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Fall'19 CSCE 629

Analysis of Algorithms

Fang Song
Texas A&M U

Lecture 14

- Dijkstra's algorithm: shortest path (non-negative weight)

Recall: shortest path problem

- Input. graph G, node s and t
- Output. $dist(s,t)$

> - Every edge has a length l_e (can be negative)
> - Length of a path $l(P) = \sum_{e \in P} l_e$
> - Distance $dist(u,v) = \min_{P:u \rightsquigarrow v} l(P)$

- Special cases
  - All edge of equal length: BFS $O(m+n)$
  - DAG: DP in topological order $O(m+n)$

  $\boxed{\text{Non-negative length: } \textbf{Dijkstra } O((m+n)\log n)}$

- General: Bellman-Ford $O(mn)$

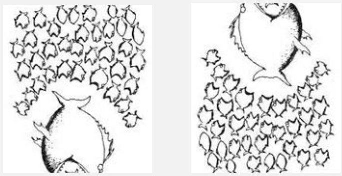
Dynamic Programing ReRecap

1. Formulate the problem recursively

- Overlapping subproblems
- May be easy to first compute optimal value & then construct an optimal solution

2. Build solutions to your recurrence

- Bottom-up: determine dependencies & find a right order (topo. order in DAG)
- Top-down: smart recursion (i.e. without repetition) by momoization



Examples

- Explicit DAG: shortest path in DAG; longest increasing subsequences (a.k.a. longest path in DAG)
- Binary choice: weighted interval scheduling
- Multi-way choice: matrix-chain mult.; longest common subsequence;
- Adding a variable: shortest path with negative length (Bellman-Ford)

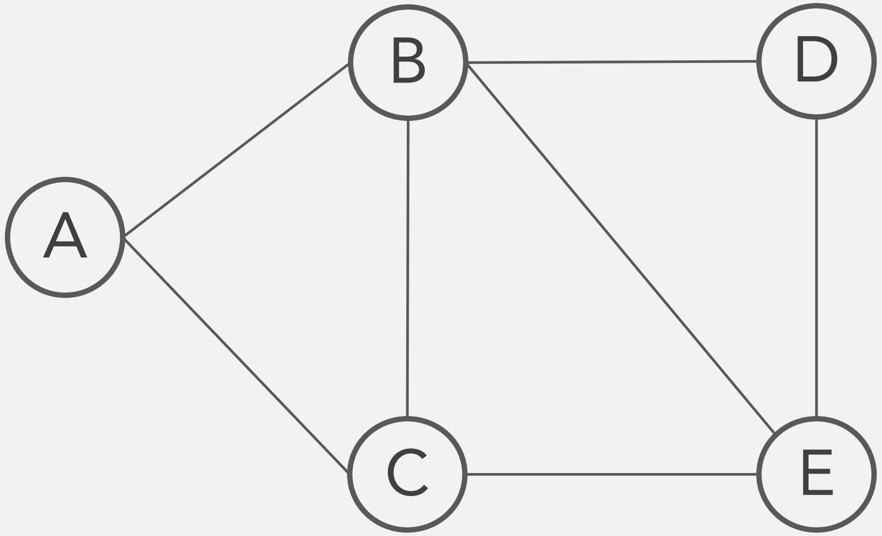
Edsger W. Dijkstra

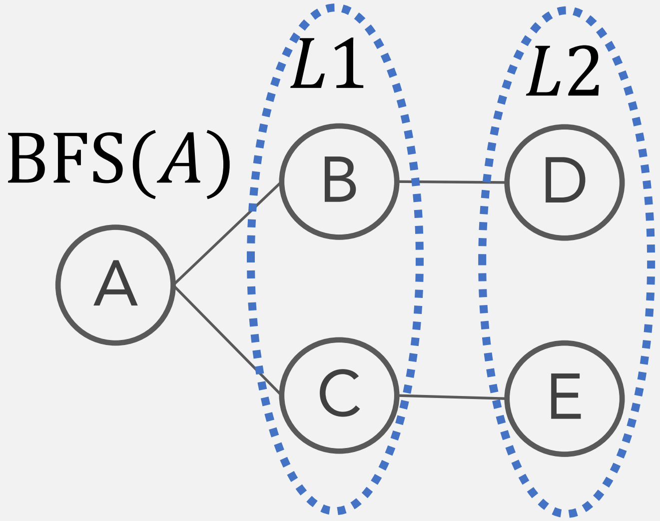
- Pioneer in graph algorithms, distributed computing, concurrent computing, programming …
- 1984 - 1999 UT Austin, TX
- 2002 passed away in Nuenen, Netherlands

