TP 1 : Index Options with Skewness and Kurtosis
Derivatives (60206A)

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1. DIVIDEND YIELD

Using the cash and carry relation,

$$f(t,T) \ = \ S(t)\ exp\,[(r\ -\ y)(T\ -\ t)]$$

we can solve for the dividend yield (*y*) by plugging in the continuously compounded and annualized risk-free rate (*r*) , the forward price (*f*) and the spot price (*S*). Doing so yields the following result, where *y* is continuously compounded and annualized :

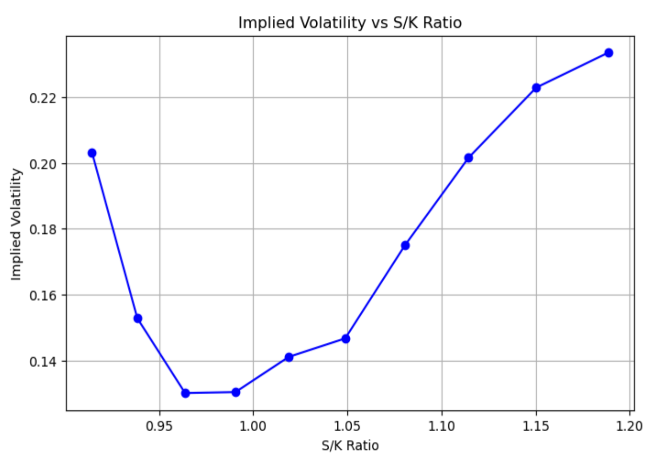
$$y \approx 2,904\%.$$

Please refer to the function *find_y (F, S, r, T, t)* for more detail on the code.

2. CLASSIC BLACK-MERTON-SCHOLES MODEL (BMS)

The BMS model relies on the assumption that asset price dynamics follow a geometric Brownian motion, with Gaussian log-returns and constant volatility. For the latter, this implies volatility must remain constant across different strikes (*K*) and maturities (*T*) for option premiums. However, this is not consistent with market behavior, and clearly displays the limitation of the BMS model in capturing the dynamics of the underlying. We can think of jumps in asset price movements, stochastic volatility, or the tendency for asset return distributions to be leptokurtic, which are not accounted for.

Our results in Graph 1 shows that implied volatility plotted against the S/K ratio (moneyness) exhibits a "smile" or "skew," meaning that options with different strikes can have significantly different implied volatilities. The inability to account for the observed smile or skew in implied volatility can result in mispricing of options, especially for those that are far out-of-the-money or deep in-the-money.



Graph 1. Implied volatility vs. S/K Ratio

	Strike	Premium	S/K	Implied Volatility
0	30	0.13	1.188667	0.233450
1	31	0.20	1.150323	0.222836
2	32	0.26	1.114375	0.201560
3	33	0.32	1.080606	0.175029
4	34	0.40	1.048824	0.146750
5	35	0.69	1.018857	0.141151
6	36	1.08	0.990556	0.130471
7	37	1.70	0.963784	0.130179
8	38	2.58	0.938421	0.152861
9	39	3.65	0.914359	0.203104

Table 1. Implied volatility for different strikes

Please refer to the function *implied_volatility (S, K, T, r, y, premiums)* for more details on the code.

3. BARONE-ADESI AND WHALEY MODEL (BA)

The BA method is a continuous-time approximation of the fair price of an American option by emphasizing the early exercise premium. In other words, the method places more weight on the intrinsic value of the option rather than its time value. This falls in line with our results in Table 2 and 3, where the BA premiums are slightly higher than the quoted puts and those of the binomial tree in exercise 4.

