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Fall'19 CSCE 629

Analysis of Algorithms

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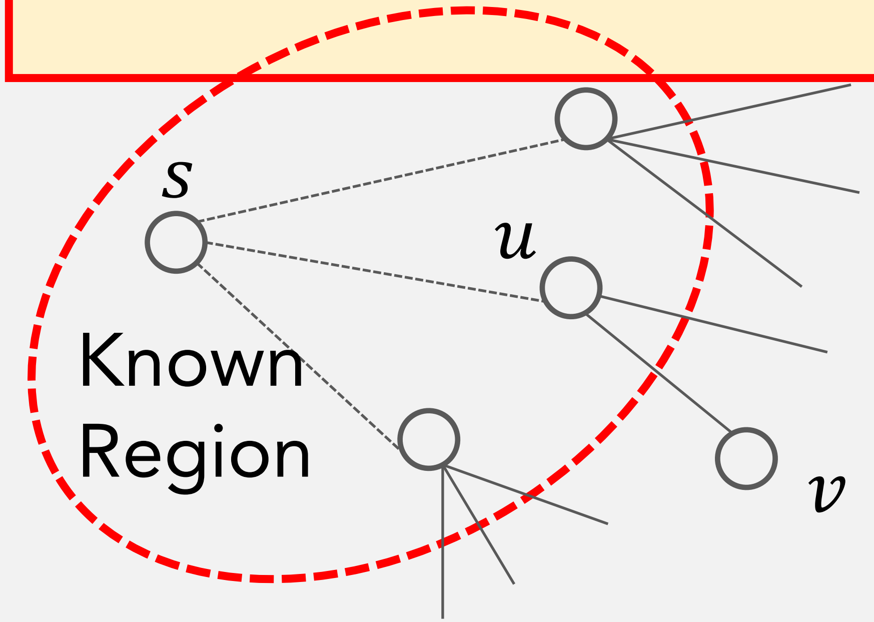
Lecture 15

- Dijkstra's algorithm cont'd
- Interval scheduling

Reflection on Dijkstra: greedy stays ahead

- Known region R: in which the shortest distance to s is known
- Growing R: adding v that has the shortest distance to s
- How to Identify v? The one that minimizes $d(u) + l(u, v)$ for $u \in R$

Shortest path to some u in known region, followed by a single edge (u, v)



Dijk(G, s) // initialize $d(s) = 0$, $d(u) = \infty$, R=∅
While R $\neq V$
    Pick $v \notin R$ w. smallest $d(v)$ // by Priority Q
    Add v to R
    For all edges $(v, w) \in E$
    If $d(v) > d(u) + l(u, v)$
        $d(v) \leftarrow d(u) + l(u, v)$

Contrast with Bellman-Ford

- Dijkstra (Greedy) $O((m+n)\log n)$

$$d(v) = \min_{u \in R} d(u) + l(u,v)$$

- Positive weight: no need to wait; more edges in a path do not help

- Bellman-Ford (Dynamic programming) $O(mn)$

$$\text{OPT}(i,v) = \min\left\{\text{OPT}(i-1,v), \min_{v \to w \in E}\{\text{OPT}(i-1,w) + l_{v\to w}\}\right\}$$

❖Global vs. Local

- Dijkstra's requires global information: known region & which to add
- Bellman-Ford uses only local knowledge of neighbors, suits distributed setting

