TP 2 : Vanilla and Barrier Options under Stochastic Volatility
Derivatives (60206A)

GROUP MEMBERS

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PRESENTED TO

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<u>PART I</u>

1. Closed-Form Heston Model

We apply the closed form solution $c(0, T, K) = S^{-yT} P_1 - K e^{-rT} P_2$ for a book of standard European call options.

	Strike Price (K)	European Call Price
0	95	8.488870
1	100	5.731202
2	105	3.671103

Table 1. Option prices for strikes {95, 100, 105} using Heston's closed-form solution.

Our results show that the value of the European call option decreases as the strike price increases. This behavior aligns with standard option pricing theory: a higher strike price reduces the intrinsic value of the call option and lowers the probability of it expiring in-the-money. The Heston model effectively captures this relationship, which accurately reflects the impact of strike prices on option values under the model's specified market conditions.

2. Monte Carlo Simulation

	Strike Price (K)	Down-and-Out Call Price	Variance
0	95	8.002482	118.213744
1	100	5.573128	83.704861
2	105	3.659962	55.631775

Table 2. Barrier call option prices for strikes {95, 100, 105} using Monte Carlo simulation.
100,000 paths simulated and H = 90.

The barrier option premiums obtained with Monte Carlo simulation are on average lower than the standard option premiums derived from the closed-form solution. This is because barrier options typically have lower premiums than standard options due to the additional trigger conditions. For a down-and-out barrier option, the payoff becomes zero if the underlying asset price breaches the barrier level before expiration, limiting the option's value and resulting in a lower price.

We note that the preferred method to value each type of option has been used to obtain the premiums : the closed-form solution is the preferred method to value standard call options, whereas Monte Carlo simulation is more suitable for path-dependent options.

Finally, our results exhibit significant variance, which is expected to increase with the inclusion of stochastic volatility under the Heston model. To mitigate this, we apply the antithetic variate technique in the next section.

2. Antithetic Variate Technique

	Strike Price (K)	Adjusted Down-and-Out Call Price	Adjusted Variance	Slope	R^2
0	95	8.023888	6.593131	0.982377	0.944124
1	100	5.592193	2.895549	0.982453	0.965360
2	105	3.676049	1.200898	0.985728	0.978392

Table 3. Option prices for strikes {95, 100, 105} using the antithetic variate technique.

The results now show significantly lower variance compared to the standard Monte Carlo simulation. Moreover, the regression lines for each strike price all have slopes and R-squared values close to 1, indicating that the pricing estimates after applying the antithetic variates closely align with the true value of the option. This suggests that the model effectively captures the underlying process of the option's price, with minimal unexplained variability, and demonstrates a strong fit to the data.