	Strike Price	Premium	American Option Premium (BA)
0	30	0.13	0.131037
1	31	0.20	0.201421
2	32	0.26	0.261818
3	33	0.32	0.322341
4	34	0.40	0.403238
5	35	0.69	0.695618
6	36	1.08	1.090104
7	37	1.70	1.718514
8	38	2.58	2.607273
9	39	3.65	3.680465

Table 2. Premiums for American puts using the Barone-Adesi Whaley approximation

To find the these premiums, we solved for the optimal exercise spot (S^{**}) in the non-linear equation,

$$K - S^{**} = p(t, T, K | S = S^{**}) - (\frac{S^{**}}{\gamma_1})(1 - e^{-y(T-t)}\Phi(-d_1(S^{**}))).$$

Then, we applied the following condition for different strikes to evaluate the American put,

$$\begin{aligned} P(t, T, K) &= K - S_t \qquad , if\ S^{**} \geq S_t \\ &= p(t, T, K) + A_1(\frac{S_t}{\gamma_1})^{\gamma_1} \qquad , if\ S^{**} < S_t \end{aligned}$$

where $A_1 = -(\frac{S^{**}}{\gamma_1})(1 - e^{-y(T-t)}\Phi(-d_1(S^{**}))$.

Please refer to the function *price_put_BA (S, K, r, y, T, t, sigma)* for more details on the code.

4. COX-ROSS-RUBINSTEIN BINOMIAL TREE MODEL (CRR)

The CRR method is a discrete-time approximation of the fair price of an American option, whose convergence depends heavily on the time steps (*N*). As *N* increases, the premiums should theoretically converge to values close to those obtained from other numerical methods. This is displayed by our results in Table 3, where the CRR premiums are very close to the BA premiums.

The discrepancies can be explained by the discretization of the tree and the BA model's capturing of the early exercise premium feature under low volatility, for deep ITM options and for strikes near maturity.

Indeed, the BA premiums are slightly higher for deep ITM options, whereas the CRR premiums are higher for deep OTM options.

	Strike Price	Premium	American Option Premium (BA)	American Option Premiun (CRR)
0	30	0.13	0.131037	0.130442
1	31	0.20	0.201421	0.200539
2	32	0.26	0.261818	0.261141
3	33	0.32	0.322341	0.321786
4	34	0.40	0.403238	0.402671
5	35	0.69	0.695618	0.695552
6	36	1.08	1.090104	1.091913
7	37	1.70	1.718514	1.723128
8	38	2.58	2.607273	2.614232
9	39	3.65	3.680465	3.688590

Table 3. Premiums for American puts using the Barone-Adesi Whaley approximation and CRR binomial tree

We obtained these results by defining the up factor (u), down factor (d) and risk-free probability (p),

$$u = e^{\sigma\sqrt{\Delta t}},\ d = \frac{1}{u},\ p = \frac{e^{rT} - d}{u - d}.$$

Then, we proceed via backwards induction with a selection criterion between the continuation value or intrinsic value of the put, where the algorithm was initialized at the penultimate date with the BMS analytical formula. Please refer to the function *american_put_option_crr (N, S, K, r, sigma, y, T)* for more details on the code.

5. CORRAD-SU BLACK-SCHOLES MODEL (SKEW-KURT)

In comparison to the classic BMS equation, the Corrado-Su BMS model includes terms which adjust for non-normal skewness and kurtosis through the added terms $skew \cdot Q_3$ and $(kurt - 3)Q_4$, respectively :

$$p = p_{BMS} + skew \cdot Q_3 + (kurt - 3)Q_4$$

In this new framework, we optimized the objective function $min\{\sigma, skew, kurt\} \sum_K (p_{obs} - p)^2$ to obtain the values for σ, $skew$, $kurt$ and the minimum objective function value in the table below.

Optimal sigma	Optimal skew	Optimal kurtosis	Optimal Value
0.168943	-0.023025	8.0	0.03579

Table 4. Values for σ, $skew$, $kurt$ and the objective function minimum

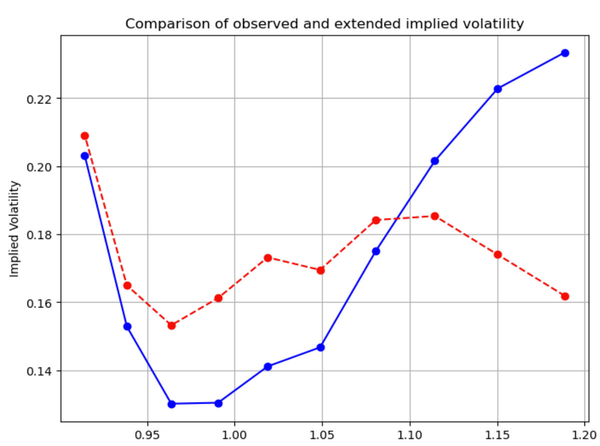
We would like to make several, short comments on the interpretability, realism and methodology of our results :

- Our σ suggests moderate volatility for the underlying, or a low expectation for significant price fluctuations. The *skew* indicates a near symmetrical return distribution with a slight tendency for negative tail events and a higher probability of downside. Lastly, the *kurt* points towards a distribution with fat tails and a higher probability of extreme events.

