Low-lying collective excited states in non-integrable pairing models based on stationary phase approximation to the path integral

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For a description of large amplitude collective motion associated with nuclear pairing, requantization of time-dependent mean-field dynamics is performed using the stationary phase approximation (SPA) to the path integral. A disadvantage of the SPA is that it is applicable only to integrable systems. We overcome this difficulty by developing a requantization approach combining the SPA with the adiabatic self-consistent collective coordinate method (ASCC+SPA). We apply the theoretical framework of ASCC+SPA to a multi-level pairing models, which is non-integrable systems, to study the nuclear pairing dynamics. The ASCC+SPA gives a reasonable description of low-lying excited 0^+ states in non-integrable pairing systems.

I. INTRODUCTION

Pairing correlation plays an important role in open-shell nuclei. Effect of the pairing is prominent in many observables for the ground states, such as odd-even mass difference, moment of inertia of rotational bands, and common quantum number $J^\pi = 0^+$ for even-even nuclei [1]. Elementary excitations associated with the pairing, pairing vibrations and pairing rotations, have been observed in a number of nuclei [2]. Most of these states, "excited" from neghboring even-even systems, are associated with the ground $J^\pi = 0^+$ states in even-even nuclei. In contrast, properties of excited $J^\pi = 0^+$ states are not clearly understood yet [3, 4], for which the pairing dynamics plays an important role in a low-energy region (a few MeV excitation) of nuclei. In this paper, we aim to understand the dynamics from pairing correlation in nuclei from microscopic view point.

The time-dependent mean-field (TDMF) theory is a standard theory to describe the dynamics of nuclei from the microscopic degrees of freedom. Inclusion of the pairing dynamcs leads to the time-dependent Hartree-Fock-Bogoliubov (TDHFB) theory, which has been utilized for a number of studies of nuclear reaction and structure [5]. The small-amplitude approximation of TDHFB with modern energy density functionals, quasiparticle random phase approximation (QRPA), has successfully reproduced properties of giant resonances in nuclei. In contrast, the QRPA description of low-lying quadrupole vibrations is not as good as the giant resonances [5]. A large-amplitude nature of the quantum shape fluctuation is supposed to be important for these low-lying collective states. Five-dimensional collective Hamiltonian (5DCH) approaches have been developed for studies of low-lying quadrupole states, in which the collective Hamiltonian is constructed from microscopic degrees of freedom using the mean-field calculation and the cranking inertia formula [**?**]. The 5DCH model is able to take into account fluctuations of the quadrupole shape degrees of freedom which are important in many nuclear low-energy phenomena, such as shape coexistence and anharmonic quadrupole vibration. However, the calculated inertia is often too small to reproduce experimental data, due to the lack of time-odd components in the cranking formula [1]. This deficiency can be remedied in the adiabatic self-consistent collective coordinate (ASCC) method [6], in which the time-odd effect is properly treated. In addition, the ASCC method enables us to identify a collective subspace of interest. The ASCC was developed from the basic idea of self-consistent collective coordinate method (SCC) by Marumori and coworkers [7]. It has been applied to the nuclear quadrupole dynamics including the shape coexistence [8–10].

TDMF (TDHFB) theory corresponds to an SPA solution in the path integral formulation [11]. It lacks a part of quantum fluctuation important in the large amplitude dynamics. To introduce the quantum fluctuation based on the TDHFB theory, the requantization is necessary [11–16]. A simple and straightforward way of requantization is the canonical quantization. This is extensively utilized for collective models in nuclear physics. For instance, the canonical quantization of the 5DCH was employed for the study of low-lying excited states in nuclei [**?**]. The similar quantization was also utilized for the pairing collective Hamiltonian [17–20]. In our previous work [21], we studied various requantization methods for the two-level pair model, to investigate low-lying excited 0^+ states. Since the collectivity is rather low in the pairing motion in nuclei, the canonical quantization often fails to produce an approximate answer to the exact solution. In contrast, the stationary phase approximation (SPA) to the path integral [22] can give quantitative results not only for the excitation energies, but also for the wave functions and two-particle-transfer strengths. The quantized states obtained in the SPA has two advantages: First, the wave functions are given directly in terms of the microscopic degrees of freedom. Second, the restoration of broken symmetries are automatic. In the pair model, the quantized states are eigenstates of the particle number operator. On the other hand, applications of the SPA has been limited to integrable systems. This is because we need to find separable periodic trajectories on a classical torus. Since the nuclear systems, of course, correspond to non-integrable systems, a straightforward application of the SPA is not possible.

In this paper, we propose a new approach of SPA applicable to the non-integrable systems, which is based on the extraction of the one-dimensional (1D) collective coordinate using the ASCC method. Since the 1D system is integrable, the collective subspace can be quantized with the SPA. The optimal degree of freedom associated with a slow collective motion is determined self-consistently inside the TDHFB space, without any assumption. Thus, our approach of ASCC+SPA to the pairing model basically consists of two steps: (1) Find a decoupled 1D collective coordinate of the pair vibration, in addition to the pair rotational degrees of freedom. (2) Apply the SPA separately to each collective mode.

The paper is organized as follows. Sec. II introduces the theoretical framework of ASCC, SPA, and their combination, ASCC+SPA. In Sec. III, we provide some details in the application of ASCC+SPA to the multi-level pairing model. We give the numerical results in Sec. IV, including neutron pair vibrations in Pb isotopes. The conclusion and future perspectives are given in Sec. V.

II. THEORETICAL FRAMEWORK

A. Adiabatic self-consistent collective coordinate method and 1D collective subspace

In this section, we first recapitulate the ASCC method to find a 1D collective coordinate, following the notation of Ref. [23].

As is seen in Sec. III, the TDHFB equations can be interpreted as the classical Hamilton's equations of motion with canonical variables $\{\xi^\alpha, \pi_\alpha\}$. Each point in the phase space (ξ^α, π_α) corresponds to a generalized Slater determinant (coherent state). Assuming slow collective motion, we expand the Hamiltonian $\mathcal{H}(\xi, \pi)$ with respect to momenta π up to second order. The TDHFB Hamiltonian is

$$\mathcal{H} = V(\xi) + \frac{1}{2}B^{\alpha\beta}(\xi)\pi_\alpha\pi_\beta \tag{II.1}$$

with the potential $V(\xi)$ and the reciprocal mass parameter $B^{\alpha\beta}(\xi)$ defined by

$$V(\xi) = \mathcal{H}(\xi, \pi = 0), \tag{II.2}$$

$$B^{\alpha\beta}(\xi) = \left.\frac{\partial^2\mathcal{H}(\xi,\pi)}{\partial\pi_\alpha\partial\pi_\beta}\right|_{\pi=0}. \tag{II.3}$$

For multi-level pairing models in Sec. III, there is a constant of motion in the TDHFB dynamics, namely the average particle number $q^n \equiv \langle N\rangle/2$. Since the particle number N is time-even Hermitian operator, we treat this as a coordinate, and its conjugate gauge angle, p_n, as a momentum. Since q^n is a constant of motion, the Hamiltonian does not depend on p_n. On the other hand, the gauge angle p_n changes in time, which corresponds to the pair rotation, a Nambu-Goldstone (NG) mode associated with the breaking of the gauge (particle-number) symmetry. We assume the existence of 2D collective subspace Σ_2 (4D phase space), described by a set of canonical variable $(q^1, q^2; p_1, p_2)$, which is well decoupled from the rest of degrees of freedom, $\{q^a, p_a\}$ with $a = 3, \cdots$. The collective Hamiltonian is given by imposing $q^a = p_a = 0$, namely by restricing the space into the collective subspace.

$$\mathcal{H}_{\rm coll}(q, p; q^n) = \bar{V}(q^1; q^n) + \frac{1}{2}\bar{B}^{11}(q^1, q^n)p_1^2. \tag{II.4}$$

Since there exist two conserved quantities, q^n and $\mathcal{H}_{\rm coll}$, this 2D system is integrable. We can treat the collective motion of (q^1, p_1) separately from the pair rotation (q^n, p_n).

