ln a}a^x+C$ （$a>0,a\neq 1$） 　　4. $\displaystyle\int e^x dx=e^x+C$

5. $\displaystyle\int\sin xdx=-\cos x+C$ 　　　　　　　6. $\displaystyle\int\cos xdx=\sin x+C$

7. $\displaystyle\int\tan xdx=-\ln|\cos x|+C$ 　　8. $\displaystyle\int\cot xdx=\ln|\sin x|+C$

9. $\displaystyle\int\sec xdx=\ln|\sec x+\tan x|+C$ 　　10. $\displaystyle\int\csc xdx=\ln|\csc x-\cot x|+C$

11. $\displaystyle\int\sec^2 xdx=\tan x+C$ 　　12. $\displaystyle\int\csc^2 xdx=-\cot x+C$

13. $\displaystyle\int\tan x\sec xdx=\sec x+C$ 　　14. $\displaystyle\int\cot x\csc xdx=-\csc x+C$

15. $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C$ （$a>0$） 　　16. $\displaystyle\int\frac{dx}{a^2+x^2}=\frac{1}{a}\arctan\frac{x}{a}+C$

17. $\displaystyle\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$ （$a>0$）

18. $\displaystyle\int\frac{dx}{\sqrt{x^2+a^2}}=\ln\left(x+\sqrt{x^2+a^2}\right)+C$ （$a>0$）

18. $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$ （$a>0$）

【例8.5】 求不定积分 $\displaystyle\int\left(x^3+e^x+3\sin x\right)dx$.

$$=\int x^3 dx+\int e^x dx+3\int\sin xdx$$

$$=\frac{1}{4}x^4+e^x-3\cos x+C.$$

【例8.6】求不定积分 $\int\left(1-\dfrac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx$.

$$= \int\left(1-\frac{1}{x^2}\right)x^{\frac{3}{4}}\,dx$$

$$= \int x^{\frac{3}{4}}\,dx - \int x^{-\frac{5}{4}}\,dx$$

$$= \frac{4}{7}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + c$$

$\left(x\cdot x^{\frac{1}{2}}\right)^{\frac{1}{2}}$
$= \left(x^{\frac{3}{2}}\right)^{\frac{1}{2}}$
$= x^{\frac{3}{4}}$

$\dfrac{x^{\frac{3}{4}}}{x^2}$ ✓

【例8.7】求不定积分 $\int\dfrac{(1-x)^2}{\sqrt{x}}\,dx$.

$$\int \frac{1-2x+x^2}{\sqrt{x}}\,dx$$

$$= \int \frac{1}{\sqrt{x}}\,dx - 2\int \sqrt{x}\,dx + \int x^{\frac{3}{2}}\,dx$$

$$= 2\sqrt{x} - 2\cdot\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c.$$

【例8.8】求不定积分 $\int\left(2^x-3^x\right)^2\,dx$.

$$= \int\left[\left(2^x\right)^2 - 2\cdot 2^x 3^x + \left(3^x\right)^2\right]\,dx$$

$$= \int 4^x\,dx - 2\int 6^x\,dx + \int 9^x\,dx$$

$$= \frac{1}{\ln 4}4^x - 2\frac{1}{\ln 6}6^x + \frac{1}{\ln 9}9^x + c.$$

【例8.9】求不定积分 $\int\tan^2 x\,dx$.

$$= \int\left(\sec^2 x - 1\right)\,dx$$

$$= \int \sec^2 x\,dx - \int 1\,dx$$

$$= \tan x - x + c.$$

$\sec^2 x \quad \csc^2 x$
$1 + \tan^2 x = \sec^2 x$ ✓
$1 + \cot^2 x = \csc^2 x$

【例8.10】求不定积分 $\int\dfrac{1+\cos^2 x}{1+\cos 2x}\,dx$.

