Requantization of time-dependent mean field for pairing collective motion in superfluid nuclei

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October 17, 2018

Contents

Chapter 1

Introduction

1.1 Pairing interaction in nuclei

Pairing correlation plays an important role near the Fermi energy. Pairing correlation is the two-body interaction which couples two identical nucleons into $J^\pi = 0^+$ state. There are many evidences of pairing correlation detected from experiment. The most clear evidence is shown in Fig. 1.1. All of the ground states of even-even Sn isotopes are $J^\pi = 0^+$ states It indicates that the ground states consist of the $J^\pi = 0^+$ pairs is more stable than other configurations. In addition, the ground states between even-even nuclei and neighborhood odd nuclei has large gaps of binding energy. In odd nuclei, the unpaired last neutron is the last single-particle level. The odd-even mass difference To break the $J^\pi = 0^+$ pair, we need large energy which is about $2\Delta \approx 24A^{-1/2}$ MeV.

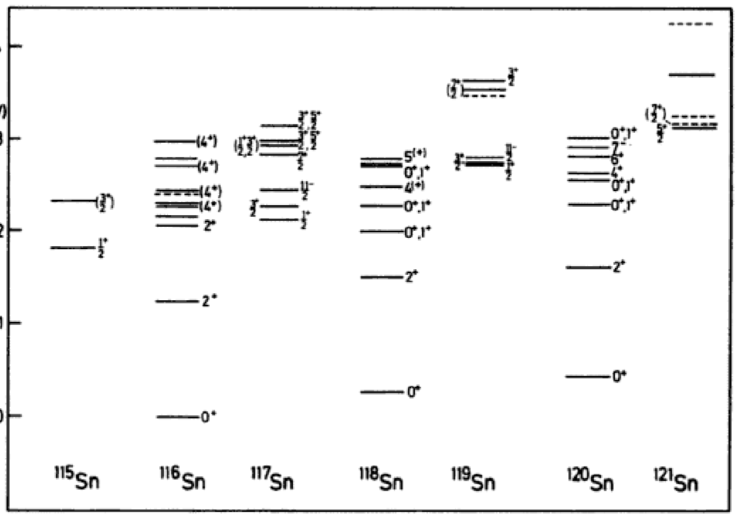

Figure 1.1: Low-lying excited states in Sn isotopes []. The absolute values of binding energy are adjusted at 0 MeV in the ground state of ^{116}Sn.

1.1.1 A subsection

More text.

Chapter 2

Pairing model

To examine the pairing dynamics influencing the structure of nuclei, we can only concentrate the nucleons near the Fermi energy. The Hamiltonian of pairing model is

$$H = \sum_{l=1}^{L} \epsilon_l n_l - g \sum_{l,l'} S_l^+ S_{l'}^- , \tag{2.1}$$

where

$$n_l = \sum_m a_{lm}^\dagger a_{lm} \tag{2.2}$$

$$S_l^+ = \sum_{m>0} a_{lm}^\dagger a_{l\overline{m}}^\dagger, \quad S_l^- = S_l^{+\dagger}. \tag{2.3}$$

In L single-particle level system, each single-particle energy ϵ_l possesses $(2\Omega_l)$-fold degeneracy ($\Omega_l = j_l + 1/2$) and $\sum_{m>0}$ indicates the summation over $m = 1/2, 3/2, \cdots$, and $\Omega_l - 1/2$. $\overline{m}$ indicates the time reversal quantum number with respect to m. The strength of pairing correlation is represented by coupling constant g. We also define a seniority ν_l, which is the number of unpaired particles in each level. The unpaired particles have no pairing interaction, and play a part in Pauli blocking effect.

In one-level system ($L = 1$), the pairing dynamics is well understood. The analytic solution of the total energy is

$$E_\nu(N) = \epsilon N - \frac{1}{4} g(N - \nu)(2\Omega - N - \nu + 2). \tag{2.4}$$

The pairing rotational band created by the pairing correlation is completely parabolic with the moment of inertia $2/g$. Also, the excited states in a specified N is only described by the quantum number ν. For example, the first excited state in even particle system is $\nu = 2$ state, which correspond to one $J^\pi = 0^+$ pair broken state.

The profound dynamics occurs in multi-level pairing model ($L \geq 2$). The pairing rotational mode becomes complex due to the shell effect. In addition, several pairing vibrational modes emerge. The pairing vibration is not single particle motion specified by ν_l, but a collective motion with the excitation of $J^\pi = 0^+$ pairs. In this chapter, we explain the numerical exact solution and derive the TDHFB dynamics in multi-level pairing Hamiltonian.