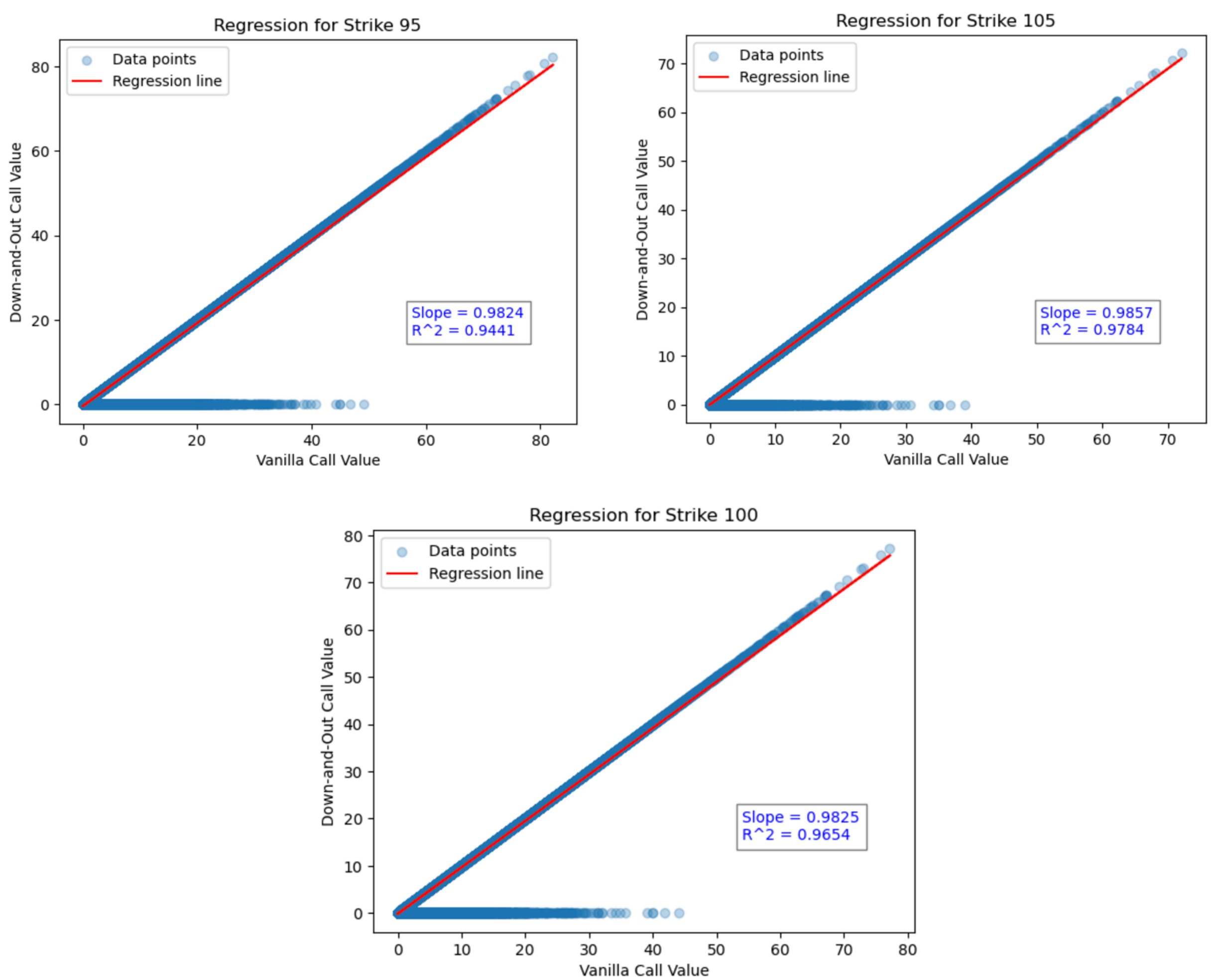

Figure 1. Regression line for strikes K = {95, 100, 105} using the antithetic variate technique. 100,000 simulations split into 40 batches of 2,500 paths. Down-and-out call paths are regressed against respective vanilla calls.

