CSCE 440/640 Quantum Algorithms

Homework 2

Texas A&M U, Spring 2019 — *Jan. 28, 2019*

Lecturer: Fang Song — *Due: Feb. 13, 2019*

Instructions. Only PDF format is accepted (type it or scan clearly). Your solutions will be graded on *correctness* and *clarity*. You should only submit work that you believe to be correct; if you cannot solve a problem completely, you will get significantly more partial credit if you clearly identify the gap(s) in your solution. For this problem set, a random subset of problems will be graded. Problems marked with "[**G**]" are required for graduate students. Undergraduate students will get bonus points for solving them.

You may collaborate with others on this problem set. However, you must ***write up your own solutions*** and ***list your collaborators*** for each problem.

1. (Quantum states and gates)
   (a) (6 points) Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
      i) Suppose we have a qubit and we first apply X and then Z. Is it equivalent to first applying Z and then X?
      ii) Suppose we have two qubits, and we apply X to both and then Z to both. Is it equivalent to applying Z to both and then applying X to both? Justify your answer.

   (b) (6 points) (SWAP gate) A SWAP gate takes two inputs a and b and outputs b and a; i.e., it swaps the values of two input registers. Show how to build a SWAP gate using only CNOT gates. (Hint: you'll need 3 of them.)

   (c) (5 points) [**G**] Show that every unitary one-qubit gate with real entries can be written as a rotation matrix, possibly preceded and followed by Z-gates. In other words, show that for every 2×2 real unitary U, there exist signs $s_1, s_2, s_3 \in \{1, -1\}$ and angle $\theta \in [0, 2\pi)$ such that

$$U = s_1 \begin{pmatrix} 1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s_3 \end{pmatrix}.$$

2. (Product states versus entangled states) In each of the following, either express the 2-qubit state as a tensor product of 1-qubit states or prove that it cannot be expressed this way.
   (a) (4 points) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$
   (b) (4 points) $\frac{3}{4}|00\rangle + \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle$

3. (Distinguishing states by local measurements) Suppose Alice and Bob are physically separated from each other, and are each given one of the qubits of some 2-qubit state. They are required to distinguish between State I and State II with only local measurements. Namely they can each perform a local (one-qubit) unitary operation and then a measurement (in the computational basis) of their own qubit. After their measurements, they can send only classical bits to each other. (This is usually referred to as LOCC: local operation and classical communication.) In each case below, either give a perfect distinguishing procedure (that never errs) or explain why there is no perfect distinguishing procedure (i.e., that for any procedure the success probability must be less than 1).

   (a) (5 points) State I: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$; State II: $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

   (b) (5 points) State I: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$; State II: $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

   (c) (5 points) **[G]** State I: $\frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$; State II: $\frac{1}{\sqrt{2}}(|00\rangle - i|11\rangle)$

4. (Linear algebra)

   (a) (15 points) (Tensor product)

      i) Show that $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$.

      ii) Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

      iii) If A and B are both invertible, show that so is $A \otimes B$.

      iv) Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

      v) Show that if A and B are unitary matrices, then so is $A \otimes B$.

   (b) (6 points) Let $U = (v_1, \ldots, v_n)$ be a unitary matrix and each $v_i \in \mathbb{C}^n$.

      i) Show that $\{v_1, \ldots, v_n\}$ form an orthonormal basis of $\mathbb{C}^n$.

      ii) Show that the eigenvalues of any unitary U are of the form $e^{i\theta}$ for some $\theta \in [0, 2\pi)$.

   (c) (4 points) Show that for any $x \in \{0,1\}^n$, $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$. $x \cdot y := \sum_{i=1}^n x_i y_i$ is the dot product over $\mathbb{Z}_2^n$.

   (d) (4 points) Let $x, y \in \{0,1\}^n$ and let $s = x \oplus y$. Show that

$$H^{\otimes n} \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle) = \frac{1}{\sqrt{2^{n-1}}} \sum_{z : z \cdot s = 0} (-1)^{x \cdot z} |z\rangle .$$

   (e) (8 points) For a vector $v = (v_0, \ldots, v_{k-1}) \in \mathbb{C}^k$, let $\|v\| := \sqrt{\sum_{i=0}^{k-1} |v_i|^2}$, which is the usual Euclidean length of v. For any $k \times k$ matrix $M \in \mathbb{C}^{k \times k}$, define its *spectral norm* $\|M\|$ as $\|M\| = \max_{|\psi\rangle} \|M|\psi\rangle\|$, where the maximum is taken over quantum states (i.e., vectors $|\psi\rangle$ such that $\||\psi\rangle\| = 1$). Define the distance between two $k \times k$ unitary matrices M_1 and M_2 as $\|M_1 - M_2\|$. Show that

      i) $\|A - B\| \le \|A - C\| + \|C - B\|$, for any three $k \times k$ matrices A, B, and C. (Namely, this distance measure satisfies the *triangle inequality*.)

ii) Show that, for any two $k \times k$ unitary matrices U_1 and U_2, and any matrix A, $\|U_1 A U_2\| = \|A\|$.