Network routing: distance-vector protocol

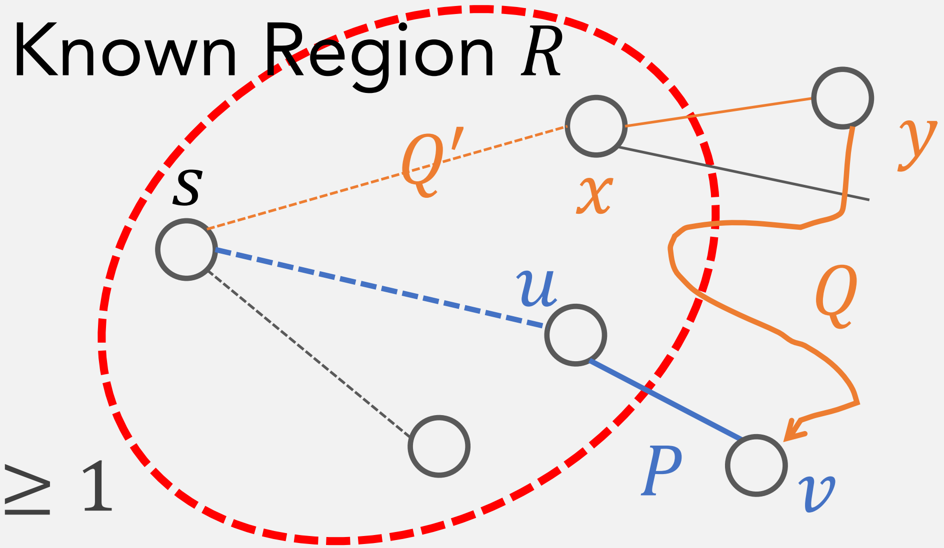
- Communication network
  - Nodes: routers
  - Edges: direct communication links
  - Cost of edge: delay on link.

naturally nonnegative, but Bellman-Ford used anyway!

- Distance-vector protocol ["routing by rumor"]
  - Each router maintains a vector of shortest-path lengths to every other node (distances) and the first hop on each path (directions).
  - Algorithm: each router performs separate computations for each potential destination node.
- Path-vector protocol: coping with dynamic costs

Correctness of Dijkstra's algorithm

Invariant. For each node $u \in R$, $d(u)$ is the length of a shortest $s - u$ path