2.1 Exact solution

The eigenvalues and eigenvectors can be solved exactly in pairing model. There are two major methods to obtain the exact solutions, diagonalization in quasispin space and Richardson equation.

2.1.1 Diagonalization in quasispin space

We introduce SU(2) quasispin operators fulfilling the follows commutation relation

$$[S_l^0, S_{l'}^+] = \delta_{ll'} S_l^+, \qquad [S_l^+, S_{l'}^-] = 2\delta_{ll'} S_l^0. \tag{2.5}$$

If we set

$$S_l^0 = \frac{1}{2}(\sum_m a_{lm}^\dagger a_{lm} - \Omega_l) \tag{2.6}$$

and S_l^+, S_l^- are the same as in (2.3), the commutation relation (2.5) is fulfilled. Using quasispin operators, the Hamiltonian can be represented by quasispin operators

$$H = \sum_l \epsilon_l (2S_l^0 + \Omega_l) - g \sum_{l,l'} S_l^+ S_{l'}^-. \tag{2.7}$$

Therefore, any states $|\psi\rangle$ belong to SU(2)$\otimes \cdots \otimes$SU(2) Hilbert space, and can be expanded in the basis $|S_1, S_1^0; \cdots ; S_l, S_l^0; \cdots\rangle$

$$|\psi\rangle = \sum_{S_1^0, \cdots S_l^0, \cdots} c_{S_1^0, \cdots S_l^0, \cdots} |S_1, S_1^0; \cdots ; S_l, S_l^0; \cdots\rangle. \tag{2.8}$$

We discuss the physical meaning of the quasispin space. Because S_l^+ and S_l^- correspond to the creation and annihilation operators with respect to the $J^\pi = 0^+$ pairs in each level, the quasispin space is decoupled from the space which the unpaired particles belong to. Also, S_l^0 is related to the occupation number in each level. $S_l^0 = -S_l$ is the minimum value corresponding no $J^\pi = 0^+$ pair in the level and $S_l^0 = S_l$ is the maximum value corresponding full of $J^\pi = 0^+$ pairs in the level. Because ν_l plays a part in Pauli blocking effect in the level, the absolute value of quasispin S_l is

$$S_l = \frac{1}{2}(\Omega_l - \nu_l). \tag{2.9}$$

The dimension of the model space is $D = \prod_l (2S_l + 1)$. However, $[H, N] = 0$ indicates that the Hamiltonian is block diagonal with respect to the total particle number N. The model space can be reduced after we extract the combination of $\{S_l^0\}$ fulfilling

$$N = \sum_l (2S_l^0 + 2S_l + \nu_l). \tag{2.10}$$

For example, we consider the dimension of the single particle levels between the magic number 50 and 82. Under spherical symmetry, we have five single particle levels, $d_{5/2}$, $g_{7/2}$, $s_{1/2}$, $d_{3/2}$, and $h_{11/2}$. The corresponding degeneracy is $\{\Omega_l\} = \{3, 4, 1, 2, 6\}$. If we only consider the case for$\nu_l = 0$, the dimension for the system is $D = 840$. On the other hand, under a specified N, the dimensions are 105 for $N = 14$ and 91 for $N = 20$.

2.1.2 Richardson equation

As mentioned above, diagonalizing the Hamiltonian is simple method to obtain the exact solution. However, diagonalization is time consuming when the model space becomes large. The calculation time is proportional to the cubes of the dimension of the model space For example, there are 16 single particle levels between the magic number 50 and 82 in deformed nuclei. Because the dimension for the system is $D = 2^{16} = 65536$, the calculation time in deformed system is $\sim 10^5$ times as much as the case in spherical system.

The alternative method to obtain the exact solution is to solve Richardson equation. The idea was introduced by R. W. Richardson [1, 2, 3]. We divide the total energy into two parts

$$E = \langle\psi|H|\psi\rangle \tag{2.11}$$

$$= \sum_l \epsilon_l \nu_l + \sum_{\alpha=1}^{M} E_\alpha. \tag{2.12}$$

The first term is the energy of unpaird particle, and the second term is the total energy of $J^\pi = 0^+$ pairs. The pair-energy E_α is derived from the non-linear equations

$$1 - g\sum_l \frac{\Omega_l - \nu_l}{2\epsilon_l - E_\alpha} + 2g\sum_{\beta(\neq\alpha)}^{M} \frac{1}{E_\beta - E_\alpha} = 0. \tag{2.13}$$

M indicates the number of $J^\pi = 0^+$ pairs in the system, and equals to the numbers of the non-linear equations. E_α can become not only real numbers, but also complex conjugate pairs when $M \geq 2$.