解：$\int \dfrac{1+\cos^2 x}{2\cos^2 x}\,dx$

$$= \frac{1}{2}\int \frac{1}{\cos^2 x}\,dx + \int \frac{1}{2}\,dx$$

$$= \frac{1}{2}\int \sec^2 x\,dx + \int \frac{1}{2}\,dx$$

$$= \frac{1}{2}\tan x + \frac{1}{2}x + c.$$

$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

$\dfrac{1}{\cos x} = \sec x$

【例8.11】求不定积分 $\displaystyle\int \frac{1}{x^2(1+x^2)}dx$.

$$= \int\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \int \frac{-1}{x^2}dx - \int \frac{1}{1+x^2}dx$$

$$= -\frac{1}{x} - \arctan x + c.$$

$$\frac{1+x^2-x^2}{x^2(1+x^2)}$$

$$= \frac{1}{x^2} - \frac{1}{1+x^2}$$

【考点5】第一类换元积分法（凑微分法）

1. 为什么要凑微分？

2. 什么是凑微分？

3. 常见的凑微分

积分求导

$$f'(x)dx = dF(x)$$

$$\square'dx = d\square$$

凑微分：求导.

已知 $\displaystyle\int f(\square)d\square$ 可积, 则

(1) $\displaystyle\int f(ax)dx = \frac{1}{a}\int f(ax)d(ax)$

(2) $\displaystyle\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b)$

(3) $\displaystyle\int f(\sqrt{x})\frac{1}{2\sqrt{x}}dx = \int f(\sqrt{x})d\sqrt{x}$

(4) $\displaystyle\int f\left(\frac{1}{x}\right)\frac{-1}{x^2}dx = \int f\left(\frac{1}{x}\right)d\frac{1}{x}$

(5) $\displaystyle\int f(e^x)e^x dx = \int f(e^x)de^x$

(6) $\displaystyle\int f(\ln x)\frac{1}{x}dx = \int f(\ln x)d\ln x$

(7) $\displaystyle\int f(x^2)\,x\,dx = \frac{1}{2}\int f(x^2)dx^2$

例1: $\boxed{\displaystyle\int \cos x\,dx = \sin x + c}$

$$\int \cos x^2\,dx^2 = \sin x^2 + c \qquad d\square \checkmark$$

$$\int \cos e^x\,de^x = \sin e^x + c \qquad dx \checkmark$$

$$\boxed{\displaystyle\int \cos \square\,d\square = \sin \square + c}$$

例2: $\cos x \, dx = d\sin x$

$\sec^2 x \, dx = d\tan x$

$\dfrac{1}{1+x^2} dx = d\arctan x$

$\dfrac{1}{2\sqrt{x}} dx = d\sqrt{x}$

$\dfrac{-1}{x^2} dx = d\dfrac{1}{x}$

凑一秒分 ✓

【例8.12】 求下列不定积分.

（1）$\displaystyle\int \sin 2x \, dx$ （2）$\displaystyle\int e^{5x-7} dx$

(1) $\dfrac{1}{2}\displaystyle\int \sin 2x \, d(2x) = -\dfrac{1}{2}\cos 2x + C.$

(2) $\dfrac{1}{5}\displaystyle\int e^{5x-7} d(5x-7) = \dfrac{1}{5} e^{5x-7} + C.$

例1: $\displaystyle\int \dfrac{1}{a^2 + x^2} dx$

$= \dfrac{1}{a^2}\displaystyle\int \dfrac{1}{1+\left(\frac{x}{a}\right)^2} dx$

$= \dfrac{1}{a^2}\cdot a\displaystyle\int \dfrac{1}{1+\left(\frac{x}{a}\right)^2} d\dfrac{x}{a}$

$= \dfrac{1}{a}\arctan\dfrac{x}{a} + C.$

$\displaystyle\int \dfrac{1}{1+x^2} dx = \arctan x + C.$

$\displaystyle\int \dfrac{1}{1+\square^2} d\square = \arctan\square + C.$

例2: $\displaystyle\int \dfrac{1}{\sqrt{a^2 - x^2}} dx \quad (a>0)$

$= \dfrac{1}{a}\displaystyle\int \dfrac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} dx$

$= \dfrac{1}{a}\cdot a\displaystyle\int \dfrac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} d\dfrac{x}{a}$

$= \arcsin\dfrac{x}{a} + C.$

$\displaystyle\int \dfrac{1}{\sqrt{1-x^2}} dx = \arcsin x + C.$

例3: $\int \dfrac{1}{x^2-a^2}dx$ ✓

$= \int \dfrac{1}{(x-a)(x+a)}dx$

$= \dfrac{1}{2a}\int\left(\dfrac{1}{x-a}-\dfrac{1}{x+a}\right)dx$

$= \dfrac{1}{2a}\left[\int\dfrac{1}{x-a}d(x-a)-\int\dfrac{1}{x+a}d(x+a)\right]$

$= \dfrac{1}{2a}\cdot\left[\ln|x-a|-\ln|x+a|\right]+C$

$= \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|+C.$

【例8.13】 求下列不定积分.