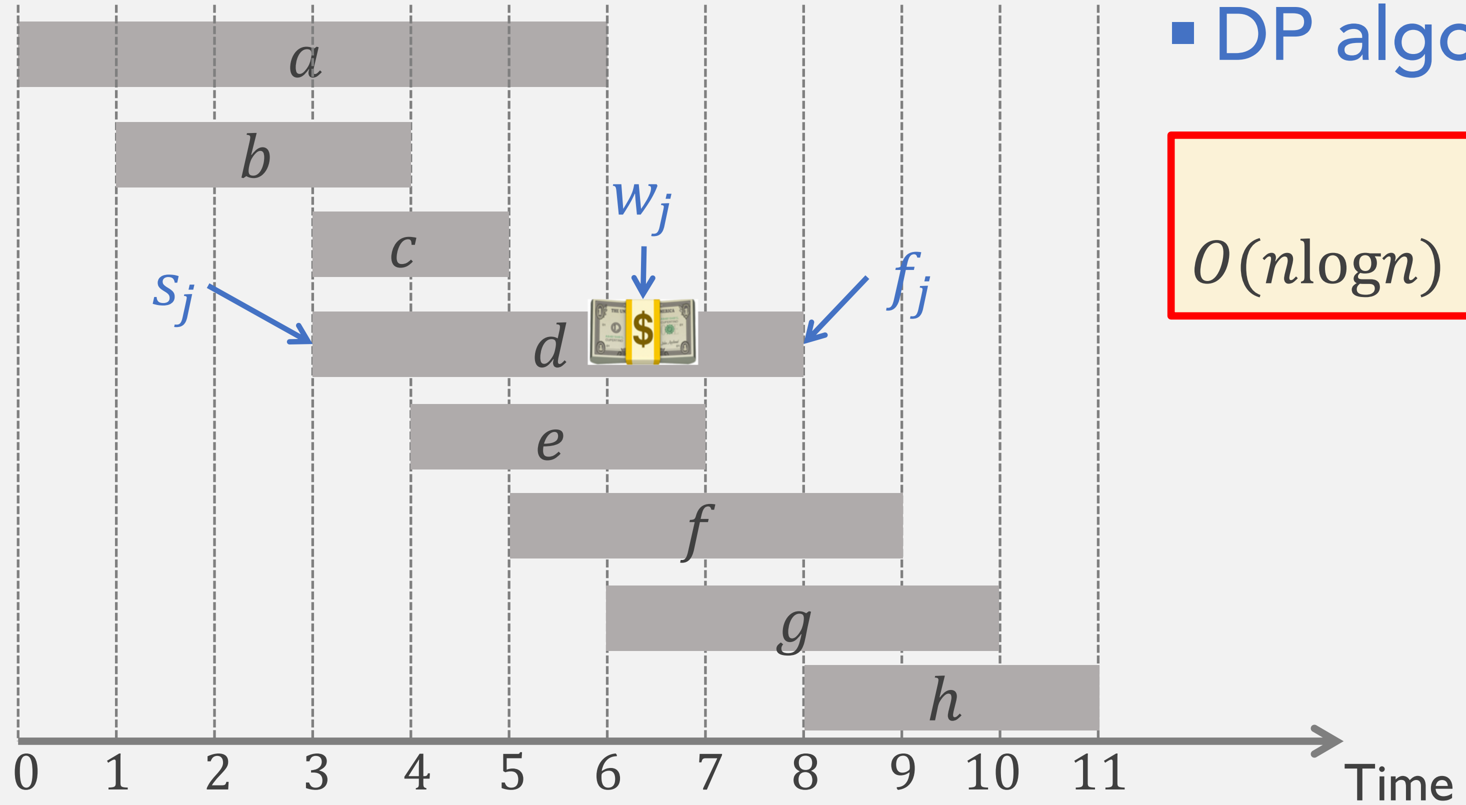


Proof. (By induction on size of R)

- Base case: $|R| = 1$ trivial
- Induction hypothesis: true for $|R| = k \geq 1$
  - Let v be the next node added to R and (u, v) be the chosen edge. Call this $s - u - v$ path P.
  - Consider any $s - v$ path Q. [Next show it's no shorter than P]
  - Let (x, y) be the first edge in Q leaving R; let Q' be the $s - x$ segment
  - $l(Q) \geq l(Q') + l(x, y) \geq d(x) + l(x, y) \geq l(P)$; because Dijkstra's picked v in this iteration (node outside R with shortest distance to s)

Recall: weighted interval scheduling

- **Input.** n jobs; job j starts at s_j, finishes at f_j, weight w_j
- **Output.** Subset of mutually compatible jobs of maximum weight



- DP algorithm $O(n \log n)$

Today
$O(n\log n)$ Greedy algorithm for $w_j = 1$.

Greedy strategies

Recall. DP recurrence. OPT(j) = value of optimal solution to jobs $1,2,\dots,j$

$$\text{OPT}(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\{\text{OPT}(j-1), w_j + \text{OPT}(pre(j))\} & \text{otherwise} \end{cases}$$

- **Greedy:** be lazy & pick the next compatible job that "looks nice"
  - Earliest start time: ascending order of s_j.
  - Earliest finish time: ascending order of f_j.
  - Shortest interval: ascending order of $f_j - s_j$.
  - Fewest conflicts: the one that conflicts the least number of jobs go first.
- **Exercise.** Find counterexamples for each strategy (if possible)

Greedy: counterexamples

☹ Earliest start time:

☹ Shortest interval:

☹ Fewest conflicts:

☺ Earliest finishing time

Greedy Algorithm: earliest finishing time

IntScheduling $(\{s_j, f_j\})$

1. Sort by finishing time so that $f_1 \leq f_2 \leq \cdots \leq f_n$ → $O(n\log n)$
2. $A \leftarrow \emptyset$ // set of selected jobs
3. For $j = 1, \ldots, n$
   If j compatible with A
   $A \leftarrow A \cup \{j\}$