In the collective Hamiltonian (II.4), the variable q^n is trivially given as the particle number, which is expanded up to the second order in momenta π,

$$q^n = \frac{\langle N\rangle}{2} = f^n(\xi) + \frac{1}{2}f^{(1)n\alpha\beta}\pi_\alpha\pi_\beta. \tag{II.5}$$

To obtain the non-trivial collective variables (q^1, p_1), we assume the point transformation*,

$$q^1 = f^1(\xi), \tag{II.6}$$

and ξ^α on the subspace Σ_2 is given as $\xi^\alpha = g^\alpha(q^1, q_n, q^a = 0)$. The momenta on Σ_2 are transformed as

$$p_1 = g^\alpha_{,1}\pi_\alpha, \quad p_n = g^\alpha_{,n}\pi_\alpha \tag{II.7}$$

$$\pi_\alpha = f^1_{,\alpha}p_1 + f^n_{,\alpha}p_n, \tag{II.8}$$

where the comma indicates the partial derivative ($f^1_{,\alpha} = \partial f^1/\partial\xi^\alpha$). The Einstein's convention for summation with respect to repeated upper and lower indices is assumed hereafter. The canonical variable condition leads to

$$f^i_{,\alpha}g^\alpha_{,j} = \delta^i_j, \tag{II.9}$$

where $i, j = 1$ and n. The collective potential $\bar{V}(q^1, q^n)$ and the collective mass parameter $\bar{B}_{11}(q^1, q^n) = (\bar{B}^{11}(q^1, q^n))^{-1}$ can be given by

$$\bar{V}(q^1, q^n) = V(\xi = g(q^1, q^n, q^a = 0)) \tag{II.10}$$

$$\bar{B}^{11}(q^1, q^n) = f^1_{,\alpha}\tilde{B}^{\alpha\beta}(\xi)f^1_{,\beta}, \tag{II.11}$$

where $\tilde{B}^{\alpha\beta}$ are defined as

$$\tilde{B}^{\alpha\beta}(\xi) = B^{\alpha\beta}(\xi) - \bar{V}_{,n}f^{(1)n\alpha\beta}(\xi). \tag{II.12}$$

Decoupling conditions for the collective subspace Σ_2 lead to the basic equations of ASCC method [6, 23], which determine tangential vectors, $f^1_{,\alpha}(\xi)$ and $g^\alpha_{,1}(q)$.

$$\delta H_M(\xi, \pi) = 0 \tag{II.13}$$

$$\mathcal{M}^\beta_\alpha f^1_{,\beta} = \omega^2 f^1_{,\alpha}, \qquad \mathcal{M}^\beta_\alpha g^\alpha_{,1} = \omega^2 g^\beta_{,1}. \tag{II.14}$$

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* We may lift the restriction to the point transformation, as Eq. (II.5) [**?**]. In this paper, we neglect these higher-order terms, such as $f^{(1)1\alpha\beta}\pi_\alpha\pi_\beta/2$.

The first equation (II.13) is called moving-frame Hartree-Fock-Bogoliubov (HFB) equation. The moving-frame Hamiltonian $\mathcal{H}_M$ is

$$\mathcal{H}_M(\xi,\pi) = \mathcal{H}(\xi,\pi) - \lambda_1 q^1(\xi) - \lambda_n q^n(\xi,\pi). \quad \text{(II.15)}$$

The second equation (II.14) is called moving-frame QRPA equation. The matrix $\mathcal{M}^\beta_\alpha$ in the moving-frame QRPA equation (II.14) can be rewritten as

$$\mathcal{M}^\beta_\alpha = \tilde{B}^{\beta\gamma}\left(V_{,\gamma\alpha} - \lambda_n f^n_{,\gamma\alpha}\right) + \frac{1}{2}\tilde{B}^{\beta\gamma}_{,\alpha} V_{,\gamma}. \quad \text{(II.16)}$$

The NG mode, $f^n_{,\alpha}$ and $g^\alpha_{,n}$, corresponds to the zero mode with $\omega^2 = 0$. Therefore, the collective mode of our interest corresponds to the mode with the lowest frequency except for the zero mode.

In practice, we obtain the collective path according to the following procedure:

1. Find the HFB minimum point ξ^α_n $(n = 0)$ by solving Eq. (II.13) with $\lambda_1 = 0$. Let us assume that this corresponds to $q^1_n = 0$.

2. Diagonalize the matrix $\mathcal{M}^\beta_\alpha$ to solve Eq. (II.14) using Eq. (II.16).

3. Move to the next meighboring point $\xi^\alpha_{n+1} = \xi^\alpha_n + d\xi^\alpha$ with $d\xi^\alpha = g^\alpha_{,1} dq^1$. This corresponds to the collective coordinate, $q^1_{n+1} = q^1_n + dq^1$.

4. At ξ^α_{n+1} (q^1_{n+1}), obtain a self-consistent solution of Eqs. (II.14) and (II.13), to determine ξ^α_{n+1}, $f^1_{,\alpha}$, and $g^\alpha_{,1}$.

5. Go back to 3 to determine the next point on the collective path.

We repeat this procedure with $dq^1 > 0$ and $dq^1 < 0$, and construct the collective path. In Step 2 and 4, we choose a mode with the lowest frequency ω^2. Note that ω^2 can be negative. In Step 4, when we solve Eq. (II.13), we use a constraint on the magnitude of $dq^1 = f^1_{,\alpha} d\xi^\alpha$. Since the normalization of $f^1_{,\alpha}$ and $g^\alpha_{,1}$ is arbitrary as far as they satisfy Eq. (II.9). We fix this scale by an additional condition of $\bar{B}^{11}(q^1) = 1$.

B. Stationary-phase approximation to the path integral

For quantization of integrable systems, we can apply the stationary phase approximation (SPA) to the path integral. In our former study [21], we have proposed and tested the SPA for an integrable pairing model. Since the collective Hamiltonian (II.4), extracted from TDHFB degrees of freedom, is integrable, kthe SPA is applicable to it. In this manner, we may apply the ASCC+SPA to non-integrable systems in general.

1. Basic idea of ASCC+SPA

Since the Hamiltonian $\mathcal{H}_{\rm coll}$ of Eq. (II.4) is separable, it is easy to find periodic trajectories on invariant tori. Since the pairing rotation corresponds to the motion of p_n with a constant q^n, all we need to do is to find classical periodic trajectories C_k in the (q^1, p_1) space (with a fixed q^n) which satisfy the Einstein-Brillouin-Keller (EBK) quantization rule with a unit of $\hbar = 1$,

$$\oint_{C_k} p_1 dq^1 = 2\pi k, \quad \text{(II.17)}$$

where k is an integer number.

At each point in the space $(q^1, q_n; p_1, p_n)$ corresponds to a generalized Slater determinant $|q^1, q^n; p_1, p_n\rangle = |\xi, \pi\rangle$ where (ξ, π) are given as $\xi^\alpha = g^\alpha(q^1, q_n, q^a = 0)$ and $\pi_\alpha = f^1_{,\alpha} p_1 + f^n_{,\alpha} p_n$. According to the SPA, the k-th excited state $|\psi_k\rangle$ is constructed from the k-th periodic trajectory C_k, given by $(q^1(t), p_1(t))$, of the Hamiltonian $\mathcal{H}_{\rm coll}$.

$$|\psi_k\rangle = \oint dp_n \oint_{C_k} \rho(q,p) dt\, |q,p\rangle\, e^{i\mathcal{T}[q,p]}, \quad \text{(II.18)}$$

where (q,p) means $(q^1, q^n; p_1, p_n)$ and the weight function $\rho(q,p)$ is given through an invariant meaure $d\mu(q,p)$ as

$$d\mu(q,p) = \rho(q,p) dE dt dq^n dp_n. \quad \text{(II.19)}$$

The invariant measure $d\mu(q,p)$ is defined by the unity condition $\int d\mu(q,p)\, |q,p\rangle\, \langle q,p| = 1$. An explicit form of $d\mu(q,p)$ for the present pairing model is presented in Eq. (III.18). The action integral $\mathcal{T}$ is defined by

$$\mathcal{T}[q,p] \equiv \int_0^t \langle q(t'), p(t')| i\hbar \frac{\partial}{\partial t'} |q(t'), p(t')\rangle\, dt'. \quad \text{(II.20)}$$

The SPA quantization is able to provide a wave function $|\psi_k\rangle$ in microscopic degrees of freedom, which is given as a superposition of generalized Slater determinants $|q,p\rangle$. In addition, the integration with respect to p_n over a circuit on torus automatically recover the broken symmetry, namely the good particle number. However, it relies on the existence of invariant tori. In the present approach of ASCC+SPA, we first derive a decoupled collective subspace Σ_2 and identify canonical variables (q,p). Because of the cyclic nature of (q^n, p_n), it is basically a 1D system and becomes integrable. In other words, we perform the torus quantization on approximate tori in the TDHFB phase space (ξ, π), which is mapped from tori in the 2D collective subspace (q,p).

