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Fall'19 CSCE 629

Analysis of Algorithms

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Lecture 13

- Bellman-Ford algorithm: Shortest path with negative weight

Credit: based on slides by A.Smith and K.Wayne

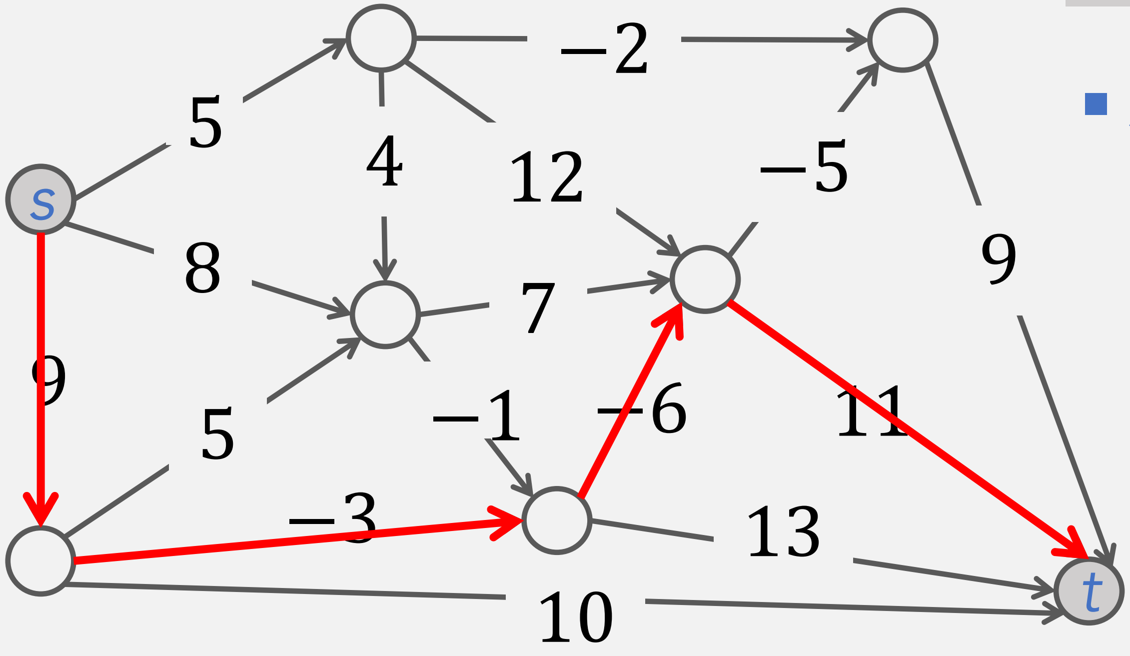
Logistics

- Write up your algorithm
  - Start with "Idea": short description of the key to your algorithm. [Your peer should be able to take it and come up with an algorithm with *a bit*, if any, thought]
  - Pseudocode: write in a nice format as you'd do in a real program, even if you are using plain-English descriptions [Your peer should be able to implement your algorithm without thinking other than implementation details]
  - Comments in your pseudocode are helpful
- When studying
  - Identify what you don't understand
  - Test yourself via (a) rederiving algorithms, proofs from class and (b) exercises in book.
  - Read alternative explanations (e.g. KT,E,DPV)

