Portland State University

W'21 CS 584/684
Algorithm Design & Analysis

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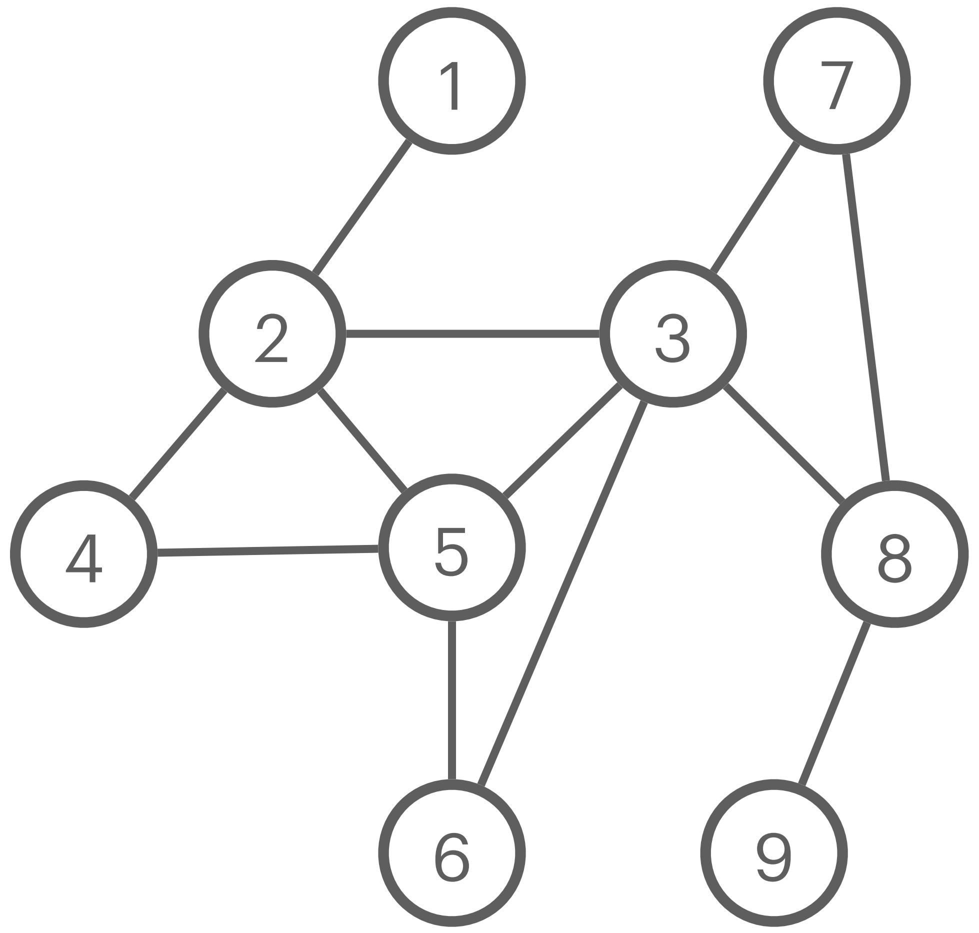
Lecture 4

- Graphs
- Graph traversal
  - BFS
  - DFS

Credit: based on slides by A. Smith & K. Wayne

Warm-up exercises

- True or False. A tree of n vertices can have n edges.
- Run BFS and DFS starting at node s, and form BFS/DFS trees. Decide if nodes 1 and 9 are connected.



Recap: BFS running time

Theorem. BFS takes $O(m+n)$ time (linear in input size).

Why not $n \cdot m$?

```
BFS(s):
// Discoverd[1,...,n] array of bits (explored or not),
initialized to all zeros.
// Queue Q ← ∅
1.  Set Discovered[s] = 1
2.  EnQ(s) // add s to Q
3.  While Q not empty  DeQ(u)
       For  each (u,v) incident to u
          If Discovered[v]=0 then
              Set Discovered[v]=1
              Add edge (u,v) to T
              EnQ(v)
```