PART II

1. Prices and Volatility of S&P 500

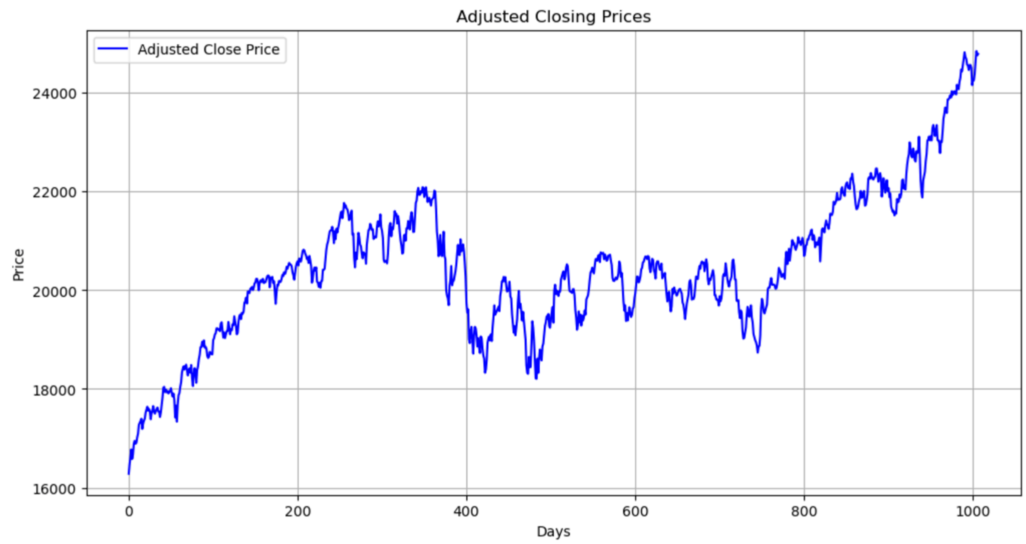

Figure 2. Adjusted closing prices for the S&P 500 from November 6, 2020 to November 11, 2024.

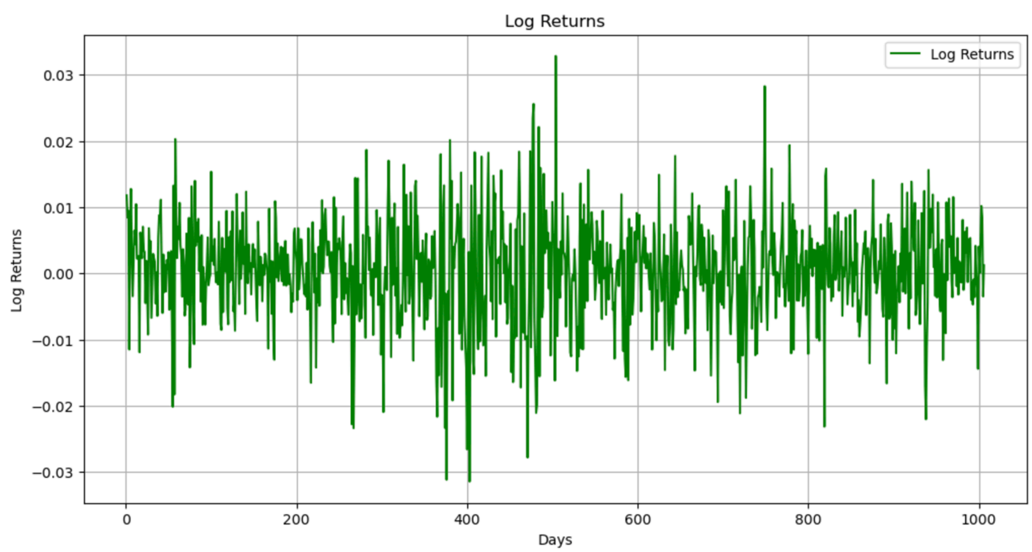

Figure 3. Log returns for the S&P 500 from November 6, 2020 to November 11, 2024.
We assumed 252 trading days per year.

The sample return volatility was found to be 12.37%. As shown in Figure 2, the adjusted closing price demonstrates a consistent upward trend over the 4-year period. Meanwhile, Figure 3 illustrates the log returns, revealing a clustered volatility pattern, with alternating periods of high and low volatility.

2. NGARCH(1,1) Model Parameters

Due to volatility clustering in the log returns, we use the NGARCH(1,1) model to fit the data, estimate its parameters (ω, α, β, γ, λ), and compare the model's unconditional volatility with the sample volatility to evaluate its effectiveness.

Estimated Parameters

	omega	alpha	beta	gamma	lambda
0	0.000014	0.010006	0.750001	0.010013	0.050064

Table 4. Maximum likelihood parameter estimates for NGARCH(1,1).
Daily risk-free rate of 2.75%/365 was used.

	Unconditional Volatility	Sample Volatility
0	0.007649	0.007793

Table 5. Unconditional volatility versus sample volatility.
Unconditional volatility obtained from parameter estimates.

The estimated parameters are as follows:

- $\omega = 0.000014$: ω represents the long-term variance. From Figure 3, we can observe that the returns fluctuate around a zero mean, indicating the market exhibits a mean-reverting nature around zero. Therefore, it is reasonable for ω to be very close to zero.

- $\alpha = 0.010006$: α represents the sensitivity of volatility to the short shocks (innovations). Here, the value suggests that while recent shocks influence volatility, their impact is moderate.

