F, 10/18/19

Fall'19 CSCE 629

Analysis of Algorithms

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Lecture 19

- Network flow

Credit: based on slides by A. Smith & K. Wayne

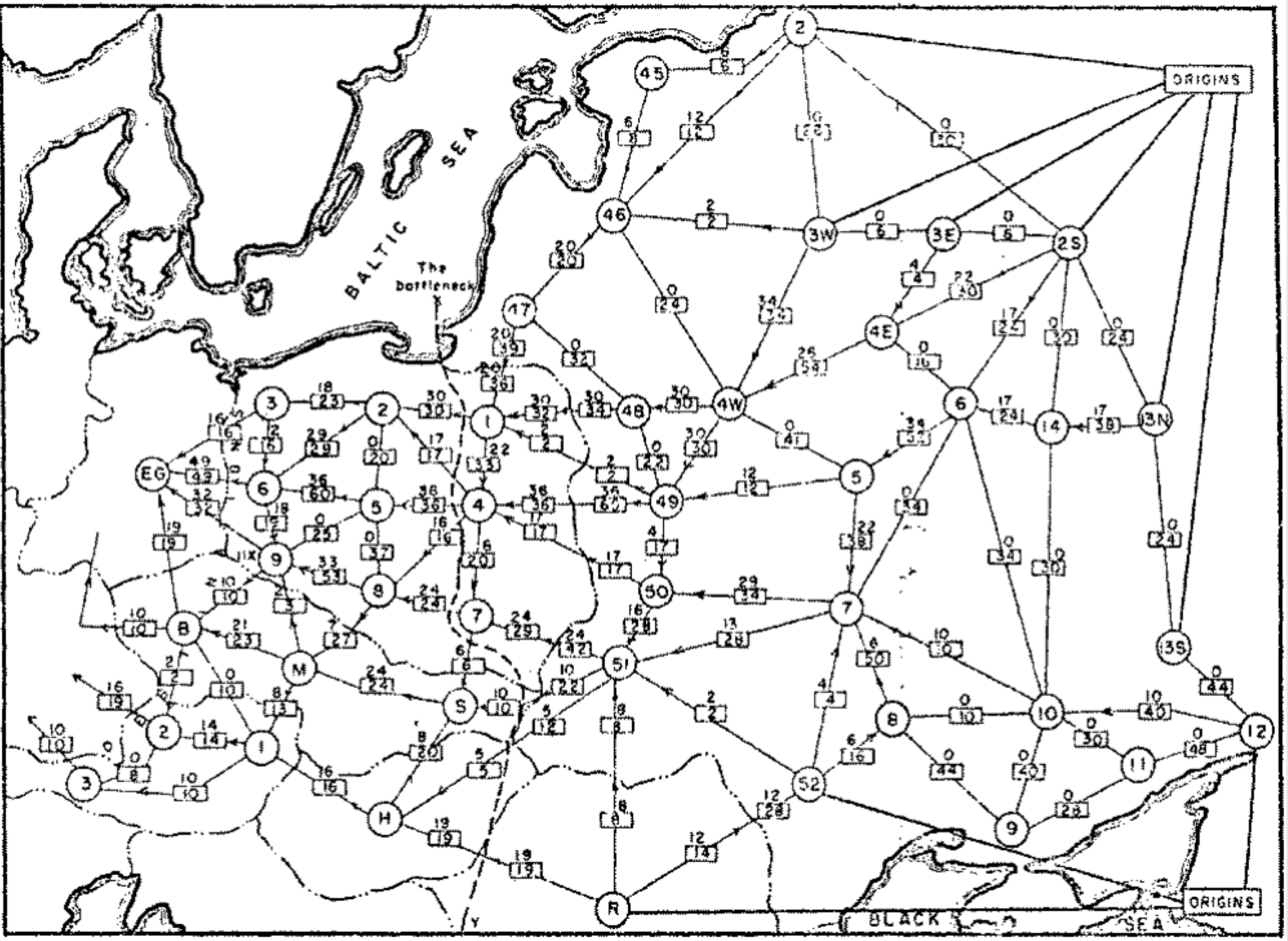


Figure 2

From Harris and Ross [1955]: Schematic diagram of the railway network of the Western Soviet Union and Eastern European countries, with a maximum flow of value 163,000 tons from Russia to Eastern Europe, and a cut of capacity 163,000 tons indicated as "The bottleneck".

Soviet Rail Network 1955

1. What is the maximum amount of stuff that could be moved from USSR into Europe?

2. What is the cheapest way to disrupt the network by blowing up train tracks (i.e., "the bottleneck")?

Schrijver, Alexander. "On the history of the transportation and maximum flow problems." Mathematical Programming 91.3 (2002): 437-445.