2.2 TDHFB dynamics

To understand the classical picture of the pairing dynamics, time-dependent mean-field (TDMF) theory is useful tool. With pairing correlation, TDMF theory attributes to TDHFB theory. The TDHFB equation can be derived from the time-dependent variational principle,

$$\delta\mathcal{S} = 0, \quad \mathcal{S} \equiv \int \langle\phi(t)|i\frac{\partial}{\partial t} - H|\phi(t)\rangle\, dt = 0, \tag{2.14}$$

where $|\phi(t)\rangle$ is the time-dependent generalized Slater determinant (coherent state) obtained from Thouless' theorem

$$|\phi(t)\rangle = \mathcal{R}\exp\left(\sum_{k<k'} Z_{kk'}(t)\beta_k^\dagger\beta_{k'}^\dagger\right)|\Phi_0\rangle\,. \tag{2.15}$$

$\beta_k^\dagger$ is quasiparticle creation operator and $\mathcal{R}$ is normalization factor. $|\Phi_0\rangle$ is generally chosen as ground state or vacuum state. $Z_{kk'}(t)$ is time-dependent complex number determined by TDHFB equation.

2.2.1 Hamilton's equation

We derive the TDHFB equation in pairing model. First, we consider the time-dependent coherent state. Because the noticeable phenomenon is the dynamics for $J^\pi = 0^+$ pairs in pairing model, we can describe the time-dependent coherent state in SU(2) quasispin space. Using Thouless' theorem, the time-dependent coherent state becomes

$$|Z(t)\rangle = \prod_l \left(1 + |Z_l(t)|^2\right)^{-S_l} \exp[Z_l(t)S_l^+]\,|0\rangle\,. \tag{2.16}$$

Here, we choose $|\Phi_0\rangle = |0\rangle$ which is vacuum (zero particle) state. In the quasispin representation, $|0\rangle = \prod_l |S_l, -S_l\rangle$. The coherent state is the extension of BCS trail wave function, which consists of the superposition of different particle number states. Z_l are complex numbers. $|Z_l| = 0$ corresponds to empty occupation and $|Z_l| = \infty$ corresponds to full occupation for $J^\pi = 0^+$ pairs.

With the transformation $Z_l = \tan\frac{\theta_l}{2}e^{-i\chi_l}$ $(0 \leq \theta \leq \pi,\ 0 \leq \chi < 2\pi)$, the Lagrangian and the expectation value of Hamiltonian become

$$\mathcal{L}(t) = \langle Z(t)|i\hbar\frac{\partial}{\partial t} - H|Z(t)\rangle = \sum_l j_l\dot{\chi}_l - \mathcal{H}(\chi, j), \tag{2.17}$$

$$\begin{aligned}\mathcal{H}(\chi, j) &\equiv \langle Z(t)|H|Z(t)\rangle \\ &= \sum_l \epsilon_l\nu_l + \sum_l 2\epsilon_l j_l - g\sum_l \left((\Omega_l - \nu_l)j_l - j_l^2 + \frac{j_l^2}{\Omega_l - \nu_l}\right) \\ &\quad - 2g\sum_{l_1<l_2}\sqrt{j_{l_1}j_{l_2}(\Omega_{l_1} - \nu_{l_1} - j_{l_1})(\Omega_{l_2} - \nu_{l_2} - j_{l_2})}\cos(\chi_{l_1} - \chi_{l_2}),\end{aligned} \tag{2.18}$$

where χ_l are the canonical coordinates and j_l are the conjugate momenta defined in

$$j_l \equiv \frac{\partial\mathcal{L}}{\partial\dot{\chi}_l} = S_l(1 - \cos\theta_l). \tag{2.19}$$

The detailed derivations are written in Appendix A. The canonical variables (χ, j) have clear physical meaning. χ_l represent a kind of gauge angle of each level, and j_l correspond to the occupation number of each level

$$\langle Z|n_l|Z\rangle = \nu_l + 2j_l. \tag{2.20}$$

From (2.19), we can find that $0 \leq j_l \leq 2S_l$, where $j_l = 0$ corresponds to empty occupation and $j_l = 2S_l$ corresponds to full occupation for $J^\pi = 0^+$ pairs. Using canonical variables, the TDHFB equations attribute to the Hamilton's equations

$$\dot{\chi}_l = \frac{\partial\mathcal{H}}{\partial j_l}, \quad \dot{j}_l = -\frac{\partial\mathcal{H}}{\partial\chi_l}. \tag{2.21}$$