- $\beta = 0.750001$: β represents the persistence of volatility over time. Here, the value suggests strong persistence, meaning that volatility changes are long-lasting. This aligns with the observed volatility clustering in Figure 3.

- $\gamma = 0.010013$: γ represents the leverage effect. The positive γ indicates that negative shocks (price drops) increase volatility more significantly than positive shocks (price rises). Here, the value suggests that the leverage effect is not particularly pronounced, which is also reflected in Figure 3, where the asymmetry between positive and negative shocks is not highly evident.

- $\lambda = 0.050064$: λ represents the nonlinearity of the model and adjusts the impact of past squared returns on volatility.

Overall, the alignment between the unconditional volatility (0.007649) and the sample volatility (0.007793), along with a persistence value close to 1, suggests that the parameter estimates and the NGARCH model effectively capture the observed data's characteristics, including its stochastic volatility dynamics.

3. Call Options on The S&P 500

	Strike Price	European Call Price
0	23000	1977.548456
1	23100	1889.257169
2	23200	1802.373235
3	23300	1716.963079
4	23400	1633.093189
5	23500	1550.939442
6	23600	1470.594573
7	23700	1392.118991
8	23800	1315.673340
9	23900	1241.361221
10	24000	1169.229800
11	24100	1099.354290
12	24200	1031.762575
13	24300	966.548987
14	24400	903.773742
15	24500	843.483826
16	24600	785.799027
17	24700	730.638654
18	24800	678.049918
19	24900	628.015644
20	25000	580.532895
21	25100	535.562276
22	25200	493.040773
23	25300	452.928130
24	25400	415.250244
25	25500	379.855842
26	25600	346.718839
27	25700	315.857054
28	25800	287.119303
29	25900	260.454465
30	26000	235.800928
31	26100	213.037992
32	26200	192.026210
33	26300	172.731249
34	26400	155.076699
35	26500	138.936155
36	26600	124.207427
37	26700	110.758735
38	26800	98.591974
39	26900	87.568101
40	27000	77.571025
41	27100	68.570031
42	27200	60.497862
43	27300	53.230260
44	27400	46.728768
45	27500	40.933933
46	27600	35.790411
47	27700	31.230722
48	27800	27.208571
49	27900	23.676607
50	28000	20.571796

Table 6. 51 call option premiums on the S&P 500 with strikes from 23,000 to 28,000 with T = 3 months.
Monte Carlo simulation was used with 100,000 paths.
NGARCH(1,1) was used to model the UA process and stochastic volatility.

The call premiums found with NGARCH decrease as the strike increases, or when the option becomes more OTM.

To verify and compare our results, we simulated option prices using the Black-Scholes model (See *Appendix*). Indeed, NGARCH call option prices are higher than the ones simulated with Black-Scholes, although not symmetrically: as the strike price increases and the options become deeper OTM, the price differences between diminishes. This can be explained by the inclusion of volatility clustering and leverage effects in NGARCH models, as opposed to Black-Scholes, which relies on the constant volatility assumption.