(1) $\displaystyle\int\dfrac{\sin\frac{1}{x}}{x^2}dx$ 　　　(2) $\displaystyle\int\dfrac{\sec^2\frac{1}{x}}{x^2}dx$

$d\dfrac{1}{x}=\dfrac{-1}{x^2}dx$

(1) $\displaystyle\int\sin\dfrac{1}{x}\cdot\dfrac{-1}{x^2}dx=-\int\sin\dfrac{1}{x}\,d\dfrac{1}{x}=\cos\dfrac{1}{x}+C.$

(2) $\displaystyle\int\sec^2\dfrac{1}{x}\cdot\dfrac{-1}{x^2}dx=-\int\sec^2\dfrac{1}{x}\,d\dfrac{1}{x}=-\tan\dfrac{1}{x}+C.$

【例8.14】 求下列不定积分.

(1) $\displaystyle\int\cos e^x\cdot e^x dx$ 　　　(2) $\displaystyle\int\dfrac{\ln x}{x}dx$

(1) $\displaystyle\int\cos e^x\,de^x=\sin e^x+C.$

(2) $\displaystyle\int\ln x\cdot\dfrac{1}{x}dx=\int\ln x\,d\ln x=\dfrac{1}{2}\ln^2 x+C.$

【例8.15】 求下列不定积分.

(1) $\displaystyle\int\dfrac{e^{\sqrt{x}}}{\sqrt{x}}dx$ $\dfrac{1}{2\sqrt{x}}dx=d\sqrt{x}$ ✓ (2) $\displaystyle\int\dfrac{\sin\sqrt{2x-1}}{\sqrt{2x-1}}dx$

$d\sqrt{x}=\dfrac{1}{2\sqrt{x}}dx$ ✓

(1) $\displaystyle\int e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}}dx=2\int e^{\sqrt{x}}\,d\sqrt{x}=2e^{\sqrt{x}}+C.$

【例8.15】求下列不定积分.

（1）$\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}}dx$ （2）$\int \dfrac{\sin\sqrt{2x-1}}{\sqrt{2x-1}}dx$

（2）$\int \sin\sqrt{2x-1}\;\boxed{\dfrac{1}{2\sqrt{2x-1}}d(2x-1)}$

$=\int \sin\sqrt{2x-1}\;d\sqrt{2x-1}$

$=-\cos\sqrt{2x-1}+C.$

$甲'd甲 = d丙$

$\dfrac{1}{2\sqrt{x}}dx = d\sqrt{x}$

$\dfrac{1}{2\sqrt{a}}da = d\sqrt{a}.$

【例8.16】求下列不定积分

（1）$\int x\sqrt{1-x^2}\,dx$ （2）$\int \dfrac{x}{\sqrt{1+x^2}}dx$

（1）$\int \sqrt{1-x^2}\,\boxed{x\,dx}=\tfrac{1}{2}\int\sqrt{1-x^2}\;d\,x^2$

$=-\tfrac{1}{2}\int\sqrt{1-x^2}\;d\left(-x^2+1\right)$

$=-\tfrac{1}{2}\cdot\tfrac{2}{3}\left(1-x^2\right)^{\frac{3}{2}}+C.$

（2）$\int \dfrac{x}{\sqrt{1+x^2}}dx=\int\dfrac{1}{\sqrt{1+x^2}}\boxed{x\,dx}$

$=\int\dfrac{1}{2\sqrt{x^2+1}}d\left(x^2+1\right)$

$=\sqrt{x^2+1}+C.$

例1. $\int \tan x\,dx$

$=\int\boxed{\dfrac{\sin x}{\cos x}}dx$

$=-\int\dfrac{1}{\cos x}d\cos x$

$=-\ln|\cos x|+C.$

例2. $\int \cot x\,dx$

$=\int\boxed{\dfrac{\cos x}{\sin x}}dx$

$=\int\dfrac{1}{\sin x}d\sin x$

$=\ln|\sin x|+C.$