

# CS 410/510 Introduction to Quantum Computing

## Homework 2

Portland State U, Spring 2017  
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Due: May 02, 2017

**Instructions.** Your solutions will be graded on *correctness* and *clarity*. You should only submit work that you believe to be correct; if you cannot solve a problem completely, you will get significantly more partial credit if you clearly identify the gap(s) in your solution. It is good practice to start any long solution with an informal (but accurate) “proof summary” that describes the main idea. For this problem set, a random subset of problems will be graded. Problems marked with “[G]” are required for graduate students. Undergraduate students will get bonus points for solving them.

You may collaborate with others on this problem set. However, you must *write up your own solutions* and *list your collaborators* for each problem.

1. (Norms) For a vector  $v = (v_0, \dots, v_{N-1}) \in \mathbb{C}^N$ , let  $\|v\| := \sqrt{\sum_{i=0}^{N-1} |v_i|^2}$ , which is the usual Euclidean length of  $v$ . For any  $N \times N$  matrix  $M \in \mathbb{C}^{n \times n}$ , define its *spectral norm*  $\|M\|$  as  $\|M\| = \max_{|\psi\rangle} \|M|\psi\rangle\|$ , where the maximum is taken over quantum states (i.e., vectors  $|\psi\rangle$  such that  $\| |\psi\rangle \| = 1$ ). Define the distance between two  $N \times N$  unitary matrices  $U_1$  and  $U_2$  as  $\|U_1 - U_2\|$ .
  - (a) (5 points) Show that  $\|A - B\| \leq \|A - C\| + \|C - B\|$ , for any three  $N \times N$  matrices  $A$ ,  $B$ , and  $C$ . (Thus, this distance measure satisfies the *triangle inequality*.)
  - (b) (5 points) Show that for any  $A$  and identity matrix  $I$ ,  $\|A \otimes I\| = \|A\|$ .
  - (c) (5 points) Show that, for any two  $N \times N$  unitary matrices  $U_1$  and  $U_2$ , and any matrix  $A$ ,  $\|U_1 A U_2\| = \|A\|$ .

Note: more generally we can define  $p$ -norms for  $v \in \mathbb{C}^N$  as  $\|v\|_p := (\sum_i |v_i|^p)^{1/p}$  for  $1 \leq p < \infty$  and  $\|v\|_\infty := \max_i \{|v_i|\}$ . The Euclidean distance is then the special case  $\|\cdot\|_2$ . These vector norms give rise to *induced norms* on matrices  $M \in \mathbb{C}^{N \times N}$  by  $\|M\|_p := \sup\{\|Mv\|_p : v \in \mathbb{C}^N, \|v\|_p = 1\}$ . Therefore the spectral norm is the induced Euclidean ( $p = 2$ ) norm.

2. (Quantum Fourier Transform)

- (a) (12 points) Let  $F_N$  denote the  $N$ -dimensional Fourier transform

$$F_N := \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)^2} \end{pmatrix}, \text{ where } \omega_N := e^{2\pi i/N} (i = \sqrt{-1})$$

(an  $N \times N$  matrix, with entry  $\frac{1}{\sqrt{N}}e^{(2\pi i/N)jk}$  position  $j, k$  for  $j, k \in \{0, 1, \dots, N-1\}$ ).

- i) Show that all rows in  $F_N$  are vectors of length 1, and any two rows are orthogonal.
  - ii) What is  $F_N^2$ ? (Hint: The matrix has a very simple form.)
  - iii) What is the minimum  $j$  such that  $F_N^j = I$  is the identity?
- (b) (5 points) In class, we computed the QFT modulo  $N = 2^n$  by a quantum circuit of size  $O(n^2)$ . Recall that it uses gates of the form

$$R_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i/2^k} \end{pmatrix}$$

for  $k \in \{2, \dots, n\}$ . Show that  $\|R_k - I\| \leq 2\pi/2^k$ , where  $I$  is the  $4 \times 4$  identity matrix. (Thus,  $R_k$  gets very close to  $I$  when  $k$  increases.)