Maximum flow and minimum cut

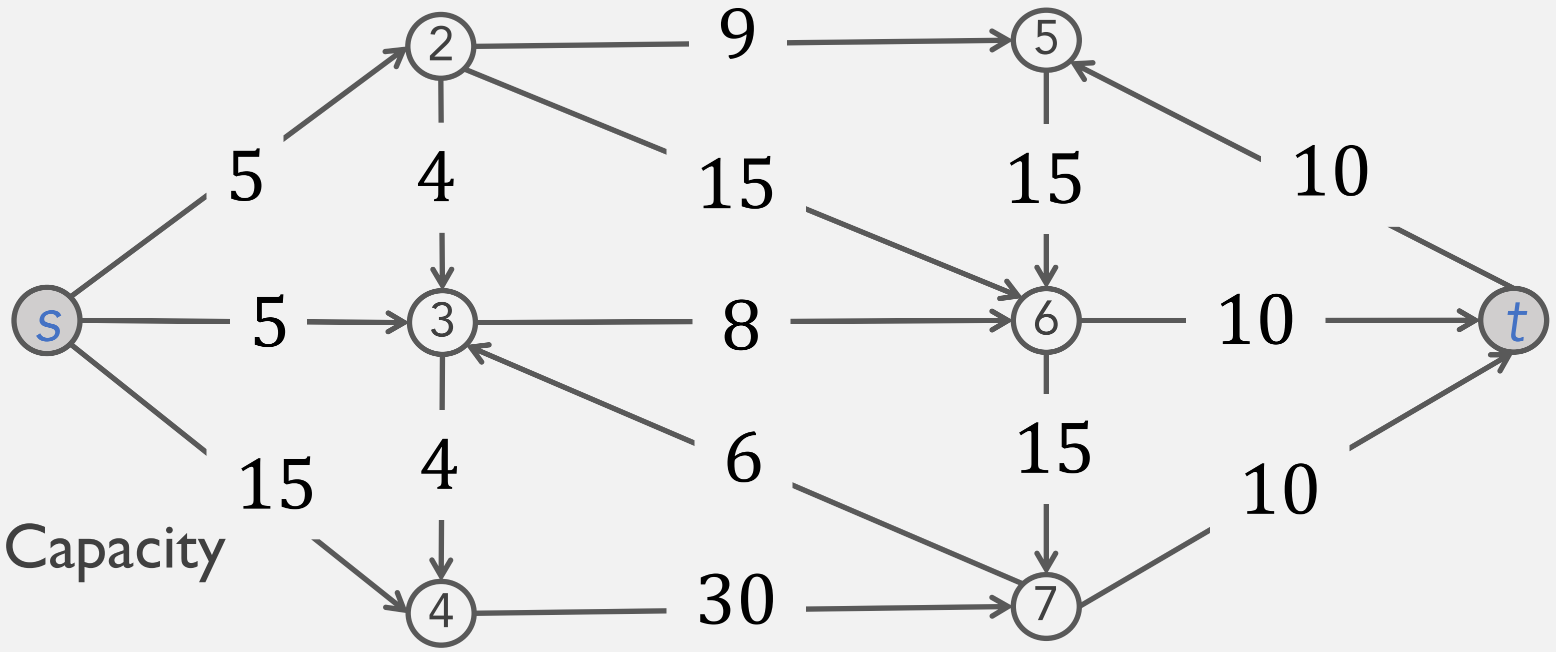
- Max flow and min cut
  - Two very rich algorithmic problems
  - Cornerstone in combinatorial optimization
  - Beautiful mathematical duality

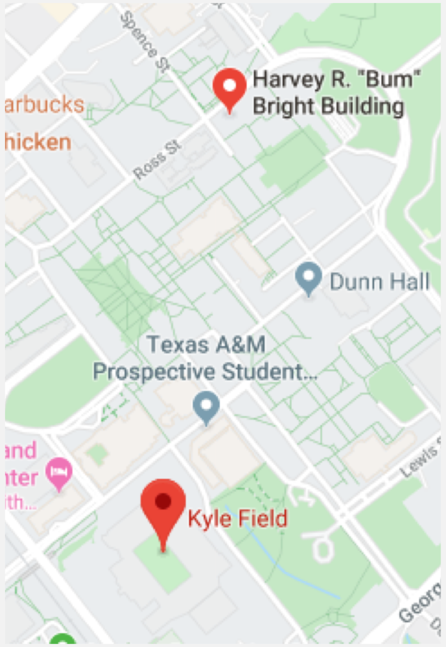
- Applications (by reductions)
  - Data mining
  - Airline scheduling
  - Bipartite matching, stable matching
  - Image segmentation, clustering, multi-camera scene reconstruction.
  - Network intrusion detection, Data privacy

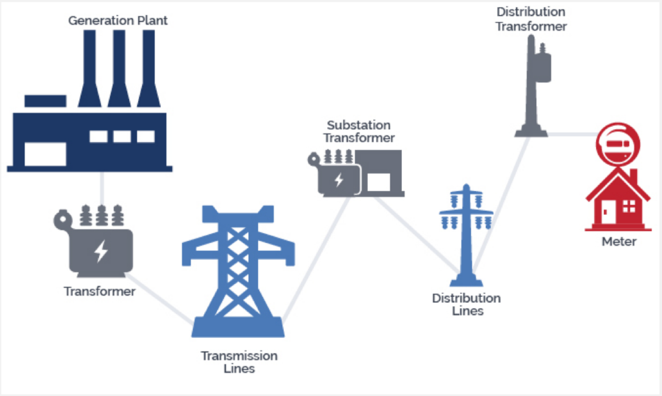
Flow network

Abstraction for material flowing through the edges

- $G = (V, E)$ directed graph, no parallel edges
- Two distinguished nodes: $s = source, t = sink$
- $\forall e \in E, c(e)$: capacity of edge e