2. Notation and practical procedure for quantization

For application of the ASCC+SPA method to the pairing model in Sec. III, we summarize some notations and procedures to obtain quantized states.

In Sec. III, the time-dependent generalized Slater determinants (coherent states) are written as $|Z\rangle$ with complex variables $Z_\alpha(t)$. The variables Z_α are transformed

into real variables $(j^\alpha, -\chi_\alpha)$ that correspond to (ξ^α, π_α) in Sec. II. χ_α and j^α correspond to the "angle" and the "number" variables, respectively. Although it is customary to take the angle as a coordinate, since the angle χ_α is time-odd quantities, we switch the coordinates and the momenta with minus signs in front of variables χ. Similarly, the gauge angle Φ and the total particle number J correspond to variables of the pair rotation, $-p_n$ and q^n, respectively.

According to the EBK quantization rule (II.17), the ground state with $k = 0$ corresponds to nothing but the HFB state with a fixed particle number $J(= q^n)$. For the k-th excited states, we perform the following calculations:

1. Obtain the 1D collective subspace with canonical variables (q^1, p_1) according to the ASCC in Sec. II A.
2. Find a trajectory $(q^1(t), p_1(t))$ which satisfies the k-th EBK quantization condition (II.17).
3. Calculate the action integral (III.16) for the k-th trajectory.
4. Using Eq. (II.18), construct the k-th excited state.

The ASCC provides the 2D collective subspace (q^1, J) and the generalized coherent states $|\Phi = 0, J; q, p = 0\rangle$. For finite values of momenta, we use Eq. (II.8) to obtain the state $|\Phi, J; q, p\rangle$.

III. PAIRING MODEL

We study low-lying excited 0^+ states in multi-level pairing model by applying the ASCC+SPA. The Hamiltonian of the pairing model is given in terms of single-particle energies ϵ_l and the pairing strength g as

$$\begin{aligned} H &= \sum_\alpha \epsilon_\alpha n_\alpha - g \sum_{\alpha,\beta} S^+_\alpha S^-_\beta \\ &= \sum_\alpha \epsilon_\alpha (2S^0_\alpha + \Omega_\alpha) - g S^+ S^-, \end{aligned} \tag{III.1}$$

where we use the SU(2) quasi-spin operators, $\boldsymbol{S} = \sum_\alpha \boldsymbol{S}_\alpha$, with

$$S^0_\alpha = \frac{1}{2}(\sum_m a^\dagger_{j_\alpha m} a_{j_\alpha m} - \Omega_\alpha), \tag{III.2}$$

$$S^+_\alpha = \sum_{m>0} a^\dagger_{j_\alpha m} a^\dagger_{j_\alpha \overline{m}}, \quad S^-_\alpha = S^{+\dagger}_\alpha. \tag{III.3}$$

Each single-particle energy ϵ_α possesses $(2\Omega_\alpha)$-fold degeneracy ($\Omega_\alpha = j_\alpha + 1/2$) and $\sum_{m>0}$ indicates the summation over $m = 1/2, 3/2, \cdots$, and $\Omega_\alpha - 1/2$. The occupation number of each level α is given by $\hat{n}_\alpha = \sum_m a^\dagger_{j_\alpha m} a_{j_\alpha m} = 2S^0_\alpha + \Omega_\alpha$. The quasi-spin operators satisfy the commutation relations

$$[S^0_\alpha, S^\pm_\beta] = \pm \delta_{\alpha\beta} S^\pm_\alpha, \quad [S^+_\alpha, S^-_\beta] = 2\delta_{\alpha\beta} S^0_\alpha. \tag{III.4}$$

The magnitude of quasi-spin for each level is $S_\alpha = \frac{1}{2}(\Omega_\alpha - \nu_\alpha)$, where ν_α is the seniority quantum number, namely the number of unpaired particle at the level α. In the present study, we only consider seniority zero states with $\nu = \sum_\alpha \nu_\alpha = 0$. The residual two-body interaction only consists of monopole pairing interaction which couples two particles to zero angular momentum. We obtain exact solutions either by solving Richardson equation [24–26] or by diagonalizing the Hamiltonian using the quasi-spin symmetry.

A. Classical form of TDHFB Hamiltonian

The time-dependent coherent state for the seniority $\nu = 0$ states ($S_\alpha = \Omega_\alpha/2$) is constructed with time-dependent complex variables $Z_\alpha(t)$, as

$$|Z(t)\rangle = \prod_\alpha \left(1 + |Z_\alpha(t)|^2\right)^{-\Omega_\alpha/2} \exp[Z_\alpha(t) S^+_\alpha]\,|0\rangle \tag{III.5}$$

where $|0\rangle$ is the vacuum (zero particle) state. The TDHFB motion is given by the time dependence of $Z_\alpha(t)$. In the SU(2) quasi-spin representation, $|0\rangle = \prod_\alpha |S_\alpha, -S_\alpha\rangle$. The coherent state $|Z(t)\rangle$ is a superposition of states with different particle numbers without unpaired particles. In the present pairing model, the coherent state is the same as the time-dependent BCS wave function with $Z_\alpha(t) = v_\alpha(t)/u_\alpha(t)$, where $(u_\alpha(t), v_\alpha(t))$ are the time-dependent BCS u, v factors.

The TDHFB equation can be derived from the time-dependent variational principle ($\hbar = 1$), $\delta\mathcal{S} = 0$, where

$$\mathcal{S} \equiv \int \mathcal{L}(t) dt = \int \langle Z(t)| i\frac{\partial}{\partial t} - H|Z(t)\rangle\, dt. \tag{III.6}$$

After transformation of the complex into real variables, $Z_\alpha = \tan\frac{\theta_\alpha}{2} e^{-i\chi_\alpha}$ $(0 \le \theta \le \pi)$, the Lagrangian $\mathcal{L}$ and the expectation value of the Hamiltonian are written as

$$\mathcal{L}(t) = \sum_\alpha \frac{\Omega_\alpha}{2}(1 - \cos\theta_\alpha)\dot{\chi_\alpha} - \mathcal{H}(Z, Z^*), \tag{III.7}$$

with

$$\begin{aligned} \mathcal{H}(Z, Z^*) &\equiv \langle Z|H|Z\rangle \\ &= \sum_\alpha \epsilon_\alpha \Omega_\alpha (1 - \cos\theta_\alpha) \\ &\quad - \frac{g}{4}\sum_\alpha \Omega_\alpha [\Omega_\alpha(1 - \cos^2\theta_\alpha) + (1 - \cos\theta_\alpha)^2] \\ &\quad - \frac{g}{4}\sum_{\alpha\neq\beta} \Omega_\alpha \Omega_\beta \sqrt{(1 - \cos^2\theta_\alpha)(1 - \cos^2\theta_\beta)}\, e^{-i(\chi_\alpha - \chi_\beta)}. \end{aligned} \tag{III.8}$$

Choose χ_α as canonical coordinates, their conjugate momenta are given by

$$j^\alpha \equiv \frac{\partial \mathcal{L}}{\partial \dot{\chi}_\alpha} = \frac{\Omega_\alpha}{2}(1 - \cos\theta_\alpha). \tag{III.9}$$

χ_α represent a kind of gauge angle of each level, and j^α correspond to the occupation number of each level, $2j^\alpha = \langle Z|\,\hat{n}_\alpha\,|Z\rangle$. As we mention in Sec. II B 2, we switch the coordinates and momenta, $(\chi_\alpha, j^\alpha) \to (j^\alpha, -\chi_\alpha)$, to make the coordinates time even. The TDHFB equation is equivalent to the classical Hamilton's equation

$$-\dot{\chi_\alpha} = -\frac{\partial \mathcal{H}}{\partial j^\alpha}, \quad \dot{j^\alpha} = \frac{\partial \mathcal{H}}{\partial(-\chi_\alpha)}. \tag{III.10}$$