Recall: shortest path problem

- **Input.** graph G, node s and t
- **Output.** $dist(s,t)$

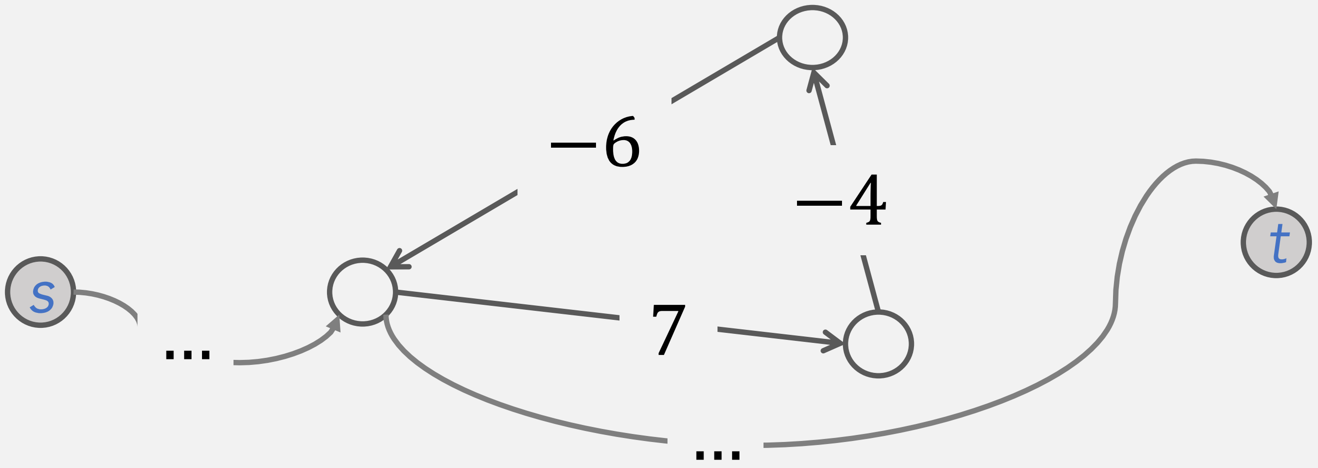
- Every edge has a length l_e (can be negative)
- Length of a path $l(P) = \sum_{e \in P} l_e$
- Distance $dist(u,v) = \min_{P: u \rightsquigarrow v} l(P)$



- **Special cases**
  - All edge of equal length: BFS $O(m+n)$
  - DAG: DP in topological order $O(m+n)$

Length of shortest path: $dist(s,t) = 9 - 3 - 6 + 11 = 11$

A technical issue: negative length cycles



- Observation.
  - If some $s \rightsquigarrow t$ path contains a negative length cycle, there does not exist a shortest $s \rightsquigarrow t$ path;
  - Otherwise there exists a simple (i.e., no repetition node) path $\leq n - 1$ edges
- For simplicity: assuming G has no NegativeLengthCycle
  - can be detected with little overhead

DP1: develop a recursion

Def. for all $i = 0, \ldots, n-1$, $v \in V$

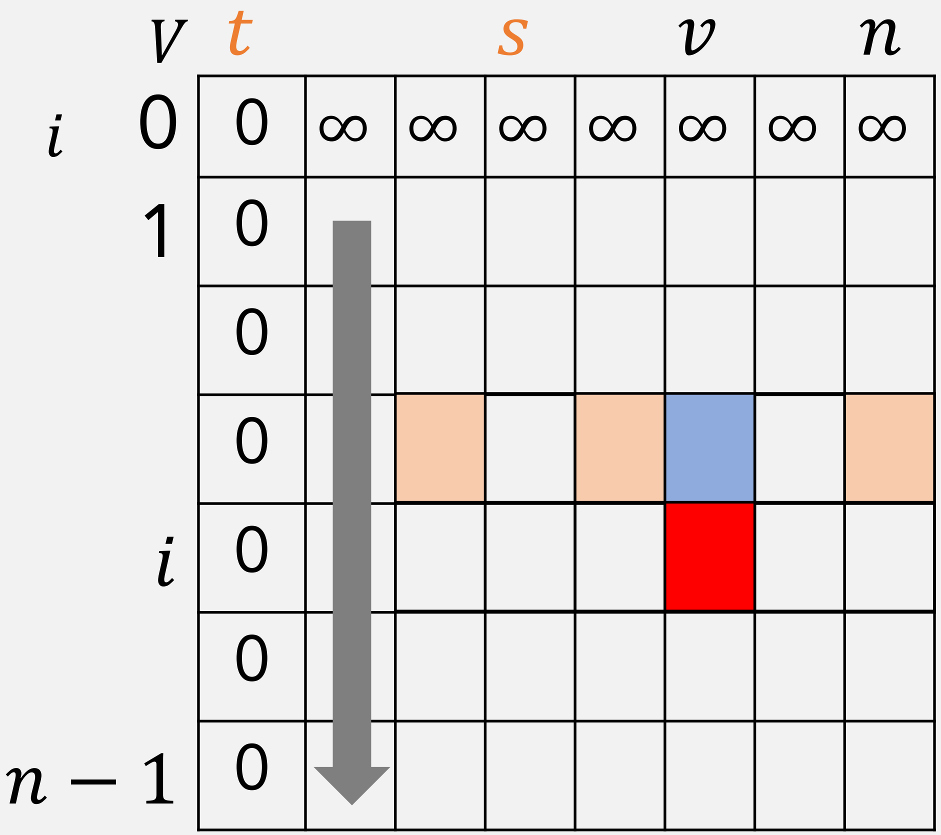
$\text{OPT}(i, v) :=$ length of shortest $v \rightsquigarrow t$ path P using $\leq i$ edges

