W, 09/25/19

Fall'19 CSCE 629

Analysis of Algorithms

Fang Song
Texas A&M U

Lecture 10

- Elements of DP
- Matrix-chain multiplication

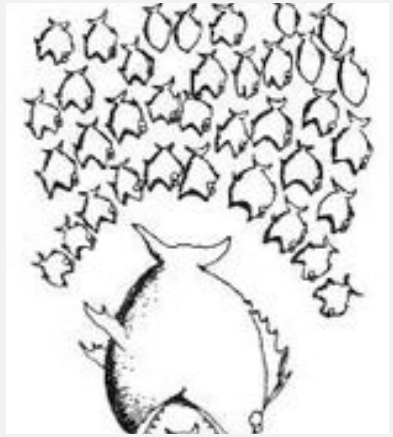
Credit: based on slides by K. Wayne

Dynamic Programming: recap

- Break up a problem into a series of overlapping subproblems
- There is an ordering on the subproblems, and a relation showing how to solve a subproblem given answers to "smaller" ones.

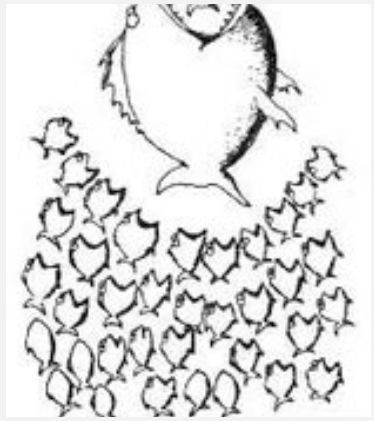
An implicit DAG: nodes=subproblems, edges = dependencies

Top-down



Credit: Mary Wootters

- **DP is about smart recursion (i.e. without repetition) by momoization**
- **Usually easy to express by building up a table iteratively**



Bottom-up

A DP recipe

1. Formulate the problem recursively (key step!)
   a) Specification. Describe what problems to solve (not how)
   b) Recursion. Give a recursive formula for the whole problem in terms answers to smaller instances of the same problem
   c) Step back and double check!

We usually go with bottom-up approach in this class

2. Build solutions to your recurrence (kinda routine)
   a) Identify subproblems
   b) Choose a memoization data structure
   c) Identify dependencies and find a good order (DAG in topological order)
   d) Write down your algorithm
   e) Analyze time (and space)
   f) Further improvements if possible

Matrix chain multiplication

In which **order** to multiply a sequence of rectangular matrices?

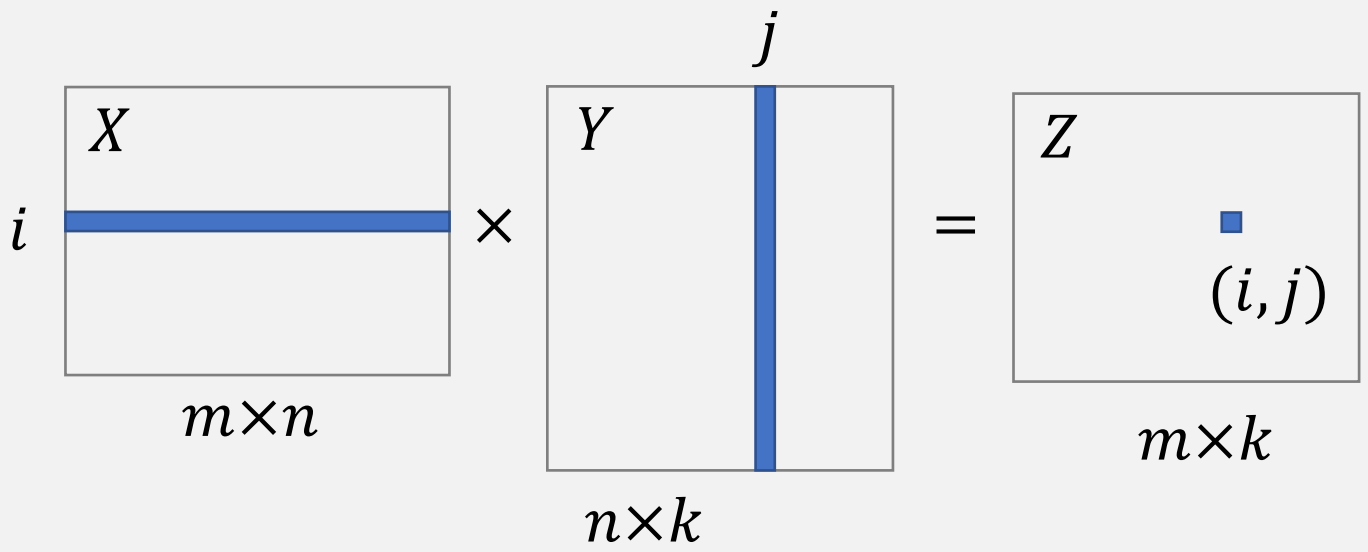
$A \times (B \times C)$ **vs.** $(A \times B) \times C$