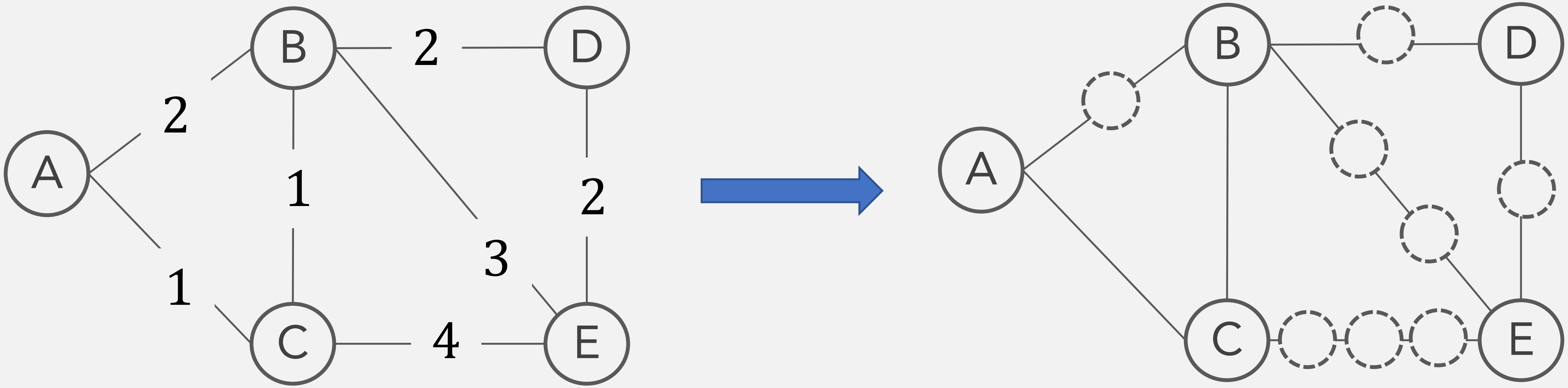
"What's the shortest way to travel from Rotterdam to Groningen? It is the algorithm for the shortest path, which I designed in about 20 minutes. One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path."

https://cacm.acm.org/magazines/2010/8/96632-an-interview-with-edsger-w-dijkstra/fulltext