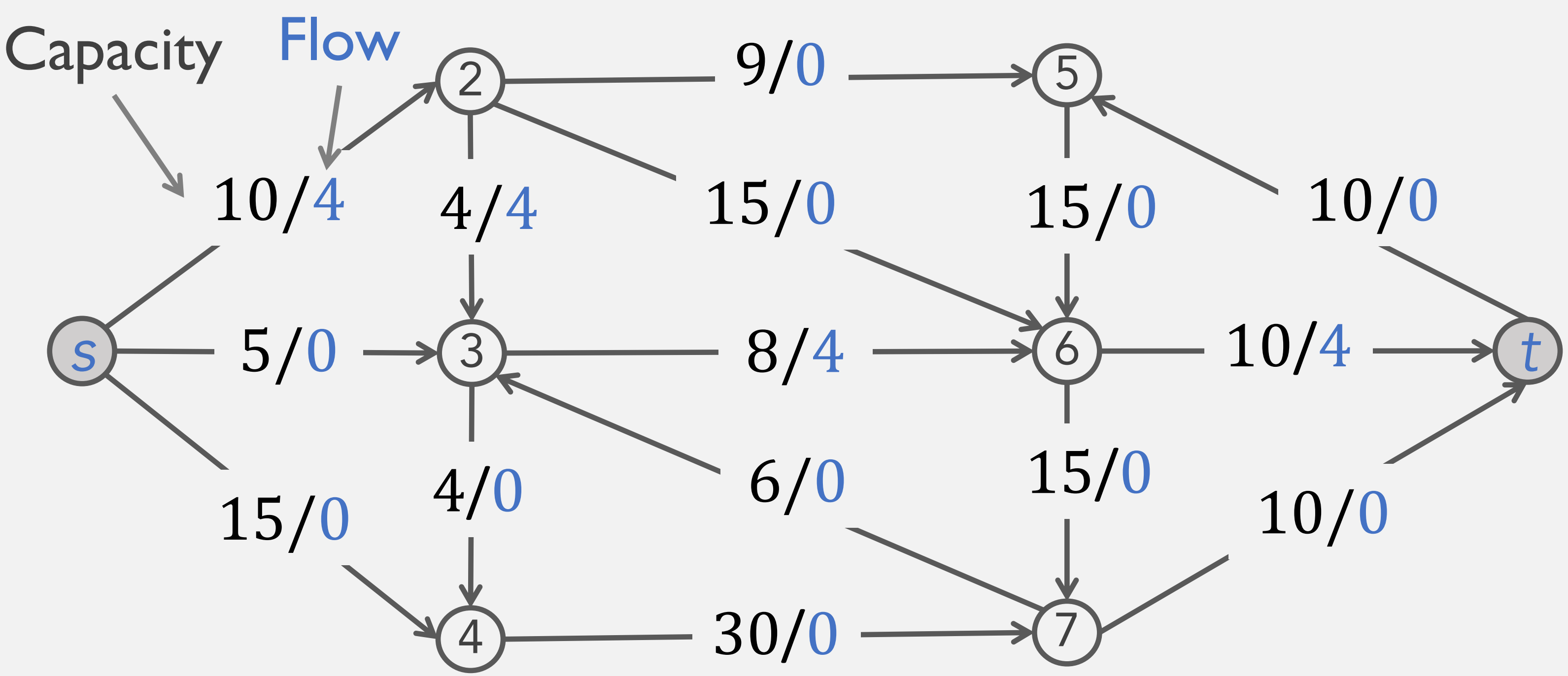


Flows

Def. An s–t **flow** is a function $f : E \to \mathbb{R}$ satisfying

- [Capacity] $\forall e \in E : 0 \le f(e) \le c(e)$
- [Conservation] $\forall v \in V \setminus \{s,t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

Def. The **value** of a flow f is $v(f) := \sum_{e \text{ out of } s} f(e)$



Value $v(f)$
$= 4 + 0 + 0 = 4$

Flows

Def. An $s{-}t$ **flow** is a function $f: E \to \mathbb{R}^+$ satisfying

- [Capacity] $\forall e \in E: 0 \leq f(e) \leq c(e)$
- [Conservation] $\forall v \in V \setminus \{s, t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