2.2.2 Constant of motion

The number of the TDHFB degrees of freedom is same as the number of the single-particle levels. However, because the Hamiltonian depends only on the relative difference in the gauge angles, the global gauge angle

$$\Phi = \frac{1}{L}\sum_l \chi_l \tag{2.22}$$

is a cyclic variable. With the definition for relative gauge angle (e.g. $\phi_l = \chi_l - \chi_1$), we can find that the conjugate momentum of Φ is the sum of j_l

$$J \equiv \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \sum_l j_l. \tag{2.23}$$

From (2.20), J is related to the total particle number

$$N = \sum_l \nu_l + 2J. \tag{2.24}$$

The Hamilton's equation for J

$$\dot{J} = -\frac{\partial\mathcal{H}}{\partial\dot{\Phi}} = 0, \quad J = \text{const} \tag{2.25}$$

indicates that the total particle number is conserved in time evolution. The canonical variable (Φ, J) describes the pairing rotation. The pairing rotational band emerges in the particle number chain $(\cdots, N-2, N, N+2, \cdots)$. The last term in (2.18) contains the higher order terms with respect to N. It related to the shell effect. In single-level system, the last term in (2.18) disappears and (2.18) attributes to (2.4). The pairing rotational band is exactly parabolic. In multi-level system, the coupling constant g and single-particle energy ϵ_l affects significantly to the last term in (2.18). The parabolic structure can be broken when the shell effect is strong.

In the analysis of TDHFB classical Hamiltonian, we can decouple the pairing rotational mode. The degrees of freedom is $L - 1$ in L level system. Therefore, the dynamics is integrable for $L = 2$ and non-integrable for $L \leq 3$ in multi-level pairing model. We discuss the integrable system and non-integrable system in Chapter 3 and 4, respectively.

Chapter 3

Requantization of TDHFB in integrable system

As discussed in Sec. 2.2.2, the TDHFB dynamics in two-level system is one-dimensional motion. We know that one-dimensional system must be integrable system. There are two new discoveries obtained from the study of the integrable system. First, the requantization method, stationary phase approximation to the path integral, can be applied straightforward to describe the collective excited 0^+ states. We find that SPA is superior to the other requantization methods, especially for the calculation of two-particle transition strength. Second, the collective coordinates are explicit in integrable system. Using the explicit collective coordinates, we can examine the performance of the collective model which assumes the pairing gap parameter as the collective coordinate. The conclusion is the pairing gap parameter can only describe a part of collective space, and the applicability of the collective model is limited.

3.1 TDHFB dynamics in two-level system

To describe the one-dimensional motion, we use the global gauge angle Φ and the relative gauge angle ϕ as canonical coordinates

$$\Phi \equiv \frac{\chi_1 + \chi_2}{2}, \qquad \phi \equiv \chi_2 - \chi_1. \tag{3.1}$$

These ranges are $0 \leq \Phi \leq 2\pi$ and $-2\pi \leq \phi \leq 2\pi$. Their conjugate momenta (J, j) are given by

$$J = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \sum_{l=1}^{2} j_l, \qquad j = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{j_2 - j_1}{2}. \tag{3.2}$$

As in (2.20), j corresponds to relative occupation number

$$j = \frac{\nu_2 - \nu_1}{2} + \frac{n_2 - n_1}{4} \tag{3.3}$$

The Hamiltonian in terms of these canonical variables $(\phi, j; \Phi, J)$ is given by

$$\begin{aligned}\mathcal{H}(\phi, j; J) = \sum_{l=1,2} \epsilon_l \nu_l + \sum_{l=1,2} 2\epsilon_l j_l - g \sum_{l=1,2} \left((\Omega_l - \nu_l) j_l - j_l^2 + \frac{j_l^2}{\Omega_l - \nu_l} \right) \\ - 2g\sqrt{j_1 j_2 (\Omega_1 - \nu_1 - j_1)(\Omega_2 - \nu_2 - j_2)} \cos\phi,\end{aligned} \tag{3.4}$$

with

$$j_1 = \frac{J}{2} - j, \qquad j_2 = \frac{J}{2} + j. \tag{3.5}$$

Because J and the total energy E constants of motion, the TDHFB trajectories with given J and E are determined in the two-dimensional phase space (ϕ, j) with the condition

$$\mathcal{H}(\phi(t), j(t); J) = E. \tag{3.6}$$

For convenience, we also define a dimensionless coupling constant in two-level system

