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Fall’19 CSCE 629

Analysis of Algorithms

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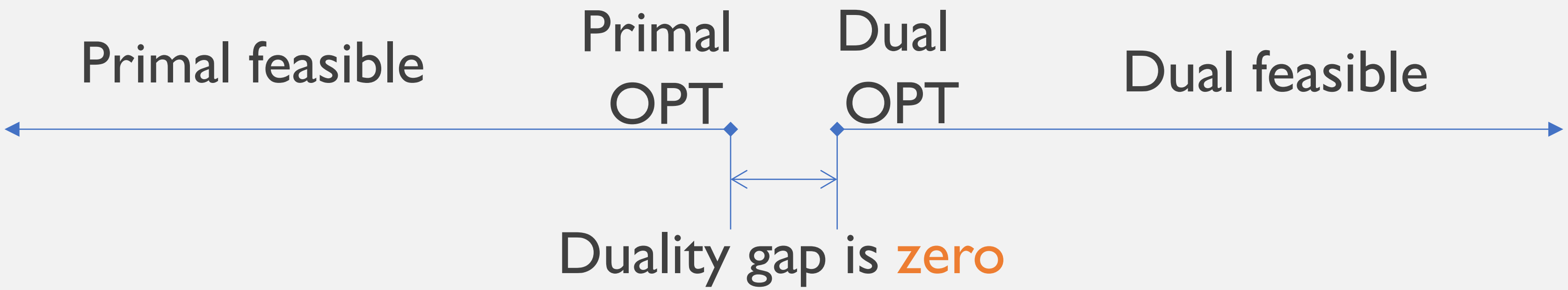
Lecture 25

- Computational intractability

Credit: based on slides by A. smith & K. Wayne

Fundamental theorem of linear programming



(Primal) Max $\boldsymbol{c}^T\boldsymbol{x}$
Subject to:
$$A\boldsymbol{x} \le \boldsymbol{b}$$
$$\boldsymbol{x} \ge \boldsymbol{0}$$

(Dual) Min $\boldsymbol{y}^T\boldsymbol{b}$
Subject to:
$$\boldsymbol{y}^T\boldsymbol{A} \ge \boldsymbol{c}^T$$
$$\boldsymbol{y} \ge \boldsymbol{0}$$

- Weak duality. If $\boldsymbol{x}$ is a feasible solution for a linear program $\sqcap$, and $\boldsymbol{y}$ is a feasible solution for its dual $\sqcup$, then $\boldsymbol{c}^T\boldsymbol{x} \le \boldsymbol{y}^T A\boldsymbol{x} \le \boldsymbol{y}^T\boldsymbol{b}$.
- Strong duality. $\sqcap$ has an optimal solution and $\boldsymbol{x}^*$ if and only if its dual $\sqcup$ has an optimal solution $\boldsymbol{y}^*$ such that $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{y}^T A\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{b}$.

Duality example

(P) Maximize: $x_1 + 5x_2$
Subject to:

$$0 \le x_1 \le 12$$
$$0 \le x_2 \le 15$$
$$x_1 + x_2 \le 24$$

(D) Minimize: $12y_1 + 15y_2 + 24y_3$
Subject to:

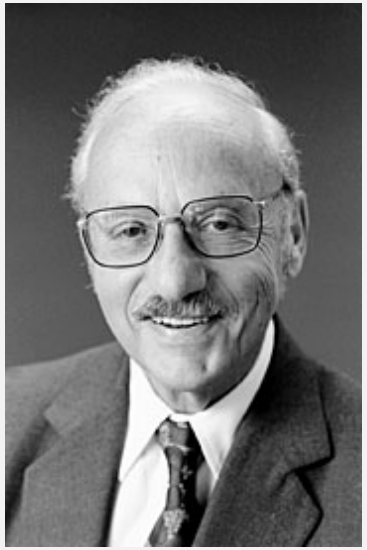
$$y_1 + y_3 \ge 1$$
$$y_2 + y_3 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

Max $= 84, x_1 = 9, x_2 = 15$

Min $= 84, y_1 = 0, y_2 = 4, y_3 = 1$

(magic) multipliers

A dialogue between Dantzig & von Neumann



George Dantzig

Let me show you my exciting finding: simplex algorithm for LP
…   …   … [next 30 mins]

Get to the point, please!

OK! Em... To be concise … [next 3 mins]

Ah, that!

[next 60 mins]
…. (convexity)… (fixed point) … (2-player game) …
so, there is duality which'd follow by my min-max theorem …

For any matrix A, $\min_x \max_y xAy = \max_y \min_x xAy$.

John von Neumann

A refection on the algorithmic journey

- So far: algorithm design triumph
  - Divide-and-conquer
  - Greedy
  - Dynamic programming
  - Linear programming (duality)
  - Local search
  - Randomization
  - …

Examples

- $O(n \log n)$ Merge sort
- $O(n \log n)$ interval scheduling
- $O(n^2)$ edit distance
- $O(n^3)$ bipartite matchin

- New goal: understand what is hard to compute

Computational intractability

- Computability: can you solve it, in principle?

Halting problem is uncomputable [Given program code, will this program terminate or loop indefinitely?]

Church-Turing Thesis. A function can be computed in any *reasonable* model of computation iff. it is computable by a **Turing machine**.

- Complexity: can you solve it, under resource constraints?