- Although the objective function can be further minimized by relaxing our current bounds $\sigma \in (0.0, 1.0)$, *skew* $\in$ (-2, 2) , *kurt* $\in$ (0, 8), we fixed these bounds to retrieve values which align with real market behavior and observed behavior on the underlying. In other words, it would be highly unusual to proceed with values of $\sigma > 0.3$, *skew* $> | 2 |$ and *kurt* > 8, even if the algorithm returns such values. Doing so would distort the pricing of the American premiums in the trinomial tree.

- These optimal values for {σ, *skew*, *kurt*} were found by first performing a global search using differential evolution . Once a suitable initial starting point for the unknown parameters were found, we then performed a local search using the L-BFGS-B method. This two step method was necessary to decrease the L-BFGS-B method's sensitivity to initialization, which would otherwise greatly affect the results. Please refer to the function *differential_evolution (objective_function_ext, bounds_ext, args=(S, K, T, r, y, premiums), seed = 777))* and the steps documented afterwards for more details on the code.

Visually, the implied volatilities extracted from the skew-kurt BMS model no longer display the "volatility smile" of the classic BMS model, as shown by Graph 2. Instead, we notice a "smoothing out" of the volatility smile into a degree 3 polynomial, especially for ATM and near-ATM options. This can be in part explained by the fact that the skew-kurt BMS model accounts for skewness and kurtosis in the return distribution of the underlying, as opposed to the classic BMS model which assumes log-normally distributed returns. When skewness is included, the return distribution becomes asymmetric, reflecting a higher probability of large upward or downward movements. On the other hand, when kurtosis is included, tail behavior is now factoried in, making extreme outcomes (both positive and negative) more likely. In short, the inclusion of these moments in the model better reflects market behavior, making the volatility smile less pronounced.



Graph 2. Observed implied volatility versus induced volatility

Graph 2 compares the implied volatility trends at different moneyness levels (S/K) between the classic BMS model and the skew-kurt BMS model. The red dashed line (---) represents the skew-kurt BMS model, while the blue solid line (—) represents the classic BMS model.

	Strikes	Moneyness	Observed_iv	Adjusted_iv	ratio
0	30	1.188667	0.233450	0.161893	0.693479
1	31	1.150323	0.222836	0.174004	0.780860
2	32	1.114375	0.201560	0.185364	0.919650
3	33	1.080606	0.175029	0.184178	1.052270
4	34	1.048824	0.146750	0.169508	1.155075
5	35	1.018857	0.141151	0.173197	1.227035
6	36	0.990556	0.130471	0.161253	1.235936
7	37	0.963784	0.130179	0.153250	1.177222
8	38	0.938421	0.152861	0.164971	1.079229
9	39	0.914359	0.203104	0.209081	1.029429

Table 5. Observed implied volatility (classic BMS) versus induced volatility (skew-kurt BMS)

6. JI-BRORSEN OPTIMIZATION OF TRINOMIAL TREE PARAMETERS

The Ji-Brorsen estimates of the trinomial tree parameters rely on matching the moments of the log-normal distribution of underlying asset price. In this framework, we obtained the values for the {*q1, q2, q3, u, m, d*} parameters and the minimum objective function below.

q1	q2	q3	u	m	d	objective_value
0.068969	0.866642	0.06439	1.010957	0.999949	0.98906	9.203407e-12

Table 6. Values for *q1, q2, q3, u, m, d* and the objective function minimum

We found these values by first performing a global search, and allowing the algorithm to randomly choose between simulated annealing or differential evolution. The goal of doing so was to avoid being trapped in local minima, since our objective function is non-linear and not particularly smooth. Once an appropriate initial starting point was obtained, we performed a local search using the SLSQP method to retrieve the optimal parameter values. In a similar fashion to exercise 5, this two step method was necessary to decrease the SLSQP method's sensitivity to initialization, which would otherwise greatly affect values found. Please refer to the function *multi_algorithm_optimization (n_starts, bounds, objective_fun, constraints, base_seed)* for more details on the code.