$$x = \frac{\epsilon_2 - \epsilon_1}{2\Omega} \tag{3.7}$$

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3.2 Requantization methods

3.2.1 Canonical quantization

3.2.2 Fourier decomposition

3.2.3 Stationary phase to the path integral

3.3 Result

3.4 Collective model treatment

Chapter 4

Requantization of TDHFB in non-integrable system

4.1 Derivation of the collective subspace in adiabatic self-consistent collective coordinate method

4.2 Application of SPA in non-integrable system

4.3 Result

Chapter 5

Discussion

Chapter 6

Conclusion

Appendix A

Detailed derivation of the action in pairing model

To obtain the TDHFB dynamics in pairing model, We need to derive the action $\mathcal{S}$ in explicit form. We give the detailed derivation in this chapter.

A.1 Properties of the spin-coherent state

The time-dependent coherent state classified into the spin-coherent state is employed in (A.19). We derive the necessary formula for the spin-coherent state in this subsection.

First, we consider the single-level case. With the commutation relation in (2.5), the SU(2) quasispin operators fulfill the following relations

$$\hat{S}^{\pm} |S, S^0\rangle = \sqrt{(S \mp S^0)(S \pm S^0 + 1)}\, |S, S^0 \pm 1\rangle \tag{A.1}$$

$$\hat{S}^0 |S, S^0\rangle = S^0 |S, S^0\rangle . \tag{A.2}$$

Using above relations, the coherent state can be expanded as

$$\begin{aligned} |Z\rangle &= (1+|Z|^2)^{-S} e^{Z\hat{S}^+} |S, S^0 = -S\rangle \\ &= (1+|Z|^2)^{-S} \sum_{n=0}^{2S} \sqrt{\frac{(2S)!}{n!(2S-n)!}} Z^n |S, -S+n\rangle . \end{aligned} \tag{A.3}$$

We can find the normalization condition is fulfilled

$$\begin{aligned} \langle Z|Z\rangle &= (1+|Z|^2)^{-2S} \sum_{n=0}^{2S} \frac{(2S)!}{n!(2S-n)!} |Z|^{2n} \\ &= (1+|Z|^2)^{-2S} \sum_{n=0}^{2S} {}_{2S}\mathrm{C}_n (|Z|^2)^n \\ &= (1+|Z|^2)^{-2S} \times (1+|Z|^2)^{2S} \\ &= 1. \end{aligned} \tag{A.4}$$

The overlap is

$$\langle \eta|Z\rangle = \frac{(1+\eta^* Z)^{2S}}{(1+|\eta|^2)^S (1+|Z|^2)^S} . \tag{A.5}$$

We also calculate the expectation values of the operators. For $\hat{S}^0$,

$$\hat{S}^0 e^{Z\hat{S}^+} |S, S^0 = -S\rangle = \sum_{n=0}^{2S} (-S+n) \sqrt{\frac{(2S)!}{n!(2S-n)!}} Z^n |S, -S+n\rangle , \tag{A.6}$$

hence,

$$\begin{aligned}
\langle Z|\hat{S}^0|Z\rangle &= (1+|Z|^2)^{-2S}\sum_{n=0}^{2S}(-S+n)\frac{(2S)!}{n!(2S-n)!}|Z|^{2n} \\
&= -S+2S\left(1-\frac{1}{1+|Z|^2}\right) \\
&= S\left(1-\frac{2}{1+|Z|^2}\right). \qquad \text{(A.7)}
\end{aligned}$$

From the first line to the second line, we used the differentiation for the both sides of

$$(1+|Z|^2)^{2S} = \sum_{n=0}^{2S} {}_{2S}\mathrm{C}_n(|Z|^2)^n \qquad \text{(A.8)}$$

with respect to $|Z|^2$. With (A.7), the expectation values of occupation number is

$$\begin{aligned}
\langle Z|\hat{n}|Z\rangle &= \langle Z|2\hat{S}^0+\Omega|Z\rangle \\
&= 2S\left(1-\frac{2}{1+|Z|^2}\right)+\Omega. \qquad \text{(A.9)}
\end{aligned}$$