- Case 1. P uses at most $i - 1$ edges
  - $\text{OPT}(i, v) := \text{OPT}(i-1, v)$
- Case 2. P uses exactly i edges
  - If (v, w) the first edge, then OPT uses (v, w) and then selects best $w \rightsquigarrow t$ path using $\leq i - 1$ edges

- Goal. Find $\text{OPT}(n-1, s)$
- Basis: $c(i, v) = 0$ or ∞ if $i = 0$
- Recursion: how to define $\text{OPT}(i, v)$ recursively?

$$\text{OPT}(i, v) = \begin{cases} 0\ if\ v = t;\ \infty \text{ if } i = 0 \\ \min\left\{\text{OPT}(i-1, v),\ \min\limits_{v \to w \in E} \{\text{OPT}(i-1, w) + l_{v \to w}\}\right\} \text{otherwise} \end{cases}$$

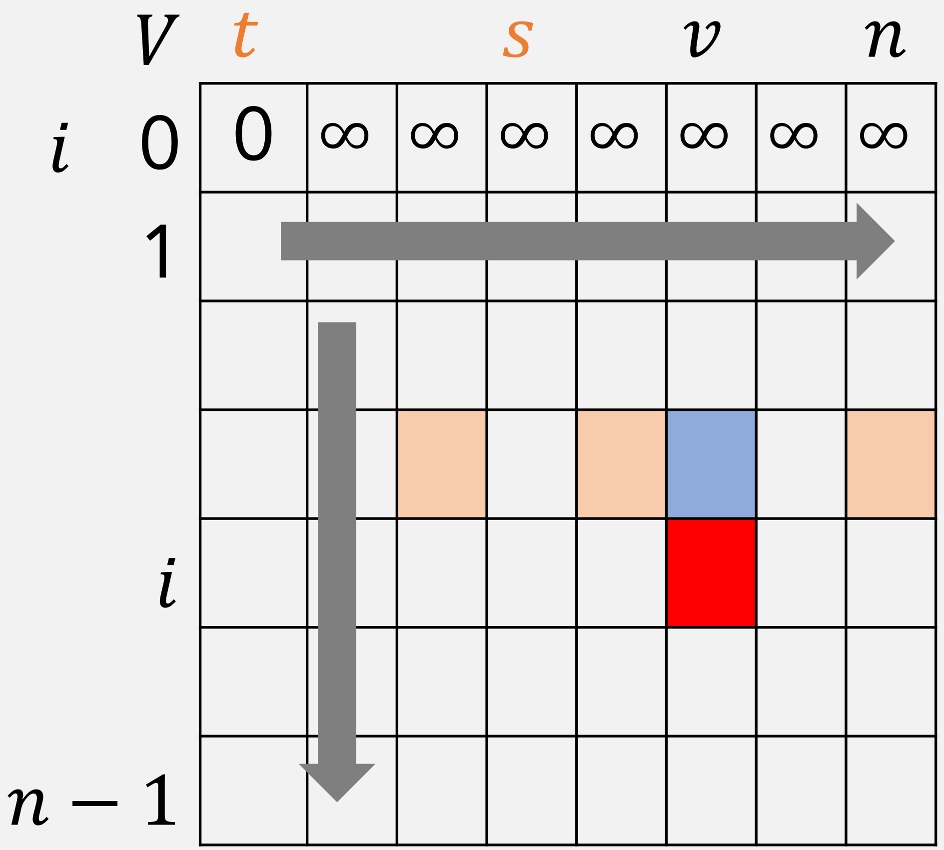
DP2: build up solutions



- Subproblems: $O(n^2)$
- Memoization data struture
  - 2-D array $M[0, \ldots n-1, v_1, \ldots, v_n]$
- Dependencies
  - Each $\mathrm{OPT}(i, v)$ depends on subproblems on the row above
- Evaluation order
  - Row by row, each row arbitrary order

$$\mathrm{OPT}(i, v) = \begin{cases} 0 \; if \; v = t; \; \infty \text{ if } i = 0 \\ \min\left\{\mathrm{OPT}(i-1, v), \min_{v \to w \in E} \{\mathrm{OPT}(i-1, w) + l_{v \to w}\}\right\} \text{otherwise} \end{cases}$$