B. Application of ASCC

We construct a 2D collective subspace Σ_2 from ASCC theory. We expand the classical Hamiltonian up to second order with respect to momenta, $-\chi_\alpha$.

$$\mathcal{H}(j,\chi) \approx V(j) + \frac{1}{2}B^{\alpha\beta}(j)\chi_\alpha\chi_\beta, \tag{III.11}$$

where potential $V(j)$ and the reciprocal mass parameter $B^{\alpha\beta}(j)$ are given as

$$\begin{aligned} V(j) =& \mathcal{H}(j,\chi=0) \\ =& \sum_\alpha 2\epsilon_\alpha j^\alpha - g\sum_\alpha\left(\Omega_\alpha j^\alpha - (j^\alpha)^2 + \frac{(j^\alpha)^2}{\Omega_\alpha}\right) \\ & - g\sum_{\alpha\neq\beta}\sqrt{j^\alpha j^\beta(\Omega_\alpha - j^\alpha)(\Omega_\beta - j^\beta)} \end{aligned} \tag{III.12}$$

$$B^{\alpha\beta}(j) = \left.\frac{\partial^2\mathcal{H}}{\partial\chi_\alpha\partial\chi_\beta}\right|_{\chi=0} \tag{III.13}$$

$$= \begin{cases} 2g\sum_{\gamma\neq\alpha}\sqrt{j^\gamma j^\alpha(\Omega_\gamma - j^\gamma)(\Omega_\alpha - j^\alpha)} & \text{for } \alpha=\beta \\ -2g\sqrt{j^\alpha j^\beta(\Omega_\alpha - j^\alpha)(\Omega_\beta - j^\beta)} & \text{for } \alpha\neq\beta \end{cases}$$

We may apply the ASCC method in Sec. II A by regarding $\xi \to j$ and $\pi \to -\chi$.

The TDHFB conserves the average total particle number N. We adopt

$$J \equiv N/2 = \sum_\alpha j^\alpha, \tag{III.14}$$

as a coordinate q^n. Since this is explicitly given as the expectation value of the particle number operator, curvature quantities, such as $f^n_{,\alpha\beta}$ and $f^{(1)n\alpha\beta}$, are explicitly calculable. On the other hand, the gauge angle $\Phi = -p_n$ is not given a priori. Since the ASCC solution provides $g^\alpha_{,n}$ as an eigenvector of Eq. (II.14), we may construct it as Eq. (II.7) in the first order in $\pi = -\chi$. We confirm that the pairing rotation corresponds to an eigenvector of Eq. (II.14) with the zero frequency $\omega^2 = 0$.

In the present pairing model, from Eq. (III.15), we find J does not dependent on χ. This means $f^{(1)n\alpha\beta} = 0$ in Eq. (**??**), thus, $\tilde{B}^{\beta\gamma} = B^{\beta\gamma}$. The second derivative of J with respect to j also vanishes, which indicates $f^I_{,\gamma\beta}$ in Eq. (II.16) is zero. In the present model, it is easy to find an explicit expression for the gauge angle Φ.

$$\Phi = \frac{1}{L}\sum_\alpha \chi_\alpha, \tag{III.15}$$

where L is the number of available shells $\alpha = 1, \cdots, L$. Its conjugate variable $\partial\mathcal{L}/\partial\dot{\Phi}$ is indeed given by J of Eq. (III.14). Again, exchanging coordinate and momentum, we have $q^n = J$ and $p_n = -\Phi$.

C. Application of SPA

After deriving the collective subspace states Σ_2, we perform the quantization according to the SPA in Sec. II B. Calculating a trajectory in the (q^1, p_1) space, we can identify a series of states $\{|\Phi, J; q^1(t), p_1(t)\rangle\}$ on the trajectory, in the form of Eq. (III.5) with parameters Z_α given at (Φ, J, q^1, p_1) and $q^a = p_a = 0$ for $a \geq 3$. Since the variables(Φ, J) and (q^1, p_1) are separable, we may take closed trajectories independently in (Φ, J) and (q^1, p_1) sectors, which we denote here as C_Φ and C_1, respectively. The action integral is given by

$$\begin{aligned} \mathcal{T}(\Phi, J; q^1, p_1) =& \int_{C_\Phi}\langle\Phi(t), J; q^1, p_1|i\frac{\partial}{\partial t}|\Phi(t), J; q^1, p_1\rangle\, dt \\ &+ \int_{C_1}\langle\Phi, J; q^1(t), p_1(t)|i\frac{\partial}{\partial t}|\Phi, J; q^1(t), p_1(t)\rangle\, dt \\ =& J\Phi + \int_{C_1}\sum_\alpha j^\alpha d\chi_\alpha \\ \equiv& \mathcal{T}_\Phi(J,\Phi) + \mathcal{T}_1(q^1, p_1; J). \end{aligned} \tag{III.16}$$

In fact, the gauge-angle dependence is formally given as

$$|\Phi, J; q^1, p_1\rangle = e^{-i\Phi\hat{N}/2}\,|J; q^1, p_q\rangle\,. \tag{III.17}$$

Then, the action for the trajectory C_1 can be also expressed as $\mathcal{T}(q^1, p_1; J) = \int_{C_1}\langle J; q^1, p_1|i\frac{\partial}{\partial t}|J; q^1, p_1\rangle\, dt$.

In the su(2) representation, the invariant measure is

$$\begin{aligned} d\mu(Z) =& \prod_\alpha \frac{\Omega_\alpha + 1}{\pi}(1 + |Z_\alpha|^2)^{-2} d\mathrm{Re}Z d\mathrm{Im}Z \\ =& \prod_\alpha \frac{-(\Omega_\alpha + 1)}{4\pi} d\cos\theta_\alpha d\chi_\alpha \\ =& \prod_\alpha \frac{1 + \Omega_\alpha^{-1}}{2\pi} d\chi_\alpha dj^\alpha \\ =& \left[\prod_\alpha \frac{1 + \Omega_\alpha^{-1}}{2\pi}\right] d\Phi dJ dq^1 dp_1 \prod_a dq^a dp_a \end{aligned} \tag{III.18}$$

where (q^a, p_a) are the intrinsic canonical variables decoupled from the collective subspace Σ_2. For the last equation in Eq. (III.18), we used the invariance of the phase space volume element in canonical transformation. According to (III.18), the weight function $\rho(q,p)$ in Eq. (II.18) is just a constant, thus, treated as the normalization of the wave function.

The coherent state $|\Phi, J; q^1, p_1\rangle = |Z\rangle$ is expanded in the su(2) quasispin basis as

$$|Z\rangle = \sum_{\{m_\alpha\}} A_m(Z)\,|\cdots; S_\alpha, -S_\alpha + m_\alpha, \cdots\rangle\,, \tag{III.19}$$

where the summation is taken over all possible combinations of integer values of $\{m_\alpha\}$.

$$\begin{aligned} A_m(Z) =& \prod_\alpha \frac{Z_\alpha^{m_\alpha}}{(1+|Z_\alpha|^2)^{\Omega_\alpha/2}\, m_\alpha!} \sqrt{\frac{(\Omega_\alpha)! m_\alpha!}{(\Omega_\alpha - m_\alpha)!}} \\ =& \prod_\alpha \left(\frac{1-\cos\theta_\alpha}{2}\right)^{m_\alpha/2} \left(\frac{1+\cos\theta_\alpha}{2}\right)^{(\Omega_\alpha - m_\alpha)/2} \\ & \times \sqrt{\frac{(\Omega_\alpha)!}{m_\alpha!(\Omega_\alpha - m_\alpha)!}} e^{-i m_\alpha \chi_\alpha}, \end{aligned} \tag{III.20}$$

where the lower index m indicates a combination of $\{m_\alpha\}$. The integer numbers m_α correspond to the number of pairs in the level α.