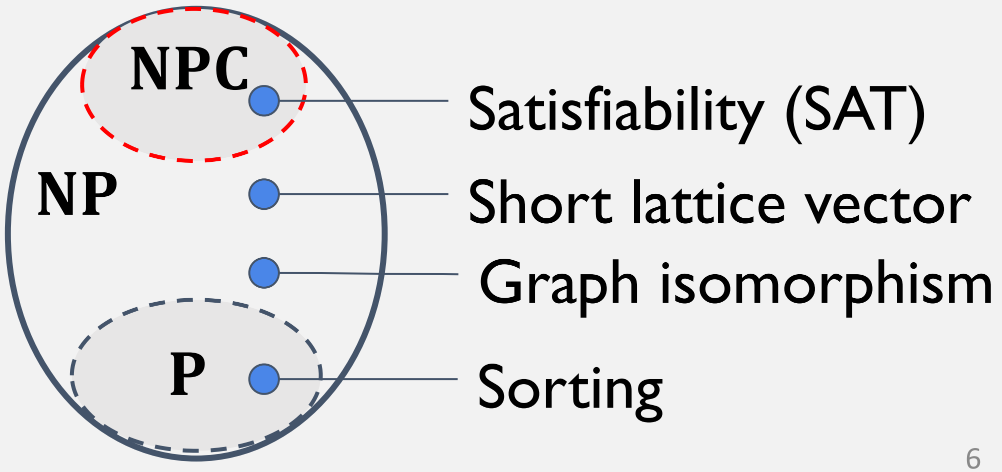
Extended Church-Turing Thesis. A function can be computed efficiently in any *reasonable* model of computation iff. it is efficiently computable by a **Turing machine**.

Disprove ECT???

Quantum supremacy using a programmable superconducting processor

Central ideas in complexity

- Poly-time as "feasible"
  - Most natural problems either are easy (e.g., n^3) or no poly-time alg. known
- Reduction : relating hardness ($A \le B \Rightarrow A$ no harder than B)
- Classify problems by "hardness"
  - $\mathbf{P}$ = {problems that are easy to answer}
  - $\mathbf{NP}$ = {problems that are easy to verify given hint} [lots of examples, stay tuned!]
  - **Complete** problems: "hardest" in a class



What'd be considered "feasible"?

Q. Which problems will we be able to solve in practice?

A. Those with poly-time algorithms. [von Neumann1953,Godel1956, Cobham1964,Edmonds1965,Rabin1966]

YES	Probably No
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover
Primality	Factoring

Classify problems

Desiderata. Classify problems as those that can be solved in polynomial-time and those that cannot.

Roughly: C program on machine with infinite memory

- Provably require exponential time.
  - Given a Turing machine, does it HALT in at most k steps?
  - Given a board position in an $n \times n$ generalization of chess, can black win?

☹Frustrating news: Huge number of fundamental problems have defied classification for decades.

- We will show: these problems are "computationally equivalent" and appear to be different manifestations of one hard problem.

Tool: polynomial-time reduction

Desiderata'. Suppose we can solve Y in poly-time. What else could we solve in polynomial time?

- Reduction. Problem X polynomial reduces to Problem Y if arbitrary instance of X can be solved using:
  - Polynomial number of standard computation steps
  - & polynomial number of calls to oracle that solves A

Notation. $X \leq_{P,Cook} Y$ (or $X \leq_P Y$)

! Mind your direction, don't confuse $X \leq_{\mathrm{P}} Y$ with $Y \leq_{\mathrm{P}} X$

N.B. We pay for time to write down instances to oracle

$\Rightarrow$ instances of Y must be of polynomial size.

What polynomial-time reductions buy us

- Design algorithms. If $X \leq_P Y$ and Y can be solved in poly-time, then X can also be solved in polynomial time.

- Establish intractability. If $X \leq_P Y$ and X cannot be solved in poly-time, then Y cannot be solved in polynomial time.

- Establish equivalence. If $X \leq_P Y$ and $X \leq_P Y$, then $X \equiv_P Y$.

Bottomline. Reductions classify problems acc. to relative difficulty

Quiz

- Which of the following poly-time reductions are known?
    A. $\text{FIND-MAX-FLOW} \leq_P \text{FIND-MIN-CUT}$
    B. $\text{FIND-MIN-CUT} \leq_P \text{FIND-MAX-FLOW}$
    C. Both A and B
    D. Neither A nor B

VALUES VS. ACTUAL FLOW/CUT

Simplification: decision problems

- Search problem. Find some structure.
  - Example. Find a minimum cut.
- Decision problem.
  - Problem X is a set of strings [e.g., strings that encode graphs containing a triangle]
  - Instance: string s [e.g., encoding of a graph]
  - YES instance: $s \in X$; NO instance: $s \notin X$
  - Algorithm A solves problem X: $A(s) = yes$ iff. $s \in X$
  - Example. Does there exist a cut of size $\leq k$?
- Self-reducibility. Bottomline. Search problem $\leq_P$ Decision version
  - Applies to all NP-complete problems in this chapter [Recall HW1]
  - Justifies our focus on decision problems

Polynomial-time transformation

- Cook reduction. Problem X polynomial reduces to Problem Y if arbitrary instance of X can be solved using:
  - Polynomial number of standard computation steps
  - & polynomial number of calls to oracle that solves A

$$X \leq_{P,Cook} Y$$

- Karp reduction. (Decision) problem X polynomial transforms to Problem Y if given any x, we can construct y with size $|y| = poly(|x|)$ such that $x \in X$ iff. $y \in Y$.

$$X \leq_{P,Karp} Y$$

N.B. Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X.

Open question. Are these two concepts the same?