$O(1)$, run once for all

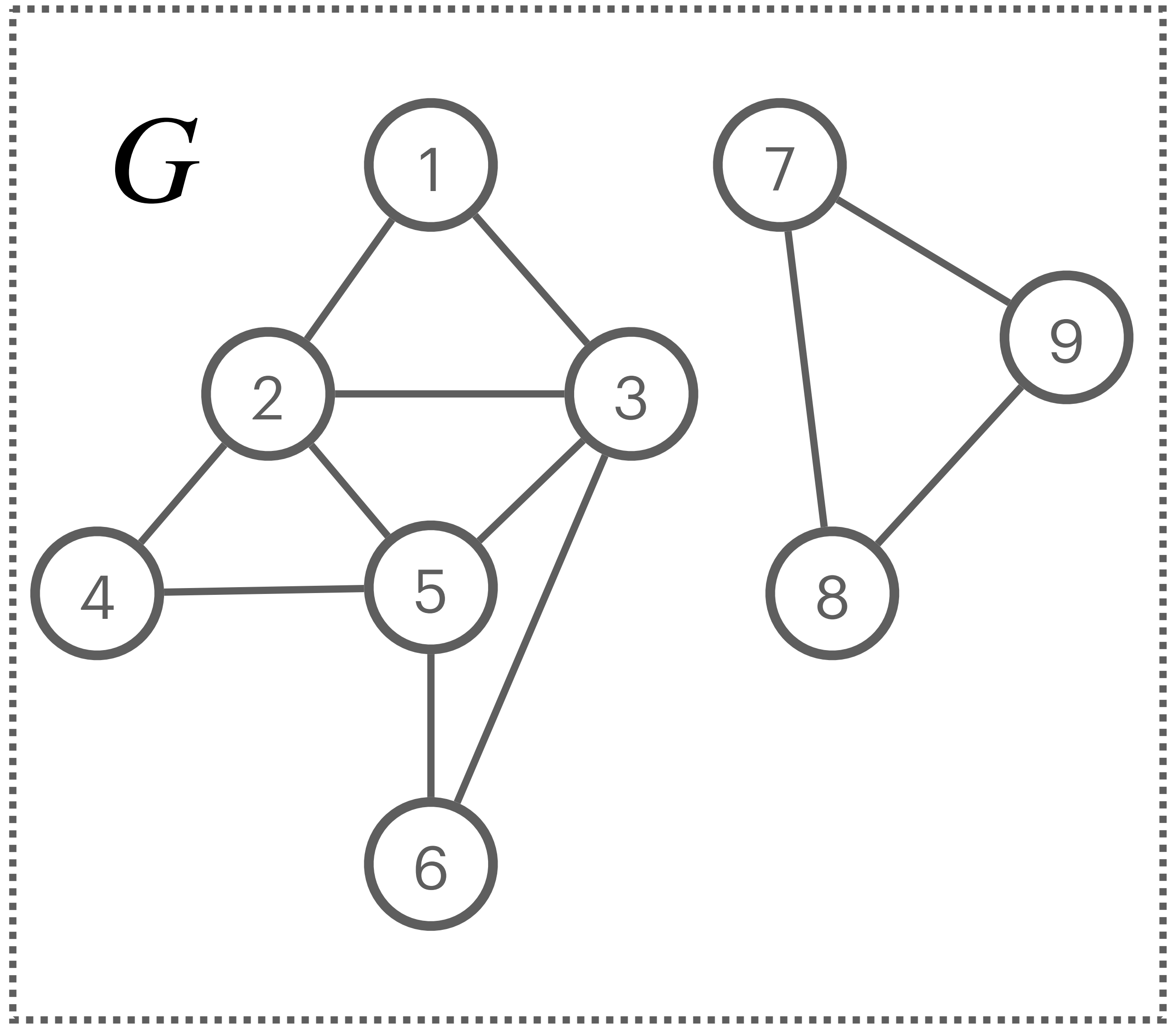
$O(1)$, run once per vertex

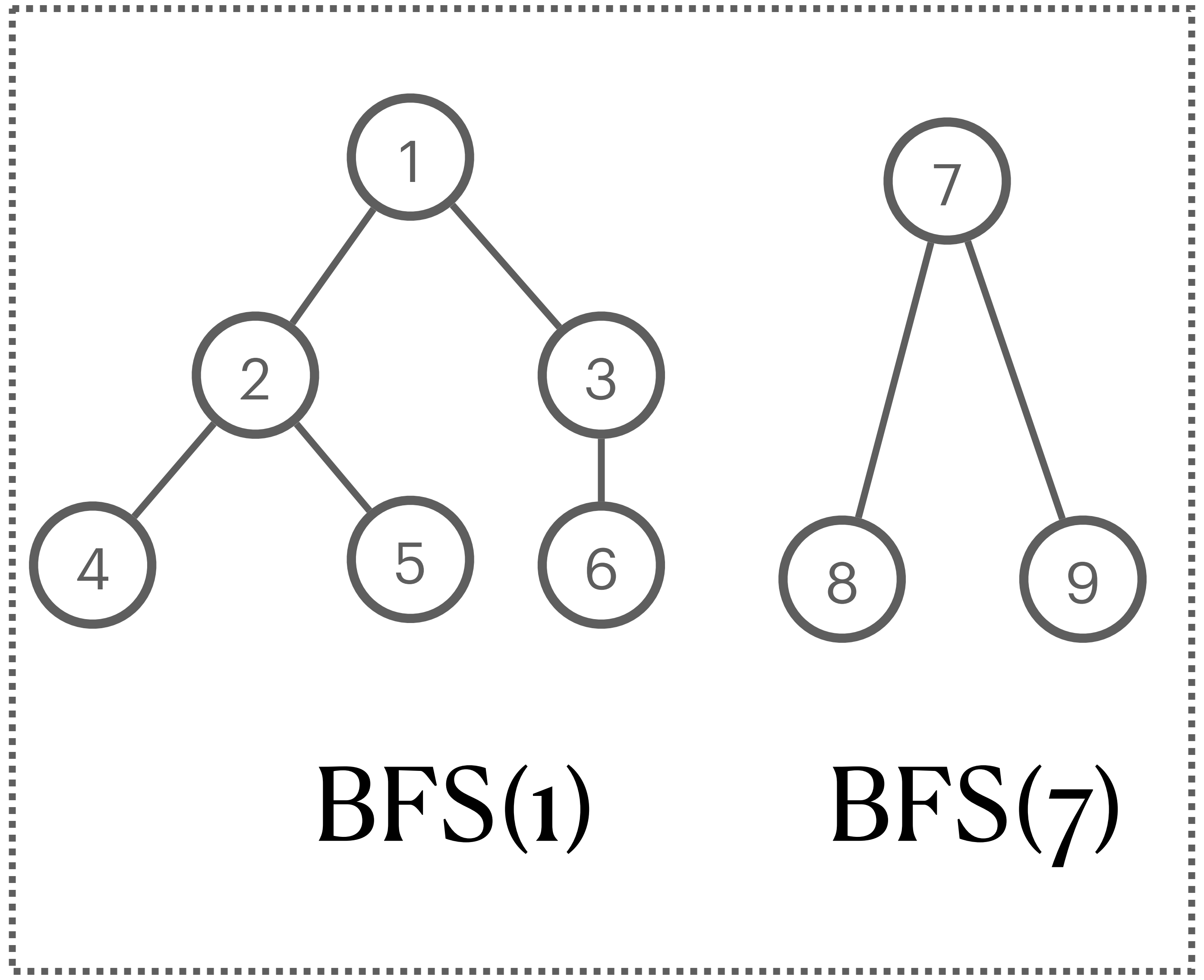
$O(1)$, run $\leq$ twice per edge

Connected components

- B/DFS tell more than s-t connectivity.

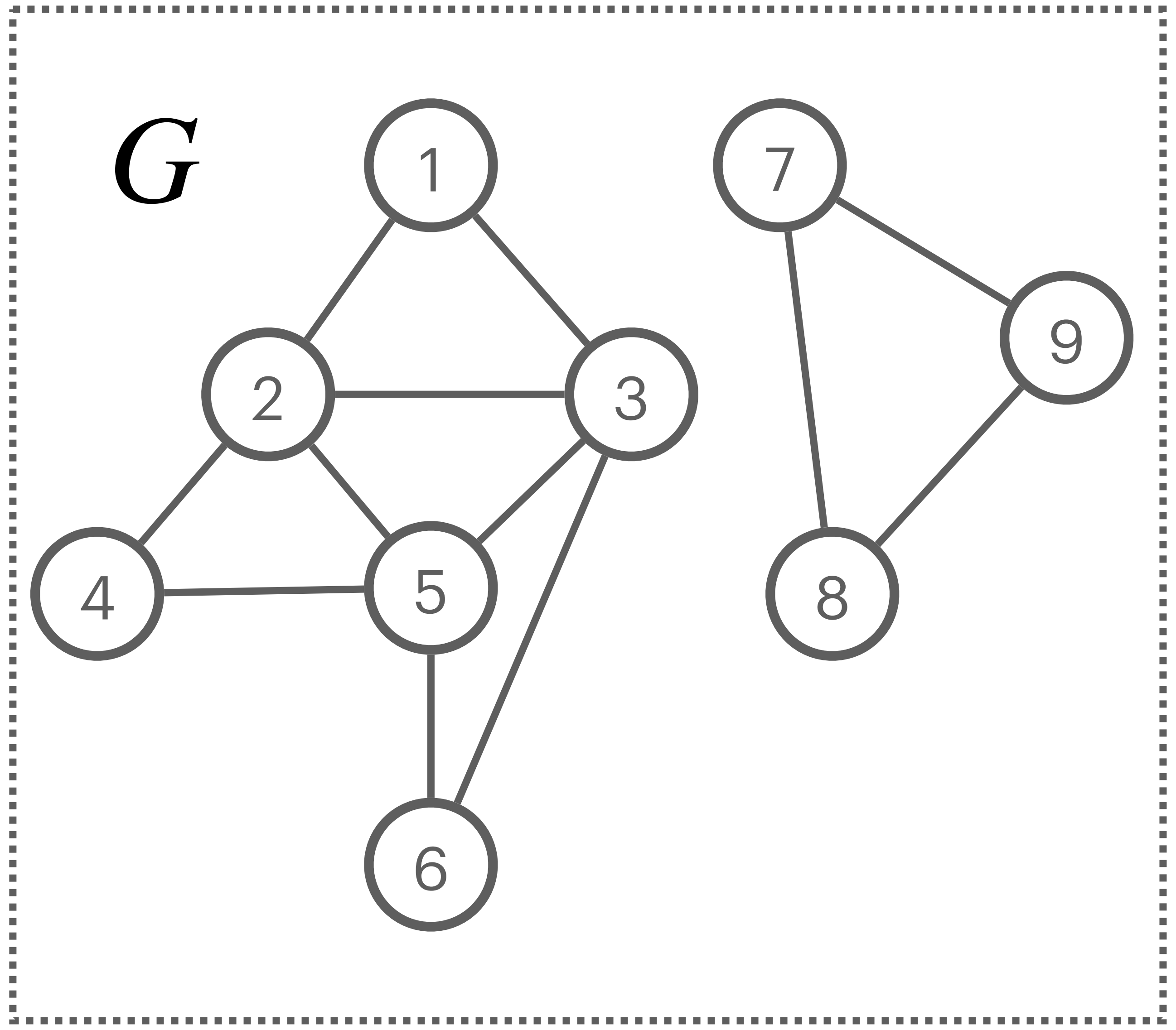
Connected component of G containing s: all nodes reachable from s.