- Both correct: associativity
- Does the order matter?
- What we care: # of multiplications of numbers

Review: matrix multiplication

- Matrices $X_{m\times n}$ and $Y_{n\times k}$

Compatible: # column(X) = # row(Y)



$$z_{ij} = \sum_{\ell=1}^{n} x_{i\ell}\, y_{\ell j}$$

n multiplications of real numbers

- Computing $Z = X_{m\times n}Y_{n\times k}$: mnk scaler multiplications

Let's forget about Strassen's divide-&-conquer algorithm for now

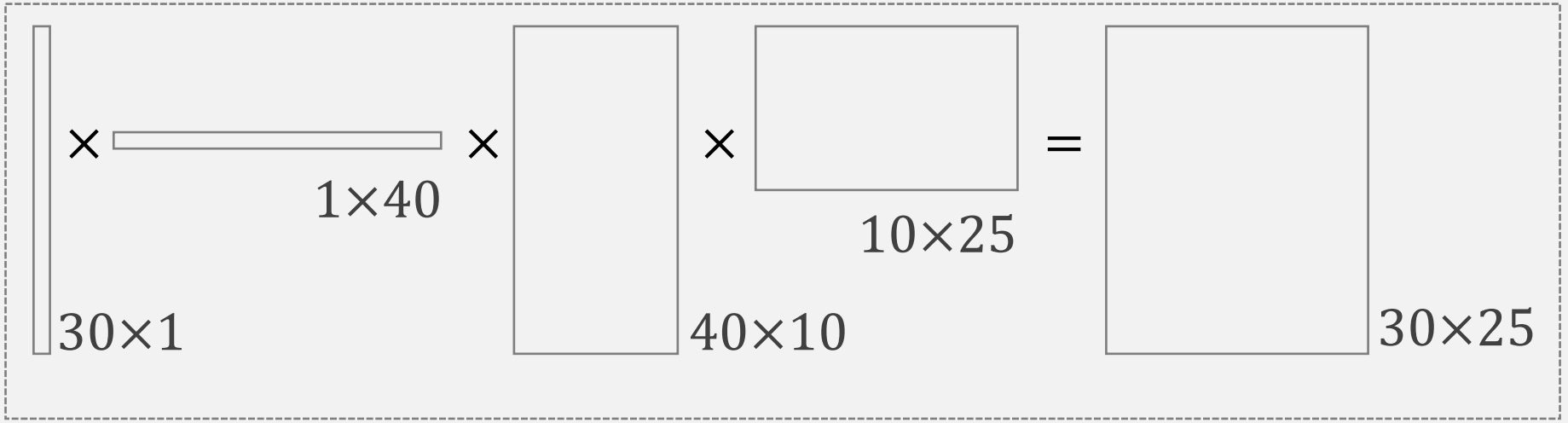
Why order matters …

$A: 30\times1$

$B: 1\times40$

$C: 40\times10$

$C: 10\times25$



$((AB)(CD))$

41200 scalar mult.

vs.

$(A((BC)D))$

1400 scalar mult.