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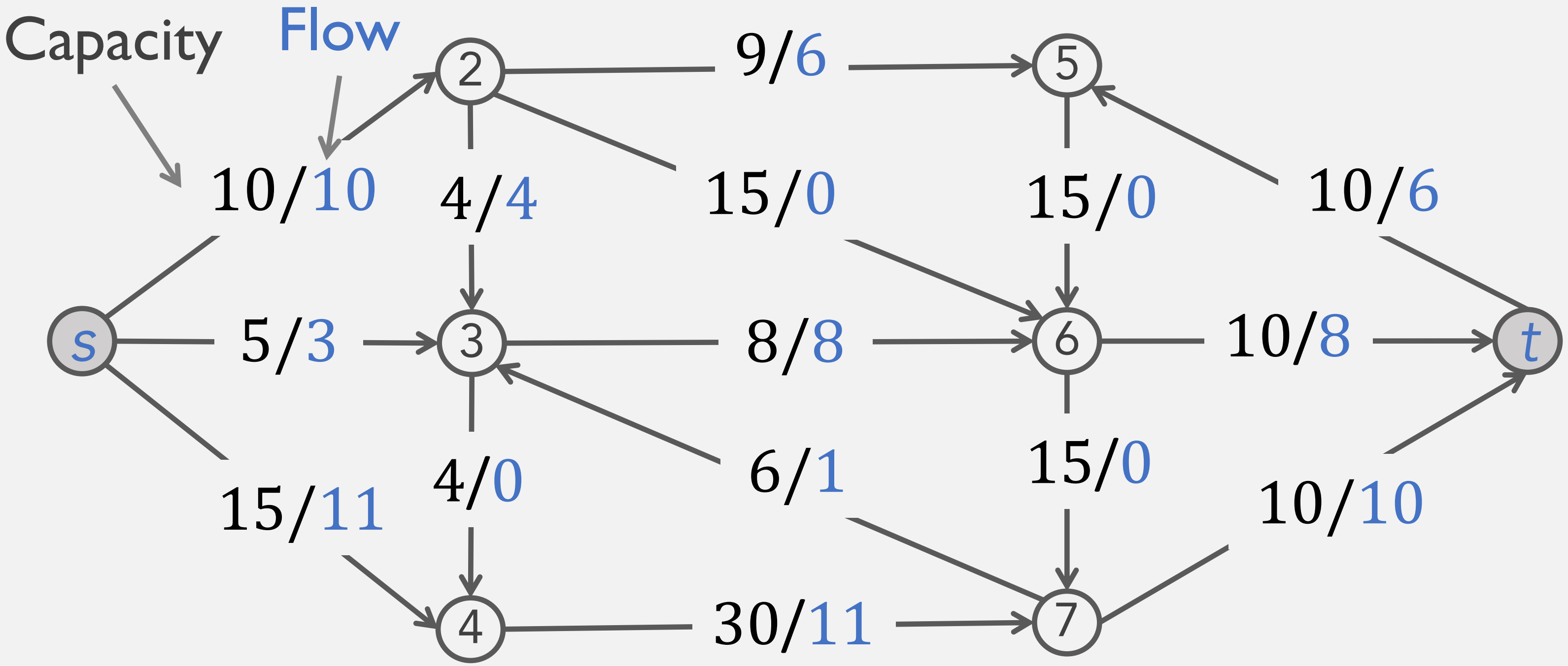
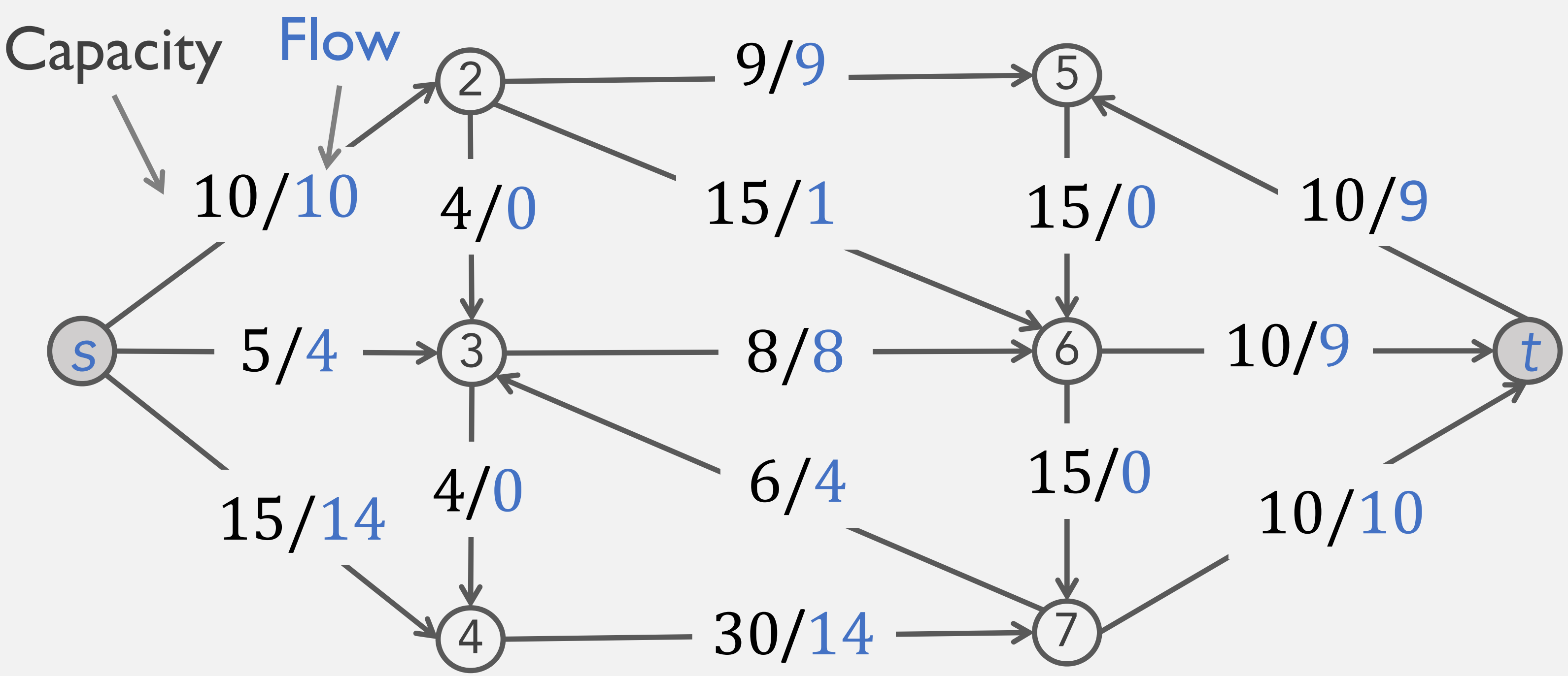