Reducing to BFS

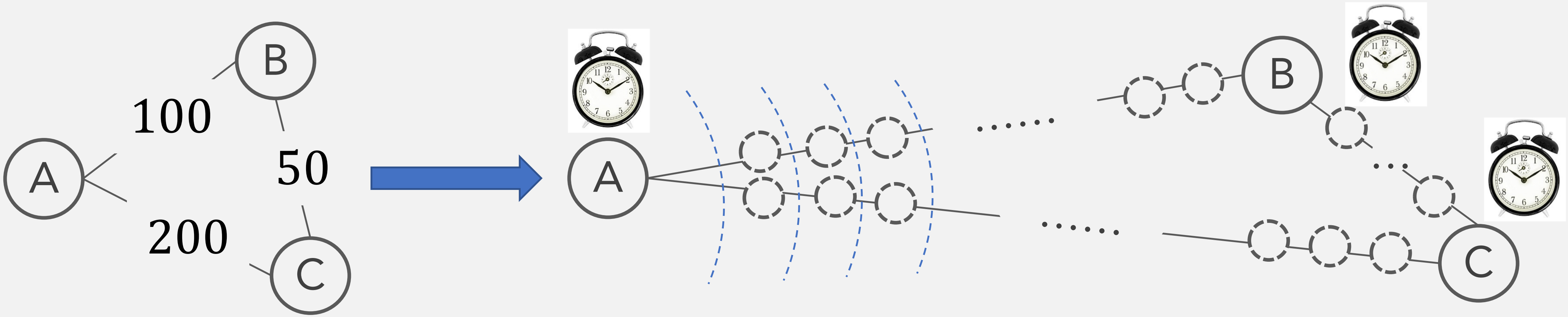






An alarm-clock algorithm

Idea. Convert G to G' by inserting dummy nodes. Run BFS on G'



AlarmSP(G, s) // set alarm clock for s at time 0

Repeat until no more alarms; Suppose next alarm goes off at T for node u

$dist(s, u) \leftarrow T$

For each neighbor v of u

If no alarm for v, set one for time $T + l(u, v)$

Else If current alarm larger, reset it for time $T + l(u, v)$

Dijkstra's algorithm: priority queue for alarms

PriorityQueue Q: set of n elements w. associated key values (alarm)

- Change-key(x). change key value of an element
- Delete-min. Return the element with smallest key, and remove it.
- Can be done in $O(\log n)$ time (by a heap)

Dijkstra(G, s) // initialize $d(s) = 0$, others $d(u) = \infty$
Make Q from V using $d(\cdot)$ as key value
While Q not empty
  $u \leftarrow$ Delete–min(Q)
  // pick node with shortest distance to s
  For all edges $(u, v) \in E$
    If $d(v) > d(u) + l(u, v)$
      $d(v) \leftarrow d(u) + l(u, v)$ and Change-key(v)

