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Fall’19 CSCE 629

Analysis of Algorithms

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Lecture 24

- Linear programming

Credit: based on slides by K. Wayne

Linear programming

- "Standard form" of an LP
  - m = # constraints, n = # decision variables. $i = 1, \dots, m, j = 1, \dots, n$
  - Input: real numbers c_j, a_{ij}, b_i
  - Output: real numbers x_j
  - Maximize linear objective function subject to linear inequalities
  - Feasible vs. optimal soln's.

Max $\sum_{j=1}^{n} c_j x_j$

Subject to: // linear constraints

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m$$

$$x_j \ge 0 \quad 1 \le j \le n$$

$\equiv$

Max $\boldsymbol{c}^T \boldsymbol{x}$

Subject to: $A\boldsymbol{x} \le \boldsymbol{b}$

$\boldsymbol{x} \ge \boldsymbol{0}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \quad \boldsymbol{0} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

Linear programming: variants

- "Slack form" of an LP: linear equalities

Max $\sum_{j=1}^{n} c_j x_j$
Subject to: // linear constraints

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m$$
$$x_j \ge 0 \quad 1 \le j \le n$$

$\Rightarrow$

Max $\sum_{j=1}^{n} c_j x_j$
Subject to: // linear constraints

$$s_i = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad 1 \le i \le m$$
$$\text{(slack vars) } s_i \ge 0 \quad 1 \le i \le m$$
$$x_j \ge 0 \quad 1 \le j \le m$$

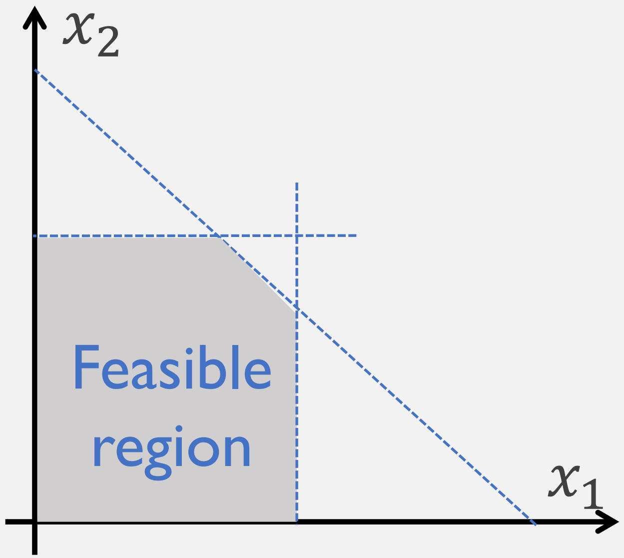
- Other equivalent variations
  - Minimization vs. maximization
  - Variables without nonnegativity constraints
  - $\ge$ vs. $\le$

Geometry of linear programming

1. Feasible

Maximize: $x_1 + 5x_2$
Subject to:

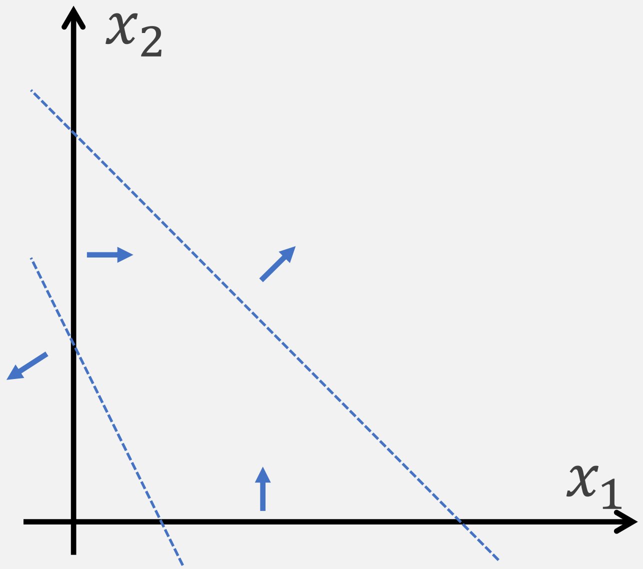
$$0 \leq x_1 \leq 12$$
$$0 \leq x_2 \leq 15$$
$$x_1 + x_2 \leq 24$$



2. Infeasible

Maximize: $x_1 - x_2$
Subject to:

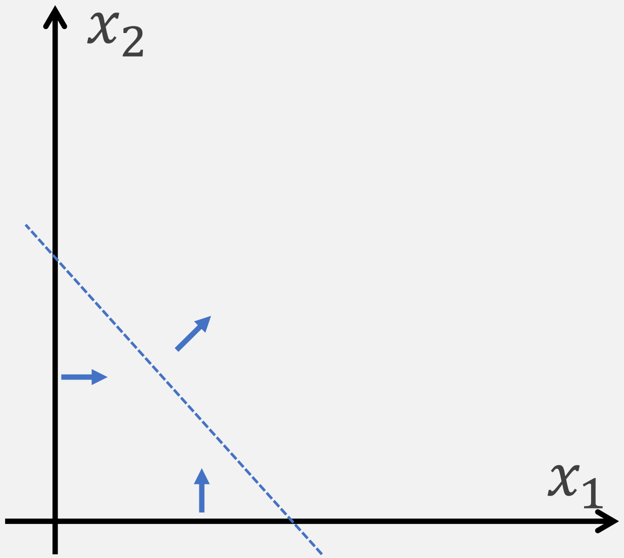
$$2x_1 + x_2 \leq 1$$
$$x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0$$



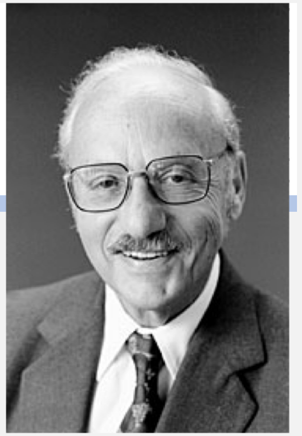
3. Unbounded

Maximize: $2x_1 + x_2$
Subject to:

$$x_1 + x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

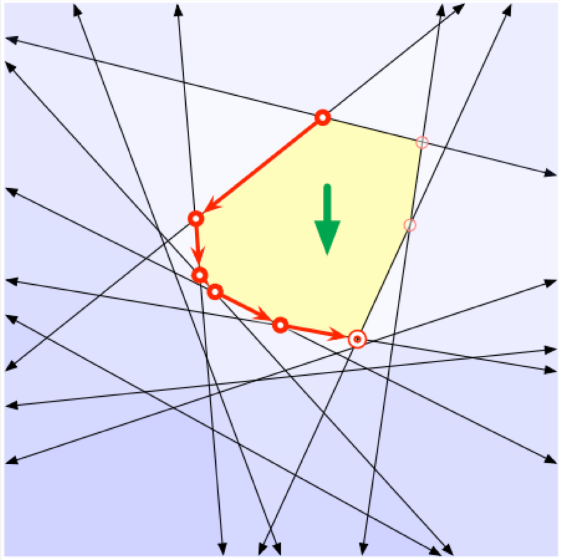


Simplex algorithm: the gist

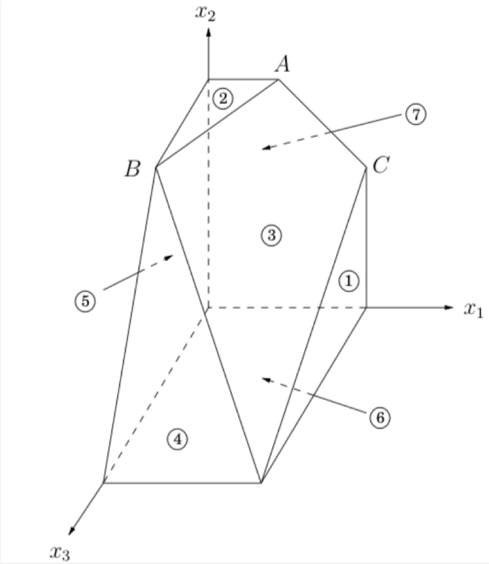


George Dantzig 1947

Let v be any vertex of the feasible region
While there is a neighbor u of v with better obj. value
    $v \leftarrow u$

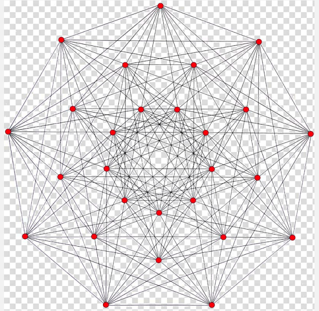


"Hill-climbing" along vertices in the polygon



3D-polyhedron defined by 7 inequalities

n variables?



- A linear eq. defines a hyperplane in R^n
- A linear ineq. defines a halfspace in R^n
- Each vertex is specified by n ineq's
- 2 vertices are neighbors if share $n-1$ defining ineq's

Simplex algorithm: the fine prints

Let v be any vertex of the feasible region
While there is a neighbor u of v with better obj. value
    $v \leftarrow u$

- How to find an initial feasible vertex?
  - Reduced to an LP and solved by simplex!
- Which neighbor to move to? (Pivot)
- Running time? [m ineq's, n variables]
  - ☹ Exponential in worst-case! $\binom{m+n}{n} \geq \left(1+\frac{m}{n}\right)^n$ verticies
  - ☺ Super fast in real world [typically terminates after at most $2(m+n)$ pivots]
- Correctness?
  - Convex polyhedron & linear objective function: local max $\equiv$ global max