Value $v(f)$
$= 10 + 3 + 11 = 24$

Maximum Flow Problem

Max flow problem: Find s-t flow of maximum value.

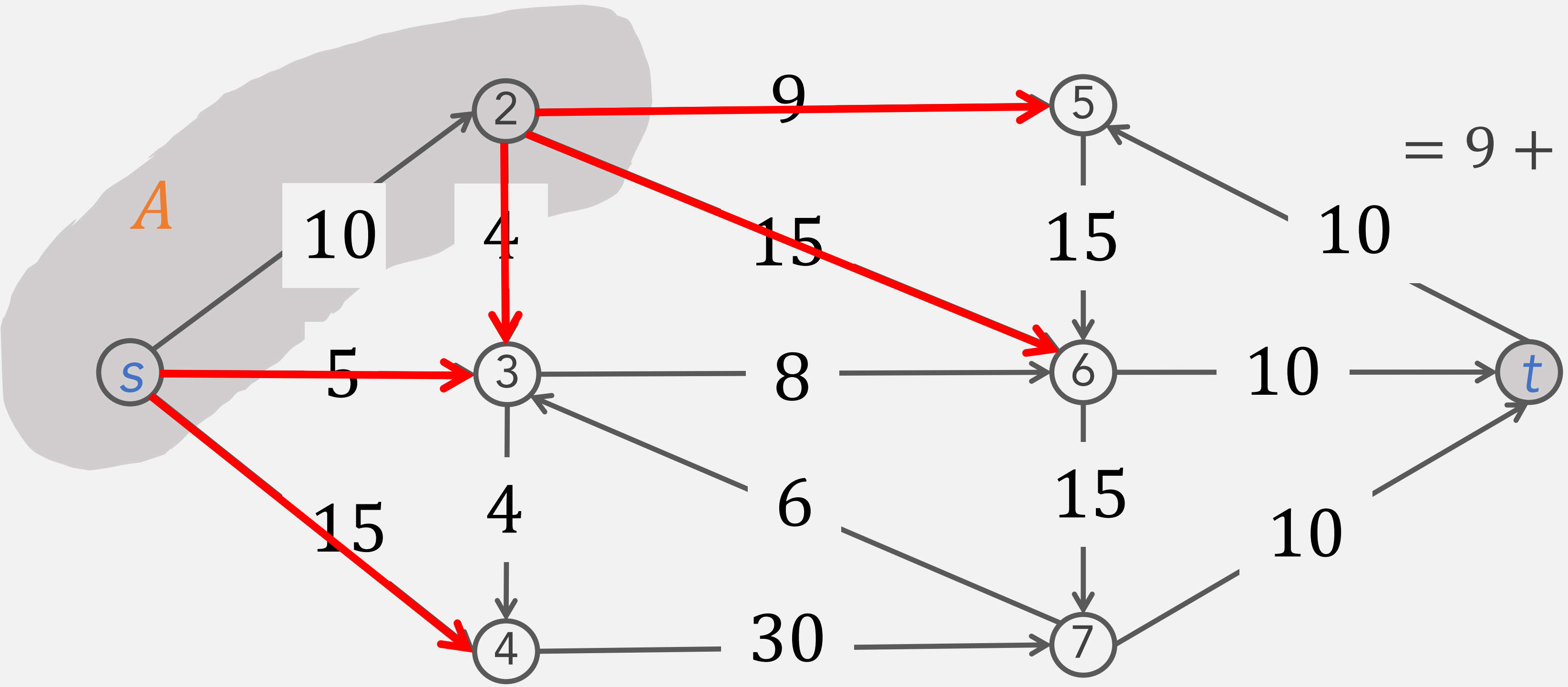
- NB. It has to be a valid flow, i.e., satisfying the two constraints



Max flow $v(f) = 28$

Cuts

- Recall: a cut is a subset of nodes
- Def. s-t cut: $(A, B \coloneqq V \backslash A)$ partition of V with $s \in A$ & $t \in B$
- Def. Capacity of cut (A, B): $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