Using Eq. (III.20), now, the k-th excited state is calculated as

$$\begin{aligned} |\psi_k\rangle \propto& \oint_{C_\Phi} d\Phi \oint_{C_1} dt\, |\Phi, J; q^1, p_1\rangle\, e^{i\mathcal{T}(\Phi, J; q^1, p_1)} \\ =& \sum_{\{m_\alpha\}} \int_0^{2\pi} d\Phi e^{i(J - \sum_\alpha m_\alpha)\Phi} \\ & \times \oint dt e^{i\mathcal{T}_1}(t) B_m(Z)\, |\cdots; S_\alpha, -S_\alpha + m_\alpha, \cdots\rangle \\ \equiv& \sum_{\{m_\alpha\}_J} C_m\, |\cdots; S_\alpha, -S_\alpha + m_\alpha, \cdots\rangle \end{aligned} \tag{III.21}$$

where $B_m(Z)$ are indentical to A_m in Eq. (III.20) but replacing χ_α by the relative angles $\phi_\alpha \equiv \chi_\alpha - \Phi$. The coefficients C_m are

$$C_m = \oint_{C_1} dt e^{i\mathcal{T}_1(t)} B_m(Z(t)) \tag{III.22}$$

In the last line of Eq. (III.21), the summation is restricted to $\{m_\alpha\}$ that satisfy $\sum_\alpha m_\alpha = J$. It is easy to find that J must be integer, according to the quantization rule (II.17) for the (J, Φ) sector.

The SPA for the ground state ($k = 0$) is given by the stationary point in the (q^1, p_1) sector, namely, the HFB state $|\Phi, J; q, p\rangle = e^{-i\Phi\hat{N}/2}\,|\mathrm{HFB}\rangle$. Nevertheless, the rotational motion in $\Phi(t)$ is present, which leads to the number quantization (projection). Therefore, Eq. (III.21) becomes

$$|\psi_{g.s.}\rangle \propto \sum_{\{m_\alpha\}} \int_0^{2\pi} d\Phi e^{i(J - \sum_\alpha m_\alpha)\Phi}\, |\mathrm{HFB}\rangle\,, \tag{III.23}$$

which is identical to the wave function of the particle number projection for HFB state.

IV. RESULTS

We apply ASCC+SPA to study the multi-level system in pairing model. The TDHFB degrees of freedom equals the number of single particle levels, including the constant of motion (pairing rotation). We consider various systems, two-level system, three-level system, and Pb isotope system in each subsection respectively. Next, we discuss the dynamics in non-integrable (more than three-level) system.

A. Confirmation in one-dimensional TDHFB system

The two-level system corresponds to one-dimensional TDHFB system after decoupling the global gauge angle Φ. We can obtain the obvious collective path from ASCC. The dynamics is exactly identical with adiabatic TDHFB (ATDHFB). We confirm that whether ASCC is identical to one-dimensional ATDHFB, and compare the difference with TDHFB, in numerical calculation.

In two-level system, the classical Hamiltonian in (III.8) is

$$\begin{aligned} \mathcal{H} =& \sum_{l=1,2} \epsilon_l \Omega_l (1 - \cos\theta_l) \\ & - \frac{g}{4} \sum_{l=1,2} \Omega_l [\Omega_l (1 - \cos^2\theta_l) + (1 - \cos\theta_l)^2] \\ & - \frac{g}{2} \Omega_1 \Omega_2 \sqrt{(1 - \cos^2\theta_{l_1})(1 - \cos^2\theta_{l_2})}\, \cos(\chi_2 - \chi_1). \end{aligned} \tag{IV.1}$$

If we define the canonical coordinate $\phi = \chi_2 - \chi_1$, it attributes to one-dimensional system. The conjugate momentum is $j = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \{\Omega_2(1 - \cos\theta_2) - \Omega_1(1 - \cos\theta_1)\}/4$. The ATDHFB indicates that the Hamiltonian can be expanded up to second order with respect to ϕ

$$\mathcal{H}(\phi, j) \approx V(j) + \frac{1}{2} B^{-1}(j) \phi^2, \tag{IV.2}$$

where $V(j) = \mathcal{H}(\phi = 0, j)$ and $B^{-1}(j) = \left.\frac{\partial^2 \mathcal{H}}{\partial \phi^2}\right|_{\phi=0}$. We know that the BCS ground state corresponds to the potential minimum point with $\phi = 0$. If the pairing correlation is strong enough to bind the excited states in the collective potential, the adiabatic approximation is expected to be well because the states are localized in small ϕ region.

We study the system with equal degeneracy $\Omega_1 = \Omega_2 = 8$, $g/(\epsilon_2 - \epsilon_1) = 0.2$, and $N = 16$. In ASCC calculation, we set the mesh size of collective coordinate $dq = 0.01$. We compared the results of $|0_2^+\rangle$ and $|0_3^+\rangle$ in three different calculations. Fig. 1 and Fig. 2 correspond to the classical informations, which are classical trajectories in phase space and action integral $\mathcal{T}$ as a function of time t. All of the mesh points obtained from ASCC are on the trajectories obtained from ATDHFB. For $|0_2^+\rangle$, the closed trajectory is well localized in the phase space and TDHFB calculation is almost the same as ASCC and ATDHFB calculations. For $|0_3^+\rangle$, the trajectory is still closed but near the transition point between closed trajectory and open trajectory. We can find ASCC and ATDHFB calculations have a small deviation from TDHFB calculation. The obtained excited 0^+ states from SPA are shown

in Fig. 3. We show the occupation probability which is decomposed into $2n$-particle-$2n$-hole components. The results from ATDHFB and ASCC are identical within numerical error. Also, TDHFB calculation is very close to other two calculations.

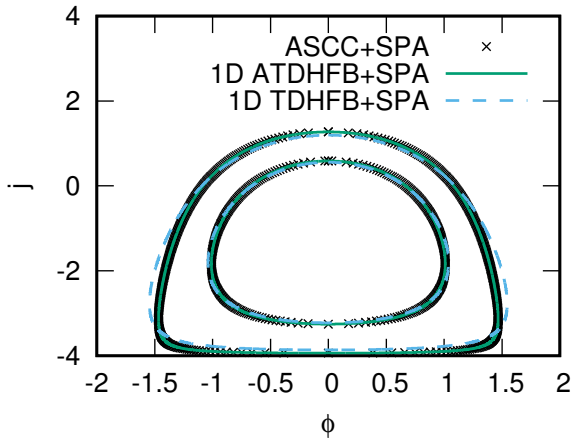

FIG. 1: Classical trajectories of $|0_2^+\rangle$ and $|0_3^+\rangle$ in two-dimensional phase space. The domain of phase space is $-\pi \leq \phi < \pi$, $-4 \leq j \leq 4$. Cross points, solid lines, and dashed lines correspond to the results from ASCC+SPA, one-dimensional ATDHFB+SPA, and one-dimensional TDHFB+SPA, respectively. In ASCC+SPA calculation, each cross point corresponds to each mesh point from ASCC calculation.

Comparing with one-dimensional TDHFB calculation, we confirmed we succeed to describe the bound collective excited states by ASCC+SPA. For the bound states, whether the adiabatic approximation included or not hardly has influence. Even for the weakly bound states, the adiabatic approximation is reasonable approximation.

B. Three-level system

The simplest non-integrable system is three-level system. We have one trivial motion and two time-dependent degrees of freedom. We study the system with $\Omega_1 = \Omega_2 = \Omega_3 = 8$, $\epsilon_1 = -1$, $\epsilon_2 = 0$, $\epsilon_3 = 1.5$, and $g = 0.2$. The phase transition occurs at $g_c = 0.058$ when $N = 2\Omega_1 = 16$. We consider the even particle number chain from $N = 14$ to $N = 24$.

From moving-frame QRPQ equation, we obtain three modes along the collective path (Fig. 4). Except zero mode, we choose the lowest mode as collective motion. Fig. 5 and Fig. 6 are occupation number in each single-particle level and collective potential in the collective path, respectively. We can find the occupation numbers become to integer at both end points. Furthermore, Hartree-Fock states always emerge at one of the end point.