Poly-time algorithms for linear programming

- Ellipsoid algorithm [Khachiyan1979]

POLYNOMIAL ALGORITHMS IN LINEAR PROGRAMMING*

L. G. KHACHIYAN

Moscow

  - A mathematical "sputnik"
  - Not competitive in practice


The New York Times

IRAN'S CIVIL GOVERNMENT OUT; HOSTAGES FACE DEATH THREAT; OIL EXPORTS BELIEVED HALTED

A Soviet Discovery Rocks World of Mathematics

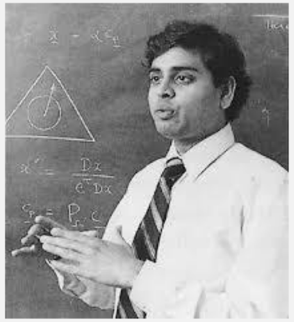

Leonid Khachiyan

- Interior point algorithm [Karmarkar1984]

A New Polynomial-Time Algorithm for Linear Programming

N. Karmarkar

AT&T Bell Laboratories
Murray Hill, New Jersey 07974

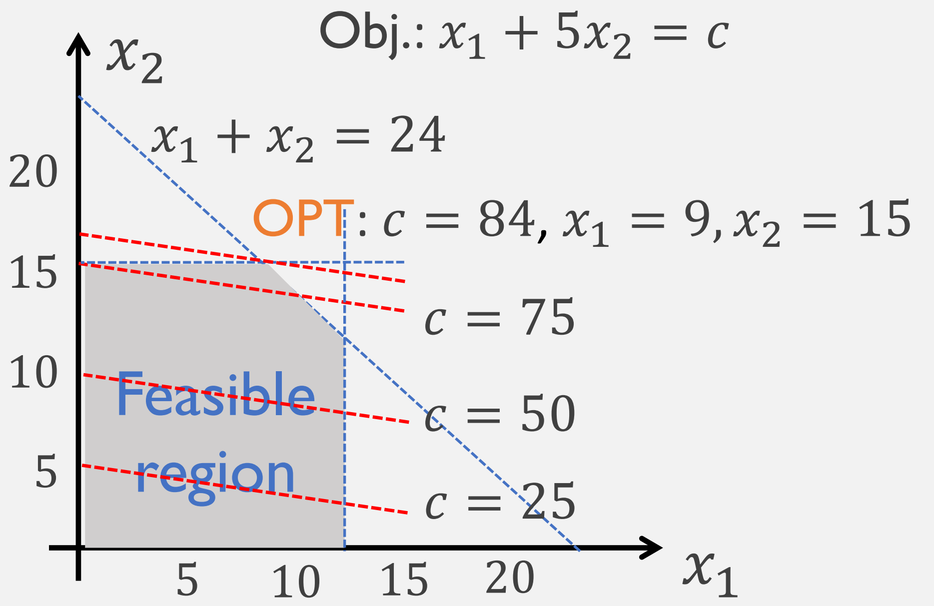

Narendra Karmarkar

N.B. Commercial solvers can solve LPs with millions of variables and tens of thousands of constraints.

How to decide optimality?

(P) Maximize: $x_1 + 5x_2$
Subject to:

$$0 \le x_1 \le 12$$
$$0 \le x_2 \le 15$$
$$x_1 + x_2 \le 24$$



Certificate: $x_1 + 5x_2 = 4 \cdot x_2 + 1 \cdot (x_1 + x_2) \le 4 \cdot 15 + 24 = 84$

How to find these (magic) multipliers?

Recall: max-flow & min-cut duality

- Weak duality (certificate of optimality) $v(f) \le cap(A, B)$

- Strong duality (max-flow min-cut theorem)

Value of max flow = capacity of min cut

Linear programming duality

(Primal) Max $\sum_{j=1}^{n} c_j x_j$
Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m$$

$$x_j \ge 0 \quad 1 \le j \le n$$

(Dual) Min $\sum_{i=1}^{m} b_i y_i$
Subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad 1 \le j \le n$$

$$y_i \ge 0 \quad 1 \le j \le m$$

(Primal) Max $\boldsymbol{c}^T \boldsymbol{x}$
Subject to:

$$A\boldsymbol{x} \le \boldsymbol{b}$$

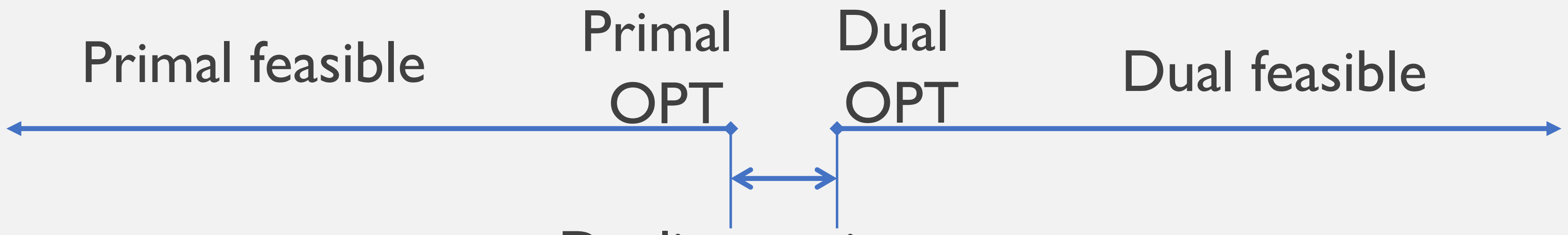
$$\boldsymbol{x} \ge \boldsymbol{0}$$

(Dual) Min $\boldsymbol{y}^T \boldsymbol{b}$
Subject to:

$$\boldsymbol{y}^T \boldsymbol{A} \ge \boldsymbol{c}^T$$

$$\boldsymbol{y} \ge \boldsymbol{0}$$

Fundamental theorem of linear programming



Duality gap is zero

(Primal) Max $\boldsymbol{c}^T\boldsymbol{x}$	(Dual) Min $\boldsymbol{y}^T\boldsymbol{b}$
Subject to:	Subject to:
$A\boldsymbol{x} \le \boldsymbol{b}$	$\boldsymbol{y}^T\boldsymbol{A} \ge \boldsymbol{c}^T$
$\boldsymbol{x} \ge \boldsymbol{0}$	$\boldsymbol{y} \ge \boldsymbol{0}$

- Weak duality. If $\boldsymbol{x}$ is a feasible solution for a linear program ⊓, and $\boldsymbol{y}$ is a feasible solution for its dual ⊔, then $\boldsymbol{c}^T\boldsymbol{x} \le \boldsymbol{y}^T A\boldsymbol{x} \le \boldsymbol{y}^T\boldsymbol{b}$.
- Strong duality. ⊓ has an optimal solution and $\boldsymbol{x}^*$ if and only if its dual ⊔ has an optimal solution $\boldsymbol{y}^*$ such that $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{y}^T A\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{b}$.

Duality example

(P) Maximize: $x_1 + 5x_2$
Subject to:

$$0 \le x_1 \le 12$$
$$0 \le x_2 \le 15$$
$$x_1 + x_2 \le 24$$

Max $= 84, x_1 = 9, x_2 = 15$

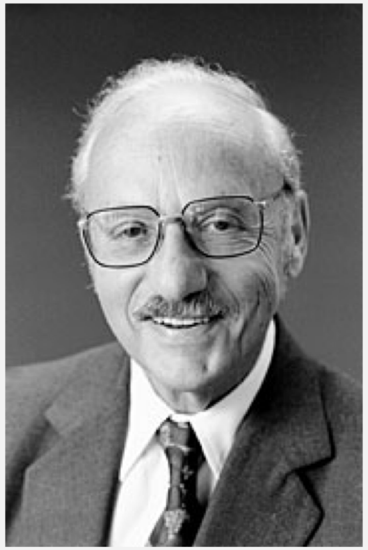
(D) Minimize: $12y_1 + 15y_2 + 24y_3$
Subject to:

$$y_1 + y_3 \ge 1$$
$$y_2 + y_3 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

Min $= 84, y_1 = 0, y_2 = 4, y_3 = 1$

(magic) multipliers

A dialogue between Dantzig & von Neumann



George Dantzig

Let me show you my exciting finding: simplex algorithm for LP
… … … [next 30 mins]

Get to the point, please!

OK! Em... There is the duality theorem … [next 3 mins]

Ah, that!

John von Neumann

[next 60 mins]
…. (convexity)… (fixed point) … (2-player game) …
so, I guess you've got a special case of my min-max theorem …

For any matrix A, $\min_x \max_y xAy = \max_y \min_x xAy$.

Exercise: Multicommodity flow

- A flow network with multiple flows (commodities)
  - $c(e)$: capacity on each edge
  - $K_i = (s_i, t_i, d_i)$: source, sink, and demand of commodity i. $i = 1, \dots, \ell$
- Goal. Decide if it is possible to accommodate all commodities

Max/min: 0

Subject to:

$$f_{ie} \geq 0, \qquad \forall e \in E$$

$$\sum_{i=1}^{\ell} f_{ie} \leq c(e), \qquad \forall e \in E$$

$$\sum\nolimits_{e \text{ into } v} f_{ie} - \sum\nolimits_{e \text{ out of } v} f_{ie} = 0, \qquad \forall v \in V \backslash \{s, t\}$$

$$\sum\nolimits_{e \text{ out of } s_i} f_{ie} - \sum\nolimits_{e \text{ into } s_i} f_{ie} = d_i, i = 1, \dots, \ell$$