$cap(A, B)$
$= 9 + 15 + 4 + 5 + 15 = 48$

Cuts

- Recall: a cut is a subset of nodes
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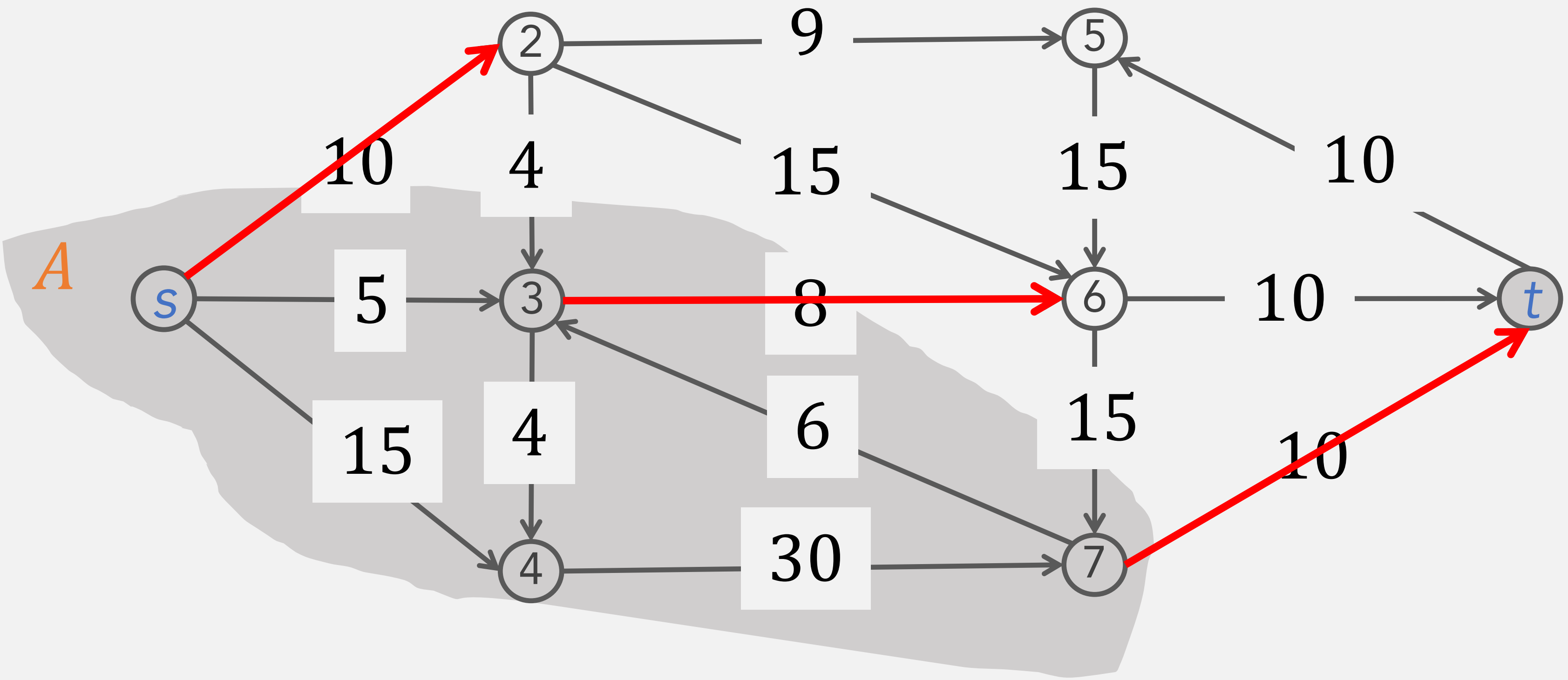
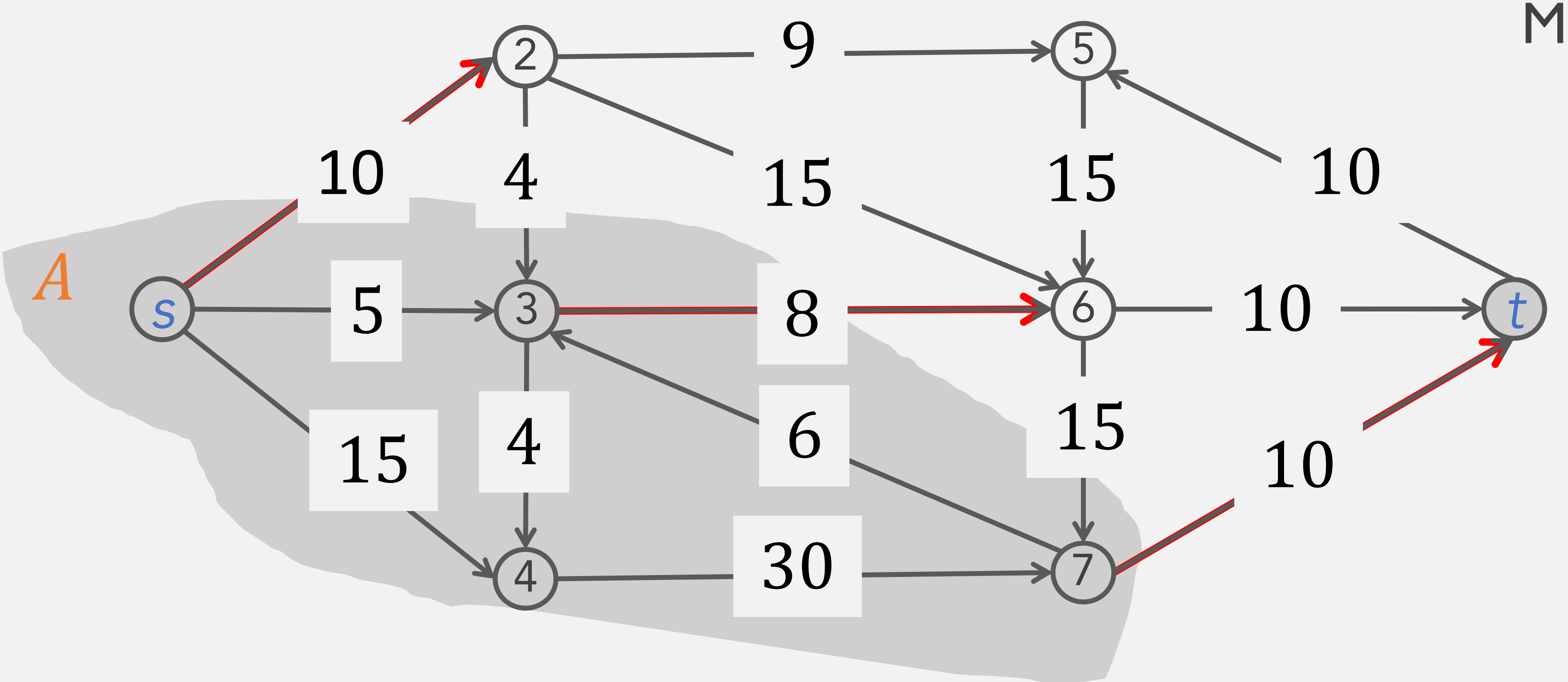
- Def. Capacity of cut (A, B): $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