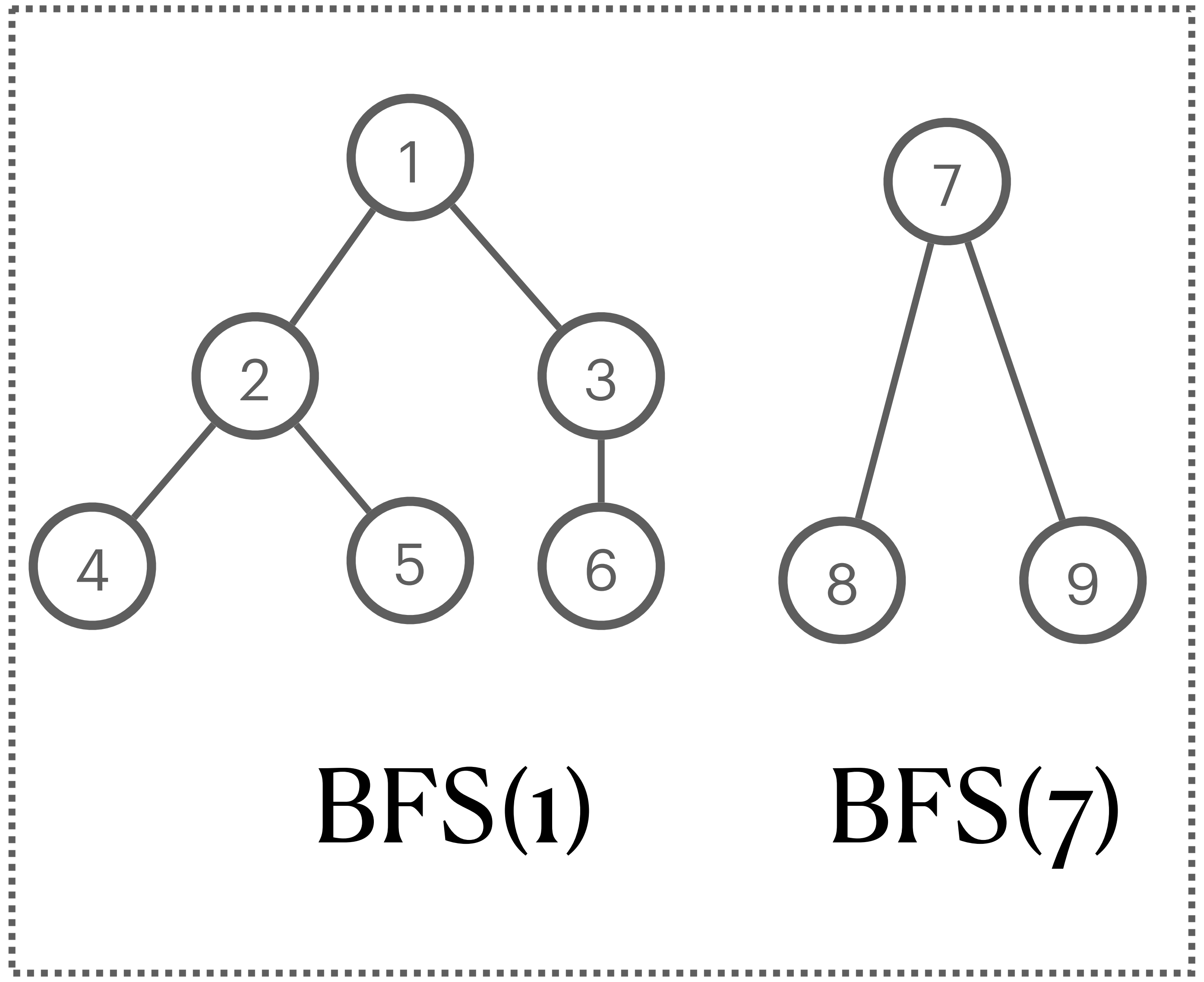




- Claim. For any tow nodes s and t, their connected components are either identical or disjoint.

The set of all connected components



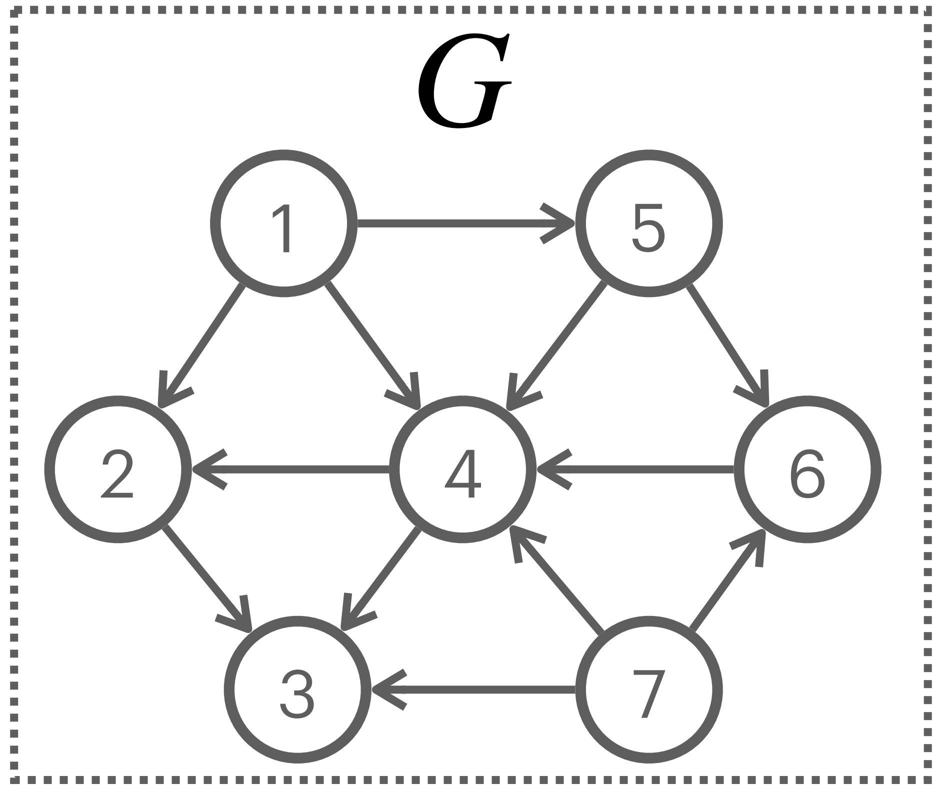


- **How to find all?**
- **How fast?**
- **Why care?**

- Iterate over V, run B/DFS.
- $\sum_i n_i + m_i = O(m+n)$.
- Basic topology about G.