The expectation values for $\hat{S}^+$ and $\hat{S}^-$ are the complex conjugate pair. For $\hat{S}^+$,

$$\hat{S}^+e^{Z\hat{S}^+}|S,S^0=-S\rangle = \sum_{n=0}^{2S}\sqrt{(2S-n)(n+1)}\sqrt{\frac{(2S)!}{n!(2S-n)!}}Z^n|S,-S+n+1\rangle. \qquad \text{(A.10)}$$

hence,

$$\begin{aligned}
\langle Z|\hat{S}^+|Z\rangle &= (1+|Z|^2)^{-2S}\sum_{n=0}^{2S}Z^n\sqrt{(2S-n)(n+1)}\sqrt{\frac{(2S)!}{n!(2S-n)!}}\times(Z^*)^{n+1}\sqrt{\frac{(2S)!}{(n+1)!(2S-n-1)!}} \\
&= (1+|Z|^2)^{-2S}Z^*\sum_{n=0}^{2S}|Z|^{2n}\frac{(2S)!}{n!(2S-n-1)!} \\
&= \frac{2SZ^*}{1+|Z|^2}, \qquad \text{(A.11)}
\end{aligned}$$

and

$$\langle Z|\hat{S}^-|Z\rangle = \langle Z|\hat{S}^+|Z\rangle^* = \frac{2SZ}{1+|Z|^2}. \qquad \text{(A.12)}$$

For the expectation value of $\hat{S}^+\hat{S}^-$,

$$\begin{aligned}
\langle Z|\hat{S}^+\hat{S}^-|Z\rangle &= (1+|Z|^2)^{-2S}\|\hat{S}^-e^{Z\hat{S}^+}|S,S^0=-S\rangle\|^2 \\
&= (1+|Z|^2)^{-2S}\sum_{n=0}^{2S}n(2S-n+1)_{2S}\mathrm{C}_n|Z|^{2n} \\
&= (1+|Z|^2)^{-2S}\left\{(2S+1)2S(1+|Z|^2)^{2S-1}|Z|^2-2S(1+|Z|^2)^{2S}\left(\frac{(2S-1)|Z|^4}{(1+|Z|^2)^2}+\frac{|Z|^2}{1+|Z|^2}\right)\right\} \\
&= 2S|Z|^2\frac{2S+|Z|^2}{(1+|Z|^2)^2}. \qquad \text{(A.13)}
\end{aligned}$$

From the second line to the third line, we need to calculate the term $\sum_{n=0}^{2S}n^2{}_{2S}\mathrm{C}_n|Z|^{2n}$. It can be obtained from the following relation

$$|Z|^2\frac{\partial}{\partial|Z|^2}|Z|^2\frac{\partial}{\partial|Z|^2}(1+|Z|^2)^{2S} = \sum_{n=0}^{2S}n^2{}_{2S}\mathrm{C}_n|Z|^{2n}. \qquad \text{(A.14)}$$

When Z is time-dependent, the expectation value of $\frac{\partial}{\partial t}$ is also important quantity.

$$\begin{aligned}
\langle Z|\frac{\partial}{\partial t}|Z\rangle &= \langle Z|\dot{Z}\frac{\partial}{\partial Z} + \dot{Z}^*\frac{\partial}{\partial Z^*}|Z\rangle \\
&= -\frac{S}{1+|Z|^2}(Z^*\dot{Z} + Z\dot{Z}^*) + \dot{Z}\,\langle Z|\hat{S}^+|Z\rangle \\
&= \frac{S}{1+|Z|^2}(Z^*\dot{Z} - Z\dot{Z}^*) \qquad \text{(A.15)}
\end{aligned}$$

From the second line to the third line, we used (A.11).

Next, we consider the multi-level case. For one-body operators, the extension from the single-level case is simple because we only sum over the index l for the single-particle levels. For two-body operators, we need to give the new derivations basically. For the expectation value of $\hat{S}^+\hat{S}^-$,

$$\langle Z|\hat{S}^+\hat{S}^-|Z\rangle = \sum_{l_1 l_2}\langle Z|\hat{S}^+_{l_2}\hat{S}^-_{l_1}|Z\rangle\,. \qquad \text{(A.16)}$$

When $l_2 = l_1$, the form is the same as in (A.13). When $l_2 \neq l_1$, from (A.11) and (A.12),

$$\langle Z|S^+_{l_2}S^-_{l_1}|Z\rangle = \frac{2S_{l_1}Z_{l_1}}{1+|Z_{l_1}|^2}\frac{2S_{l_2}Z^*_{l_2}}{1+|Z_{l_2}|^2}. \qquad \text{(A.17)}$$

Therefore,

$$\langle Z|\hat{S}^+\hat{S}^-|Z\rangle = \sum_l 2S_l|Z_l|^2\frac{2S_l+|Z_l|^2}{(1+|Z_l|^2)^2} + \sum_{l_1\neq l_2}\frac{2S_{l_1}Z_{l_1}}{1+|Z_{l_1}|^2}\frac{2S_{l_2}Z^*_{l_2}}{1+|Z_{l_2}|^2}. \qquad \text{(A.18)}$$

We conclude the useful formula for the spin-coherent state as follows.