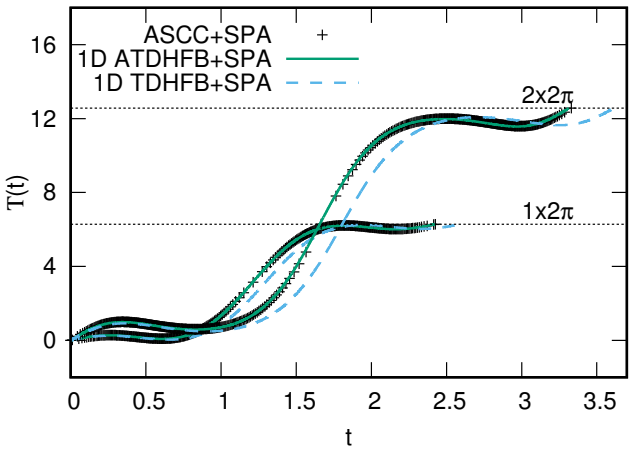

FIG. 2: Action integrals of $|0_2^+\rangle$ and $|0_3^+\rangle$ with respect to time t.Cross points, solid lines, and dashed lines correspond to the results from ASCC+SPA, one-dimensional ATDHFB+SPA, and one-dimensional TDHFB+SPA, respectively. Dotted lines are the values correspond to EBK quantization condition for $|0_2^+\rangle$ and $|0_3^+\rangle$. Based on Fig. 1, we calculated the action integrals on each trajectory from $(\phi, j) = (0, j_{\max})$ in clockwise direction.

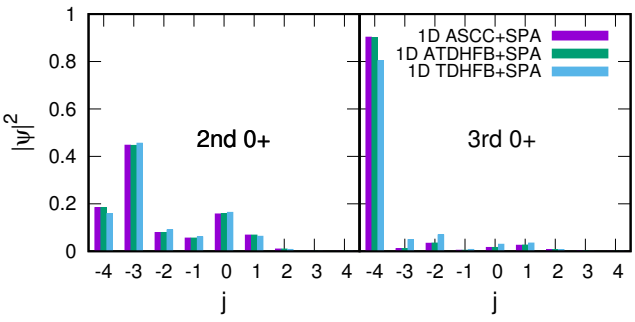

FIG. 3: Occupation probabilities of $|0_2^+\rangle$ and $|0_3^+\rangle$ as a function of j. The two vertical bars at each j from the left to the right represent the squared components of the wave functions from ASCC+SPA, ATDHFB+SPA and TDHFB+SPA calculations, respectively. The left end of the horizontal axis at $j = j_{\min}$ corresponds to a component with $(n_1, n_2) = (N, 0)$. The next at $j = j_{\min} + 1$ corresponds to the one with $(n_1, n_2) = (N - 2, 2)$, and so on.

Based on the information from ASCC calculation, we can obtain the excited states from SPA. Table I shows the excitation energies of first and second excited states in the collective subspace. Up to the second excitation, all of the energies are in their energy pockets shown in Fig. 6. Comparing the result from ASCC+SPA with

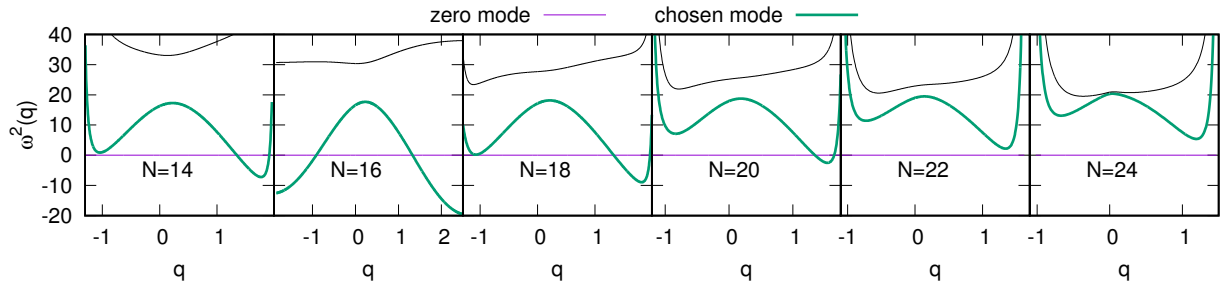

FIG. 4: Eigenvalues of moving-frame QRPA equation with respect to the collective coordinate q, from $N = 14$ to $N = 24$. Purple lines are spurious modes and green lines are chosen modes corresponding to the collective coordinate. In each panel, both edges correspond to the end points of the collective coordinate.

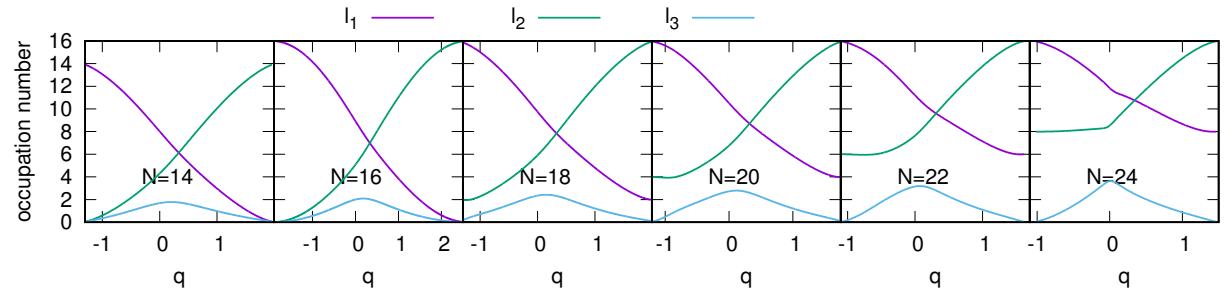

FIG. 5: Occupation numbers in each single-particle level with respect to the collective coordinate q, from $N = 14$ to $N = 24$. At the left end point of the collective coordinate in each panel, the configurations correspond to Hartree-Fock states.

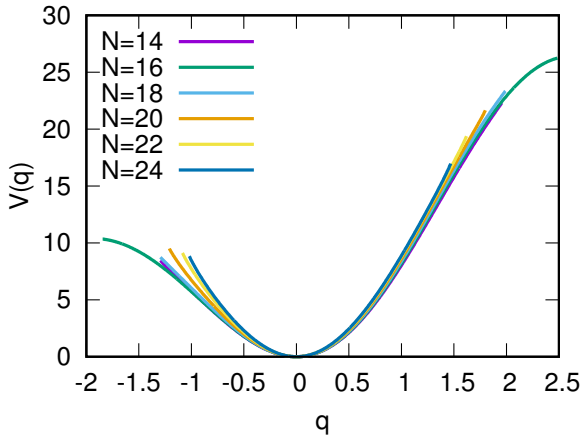

FIG. 6: Collective potential obtained from ASCC. We adjusted the energy minimum point as $V = 0$.

exact solution, we can find all of the values are nicely reproduced. The slightly difference is that the values from ASCC+SPA are about 5% smaller than exact solution. Using the wave function of excited states, the pair additional transition strength is shown in Fig. 7. For intraband transition, comparing ASCC+SPA with exact solution, $B(P_{\rm ad}; 0_1^+ \to 0_1^+)$ are almost identical and $B(P_{\rm ad}; 0_2^+ \to 0_2^+)$ are about $10\% \sim 20\%$ small. For interband transition, the strength from ASCC+SPA is much smaller than exact solution. It is difficult to reproduce the absolute value of exact solution. However, because the pairing correlation is enough strong in this case, the strength of intraband transition is dominant compared with interband transition. We can find that $B(P_{\rm ad}; 0_1^+ \to 0_2^+)$ and $B(P_{\rm ad}; 0_2^+ \to 0_1^+)$ are only about 1% of $B(P_{\rm ad}; 0_1^+ \to 0_1^+)$ and $B(P_{\rm ad}; 0_2^+ \to 0_2^+)$. Therefore, the interband transition can be regarded as reasonable. From all above results, we can conclude that ASCC+SPA reproduces exact solution quantitatively.

We need to discuss about the second excited state in the collective path corresponds to not $|0_3^+\rangle$ and others, but $|0_4^+\rangle$. In pair additional transition $B(P_{\rm ad}; k \to k')$ from exact solution, $B(P_{\rm ad}; 0_2^+ \to 0_3^+)$ is $10 \sim 100$ times smaller than $B(P_{\rm ad}; 0_1^+ \to 0_3^+)$, and $B(P_{\rm ad}; 0_1^+ \to 0_2^+)$ is in the same order with $B(P_{\rm ad}; 0_1^+ \to 0_3^+)$. It indicates that $|0_3^+\rangle$ is also one-phonon state which belongs to the other collective path (black line in Fig. 4). About two-phonon states, we can easily find the candidates of two-phonon states from excitation energies and $B(P_{\rm ad}; 0_1^+ \to 0_{ex}^+)$. In the candidates, $|0_4^+\rangle$ is the lowest two-phonon state and $B(P_{\rm ad}; 0_2^+ \to 0_4^+)$ is much larger than $B(P_{\rm ad}; 0_3^+ \to 0_4^+)$. Therefore, $|0_4^+\rangle$ is considered to be most appropriate corresponding to the second excited

state in ASCC+SPA calculation.