$$cap(A, B)$$
$$= 10 + 8 + 10 = 28$$

Minimum Cut Problem

Min cut problem: Find s-t cut of minimum capacity value.



Min cut $cap(A, B) = 28$

Max flow

Min cut

How do they relate?

Flow value lemma

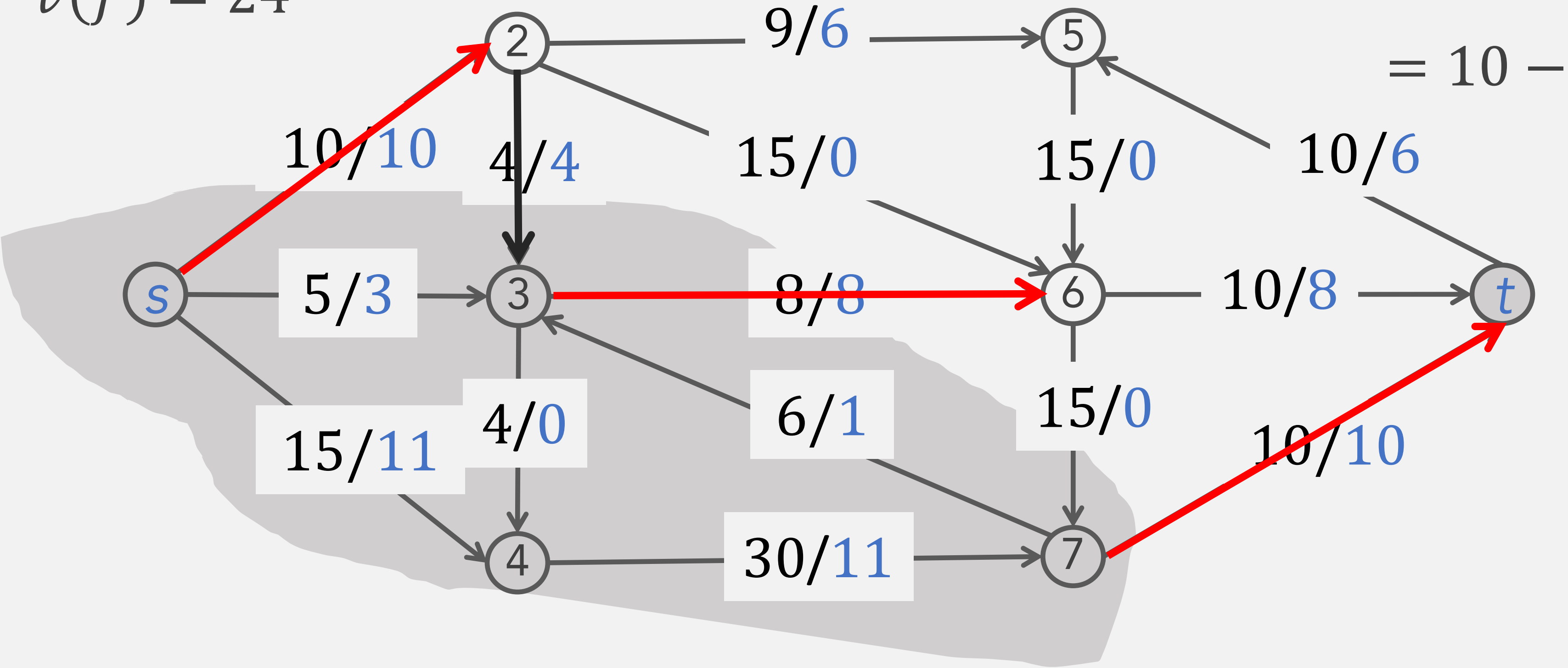
Flow-value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the net flow across the cut is equal to the amount leaving s (i.e., value of flow).