4. Volatility Smile

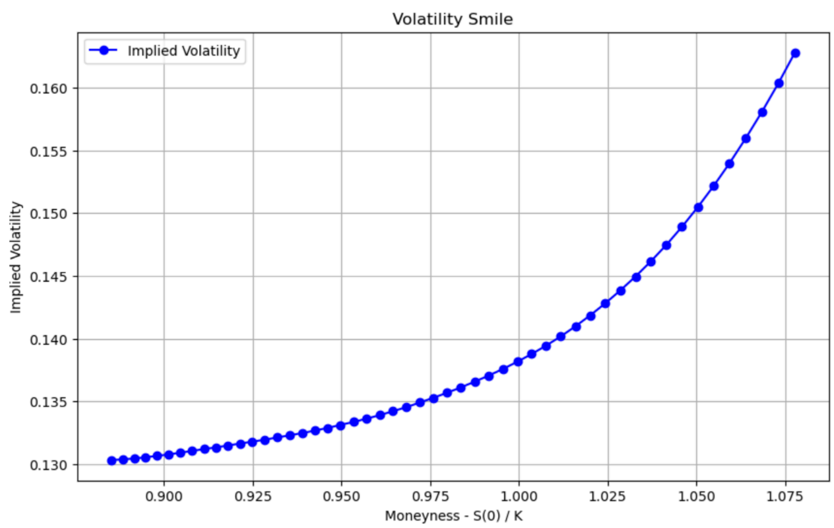

Figure 4. Volatility smile for 51 call options on the S&P 500 with strikes from 23,000 to 28,000 with T = 3 months. Monte Carlo simulation for 100,000 paths was used to generate premiums. NGARCH(1,1) was used to model the UA process and stochastic volatility.

The implied volatility decreases for deep OTM calls and increases for deep ITM calls, resulting in an asymmetric smile. Indeed, the NGARCH model captures the skew more realistically, reflecting market behaviors such as risk asymmetry and heavy-tailed returns. Based on empirical data, a forward skew is common in index options markets during periods of upward trending expectations or moderate bullish sentiment.

Appendix

	Strike	NGARCH Price	BS Price	Difference
0	23000	1977.548456	1870.625860	106.922596
1	23100	1889.257169	1783.234273	106.022896
2	23200	1802.373235	1697.349919	105.023316
3	23300	1716.963079	1613.083259	103.879820
4	23400	1633.093189	1530.543942	102.549247
5	23500	1550.939442	1449.839701	101.099741
6	23600	1470.594573	1371.075236	99.519337
7	23700	1392.118991	1294.351086	97.767905
8	23800	1315.673340	1219.762530	95.910810
9	23900	1241.361221	1147.398523	93.962698
10	24000	1169.229800	1077.340694	91.889106
11	24100	1099.354290	1009.662428	89.691862
12	24200	1031.762575	944.428045	87.334530
13	24300	966.548987	881.692090	84.856897
14	24400	903.773742	821.498753	82.274989
15	24500	843.483826	763.881426	79.602400
16	24600	785.799027	708.862400	76.936627
17	24700	730.638654	656.452713	74.185941
18	24800	678.049918	606.652145	71.397773
19	24900	628.015644	559.449352	68.566291
20	25000	580.532895	514.822150	65.710745
21	25100	535.562276	472.737916	62.824360
22	25200	493.040773	433.154119	59.886654
23	25300	452.928130	396.018953	56.909178
24	25400	415.250244	361.272059	53.978185
25	25500	379.855842	328.845329	51.010513
26	25600	346.718839	298.663761	48.055078
27	25700	315.857054	270.646366	45.210688
28	25800	287.119303	244.707089	42.412214
29	25900	260.454465	220.755752	39.698713
30	26000	235.800928	198.698981	37.101947
31	26100	213.037992	178.441116	34.596876
32	26200	192.026210	159.885092	32.141118
33	26300	172.731249	142.933271	29.797977
34	26400	155.076699	127.488224	27.588474
35	26500	138.936155	113.453448	25.482707
36	26600	124.207427	100.734022	23.473405
37	26700	110.758735	89.237187	21.521548
38	26800	98.591974	78.872857	19.719116
39	26900	87.568101	69.554054	18.014047
40	27000	77.571025	61.197265	16.373760
41	27100	68.570031	53.722740	14.847291
42	27200	60.497862	47.054706	13.443156
43	27300	53.230260	41.121529	12.108731
44	27400	46.728768	35.855810	10.872958
45	27500	40.933933	31.194426	9.739506
46	27600	35.790411	27.078524	8.711887
47	27700	31.230722	23.453469	7.777253
48	27800	27.208571	20.268757	6.939814
49	27900	23.676607	17.477895	6.198713
50	28000	20.571796	15.038252	5.533544

Table 6. Comparison between Black-Scholes versus NGARCH(1,1) call prices on the S&P 500 with strikes from 23,000 to 28,000. T = 3 months.

References

Dorion, Christian, and Pascal François. 2024. *Derivatives*. HEC Montreal & Fellow of the Canadian Derivatives Institute. Preliminary Draft, Fall 2024 Semester.

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