5. (Errors in randomized algorithms) Suppose you want to write a computer program C to compute a Boolean function $f : \{0,1\}^n \to \{0,1\}$, mapping n bits to 1 bit. If C is a deterministic algorithm, then "C successfully computes f" has a clear meaning that that $C(x) = f(x)$ for all inputs $x \in \{0,1\}^n$. But what if C is a probabilistic algorithm?

   (a) (8 points) The best thing is if C is a *zero-error* algorithm with failure probability p. Namely

   - on every input x, the output of $C(x)$ is either $f(x)$ or $\perp$ (denoting failure).
   - on every input x we have $\Pr[C(x) = \perp] \le p$ (NB. the probability is only over the internal randomness of C, not the random choice of x.).

   i) If you have a zero-error algorithm C for f with failure probability 90%, show how to convert it to a zero-error algorithm C' with failure probability at most 2^{-500}. The "slowdown" should only be a factor of a few thousand.

   ii) Alternatively, show how to convert C to an algorithm C'' for f which: (i) always outputs the correct answer, meaning $C''(x) = f(x)$ for all x; (ii) has expected running time only a few powers of 2 worse than that of C. (Hint: look up the mean of a geometric random variable.)

   (b) (5 points) The second best thing is if C is a one-sided error algorithm for f, with failure probability p. There are two kinds of such algorithms, "no-false-positives" and "no-false-negatives". For simplicity, let's just consider "no false-negatives" (the other case is symmetric);

   - on every input x, the output $C(x)$ is either 0 or 1;
   - on every input x such that $f(x) = 1$, the output $C(x)$ is also 1;
   - on every input x such that $f(x) = 0$, we have $\Pr[C(x) = 1] \le p$.

   Show how to convert a no-false-negatives algorithm C for f with failure probability 90% to another no-false-negatives algorithm C' for f with failure probability at most 2^{-500}. The "slowdown" should only be a factor of a few thousand.

   (c) (5 points) The third possibility (which is rare in practice) is if C is a two-sided error algorithm for f, with failure probability p. Namely,

   - on every input x, the output $C(x)$ is either 0 or 1.
   - on every input x, we have $\Pr[C(x) \neq f(x)] \le p$.

   If you have a two-sided error algorithm C for f with failure probability 40%, show how to convert it to a two-sided error algorithm C' for f with failure probability at most 2^{-500}. The "slowdown" should only be a factor of a few dozen thousand. (Hint: look up the Chernoff bound.)

6. (Simple search algorithms) In the context of this question, we are interested in exact solutions (with failure probability zero).

(a) (6 points) (1-out-of-4 search) Consider a black-box function $f : \{0,1\}^2 \to \{0,1\}$ with the property that there is a unique $x \in \{0,1\}^2$ such that $f(x) = 1$ and the goal is to determine x. How many classical queries are necessary to solve this problem? Design a quantum algorithm that finds x using 1 quantum query.

(b) (6 points) **[G]** (2-out-of-4 search) Given a black-box for a function $f : \{0,1\}^2 \to \{0,1\}$ with exactly two $x \in \{0,1\}^2$ such that $f(x) = 1$ and the goal is to determine both x's. Prove that 3 classical queries are necessary to solve this problem and that 2 quantum queries are sufficient to solve this problem.

7. (Simulating classical circuits) Let $f : \{0,1\}^2 \to \{0,1\}^2$ be a function such that $f(ab) = 0$ if $a = b = 1$ and $f(ab) = 1$ otherwise.

   (a) (3 points) Design a circuit using your favorite gate set (e.g., AND, OR, NOT) to compute f.

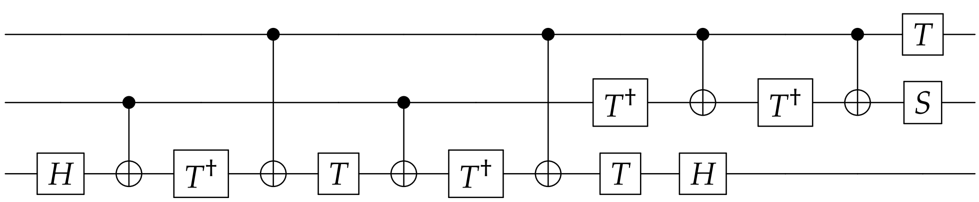
   (b) (3 points) Turn your circuit into a reversible circuit using Toffoli gate $T : a, b, c \mapsto a, b, a \wedge b \oplus c$ and other reversible gates. You may need to introduce ancilla bits

   (c) (4 points) Turn your reversible circuit into a unitary quantum circuit that implements the unitary $U_f : |x\rangle|y\rangle \mapsto |x\rangle|f(x) \oplus y\rangle$.

8. (Playing with quantum circuits)

   (a) (Exercise) Play around both the graphic composer and QASM editor IBM Q experience (or some other tools, e.g., Quirky and Quantum playground). Test the teleportation protocol.

   (b) (10 points) Determine the behavior of the following quantum circuit by implementing it (in graphic interface or programming it): You'll want to precede



   this circuit by all 8 possible ways of doing or not doing NOT gates on the relevant 3 qubits, so as to see what this circuit does to each of the basic states $|000\rangle, |001\rangle, \ldots, |111\rangle$.

9. (Watch in your leisure time. No grades.)

   (a) Many Worlds Interpretation.

   (b) Fast Fourier Transform.

   (c) The enormity of the number 2^{256}.