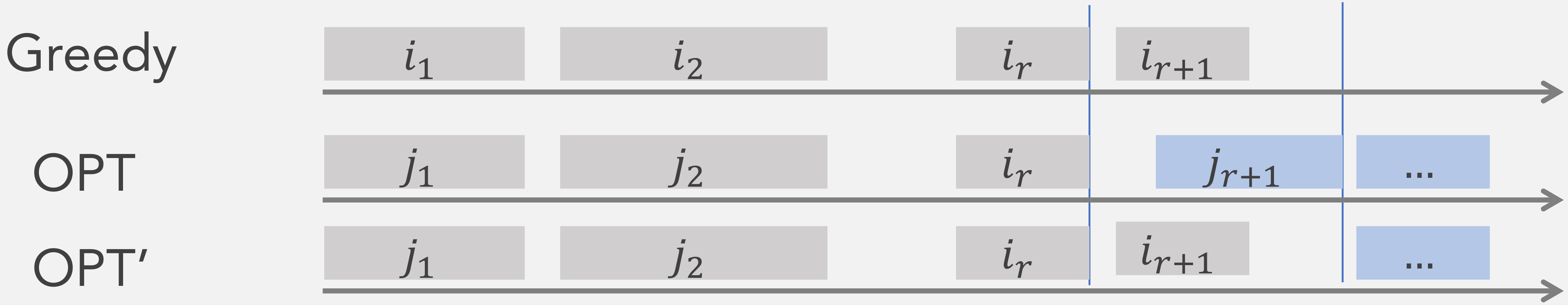
$n \times O(1)$

- Running time: $O(n \log n)$
- Correctness: proof by contradiction
  - Suppose greedy is not optimal
  - Consider an optimal strategy: one that agrees with Greedy for as many initial jobs as possible
  - Look at the first place that they differ: show a new optimal that agrees with greedy more

Greedy Algorithm: correctness

Proof (by contradiction): Suppose greedy is not optimal

- Let $i_1, i_2, \ldots, i_k$ denote set of jobs selected by greedy
- Let $j_1, j_2, \ldots, j_m$ be set of jobs in the optimal solution OPT where $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r
- Sub i_{r+1} for j_{r+1} in OPT: still feasible and optimal (OPT'); but agrees with Greedy at $r + 1$ positions; contradicts the maximality of r



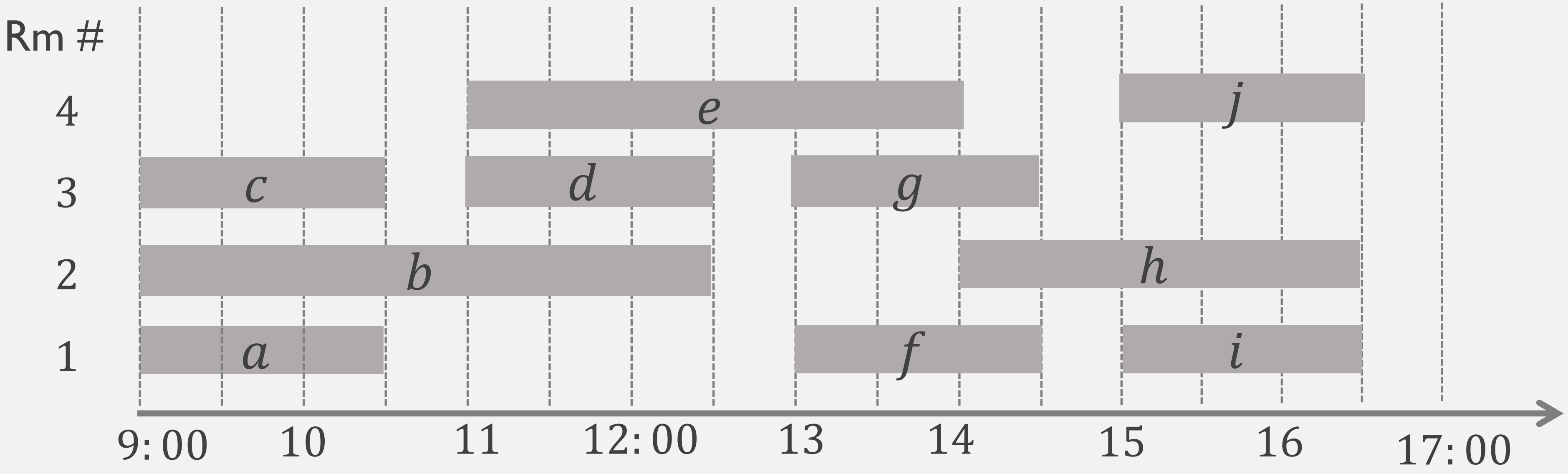
Interval Partitioning Problem

Scheduling classes

- Input. Lectures $\{s_j, f_j\}$
- Output. Minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Can you do better?