$O(n\log n)$

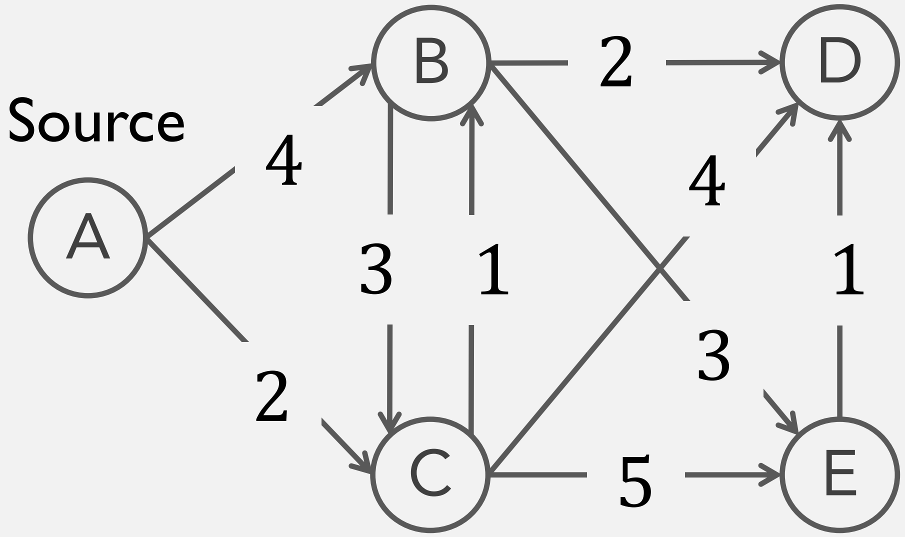
$O(m\log n)$

Dijkstra
$O((m + n)\log n)$

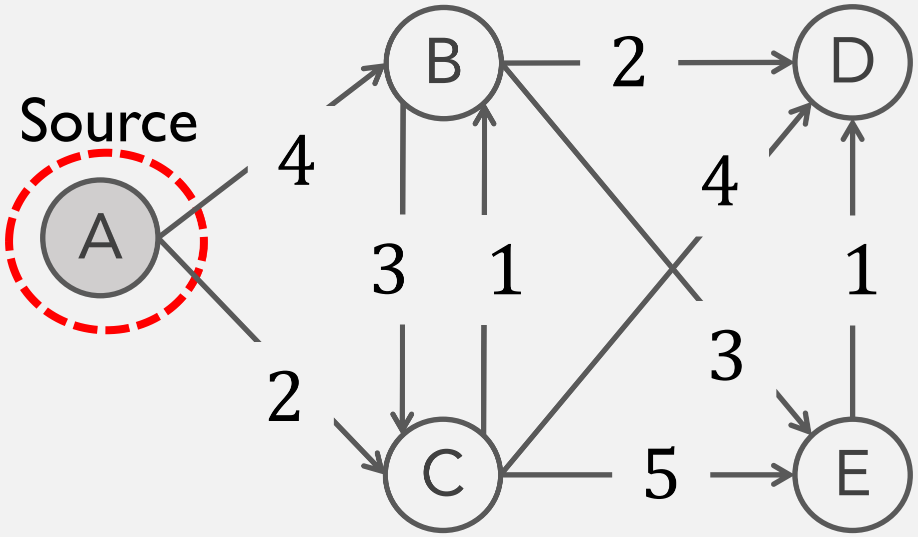
Further improvement possible by Fibonacci heap [More to come]