	N	14	16	18	20	22	24
0_2^+ one	Exact	4.09	4.13	4.20	4.30	4.44	4.60
phonon	ASCC+SPA	3.87	3.90	3.97	4.09	4.23	4.33
0_4^+ two	Exact	7.65	7.71	7.88	8.15	8.49	8.74
phonon	ASCC+SPA	7.42	7.42	7.60	7.92	8.26	8.47

TABLE I: Excitation energies of one-phonon and two-phonon states from exact solution and ASCC+SPA calculation. In a chosen collective subspace, one-phonon and two-phonon states correspond to the first and second excitation, respectively. In the whole Hilbert space, one-phonon and two-phonon states correspond to $|0_2^+\rangle$ and $|0_4^+\rangle$, respectively.

C. Pb isotopes

With pairing model, we study neutron's pairing vibration in Pb isotope. We prepare the neutron's single-particle level between the magic number 82 and 126 (Table II). The coupling constant $g = 0.138$(MeV) is determined to reproduce the experimental pairing gap $\Delta(N) = \frac{(-1)^{N+1}}{2}(B(N+1) - 2B(N) + B(N-1))$ of ^{192}Pb. We consider the even-even system from ^{188}Pb to ^{194}Pb.

orbit	$h_{9/2}$	$f_{7/2}$	$i_{13/2}$	$p_{3/2}$	$f_{5/2}$	$p_{1/2}$
energy(MeV)	−10.94	−10.69	−8.74	−8.44	−8.16	−7.45

TABLE II: Single-particle levels of Pb isotopes used in the calculation. The orbits are obtained from spherical Woods-Saxon potential with spin-orbit coupling.

We have six TDHFB degrees of freedom in the system. Fig. 8 shows the eigenvalue of moving-frame QRPA equation. We can find there are one spurious mode and five vibrational modes. The collective modes we focus on did not cross with other modes in all panels, while the spacings of the collective modes and the second vibrational modes are small, especially for ^{194}Pb. Under the chosen collective modes, the occupation number is shown in Fig. 9. As with three-level system, the left end points correspond to HF state in all panels. The last neutrons are in $i_{13/2}$ from ^{188}Pb to ^{194}Pb. In the collective path, we can find $i_{13/2}$ and $p_{3/2}$ orbits change significantly, while other orbits don't change so much. The collective potential for each isotope is shown in Fig. 10. The depths of them are about $2.5 \sim 3.5$(MeV) only enough to bind $|0_2^+\rangle$, one-phonon states. We can consider that the collectivity of pairing is small for excited states, even though their ground states are superfluid states.

We show the results of $|0_2^+\rangle$, one-phonon state in the collective coordinate, from SPA. Table II shows the excitation energy. The value is nothing to do with experimental value because pairing model is too simple to describe realistic nuclear interaction. We are only interested in the comparison between exact solution and ASCC+SPA. We can find the excitation energy from ASCC+SPA reproduce the exact solution as well as in three-level system. The pair additional transition strength is shown in Fig. 11. The feature is similar to the case in three-level system, dominant intraband transition and the accuracy from ASCC+SPA. The deviation of $B(P_{\rm ad}; 0_2^+ \to 0_2^+)$ has about 25% deficiency compared with exact solution. Therefore, ASCC+SPA gives reasonable results in realistic single-particle level systems.

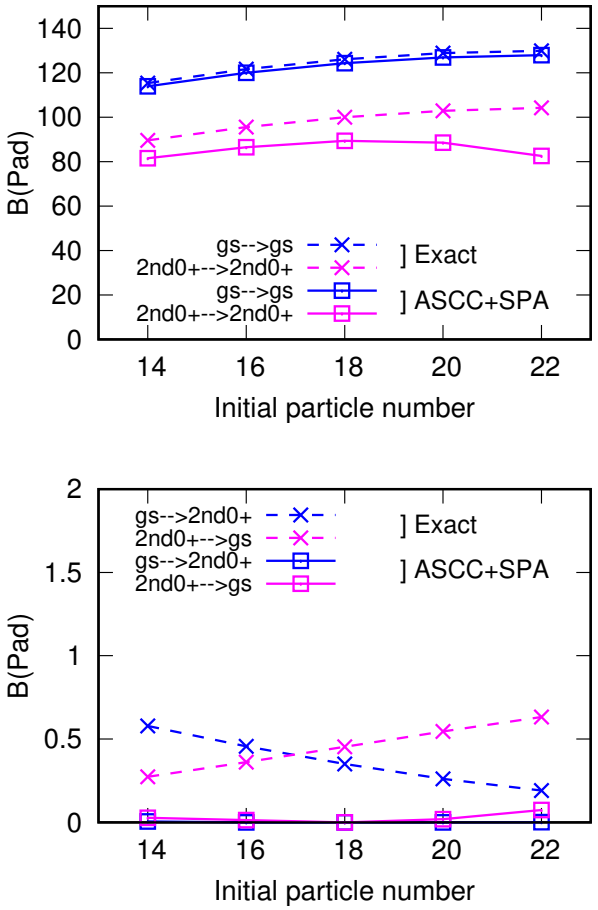

FIG. 7: The strength of pair additional transition $B(P_{\rm ad}; k \to k') = |\langle N+2, k'|S^+|N, k\rangle|^2$, from $N = 14$ to $N = 22$. Dashed lines are exact solution and solid lines are ASCC+SPA calculation. Horizontal line shows the particle number of the initial states. Upper panel shows the intraband transitions of $|0_1^+\rangle \to |0_1^+\rangle$ and $|0_2^+\rangle \to |0_2^+\rangle$, while lower panel shows the interband transition of $|0_1^+\rangle \to |0_2^+\rangle$ and $|0_2^+\rangle \to |0_1^+\rangle$.

Finally, we discuss about the validity of the collective treatment of the pairing. We know that the 5D collective model described by the quadrupole deformation parameters $\alpha_{2\mu}(\mu = \pm 2, \pm 1, 0)$ is widely applied to ana-

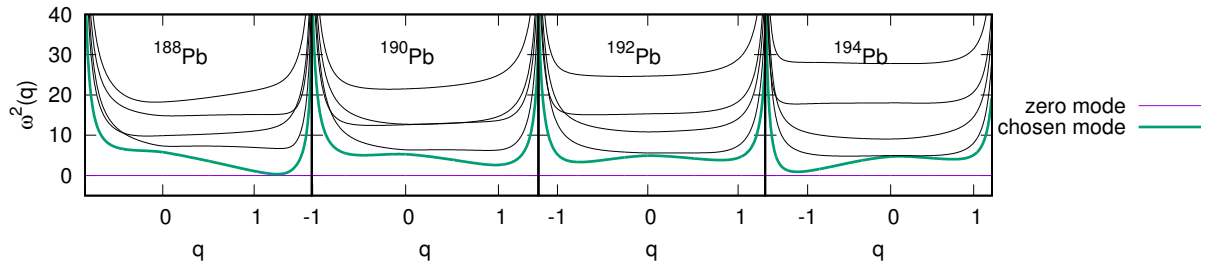

FIG. 8: The same as Fig. 4 but for Pb isotopes.