Directed graphs

- A directed graph $G = (V, E)$
  - Edge $u \to v$ leaves node u and enters node v.
  - Adjacency matrix: asymmetric
  - Adjacency list: track outgoing edges (or two for in and out)



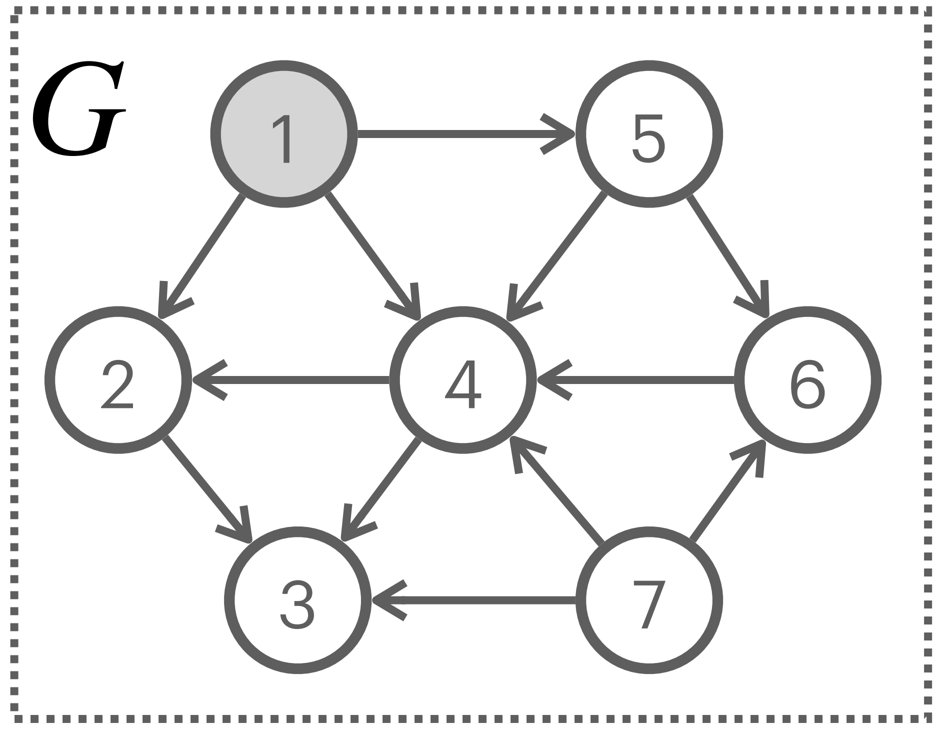
… $Adj_{\text{out}}[2] = \{3\}, Adj_{\text{in}} = \{1,4\}$

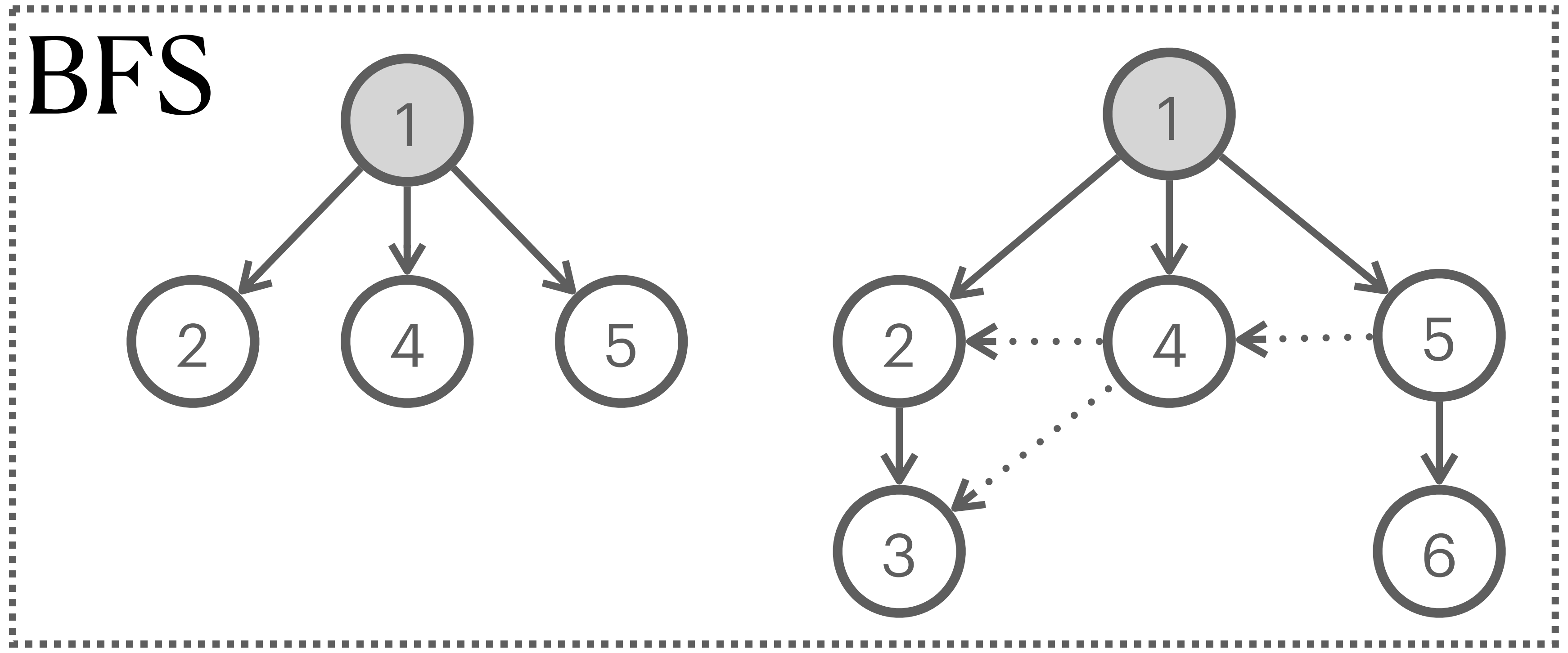
- Examples.

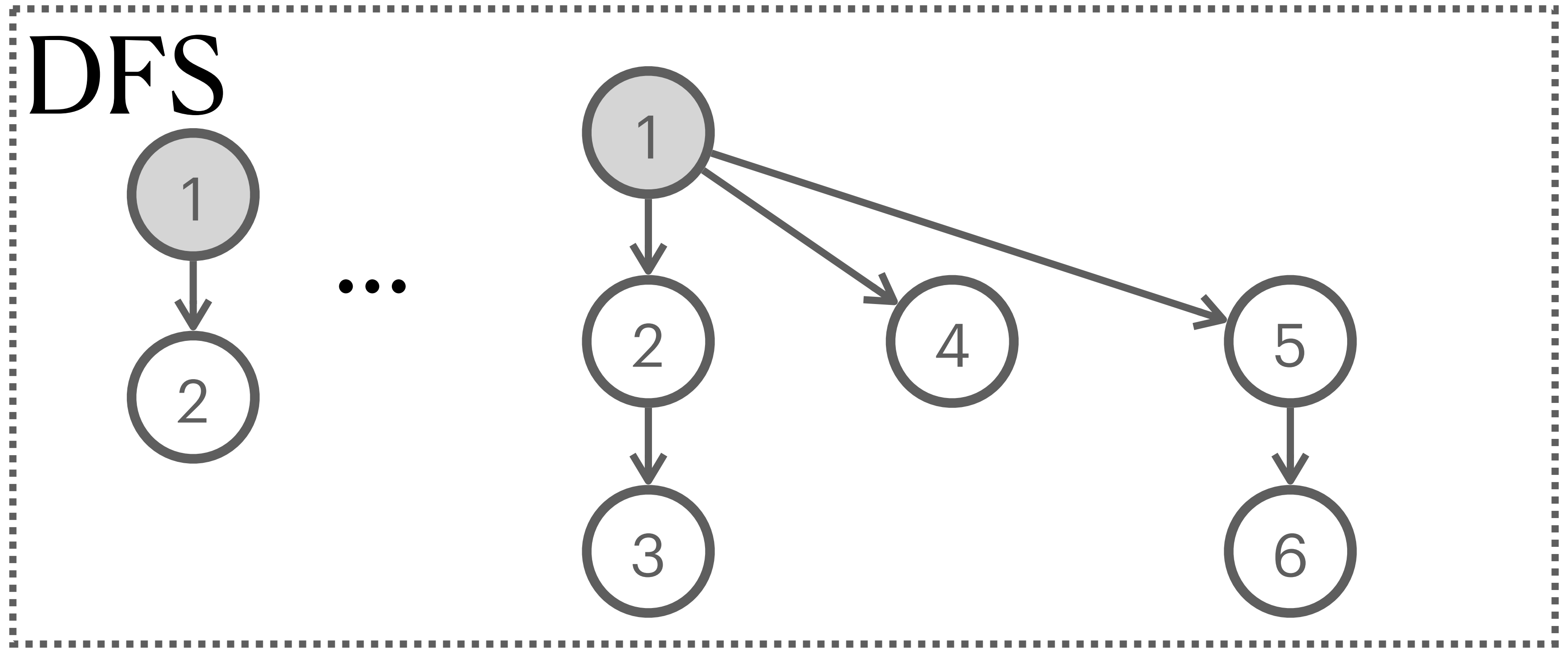
Directed graph	Node	Directed edges
Transportation	Intersections	One-way street
Social network	People	Following
Web	Webpage	Hyperlink
Citation	Article	Citing