Matrix chain order problem

- Input. Matrices $A_1, \ldots, A_n$
  - A_i size $d_{i-1} \times d_i$
- Output. Optimal order for computing $\Pi_i A_i$
  - Minimum # of scalar multiplications
- Brute-force

$$P(1) = 1, P(n) = \sum_{k=1}^{n-1} P(k)P(n-k)$$

  - Exercise. Prove $P(n) = \Omega(2^n)$

DP1: develop a recursion

- Input. Matrices $A_1, \ldots, A_n$
  - A_i size $d_{i-1} \times d_i$
- Output. Optimal order for computing $\Pi_i A_i$
  - Minimum # of scalar multiplications

1a. specification

Def. $M(i,j)$ = min # of mult. needed to compute $P_{ij} := A_i A_{i+1} \ldots A_j$

- Goal. Find $M(1,n)$
- Basis: $M(i,i) = 0$
- Recursion: how to define $M(i,j)$ recursively?

1b. b recursion

DP1: develop a recursion

- Assuming optimal order divides at k $A_i \dots A_j$

$$(P_{ik})_{d_{i\_1} \times d_k} \quad (A_i \dots A_k)(A_{k+1} \dots A_j) \quad (P_{k+1j})_{d_k \times d_j}$$

$M(i,k)$ $M(k+1,j)$

  - Cost to compute P_{ik}: $M(i,k)$
  - Cost to compute P_{k+1j}: $M(k+1,j)$
  - Cost to compute: $P_{ik} \times P_{kj}$: $d_{i-1} \times d_k \times d_j$

computing $Z = X_{m \times n} Y_{n \times k}$: mnk scaler multiplications

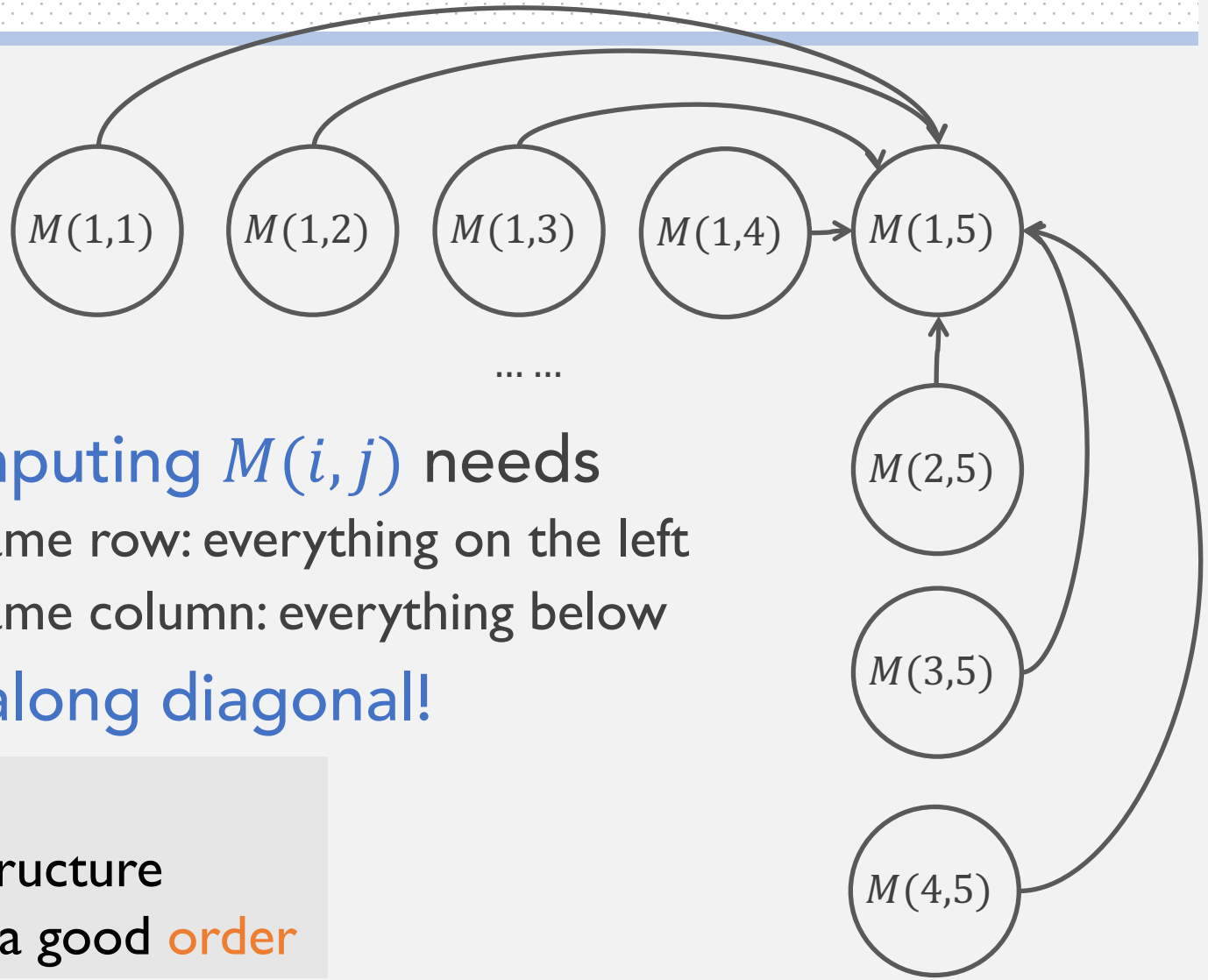
$$M(i,j) = \begin{cases} 0 & \text{if } i = j \\ \max\limits_{i \le k < j} \{M(i,k) + M(k+1,j) + d_{i-1} d_k d_j\} & \text{otherwise} \end{cases}$$