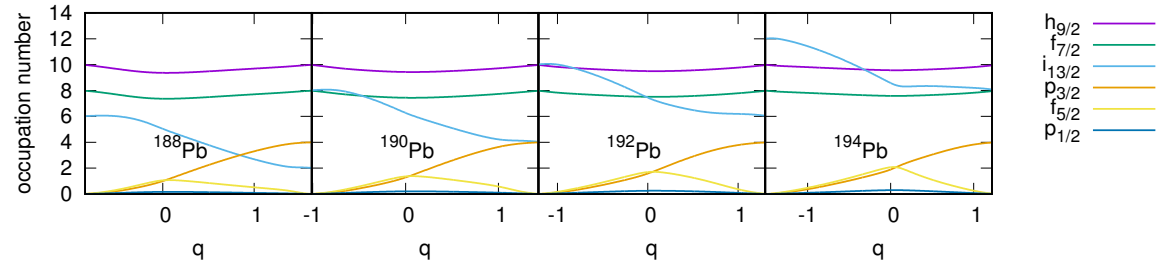

FIG. 9: The same as Fig. 5 but for Pb isotopes.

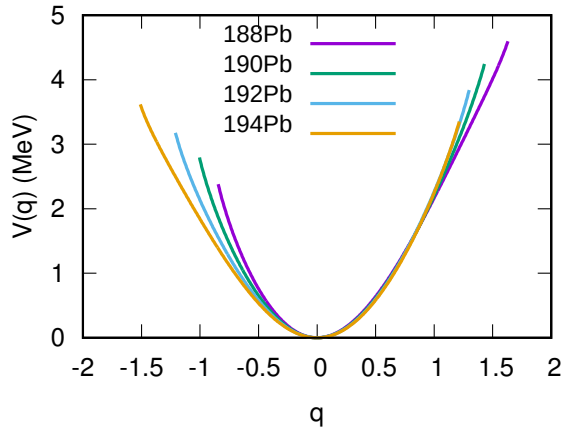

FIG. 10: The same as Fig. 6 but for Pb isotopes.

	^{186}Pb	^{188}Pb	^{190}Pb	^{192}Pb	^{194}Pb	^{196}Pb
Exact	2.58	2.44	2.34	2.25	2.20	2.15
ASCC+SPA	unbound	2.31	2.21	2.12	2.04	unbound

TABLE III: The same as Table. I but for Pb isotopes. The energies are given in units of MeV.

lyze experimental data. As with the 5D collective model, pairing can also be supposed as a sort of deformation. The collective model is described by the pair deformation parameters pairing gap Δ and global gauge angle Φ. The utilization of the collective model for nuclear pairing dynamics was done in several references[**?**]. In our previous study, however, we find that the pairing gap Δ is not proper to be the collective coordinate in two-level pairing model [**?**]. In multi-level system, we can investigate the relations between the pairing gap and the collective coordinate. The pairing gap Δ is defined as

$$\begin{aligned}\Delta(q) &\equiv g\,\langle\Phi, J; q, p|S^-|\Phi, J; q, p\rangle\big|_{\Phi=p=0} \\ &= g\sum_l \sqrt{j_l(\Omega_l - j_l)}. \end{aligned} \tag{IV.3}$$

Fig. 12 shows the pairing gap of Δ in ^{192}Pb as a function of the collective coordinate q. We can see that the peak is near $q = 0$ and both end points have small values. Because only one orbit near the Fermi energy ($i_{13/2}$ orbit for both end points) contributes the value of pairing gap, the value at both end points are small compared with the value around the ground state. There is no one-to-one correspondence between Δ and q not only in ^{192}Pb, but also in all multi-level systems. Therefore, the pairing gap Δ cannot properly describe the pairing dynamics.

V. CONCLUSION AND DISCUSSION

Based on our previous work which demonstrated the accuracy of SPA for the requantization of TDHFB dynamics, we considered the application of SPA to non-integrable system. The one of the possible way to

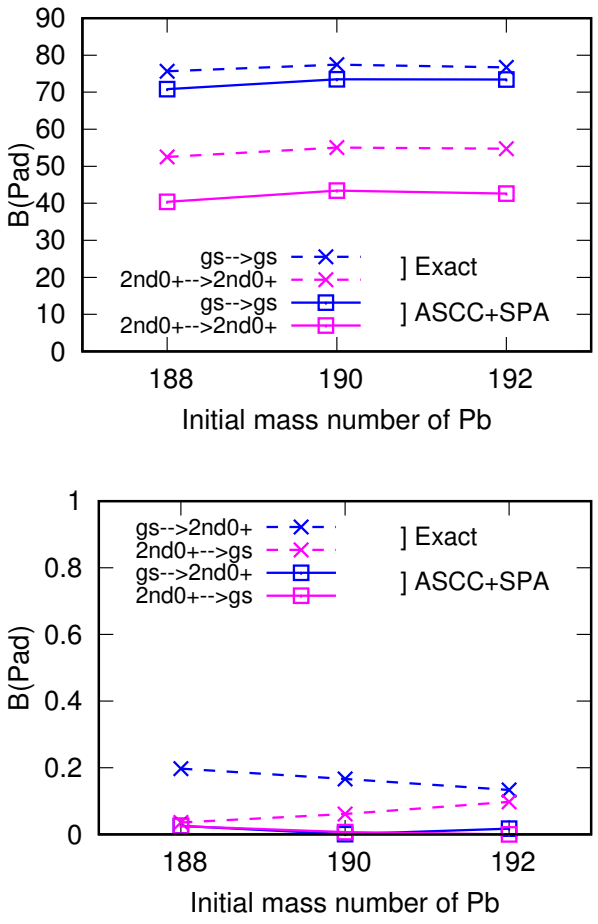

FIG. 11: The same as Fig. 7 but for Pb isotopes.

extend SPA into non-integrable system is to combine with ASCC. We constructed the theoretical framework of ASCC+SPA for bound states, which is bound in the collective potential from ASCC.

We applied ASCC+SPA into multi-level pairing model. Because the global gauge angle Φ is a cyclic variable, the simplest non-integrable system is three level system. We investigated three level system and more realistic system, neutron's single-particle levels from Pb isotope. In both cases, the low-lying excited 0^+ states from ASCC+SPA reproduce exact solution well not only for excitation energies but also for wave functions. In ASCC+SPA, the pair transition calculation has no difficulty because we don't need to requantize the pair transition operators. This point overcomes the disadvantage in canonical requantization.

While the basic theoretical framework of ASCC+SPA was constructed, there are several problems remaining. Up to now, ASCC+SPA is only available when the excited states are bound in the pocket of collective potential (See Fig. 6 and 10). For example, we cannot calculate $|0_2^+\rangle$ in ^{186}Pb because the potential is too shallow to bound the excited state. The difficulty for the application to the unbound states is that the wave function from

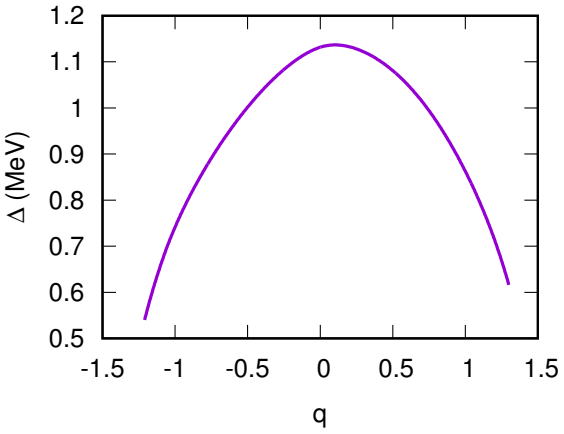

FIG. 12: Pairing gap as a function of collective coordinate in ^{192}Pb.

SPA is not stationary state because the classical trajectory in phase space is not periodic. It is necessary to find some proper boundary conditions in the collective subspace. The second problem is about the crossing of the eigenvalues from moving-frame QRPA equation in the collective path. No problem occurs when two completely orthogonal modes crossing, such as rotational mode and vibrational mode (e.g. N=14,16,18,20 in Fig. 4). However, if two vibrational modes not orthogonal with each other become the same eigenvalues, the ASCC calculation stops. Actually, the two eigenvalues become complex conjugate values and the next mesh point of collective path cannot be decided properly. Such case is occurred in ^{196}Pb. The eigenvalues crossing problem supposed to be the rare case because ^{196}Pb system is the only case including the previous studies about the application of ASCC. In addition, we need to consider the multi energy minimum pockets in the collective potential [9, 10]. Such case doesn't appear in pairing model. When the excitation energy is below the multi minimum pocket, the EBK quantization condition is not unique. With these problems, the theoretical framework of ASCC+SPA need to be developed in the next step.

ACKNOWLEDGMENTS

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