Connectivity in directed graphs

- **Directed reachability.** **Find all nodes reachable from a node s.**
  - BFS/DFS apply.
  - $s \rightsquigarrow t$: there is a path from s to t. Need not be $t \rightsquigarrow s$.







- **Application: web crawler.**
  - Start from web page s. Find all web pages linked from s, via one or more hops.

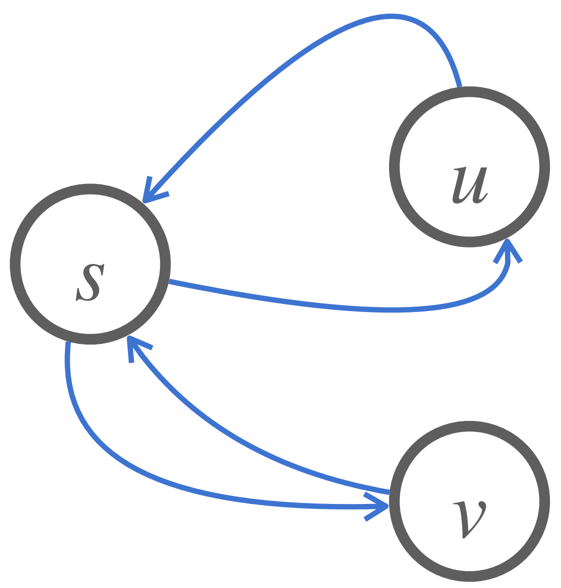
Strong connectivity

- **Def.** u and v are mutually reachable ($u \leftrightsquigarrow v$)
- **Observation.** If $u \leftrightsquigarrow v$ and $v \leftrightsquigarrow w$, then $u \leftrightsquigarrow w$.

Def. A graph is strongly connected if every pair of nodes is mutually reachable.

Lemma. Let s be any node. G is strongly connected iff. every node is reachable from s, and s is reachable from every node.

- **Proof.** [Show both "if" and "only if"]
  - $\Rightarrow$ (only if) By definition of "strongly connected".
  - $\Leftarrow$ (if) for any two nodes u, v: $u \rightsquigarrow v$ by following $u \rightsquigarrow s$ then $s \rightsquigarrow v$.
    $v \rightsquigarrow u$ by following $v \rightsquigarrow s$ then $s \rightsquigarrow u$.

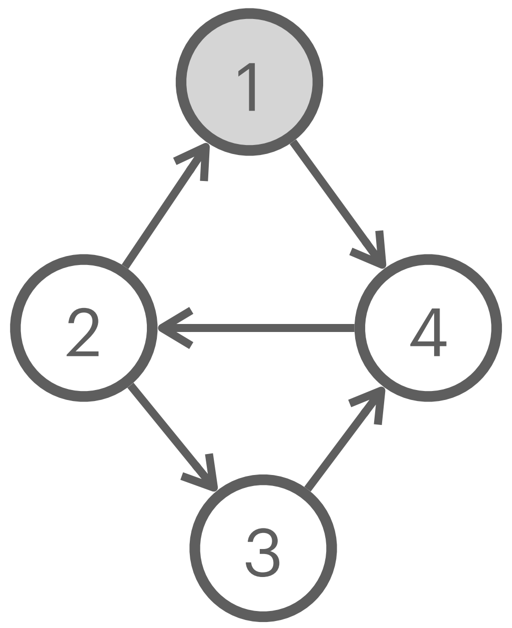


Testing strong connectivity

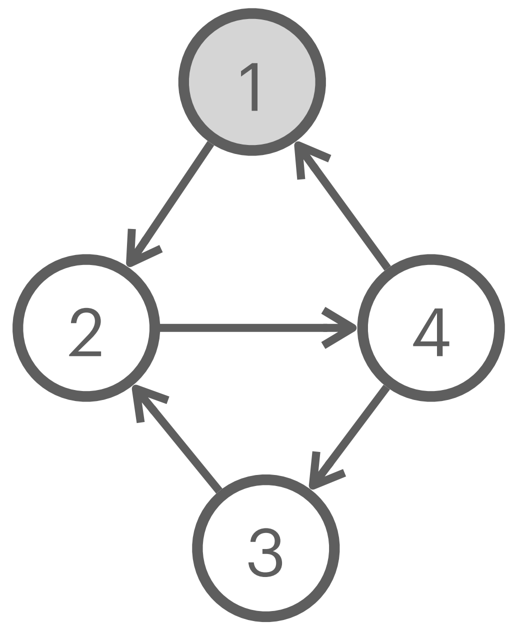
Theorem. Theres is an $O(m + n)$ time algorithm that determines if G is strongly connected.

- **Proof.** [construction of an algorithm. Fill in the analysis on your own.]

G



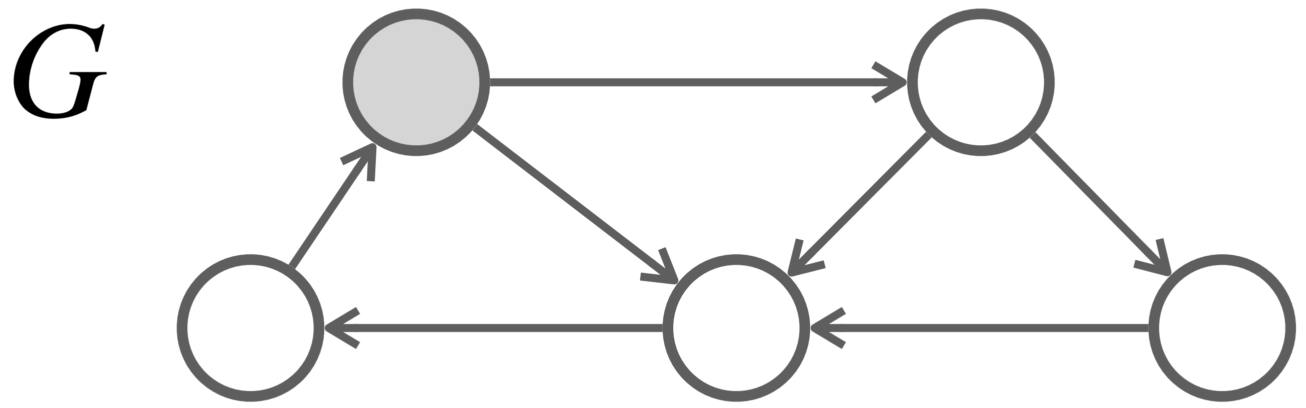
G^{rev}: reverse orientation of all edges in G.