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

$v(f) = 24$

Net flow

$= 10 - 4 + 8 - 0 + 10 = 24$



Flow value lemma: proof

Flow-value lemma. Let f be any flow, and let (A, B) be any s-t cut.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

Proof.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$ // definition

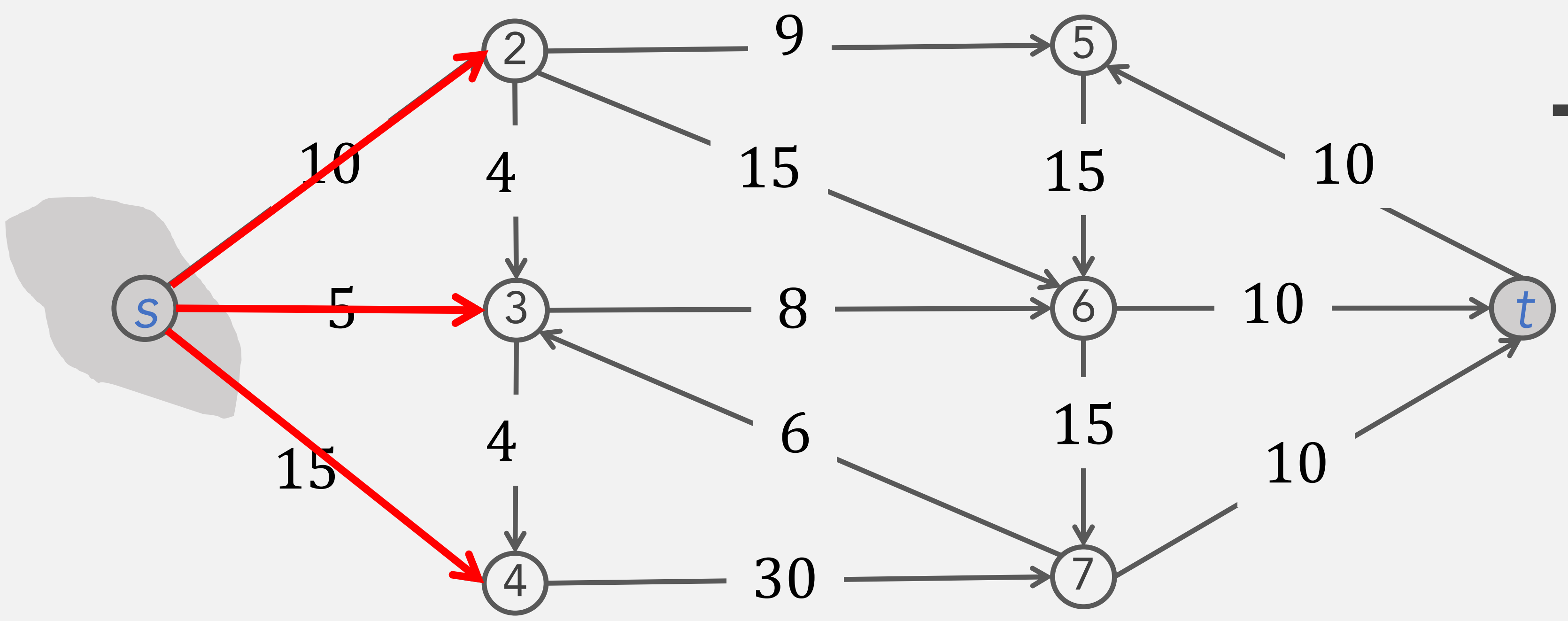
$$= \sum_{v \in A} \left(\sum_{e \text{ out of } s} f(e) - \sum_{e \text{ out of } s} f(e) \right)$$ // all but $v = s$ are 0 by conservation

$$= \sum_{e \text{ outof } A} f(e) - \sum_{e \text{ into } A} f(e))$$

Weak duality

Weak duality. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

$$v(f) \leq cap(A, B)$$



Capacity $= 30$

➔ Any flow, value ≤ 30

Weak duality: proof

Weak duality. Let f be any flow, and let (A, B) be any s-t cut.

$$v(f) \leq cap(A, B)$$

Proof.

$$v(f) = \sum_{e \text{ outof } A} f(e) - \sum_{e \text{ into } A} f(e)$$ // flow value lemma

$$\leq \sum_{e \text{ outof } A} f(e)$$

$$\leq \sum_{e \text{ outof } A} c(e)$$ // capacity constraint

$$= cap(A, B)$$ // definition of capacity