10 lectures scheduled in 4 classrooms

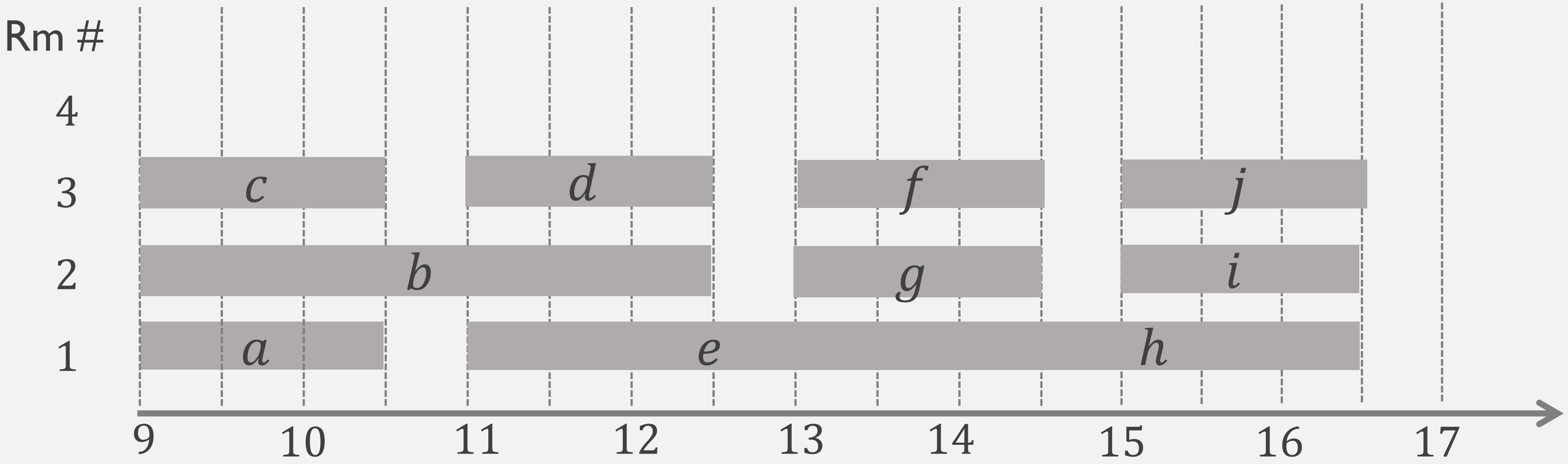


Interval Partitioning Problem

Scheduling classes

- Input. Lectures $\{s_j, f_j\}$
- Output. Minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

YES! 10 lectures scheduled in 3 classrooms



Greedy algorithm

- Idea. Sort lectures in increasing order of start time: assign lecture to any compatible classroom.

IntPartition$(\{s_j, f_j\})$ // $r \leftarrow 0$ # of allocated rooms

1. Sort by starting time so that $s_1 \le s_2 \le \cdots \le s_n$
2. For $j = 1, \ldots, n$
   - If j compatible with some classroom k
     - Schedule j in room k
   - Else allocate new classroom $r + 1$
     - Schedule j in room $r + 1$
     - $r \leftarrow r + 1$

How to do it in $O(\log r)$

OBS. # rm needed $\ge$ depth of input intervals (i.e., Max. number of lectures that overlap)

- Running time. $O(n \log n)$
- Optimality. #Rm allocated = depth of input intervals