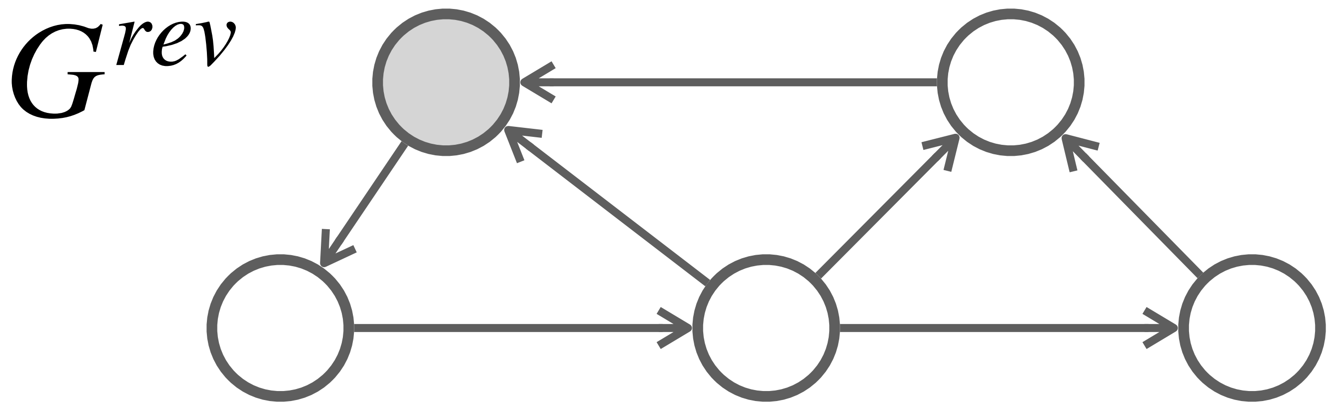


1. Pick any node s.
2. Run **BFS** from s on G.
3. Run **BFS** from s on G^{rev}.
4. Return true if all nodes reached in both **BFS** runs.

Exercise

- Determine if the graph is strongly connected.

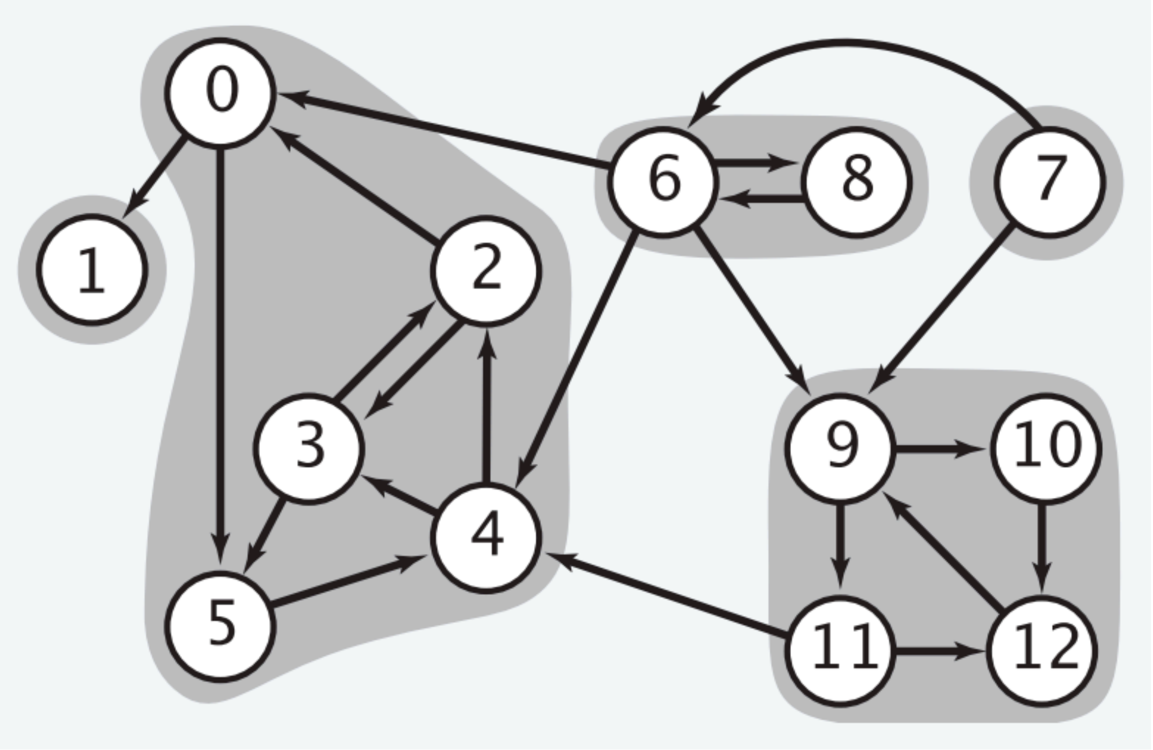




Strong (connected) components

- Def. A strong component is a maximal subset of mutually reachable nodes.
- Obs. For any two nodes s and t in a directed graph, their strong components are either identical or disjoint.

Theorem. Theres is an $O(m + n)$ time algorithm that finds all strong components.



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DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

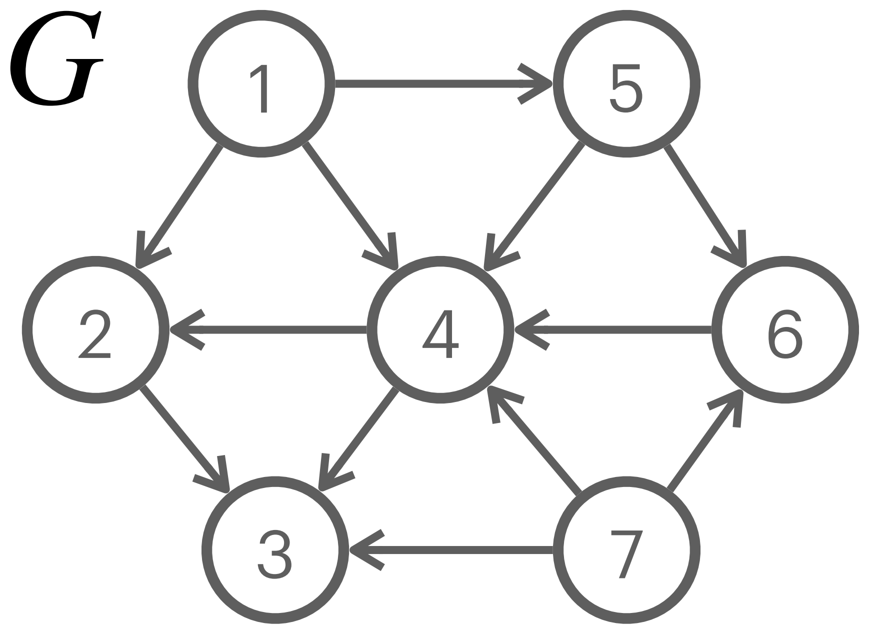
ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by