- Spin-coherent state

$$|Z\rangle = \prod_l \left(1+|Z_l|^2\right)^{-S_l} e^{Z_l\hat{S}^+_l}\,|S_l, S^0_l = -S_l\rangle\,. \qquad \text{(A.19)}$$

- Normalization

$$\langle Z|Z\rangle = 1 \qquad \text{(A.20)}$$

- Overlap

$$\langle\eta|Z\rangle = \prod_l \frac{(1+\eta^*_l Z_l)^{2S_l}}{(1+|\eta_l|^2)^{S_l}(1+|Z_l|^2)^{S_l}}. \qquad \text{(A.21)}$$

- Expectation value of $\hat{S}^0$

$$\langle Z|\hat{S}^0|Z\rangle = \sum_l S_l\left(1-\frac{2}{1+|Z_l|^2}\right) \qquad \text{(A.22)}$$

- Expectation value of $\hat{S}^+$

$$\langle Z|\hat{S}^+|Z\rangle = \sum_l \frac{2S_lZ^*_l}{1+|Z_l|^2} \qquad \text{(A.23)}$$

- Expectation value of $\hat{S}^-$

$$\langle Z|\hat{S}^-|Z\rangle = \sum_l \frac{2S_lZ_l}{1+|Z_l|^2} \qquad \text{(A.24)}$$

- Expectation value of $\hat{S}^{+}\hat{S}^{-}$

$$\langle Z|\hat{S}^{+}\hat{S}^{-}|Z\rangle = \sum_{l} 2S_l|Z_l|^2 \frac{2S_l+|Z_l|^2}{(1+|Z_l|^2)^2} + \sum_{l_1\neq l_2} \frac{2S_{l_1}Z_{l_1}}{1+|Z_{l_1}|^2}\frac{2S_{l_2}Z^*_{l_2}}{1+|Z_{l_2}|^2} \tag{A.25}$$

- Expectation value of $\frac{\partial}{\partial t}$

$$\langle Z|\frac{\partial}{\partial t}|Z\rangle = \sum_{l} \frac{S_l}{1+|Z_l|^2}(Z^*_l\dot{Z}_l - Z_l\dot{Z}^*_l) \tag{A.26}$$

A.2 Derivation of the path integral in spin-coherent state

We derive the path integral formulation for SU(2) spin-coherent state. First, we consider the case of one degree of freedom corresponding single-level system. Using the coherent state $|Z\rangle$, the time development from t' to t'' of the system is described by the propagator

$$\begin{aligned} K(Z'',t'';Z',t') &= \langle Z''|\exp\left\{-\frac{i}{\hbar}H(t''-t')\right\}|Z'\rangle &\text{(A.27)}\\ &= \lim_{n\to\infty}\int\prod_{k=1}^{n-1}d\mu(Z_k)\prod_{k=1}^{n}\langle Z_k|\exp\left\{-\frac{i}{\hbar}H\epsilon\right\}|Z_{k-1}\rangle\,, &\text{(A.28)}\end{aligned}$$

where we set $Z_n = Z''$, $Z_0 = Z'$, and $\epsilon = \frac{t''-t'}{n}$. Because ϵ is infinitesimal quantity, we can expand (A.28) with respect to ϵ up to first order

$$\begin{aligned}\langle Z_k|\exp\left\{-\frac{i}{\hbar}H\epsilon\right\}|Z_{k-1}\rangle &\approx \langle Z_k|1-\frac{i}{\hbar}\epsilon H|Z_{k-1}\rangle\\ &\approx \langle Z_k|Z_{k-1}\rangle\exp\left\{\frac{i}{\hbar}\epsilon\frac{\langle Z_k|H|Z_{k-1}\rangle}{\langle Z_k|Z_{k-1}\rangle}\right\}\\ &\approx \langle Z_k|Z_{k-1}\rangle\exp\left\{\frac{i}{\hbar}\epsilon\,\langle Z_k|H|Z_{k-1}\rangle\right\}. &\text{(A.29)}\end{aligned}$$