DP2: build up solutions



<u>***SPLen(G, t)***</u>

// $M[i, v]$ memoize subproblem values

// $M[0, t] = 0$ and $M[0, v] = \infty$ otherwise

For $i = 1, \ldots, n-1$ // row by row

    For $v \in V$ // any order

        $M[i, v] \leftarrow M[i-1, v]$ // case 1

        For edge $(v \rightarrow w) \in E$ // case 2

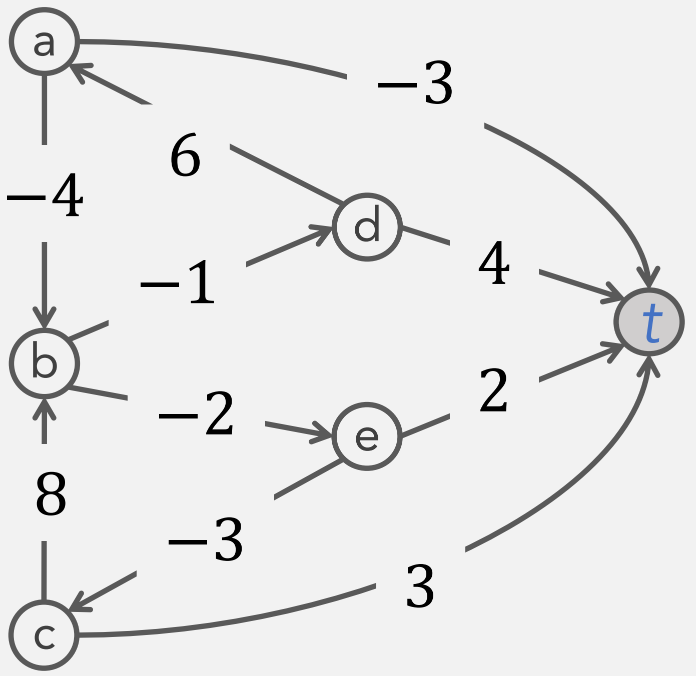
            $M[i, v]$

            $\leftarrow \min\{M[i, v], M[i-1, w] + l_{vw}\}$

This actually gives us $dist(v, t)$ for all $v \in V$

- **Analysis:** $O(n^2)$ space; $O(mn)$ time [visit all edges for each i]
- **Finding a shortest path:** maintain a "successor" for each entry

Example



V	t	a	b	c	d	e
i 0	0	∞	∞	∞	∞	∞
1	0	-3	∞	3		
2	0					
3	0					
4	0					
5	0					

0	∞	∞	∞	∞	∞
0	-3	∞	3	4	2
0	-3	0	3	3	0
0	-4	-2	3	3	0
0	-6	-2	3	2	0
0	-6	-2	3	0	0

For $i = 1, \dots, n-1$ // row by row
    For $v \in V$ // any order
        $M[i, v] \leftarrow M[i-1, v]$ // case 1
        For edge $(v \to w) \in E$ // case 2
            $M[i, v] \leftarrow \min\{M[i, v], M[i-1, w] + l_{vw}\}$

A simple but impactful improvement

Maintain only one array $M[v] =$ length of shortest $v \rightsquigarrow t$ path found so far.
No need to check edge (v, w) unless $M[w]$ changed in previous iteration.

Theorem. Throughout the algorithm, $M[v]$ is length of some $v \rightsquigarrow t$ path, and after i rounds of updates, the value $M[v]$ is no larger than the length of shortest $v \rightsquigarrow t$ path using $\leq i$ edges.

- Memory: $O(m + n)$.
- Running time: $O(mn)$ worst case, but faster in practice.

➔Bellman-Ford algorithm: efficient implementation

- Application: (distance-vector) routing protocol on Internet [more to come]

Single-source shortest paths with negative weights

Year	Worst case	Discovered by
1955	$O(n^4)$	Shimbel
1956	$O(mn^2W)$	Ford
1958	$O(mn)$	Bellman, Moore
1983	$O(n^{3/4}m\log W)$	Gabow
1989	$O(mn^{1/2}\log(nW))$	Gabow-Tarjan
1993	$O(mn^{1/2}\log W)$	Goldberg
2005	$O(n^{2.38}W)$	Sankowsi, Yuster-Zwich
2016	$O(n^{10/7}\log W)$	Cohen-Madry-Sankowski-Vladu
20XX	???	You???

weights between $[-W, W]$