- (c) (8 points) Here we compute an *approximation* of this QFT within  $\varepsilon$  by a quantum circuit of size  $O(n \log(n/\varepsilon))$ . The idea is to start with the  $O(n^2)$  circuit and then remove some of its  $R_k$  gates (it is equivalent to changing the  $R_k$  gate to identity gate). Removing an  $R_k$  gate makes the circuit smaller but also changes the unitary transformation, but if  $k$  is large then from above we can deduce that removing a  $P_k$  gate changes the unitary transformation by only a small amount. Show how to use this approach to obtain a quantum circuit of size  $O(n \log(n/\varepsilon))$  that computes a unitary transformation  $\tilde{F}_N$  such that  $\|\tilde{F}_N F_N\| \leq \varepsilon$ . (Hint: Try removing all  $R_k$  gates where  $k \geq t$ , for some carefully chosen threshold  $t$ . The properties of our distance measure from the previous question should be useful for your analysis here.)
3. (Square root of a quantum operation) Let  $U$  be a unitary quantum circuit on  $n$  qubits. In this problem, we want to construct another circuit that computes a square root of  $U$  (i.e., a unitary  $V$  such that  $V^2 = U$ ).
- (a) (5 points) Let  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  be the matrix for a 1-qubit NOT gate. Find a  $2 \times 2$  matrix  $V$  (i.e., 1-qubit gate) such that  $V^2 = X$ .
  - (b) (5 points) Suppose we construct  $V$  by simply taking the square root of each gate  $U_i$  in circuit  $U$ , does this work, i.e. is  $V^2 = U$ ? Justify your answer.
  - (c) (20 points) We explore a strategy of implementing  $V$  using the *phase estimation* algorithm. Suppose  $U$  is constituted by  $s$  two-qubit gates. We study a simple case here. Let  $\{|\psi_x\rangle : x \in \{0, \dots, 2^n - 1\}\}$  be a set of orthonormal eigenvectors of  $U$  with eigenvalues in  $\{\pm 1, \pm i\}$ . Namely  $U|\psi_x\rangle = i^{\phi_x}|\psi_x\rangle$  with  $\phi_x \in \{0, 1, 2, 3\}$ . We outline a construction of  $V$  as follows such that  $V|\psi_x\rangle = \omega^{\phi_x}|\psi_x\rangle$  where  $\omega = e^{2\pi i/8}$ .

- Construct a generalized-control- $U$ , with two control-qubits, i.e.,  $|ab\rangle \otimes |c\rangle \mapsto |ab\rangle \otimes U^{ab}|c\rangle$  (applying  $U$  iff. both  $a = 1$  and  $b = 1$ ).
- Then apply the phase-estimation algorithm to this controlled- $U$  gate, which results in a circuit that computes  $ab = \phi_x$ , in two ancillary qubits for any input  $|\psi_x\rangle$ .
- Apply gates to those two ancillary qubits to induce the mapping  $|ab\rangle \mapsto \omega^{ab}|ab\rangle$ .
- Then apply the inverse of the phase-estimation circuit.

Answer the following questions:

- i) Explain how to construct a circuit computing the two-qubit controlled- $U$  operation using 3s 3-qubit gates.
- ii) Explain how to construct a circuit computing  $ab = \phi_x$  on input  $|\phi_x\rangle$  using 3s 3-qubit gates, one 2-qubit gate, and four 1-qubit gates.
- iii) Give a quantum circuit consisting of two 1-qubit gates that maps each basis state  $|ab\rangle$  to  $\omega^{ab}|ab\rangle$ .
- iv) Verify that the construction  $V$  is correct, i.e.,  $V|\psi_x\rangle = \omega^{\phi_x}|\psi_x\rangle$ . Explain why this implies  $V^2 = U$ , namely  $V$  computes the square root of  $U$  on any input state  $|\psi\rangle$ .

Note: the total gate cost is 6s 3-qubit gates plus two 2-qubit gates plus eight 1-qubit gates. This can be converted into a circuit consisting of  $O(s)$  2-qubit gates, not much more than the original circuit.