In addition, we expand the overlap up to first order

$$\begin{aligned}\langle Z_k|Z_{k-1}\rangle &= \exp(\log\kappa(Z_k,Z^*_k;Z_{k-1},Z^*_{k-1}))\\ &\approx \exp\left\{\frac{i}{\hbar}\left(\frac{\hbar}{i}\left(\frac{\partial\kappa}{\partial Z_k}\Delta Z_k+\frac{\partial\kappa}{\partial Z^*_k}\Delta Z^*_k\right)\right)\right\}, &\text{(A.30)}\end{aligned}$$

where we set $\Delta Z_k = Z_k - Z_{k-1} = \dot{Z}_k\epsilon$. Using (A.21), the differentiations of κ become

$$\left.\frac{\partial\kappa}{\partial Z_k}\right|_{Z_k=Z_{k-1}} = -\frac{SZ^*_k}{1+|Z_k|^2} \tag{A.31}$$

$$\left.\frac{\partial\kappa}{\partial Z^*_k}\right|_{Z^*_k=Z^*_{k-1}} = \frac{SZ_k}{1+|Z_k|^2} \tag{A.32}$$

Therefore, the propagator is written in path integral formulation

$$K(Z'',t'';Z',t') = \int \mathcal{D}\mu(Z(t)) e^{\frac{i}{\hbar}\mathcal{S}} \tag{A.33}$$

$$\mathcal{D}\mu(Z(t)) = \lim_{n\to\infty} \prod_{k=1}^{n-1} d\mu(Z_k) \tag{A.34}$$

$$\mathcal{S} = \int dt \left\{ \frac{i\hbar S}{1+|Z|^2}(Z^*\dot{Z} - Z\dot{Z}^*) - \langle Z|H|Z\rangle \right\} \tag{A.35}$$

Of course, the action $\mathcal{S}$ is identical to the definition

$$\mathcal{S} = \int dt \, \langle Z| i\hbar \frac{\partial}{\partial t} - H |Z\rangle \, . \tag{A.36}$$

It can be easily checked by using (A.26).

The extension to the multi-level system is straightforward. The action $\mathcal{S}$ becomes

$$\mathcal{S} = \int \mathcal{L}(t)dt, \quad \mathcal{L}(t) = i\hbar \sum_l \frac{S_l}{1+|Z_l|^2}(Z_l^*\dot{Z}_l - Z_l\dot{Z}_l^*) - \langle Z|H|Z\rangle \tag{A.37}$$

A.3 Action in canonical variables representation

We derived the action in the previous section. Using the formula in Sec. A.1, the expectation value of pairing Hamiltonian is

$$\begin{aligned}\langle Z|H|Z\rangle =& \sum_l \epsilon_l \left\{ 2S_l \left(1 - \frac{2}{1+|Z_l|^2}\right) + \Omega_l \right\} \\ & - g\sum_l 2S_l|Z_l|^2 \frac{2S_l + |Z_l|^2}{(1+|Z_l|^2)^2} + \sum_{l_1\neq l_2} \frac{2S_{l_1}Z_{l_1}}{1+|Z_{l_1}|^2}\frac{2S_{l_2}Z_{l_2}^*}{1+|Z_{l_2}|^2}.\end{aligned} \tag{A.38}$$

To obtain more tractable form of the action, we transform the complex variables Z_l into real variables (θ_l, χ_l) by $Z_l = \tan\frac{\theta_l}{2} e^{-i\chi_l}$ ($0 \le \theta \le \pi$, $0 \le \chi < 2\pi$). The Lagrangian becomes

$$\mathcal{L}(t) = \sum_l S_l(1-\cos\theta_l)\dot{\chi}_l - \langle Z|H|Z\rangle \, . \tag{A.39}$$

If we choose the χ_l as canonical coordinates, the conjugate momenta j_l are given by

$$j_l \equiv \frac{\partial \mathcal{L}}{\partial \dot{\chi}_l} = S_l(1-\cos\theta_l). \tag{A.40}$$

Therefore, the Lagrangian in canonical variables representation becomes

$$\mathcal{L}(t) = \sum_l j_l\dot{\chi}_l - \mathcal{H}(\chi, j), \tag{A.41}$$

$$\begin{aligned}\mathcal{H}(\chi,j) \equiv& \langle Z|H|Z\rangle \\ =& \sum_l \epsilon_l\nu_l + \sum_l 2\epsilon_l j_l - g\sum_l \left((\Omega_l - \nu_l)j_l - j_l^2 + \frac{j_l^2}{\Omega_l - \nu_l} \right) \\ & - g\sum_{l_1\le l_2} \sqrt{j_{l_1}j_{l_2}(\Omega_{l_1} - \nu_{l_1} - j_{l_1})(\Omega_{l_2} - \nu_{l_2} - j_{l_2})} \cos(\chi_{l_1} - \chi_{l_2}).\end{aligned} \tag{A.42}$$

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