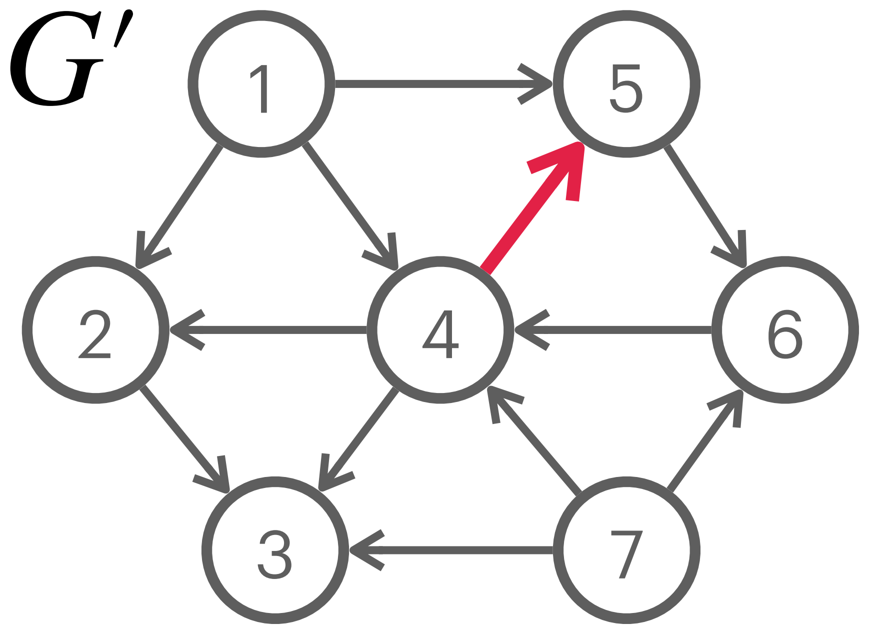
Directed acyclic graphs (DAG)

- **Def. A DAG is a directed graph that contains no directed cycles.**

G



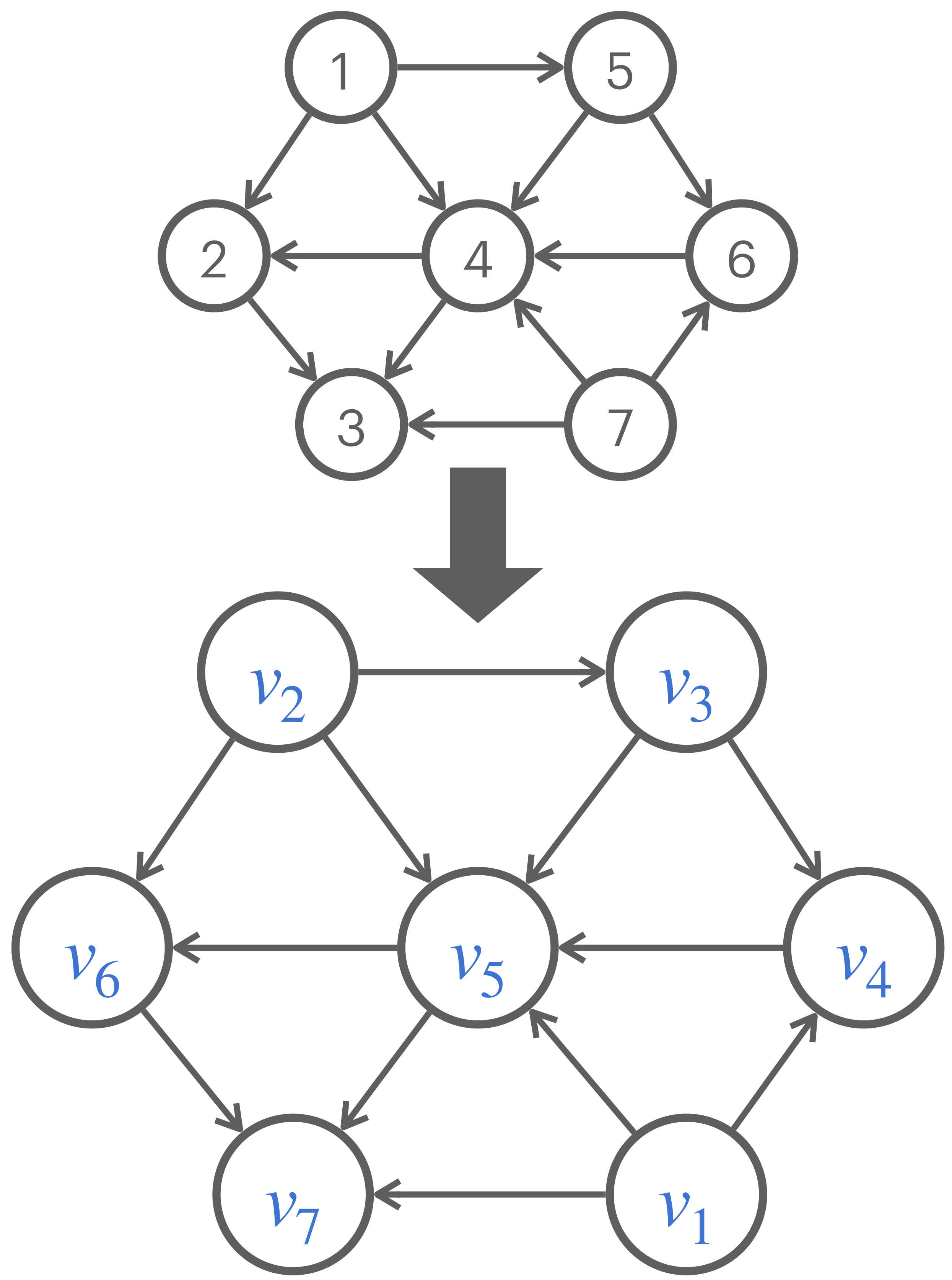
G'



- **Application: precedence constraints.**
  - Course prerequisite: 350 must be taken before 584/684.
  - Compilation: module i must be complied before j.
  - Pipeline of computing jobs: output of job i determines input of job j.

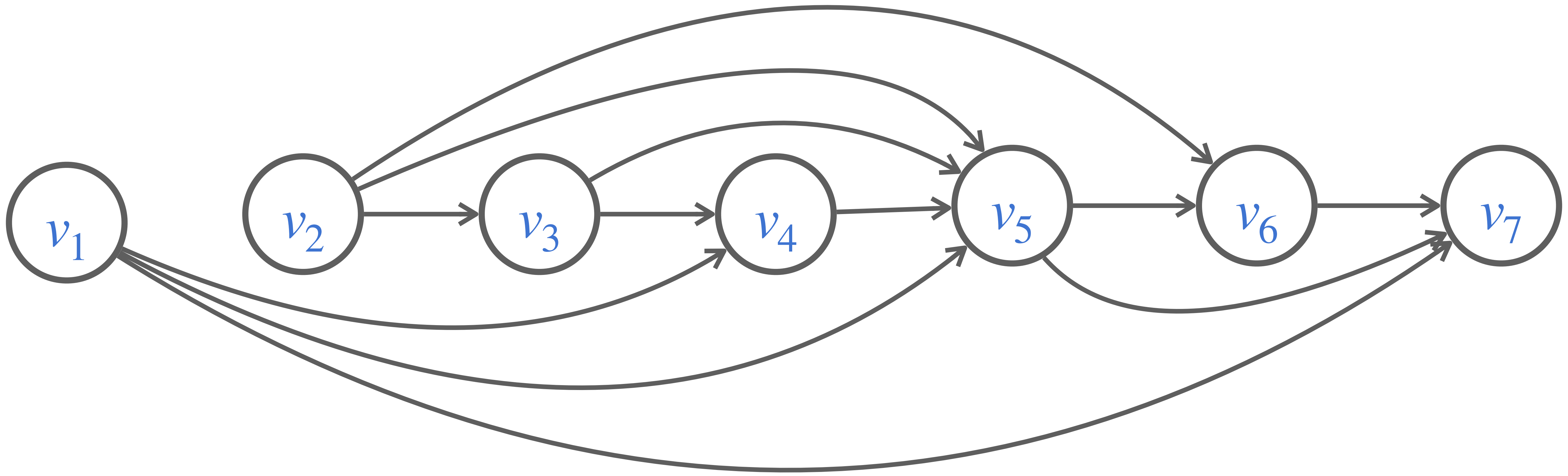
Topological order

- **Def.** A topological order of a directed graph is an ordering of its nodes $v_1, \ldots, v_n$, so that for every edge $v_i \rightarrow v_j$ we have $i < j$.