N.B. BFS uses ordinary Queue. Dijkstra = BFS+Priority Queue

Demo



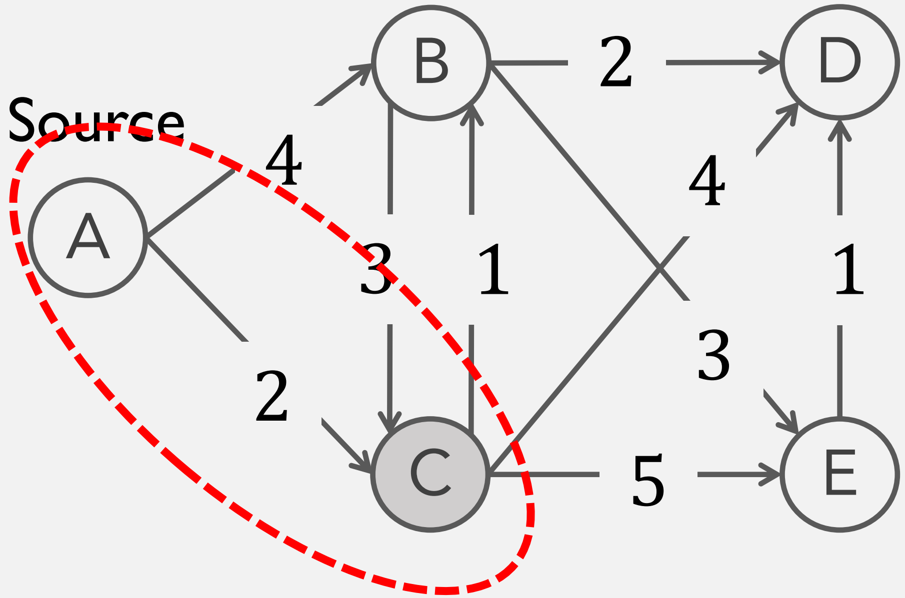
	d
A	0
B	∞
C	∞
D	∞
E	∞



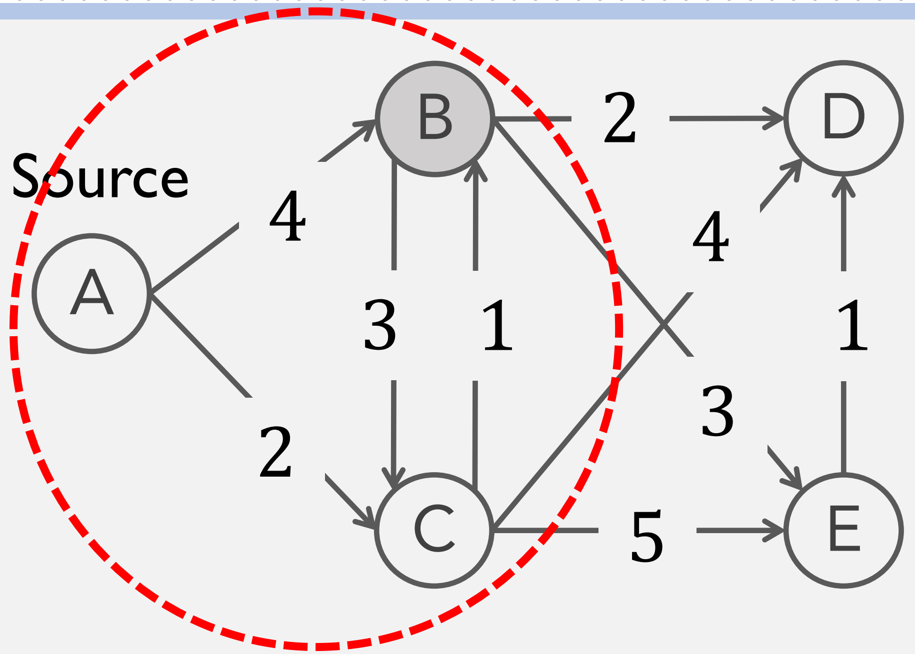
	d
A	0
B	4
C	2
D	∞
E	∞

While Q not empty
  $u \leftarrow$ Delete–min(Q)
  // pick node with shortest distance to s
  For all edges $(u, v) \in E$
  If $d(v) > d(u) + l(u, v)$
    $d(v) \leftarrow d(u) + l(u, v)$ and Change–key(v)