- How many subproblems in total? $O(n^2)$

DP2: build solutions bottom up

M	1	2	3	4	5
1					
2	X				
3	X	X			
4	X	X	X		
5	X	X	X	X	

4 3 2 1 0



- Computing $M(i,j)$ needs
  - Same row: everything on the left
  - Same column: everything below
- Go along diagonal!

2a. Identify subproblems
2b. Choose a memoization data structure
2c. Identify dependencies and find a good order

DP2: build solutions bottom up

M	1	2	3	4	5
1	0				
2	X	0			
3	X	X	0		
4	X	X	X	0	
5	X	X	X	X	0

1 2 3 4

MatrixChain (n)

// $M(i,j)$ memoize subproblem values

For $i = 2, \dots, n$

  $M[i,i] \leftarrow 0$

For $l = 1, \dots, n-1$ // diagonals

  For $i = 1, \dots, n-l$ //row

    $j = i + l$ // the column of row i on l-th diag.

    $M[i,j] \leftarrow \infty$

    For $k = i, \dots j-1$

      $M[i,j]$

      $= min\{M[i,j], M[i,k] + M[k+1,j], d_{i-1}d_k d_j\}$

2d. Write down your algorithm

2e. Analyze time (and space)

- Running time: $O(n^3)$

Example

	1	2	3	4
1	0	1200	700	1400
2	X	0	400	650
3	X	X	0	10,000
4	X	X	X	0

$A_1: 30\times1$

$A_2: 1\times40$

$A_3: 40\times10$

$A_4: 10\times25$

$$M(1,2) = 30\times1\times40$$

$$M(2,3) = 1\times40\times10$$

$$M(3,4) = 40\times10\times25$$

$$M(1,3) = \min\{M(1,2) + 30\times40\times10,\ M(2,3) + 30\times1\times10\}$$

$$M(2,4) = \min\{M(2,3) + 1\times10\times25,\ M(3,4) + 1\times40\times25\}$$

$$M(1,4) = \min\{\dots\}$$

DP3: constructing an optimal solution

MatrixChain (n)

// $M(i,j)$ memoize subproblem values

// $S[i,j]$ memoize optimal split index

……

For $l =$

  For $i = 1, \ldots, n - l$

    $j = i + l$ // the column of row i on l-th diag.

    For $k = i, \ldots j - 1$

      $M[i,j] = min\{M[i,j], M[i,k] + M[k+1,j], d_{i-1}d_k d_j\}$

    Record the optimal k: $S[i,j] \leftarrow k$

- Exercise. Find the optimal order of multiplication from S