A topological order

All edges go from left to right



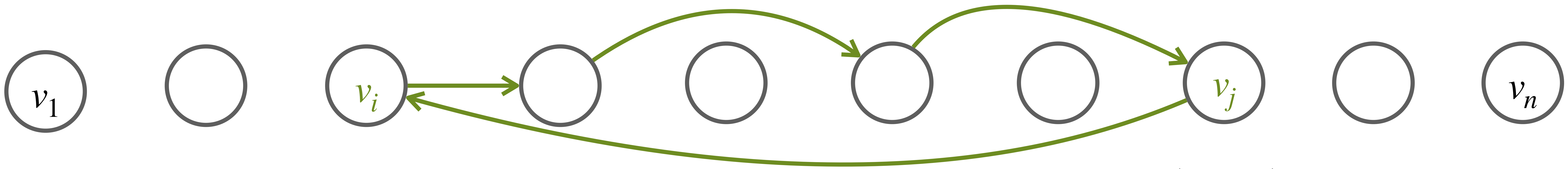
1. If G has a topological order, is G necessarily a DAG?

2. Does every DAG have a topological order?

Q1: If G has a topological order, is G necessarily a DAG?

Lemma 1. If G has a topological order, then G is a DAG.

- **Proof** [by contradiction]
  - Suppose G has topological order $v_1, \ldots, v_n$; and G also has a directed cycle C.
  - Let v_i be the lowest-indexed node in C, v_j be the node just before v_i in C.
  - Then $v_j \to v_i$ is an edge & by our choice $i < j$.
  - But since $v_1, \ldots, v_n$ is a topological order, if $v_j \to v_i$ is an edge, then $j < i$.
  - Contradiction!



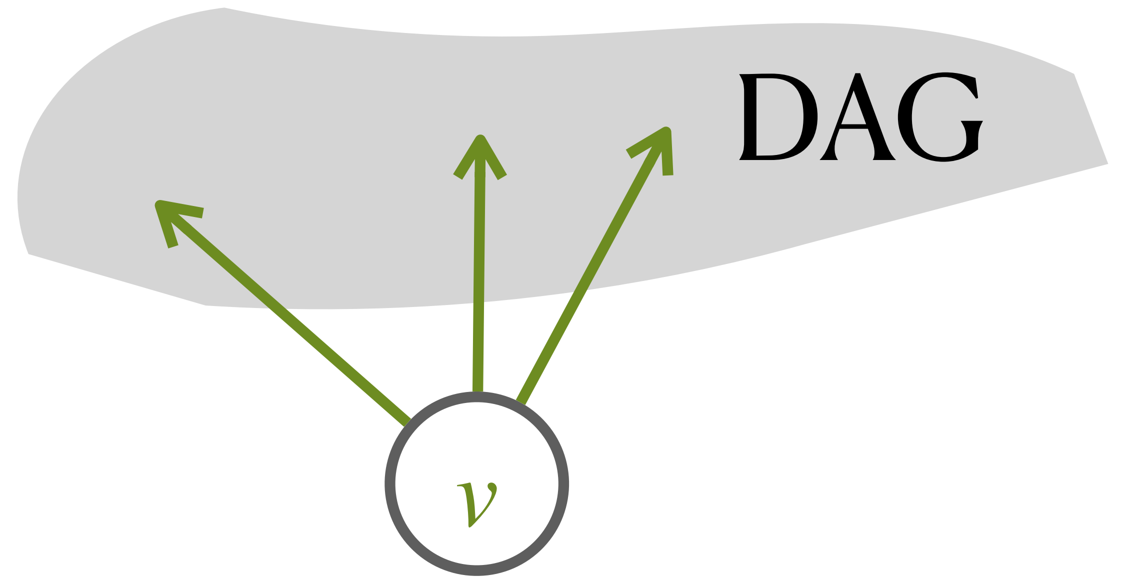
Directed cycle C

Q2: Dose every DAG have a topological order?

Lemma 2. A DAG G has a node with no entering edges.

Corollary. If G is a DAG, then G has a topological order.

- **Proof** of corollary given Lemma 1 [by induction on number of nodes]
  - Base case: true if $n = 1$.
  - Given a DAG on $n > 1$ nodes, find a node v with no entering edges [Lemma 1].
    - $G - \{v\}$ is a DAG, since deleting v cannot create cycles.
  - Induction hypothesis, $G - \{v\}$ (with $n - 1$ nodes) has a topological order.
  - Place v first then append nodes of $G - \{v\}$ in topological order [valid because v has no entering edges].



Topological sorting algorithm

TopSort(G):
// count(w)= remaining number of incoming edges
// S = set of remaining nodes with no incoming edges
// $V[1,\ldots,n]$ topological order

1. Initialize S and $Count(\cdot)$ for all nodes
2. For $v \in S$

    Append v to V

    For all w with $v \to w$ // delete v from G

        $Count(w) --$

        If $Count(w) == 0$ add w to S

$O(n+m)$, a single scan of adjacency list

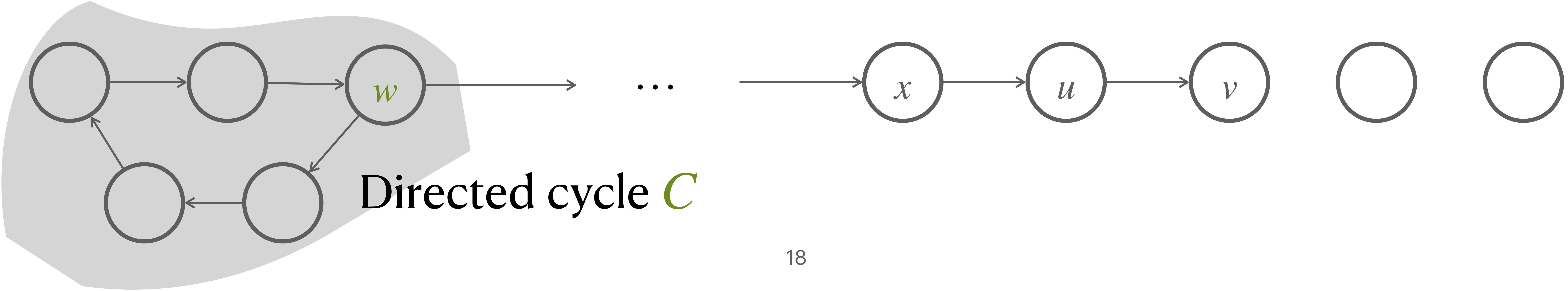
$O(1)$, run once per edge

Theorem. TopSort computes a topological order in $O(n+m)$ time.

Completing the proof

Lemma 1. A DAG G has a node with no entering edges.

- **Proof [by contradiction]**
  - Suppose G is a DAG, and every node has at least one entering edge.
  - Pick any node v, and follow edges backwards from v. Repeat till we visit a node, say w, twice. ($v \leftarrow u \leftarrow x \ldots \leftarrow w \ldots \leftarrow w$)
  - Let C be the sequence of nodes between successive visits to w.
  - C is a cycle. Contradiction!



Scratch