Demo



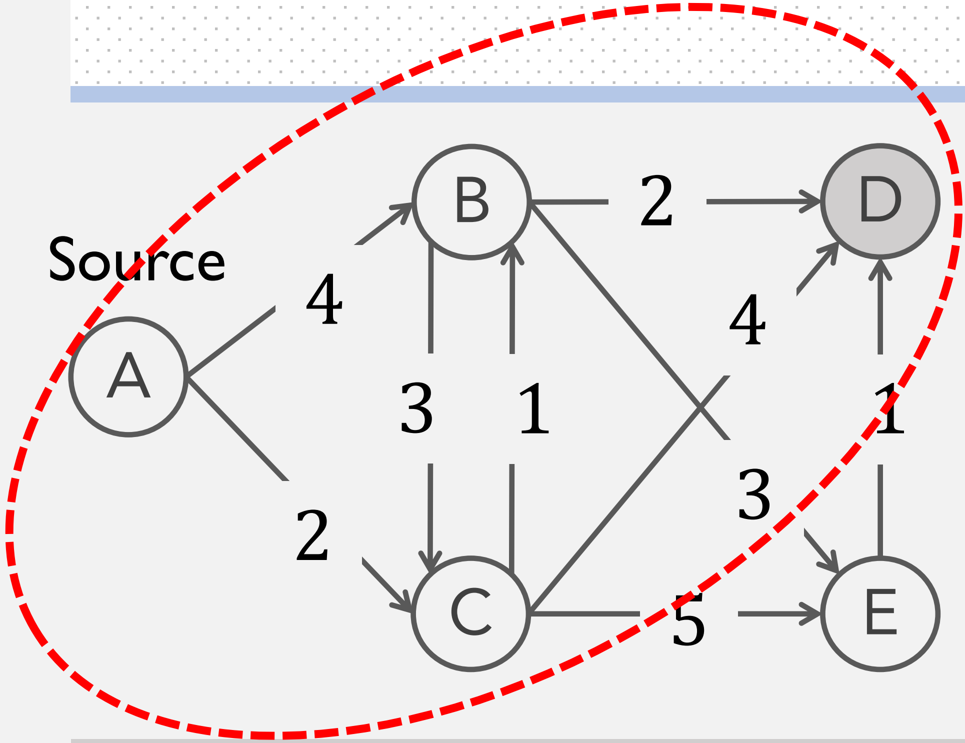
	d
A	0
B	3
C	3
D	6
E	7



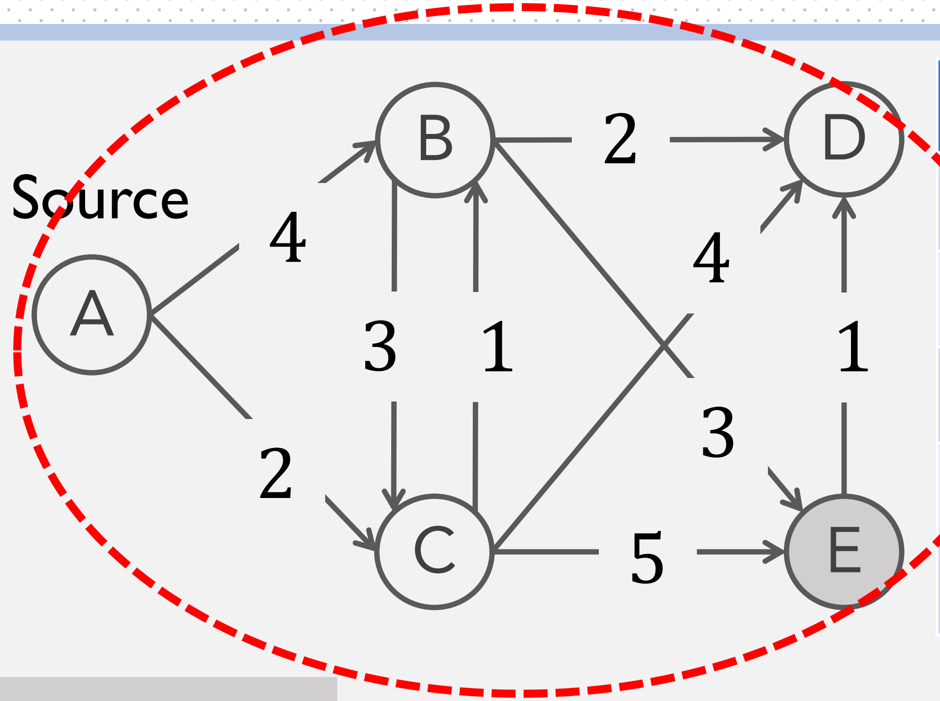
	d
A	0
B	3
C	3
D	5
E	6

While Q not empty
  $u \leftarrow$ Delete–min(Q)
  // pick node with shortest distance to s
  For all edges $(u, v) \in E$
  If $d(v) > d(u) + l(u, v)$
    $d(v) \leftarrow d(u) + l(u, v)$ and Change-key(v)

Demo



	d
A	0
B	3
C	3
D	5
E	6



	d
A	0
B	3
C	3
D	5
E	6

While Q not empty

    $u \leftarrow$ Delete–min(Q)

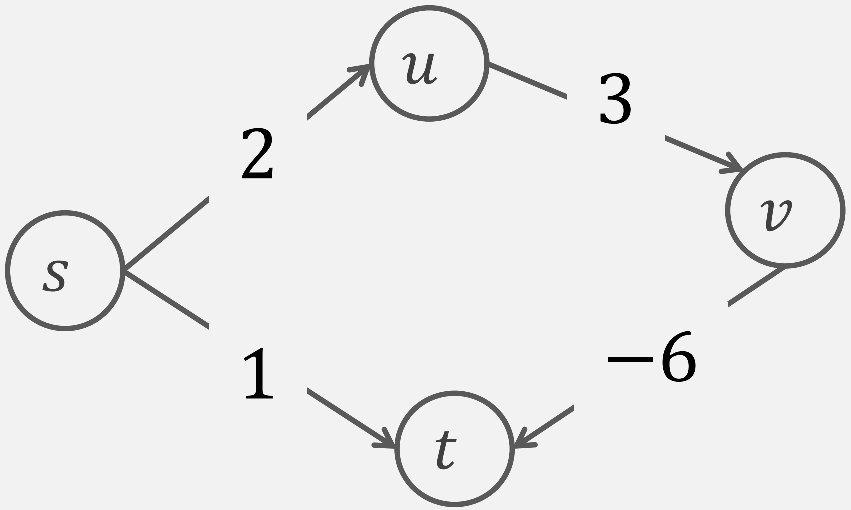
    // pick node with shortest distance to s

    For all edges $(u, v) \in E$

    If $d(v) > d(u) + l(u, v)$

        $d(v) \leftarrow d(u) + l(u, v)$ and Change-key(v)

How it fails on negative lengths?