It must be noted that the values for {*q1, q2, q3, u, m, d*} suffer from propagation error (i.e. they are dependent on the optimized values for σ, $skew$ and $kurt$ in exercise 5) and from optimization error due to numerical methods. For the latter, reduce the algorithm' sensitivity to initialization in the method outlined previously.

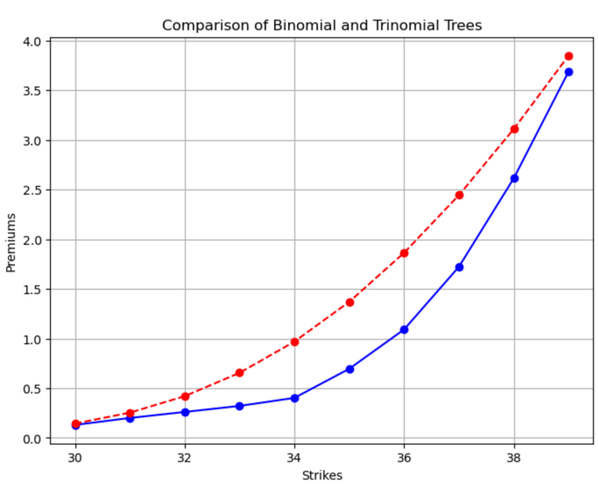
7. TRINOMIAL TREE (TT)

The accuracy of the TT model should be more precise than that of the CRR model with lesser time steps, since it captures the behavior of the underlying more precisely with the middle node (m), it more closely resembles the continuous-time nature of asset price movements, and handles early exercise decisions at each node. Hence, its convergence to the fair value of the American put is smoother and faster than that of the CRR model.

As displayed in Table 7 and Graph 3, the TT premiums are slightly higher than the CRR premiums, which can be attributed to inclusion of kurtosis and skewness in the TT model. In particular, higher kurtosis ($kurt$ = 8.0) indicates that the market perceives an increased likelihood of extreme movements in the asset's price. Additionally, the negative skewness (skew = -0.023025) implies a higher likelihood for downward movements in the underlying price as opposed to upward movements. Together, these factors point toward a greater likelihood of extreme downward price movements. In the case of an American put option, the lower the future price, the higher the premium will be. Indeed, this aligns with our results, i.e. TT premiums are higher than CRR premiums.

	Strike Price	Premium	American Option Premiun (BAW)	American Option Premiun (CRR)	American Option Premiun (Adjusted_BMS)	American Option Premiun (TRI)
0	30	0.13	0.131037	0.130442	0.147399	0.144074
1	31	0.20	0.201421	0.200539	0.188767	0.254100
2	32	0.26	0.261818	0.261141	0.223113	0.420050
3	33	0.32	0.322341	0.321786	0.273639	0.655057
4	34	0.40	0.403238	0.402671	0.397550	0.969517
5	35	0.69	0.695618	0.695552	0.665696	1.370074
6	36	1.08	1.090104	1.091913	1.127128	1.863836
7	37	1.70	1.718514	1.723128	1.784534	2.444281
8	38	2.58	2.607273	2.614232	2.596820	3.110173
9	39	3.65	3.680465	3.688590	3.503015	3.849881

Table 7. American put premiums obtained from BA, CRR, adjusted skew-kurt BMS, TT models



Graph 3. TT premiums plotted against CRR premiums for American puts

Graph 3 compares the premiums at different strike levels (K) between the binomial model and the trinomial model. The red dashed line (---) represents the trinomial model, while the blue solid line (—) represents the binomial model. From the graph, we can clearly see that the premiums obtained from the trinomial model are higher than those from the binomial model.

References

Dorion, Christian, and Pascal François. 2024. *Derivatives*. HEC Montreal & Fellow of the Canadian Derivatives Institute. Preliminary Draft, Fall 2024 Semester.

François, Pascal. 2024. *Index Options with Skewness and Kurtosis*. TP 1 – Derivatives (60206A), HEC Montréal. Fall 2024 Semester.