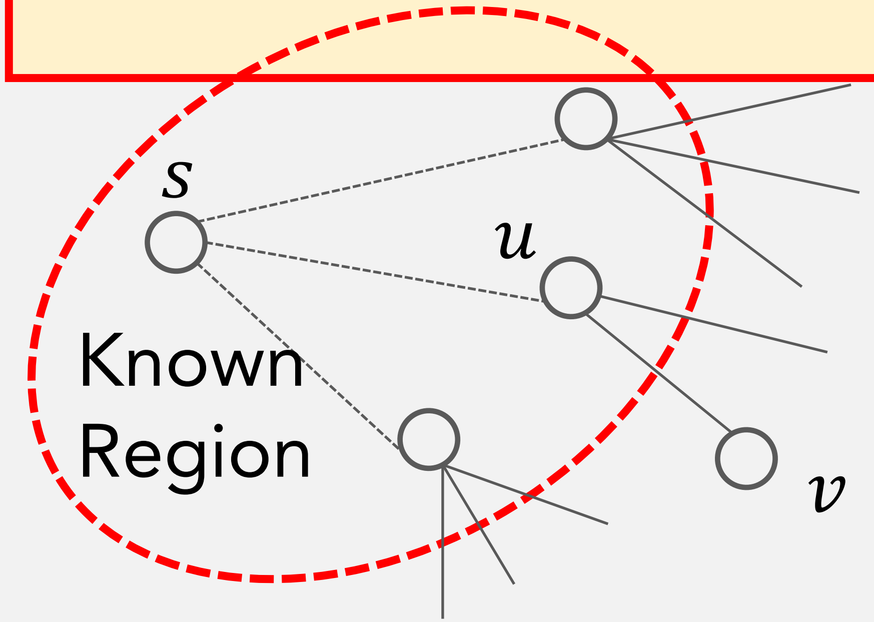


Jumping to a short one too early!

Reflection on Dijkstra: greedy stays ahead

- Known region R: in which the shortest distance to s is known
- Growing R: adding v that has the shortest distance to s
- How to Identify v? The one that minimizes $d(u) + l(u, v)$ for $u \in R$

Shortest path to some u in known region, followed by a single edge (u, v)



Dijk(G, s) // initialize $d(s) = 0$, $d(u) = \infty$, R=∅
While R $\neq V$
    Pick $v \notin R$ w. smallest $d(u)$ // by Priority Q
    Add v to R
    For all edges $(v, w) \in E$
    If $d(v) > d(u) + l(u, v)$
        $d(v) \leftarrow d(u) + l(u, v)$

Contrast with Bellman-Ford

- Dijkstra (Greedy)

$$d(v) = \min_{u \in R} d(u) + l(u, v)$$

  - Positive weight: no need to wait; more edges in a path do not help

- Bellman-Ford (Dynamic programming)

$$\mathrm{OPT}(i, v) = \min\left\{\mathrm{OPT}(i-1, v), \min_{v \to w \in E} \{\mathrm{OPT}(i-1, w) + l_{v \to w}\}\right\}$$

❖Global vs. Local

- Dijkstra's requires global information: known region & which to add
- Bellman-Ford uses only local knowledge of neighbors, suits distributed setting

Network routing: distance-vector protocol

- Communication network
  - Nodes: routers
  - Edges: direct communication links
  - Cost of edge: delay on link.

naturally nonnegative, but Bellman-Ford used anyway!

- Distance-vector protocol ["routing by rumor"]
  - Each router maintains a vector of shortest-path lengths to every other node (distances) and the first hop on each path (directions).
  - Algorithm: each router performs n separate computations, one for each potential destination node.
- Path-vector protocol: coping with dynamic costs